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TOPICS IN NONLINEAR PDEs: FROM MEAN FIELD GAMES TO PROBLEMS MODELED ON HÖRMANDER VECTOR FIELDS

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Abstract

This thesis focuses on qualitative and quantitative aspects of some nonlinear PDEs arising in optimal control and differential games, ranging from regularity issues to maximum principles. More precisely, it is concerned with the analysis of some fully nonlinear second order degenerate PDEs over Hörmander vector fields that can be written in Hamilton-Jacobi-Bellman and Isaacs form and those arising in the recent theory of Mean Field Games, where the prototype model is described by a coupled system of PDEs involving a backward Hamilton-Jacobi and a forward Fokker-Planck equation. The thesis is divided in three parts.

The first part is devoted to analyze strong maximum principles for fully nonlinear second order degenerate PDEs structured on Hörmander vector fields, having as a particular example fully nonlinear subelliptic PDEs on Carnot groups. These results are achieved by introducing a notion of subunit vector field for these nonlinear degenerate operators in the spirit of the seminal works on linear equations. As a byproduct, we then prove some new strong comparison principles for equations that can be written in Hamilton-Jacobi-Bellman form and Liouville theorems for some second order fully nonlinear degenerate PDEs.

The second part of the thesis deals with time-dependent fractional Mean Field Game systems. These equations arise when the dynamics of the average player is described by a stable Lévy process to which corresponds a fractional Laplacian as diffusion operator. More precisely, we establish existence and uniqueness of solutions to such systems of PDEs with regularizing coupling among the equations for every order of the fractional Laplacian $s \in (0,1)$. The existence of solutions is addressed via the vanishing viscosity method and we prove that in the subcritical regime the equations are satisfied in classical sense, while if $s \leq 1/2$ we find weak energy solutions. To this aim, we develop an appropriate functional setting based on parabolic Bessel potential spaces. We finally show uniqueness of solutions both under the Lasry-Lions monotonicity condition and for short time horizons.

The last part focuses on the regularizing effect of evolutive Hamilton-Jacobi equations with Hamiltonian having superlinear growth in the gradient and unbounded right-hand side. In particular, the analysis is performed both for viscous Hamilton-Jacobi equations and its fractional counterpart in the subcritical regime via a duality method. The results are accomplished exploiting the regularity of solutions to Fokker-Planck-type PDEs with rough velocity fields in parabolic Sobolev and Bessel potential spaces respectively.

Contents

Notations

$\mathbb{R}^d$	The d-dimensional Euclidean space, $d \geq 1$.
$\mathbb{T}^d$	The d-dimensional flat torus $\mathbb{R}^d/\mathbb{Z}^d$, $d \geq 1$.
$B(x,r)$	The open ball of radius r centered at x.
Ω	will always be a bounded domain of $\mathbb{R}^d$.
$\partial\Omega$	boundary of Ω, namely $\partial\Omega = \overline{\Omega}\backslash\Omega$.
$\mathrm{USC}(\Omega)$	The space of upper semicontinuous functions on Ω.
$\mathrm{LSC}(\Omega)$	The space of lower semicontinuous functions on Ω.
$C(\Omega)$	The space of continuous functions on Ω.
$C^k(\Omega)$	The space of continuous functions on Ω with continuous derivatives of order j, $j = 1, \dots, k$.
$\mathcal{P}(\Omega)$	The space of functions $m \in L^1(\Omega)$ such that $\int_\Omega m = 1$ (probability densities).
$\partial_i f$, Df	Partial derivatives with respect to the i-th variable and gradient vector of f.
$\partial_t f$	Partial derivative with respect to the time-variable.
$D^2 f$	Hessian matrix of f.
$\mathrm{Tr}(A)$	Trace of a square matrix A
$v \otimes w$	matrix obtained by the tensor product $\otimes : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ of two vectors $v, w \in \mathbb{R}^d$ whose elements are $v \otimes w = (v_i w_j)_{i,j=1}^d$.
$\mathcal{S}_d$	The space of square $d \times d$ symmetric matrices with real entries.
$e_i(A)$	The i-th eigenvalue of $A \in \mathcal{S}_d$, $i = 1, \dots, d$, with ordering $e_1(A) \leq \dots \leq e_d(A)$. Sometimes we will drop the matrix variable inside the brackets when it results implicit from the context.
a.e.	almost everywhere
$C_0^\infty(X)$	space of smooth functions with compact support on X.
$\chi_A(x)$	characteristic function of A defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$
$\mathcal{L}(X,Y)$	Banach space of linear continuous operators from the Banach space X to the Banach space Y equipped with the norm topology. When $X = Y$ we only write $\mathcal{L}(X)$.
X'	dual of the space X
$\langle x', x\rangle_{X',X}$	duality product of $x' \in X'$ and $x \in X$.
$X \hookrightarrow Y$	if $X \subset Y$ with continuous injection.
$\mathcal{F}$	Fourier transform $\mathcal{F}u(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} u(x)dx$
$L^p(\Omega)$	Banach spaces of (classes) of measurable functions $f : \Omega \to \mathbb{R}$ such that $\|f\|_p < \infty$ with $\|u\|_p := \left(\int_\Omega f(x)^p\,dx\right)^{\frac{1}{p}}$ if $p < \infty$ and $\|f\|_\infty := \mathrm{ess\,sup}_\Omega f$ if $p = \infty$.

$D(A)$	Domain of the operator $A : D(A) \to X$, namely a linear subspace of X equipped with the graph norm $\|x\|_A = \|x\| + \|Ax\|$.
$(X,Y)_{\theta,p}$	the real interpolation spaces.
$J_\theta(X,Y)$	the class J_θ between X and Y.
$K(t,x,X,Y)$	the K function.
$K_\theta(X,Y)$	the class K_θ between X and Y.
$C^\alpha(\mathbb{T}^d)$	the space of α-Hölder continuous functions on $\mathbb{T}^d$.
$B^\mu_{p,q}(\Omega)$	the Besov spaces.
$H^\mu_p(\Omega)$	the Bessel potential spaces.
$W^{\mu,p}(\Omega)$	the fractional Sobolev spaces.
$W^{1,p}(I;X)$	the vector-valued Sobolev spaces, $I \subset \mathbb{R}$ open set.
$C(I;X)$	Space of continuous functions $u : I \to X$, $I \subseteq \mathbb{R}$, equipped with the norm $\|u\|_{C(I;X)} := \max_{t\in I} \|u(t)\|_X$.
$L^p(I;X)$	Space of all measurable functions $u : I \to X$, $I \subset \mathbb{R}$ open set, with respect to the equivalence relation $f \sim g \iff f(t) = g(t)$ for a.e. $t \in I$, such that $t \longmapsto \|u(t)\|_X$ belong to $L^p(I)$. It is endowed with the norm $\|u\|_{L^p(I;X)} := \left(\int_I \|u(t)\|^p_X \, dx\right)^{\frac{1}{p}}$ if $p < \infty$ and $\|u\|_\infty :=$ ess $\sup_{t\in I}\|f(t)\|_X$ if $p = \infty$.
$\mathcal{H}^\mu_p(Q_T)$	space of measurable functions $u \in L^p(0,T; H^\mu_p(\Omega))$ with $\partial_t u \in L^p(0,T; H^{\mu-2}_p(\Omega))$, being $Q_T = \Omega \times (0,T)$ and $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$. When $\mu = 2$ the space $\mathcal{H}^2_p \simeq W^{2,1}_p$.
$\mathcal{H}^{\mu,s}_p(Q_T)$	space of measurable functions $u \in L^p(0,T; H^\mu_p(\Omega))$ with $\partial_t u \in L^p(0,T; H^{\mu-2s}_p(\Omega))$, $s \in (0,1)$.

Some useful inequalities

Space-time Hölder's inequality: Let $I \subseteq \mathbb{R}$, X be a Banach space and denote by X' its dual. If $f \in L^p(I;X)$ and $g \in L^q(I;X')$ with $1/p + 1/q = 1/r$ and $u(t) = \langle f(t), g(t) \rangle_{X',X}$, then $u(t) \in L^r(I)$ and

$$\|u\|_{L^r(I)} \leq \|f\|_{L^p(I;X)} \|g\|_{L^q(I;X')}$$

Generalized Young's inequality: For $p \in (1, \infty)$ and $p' = p/(p-1)$ and any positive $\epsilon > 0$ we have

$$ab \leq \epsilon^p \frac{a^p}{p} + \frac{1}{\epsilon^{p'}} \frac{b^{p'}}{p'} \ \forall a, b > 0 \ .$$

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Introduction

This thesis collects new developments on the analysis of some nonlinear elliptic and parabolic partial differential equations (briefly PDEs) arising in optimal control problems and differential games. More precisely, the manuscript is divided in three parts, each of which is divided into chapters corresponding to different papers as follows:
Part I:

- M. Bardi and A. Goffi, *New strong maximum and comparison principles for fully nonlinear degenerate elliptic PDEs*, Calc. Var. Partial Differential Equations 58 (184), 2019.
- M. Bardi and A. Goffi, *A note on the strong maximum principle for fully nonlinear parabolic PDEs*, forthcoming.
- M. Bardi and A. Goffi, *Liouville results for fully nonlinear equations modeled on Hörmander vector fields*, forthcoming.

Part II:

- M. Cirant and A. Goffi, *On the existence and uniqueness of solutions to time-dependent fractional MFG*, SIAM J. Math. Anal. 51 (2) 913-954.

Part III:

- M. Cirant and A. Goffi, *Lipschitz regularity for viscous Hamilton-Jacobi equations with L^p terms*, arXiv:1812.03706, submitted.
- A. Goffi, *Transport equations with nonlocal diffusion and applications to Hamilton-Jacobi equations*, forthcoming.

More precisely, the first part is devoted to analyze strong maximum principles for some fully nonlinear second order degenerate equations with the aim of providing also some applications to comparison principles and Liouville-type results. It is independent from the others, being mainly based on viscosity solutions' theory for such nonlinear PDEs. The second part contains only one wide chapter concerning the analysis of evolutive Mean Field Game (MFG) systems driven by fractional diffusion and three related appendices (Appendix A, Appendix B and Appendix C) regarding some regularity aspects for parabolic Hölder and Bessel potential spaces in the periodic setting for the fractional heat operator, together with chain and product rules on fractional spaces and some embedding theorems for the aforementioned Bessel functional classes. The last part is devoted to study regularity issues for Hamilton-Jacobi equations with classical and nonlocal diffusion with rough terms via duality methods.

Part I- Strong maximum principles for fully nonlinear degenerate equations and Liouville theorems for subelliptic problems

The Strong Maximum Principle (SMP) for elliptic equations goes back to the seminal work by E. Hopf and, since then, it has had an increasing interest for nonlinear equations, see e.g. the treatise [122] and the references therein. Such results heavily rely on the uniform ellipticity of the operator rather than the regularity of the coefficients and the particular structure of the PDE. In the case of the second order equation

$$Lu(x) = -\sum_{i,j} a_{ij}(x)\partial_{ij}u + \sum_i b_i(x)\partial_i u + c(x)u = 0$$

with $A = (a_{ij}(x)) : \Omega \to \mathcal{S}_d$, $A > 0$, $b : \Omega \to \mathbb{R}^d$ bounded and continuous and c nonnegative and bounded, E. Hopf proved the SMP via the so-called Boundary Point Lemma. In particular, the latter states that if $u \in C^1(\overline{\Omega})$, $\partial\Omega$ is smooth, $Lu \leq 0$ in Ω and u attains a nonnegative maximum at some point $x_0 \in \partial\Omega$, then $\frac{\partial u}{\partial \nu}(x_0) > 0$ for any vector ν pointing outward from Ω at x_0, provided that $\partial\Omega$ satisfies an interior sphere condition at x_0 (see [122, Lemma 3.4]). As an immediate consequence, one gets the SMP, which asserts that a subsolution to a homogeneous equation $Lu = 0$ in an open connected set $\Omega \subseteq \mathbb{R}^d$ that attains a nonnegative maximum at an interior point $x_0 \in \Omega$ must be constant, under the right choice of the sign of the coefficient of the zero-th order term [122, Theorem 3.5].

In the case of degenerate elliptic operators, the way to deduce the SMP is more delicate and the problem can be formulated as follows. Let $x_0 \in \Omega$ be given.

Is it always possible to determine a subset $D(x_0)$ of Ω such that if u is a subsolution to a degenerate elliptic equation and u has a local maximum at x_0, then $u \equiv u(x_0)$ throughout $D(x_0)$?

The answer is yes. However, it turns out that the set $D(x_0) := \mathrm{Prop}(x_0)$, usually called *propagation set*, does not necessarily coincide with the whole Ω as in the uniformly elliptic case. The seminal contributions on degenerate elliptic linear equations are due to J.-M. Bony [52], D. W. Stroock and S.R.S. Varadhan [225]. They consider the above linear equation $Lu(x) = 0$ with smooth coefficients, $A \geq 0$ (i.e. positive semidefinite) and $c \geq 0$. Labeling by X_j, $j = 1, ..., m$, the $j - th$ column of A, one defines the so-called *drift vector field* (also named *subprincipal part* of the operator L) as

$$X_0 := \sum_{i=1}^{d} (b_i(x) - \partial_i a_{ij}(x))\partial_i \ .$$

C. D. Hill [135] described $\mathrm{Prop}(x_0)$ in terms of all points that can be reached from x_0 following a finite number of trajectories of X_j backward and forward in time and of X_0 backward in time (we recall that a drift trajectory is a curve $\theta : [t_1, t_2] \to \Omega$ such that $\theta'(t) = X_0(\theta(t))$ on $[t_1, t_2]$ oriented for increasing time). J.-M. Bony characterized the propagation set $\mathrm{Prop}(x_0)$ for operators satisfying the Hörmander condition [52, Corollary 3.1], saying that the Lie algebra $\mathcal{L}(X_1, ..., X_m)$ generated by the vector fields X_j has full rank at every point of Ω, showing that $\mathrm{Prop}(x_0) = \Omega$, and hence the validity of the SMP for classical subsolutions to $Lu = 0$. D. W. Stroock and S.R.S. Varadhan proposed a description of the propagation set via probabilistic methods. Our result is instead inspired by the work of K. Taira, who proved by purely analytical methods a characterization in terms of the *subunit vector fields* associated to the linear operator L. According to C. Fefferman and D.H. Phong [111], a subunit vector field Z associated to the linear operator $-\mathrm{Tr}(A(x)D^2u)$ verifies the inequality $A - Z \otimes Z \geq 0$. K. Taira proved that the propagation set can be described in terms of all points that can be reached from x_0 following a finite number of trajectories of the subunit vector fields backward and forward in time and of the drift vector field X_0 backward in time (cf [226, Theorem 7.2.1]). See also Section 2.1 for further details and references.

In the context of viscosity subsolutions to second order fully nonlinear uniformly elliptic equations, the SMP was proved by L. Caffarelli and X. Cabré [62] as a consequence of the Harnack inequality. Under lower ellipticity assumptions it was derived in a more direct way in [148] (in a weaker form) and [21]. Control theoretic and probabilistic descriptions of the propagation set for Hamilton-Jacobi-Bellman equations were given in [22] and [23]. Our SMP for such equations, Corollary 2.32, is derived in a simpler way and extends also to Isaacs equations, see Section 3.4.3. Here, we are interested in fully nonlinear equations of the general form

$$F(x, u, Du, D^2u) = 0 \ ,$$

where $x \in \Omega$ and u is a function defined in Ω and $F(x, r, p, X)$ is a real-valued function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d$. In particular, we will focus on some specific PDEs of this form structured over Hörmander vector fields, which indeed are degenerate if represented in Euclidean coordinates. A particular case of vector fields satisfying the Hörmander condition are those families generating a Carnot group. The theory of such fully nonlinear PDEs, usually named *subelliptic*, began with [179] and [40], see also [34, 42, 234] and our Corollary 2.7 seems to be the first Strong Maximum Principle for such equations.

As announced, here we review and extend the concept of subunit vector field to fully nonlinear operators $F = F(x, r, p, X)$ (see Definition 2.3), where F is a real-valued function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d$, providing a nonlinear analogue to the description of the propagation set (see Theorem 2.4). This implies the SMP for subunit vector fields associated to F satisfying the Hörmander condition, see Corollary 2.6. Our results cover various fully nonlinear subelliptic equations arising in stochastic control problems. In particular, our main examples are the Hamilton-Jacobi-Bellman (HJB) and Isaacs (HJI) equations coming from stochastic control and differential games whose dynamics is described by the stochastic differential equation (SDE)

$$dX_t = b(X_t, \alpha, \beta)dt + \sigma(X_t, \alpha, \beta)dB_t$$

with σ taking values in $\mathbb{R}^{d\times m}$, B_t standing for a m-dimensional Brownian motion and α, β taking value in some compact sets A and B respectively. Here, if one considers a running cost functional $l(X_t, \alpha, \beta)$ and a discount rate $c(X_t, \alpha, \beta)$, then the PDE associated to the value function turns out to be of the form

$$\max_{\alpha\in A}\min_{\beta\in B}\{-\mathrm{Tr}(\sigma\sigma^T D^2u) - b\cdot Du + cu - l\} = 0 \ .$$

The particular case in which A is a singleton and $\sigma(x,\beta) = \sigma(x)\beta$ with $\beta \in \{\beta \in \mathcal{S}_m : \sqrt{\lambda}I \leq \beta \leq \sqrt{\Lambda}I\}$ leads to a fully nonlinear equation driven by the so-called Pucci's minimal operator $\mathcal{M}^-$, and with the maximal operator $\mathcal{M}^+$ simply by reversing the roles of the controls. Moreover, the case of uncontrolled diffusion matrix $\sigma = \sigma(x)$ leads to a quasi-linear subelliptic equation. Our properties turn out to be new even for some of these quasi-linear PDEs, including the subelliptic p- and ∞- Laplace equations. In particular, our results in Section 3.4.2 for Hamilton-Jacobi-Bellman equations improve upon [22, 23], giving a more direct characterization of the strong maximum and minimum principles, while those in Section 3.4.3 for Hamilton-Jacobi-Isaacs equations, which are neither convex nor concave, seems to be not yet explicitly written down anywhere in the literature, although they can be obtained via similar viscosity arguments used in the aforementioned contributions by M. Bardi and F. Da Lio. Still, our results are completely new for all classes of nonlinear subelliptic PDEs modeled on families of vector fields $X_1, ..., X_m$ satisfying the Hörmander condition, that appear in the compact form

$$G(x, u, D_{\mathcal{X}}u, (D^2_{\mathcal{X}}u)^*) = 0 \text{ in } \Omega \ ,$$

where $(D^2_{\mathcal{X}}u)^*$ is the symmetrized Hessian matrix of $D^2_{\mathcal{X}}u = X_iX_ju$, $i,j = 1, ..., m$, $D_{\mathcal{X}}u$ is the intrinsic gradient and $G : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^m \times \mathcal{S}_m \to \mathbb{R}$ is proper. See Subsection 1.0.5 and Subsection 2.4.1 for explicit examples and properties.
An immediate consequence of the SMP for linear equations is the so-called Strong Comparison Principle. In the case of linear operators it can be stated as follows: let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $Lu \leq 0$ and $Lv \geq 0$ in a open connected set Ω. If $u \leq v$ in Ω, then either $u \equiv v$ or $u < v$ in Ω. In the fully nonlinear framework, such property was found by N.S. Trudinger [233] for Lipschitz viscosity solutions of uniformly elliptic equations. However, very little is known for degenerate equations: they concern particular PDEs motivated by geometric problems [121, 189, 170, 67] and are quite different from our Theorem 2.41. On the other hand, the literature on the (weak) Comparison Principle is huge: the results are very general if F is strictly proper (i.e., strictly increasing in r) since they include first order equations, see [96, 15]. Under the mere properness (see (i) in Section 2.2), instead, some ellipticity is needed and the minimal conditions are an open problem, see [143, 29, 148, 149], and [26, 40, 180, 25, 41] for equations involving Hörmander vector fields, see also the references therein. Our Corollary 2.42 completes the results of [25].
We apply similar strategies to analyze the SMP for the evolutive operator $\partial_t + F$, following the seminal work [191] (see also [117, 116]) and then adapted in the context of viscosity solutions' to fully nonlinear parabolic problems in [100] (see also [60]). In Chapter 3 we review and generalize the results in [100].

The classical Liouville theorem for harmonic functions on the whole space states that the only harmonic functions in $\mathbb{R}^d$ bounded from above or below are constants, and it is a consequence of the Harnack inequality (see e.g. [122]). Such result actually holds for classical solutions to more general uniformly elliptic equations. The Liouville property holds also in the much larger class of merely subharmonic functions (i.e. subsolutions) if the space dimension is $d = 2$, by exploiting the behavior of the fundamental solution $\log|x|$ and using the Hadamard Three-Circle Theorem (see, e.g., [197, Theorem 2.29] and Theorem 4.1 below for a different proof). However, this result crucially fails in higher dimensions $d \geq 3$. Indeed, straightforward computations show that the functions $u_1(x) := -(1 + |x|^2)^{-1/2}$ and $u_2(x) := -(1 + |x|^2)^{-1}$ are nonpositive nontrivial subharmonic functions in $\mathbb{R}^3$ and, respectively, in $\mathbb{R}^d$ with $d \geq 4$.

Linear degenerate elliptic equations are studied in [51, Section 5.8], which gives Liouville theorems for solutions to equations driven by sub-Laplacians; they are again deduced from a suitable Harnack inequality. In the case of subsolutions (or supersolutions) to $-\Delta_{\mathcal{X}} u = 0$ in $\mathbb{R}^m$, where $\mathcal{X} = \{X_1, ..., X_m\}$ is a system of vector fields satisfying the Hörmander condition (e.g. those generating a Carnot group, the simplest and most popular instance being the Heisenberg vector fields) we give simple explicit examples of bounded non-constant classical sub- and supersolutions of the sub-Laplace equation in any Heisenberg group $\mathbb{H}^d$, see Section 4.3, and in the Grushin plane, see Section 4.5, showing the failure of the Liouville property in this setting. However, it can be recovered for super polyharmonic functions, i.e. solutions to $(-1)^r \Delta^r_{\mathbb{H}^d} u \geq 0$ in $\mathbb{R}^{2d+1}$ for $r = 1, ..., k$, $k > 1$, if $2k \geq Q$, $Q = 2d + 2$ being the homogeneous dimension of the Heisenberg group, see [44, Theorem 1.6].

Liouville theorems were then widely investigated in the context of semilinear elliptic equations and we refer to the survey [68] and the references therein. We quote also [153, 152, 69, 45] for PDEs on Carnot groups and [2] for some results on the Heisenberg p-Laplacian equation.

In this manuscript, we are mostly interested in Liouville properties for viscosity sub- and supersolutions of fully nonlinear degenerate elliptic equations. In the case of uniformly elliptic equations of the form

$$F(x, D^2u) = 0 \text{ in } \mathbb{R}^d \tag{1}$$

it was proved in [62, Remark 4.2.4] that continuous viscosity solutions either bounded from above or below are constants. This is a consequence of the Harnack inequality combined with the comparison with Pucci's extremal operators

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq F(x, D^2u) - F(x, 0) \leq \mathcal{M}^+_{\lambda,\Lambda}(D^2u), \tag{2}$$

as usual in the theory of fully nonlinear second order uniformly elliptic equations. Further related results for solutions to Hessian PDEs of the form $F(D^2u) = 0$ can be found in [190, Section 1.7] and [9, Theorem 1.7]. We remark that to get Liouville properties for solutions to Hessian equations of the form $F(D^2u) = 0$, the assumption $F(0) = 0$ is crucial and cannot be dropped when $d \geq 5$ (see [190, Section 1.7]), although it is conjectured that for lower dimensions $d \leq 4$ the Liouville property should hold without having this hypothesis in force (cf [190, Conjecture 1.7.1])

The first results for mere sub- or supersolutions of (1) are due to A. Cutrì and F. Leoni [98]. They proved that if $u \in C(\mathbb{R}^d)$ is either bounded below and satisfying

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \leq 0 \text{ in } \mathbb{R}^d \tag{3}$$

in viscosity sense, or bounded above and satisfying

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \geq 0 \text{ in } \mathbb{R}^d \tag{4}$$

in viscosity sense, then u is constant provided that $d \leq \frac{\Lambda}{\lambda}+1$, $\mathcal{M}^\pm_{\lambda,\Lambda}$ standing for the Pucci's extremal operators with parameters $\Lambda \geq \lambda > 0$ [62, 201]. This can be seen as the fully nonlinear analogue of the Liouville theorem for subharmonic functions, since when $\lambda = \Lambda$ one gets the Laplacian (up to constants) and the constraint reads $d \leq 2$. Such conditions are known to be sharp: examples of nontrivial solutions to Pucci's extremal equations when $d > \frac{\Lambda}{\lambda} + 1$ can be found in [98, Remark 2] and will be recalled in Section 4.3.1.

This result was extended to the Heisenberg group $\mathbb{H}^d$ in [99, Theorem 5.2] for the inequalities (3) and (4) on $\mathbb{R}^{2d+1}$ with D^2u replaced by $D^2_{\mathbb{H}^d}u$. Here the condition $d \leq \frac{\Lambda}{\lambda} + 1$ is replaced by $Q \leq \frac{\Lambda}{\lambda} + 1$, Q being the aforementioned homogeneous dimension of the Heisenberg group. A counterexample to the Liouville property when $Q > \frac{\Lambda}{\lambda}+1$ is in Section 4.3.1 and this seems to be new to our knowledge. This is consistent with the failure of Liouville properties for subharmonic functions in the Heisenberg group, which can be formally seen via the fundamental solution (see e.g. [51]); however, in Section 4.3.1 we provide a new explicit counterexample built on "radial" functions to show that the Liouville property is false in the linear case.

Liouville results were then found in the fully nonlinear case in [98] by adding a semilinear perturbation term to the fully nonlinear uniformly elliptic operator. More precisely, it is proved that there exists a number $\bar{p} > 0$, depending on λ/Λ and d, such that the only nonnegative viscosity supersolution u to

$$F(x, D^2u) + u^p = 0 \text{ in } \mathbb{R}^d,$$

with $F(x, 0) = 0$ and $p \in (0, \bar{p})$, is $u \equiv 0$. The results were then generalized in [8]. Such properties for equations involving gradient terms were first investigated in [70] for PDEs of the form $F(x, D^2u) + g(|x|)|Du| + h(x)u^p = 0$, specifically by adding a sublinear first order assumption on the gradient term, that is assuming that the drift term g is bounded and such that

$$-\frac{\Lambda(d-1)}{|x|} \leq g(|x|) \leq \frac{\lambda - \Lambda(d-1)}{|x|}$$

for $|x|$ large. We refer also to [210], [195] and [85] for Liouville results to fully nonlinear PDEs with gradient dependence.

A new approach to Liouville properties for sub- and supersolutions of Hamilton-Jacobi-Bellman equations involving operators of Ornstein-Uhlenbeck type was initiated in [17], based on the strong maximum principle and the existence of a sort of Lyapunov function for the equation, that is a sub- and supersolution to the equation respectively that blow-up at infinity. This leads to assumptions on the sign of the coefficients of the first and zero-th order terms, and on their size, that must be large enough for large $|x|$, contrary to the results quoted above, see [16]. Fully nonlinear uniformly elliptic equations $F(x, u, Du, D^2u) = 0$ are treated in [16], a linear degenerate case in [181] and some quasilinear hypoelliptic equations in [16]. A particular result of [16] concerns the inequalities

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq 0 \text{ in } \mathbb{R}^d, \qquad \mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq 0 \text{ in } \mathbb{R}^d,$$

and states that $u \in C(\mathbb{R}^d)$ either bounded above and solving the former, or bounded below and satisfying the latter, must be constant if $d \leq \frac{\lambda}{\Lambda}+1$. This complements the result of [98] on (3) and (4); note that the restriction on d is more stringent now but is still sharp for the Laplacian, $\lambda = \Lambda$. Still, we remark that this second constraint on d is sharp, see Section 4.3.1. Note also that this result fits better the treatment of uniformly elliptic equations via the inequalities (2).

Chapter 4 develops Liouville-type results in the spirit of [16] for fully nonlinear inequalities involving the intrinsic (or horizontal) gradient and intrinsic Hessian

$$D_{\mathcal{X}}u = (X_1 u, ..., X_m u), \quad (D^2_{\mathcal{X}}u)_{ij} = X_i(X_j u),$$

associated to a given family $\mathcal{X} = (X_1, ..., X_m)$ of $C^{1,1}$ vector fields satisfying the Hörmander condition. Our main motivation are subelliptic equations of the form

$$G(x, u, D_{\mathcal{X}}u, (D^2_{\mathcal{X}}u)^*) = 0 \text{ in } \mathbb{R}^d \ , \tag{5}$$

where Y^* is the symmetrized matrix of Y and $G : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^m \times \mathcal{S}_m \to \mathbb{R}$ is proper. We will give some sufficient conditions for the Liouville properties

any subsolution (resp. supersolution) of (5) bounded from above (resp. below) is a constant.

The main assumption on G is a comparison with Pucci's extremal equations asociated with the vector fields $\mathcal{X}$

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*) \leq G(x, u, D_{\mathcal{X}}u, (D^2_{\mathcal{X}}u)^*) - G(x, u, D_{\mathcal{X}}u, 0) \leq \mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*) \ .$$

Then a subsolution of (5) is also a subsolution to

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*) + G(x, u, D_{\mathcal{X}}u, 0) = 0,$$

and a supersolution of (5) solves a similar inequality for the maximal operator $\mathcal{M}^+$. A mild subadditivity condition is also required for G.

The main new tools we will use in Chapter 4 are the strong maximum and minimum principles for fully nonlinear subelliptic equations obtained in Chapter 2 via the generalization of the concept of subunit vector field to the fully nonlinear setting (cf Definition 2.3). Moreover, we use suitable homogeneous norms associated to the vector fields $\mathcal{X}$ to build appropriate Lyapunov functions. The results are made explicit in three cases: the Heisenberg group, free step 2 Carnot groups with r-generators [120, 55], and Grushin-type geometries, where no group structure is available.

An example of our results, in the case of the Heisenberg group $\mathbb{H}^d \simeq \mathbb{R}^{2d+1}$, is the following: if

$$G(x, r, p, X) \geq \mathcal{M}^-_{\lambda,\Lambda}(X) + \inf_{\alpha \in A}\{c^\alpha(x) r - b^\alpha(x) \cdot p\} \ ,$$

we prove the Liouville property for subsolutions under the condition

$$\sup_{\alpha \in A}\{b^\alpha(x) \cdot \frac{\eta}{|x_H|^2} - c^\alpha(x) \frac{\rho^4}{|x_H|^2} \log \rho\} \leq \lambda - \Lambda(Q-1)$$

for large x, where $x_H := (x_1, ..., x_{2d})$, b^α takes values in $\mathbb{R}^{2d}$, $\eta \in \mathbb{R}^{2d}$ is defined by $\eta_i = x_i|x_H|^2 + x_{i+d}x_{2d+1}$, $\eta_{i+d} = x_i|x_H|^2 - x_i x_{2d+1}$, for $i = 1, ..., d$, ρ is a suitable 1-homogeneous norm with respect to the dilations of the group, and $Q = 2d + 2$ is the homogeneous dimension of $\mathbb{H}^d$. This condition is satisfied if $c^\alpha \geq 0$ and $b^\alpha(x) \cdot \eta \leq 0$ for x large, and under suitable growth conditions at infinity. Note also that for subsolutions of $\mathcal{M}^-_{\lambda,\Lambda}(D^2_{\mathbb{H}^d}u) \leq 0$ the condition with $c^\alpha \equiv 0$, $b^\alpha \equiv 0$ becomes $Q \leq \frac{\lambda}{\Lambda} + 1$, as expected from the Euclidean case treated in [16] and recalled above (cf. also the aforementioned result for $\mathcal{M}^+_{\lambda,\Lambda}(D^2_{\mathbb{H}^d}u) \leq 0$ in [99]). In Section 4.3.1 we give an example showing that this condition is sharp.

Our study is motivated by applications of Liouville properties to various issues, such as ergodic problems, large time stabilization in parabolic equations (see e.g. [16, Section 5-6]), and regularity theory for fully nonlinear second order PDEs [138]. As for fully nonlinear degenerate equations, we also mention the recent paper [169] in the context of PDEs arising in conformal geometry and [46] for Liouville properties of solutions to degenerate versions of Pucci's extremal equations.

Part II- Time-dependent fractional Mean Field Games

In the second part of this manuscript, we focus on evolutive systems arising in the recent theory of MFGs, which was developed almost simultaneously by P.-L. Lions and J.-M. Lasry [163] and M. Huang, P. Caines and R. Malhamé [137], and aiming at describing Nash equilibria in differential games with infinitely many players, each of whom having a negligible impact on the overall system. In particular, the heuristic interpretation of these models is the following: each player controls the dynamics described by the following SDE

$$dX_t = \alpha_t\, dt + \sqrt{2}dB_t \ , X_0 = x \ ,$$

where B_t stands for a classical Brownian motion, through the control α_t. The goal of each agent is to minimize, over $\alpha_t \in \mathcal{A}$, $\mathcal{A}$ being the set of controls, the cost

$$J(x,t,\alpha) = \mathbb{E}\left[\int_t^T L(X_\tau, m(\tau), \alpha_\tau)d\tau + u_T(x)\right] \ ,$$

where m denotes the density of the players, which in turn evolves according to a transport equation, as described in the next lines. By classical dynamic programming arguments it turns out that the value function associated to each player, namely $u(x,t) := \inf_{\alpha\in\mathcal{A}} J(x,t,\alpha)$, solves a viscous Hamilton-Jacobi-type equation. Roughly speaking, the value function indicates how the agent should choose his/her control in order to behave in an optimal way. At least formally, by verification arguments one easily finds an optimal feedback $\alpha^* = -D_pH(x, Du)$, which depends on the (evolutive) family of probability measures $\{m(t)\}$. Under the assumptions that each player controls the same dynamics and minimizes the same cost, owing to the optimal control α^*, the population density evolves according to the previous SDE with drift $\alpha = \alpha^*$, leading to a Fokker-Planck type equation describing the collective behavior of the agents. As a byproduct, coupling the above PDEs, the classical MFG system takes the form

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = F[m(t)](x) & \text{in } Q_T = \mathbb{T}^d \times (0,T), \\ \partial_t m - \Delta m - \mathrm{div}(m D_p H(x, Du)) = 0 & \text{in } Q_T = \mathbb{T}^d \times (0,T), \\ m(x,0) = m_0(x),\ u(x,T) = u_T(x) & \text{in } \mathbb{T}^d\ , \end{cases}$$

even though, usually, a further coupling through the terminal data $u(x,T)$ is present in the standard theory (which we avoid here for the sake of presentation). Recently, this theory has been spread in several different directions, which regards on the one hand analytic and probabilistic issues (see [78, 79, 80]) and, on the other hand, applications to engineering, finance and social sciences, among others. From the PDE viewpoint, the analysis of such models has been carried out either when the dynamics of the average player is driven by standard diffusions (see for example [127, 163]), possibly degenerate [73], or first order (deterministic) systems (see e.g. [72, 75]). The purpose of this part is thus to analyze the intermediate situation where the dynamics of agents is driven by a jump-diffusion process. More precisely, we study a time-dependent model situation in which the underlying dynamics is driven by a $2s$-stable Lévy process, which gives rise to a fractional Laplacian as a diffusion operator. We aim at providing an analytical model to study more general PDEs and systems driven by integro-differential operators (see e.g. [83, Section 5] for some MFG models in this direction). More precisely, the system we are going to analyze in Chapter 5 is of the form

$$\begin{cases} -\partial_t u + (-\Delta)^s u + H(x, Du) = F[m(t)](x) & \text{in } Q_T \\ \partial_t m + (-\Delta)^s m - \mathrm{div}(m D_p H(x, Du)) = 0 & \text{in } Q_T \\ m(x,0) = m_0(x),\ u(x,T) = u_T(x) & \text{in } \mathbb{T}^d\ , \end{cases}$$

where H is a superlinear Hamiltonian in the gradient Du (see assumptions (H1F)-(H5F) below) behaving like $|Du|^\gamma$, $\gamma > 1$, F is a smoothing operator, m_0, u_T are sufficiently smooth data and $(-\Delta)^s$ is a fractional Laplacian of order $s \in (0,1)$. Its stationary counterpart has been recently analyzed in [81], using different techniques than those we are going to present in this manuscript. We further mention the recent analysis [65] of MFG systems with time-fractional derivatives arising from subdiffusive dynamics of the players.
Lévy processes meet a variety of challenging topics ranging from financial modeling (see e.g. the monograph [94]) to Physics and Biology among others. We refer to [38, 211] for a comprehensive treatment of stable-like processes, to the monograph [6] for a more general analysis on jump-type processes, the nice survey [5] and [208, 207] and references therein for further research directions and applications to PDEs.
The main differences and novelties with respect to the aforementioned well-known works on the analysis of MFG systems rely on the analytical methods used to study the regularity of solutions. The presence of the time derivative and the fact that the dynamics is driven by a nonlocal operator make necessary to analyze in depth the regularity of the fractional heat operator $\partial_t + (-\Delta)^s$ on parabolic Hölder and Sobolev spaces. Therefore, our main contribution is to first develop a careful analysis of fractional heat equations in the L^p setting, whence we systematically treat spaces

of the form

$$\mathcal{H}_p^{\mu}(Q_T) = \mathcal{H}_p^{\mu;s}(\Omega \times (0,T)) = \{u \in L^p(0,T;H_p^{\mu}(\Omega))\ ,\partial_t u \in L^p(0,T;H_p^{\mu-2s}(\Omega))\},$$

where H_p^{μ} denotes the space of Bessel potentials, providing the fractional counterpart of the analysis of the more popular parabolic Sobolev spaces $W_p^{2k,k}$, $k \in \mathbb{N}$, see [159]. Here, we are inspired by some results that appeared in the context of stochastic partial differential equations, and we prove embedding theorems for $\mathcal{H}_p^{\mu}(Q_T)$ that, apart from their own interest, play a key role in the analysis of our system of PDEs. Our results are consistent with the classical ones already known in the literature (see e.g. [159] and Appendix C). We refer to [84] for some discussions on $\mathcal{H}_p^{\mu,s}(\mathbb{R}^d \times (0,T))$, and [155] and references therein for the case $s = 1$. We also mention that some of the embeddings and maximal L^p-regularity results we obtain here can be deduced through methods for abstract evolution equations and we refer the interested reader to [199], see also the references therein. Our aim is to give a PDE-oriented proof of the results by also mixing some ingredients from interpolation theory in Banach spaces with the purpose of providing a more transparent and self-contained treatment.
As for parabolic Hölder's spaces and Schauder's type estimates, the theory is still incomplete and partially developed only during the last years. Some results in these directions can be found in [59, 114, 209, 133]. However, we provide a study of Schauder's type regularity by using interpolation theory in Banach spaces in Appendix B, following classical works on abstract Cauchy problems, see e.g. [178, 177, 220]. Here, the analysis is carried out on the periodic setting $\Omega = \mathbb{T}^d$ in order to exploit the compactness of the state space and we point out that a functional treatment of the Sobolev classes $\mathcal{H}_p^{\mu}$ on $\mathbb{R}^d \times (0,T)$ can be obtained via the very same schemes. Moreover, in order to study the regularity of the solutions in the subcritical regime $s > 1/2$, we will need some product and chain rules on Bessel potential spaces on the torus which, up to our knowledge, were not available in literature and are useful in various other fields of the analysis of PDEs (like Korteweg-de Vries, Schrödinger equations,...). Their proofs require transference arguments from $\mathbb{R}^d$ to $\mathbb{T}^d$ and harmonic analysis' tools which we describe in details in Appendix A.
We then provide existence of solutions via the vanishing viscosity method and using the classical fixed point strategy [71]. Such procedure turns out to heavily depend on a priori bounds for solutions of both equations. More precisely, we use the recent nonlinear adjoint method introduced by L.C. Evans [107] (which will be matter of further investigation in Part III), to deduce semiconcavity bounds for the viscous-fractional Hamilton-Jacobi equation, which are independent of the viscosity parameter. This in turn ensures Lipschitz bounds, essential to prove the existence of smooth solutions for HJ equations with coercive Hamiltonian. We point out that this duality method was introduced to study more deeply the vanishing viscosity process, and the gradient shock structures, i.e. the structure of the singularities, to solutions of non-convex Hamilton-Jacobi equations. Furthermore, it has been extensively used in the analysis of asymptotic problems for viscous and degenerate Hamilton-Jacobi equations [64, 230, 187] (see also the references therein), and to study differentiability properties of solutions to ∞-Laplacian PDEs, see [109].
Uniqueness of solutions for such systems in general is not always expected. Under

the classical Lasry-Lions monotonicity condition (see [163]) the result continue to hold for every $s \in (0,1)$ and is consistent with the qualitative properties of solutions to the PDE system in the borderline cases $s = 1$ and $s = 0$.
Lately, there have been an increasing interest in uniqueness criteria for short time horizons. The first results were announced in the recorded video lectures by P.-L. Lions at Collège de France and first attacked by M. Bardi and M. Fisher [24] and M. Bardi and M. Cirant [19] via energy methods for the continuous models (see also [124] for an earlier result for finite-state MFGs and [24, Remark 4.13] for other related references). However, one could expect the validity of such properties via contraction mapping principle and methods for abstract evolution equations, which is in fact a quite common approach in the framework of evolutive nonlinear PDEs (see [229] and the recent work [91] for systems related to the mean-field equations). Nonetheless, the subtle coupling among the PDEs and their backward-forward structure determine some new difficulties when applying such procedure. Here, we exploit the aforementioned product rules on the spaces of Bessel potentials and the representation of the PDE system in a forward-forward form via the variation of constants formula to deduce such short-time uniqueness results when $s \in (1/2, 1)$. The same kind of strategy has been implemented in [92] and [91]. However, we point out that uniqueness of solutions under this regime is at this stage open when $s \in (0, 1/2]$, since the semigroup approach on Bessel potential spaces crucially fails: heuristically, this is due to the fact that, somehow, under this regime the diffusion is deteriorating and one looses the crucial decay estimates which allow the machinery to work. Anyhow, we believe that short-time uniqueness in the case of supercritical and critical diffusion can be both obtained in view of the recent developments for first order MFG systems [118, 183]. We highlight that our uniqueness result for short-time horizons requires some additional smoothness assumptions on the Hamiltonian compared to [24], and thus is closer to that appeared in [92] with respect to the regularity requirements. Our regularity hypotheses are crucial to run the arguments via contraction mapping theorem through chain and composition rules in fractional Sobolev classes. However, we remark that the energy methods developed in [24, 19] do not directly apply to the fractional framework due to the gap between the fractional derivatives and the divergence operator, and this is the main reason of our extra regularity assumptions.

Part III-Lipschitz regularity to Hamilton-Jacobi equations with rough data

It is well-known that bounded solutions of the heat equation posed on the whole space $\mathbb{R}^d$ starting from a bounded initial data become immediately Lipschitz continuous as soon as $t > 0$ with a global Lipschitz estimate of the form $\|Du(\cdot,t)\|_\infty \lesssim (1/\sqrt{t})\|u_0\|_\infty$ (see e.g. [229, p.35]). The aim of this last part is to address the same question for viscous and fractional Hamilton-Jacobi (briefly HJ) equations

$$\begin{cases} \partial_t u(x,t) + \mathcal{A}u(x,t) + H(x, Du(x,t)) = f(x,t) & \text{in } Q_T = \mathbb{T}^d \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}^d, \end{cases} \tag{6}$$

with unbounded right-hand side f, where H is superlinear in Du and the diffusion operator A will be replaced by $-\sum_{i,j} a_{ij}(x,t)\partial_{ij}$ in Chapter 6 (under suitable regularity assumption on a_{ij} that we will outline below) and $(-\Delta)^s$ in Chapter 7. The precise statement is the following: there exists a positive function $D : (0, T] \to [0, \infty)$ such that

$$\|u(\cdot, t)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq D(t)\|u_0\|_{L^\infty(\mathbb{T}^d)} \ , t \in (0, T] \ .$$

More precisely, we seek to show that weak solutions (in a suitable sense) with bounded initial data u_0 to (6) become Lipschitz continuous at positive times and satisfy the above "decay" estimate.
The motivation for such work is twofold. On one hand, this is motivated by a remarkable result by P.-L. Lions [175], that states Lipschitz regularity of solutions to the *stationary* counterpart of the above HJ equation for $A = I$, $f \in L^q$, $q > d$ and *any* $\gamma > 1$, namely for the simpler viscous equation

$$-\Delta u + |Du|^\gamma = f(x) \text{ on } \mathbb{T}^d \ . \tag{7}$$

In [175], a refinement of the classical Bernstein method that exploits both diffusion and coercivity is developed, but unfortunately it does not seem to generalize to time-dependent problems like (6). This procedure, modeled on the Bernstein initial idea for linear PDEs [37], consists in looking at the equation satisfied by $w = |Du|^2$. Owing to simple computations, one easily finds

$$w_{x_j} = 2Du \cdot Du_{x_j} \ , \ \Delta w = 2[|D^2u|^2 + Du \cdot D(\Delta u)] \ .$$

Differentiating then the equation (7) with respect to x_j and summing, one obtains the following PDE satisfied by w

$$-\Delta w + \gamma|Du|^{\gamma-2}Du \cdot Dw + 2|D^2u|^2 = 2Df \cdot Du \ .$$

The idea developed in [175] for smooth solutions is basically based on multiplying the above equation by w^p for some $p > 1$ large to be determined, and integrating in space. Roughly speaking, the third term (see [88]) gives rise to the integral

$$\int_\Omega |\Delta u|^2 w^p \, dx \ .$$

At this stage, the key tool is to use the above integral to get additional "coercivity" by plugging the equation via the inequality

$$|\Delta u|^2 \geq C_1|Du|^{2\gamma} - C_2 f^2$$

for some $C_1, C_2 > 0$, and handle all the terms by a delicate combination of Sobolev, Young and Hölder inequalities in order to conclude the gradient bound. However, if one tries to adapt the same procedure for the evolutive problem, mixed integral terms involving $\partial_t w$ and suitable powers of w appear, and, unfortunately, it is not clear how to handle them.
Typically, in the framework of quasi-linear equations with superlinear growth γ in Du two regimes are identified, namely the sub-quadratic $\gamma < 2$ and the superquadratic growth $\gamma > 2$. Here, we have in mind Hamiltonians of the form

$$H(x, p) = h(x)|p|^\gamma + b(x) \cdot p, \tag{8}$$

for some $h, b \in C^1(\mathbb{T}^d)$, $\gamma > 1$ and $0 < h_0 \le h(x)$. For $f \in L^\infty$, Lipschitz (and further) regularity of solutions for quasi-linear equations of the form (6.1) goes back to classical literature, see e.g. [159].
On the other hand, in the super-quadratic case $\gamma > 2$ the diffusion term is considered "weaker", and thus typically regarded as a perturbation of a first-order HJ equation. In this direction, Hölder and Sobolev regularity results with possibly unbounded f have been obtained in [77, 75] (where a_{ij} can indeed be degenerate). We refer also to [224, 82] for a different approach in the viscous case and to Section 6.3 for a brief survey on the literature and the techniques used to derive gradient bounds for such nonlinear PDEs. This different regimes can be easily detected by performing a classical L^∞-scaling argument: by setting $v(x,t) = u(\varepsilon x, \varepsilon^\gamma t)$, one finds the following PDE satisfied by v

$$\partial_t v - \varepsilon^{\gamma-2}\Delta v + |Dv|^\gamma = \varepsilon^\gamma f(\varepsilon x, \varepsilon^\gamma t) \ .$$

By direct inspection, one observes that the equation can be typically considered as a perturbation of the heat equation in the subquadratic regime, while in the case $\gamma > 2$ the usual approach is to regard the Laplacian as a perturbation of a first order equation, due to the fact that at small scales the diffusion is weaker than the gradient terms. However, when a nondegenerate diffusion is in force, one expects a better regularization effect, even when $\gamma > 2$, in the spirit of the elliptic results in [175]: in fact, by performing a $W^{1,\infty}$ parabolic scaling, one immediately sees that $w(x,t) = \varepsilon^{-1}u(\varepsilon x, \varepsilon^2 t)$ solves

$$\partial_t w - \Delta w + \varepsilon |Dw|^\gamma = \varepsilon f(\varepsilon x, \varepsilon^2 t) =: g_\varepsilon(x,t) \ .$$

Here, one notices that the space-time L^q norm of g_ε is invariant under the previous scaling precisely when $q = d + 2$, which is indeed the threshold we will meet throughout our analysis and above which we see that solutions to (6) exhibit a further Lipschitz regularization effect. To overcome the aforementioned difficulties of [175] in the evolutive framework, we perform our analysis via a duality approach. The study of linear equations through their duals (adjoint) is a classical matter, which has been recently explored in the nonlinear framework of HJ equations by L.C. Evans [107]. Its applications to viscous HJ equations, appearing in particular in so-called MFG systems, have been then investigated in a series of papers by D. Gomes and collaborators, see [127] and references therein. Lipschitz bounds of solutions to equations of the form (6) with unbounded or rough data have been in particular considered in [123, 128]. In these works, limitations on the regularity of u itself (it is typically smooth), on the growth of H (more precisely the growth $\gamma < 3$), or on d are imposed. Here, we obtain results for all $\gamma > 1$ and $d \in \mathbb{N}$, and for weak solutions to (6). The regularization effect is based both on the non-degeneracy of the diffusion operator and on the strong coercivity assumption of the Hamiltonian H with respect to Du. Up to our knowledge, the results we are going to present improve in several directions the known literature on the subject. Specifically, we are able to handle right-hand sides unbounded both in space and time, unlike the quoted contributions in the context of MFGs, and we provide a result which is completely new when $\gamma \ge 3$, see Section 6.3 for additional comments on the literature.

From the modeling viewpoint, a further motivation of our analysis comes indeed from the theory of MFGs [162], where HJ equations of the form (6) appear naturally, and, as outlined above, describe the value function of a typical player in a differential game involving a large population of agents. Here, f is a coupling term that may belong to a Lebesgue space. An important point in such systems is to prove boundedness of the gradient of u, that is crucial both for PDE purposes, since it implies that the mean-field equations are satisfied in a stronger sense, and also guarantees the boundedness of the optimal control-velocity of players at equilibrium and regularity of their distribution. It is worth noting that MFG systems naturally exhibit the presence of an HJ equation and its (dual) Fokker-Planck: this feature somehow inspired the methods by duality presented in this thesis. The regularity results appearing here would be crucial to study the regularity of MFG systems with power-like couplings [73, 92], which is not completely understood even in the stationary case, see the introduction in [90].

More precisely, in Chapter 6 we prove our Lipschitz regularity result for equations with general second order diffusion operators $\partial_t - a_{ij}(x,t)\partial_{ij}$. We first seek to prove the regularization effect for weak energy solutions to (6) when f in $L^q(Q_T)$ with

$$q > d+2 \text{ and } q \geq \frac{d+2}{\gamma'-1} \ .$$

Note that for $\gamma \leq 2$ this condition reads as $q > d+2$ and it can be regarded as the parabolic analogue of the result in [175] presented before. Then, for classical solutions we use a dual version of the Bernstein method to improve our condition to

$$q > d+2 \text{ and } q \geq \frac{d+2}{2(\gamma'-1)} \ ,$$

which reads as the parabolic Lions' condition $q > d+2$ as soon as $\gamma \leq 3$. As a byproduct, we get a maximal L^q regularity result for the viscous HJ equation, giving a first attempt to generalize Lions' results to the evolutive setting.
A fundamental step towards this result is the analysis of the dual (Fokker-Planck equation)

$$-\partial_t\rho + \mathcal{A}^*\rho + \operatorname{div}(b(x,t)\rho) = 0 \text{ in } Q_\tau := \mathbb{T}^d \times (0,\tau) \ , \tag{9}$$

$\mathcal{A}^*$ standing for the formal adjoint of A, when the drift is assumed to have enough Lebesgue integrability. The basic idea behind the proof is the following. If u is a solution to (6.1) with $\mathcal{A} = -\Delta$, then any directional derivative $v = \partial_\xi u$ satisfies a linearized equation of the form

$$\partial_t v - \Delta v + D_p H(x, Du) \cdot Dv = \partial_\xi f \ .$$

Then, one tests the equation against a solution of the backward equation (9) with drift $b(x,t) = -D_p H(x, Du)$, which develops a Dirac mass at the terminal time $t = \tau$, and integrates by parts to get the estimate (see Section 6.2 for further details).
As for the adjoint problem, when $\mathcal{A} = -\Delta$, rephrasing the transport equation (9) on $\mathbb{R}^d \times (0,T)$, one immediately notices that the above equation has a natural scaling. Indeed, if ρ is a solution to (9), then $\rho_\lambda(x,t) := \rho(\lambda x, \lambda^2 t)$ solves a transport-diffusion

equation with scaled drift $b_\lambda(x,t) := \lambda b(\lambda x, \lambda^2 t)$. In particular, it can be observed that the space $L^{\mathcal{Q}}(L^{\mathcal{P}}) = L_t^{\mathcal{Q}}(L_x^{\mathcal{P}})$ is invariant under the previous scaling of the velocity field b precisely when $d/(2\mathcal{P}) + 1/\mathcal{Q} = 1/2$. Then, we say that a Banach space endowed with norm $\|\cdot\|_X$ is called *critical* if $\|\rho_\lambda\|_X = \|\rho\|_X$ for every $\lambda > 0$, subcritical when $\|\rho_\lambda\|_X \to 0$ as $\lambda \to 0$ and supercritical in the case $\|\rho_\lambda\|_X \to \infty$ as $\lambda \to 0$ (meaning that if one zooms in at a point, i.e. $\lambda \to 0$, then the bound on the drift becomes better, invariant or worse respectively). Therefore, here the subcritical space turns out to be the mixed space $L^{\mathcal{Q}}(L^{\mathcal{P}})$ when $\mathcal{P}, \mathcal{Q}$ meet the inequality

$$\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} \leq \frac{1}{2} \ .$$

Such condition is sometimes called "Aronson-Serrin" interpolated condition and goes back to the earlier works [159, 11]. In particular, within the critical regime with respect to the parabolic scaling, one usually expects some $L^\infty(L^p)$ bounds, while in the subcritical regime L^∞ estimates (and even Hölder continuity, see e.g. [50] and references therein). On the contrary, when dealing with supercritical spaces, space-time unboundedness may occur (see e.g. [39]). Anyhow, the importance of this condition is twofold: on one hand it guarantees the well-posedness of the adjoint problem and, on the other hand, it is also crucial to ensure the uniqueness of weak solutions to HJ equations. Roughly speaking, under the above interpolated condition one can regard the transport term as a lower order perturbation of the heat equation. When $\mathcal{A} = (-\Delta)^s$, the regularity of solutions of the dual equation under rough conditions of the drift is far from being complete. Therefore, we discuss properties of fractional transport equations of the form

$$-\partial_t \rho + (-\Delta)^s \rho + \operatorname{div}(b(x,t)\rho) = 0 \text{ in } Q_\tau$$

equipped with terminal data $\rho(x,\tau) = \rho_\tau(x)$ in $\mathbb{T}^d$ and rough velocity field in $b \in L^{\mathcal{Q}}(L^{\mathcal{P}})$ spaces. Under the classical incompressible condition $\operatorname{div}(b) = 0$, typical of fluid dynamics settings, the previous PDE is formally equivalent to $-\partial_t\rho + (-\Delta)^s\rho + b \cdot D\rho = 0$: in this framework some well-posedness and integrability estimates for subcritical fractional parabolic equations with drift terms have been established in the context of Surface Quasi-Geostrophic (SQG) equations (see e.g. [235, 93, 63, 218] and the references therein). We also refer the reader to the survey at the beginning of the paper [188] for the extremal regimes $s = 0$ and $s = 1$. In particular, one immediately realizes that on $\mathbb{R}^d \times (0,T)$ the equation is invariant under the scalings $\rho_\lambda(x,t) := \rho(\lambda x, \lambda^{2s} t)$ and $b_\lambda(x,t) := \lambda^{2s-1} b(\lambda x, \lambda^{2s} t)$. Even in this case, when $s \in (1/2, 1)$ the subcritical space turns out to be a mixed space $L^{\mathcal{Q}}(L^{\mathcal{P}})$ when the exponents $\mathcal{P} \geq d/(2s-1)$ and $\mathcal{Q} \geq 2s/(2s-1)$ fulfill the condition

$$\frac{d}{2s\mathcal{P}} + \frac{1}{\mathcal{Q}} \leq \frac{2s-1}{2s} \ ,$$

which can be seen as the fractional counterpart of the above Aronson-Serrin interpolated condition met in the viscous problem, $s = 1$. No results in this fractional setting can be tracked back to our knowledge under this general conditions on the drift when, in addition, no information on its divergence is available. The above

fractional range for the exponents $\mathcal{P}$ and $\mathcal{Q}$ can be found in [167] for fractional heat equations, [142, Example 3] in the study of fundamental solutions to time-dependent gradient perturbation of the fractional Laplacian.
As a byproduct, in the case of subcritical fractional diffusion, we deduce the result for weak solutions to fractional HJ equations with coercive Hamiltonian in Du exploiting the analysis of parabolic spaces developed in Part II.
Anyhow, by assuming some additional fractional regularity hypothesis on the right-hand side, that is imposing $f \in L^q(0,T; H_q^{2-2s}(\mathbb{T}^d))$ for

$$q > d + 2s \text{ and } q \geq \frac{d+2s}{(2s-1)(\gamma'-1)} \ ,$$

we are able to show a Lipschitz regularization effect. This additional integrability hypothesis on the right-hand side is required since in this manuscript we are merely able to estimate fractional derivatives of ρ in Lebesgue spaces by means of (fractional) parabolic Calderón-Zygmund theory (see Section 7.3.1 for additional comments on the integrability assumptions).
We further emphasize that the approach carried out either in [175, 27] via refinements of the Bernstein method or by coupling the duality method and the integral Bernstein method (cf Chapter 6) cannot be directly reproduced due to the nonlocality of the operator. These latter phenomena will be matter of further investigation, together with a treatment of the stationary problem.
We finally conclude saying that a crucial point to achieve these estimates is an a priori information on the crossed quantity

$$\iint |D_pH(Du)|^{\gamma'} \rho \, dxdt,$$

that is obtained using a sort of duality between (6) and its adjoint, and has a very precise meaning in terms of optimality in stochastic control problems. Indeed, recalling the fact that ρ is the distribution law of X_t, one has that if $b(\cdot,t) = -D_pH(\cdot, Du(\cdot,t))$ is the optimal drift, then

$$\iint |D_pH(x,Du)|^{\gamma'} \rho \, dxdt \int_0^T \int_\Omega L(x, -D_pH(x,Du))\rho \, dxdt$$
$$= \int_0^T \int_\Omega \rho(D_pH(x,Du)\cdot Du - H(x,Du)) \, dxdt \simeq \mathbb{E} \int_0^T |D_pH(X_t, Du(X_t,t))|^{\gamma'} \, dt \ ,$$

and thus an a priori bound on such quantity highlights that the drift has $L^{\gamma'}$-regularity along the trajectory of the associated stochastic dynamics. This is a quite common condition appearing in regularity and uniqueness issues for Fokker-Planck equations (see [194, 50, 185, 49] and references therein).

Part I

Strong maximum principles for fully nonlinear degenerate PDEs via subunit vector fields and applications

Chapter 1

Few basic facts on viscosity solutions and Carnot groups

1.0.1 Viscosity solutions to fully nonlinear PDEs

This subsection is devoted to collect some basic notions on viscosity solutions' theory for fully nonlinear second order PDEs. For a more detailed treatment we refer the reader to [20, 96] for second order problems and [15, 1] for first order equations.
Throughout this part we are interested in fully nonlinear second order PDEs of the form

$$F(x, u, Du, D^2u) = 0 \text{ in } \Omega \tag{1.1}$$

where $x \in \Omega$, u is a function defined in Ω and $F(x, r, p, X)$ is a real-valued function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d$. and their evolutive counterpart

$$\partial_t u + F(x, t, u, Du, D^2u) = 0 \text{ in } \Omega \times (0, T)$$

where $(x, t) \in \Omega \times (0, T)$, u is a function defined in $\Omega \times (0, T)$ and $F(x, t.r, p, X)$ is a real-valued function defined in $\Omega \times (0, T) \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d$. The usual ordering is used on $\mathcal{S}_d$, i.e. $Y \leq X$ means that $X - Y$ is nonnegative semidefinite. We give some notions of ellipticity that we will meet within the first part of the thesis.

Definition 1.1. *We say that the operator $F = F(x, r, p, M)$ is uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda$ if*

$$\lambda \text{Tr}(N) \leq F(x, r, p, M) - F(x, r, p, M + N) \leq \Lambda \text{Tr}(N)$$

for every $x \in \Omega$, $r \in \mathbb{R}$, $p \in \mathbb{R}^d$ and $M, N \in \mathcal{S}_d$ with $N \geq 0$.
We say that F is proper if

$$F(x, r, p, M) \leq F(x, s, p, N) \ , r \leq s \text{ and } N \leq M \ .$$

and degenerate if

$$F(x, r, p, M) \leq F(x, r, p, N) \ , N \leq M$$

Analogous definitions can be given for the time-dependent operator $\partial_t + F$ (see e.g. [96, Section 8] and [100]). When dealing with such nonlinear operators the typical framework is that of viscosity solutions. The basic idea behind this concept is to extend the notion of sub- and supersolution for classical linear operators to a larger set of suitable non-smooth functions so that the classical maximum principle is preserved. For instance, in the case of the Laplace equation one can prove that u is subharmonic if and only if for all $x \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at x, then $-\Delta\varphi \leq 0$ and, consistently, the idea is to take this as definition of viscosity solution for such nonlinear equations, as we shall see below in the next definition. This is somehow reminiscent of the classical Perron's method [122] for elliptic equations. We have the following

Definition 1.2. *(1) We say that a function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of (1.1) (equivalently a viscosity solution of $F \leq 0$) in Ω if $u \in \mathrm{USC}(\Omega)$ and for every $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at x_0, then*

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0 \ . \tag{1.2}$$

(2) We say that a function $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of (1.1) (equivalently a viscosity solution of $F \geq 0$) in Ω if $u \in \mathrm{LSC}(\Omega)$ and for every $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at x_0, then

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \ . \tag{1.3}$$

(3) Finally, a function $u : \Omega \to \mathbb{R}$ is called a viscosity solution of (1.1) *in Ω if it is both a viscosity subsolution and a viscosity supersolution.*

We also say that $F(x, u, Du, D^2u) \leq (\geq, =)\ 0$ *is satisfied in the viscosity sense* in Ω if u is a viscosity subsolution (resp. supersolution, solution) of (1.1) in Ω. Notice that a viscosity solution is a continuous function since it is both upper and lower semicontinuous.

Remark 1.3. One can also define viscosity solutions via the so-called semi-jets. However, since we will not use such tool here, we prefer to skip the details, referring, among others, to [96]. In addition, as we will see in the next chapter, when dealing with PDEs modeled on the p-Laplacian, one needs to slightly revisit the above definition (see Chapter 2).

We now give some examples of fully nonlinear elliptic PDEs that we will meet throughout the thesis, referring, among others, to the monographs [62, 190] for other interesting examples and the expository paper [57] for a gentle introduction to the subject.

Example 1.4 (Linear elliptic equations). Consider the equation

$$-\sum_{ij} a_{ij}(x)\partial_{ij}u + \sum_i b_i(x)\partial_i u + c(x)u = f(x) \ .$$

The corresponding operator is given by $F(x, r, p, X) = -\mathrm{Tr}(A(x)D^2u(x)) + b(x) \cdot Du + c(x)u - f(x)$ and we observe that F is degenerate elliptic if and only if $A \geq 0$. By taking $a_{ij} = \delta_{ij}$, $b \equiv 0$ and $c = f = 0$ one recovers the classical Laplace equation.

Example 1.5 (Quasilinear elliptic equations in divergence form). Such equations appear in the general form

$$-\sum_i \partial_i(a_i(x, Du)) + b(x, u, Du) = 0 \ .$$

If the coefficients of the above PDE are differentiable, one can rewrite the operator as

$$F(x, r, p, X) = -\mathrm{Tr}(D_p a(x, p)X) + b(x, r, p) - \sum_i \partial_i(a_i(x, p)) \ .$$

A well-known example we will discuss within Part I is the m-Laplacian equation

$$-\Delta_p u := -\mathrm{div}(|Du|^{p-2}Du) = 0 \ ,$$

where $a(x, p) = |p|^{p-2}p$, $p \geq 1$. Note that for $p = 1$ one gets the mean curvature operator, while for $p = 2$ the classical Laplace equation. In the limit $p = \infty$ one obtains the so-called ∞-Laplacian

$$-\Delta_\infty u = \sum_{ij} \partial_i u \partial_j u \partial_{ij} u \ .$$

For the applications of these equations in the borderline cases in the context of differential games we refer to [106].

Example 1.6 (Quasilinear elliptic equations in nondivergence form). An equation of the form

$$-\sum_{ij} a_{ij}(x)\partial_{ij}u(x) + b(x, u, Du) = 0$$

covers all the previous examples as special cases. A relevant proptotype is the viscous Hamilton-Jacobi equation

$$-\sigma\Delta u + H(x, u, Du) = 0$$

which appears, among others, in models related to the recent theory of Mean Field Games developed by J.-M. Lasry and P.-L. Lions [163] that we discuss in Part II, when the diffusion operator $-\Delta$ is replaced by its fractional power $(-\Delta)^s$, $s \in (0, 1)$, and in this case we say that the equation is a quasilinear integro-differential equation, see [31] (and references therein) where viscosity solutions for such PDEs are defined and used, and Part III in their evolutive form.

Example 1.7 (Pucci's equations). Pucci's equations are the simplest examples of PDEs that can be written in Hamilton-Jacobi-Bellman form and represent the cornerstone to analyze fully nonlinear (uniformly elliptic) second order PDEs. Such "extremal" equations were introduced by C. Pucci in [201] in d dimension (see also the earlier work in the plane [200]). These operators were defined in the following way: let $L_A(M) = -\mathrm{Tr}(AM)$, $M \in \mathcal{S}_d$ and $\mathcal{B}_\alpha$, $\alpha > 0$ be the class of matrices

$$\mathcal{B}_\alpha := \{A \in \mathcal{S}_d : A\xi \cdot \xi \geq \alpha|\xi|^2, \mathrm{Tr}(A) = 1, \forall \xi \in \mathbb{R}^d\} \ .$$

We define

$$\mathcal{P}^+_\alpha(M) = \sup_{A \in \mathcal{B}_\alpha} L_A M \tag{1.4}$$

and

$$\mathcal{P}^-_\alpha(M) = \inf_{A\in\mathcal{B}_\alpha} L_A M \ . \tag{1.5}$$

As pointed out in [201] (see also [122, Chapter 17]), one can immediately check the validity of the representation formulas

$$\mathcal{P}^+_\alpha(M) = -\alpha\sum_{k=2}^{d} e_k - [1-(d-1)\alpha]e_1 = -\alpha\mathrm{Tr}(M) - (1-d\alpha)e_1 \tag{1.6}$$

and

$$\mathcal{P}^-_\alpha(M) = -\alpha\sum_{k=1}^{d-1} e_k - [1-(d-1)\alpha]e_d = -\alpha\mathrm{Tr}(M) - (1-d\alpha)e_d \tag{1.7}$$

for any $M \in \mathcal{S}_d$, where $e_1 \leq ... \leq e_d$ are the ordered eigenvalues of the matrix M. Such operators were then analyzed in the parabolic framework in [13].
These nonlinear operators defined over a different class of matrices were then revisited by L. Caffarelli and X. Cabré to study properties of fully nonlinear PDEs (see [62]). In particular, let

$$\mathcal{A}_{\lambda,\Lambda} := \{A \in \mathcal{S}_d : \lambda|\xi|^2 \leq A\xi\cdot\xi \leq \Lambda|\xi|^2 \ , \forall \xi \in \mathbb{R}^d\} \ .$$

The so-called *Pucci's extremal operators* on symmetric matrices $M \in \mathcal{S}_d$ are defined as

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} L_A M \tag{1.8}$$

and

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} L_A M \ . \tag{1.9}$$

In particular, $\mathcal{M}^+_{\lambda,\Lambda}$ and $\mathcal{M}^-_{\lambda,\Lambda}$ are respectively called the *maximal* and the *minimal Pucci's operator*. Even in this case, one can check (see [62, Section 2.2]) that the following hold for every $M \in \mathcal{S}_d$

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = -\Lambda\sum_{e_k<0} e_k - \lambda\sum_{e_k>0} e_k \tag{1.10}$$

and

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = -\Lambda\sum_{e_k>0} e_k - \lambda\sum_{e_k<0} e_k \ . \tag{1.11}$$

The usefulness of the latter operators arises when dealing with uniformly elliptic fully nonlinear second order equations, since they allow to transfer properties of solutions from the fully nonlinear operator F to sub- and supersolutions of equations driven by the extremal operators $\mathcal{M}^\pm$. In fact one can prove the following easy characterization that stems from Definition 1.1 (see e.g. [62, Lemma 2.2]).

Proposition 1.8. *The following are equivalent*

(i) F is uniformly elliptic with ellipticity constants λ and Λ with $0 < \lambda \leq \Lambda$, i.e. for every $M, N \in \mathcal{S}_d$, $N \geq 0$ and $(x, r, p) \in \Omega\times\mathbb{R}\times\mathbb{R}^d$ we have

$$\lambda\mathrm{Tr}(N) \leq F(x, r, p, M) - F(x, r, p, M + N) \leq \Lambda\mathrm{Tr}(N) \ .$$

(ii) $F(x,r,p,M) - F(x,r,p,M-N) \leq \Lambda\mathrm{Tr}(N^-) - \lambda\mathrm{Tr}(N^+)$ *for every* $M, N \in \mathcal{S}_d$ *and* $(x,r,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^d$*, where* N^+ *and* N^- *stands for the positive and negative part of* N *respectively.*

(iii) $\mathcal{M}^-(M-N) \leq F(x,r,p,M) - F(x,r,p,N) \leq \mathcal{M}^+(M-N)$ *for every* $M, N \in \mathcal{S}_d$ *and* $(x,r,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^d$*.*

Such correspondence turns out to be useful even for some degenerate equations (see the next Chapter 4 where it will be applied to equations driven by Pucci's subelliptic operators), and it is actually a standard way to analyze qualitative and quantitative properties of viscosity solutions to fully nonlinear equations [62]. Moreover, one can prove that the following inequalities are in force

$$\mathcal{P}^+_\lambda(M) \leq \mathcal{M}^+_{\lambda,\lambda+(1-d\lambda)}(M) \text{ and } \mathcal{P}^-_\lambda(M) \geq \mathcal{M}^-_{\lambda,\lambda+(1-d\lambda)}(M) \ .$$

We will meet these relations among the extremal operators during the treatment of Liouville theorems for PDEs over Hörmander vector fields in Chapter 4.

Example 1.9 (Hamilton-Jacobi-Bellman and Isaacs equations). When the family $\mathcal{A}_{\lambda,\Lambda}$ is replaced by an arbitrary family of linear elliptic operators L^α, $\alpha \in \mathcal{A}$ is any set, one obtains the so-called Hamilton-Jacobi-Bellman equation, arising in stochastic control problems

$$\inf_\alpha \{L^\alpha u(x) - f^\alpha(x)\} = 0$$

and

$$\sup_\alpha \{L^\alpha u(x) - f^\alpha(x)\} = 0 \ .$$

In particular let us recall that any fully nonlinear equation $F = F(x, u, Du, D^2u)$ which is concave or convex in (u, Du, D^2u) can be recasted in one of the above formulations respectively by means of the Legendre transform. Other important fully nonlinear second order PDEs are the Isaacs equations coming from differential games

$$\sup_\alpha \inf_\beta \{L^{\alpha,\beta} u(x) - f^{\alpha,\beta}(x)\} = 0$$

and

$$\inf_\beta \sup_\alpha \{L^{\alpha,\beta} u(x) - f^{\alpha,\beta}(x)\} = 0 \ .$$

where α, β belong to arbitrary sets $\mathcal{A}$ and $\mathcal{B}$ respectively. We remind the reader that every fully nonlinear uniformly elliptic operator can be written in Isaacs form [58, Remark 1.5].

Example 1.10 (Subelliptic PDEs). One may replace the Euclidean gradient Du, and hence classical derivatives ∂_i, by a suitable family of vector fields $\mathcal{X} = \{X_1, ..., X_m\}$, $m \leq d$, which do not necessarily commute, leading to consider PDEs modeled on the so-called horizontal gradient $D_\mathcal{X} u = (X_1 u, ..., X_m u)$. Analogously, one defines the symmetrized horizontal Hessian $(D^2_\mathcal{X} u)^*$ (see Section 1.0.5 below). These considerations leads to fully nonlinear PDEs of the form

$$F(x, u, D_\mathcal{X} u, (D^2_\mathcal{X} u)^*) = 0 \ ,$$

where F is a real-valued operator defined on $\Omega \times \mathbb{R} \times \mathbb{R}^m \times \mathcal{S}_m$, $\Omega \subseteq \mathbb{R}^d$, for some $m \leq d$. Typically, this is the case of PDEs over Carnot groups and other particular sub-Riemannian geometries that will be the matter of Part I (here m is the dimension of the horizontal layer in the case of Carnot groups, while $d = m = 2$ in the Grushin plane, see below)

Example 1.11. Other important examples of fully nonlinear second order PDEs are Monge-Ampére equations and Hessian equations. For these and other fully nonlinear interesting examples we refer to [96, 190]. We finally mention the recent analysis on "truncated" Laplacian-type operators that appear as sums of the first (or last) $k < d$ eigenvalues of the Hessian matrix (see [46] and references therein), which therefore fall within the latter class of equations.

1.0.2 Carnot groups

In this section we collect some standard facts on Carnot groups. Here and in the sequel we will take for granted some standard definitions, referring to [51] for more details. We recall for the convenience of the reader that a Lie algebra is a vector space V endowed with a Lie bracket, that is a bilinear operation $[\cdot,\cdot] : V \times V \to V$, and satisfies $[x, x] = 0$ for every $x \in V$ and the Jacobi identity $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in V$.

Definition 1.12. *A Carnot group $\mathbb{G}$ of step r is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ is stratified of step r, namely there exist linear subspaces $V_1, ..., V_r$ of $\mathfrak{g}$ such that*

$$\mathfrak{g} = V_1 \oplus ... \oplus V_r$$

with

$$[V_1, V_i] = V_{i+1} \ 1 \leq i \leq r-1 \ \textit{and} \ [V_1, V_r] = \{0\} \ ,$$

where $[V_1, V_i] = \mathrm{Span}\{[a, b] : a \in V_1, b \in V_i\}$. In particular, the subspace V_1 is called horizontal layer and its elements are called left invariant vector fields. The rank of $\mathbb{G}$ is $\dim(V_1)$.

One can prove that (see [51, Proposition 1.1.7]) $[V_i, V_j] \subset V_{i+j}$ if $i + j \leq r$ and $[V_i, V_j] = \{0\}$ otherwise. One can then identify $\mathfrak{g}$ with $\mathbb{R}^d$ via the so-called exponential map $\exp : \mathfrak{g} \to \mathbb{G}$, which turns out to be a diffeomorphism. Given a basis $X_1, ..., X_d$ adapted to the stratification, any $x \in \mathbb{G}$ can be written in a unique way as

$$x = \exp(x_1X_1 + ... + x_dX_d)$$

and we thus identify $x \in \mathbb{G}$ with $(x_1, ..., x_d) \in \mathbb{R}^d$ and $\mathbb{G}$ with $(\mathbb{R}^d, \circ)$, where the group law in these coordinates is determined via the well-known Baker-Campbell-Hausdorff formula.
We also say that a curve $\gamma : [a, b] \to \mathbb{G}$ is absolutely continuous if it is absolutely continuous as a curve into $\mathbb{R}^d$. Fix now an orthonormal basis $X_1, ..., X_m$ of the first layer V_1. We have the following

Definition 1.13. *An absolutely continuous curve $\gamma : [a,b] \to \mathbb{G}$ is horizontal if there exists $\alpha_1, ..., \alpha_m \in L^1([a,b])$ such that*

$$\gamma'(t) = \sum_{j=1}^{m} \alpha_j(t) X_j(\gamma(t)) \text{ for almost every } t \in [a,b]$$

The length of the curve is given by $L_{\mathbb{G}}(\gamma) = \int_a^b |\alpha|$.

A well-known result by Chow states that any two point in a Carnot group can be connected by a horizontal curve. Hence, the following definition turns out to be well-posed

Definition 1.14. *The Carnot-Carathéodory (CC) distance between $x, y \in \mathbb{G}$ is defined by*

$$d_{CC}(x,y) = \inf\{L_{\mathbb{G}}(\gamma) : \gamma \text{ is a horizontal curve joining } x \text{ to } y\}$$

In terms of the layer decomposition of $\mathbb{G}$, one defines a one-parameter family of dilations δ_λ on $\mathbb{G}$ by setting for $x = x_1 + ... + x_r$, $x_j \in \mathbb{R}^{n_j}$, where n_j stands for the dimension of the j-th layer,

$$\delta_\lambda(x) = \sum_{j=1}^{r} \lambda^j x_j \ .$$

Moreover, for any $x \in \mathbb{G}$, the Jacobian of the map $x \longmapsto \delta_\lambda(x)$ coincides with $\lambda^{\mathcal{Q}}$, where $\mathcal{Q} = \sum_{j=1}^{r} j n_j$ is the so-called homogeneous dimension. Using such family of dilations, one can define a norm on $\mathbb{G}$ given by $\|x\|_C := d_{CC}(0,x)$. However, one can introduce on $\mathbb{G}$ a new norm equivalent to the Carnot-Carathéodory norm $\|\cdot\|_C$ which is more suited for computational purposes (see Chapter 4) and typically called *homogeneou norm.* More precisely, (see [51, Section 5.1] for further details) a homogeneous norm on $\mathbb{G}$ is a mapping $x \longmapsto \rho(x)$ from $\mathbb{G}$ to $\mathbb{R}^+$ such that the following properties hold true:

(i) $x \longmapsto \rho(x)$ is continuous on $\mathbb{G}$ and smooth on $\mathbb{G}\backslash\{0\}$;

(ii) $\rho(x) = 0$ if and only if $x = 0$;

(iii) $\rho(x) = \rho(-x)$;

(iv) $\rho(\delta_\lambda(x)) = \lambda\rho(x)$ for every $\lambda > 0$.

Example 1.15. Let $\mathbb{G}$ be a Carnot group with stratification $V_1, ..., V_r$. We define a homogeneous norm on $\mathbb{G}$ via the stratification as

$$\rho(x) := \left(\sum_{i=1}^{r} |x_i|^{\frac{2r!}{i}} \right)^{\frac{1}{2r!}} ,$$

where $|x_i|$ is the k-dimensional Euclidean norm defined on the vector space V_i.

One can show that all homogeneous norms on Carnot groups are equivalent [51, Proposition 5.1.4] and they satisfy pseudo-triangle inequalities [51, Proposition 5.1.7 and Proposition 5.1.8].

1.0.3 Examples of Carnot groups: The Heisenberg group and free step-2 Carnot groups

In this section we briefly recall some standard facts on Carnot groups, specifically we discuss the Heisenberg group and free Carnot groups.
The Heisenberg group $\mathbb{H}^d$ can be identified with $(\mathbb{R}^{2d+1}, \circ)$, where $2d+1$ stands for the topological dimension and the group law $\circ$ is defined by

$$x \circ y = \left(x_1 + y_1, ..., x_{2d} + y_{2d}, x_{2d+1} + y_{2d+1} + 2 \sum_{i=1}^{d} (x_i y_{i+d} - x_{i+d} y_i) \right) .$$

The d-dimensional Heisenberg algebra is the Lie algebra spanned by the vector fields

$$X_i = \partial_i + 2x_{i+d}\partial_{2d+1} ,$$

$$X_{i+d} = \partial_{i+d} - 2x_i\partial_{2d+1} ,$$

for $i = 1, ..., d$ and x denotes a point of $\mathbb{R}^{2d+1}$. Such vector fields satisfy the commutation relations

$$[X_i, X_{i+d}] = -4\partial_{2d+1} \text{ and } [X_i, X_j] = 0 \text{ for all } j \neq i + d,\, i \in \{1, ..., d\} .$$

Following Example 1.15, the corresponding homogeneous norm for such structure can be defined as

$$\rho(x) = \left(\left(\sum_{i=1}^{2d} (x_i)^2 \right)^2 + x_{2d+1}^2 \right)^{\frac{1}{4}} . \tag{1.12}$$

We now turn our attention to free step-2 Carnot groups following [51, Section 14.1], see also [164]. Such structures appeared first in [120] and later in the context of control problems in [55]. We first present a more abstract definition and then we give the representation in coordinates. We start by recalling the following

Definition 1.16. *Let $r \geq 2$ and $s \geq 1$ be integers. We say that $\mathcal{F}_{r,s}$ is the free nilpotent Lee algebra with r generators $x_1, ..., x_r$. of step s if*

- *$\mathcal{F}_{r,s}$ is a Lie algebra generated by $x_1, ..., x_r$.*

- *$\mathcal{F}_{r,s}$ is nilpotent of step s.*

- *For every Lie algebra $\mathfrak{g}$ that is nilpotent of step s and for every map Φ : $\{x_1, ..., x_r\} \to \mathfrak{g}$, there is an homomorphism of Lie algebras $\tilde{\Phi} : \mathcal{F}_{r,s} \to \mathfrak{g}$ that extends Φ, and moreover it is unique.*

Definition 1.17. *A Free Carnot group is a Carnot group whose Lie algebra is isomorphic to a free nilpotent Lie algebra $\mathcal{F}_{r,s}$ for some $r \geq 2$ and $s \geq 1$. Moreover, the horizontal layer of the free Carnot group is isomorphic to the span of the generators of $\mathcal{F}_{r,s}$.*

We now give a representation of free Carnot groups of step 2 via exponential coordinates, which will be useful to deduce our sufficient conditions for Liouville theorems in Chapter 4.
More precisely, fix an integer $r \geq 2$ and denote by $d = r + \frac{r(r-1)}{2}$. In $\mathbb{R}^d$, let us denote the coordinates of the first layer by x_i, $1 \leq i \leq r$ and that of the second layer by t_{ij}, $1 \leq j < i \leq r$. Let ∂_i and ∂_{ij} the standard basis vectors in this coordinate system. Denote the d vector fields on $\mathbb{R}^d$ by

$$X_k := \partial_k + 2\left(\sum_{j>k} x_j \partial_{jk} - \sum_{j<k} x_j \partial_{kj}\right) \text{ if } 1 \leq k \leq r \ ,$$

$$X_{kj} := \partial_{kj} \text{ if } 1 \leq j < k \leq r \ .$$

The Carnot structure of $\mathbb{G}_r$ is given by

$$V_1 = \text{Span}\{X_k : 1 \leq k \leq r\} \text{ and } V_2 = \text{Span}\{X_{kj} : 1 \leq j < k \leq r\}$$

The commutation relations for $1 \leq j < k \leq r$ and $1 \leq i \leq r$ are given by

$$[X_k, X_j] = 4X_{kj} \text{ and } [X_i, X_{kj}] = 0$$

and denote by

$$(x_H, x_V) = (x_1, ..., x_r, t_{r,1}, ..., t_{r,r-1}) \in \mathbb{R}^r \times \mathbb{R}^{\frac{r(r-1)}{2}}$$

Definition 1.18. *The free Carnot group of step 2 and r generators is $\mathbb{G}_r \equiv (\mathbb{R}^d, \bullet)$, where the group law is defined as*

$$(x \bullet y)_k = x_k + y_k \textit{ if } 1 \leq k \leq r \ ,$$

$$(x \bullet y)_{ij} = x_{ij} + y_{ij} + 2(x_i y_j - y_i x_j) \textit{ if } 1 \leq j < k \leq r \ .$$

We also remark that free Carnot groups of step 2 are those that are isomorphic to a Carnot group $\mathbb{G}_r$ for some r (see again [51] and references therein).
According to [54, Section 3.4.2], free groups are those whose system of vector fields $X_1, ..., X_m$ generating the algebra is free up to step r. Roughly speaking, this happens when the X_i's and their commutators up to step r do not satisfy any linear relation except the ones holding by the properties of the Lie algebra, namely the antisymmetry of the Lie bracket and the Jacobi identity.

Remark 1.19. Observe that the free Carnot group of step 2 coincides with the Heisenberg group only in one dimension, i.e. $\mathbb{H}^1 \equiv (\mathbb{R}^3, \circ)$. Indeed for $r = 2d$ generators, we have a free Carnot group of step 2 if and only if the following equality holds

$$2d + 1 = 2d + \frac{2d(2d-1)}{2} \ ,$$

which is fulfilled only when $d = 1$. This can be also seen by the commutation relation between the vector fields, see [54, Section 3.4.2 Example 46]. In fact the vector fields X, Y in $\mathbb{H}^1$ are free up to step 2 since X,Y and $[X, Y]$ are linearly independent. This does not happen for $\mathbb{H}^2$, since $[X_1, X_3] = [X_2, X_4]$, which is a nontrivial relation among the commutators of step 2.

Remark 1.20. The vector fields inducing the Grushin plane are not free of step 2 since, as we will see in the next Subsection 1.0.4, it happens that $Y = x[X, Y]$, which is a nontrivial relation between a generator and a commutator.

For later purposes, according to Definition 1.15 we recall here that the homogeneous norm is defined by

$$\rho(x) = \left((x_1^2 + ... + x_r^2)^2 + t_{2,1}^2 + ... + t_{r,r-1}^2\right)^{\frac{1}{4}} \ . \tag{1.13}$$

1.0.4 Sub-riemannian geometries of Grushin-type

We now discuss an example of sub-Riemannian geometry which does not fall within the previous theory of nilpotent stratified Lie groups. Grushin-type geometries are defined on $\mathbb{R}^d = \mathbb{R}^p \times \mathbb{R}^q$, $d = p + q \geq 2$, and induced by the vector fields

$$X_i = \partial_{x_i} \ , 1 \leq i \leq p \ ; Y_j = |x|^\alpha \partial_{y_j} \ , 1 \leq j \leq q \ .$$

for $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$. The Grushin plane corresponds to $d = 2$, $p = q = \alpha = 1$, that is $\mathbb{R}^2$ equipped with the basis of vector fields

$$X = \partial_x \text{ and } Y = x\partial_y \ .$$

Note that the vector fields X, Y are not left invariant with respect to any group law on $\mathbb{R}^2$ [51, Proposition 1.2.13] and hence Grushin-type geometries cannot be endowed with any group structure. However, one can easily check that X and Y satisfy the Hörmander condition (cf Definition 2.5) since at the origin $(0, 0)$ we have $\text{Span}(X, Y) = \text{Span}(X) \neq \mathbb{R}^2$, but $[X, Y] = \partial_y$ and hence $\text{Span}(X, Y, [X, Y]) = \mathbb{R}^2$ at any point $(x, y) \in \mathbb{R}^2$. Even in this case one can define a homogeneous norm similarly to the one considered in the Heisenberg group, although a group structure is not available. For $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$ one defines the homogeneous norm

$$\rho((x, y)) = (|x|^{2(1+\alpha)} + (1 + \alpha)^2 |y|^2)^{\frac{1}{2(1+\alpha)}} \ ,$$

which turns out to be homogeneous of degree one with respect to the dilations $\delta_\lambda((x, y)) = (\lambda x, \lambda^{1+\alpha} y)$ and reduces to

$$\rho((x, y)) = (x^4 + 4y^2)^{\frac{1}{4}}$$

in the case of the Grushin plane, where $(x, y) \in \mathbb{R}^2$ and $\alpha = 1$.

1.0.5 On subelliptic equations

In this section we recall some standard notions one needs to deal with subelliptic PDEs. Let $X_1, ..., X_m$ be a system of vector $C^{1,1}$ vector fields. We give the following

Definition 1.21. *Let $u : \Omega \to \mathbb{R}$, $x \in \Omega$. The horizontal gradient of u at x is defined as*

$$D_{\mathcal{X}} u(x) = (X_1 u(x), ..., X_m u(x)) \in \mathbb{R}^m \ .$$

The symmetrized horizontal Hessian of u is the $\mathcal{S}_m$ matrix whose elements are given by

$$(D^2_{\mathcal{X}} u(x))^*_{i,j} = \left[\frac{X_i(X_j u(x)) + X_j(X_i u(x))}{2}\right] \text{ for } i, j = 1, ..., m$$

Remark 1.22. We point out that the intrinsic Hessian, whose elements are $X_i(X_j u)(x)$, is different from the symmetrized horizontal Hessian defined above, since the vector fields do not commute. Indeed

$$X_i(X_j u(x)) = \frac{X_i(X_j u(x)) + X_j(X_i u(x))}{2} + \frac{1}{2}[X_i, X_j]u(x) \ .$$

We will test our results, especially in Chapter 4, to equations of the form

$$G(x, u, D_{\mathcal{X}} u(x), (D^2_{\mathcal{X}} u(x))^*) = 0 \ .$$

Typically we will consider those operators G satisfying a properly rescaled uniform ellipticity condition, according to the next Definition.

Definition 1.23. *We say that the operator G is uniformly subelliptic with ellipticity constants $0 < \lambda \leq \Lambda$ if*

$$\lambda \mathrm{Tr}(Y) \leq G(x, r, p, X) - G(x, r, p, X + Y) \leq \Lambda \mathrm{Tr}(Y)$$

for every $x \in \overline{\Omega}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^m$ and $X, Y \in \mathcal{S}_m$ with $Y \geq 0$.

Similarly to the Euclidean case one can characterize the above property in terms of (degenerate) Pucci's extremal operators.

Lemma 1.24. *The following are equivalent*

(i) G is uniformly subelliptic with ellipticity constants $0 < \lambda \leq \Lambda$.

(ii) $\mathcal{M}^-_{\lambda,\Lambda}(M-N) \leq G(x, r, p, M) - G(x, r, p, N) \leq \mathcal{M}^+_{\lambda,\Lambda}(M-N)$ for every $M, N \in \mathcal{S}_m$ and $(x, r, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m$.

Proof. We use the well-known fact that any symmetric matrix $N \in \mathcal{S}_m$ can be uniquely decomposed as the sum of two nonnegative symmetric matrices with null product, i.e. there exist $N^+, N^- \geq 0$ such that $N^+ N^- = 0$ and $N = N^+ - N^-$ (which we call the positive and negative part of N). Owing to this property, one easily shows that (i) (i.e using the inequalities in Definition 1.23) implies

$$G(x, r, p, M) - G(x, r, p, M - N) \leq \Lambda \mathrm{Tr}(N^-) - \lambda \mathrm{Tr}(N^+)$$

for every $M, N \in \mathcal{S}_m$ and $(x, r, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^m$, and consequently the right-hand side of the inequality stated in (ii). Indeed

$$\begin{aligned} G(x, r, p, M) - G(x, r, p, M - N) &= G(x, r, p, M) - G(x, r, p, M + N^-) \\ &+ G(x, r, p, M + N^-) - G(x, r, p, M - N^+ + N^-) \leq \Lambda \mathrm{Tr}(N^-) - \lambda \mathrm{Tr}(N^+) \ . \end{aligned}$$

Similarly, one has

$$G(x, r, p, M) - G(x, r, p, M - N) \geq \lambda \mathrm{Tr}(N^-) - \Lambda \mathrm{Tr}(N^+) \ .$$

Then, using the definition of Pucci's extremal operators one immediately obtains that the above properties implies (ii). The proof of the fact that (ii) implies (i) is straightforward. $\square$

Remark 1.25. By taking $N = 0$ in Lemma 1.24-(ii), one immediately realizes that

$$\mathcal{M}^-_{\lambda,\Lambda}(M) \le G(x, r, p, M) - G(x, r, p, 0) \le \mathcal{M}^+_{\lambda,\Lambda}(M) \ .$$

Such equations with underlying subelliptic structure fall within the theory of viscosity solutions (cf [179]). Another way to look at such equations is to exploit their representation in Euclidean coordinates. To this aim, let $\sigma \in \mathbb{R}^{d\times m}$ be the matrix whose columns are the coefficients of the vector fields $X_1, ..., X_m$ with respect to the standard basis of $\mathbb{R}^d$. For any sufficiently smooth function u we have

$$D_{\mathcal{X}} u = \sigma^T(x) Du$$

and

$$(D^2_{\mathcal{X}} u(x))^* = \sigma^T(x) D^2 u \sigma(x) + g(x, Du) \ ,$$

where $g(x, p)$ is a $m \times m$ matrix whose elements are

$$g_{ij}(x, p) = \left(\frac{D\sigma^j(x)\sigma^i(x) + D\sigma^j(x)\sigma^i(x)}{2} \right) \cdot p \ ,$$

where the σ^j's are the columns of σ. Note that the first order term g is null for Carnot groups of step 2. Simple examples where the symmetrized horizontal Hessian contains also first order terms are the Engel group (which is a Carnot group of step 3, see e.g. [179, Example 3]), and the vector fields inducing the Grushin plane (see Chapter 2).

Chapter 2

Strong maximum principles for fully nonlinear degenerate elliptic PDEs

Maximum principles are among the most powerful tools in the study of elliptic and parabolic PDEs. In particular, they allow to deduce several quantitative results such as a priori estimates, and also uniqueness and stability theorems without knowing a priori the explicit form of the solution. As we shall see, various forms of maximum principles are linked to what are known in literature as comparison principles, and also some important qualitative properties such as Liouville theorems (that we analyze in Chapter 4). In particular, throughout this chapter we will deduce a form of the so-called *strong maximum principle* for functions satisfying suitable nonlinear "differential inequalities" that we will make precise below, and apply the results to study some comparison theorems. We are mainly interested in fully nonlinear second order degenerate elliptic equations arising in the context of stochastic control and differential games, among which HJB and HJI equations. Our results turn out to be new even for some quasi-linear equations modeled on the horizontal p- and ∞-Laplacian.

2.1 Propagation of maxima for linear degenerate equations: a survey

The purpose of this introductory section is to review some classical results for the propagation of maxima of linear degenerate PDEs. Let us start recalling that the strong maximum principle for the Laplace equation asserts that if $u \in C^2(\Omega)$, Ω being a connected open set of $\mathbb{R}^d$, solves $-\Delta u \leq 0$ and u takes its largest value at a point $x_0 \in \Omega$, then u is constant.
Let us turn now to a more general linear elliptic operator of the form

$$\mathcal{L}u := -\mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du$$

and assume that $A \geq 0$ (i.e. the equation can be degenerate), $A \in C^2(\mathbb{R}^d)$ with bounded derivatives, $b : \mathbb{R}^d \to \mathbb{R}^d$ is C^1 with bounded derivatives. As announced in the introduction, the problem of propagation of maxima can be formulated as follows

Problem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be open and connected and $x_0 \in \Omega$. Determine the largest connected, relatively closed subset $D(x_0)$ of Ω containing x_0, such that if $u \in C^2(\Omega)$ satisfies $\mathcal{L}u \leq 0$ in Ω and u attains its maximum at x_0, then u is constant throughout $D(x_0)$.*

The set $D(x_0)$ is usually named *propagation set* of $x_0 \in \Omega$ and will be henceforth denoted by $\mathrm{Prop}(x_0)$.
We now remind the coordinate-free description proposed by K. Taira [226], and obtained through analytical methods via the notion of subunit vector field for linear operators that we will extend to the fully nonlinear framework in the next section.
We recall that a vector field Z is subunit for $-\mathrm{Tr}(A(x)D^2u)$, or for the matrix $A \geq 0$, at a point x if $A - Z \otimes Z \geq 0$, i.e.

$$\xi^T A(x)\xi \geq |Z(x) \cdot \xi|^2 \ \forall \xi \in \mathbb{R}^d \ .$$

This concept was introduced by C. Fefferman and D.H. Phong in [111, Section 6.9-Theorem 6.9.4]. We highlight that this notion is coordinate free, in the sense that one can always diagonalize the matrix A at x so that $A(x) = (\lambda_i \delta_{ij})_{ij}$ with $\lambda_1, ..., \lambda_k > 0$ and $\lambda_{k+1} = ... = \lambda_d = 0$, where $k = \mathrm{rank}(A)$. As we shall see in the next Lemma 2.8, it is possible to prove that Z is subunit for A if and only if it is contained in the following k-dimensional ellipsoid

$$\left\{ \eta \in \mathbb{R}^d : \sum_{i=1}^{k} \frac{\eta_i^2}{\lambda_i} \leq 1 \ , \eta_{k+1} = ... = \eta_d = 0 \right\} \ .$$

A subunit trajectory is a Lipschitz path $\theta : [t_1, t_2] \to \Omega$ such that the tangent vector $\theta'(t)$ is subunit for A at $\theta(t)$ for almost every t. We also note that subunit trajectories are not oriented, i.e. if $\theta'(t)$ is subunit for A, $-\theta'(t)$ is subunit as well. Let also

$$X_0 := \sum_{i=1}^{d} (b_i(x) - \partial_i a_{ij}(x))\partial_i$$

be the so-called drift vector field. A drift trajectory is a curve $\theta : [t_1, t_2] \to \Omega$ such that $\theta'(t) = X_0(\theta(t))$ on $[t_1, t_2]$ oriented for increasing time. The main result on the characterization of the propagation set for linear degenerate equations can be stated as follows and was proved by K. Taira [226, Theorem 7.2.1].

Theorem 2.2. *The propagation set* $\mathrm{Prop}(x_0)$ *of $x_0 \in \Omega$ contains the closure in Ω of all points $y \in \Omega$ that can be reached from x following a finite number of subunit and drift trajectories.*

This result highlights the mechanism of propagation of maxima and why the strong maximum principle holds true for the Laplace equation (or generally, uniformly elliptic equations). Roughly speaking, it is saying that if A is nondegenerate, i.e. $k = \mathrm{rank}(A) = d$, then the maximum propagates in a neighborhood of x_0, but when A is degenerate, the maximum propagates only in a small ellipsoid of dimension k and in the direction of the vector field X_0.
In addition, D. W. Stroock and S.R. S. Varadhan [225] gave a (non-coordinate free)

characterization of the propagation set as a consequence of their results on the support of the diffusion process corresponding to the above linear operator. As it is proved in [226, Theorem 7.2.2], the propagation set in Theorem 2.2 coincides with the Stroock-Varadhan's characterization.
In the case when the operator is written as a sum of square of smooth vector fields $Y_1, ..., Y_m$, perturbed by a smooth vector field Y_0, i.e. $Lu := -\sum_{i=1}^m Y_i^2 u - Y_0$, J.-M. Bony [52] proved that the maximum propagates along the so-called Hill's diffusion and drift trajectories, whenever the Lie algebra generated by the vector fields has constant rank throughout Ω. We recall that a Hill's diffusion trajectory is a curve $\theta : [t_1, t_2] \to \Omega$ such that $\theta'(t) = Y_k(\theta(t))$ with $\theta'(t) \neq 0$ on $[t_1, t_2]$, while Hill's drift trajectories are defined by replacing Y_k with Y_0, see [135]. In particular, the propagation set in Theorem 2.2 coincides with that of J.-M. Bony and C. D. Hill, see [226, Theorem 7.2.4].
The results presented above are our starting points to develop the analysis on strong maximum principles for fully nonlinear degenerate equations, and our results can be seen as nonlinear degenerate extensions to these fundamental contributions.

2.2 Results, basic notions and standing assumptions

Our aim is to investigate the validity of SMPs and some Strong Comparison Principles for semicontinuous viscosity subsolutions and supersolutions of fully nonlinear second order PDEs

$$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega , \tag{2.1}$$

where $F : \overline{\Omega} \times \mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\}) \times \mathcal{S}_d \to \mathbb{R}$, Ω is an open connected set of $\mathbb{R}^d$ and $\mathcal{S}_d$ is the set of $d \times d$ symmetric matrices. Our basic assumptions are

(i) F is *lower semicontinuous* and *proper* in the sense of [96], i.e.

$$F(x, r, p, X) \leq F(x, s, p, Y) , \quad \text{if } r \leq s \ , \ Y \leq X \ ;$$

(ii) (*Scaling*) for some $\phi : (0, 1] \to (0, +\infty)$, F satisfies

$$F(x, \xi s, \xi p, \xi X) \geq \ \phi(\xi) F(x, s, p, X)$$

for all $\xi \in (0, 1]$, $s \in [-1, 0]$, $x \in \Omega$, $p \in \mathbb{R}^d \backslash \{0\}$, and $X \in \mathcal{S}_d$;

where $Y \leq X$ means that $X - Y$ is nonnegative semidefinite, the usual ordering in $\mathcal{S}_d$. Moreover we assume that the operator F is nondegenerate elliptic in the direction of some rank-one matrices identified by the next definition.

Definition 2.3. *$Z \in \mathbb{R}^d$ is a generalized* subunit vector *(briefly, SV) for F at $x \in \Omega$ if*

$$\sup_{\gamma > 0} F(x, 0, p, I - \gamma p \otimes p) > 0 \quad \forall p \in \mathbb{R}^d \ \text{ such that } \ Z \cdot p \neq 0;$$

$Z : \Omega \to \mathbb{R}^d$ is a subunit vector field *(briefly, SVF) if $Z(x)$ is SV for F at x for every $x \in \Omega$.*

The name is motivated by the the notion introduced by C. Fefferman and D.H. Phong [111] for linear operators

$$F(x, D^2u(x)) := -\mathrm{Tr}(A(x)D^2u(x)) \tag{2.2}$$

recalled in the previous section. It is easy to show that a classical subunit vector is a generalized SV in our sense, and that if Z is a SV according to Definition 2.3, with F linear, then rZ is subunit for the matrix A for all $r > 0$ small enough, see Section 2.3.1.

Our first result concerns the propagation of maxima of a subsolution to (2.1) along the trajectories of a subunit vector field.

Theorem 2.4. *Assume F satisfies* (i), (ii)*, and it has a locally Lipschitz subunit vector field Z. Suppose $u \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of* (2.1) *attaining a nonnegative maximum at $x_0 \in \Omega$. Then $u(x) = u(x_0) = \max_\Omega u$ for all $x = y(s)$ for some $s \in \mathbb{R}$, where $y'(t) = Z(y(t))$ and $y(0) = x_0$.*

If F has more than one SVF, say a family Z_i, $i = 1, \ldots, m$, we can piece together their trajectories to find a larger set of propagation of the maximum. It is natural to consider the control system

$$y'(t) = \sum_{i=1}^{m} Z_i(y(t))\beta_i(t) \ , \tag{2.3}$$

where the controls β_i are measurable functions taking values in a fixed neighborhood of 0. If this system has the property of *bounded time controllability* , namely

$$\forall\, x_0, x_1 \in \Omega \quad \exists \quad \text{a trajectory } y(\cdot) \text{ of (2.3) with } y(0) = x_0, y(s) = x_1, \qquad y(t) \in \Omega \ \forall\, t \in [0, s], \tag{BTC}$$

then a nonegative maximum of the subsolution u propagates to all Ω, and therefore u is constant. A classical sufficient condition for (BTC), for vector fields smooth enough, is the *Hörmander condition* [136], see also Remark 2.16 below

Definition 2.5 (Hörmander condition)**.** *The C^∞ vector fields $Z_1, ..., Z_m$ are said to satisfy the Hörmander condition if $Z_1, ..., Z_m$ and their commutators span $\mathbb{R}^d$ at each point of Ω.*

Then we have the following

Corollary 2.6 (Strong Maximum Principle)**.** *Assume* (i), (ii)*, and the existence of subunit vector fields Z_i, $i = 1, ..., m$, of F satisfying the Hörmander condition. Then any viscosity subsolution of* (2.1) *attaining a nonnegative maximum in Ω is constant.*

This result is a generalization to fully nonlinear equations of the classical maximum principle of Bony [52] for smooth subsolutions of linear equations (see also [226]).

Our main application concerns fully nonlinear subelliptic equations, as defined by Manfredi [179]. Given a family $\mathcal{X} = (X_1, ..., X_m)$ of $C^{1,1}$ vector fields in Ω one defines the intrinsic (or horizontal) gradient and intrinsic Hessian as

$$D_\mathcal{X} u = (X_1 u, ..., X_m u), \quad (D^2_\mathcal{X} u)_{ij} = X_i(X_j u).$$

A subelliptic equation has the form

$$G(x, u, D_{\mathcal{X}}u, (D^2_{\mathcal{X}}u)^*) = 0 \ , \tag{2.4}$$

where Y^* is the symmetrized matrix of Y and $G : \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^m \backslash \{0\}) \times \mathcal{S}_m \to \mathbb{R}$ satisfies at least (i). We assume that G is *elliptic for any x and p fixed* in the following sense:

$$\sup_{\gamma>0} G(x, 0, q, X - \gamma q \otimes q) > 0 \quad \forall\, x \in \Omega,\ q \in \mathbb{R}^m,\ q \neq 0,\ X \in \mathcal{S}_m. \tag{2.5}$$

By rewriting the equation (2.4) in Euclidean coordinates we find an equivalent equation of the form (5.47) with F having $X_1, ..., X_m$ as subunit vector fields. As a consequence we will prove the following SMP for fully nonlinear subelliptic problems:

Corollary 2.7. *Assume G verifies* (i), (ii)*, and* (2.5)*, and the vector fields $X_1, ..., X_m$ satisfy the Hörmander condition. Then any viscosity subsolution of* (2.4) *attaining a nonnegative maximum in Ω is constant.*

In Section 2.4.1 we give several examples of operators satisfying the assumptions of this result, including the p-Laplacian, the ∞-Laplacian, and Pucci's extremal operators associated to Hörmander vector fields. Let us recall that the generators of stratified Lie groups, or Carnot groups, satisfy the Hörmander property. Many examples of such sub-Riemannian structures can be found in [51], the most famous being the Heisenberg group, Example 2.26 and Subsection 1.0.2. Therefore the last Corollary applies to a large number of degenerate elliptic PDEs. In Section 2.4 we also give applications to Hamilton-Jacobi-Bellman and Isaacs equations.

Next we make an application to a Strong Comparison Principle, that is, the following property:

(SCP) *if u and v are a sub- and supersolution of* (5.47) *and $u - v$ attains a nonnegative maximum in Ω, then $u \equiv v$+constant.*

If Ω is bounded the SCP implies the usual (weak) Comparison Principle, namely, $u \leq v$ in Ω if in addition $u \in \mathrm{USC}(\overline{\Omega})$, $v \in \mathrm{LSC}(\overline{\Omega})$, and $u \leq v$ in $\partial\Omega$. For a class of equations that can be written in Hamilton-Jacobi-Bellman form we can show that $w := u - v$ is a subsolution of a homogeneous PDE $F_0(x, w, Dw, D^2w) = 0$ satisfying the SMP, and therefore we deduce immediately the SCP. A model problem is the equation

$$\mathcal{M}^+((D^2_{\mathcal{X}}u)^*) + H(x, Du) = 0, \tag{2.6}$$

where $\mathcal{M}^+$ denotes the Pucci's maximal operator (see Section 2.4.1 for the definition), $\mathcal{X} = (X_1, ..., X_m)$ are Hörmander vector fields, and $H(x, p) = \sup_\alpha \{p \cdot b^\alpha(x) + f^\alpha(x)\}$ with data b^α, f^α bounded and Lipschitz uniformly in α. Remarkably, this result implies the (weak) Comparison Principle also in some cases for which it was not yet known, see Section 2.5.

The plan for this chapter is the following. In Section 2.3 we prove a geometric property of the propagation set of an interior maximum in terms of SV and deduce the connection with the controllability of system (2.3), as well as a Hopf boundary lemma. Then we get some strong maximum and minimum principles. Section 2.4

presents the applications to some subelliptic nonlinear equations associated to a family of vector fields, to HJB and HJI equations, and some other examples. All these results are new, except for the Euclidean case, i.e., when $\mathcal{X}$ is a basis of $\mathbb{R}^d$. Finally, in Section 2.5 we prove the Strong Comparison Principle and give some examples.

2.3 Strong Maximum and Minimum Principles

2.3.1 Definitions and preliminaries

We begin by comparing our Definition 2.3 of subunit vector for the operator F with the classical one given by Fefferman-Phong for linear operators (2.2). We recall that a vector Z is subunit for A at a point x, that we freeze and do not display, if $A \geq Z \otimes Z(x)$. Then

$$F(x,0,p,I-\gamma p\otimes p) = -\mathrm{Tr}A+\gamma p\cdot Ap \geq -\mathrm{Tr}A+\gamma \sum_{i,j} Z_iZ_jp_jp_i = -\mathrm{Tr}A+\gamma|Z(x)\cdot p|^2$$

which can be made positive for γ large enough if $Z\cdot p \neq 0$. As a partial converse we can prove the following.

Lemma 2.8. *If Z is a SV at x for F linear* (2.2)*, then rZ is subunit for $A(x)$ for some $r>0$.*

Proof. In view of Definition 2.3, one easily observes that Z is SV if and only if

$$\sum_{i,j} a_{ij}p_ip_j = \mathrm{Tr}(Ap\otimes p) > 0 \text{ for all } p \text{ such that } p\cdot Z \neq 0.$$

Set $k = \mathrm{rank}(A)$. Then, one may always diagonalize the matrix A in order to have that

$$a_{ij} = \lambda_i\delta_{ij}\ ,\ \lambda_i > 0 \text{ for } i = 1,...k\ ,\ \lambda_i = 0 \text{ for } i = k+1,...,d\ ,$$

so the above condition reads

$$\sum_i \lambda_i p_i^2 > 0 \text{ for all } p \text{ such that } p\cdot Z \neq 0\ . \tag{2.7}$$

One can check the following easy characterization [226]: Z is subunit for A if and only if rZ is contained in the following ellipsoid

$$E := \left\{\eta\in\mathbb{R}^d : \sum_{i=1}^k \frac{\eta_i^2}{\lambda_i} \leq 1\ , \eta_{k+1} = ... = \eta_d = 0\right\}$$

for some small r. Then, if rZ does not belong to E there exists a component $Z_j \neq 0$ with $j = k+1,...,d$, since, up to rescaling, the condition $\sum_{i=1}^k \frac{\eta_i^2}{\lambda_i} \leq 1$ is always satisfied. Thus, by taking $p = e_j$ it follows that $p\cdot Z \neq 0$, but $\sum_i \lambda_i p_i^2 = 0$, a contradiction with (2.7). □

Example 2.9. It is easy to check, by means of Cauchy-Schwarz inequality, that the columns of a positive semidefinite matrix A are subunit vectors after multiplication by a sufficiently small constant. Moreover, if A can be decomposed as $A = \sigma\sigma^T$ with $\sigma \in \mathbb{R}^{d\times m}$, then the columns of σ are subunit vectors for A (see, e.g., [23, Example 2.2-2.3]).

Since equation (2.1) can be singular at $p = 0$, e.g. those involving the p-Laplacian, the notion of viscosity solution is slightly weakened with respect to the classical one [96], as follows:

Definition 2.10. *A function* $u \in \mathrm{USC}(\Omega)$ *(resp.* $\mathrm{LSC}(\Omega)$*) is a viscosity subsolution (resp. supersolution) of the* (2.1) *in* Ω *if, for every* $\varphi \in C^2(\Omega)$ *and* x *maximum (resp. minimum) point of* $u - \varphi$ *such that* $D\varphi(x) \neq 0$

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0 \ (\textit{resp.} \geq 0) \ .$$

We recall that the notion of viscosity solution for fully nonlinear PDEs is consistent with that of classical C^2 solution by standard arguments. In the case of the p-Laplace equation, the equivalence of the above definition to that of p-harmonic functions can be found in [145, Theorem 2.7 and Corollary 2.8]. From now on all sub- and supersolutions will be meant in the viscosity sense.

We define the *Propagation set* of a viscosity subsolution u of (2.1) attaining a nonnegative maximum at $x \in \Omega$ as

$$\mathrm{Prop}(x, u) := \{y \in \Omega : u(y) = u(x) = \max_{\Omega} u\}.$$

We will need the notion of generalized exterior normal, also called Bony normal or proximal normal (see, e.g., [52] or [15, Definition 2.17]):

Definition 2.11. *A unit vector* ν *is a* generalized exterior normal *to a nonempty set* $K \subseteq \mathbb{R}^d$ *at* $z \in \partial K$ *if there is a ball outside* K *centered at* $z + t\nu$ *for some* $t > 0$ *touching* $\overline{K}$ *precisely at* z*, i.e.* $\overline{B}(z + t\nu, t) \cap \overline{K} = \{z\}$*. Then we write that* $\nu \perp K$ *at* z*, and we use also the notation*

$$K^* := \{z \in \partial K : \textit{there exists } \nu \perp K \textit{ at } z\} \ .$$

As in the classical paper of Bony [52] we will use a geometric characterization of invariant sets for the control system (2.3), that we recall next. We consider as admissible the control functions $\beta = (\beta_1, ..., \beta_m) : [0, +\infty) \to \mathbb{R}^m$ in the set

$$\mathcal{B} := \{\beta : \sum_{i=1}^{m} \beta_i^2(t) \leq 1 \text{ and } \beta_i \text{ is measurable } \forall\, i = 1, ..., m\},$$

and denote with $y_x(\cdot, \beta)$ the solution of the system (2.3) with initial condition $y(0) = x$, which exists at least locally if the vector fields $Z_i : \overline{\Omega} \to \mathbb{R}^d$ are locally Lipschitz. A set $K \subseteq \overline{\Omega}$ is *invariant for the system* (2.3) if for all $x \in K$, $\beta \in \mathcal{B}$ and $\tau > 0$ such that the solution $y_x(\cdot, \beta)$ exists in $[0, \tau)$, we have $y_x(t, \beta) \in K$ for all $t \in [0, \tau)$.

Theorem 2.12. *Let $Z_i : \overline{\Omega} \to \mathbb{R}^d$ be locally Lipschitz and $K \neq \emptyset$ be a relatively closed subset of Ω. If for all $x \in K^* \cap \Omega$ and for all $\nu \perp K$ at x*

$$Z_i(x) \cdot \nu = 0 \quad \forall\, i = 1, \dots, m, \tag{2.8}$$

then K is invariant for (2.3).

Proof. We can repeat the proof of [22, Theorem 2.1], which combines the classical result for $\Omega = \mathbb{R}^d$ with a localization argument. Then it is easy to see that it is enough to assume (2.8) at points $x \in \partial K \cap \Omega$. □

2.3.2 Propagation of maxima

We first give a technical result providing a crucial geometric property of the propagation set.

Proposition 2.13. *Let u be a viscosity subsolution of* (2.1) *that achieves a nonnegative maximum at $x \in \Omega$. Assume that* (i)-(ii) *hold and F has a subunit vector field as in Definition 2.3. Then $K := \mathrm{Prop}(x, u)$ is such that for every $z \in K^* \cap \Omega$ and for every $\nu \perp K$ at z we have $Z \cdot \nu = 0$ for every subunit vector of F at z.*

Proof. We fix $z \in \partial K \cap \Omega$ and $\nu \perp K$ at z. Arguing by contradiction, we assume there exists a subunit vector $\bar{Z}$ at z such that $\bar{Z} \cdot \nu \neq 0$. By definition of normal we can take $R > 0$ and $y = z + R\frac{\nu}{|\nu|}$ such that $B(y, R) \subseteq \Omega \backslash K$. We divide the proof in two steps.

Step 1. We claim that there exist $r > 0$ and a function $v \in C^2(\mathbb{R}^d)$ such that

$$F(x, v(x), Dv(x), D^2v(x)) \geq C > 0 \text{ for every } x \in B(z, r)\ ,$$

with the properties $v(z) = 0$, $-1 < v < 0$ in $B(y, R)$ and $v > 0$ outside $B(y, R)$. To see this, consider

$$v(x) = e^{-\gamma R^2} - e^{-\gamma |x-y|^2}\ . \tag{2.9}$$

Note that $v \equiv 0$ on $\partial B(y, R)$ (which gives $v(z) = 0$) and $v > 0$ outside $B(y, R)$. Moreover $-1 < v < 0$ in $B(y, R)$. By direct computations we have

$$Dv(x) = 2\gamma e^{-\gamma |x-y|^2}(x - y)$$

and

$$D^2v(x) = 2\gamma e^{-\gamma |x-y|^2}(I - 2\gamma(x - y) \otimes (x - y))\ .$$

Now, using that $z - y = -\nu$ and the scaling property (ii) we have

$$\begin{aligned} F(z, v(z), Dv(z), D^2v(z)) &= F(z, 0, 2\gamma e^{-\gamma R^2}(-\nu), 2\gamma e^{-\gamma R^2}(I - 2\gamma \nu \otimes \nu)) \\ &\geq \phi(2\gamma e^{-\gamma R^2}) F(z, 0, -\nu, I - 2\gamma \nu \otimes \nu)\ . \end{aligned} \tag{2.10}$$

By the definition of subunit vector at z and $\bar{Z} \cdot \nu \neq 0$ we obtain

$$F(z, 0, -\nu, I - 2\gamma \nu \otimes \nu) > 0$$

for some $\gamma > 0$. Then (2.10) and $\phi(\xi) > 0$ for all $\xi > 0$ give

$$F(z, v(z), Dv(z), D^2v(z)) > 0 \ .$$

Since F is lower semicontinuous we can conclude that there exists $r > 0$ such that

$$F(x, v(x), Dv(x), D^2v(x)) \geq C > 0 \text{ for every } x \in B(z, r) \ . \tag{2.11}$$

Step 2. We claim now that there exists $\epsilon > 0$ such that $u(x) - u(z) \leq \epsilon v(x)$ in $X := B(z, r) \cap B(y, R)$.
Let us choose $\epsilon > 0$ small enough such that $u(x) - u(z) \leq \epsilon v(x)$ for every $x \in \partial X$. To prove that the inequality holds on the whole X, suppose by contradiction that there exists $\bar{x} \in X$ such that $u(\bar{x}) - u(z) - \epsilon v(\bar{x}) = \max_X(u - u(z) - \epsilon v) > 0$. Since ϵv is smooth in $\mathbb{R}^d$, using that $u - u(z)$ is a viscosity subsolution of (2.1) and the scaling property (ii) we get

$$\phi(\epsilon)F(\bar{x}, v(\bar{x}), Dv(\bar{x}), D^2v(\bar{x})) \leq F(\bar{x}, \epsilon v(\bar{x}), \epsilon Dv(\bar{x}), \epsilon D^2v(\bar{x})) \leq 0$$

which contradicts (2.11) because $\phi > 0$.
Then $u(x) - \epsilon v(x) \leq u(z)$ and $u(z) - \epsilon v(z) = u(z)$ since $v(z) = 0$. Therefore, the function $\Phi(x) := u(x) - \epsilon v(x)$ has a maximum at z in X. Moreover, in $B(z, r)\backslash X$, we have $v \geq 0$ and $u(x) - \epsilon v(x) \leq u(x) \leq u(z)$. As a consequence the function $\Phi(x)$ has a maximum in $B(z, r)$ at z. Since $\epsilon v \in C^\infty(\mathbb{R}^d)$, F is proper, using also the definition of viscosity subsolution and (ii), we get

$$\phi(\epsilon)F(z, v(z), Dv(z), D^2v(z)) \leq F(z, u(z), \epsilon Dv(z), \epsilon D^2v(z)) \leq 0,$$

a contradiction with (2.11). □

Our main result is the following, containing Theorem 2.4 as a special case.

Theorem 2.14. *Let u be a viscosity subsolution of* (2.1) *that achieves a nonnegative maximum at $x \in \Omega$. Assume that* (i)-(ii) *hold and F has locally Lipschitz continuous subunit vector fields $Z_i : \overline{\Omega} \to \mathbb{R}^d$, $i = 1, ..., m$. Then* $\text{Prop}(x, u)$ *contains all the points reachable by the system* (2.3) *starting at x, i.e., if $y = y_x(t, \beta)$ for some $t > 0, \beta \in \mathcal{B}$, then $y \in$* $\text{Prop}(x, u)$.

Proof. If $\text{Prop}(x, u) = \Omega$ the conclusion is true. Otherwise, for all $z \in \partial \,\text{Prop}(x, u) \cap \Omega$ Proposition 2.13 implies $Z_i(z) \cdot \nu = 0$ for all $\nu \perp \text{Prop}(x, u)$ at z and $i = 1, \ldots, m$. Then Theorem 2.12 ensures the invariance of $\text{Prop}(x, u)$ for the system (2.3), and therefore all trajectories starting at x remain forever in $\text{Prop}(x, u)$. □

Corollary 2.15 (Strong Maximum Principle)**.** *In addition to the assumptions of Theorem 2.14 suppose the system* (2.3) *satisfies the bounded time controllability property* (BTC)*. Then u is constant.*

Proof. If (BTC) holds then any point of Ω is reachable by the system (2.3) starting at x. Then Theorem 2.14 gives $\text{Prop}(x, u) = \Omega$. □

Remark 2.16. Before proving Corollary 2.6 we recall that the classical Hörmander condition requires that

(H) *the vector fields Z_i, $i = 1, ..., m$, are C^∞ and the Lie algebra generated by them has full rank d at each point of Ω.*

The smoothness requirement on Z_i can be reduced to C^k for a suitable k and considerably more if the Lie brackets are interpreted in a generalized sense, see [112] and the references therein.

Proof of Corollary 2.6. By the classical Chow-Rashevskii theorem in sub-Riemannian geometry and its control-theoretic version (see, e,g, [15, Lemma IV.1.19]), for any $z \in \Omega$ the set of points reachable from z by the system contains a neighborhood of z. Since $u \in \mathrm{USC}(\Omega)$, $K = \mathrm{Prop}(x, u) = \{y \in \Omega : u(y) = \max u\}$ is relatively closed. Then Ω connected implies that either $K = \Omega$ or K is not relatively open. In the latter case there would be $z \in K$ with no neighborhood contained in K, a contradiction with Theorem 2.14. Then $K = \Omega$. $\square$

Remark 2.17. Note that the existence of a SV at x for F and the scaling property (ii) imply

$$\limsup_{(s,p,X)\to(0,0,0)} F(x, s, p, X) \geq 0\,,$$

a weaker condition than $F(x, 0, 0, 0) \geq 0$ used in [148].

Remark 2.18. It is easy to see from the proof of Proposition 2.13 that the function ϕ in the scaling property (ii) can be allowed to depend also on x, s, p, and X. What is really needed is that $F(x, s, p, X) > 0$ implies $F(x, \xi s, \xi p, \xi X) > 0$ for all $\xi \in (0, 1]$ and all x, s, p, X.

Remark 2.19. In all the previous results the scaling assumption (ii) on F can be avoided if there is $\tilde{F}$ satisfying all conditions and approximating F in the sense that

$$F(x, \epsilon s, \epsilon p, \epsilon X) \geq \tilde{F}(x, \epsilon s, \epsilon p, \epsilon X) + \phi(\epsilon)\psi(\epsilon)$$

with $\lim_{\epsilon\to 0^+} \psi(\epsilon) = 0$. Indeed, in the proof of Proposition 2.13 one can see that (2.11) still holds under this assumption (cf. [21]).

We end this section with

Lemma 2.20 (Hopf boundary lemma)**.** *Let $U \subseteq \Omega$ be an open set, $x_0 \in \partial U$, $u \in \mathrm{USC}(U \cup \{x_0\})$ be a viscosity subsolution of* (2.1) *in U such that*

(a) $u(x_0) > u(x)$ for every $x \in U$ and $u(x_0) \geq 0$;

(b) there exists a ball $B := B(y, R)$ such that $B \subseteq U$ and $\overline{B} \cap \partial U = \{x_0\}$.

Assume that F satisfies (i)-(ii) and there exists a SV Z for F such that $p := x_0 - y$ satisfies $p \cdot Z \neq 0$. Then, for any $w \in \mathbb{R}^d$ such that $w \cdot p < 0$, we have

$$\limsup_{\tau\to 0^+} \frac{u(x_0 + \tau w) - u(x_0)}{\tau} < 0$$

Proof. As in Step 1 of Proposition 2.13 we define v as in (2.9), which turns out to be a strict classical supersolution in $\overline{X} := B \cap B(x_0, r)$ for a suitably small $r > 0$ because $p \cdot Z \neq 0$. Then, arguing as in Step 2 of Proposition 2.13 one proves that $u(x) - u(x_0) \leq \epsilon v(x)$ for every $x \in \overline{X}$. To conclude, it is then sufficient to observe that, for any $w \in \mathbb{R}^d$ such that $w \cdot p < 0$, one has

$$\limsup_{\tau \to 0^+} \frac{u(x_0 + \tau w) - u(x_0)}{\tau} \leq \varepsilon Dv(x_0) \cdot w = 2\gamma e^{-\gamma |x_0 - y|^2} p \cdot w < 0 \ .$$

□

2.3.3 Propagation of minima

Various Strong Minimum Principles for (viscosity) supersolutions of (2.1) can be easily derived from the results of the previous section by recalling that $v \in \mathrm{LSC}(\Omega)$ is a supersolution of (2.1) if and only if $u = -v$ is a subsolution of

$$-F(x, -u, -Du, -D^2 v) = 0 \quad \text{in } \ \Omega.$$

Therefore one can read properties of the minima of v from the preceding results by applying them to u and

$$F^-(x, r, p, X) := -F(x, -r, -p, -X).$$

Let us make explicit the assumptions on F that imply a Strong Minimum Principle. First we replace (i)-(ii) by

(i') $F : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \backslash \{0\} \times \mathcal{S}_d \to \mathbb{R}$ is upper semicontinuous and proper.

(ii') For some $\phi > 0$ the operator satisfies $F(x, \xi s, \xi p, \xi X) \leq \ \phi(\xi) F(x, s, p, X)$ for all $\xi \in (0, 1]$ and $s \in [0, 1]$.

Moreover, a vector Z is a subunit vector for F^- at x if and only if

$$\inf_{\gamma > 0} F(x, 0, p, \gamma p \otimes p - I) < 0 \quad \forall p \in \mathbb{R}^d \ \text{ such that } \ Z \cdot p \neq 0. \tag{2.12}$$

Now we can easily get the following properties of minima.

Corollary 2.21. *Let $v \in \mathrm{LSC}(\Omega)$ be a viscosity supersolution of* (2.1) *that achieves a nonnegative minimum at $x \in \Omega$. Assume that* (i')-(ii') *hold and $Z_i : \overline{\Omega} \to \mathbb{R}^d$, $i = 1, ..., m$, are locally Lipschitz subunit vector fields of F^-, i.e., at each $x \in \Omega$ $Z_i(x)$ verifies* (2.12)*. Then $v(y) = v(x) = \min_\Omega v$ for all points y reachable by the system* (2.3) *starting at x.*

Corollary 2.22 (Strong Minimum Principle)**.** *In addition to the assumptions of Corollary 2.22 suppose the system* (2.3) *satisfies the bounded time controllability property* (BTC)*. Then v is constant. This holds in particular if the fields Z_i, $i = 1, ..., m$, verify the Hörmander condition.*

2.4 Some applications

2.4.1 Fully Nonlinear Subelliptic Equations

Our main application concerns fully nonlinear subelliptic equations. In this framework, one is given a family $\mathcal{X} = (X_1, ..., X_m)$ of $C^{1,1}$ vector fields defined in $\overline{\Omega}$. The intrinsic gradient and intrinsic Hessian are defined as $D_{\mathcal{X}}u = (X_1 u, ..., X_m u)$ and $(D^2_{\mathcal{X}}u)_{ij} = X_i(X_j u)$. After choosing a base in Euclidean space we write $X_j = \sigma^j \cdot D$, with $\sigma^j : \overline{\Omega} \to \mathbb{R}^d$, and $\sigma = \sigma(x) = [\sigma^1(x), ..., \sigma^m(x)] \in \mathbb{R}^{d\times m}$. Then

$$D_{\mathcal{X}}u = \sigma^T Du = (\sigma^1 \cdot Du, ..., \sigma^m \cdot Du)$$

and

$$X_i(X_j u) = (\sigma^T D^2 u\ \sigma)_{ij} + (D\sigma^j\ \sigma^i) \cdot Du\ .$$

Denote by Y^* the symmetrized matrix of Y. By the chain rule (see, e.g., [34, Lemma 3]) one can obtain that for $u \in C^2$

$$(D^2_{\mathcal{X}}u)^* = \sigma^T D^2 u\sigma + g(x, Du)\ ,$$

where the correction term g is

$$(g(x, P))_{ij} = \frac{1}{2}[(D\sigma^j\ \sigma^i) \cdot p + (D\sigma^i\ \sigma^j) \cdot p]\ .$$

Then the subelliptic equation (2.4) can be written as

$$G(x, u, \sigma^T(x)Du, \sigma^T(x)D^2u\sigma(x) + g(x, Du)) = 0\ , \tag{2.13}$$

which is of the form (5.47) if we define

$$F(x, r, p, X) := G(x, r, \sigma^T(x)p, \sigma^T(x)X\sigma(x) + g(x, p))\ . \tag{2.14}$$

Lemma 2.23. *If G satisfies properties* (i), (ii) *and* (2.5)*, then F satisfies properties* (i) *and* (ii) *and the vector fields σ^i are subunit for F in the sense of Definition 2.3.*

Proof. (i) holds because $X \le Y$ implies $\sigma^T(x)X\sigma(x) \le \sigma^T(x)Y\sigma(x)$, so F is proper.

(ii) holds for F if it does for G because $g(x, p)$ is positively 1-homogeneous in the variable p.

To prove that any X_i is SV for F we use property (2.5) of G with $q = \sigma^T(x)p$, $X = \sigma^T\sigma + g$ to get

$$F(x, 0, p, I-\gamma p\otimes p) = G(x, 0, \sigma(x)^T p, \sigma^T(x)I\sigma(x)-\gamma(\sigma^T(x)p)\otimes(\sigma^T(x)p)+g(x, p)) > 0$$

for some $\gamma > 0$ if $\sigma^i(x) \cdot p \neq 0$. $\square$

This Lemma and Theorem 2.14 give the following propagation of maxima and SMP.

Corollary 2.24. *Assume G verifies* (i), (ii), *and* (2.5)*, and let u be a subsolution of* (2.4) *or, equivalently,* (2.13)*, attaining a maximum at $x \in \Omega$. Then* $\mathrm{Prop}(x, u)$ *contains all the points reachable from x by the system* (2.3) *with $Z_i = X_i$. In particular if the property* (BTC) *holds for such system then u is constant.*

From this we get immediately the Strong Maximum Principle for subelliptic equations with the Hörmander condition, Corollary 2.7, as in the proof of Corollary 2.6.

Example 2.25. A very simple example in $\mathbb{R}^2$ of vector fields that fail to span all $\mathbb{R}^2$ at some point but satisfy the Hörmander condition are the Grushin vector fields, namely,

$$\sigma_{\mathcal{G}}(x) = \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix} .$$

In this case the symmetrized horizontal Hessian is given by

$$(D^2_{\mathcal{X}}u)^* = \sigma^T_{\mathcal{G}}(x) D^2 u \sigma_{\mathcal{G}}(x) + g(x, Du) = \begin{pmatrix} u_{x_1x_1} & x_1 u_{x_1x_2} + \frac{u_{x_2}}{2} \\ x_1 u_{x_1x_2} + \frac{u_{x_2}}{2} & x_1^2 u_{x_2x_2} \end{pmatrix} .$$

Example 2.26. The most studied examples of vector fields satisfying the Hörmander condition are the generators of a Carnot group: see the treatise [51] for a comprehensive introduction and for the theory of linear subelliptic equations in such groups. The simplest prototype of Carnot group is the Heisenberg group $\mathbb{H}^1$ in $\mathbb{R}^3$ whose generators are

$$\sigma_{\mathbb{H}^1}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x_2 & -2x_1 \end{pmatrix} .$$

Here the correction term of the Hessian is $g \equiv 0$, and this occurs for all groups of step 2. An example of Carnot group of step 3 where $g(x,p) \neq 0$ is the Engel group, see e.g. [34, Example 3].

Next we list some examples of equations of the form

$$c(x)|u|^{k-1}u - a(x)E(D_{\mathcal{X}}u, (D^2_{\mathcal{X}}u)^*) = 0 \tag{2.15}$$

where we assume $E : \mathbb{R}^d \backslash \{0\} \times \mathbb{R}^{m\times m}$ is positively homogeneous of degree $\alpha \geq 0$, c, a are continuous and satisfy

$$c \geq 0 \ , \ a > 0 \ , \quad \text{and} \quad \text{either } c = 0 \text{ or } \alpha \leq k \ , k > 0 \ . \tag{2.16}$$

We give some examples of operators E for which the SMP and Strong Minimum Principle for equation (2.15) are known to hold in the Euclidean case, i.e., if the fields $\mathcal{X}$ are the canonical basis of $\mathbb{R}^d$, see [21]. Our contribution is that they hold for Hörmander vector fields as well.

Example 2.27. The *subelliptic* ∞*-Laplacian* [40, 42, 234] is

$$-\Delta_{\mathcal{X},\infty}u = -D_{\mathcal{X}}u \cdot (D^2_{\mathcal{X}}u)^* D_{\mathcal{X}}u$$

where $E = -p \cdot Xp$ is homogeneous of degree $\alpha = 3$ and (2.5) is satisfied because

$$E(q, X - \gamma q \otimes q) = -q \cdot Xq + \gamma |q|^4 .$$

Note that the associated operator F satisfies also the condition (2.12). Then the equation (2.15) with E the ∞-Laplacian satisfies both the SMP and the Strong Minimum Principle.

Example 2.28. A generalization of the previous example (considered in [43] for the evolutive case) is

$$-\Delta^h_{\mathcal{X},\infty} u = -|D_{\mathcal{X}} u|^{h-3} (D^2_{\mathcal{X}} u)^* D_{\mathcal{X}} u \cdot D_{\mathcal{X}} u$$

with $h \geq 0$, where E is homogeneous of degree h and satisfies (2.5) because

$$E(q, X - \gamma q \otimes q) = E(q, X) + \gamma |q|^{h+1} \ .$$

Example 2.29. The *subelliptic p-Laplacian*, $m > 1$, is

$$-\Delta_{\mathcal{X},p} u := -\mathrm{div}_{\mathcal{X}}(|D_{\mathcal{X}} u|^{p-2} D_{\mathcal{X}} u) = -(|D_{\mathcal{X}} u|^{p-2} \Delta_{\mathcal{X}} u + (p-2)|D_{\mathcal{X}} u|^{p-4} \Delta_{\mathcal{X},\infty} u)$$

where $\Delta_{\mathcal{X}} u := \mathrm{Tr}(D^2_{\mathcal{X}} u)$ is the sub-Laplacian. Here E is homogeneous of degree $\alpha = p - 1$ and (2.5) holds because

$$E(q, X - \gamma q \otimes q) = E(q, X) + \gamma |q|^p (p-1).$$

Similarly one checks (2.12). Recently the SMP and a Strong Comparison Principle were proved in [67] for weak C^1 solution of similar equations involving the subelliptic m-Laplacian.

Since the p-Laplacian is in divergence form the natural notion of solution for $-\Delta_{\mathcal{X},p} u = 0$ is variational. The equivalence of solutions in Sobolev spaces with viscosity solutions was shown by T. Bieske [41] in Carnot groups. For this homogeneous equation the SMP can also be deduced from the Harnack inequality, see the references in [67].

Example 2.30. For fixed $0 < \lambda \leq \Lambda$, the *Pucci's extremal operators* on symmetric matrices $M \in \mathcal{S}_d$ are

$$\mathcal{M}^+(M) := -\lambda \sum_{e_k > 0} e_k - \Lambda \sum_{e_k < 0} e_k = \sup\{-\mathrm{Tr}(AM) : A \in \mathcal{S}_d,\ \lambda I \leq A \leq \Lambda I\} \quad (2.17)$$

$$\mathcal{M}^-(M) = -\Lambda \sum_{e_k > 0} e_k - \lambda \sum_{e_k < 0} e_k = \inf\{-\mathrm{Tr}(AM) : A \in \mathcal{S}_d,\ \lambda I \leq A \leq \Lambda I\}. \quad (2.18)$$

They are 1-homogeneous and satisfy (2.5) because

$$\mathcal{M}^+(X - \gamma q \otimes q) \geq \mathcal{M}^-(X - \gamma q \otimes q) \geq \mathcal{M}^-(X) - \lambda \gamma |q|^2.$$

If we take a *subelliptic Pucci's operator* $E((D^2_{\mathcal{X}} u)^*) = \mathcal{M}^+((D^2_{\mathcal{X}} u)^*)$ then the equation (2.15) satisfy the SMP and the Strong Minimum principle, and the same holds if $\mathcal{M}^+$ is replaced by $\mathcal{M}^-$.

2.4.2 Hamilton-Jacobi-Bellman Equations

We are given a family of linear degenerate elliptic operators

$$L^\alpha u := -\mathrm{Tr}(A^\alpha(x) D^2 u) - b^\alpha(x) \cdot Du + c^\alpha(x) u \quad (2.19)$$

where the parameter α takes values in a given set, $A^\alpha(x) \geq 0$ and $c^\alpha(x) \geq 0$ for all x and α. The H-J-B operators are

$$F_s(x,u,Du,D^2u) := \sup_\alpha L^\alpha u\,, \qquad F_i(x,u,Du,D^2u) := \inf_\alpha L^\alpha u \tag{2.20}$$

and we assume that $F_s(x,r,p,X), F_i(x,r,p,X)$ are finite and continuous for all entries. They are clearly proper and positively 1-homogeneous. We can characterize the subunit vectors of these operators as follows.

Lemma 2.31. *Let $Z \in \mathbb{R}^d$ and $x \in \Omega$.*
i) Z is SV for F_i at x if and only if Z is subunit for all the matrices $A^\alpha(x)$, i.e., $A^\alpha(x) \geq Z \otimes Z$ for all α;
ii) Z is SV for F_s at x if there exists $\bar\alpha$ such that Z is subunit for the matrix $A^{\bar\alpha}(x)$.

Proof. i) First suppose $A^\alpha(x) \geq Z \otimes Z$ for all α. Then, for $p \cdot Z \neq 0$ and γ large enough,

$$\begin{aligned} F_i(x,0,p,I-\gamma p\otimes p) &= \inf_\alpha\{-\mathrm{Tr}A^\alpha(x) + \gamma p\cdot A^\alpha(x)p - b^\alpha(x)\cdot p\} \\ &\geq \inf_\alpha\{-\mathrm{Tr}A^\alpha(x) - b^\alpha(x)\cdot p\} + \gamma|Z\cdot p|^2 > 0\,. \end{aligned}$$

Viceversa, suppose Z is not a subunit vector of $A^{\bar\alpha}(x)$. Then there exist $\bar p$ such that $\bar p \cdot Z \neq 0$ and $\bar p \cdot A^{\bar\alpha}(x)\bar p = 0$. Then, for any $\eta \in \mathbb{R}$ and $\gamma > 0$

$$F_i(x,0,\eta\bar p, I-\gamma\eta^2\bar p\otimes\bar p) \leq -\mathrm{Tr}A^{\bar\alpha}(x) - \eta b^{\bar\alpha}(x)\cdot\bar p \leq -\eta b^{\bar\alpha}(x)\cdot\bar p\,.$$

But the right hand side is ≤ 0 by choosing $\eta = \mathrm{sign}(b^{\bar\alpha}\cdot\bar p)$, and so Z is not SV for F_i.

ii) Suppose $A^{\bar\alpha}(x) \geq Z \otimes Z$. Then, for $p\cdot Z \neq 0$ and γ large enough

$$\begin{aligned} F_s(x,0,p,I-\gamma p\otimes p) &= \sup_\alpha\{-\mathrm{Tr}A^\alpha(x) + \gamma p\cdot A^\alpha(x)p - b^\alpha(x)\cdot p\} \\ &\geq -\mathrm{Tr}A^{\bar\alpha}(x) + \gamma|Z\cdot p|^2 - b^{\bar\alpha}(x)\cdot p > 0\,. \end{aligned}$$

□

The results of sections 2.3.2 and 2.3.3 combined with this Lemma give informations on the sets of propagation of maxima and minima of sub- and supersolutions. This was studied in detail in the papers by M. Bardi and F. Da Lio [22, 23] using also tools from diffusion processes and differential games. Therefore we only point out explicitly a SMP for the concave H-J-B operator F_i that we will exploit in Section 2.5. Its proof is an immediate consequence of Corollary 2.15 and Lemma 2.31, and therefore it is more direct than the one in [23]. We also give a Strong Minimum Principle for the convex operator F_s following from Corollary 2.22.

Corollary 2.32. *Assume $Z_i : \overline{\Omega} \to \mathbb{R}^d$, $i = 1, ..., m$, are locally Lipschitz vector fields such that*

$$A^\alpha(x) \geq Z_i(x) \otimes Z_i(x) \quad \textit{for all } \alpha,\ i, \textit{ and } x\,,$$

and the system (2.3) *satisfies the bounded time controllability property* (BTC)*. Then*
i) any subsolution of $\inf_\alpha L^\alpha u = 0$ attaining a maximum in Ω is constant,
ii) any supersolution of $\sup_\alpha L^\alpha u = 0$ attaining a minimum in Ω is constant.

2.4.3 Hamilton-Jacobi-Isaacs Equations

Now, we are given a two-parameter family of linear degenerate elliptic operators

$$L^{\alpha,\beta}u := -\mathrm{Tr}(A^{\alpha,\beta}(x)D^2u) - b^{\alpha,\beta}(x)\cdot Du + c^{\alpha,\beta}(x)u$$

where the parameters α, β take values in two given sets, $A^{\alpha,\beta}(x) \geq 0$ and $c^{\alpha,\beta}(x) \geq 0$ for all x, α, β. The Hamilton-Jacobi-Isaacs (briefly, H-J-I) operators are

$$F_-(x,u,Du,D^2u) := \sup_\beta \inf_\alpha L^{\alpha,\beta}u\,, \qquad F_+(x,u,Du,D^2u) := \inf_\alpha \sup_\beta L^{\alpha,\beta}u$$

and we assume that $F_-(x,r,p,X), F_+(x,r,p,X)$ are finite and continuous for all entries $(x,r,p,X) \in \overline{\Omega} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d$. They are clearly proper and positively 1-homogeneous. We can find subunit vectors of these operators following the arguments of Lemma 2.31.

Lemma 2.33. *Let $Z \in \mathbb{R}^d$ and $x \in \Omega$.*
i) Z is SV for F_- at x if there exists $\bar\beta$ such that $A^{\alpha,\bar\beta}(x) \geq Z \otimes Z$ for all α;
ii) Z is SV for F_+ at x if for all α there exists $\beta(\alpha)$ such that $A^{\alpha,\beta(\alpha)}(x) \geq Z \otimes Z$.

Then we get the following SMP for the H-J-I equations.

Corollary 2.34. *Assume $Z_i : \overline{\Omega} \to \mathbb{R}^d$, $i = 1, ..., m$, are locally Lipschitz vector fields such that the system* (2.3) *satisfies the bounded time controllability property* (BTC). *Then*
i) if there exists $\bar\beta$ such that

$$A^{\alpha,\bar\beta}(x) \geq Z_i(x) \otimes Z_i(x) \quad \textit{for all } \alpha,\ i,\ \textit{and } x\,,$$

then any subsolution of $\sup_\beta \inf_\alpha L^{\alpha,\beta}u = 0$ attaining a maximum in Ω is constant;
ii) if for all α there exists $\beta(\alpha)$ such that

$$A^{\alpha,\beta(\alpha)}(x) \geq Z_i(x) \otimes Z_i(x) \quad \textit{for all } i \textit{ and } x\,,$$

then any subsolution of $\inf_\alpha \sup_\beta L^{\alpha,\beta}u = 0$ attaining a maximum in Ω is constant.

Sufficient conditions for the Strong Minimum Principle can be easily found in the same way, as follows.

Corollary 2.35. *Assume $Z_i : \overline{\Omega} \to \mathbb{R}^d$, $i = 1, ..., m$, are locally Lipschitz vector fields such that the system* (2.3) *satisfies the bounded time controllability property* (BTC). *Then*
i) if for all β there exists $\alpha(\beta)$ such that

$$A^{\alpha(\beta),\beta}(x) \geq Z_i(x) \otimes Z_i(x) \quad \textit{for all } i \textit{ and } x\,,$$

then any supersolution of $\sup_\beta \inf_\alpha L^{\alpha,\beta}u = 0$ attaining a minimum in Ω is constant;
ii) if there exists $\bar\alpha$ such that

$$A^{\bar\alpha,\beta}(x) \geq Z_i(x) \otimes Z_i(x) \quad \textit{for all } \beta,\ i,\ \textit{and } x\,,$$

then any supersolution of $\inf_\alpha \sup_\beta L^{\alpha,\beta}u = 0$ attaining a minimum in Ω is constant.

Example 2.36. If $\mathcal{X} = (X_1, ..., X_m)$ are $C^{1,1}$ vector fields on $\overline{\Omega}$ satisfying (BTC), $a, b \in C(\overline{\Omega})$ are nonnegative, and $\mathcal{M}^+, \mathcal{M}^-$ denote the Pucci's extremal operators, then the equation

$$a(x)\mathcal{M}^+((D^2_{\mathcal{X}}u)^*) + b(x)\mathcal{M}^-((D^2_{\mathcal{X}}u)^*) = 0$$

is of H-J-I form and satisfies both the SMP and the Strong Minimum Principle.

2.4.4 Other examples and remarks

All the examples of the previous sections satisfy the following property, stronger than Definition 2.3,

$$\lim_{\gamma\to+\infty} F(x, 0, p, I - \gamma p \otimes p) = +\infty \quad \forall p \in \mathbb{R}^d \text{ such that } Z \cdot p \neq 0. \tag{2.21}$$

If F has a SV Z at x satisfying (2.21), then clearly Z is a SV at x also for any perturbation of F with first or zero-th order terms

$$\bar{F}(x, r, p, X) = F(x, r, p, X) + H(x, r, p).$$

As a consequence, if F satisfies a SMP and H is lower semicontinuous, non-decreasing in r, and satisfies (ii) with the same ϕ as F, then $\bar{F}$ satisfies the same SMP as F.

Example 2.37. Consider the following perturbation of a Pucci's subelliptic equation associated to Hörmander vector fields $\mathcal{X}$

$$c(x)|u|^{k-1}u - a(x)\mathcal{M}^+((D^2_{\mathcal{X}}u)^*) + H(x, Du) = 0\,,$$

where c, a, H are continuous and satisfy

$$c \geq 0\ ,\ a > 0\ ,\ \text{ either } c = 0 \text{ or } 1 \leq k\ ,\ H(x, \xi p) = \xi H(x, p)\ \forall\, \xi > 0\,.$$

Then the SMP and the Strong Minimum Principle hold, and the same is true if $\mathcal{M}^+$ is replaced by $\mathcal{M}^-$.

Next we give an example of operator that satisfies SMP but whose SV do not satisfy the stronger property (2.21).

Example 2.38. Consider the equation

$$\frac{-\Delta u}{1 + |\Delta u|} + f(x) = 0. \tag{2.22}$$

It is easy to see that $F(x, X) = -\mathrm{Tr}X/(1 + |\mathrm{Tr}X|) + f(x)$ satisfies condition (i), and also the scaling condition (ii) if $f(x) \geq 0$, by taking $\phi(\xi) = 1$ if $\mathrm{Tr}X \geq 0$ and $\phi(\xi) = \xi$ if $\mathrm{Tr}X < 0$. Moreover

$$\lim_{\gamma\to+\infty} F(x, 0, p, I - \gamma p \otimes p) = 1 + f(x) \quad \forall p \in \mathbb{R}^d\,,$$

so any vector $Z \in \mathbb{R}^d$ is SV for F at x if $f(x) > -1$. Then for $f \geq 0$ the equation satisfies the SMP by Remark 2.18. However the stronger property (2.21) is not verified for any $Z \in \mathbb{R}^d$.

Example 2.39. (A counterexample from [148]) Consider equation (2.22) with $f(x) = 0$ for all $x \neq 0$ and $f(0) = -1$. Then (i) holds everywhere, whereas (ii) and the existence of SVs fail only at $x = 0$. The SMP is violated by the subsolution $u(x) = 0$ for all $x \neq 0$ and $u(0) = 1$.

2.5 Strong Comparison Principles

In this section, we consider non-homogeneous equations that can be written in H-J-Bellman form, namely

$$\inf_{\alpha}\{L^\alpha u - f^\alpha(x)\} = 0 \quad \text{in } \Omega \tag{2.23}$$

$$\sup_{\alpha}\{L^\alpha u - f^\alpha(x)\} = 0 \quad \text{in } \Omega \tag{2.24}$$

where L^α are the linear operators defined in (2.19). We recall that F_i and F_s defined in (3.12) are the 1-homogeneous operators obtained by setting $f^\alpha = 0$ in the operator of the equation (2.23) and (2.24), respectively. We say that a PDE satisfies the *Comparison Principle in a ball* $B(x,r)$ if for any subsolution u and supersolution v in $B(x,r)$ such that $u \le v$ on $\partial B(x,r)$ we have $u \le v$ on $\overline{B}(x,r)$. We will denote

$$F(x,r,p,X) := \inf_{\alpha}\{-\mathrm{Tr}(A^\alpha(x)X) - b^\alpha(x)\cdot p + c^\alpha(x)r - f^\alpha(x)\}$$

Lemma 2.40. *Let $u \in \mathrm{USC}(\Omega)$, $v \in \mathrm{LSC}(\Omega)$ be, respectively, a sub- and a supersolution of* (2.23)*. Assume that for some $\bar r$ the equation* (2.23) *satisfies the Comparison Principle in $B(x,r)$ for all $0 < r < \bar r$, and that F_i is continuous and verifies the SMP. If $u - v$ attains a nonnegative maximum in Ω, then $u \equiv v$+constant.*

Proof. We claim that $w = u - v$ is a subsolution of $F_i(x,w,Dw,D^2w) = 0$. This is easily seen if u, v are smooth because

$$\inf_{\alpha}\{L^\alpha(u-v)\} \le \inf_{\alpha}\{L^\alpha u - f^\alpha(x) - \inf_{\alpha'}[L^{\alpha'}u - f^{\alpha'}(x)]\} \le 0\,.$$

However, handling the viscosity subsolution property requires more care and the use of the local Comparison Principle. Once the claim is proved the conclusion of the lemma is immediately achieved by the SMP for F_i.

We use the compact notations $F[z], F_i[z]$ to denote, respectively, $F(x,z,Dz,D^2z)$ and $F_i(x,z,Dz,D^2z)$. Let $\bar x \in \Omega$ and φ be a smooth function such that $(w-\varphi)(\bar x) = 0$ and $w - \varphi$ has a strict maximum at $\bar x$. Let us argue by contradiction, assuming that $F_i[\varphi(\bar x)] > 0$. We first observe that, by the continuity of F_i, there exists $\delta > 0$ such that

$$F_i(\bar x, \varphi(\bar x) - \delta, D\varphi(\bar x), D^2\varphi(\bar x)) > 0\ .$$

Therefore, using the continuity of F_i and the smoothness of φ, we get the existence of r such that

$$F_i[\varphi - \delta] > 0 \text{ in } B(\bar x, r)\ .$$

Since $w - \varphi$ attains a strict maximum at $\bar x$, there exists $0 < \eta < \delta$ such that $w - \varphi \le -\eta < 0$ on $\partial B(\bar x, r)$. We now claim that $v + \varphi - \eta$ satisfies $F[v+\varphi-\eta] \ge 0$ in $B(\bar x, r)$. To this aim, take $\tilde x \in B(\bar x, r)$ and ψ smooth such that $v + \varphi - \eta - \psi$ has a minimum at $\tilde x$. Using that v is a supersolution of (2.23), denoting by $\tilde L^\alpha u := -\mathrm{Tr}(A^\alpha(x)D^2u) - b^\alpha(x)\cdot Du$, we obtain

$$\begin{aligned} 0 \le F[\psi(\tilde x) - \varphi(\tilde x) + \eta] &= \inf_{\alpha}\{\tilde L^\alpha\psi(\tilde x) - \tilde L^\alpha\varphi(\tilde x) + c^\alpha(\tilde x)(\psi(\tilde x) - \varphi(\tilde x) + \eta) - f^\alpha(\tilde x)\} \\ &\le \inf_{\alpha}\{\tilde L^\alpha\psi(\tilde x) + c^\alpha(\tilde x)\psi(\tilde x) - f^\alpha(\tilde x)\} - \inf_{\alpha}\{\tilde L^\alpha\varphi(\tilde x) + c^\alpha(\tilde x)(\varphi(\tilde x) - \eta)\} \\ &= F[\psi(\tilde x)] - F_i[\varphi(\tilde x) - \eta] < F[\psi(\tilde x)]\ . \end{aligned}$$

This proves the claim that $v+\varphi-\eta$ is a supersolution of (2.23) in $B(\bar{x},r)$. Now, since $u \le v+\varphi-\eta$ on $\partial B(\bar{x},r)$, the (local) Comparison Principle yields $u \le v+\varphi-\eta$ in $B(\bar{x},r)$, in contradiction with the fact that $u(\bar{x}) = v(\bar{x})+\varphi(\bar{x})$. □

Now we can prove the second main result of Chapter 2. We will make the following standard assumptions on the coefficients of F:

$$A^\alpha(x) = \sigma^\alpha(x)(\sigma^\alpha(x))^T\,, \quad \sigma^\alpha : \overline{\Omega} \to \{d \times m \text{ matrices }\} \tag{2.25}$$

$$\sigma^\alpha \text{ and } b^\alpha : \overline{\Omega} \to \mathbb{R}^d \ \text{ locally Lipschitz in } x \text{ uniformly in } \alpha\,; \tag{2.26}$$

$$c^\alpha \ge 0\,, \quad c^\alpha \text{ and } f^\alpha \text{ continuous in } x \in \overline{\Omega} \text{ uniformly in } \alpha\,. \tag{2.27}$$

Theorem 2.41. *Assume* (2.25), (2.26), (2.27), *and the existence of vector fields* $Z_i : \overline{\Omega} \to \mathbb{R}^d$, $i = 1, ..., m$, *satisfying the Hörmander condition* (H) *and such that*

$$A^\alpha(x) \ge Z_i(x) \otimes Z_i(x) \quad \textit{for all } \alpha,\ i,\ \textit{and } x\,.$$

If $u \in \mathrm{USC}(\Omega)$, $v \in \mathrm{LSC}(\Omega)$ *are, respectively, a sub- and a supersolution of* (2.23) *and* $u - v$ *attains a nonnegative maximum in* Ω, *then* $u \equiv v+$*constant.*

Proof. Under the current assumptions F is finite and continuous in $\overline{\Omega} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d$ and it is proper. The homogeneous operator F_i satisfies the SMP by Corollary 2.32.

Note that F satisfies the Lipschitz property in p in any compact subset $K \subset \Omega$:

$$|F(x,r,p,X) - F(x,r,q,X)| \le L_K|p-q|\,, \quad \forall\, x \in K. \tag{2.28}$$

Moreover there is $\eta \in C(\overline{\Omega})$, $\eta > 0$, such that

$$F(x,r,p,X+sI) \le F(x,r,p,X) - \eta(x)s, \quad \forall\, s > 0. \tag{2.29}$$

In fact, $\mathrm{Tr}(A^\alpha(x)I) \ge \mathrm{Tr}(Z_i(x) \otimes Z_i(x)) = |Z_i(x)|^2$ for all i, and so

$$\eta(x) := \frac{1}{m}\sum_{i=1}^m |Z_i(x)|^2$$

does the job, because the Hörmander condition prevents that all Z_i vanish at the same point.

By standard viscosity theory [96] the equation (2.23) verifies the Comparison Principle between a supersolution v and a strict subsolution, say u_ϵ, in a ball $B(x,\bar{r}) \subseteq \Omega$ for some $\bar{r} > 0$. More precisely, u_ϵ is an upper semicontinuous function in $\overline{B}(x,\bar{r})$ such that

$$F(x, u_\epsilon, Du_\epsilon, D^2u_\epsilon) \le \alpha(x) \text{ in } B(x,\bar{r})$$

with $\alpha \in C(\overline{B}(x,\bar{r}))$ and $\alpha < 0$. If, in addition, $u_\epsilon \to u$ for all x as ϵ approaches to 0, then one immediately concludes $u \le v$ in $B(x,\bar{r})$. Next we show that the Comparison Principle holds in all sufficiently small balls, following an argument in [25]. To this aim, fix $\bar{x} \in \Omega$, $r_1 > 0$ such that $\overline{B}(\bar{x},r_1) \subseteq \Omega$, and let $\bar{\eta} := \min_{\overline{B}(\bar{x},r_1)} \eta > 0$. We choose $0 < \delta < \bar{\eta}$ and

$$\bar{r} := \min\left(\frac{\bar{\eta}-\delta}{L_K}, r_1\right), \quad K := \overline{B}(\bar{x},r_1).$$

Consider the function

$$u_\epsilon(x) = u(x) + \epsilon(e^{\frac{|x-\bar{x}|^2}{2}} - \lambda),\ x \in B(\bar{x}, \bar{r})\ .$$

We claim that u_ϵ is a strict subsolution in $B(\bar{x}, \bar{r})$ for λ sufficiently large independent of ϵ. Let us take $\lambda \geq e^{\frac{|x-\bar{x}|^2}{2}}$ for every $x \in B(\bar{x}, \bar{r})$ so that $u_\epsilon \leq u$. Straightforward computations yield

$$\partial_i(u_\epsilon) = \partial_i u + \epsilon(x_i - \bar{x})e^{\frac{|x-\bar{x}|^2}{2}}$$

and

$$\partial_{ij}(u_\epsilon) = \partial_{ij}u + \epsilon(\delta_{ij} + (x_i - \bar{x})(x_j - \bar{x}))e^{\frac{|x-\bar{x}|^2}{2}}$$

so that

$$D^2 u_\epsilon = D^2 u + \epsilon(I + (x - \bar{x}) \otimes (x - \bar{x}))e^{\frac{|x-\bar{x}|^2}{2}} \geq D^2 u + \epsilon e^{\frac{|x-\bar{x}|^2}{2}} I$$

Since F is proper and $u_\epsilon \leq u$, one obtains

$$F(x, u_\epsilon, Du_\epsilon, D^2 u_\epsilon) \leq F(x, u, Du + \epsilon(x - \bar{x})e^{\frac{|x-\bar{x}|^2}{2}}, D^2 u + \epsilon e^{\frac{|x-\bar{x}|^2}{2}} I)$$

Combining (2.28) and (2.29), one immediately gets

$$\begin{aligned} F(x, u, Du + \epsilon(x - \bar{x})e^{\frac{|x-\bar{x}|^2}{2}}, D^2 u + \epsilon e^{\frac{|x-\bar{x}|^2}{2}} I) &\leq F(x, u, Du, D^2 u) \\ &\quad + \epsilon e^{\frac{|x-\bar{x}|^2}{2}}(L_K|x - \bar{x}| - \eta(x)) \end{aligned}$$

Using that u is a subsolution and $x \in B(\bar{x}, \bar{r})$, by the above choice of $\bar{r}$ we conclude

$$F(x, u_\epsilon, Du_\epsilon, D^2 u_\epsilon) \leq -\epsilon e^{\frac{|x-\bar{x}|^2}{2}} \delta =: \alpha(x)\ ,$$

as desired. □

Corollary 2.42. *Under the assumptions of Theorem 2.41 and for bounded Ω, if $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ are, respectively, a sub- and a supersolution of* (2.23) *such that $u \leq v$ in $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$. Moreover, if $u(x) = v(x)$ for some $x \in \Omega$ then $u \equiv v$.*

Proof. If $\max_{\overline{\Omega}}(u - v)$ is negative or attained on $\partial\Omega$ the first conclusion is achieved. Otherwise we can apply Theorem 2.41 and get $u(x) - v(x) = k$ for all $x \in \Omega$. Then, for $y \in \partial\Omega$,

$$k \leq \limsup_{x \to y}(u(x) - v(x)) \leq u(y) - v(y) \leq 0,$$

which gives $u \leq v$. Then the last statement follows from Theorem 2.41. □

Remark 2.43. The last two results hold also for the equation (2.24) with convex instead of concave operator. In fact $z = v - u$ is a supersolution of $F_s(x, z, Dz, D^2 z) = 0$ and we apply the Strong Minimum Principle of Corollary 2.32 *ii)* to this equation.

Example 2.44. Theorem 2.41 and Corollary 2.42 apply to the quasilinear equations

$$-\mathrm{Tr}(A(x)(D^2_{\mathcal{X}}u)^*) + H(x,u,Du) = 0,$$

where either $H = H_i$ or $H = H_s$ with

$$H_i(x,r,p) := \inf_{\alpha}\{-b^\alpha(x)\cdot p + c^\alpha(x)r - f^\alpha(x)\},$$

$$H_s(x,r,p) := \sup_{\alpha}\{-b^\alpha(x)\cdot p + c^\alpha(x)r - f^\alpha(x)\},$$

the vector fields are $\mathcal{X} = (Z_1, ..., Z_m)$, and the coefficients $A, b^\alpha, c^\alpha, f^\alpha$ satisfy (2.26) and (2.27). Also the weak Comparison principle, i.e., the first statement of Corollary 2.42, is new for these equations, since the results of [25] cover either the case of a Hamiltonian H depending only on the horizontal gradient $D_{\mathcal{X}}u$, or the case where the Lipschitz constant of H w.r.t. p and the diameter of $\overline{\Omega}$ are small compared to $\min_{\overline{\Omega}}\sum_i |Z_i|^2/m$ (however, in [25] H is not necessarily concave or convex in p).

Example 2.45. All the statements of the previous example hold word by word also for the fully nonlinear equations

$$\mathcal{M}^-((D^2_{\mathcal{X}}u)^*) + H_i(x,u,Du) = 0,$$

$$\mathcal{M}^+((D^2_{\mathcal{X}}u)^*) + H_s(x,u,Du) = 0.$$

Chapter 3

Strong maximum principles for fully nonlinear degenerate parabolic PDEs

The main goal of this chapter is to extend the results of Chapter 2 to the parabolic framework. We thus focus on the validity of the SMP for fully nonlinear parabolic equations of the form

$$\partial_t u + F(x, t, u, Du, D^2 u) = 0 \text{ in } \mathcal{O} \ , \tag{3.1}$$

where $\mathcal{O} := \Omega \times (0, T) \subseteq \mathbb{R}^{d+1}$, with Ω bounded open set of $\mathbb{R}^d$. To compare our results with those analyzed in the existing literature, it turns out to be useful to rewrite the operator appearing in (3.1) in the more general form

$$\overline{F}(x, t, u, \partial_t u, D_x u, D^2_{xt} u) = 0 \ , \tag{3.2}$$

see e.g. [100, 132]. By strong maximum and minimum principle we mean the following properties: any $u \in \mathrm{USC}(\overline{\Omega} \times [0, T])$ (resp. $v \in \mathrm{LSC}(\overline{\Omega} \times [0, T])$) viscosity subsolution (supersolution) to (3.1) attaining a nonnegative maximum (nonpositive minimum) at $(x_0, t_0) \in \Omega \times (0, T]$, is constant in $\overline{\Omega} \times [0, t_0]$.
We will work under the same kind of assumptions imposed in the elliptic framework adding suitable modifications to deal with the time variable. More precisely, as for the SMP, we suppose

(i) F is lower semicontinuous and proper in the sense of [96], i.e.

$$F(x, t, r, p, X) \leq F(x, t, s, p, Y) \ , r \leq s \ , Y \leq X \ .$$

for every $(x, t) \in \overline{\Omega} \times [0, T]$, $r \in \mathbb{R}$, $p \in \mathbb{R}^d \backslash \{0\}$ and $X, Y \in \mathcal{S}_d$.

(ii) (*Scaling*) For some $\phi > 0$ the operator satisfies

$$\xi s + F(x, t, \xi r, \xi p, \xi X) \geq \ \phi(\xi)(s + F(x, t, r, p, X))$$

for all $\xi \in (0, 1]$ and $r \in [-1, 0]$ and for every $(x, t) \in \Omega \times (0, T)$, $p \in \mathbb{R}^d \backslash \{0\}$ and $X \in \mathcal{S}_d$.

We refer to remark 3.11 for the assumptions needed to get the strong minimum principle. Analogously to the elliptic case, the idea is to characterize the set of propagation of maxima of a (viscosity) subsolution to (3.1) in terms of subunit vector fields associated to the fully nonlinear operator $\partial_t + F$. To this aim, we note that Definition 2.3 extends to the evolutive framework as follows.

Definition 3.1. *$Z \in \mathbb{R}^{d+1}$ is a generalized subunit vector field (briefly SV) for the parabolic operator $\overline{F}$ at $(x,t) \in \mathcal{O}$ if*

$$\sup_{\gamma>0} \overline{F}(x,t,0,p_t,p,I-\gamma(p,p_t)\otimes(p,p_t)) > 0$$

for every $(p,p_t) \in \mathbb{R}^{d+1}$ such that $(p,p_t)\cdot Z \neq 0$. Accordingly, a subunit vector $Z \in \mathbb{R}^{d+1}$ is a generalized subunit vector for the parabolic operator $\partial_t + F$ in (3.1) at $(x,t) \in \mathcal{O}$ if

$$\sup_{\gamma>0}\{p_t + F(x,t,0,p,I-\gamma p\otimes p)\} > 0 \ ,$$

for every $(p,p_t) \in \mathbb{R}^{d+1}$ such that $(p,p_t)\cdot Z \neq 0$.

We point out that all the proofs can be formulated even for the general operator $\overline{F}$, but for the sake of simplicity and since our main examples will appear in the evolution form (3.1), we provide all the treatment for the evolutive operator $\partial_t + F$ only. We recall that one of the main purposes to describe the SMP for the operator $\overline{F}$ in [100] was to embrace the Levi operator, which in fact appear in the form (3.2), being defined as $\widetilde{F}(x,y,r,s,p,X) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{S}_3 \to \mathbb{R}$ via

$$\widetilde{F}_1(x,y,r,s,p,X) = -\mathrm{Tr}(A(s,p)X) + k(x,y,t)(1+|p|^2)^{\frac{3}{2}} \ ,$$

where k is a nonnegative continuous function and

$$A(s,p) = \begin{pmatrix} 1+s^2 & 0 & -sp_1+sp_2 \\ 0 & 1+s^2 & -sp_2-p_1 \\ -sp_1+p_2 & -sp_2-p_1 & p_1^2+p_2^2 \end{pmatrix} \ .$$

The classical strong maximum principle for linear parabolic equations (see [191, 117] for uniformly parabolic operators) states that if a subsolution to

$$\partial_t u + Lu = \partial_t u - \mathrm{Tr}(A(x,t)D^2u) + b(x,t)\cdot Du + c(x,t)u = 0 \text{ in } \Omega\times(0,T) \ ,$$

where $\Omega\times(0,T)$ is a domain of $\mathbb{R}^{d+1}$, $A : \Omega\times(0,T) \to \mathbb{R}^{d\times d}$, $b : \Omega\times(0,T) \to \mathbb{R}^d$ and $c \geq 0$ are continuous, attains a nonnegative maximum at some point $(x_0,t_0) \in \mathcal{O}$, then the maximum is achieved on every curve in $\mathcal{O}$ along which the time-coordinate is nondecreasing.

Following the seminal paper [191] (see also [117, Chapter 2]), we now set $Q_T = \Omega\times(0,T]$ and for any point $P_0 = (x_0,t_0) \in Q_T$, we denote by $S(P_0)$ the set of all points $Q \in Q_T$ which can be connected to P_0 by a simple continuous curve in Q_T along which the t-coordinate is nondecreasing from Q to P_0 and by $C(P_0)$ the component of $\Omega\times\{t=t_0\}$ which contains P_0. In the case that there is a propagation of maxima in $C(P_0)$ we say that there is a *horizontal propagation*, while in case the

maximum propagates in $S(P_0)$ we say that a *vertical propagation* of maxima occurs. Note that $C(P_0) \subset S(P_0)$. Similar strategies were implemented in the viscosity solutions' framework in [100, 60, 132]. See also [87] for the case of fully nonlinear integro-differential equations and [108] for reaction-diffusion systems of PDEs.
Our main result reads as follows

Theorem 3.2. *Let u be a viscosity subsolution of* (3.1) *attaining a nonnegative maximum at $P_0 = (x_0, t_0) \in \mathcal{O}$. Assume F satisfies (i)-(ii) and there exist SVs for $\partial_t + F$, $Z_1, ..., Z_m$, satisfying the Hörmander's condition and $F(x_0, t_0, 0, 0, 0) \geq 0$. Then we have both horizontal and vertical propagation of maxima, that is $u = u(P_0)$ is constant in $S(P_0)$.*

This result is accomplished by studying separately the propagation of maxima in $C(P_0)$ and $S(P_0)$. We will see that the additional assumption $F(x_0, t_0, 0, 0, 0) \geq 0$ will be required for the vertical propagation only and this is consistent with the results of Chapter 2, where one needs (i)-(ii) and the existence of SVs for the operator. We are mainly interested in evolutive fully nonlinear equations modeled on Hörmander vector fields, among which the parabolic versions of the operators treated in Chapter 2 (i.e. PDEs on Carnot groups, Grushin plane,...). Given a family $\mathcal{X} = (X_1, ..., X_m)$ of $C^{1,1}$ vector fields in $\mathcal{O}$, as outlined in Section 1.0.5, one defines the intrinsic (or horizontal) gradient $D_{\mathcal{X}} u$ and the symmetrized horizontal Hessian $(D^2_{\mathcal{X}} u)^*$. The model fully nonlinear parabolic equation in this setting reads as

$$\partial_t u + G(x, t, u, D_{\mathcal{X}} u, (D^2_{\mathcal{X}} u)^*) = 0 \ , \tag{3.3}$$

which can be recasted in Euclidean coordinates by writing

$$\partial_t u + F(x, t, u, \sigma^T(x) Du, \sigma^T(x) D^2 u \sigma(x)) = 0 \ . \tag{3.4}$$

We say that the evolutive operator $\partial_t + G$ is parabolic in the following sense: for any (x, t) and $q \in \mathbb{R}^m$ fixed

$$\sup_{\gamma > 0} \{G(x, t, 0, q, X - \gamma q \otimes q)\} > 0 \tag{3.5}$$

for every $(x, t) \in \mathcal{O}$, $q \in \mathbb{R}^m, q \neq 0$ and $X \in \mathcal{S}_m$. Similarly to the elliptic setting, our result reads as follows

Corollary 3.3. *Assume that the parabolic operator G verifies (i)-(ii) and* (3.5) *and the vector fields $X_1, ..., X_m$ satisfy the Hörmander condition. Then, for every viscosity subsolution of* (3.4) *attaining a nonnegative maximum in $\mathcal{O}$ we have both horizontal and vertical propagation of maxima.*

Clearly, all the results of Section 2.3.1 regarding subunit vector fields can be reformulated exactly in the same manner allowing the time-dependence. We finally recall that the SMP for linear evolutive PDEs modeled on Hörmander vector fields can be deduced by Bony's seminal paper, as pointed out in [151, Proposition 2.4]. We remark that very little is known in the context of general fully nonlinear evolutive PDEs over Hörmander's vector fields and the results of this chapter seem to be the first ones dealing with such parabolic subelliptic equations. However, some of the examples we are going to present can be obtained by using the ideas established in [21, 22, 23, 100] and, specifically, some results in the context of HJB and HJI equations regarding propagation of maxima and minima have already appeared in [100]. However, as announced, our results are obtained via a generalization of the concept of subunit vector fields in the parabolic framework and the characterization we propose, together with the examples in the subelliptic setting, seem to be new and this is the main difference with respect to the known contributions in the parabolic literature [100, 132]. This chapter is organized as follows. In the forthcoming section we review the seminal results of L. Nirenberg and A. Friedman for linear uniformly parabolic equations, in Section 3.2 we prove the horizontal propagation of maxima, while Section 3.3 is devoted to the vertical propagation. Finally, Section 3.4 collects some applications of the results of the previous sections.

3.1 Propagation of maxima in the linear case: a survey

In this section we highlight the main ideas presented by L. Nirenberg regarding the mechanism of the propagation of maxima for linear uniformly parabolic equations. Consider the linear parabolic equation

$$\partial_t u - \mathrm{Tr}(A(x,t)D^2u) + b(x,t) \cdot Du + c(x,t)u = 0 \tag{3.6}$$

posed in $\Omega_T := \Omega \times (0,T)$, Ω being a connected open set of $\mathbb{R}^d$, and assume that $A : \Omega_T \to \mathcal{S}_d$ is bounded and continuous, $b : \Omega_T \to \mathbb{R}^d$ bounded and continuous and $c : \Omega_T \to \mathbb{R}$ bounded. Assume further that the above evolution operator is uniformly parabolic, i.e. there exists $\alpha > 0$ such that

$$A(x,t) \geq \alpha I \quad \forall (x,t) \in \Omega_T \ .$$

The following theorem is proved in [191] for the two-dimensional heat equation and [117] for the general case

Theorem 3.4. *Let $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ be a subsolution to* (3.6) *and let $c \geq 0$.*

- *If $c \equiv 0$ and there exists $P_0 = (x_0, t_0) \in \Omega_T$ such that u attains its greatest value M at $(x_0, t_0) \in \Omega_T$, then $u \equiv M$ on $S(P_0)$.*
- *If $c \neq 0$ and there exists $P_0 = (x_0, t_0) \in \Omega_T$ such that u attains its nonnegative greatest value M at $(x_0, t_0) \in \Omega_T$, then $u \equiv M$ on $S(P_0)$.*

As announced, the proof of this result relies on proving the propagation of maxima on the horizontal component $C(P_0)$, i.e. along $\Omega \times \{t = t_0\}$, and then the vertical propagation, namely the propagation of maxima over $S(P_0)$, that we briefly describe below.

- *Horizontal propagation. Step 1.* The proof is usually done by contradiction, assuming that there exists a point $P_1 = (x_1, t_0) \in \Omega \times \{t = t_0\}$ such that $u(P_1) < u(P_0)$. By classical geometric arguments, one can find a closed ellipsoid

E such that $(\bar{x}, t_0) \in C(P_0)$, $u < u(P_0)$ in the interior of E with $P^* = (x^*, t^*) \in \partial E$ satisfying $u(P^*) = u(P_0)$, $\bar{x} \neq x^*$ and $P^* \notin \partial\Omega \times (0, T)$. One can then determine a $(d+1)$-dimensional ball $B = B((\tilde{x}, \tilde{t}), R) \subseteq E$ with $\partial B \cap \partial E = \{(x^*, t^*)\}$.
Step 2. Find a classical strict supersolution in B such that $v < 0$ in the interior of E, $v > 0$ outside E and $v = 0$ on ∂E. Such v can be found among barrier-type functions, as in Proposition 2.13- Step 1. To find a contradiction, one notices that the function $w = u - \varepsilon v$ solves $\partial_t w - \mathrm{Tr}(A(x,t)D^2 w) + b(x,t) \cdot Dw + c(x,t)w < 0$ and w attains its maximum at an interior point of B, which would be impossible.

- *Vertical propagation.* The first part of the proof relies on proving a local vertical propagation, i.e. if R is a rectangle of the form $\{(x,t) : x_0^i - a^i \leq x^i \leq x_0^i + a^i,\ t_0 - a_0 \leq t \leq t_0\} \subseteq \overline{\Omega} \times [0, t_0]$ with a_i small enough, then for any $R \subseteq \Omega \times (0, t_0]$, the rectangle R minus its top face contains a point $P_R \neq P_0$ where the maximum propagates, i.e. such that $u(P_R) = u(P_0)$. The proof of this results is mainly based on geometric and contradiction arguments as in the horizontal propagation. One finally shows indeed that the maximum propagates over all $S(P_0)$.

In the case of degenerate equations, we remark that the Bony's strong maximum principle for operators written as a sum of squared vector fields easily applies, for instance, to the sub-parabolic heat equation $\partial_t u - \sum_{i=1}^m X_i^2 u = 0$.
The main issue in the program we develop in this chapter is to adapt the aforementioned geometric constructions to weak solutions (in the viscosity sense) of fully nonlinear parabolic equations, similarly to the elliptic counterpart developed in Chapter 2.

3.2 Horizontal propagation

As usual, we denote

$$K^* := \{z \in \partial K : \text{there exists } \nu \perp K \text{ at } z\},$$

where ν denotes the Bony's normal and $K \subseteq \mathbb{R}^{d+1}$ a compact set as in Chapter 2. We prove the following results saying that if u is a viscosity subsolution to (3.1) attaining a nonnegative maximum at some point $P_0 \in Q_T$, then the subunit vector fields associated to the operator F are orthogonal to the proximal normal of the set of maxima of u.

Proposition 3.5. *Let $u \in \mathrm{USC}(Q_T)$ be a viscosity subsolution of (3.1) that attains a nonnegative maximum at some point $P_0 = (x_0, t_0) \in Q_T$. Suppose that (i)-(ii) hold and $\partial_t + F$ has a subunit vector field as in Definition 3.1. Then $K := \mathrm{Prop}((x_0, t_0))$ is such that for any $z = (x^*, t^*) \in K^* \cap Q_T$ and $(\nu, \nu_t) \perp K$ at z we have $Z \cdot (\nu, \nu_t) = 0$ for every subunit vector of $\partial_t + F$ at z.*

Proof. Let $z \in \partial K$ and $\nu \perp K$ at z. We argue by contradiction, assuming that there exists a subunit vector $\overline{Z}$ such that $\overline{Z} \cdot \nu \neq 0$. By definition of proximal normal, there exists $\tilde{y} = (\tilde{x}, \tilde{t}) \in Q_T$ and $R > 0$ such that $B(\tilde{y}, R) \cap K = \{z\}$ (hence $\tilde{y} = z + R\frac{(\nu,\nu_t)}{|(\nu,\nu_t)|}$). As in the elliptic case (cf Proposition 2.13) we divide the proof in two steps

Step 1. We claim that there exist $r > 0$ and a function $v \in C^2(E_T)$, $E_T := \mathbb{R}^d \times (0, T)$, such that

$$\partial_t v + F(x, t, v(x,t), Dv(x,t), D^2v(x,t)) \geq C > 0 \ \forall (x,t) \in B(z,r) \ ,$$

$v(z) = 0$, $-1 < v < 0$ in $B(y, R)$ and $v > 0$ outside $B(y, R)$.
Let $y = (x, t)$ and consider

$$v(y) = e^{-\gamma R^2} - e^{-\gamma |y - \tilde{y}|^2} \ .$$

Direct calculations yields

$$\partial_t v(y) = 2\gamma e^{-\gamma|y-\tilde{y}|^2}(t - \tilde{t}) \ ,$$

$$Dv(y) = 2\gamma e^{-\gamma|y-\tilde{y}|^2}(x - \tilde{x})$$

and

$$D^2v(y) = 2\gamma e^{-\gamma|y-\tilde{y}|^2}(I - 2\gamma(x - \tilde{x}) \otimes (x - \tilde{x})) \ .$$

Now, using that $z - \tilde{y} = -(\nu, \nu_t)$ and the scaling property (ii) we have

$$\begin{aligned}\partial_t v(z) + F(z, v(z), Dv(z), D^2v(z)) &= \partial_t v(x^*, t^*) \\ &\quad + F(x^*, t^*, v(x^*, t^*), Dv(x^*, t^*), D^2v(x^*, t^*)) \\ &= 2\gamma e^{-\gamma R^2}(-\nu_t) + F(x^*, t^*, 0, 2\gamma e^{-\gamma R^2}(-\nu), 2\gamma e^{-\gamma R^2}(I - 2\gamma\nu \otimes \nu)) \\ &\geq \phi(2\gamma e^{-\gamma R^2})(-\nu_t + F(x^*, t^*, 0, -\nu, I - 2\gamma\nu \otimes \nu))\end{aligned}$$

where $\phi > 0$. Now the definition of subunit vector at z and the fact that $\overline{Z} \cdot (\nu, \nu_t) \neq 0$ imply that

$$-\nu_t + F(x^*, t^*, 0, -\nu, I - 2\gamma\nu \otimes \nu) > 0$$

for some $\gamma > 0$, giving thus

$$\partial_t v(z) + F(x, t, v(z), Dv(z), D^2v(z)) > 0 \ .$$

By the lower semicontinuity (i), there exists $r \in (0, R)$ such that

$$\partial_t v(x,t) + F(x, t, v(x,t), Dv(x,t), D^2v(x,t)) \geq C > 0 \text{ for every } (x,t) \in B(z,r) \ . \tag{3.7}$$

Step 2. We claim now that for ϵ sufficiently small $u(y) - u(z) \leq \epsilon v(y)$ for every $y \in X := B(z,r) \cap B(y, R)$.
To this aim, choose ϵ sufficiently small so that $u(y) - u(z) \leq \epsilon v(y)$ for every $y \in \partial X$. To prove that the inequality holds on X, suppose by contradiction that there exists $(\bar{x}, \bar{t}) \in X$ such that $u(\bar{x}, \bar{t}) - u(z) - \epsilon v(\bar{x}, \bar{t}) = \max_X(u - u(z) - \epsilon v) > 0$. Since

$\epsilon v \in C^\infty(E_T)$, using that $u - u(z)$ is a viscosity subsolution and the scaling property (ii) we get

$$\begin{aligned}\phi(\epsilon)(\partial_t v(\bar{x},\bar{t}) + F(\bar{x},\bar{t},v(\bar{x},\bar{t}),\partial_t v(\bar{x},\bar{t}),Dv(\bar{x},\bar{t}),D^2v(\bar{x},\bar{t})))\\ \leq \epsilon\partial_t v(\bar{x},\bar{t}) + F(\bar{x},\bar{t},\epsilon v(\bar{x},\bar{t}),\epsilon Dv(\bar{x},\bar{t}),\epsilon D^2 v(\bar{x},\bar{t})) \leq 0\ ,\end{aligned}$$

a contradiction with (3.7) since $\phi > 0$. This implies that the function $\Phi(y,s) := u(y,s) - \varepsilon v(y,s)$ has a maximum at z in $B(z,r)$. Since $\epsilon v \in C^\infty(E_T)$, F is proper, using also the definition of viscosity subsolution and (ii) we get

$$\begin{aligned}\phi(\epsilon)(\partial_t v(z) + F(x,t,v(z),Dv(z),D^2v(z))) \leq \epsilon\partial_t v(z) + F(z,\epsilon v(z),\epsilon Dv(z),\epsilon D^2v(z))\\ \leq \epsilon\partial_t v(z) + F(z,u(z),\epsilon Dv(z),\epsilon D^2v(z)) \leq 0\ ,\end{aligned}$$

again a contradiction with (3.7). □

Corollary 3.6 (Horizontal propagation of maxima)**.** *Assume (i)-(ii) and the existence of subunit vector fields $Z_i \in \mathbb{R}^{d+1}$, $i = 1,...,m$ of F satisfying the Hörmander condition. Let $u \in \mathrm{USC}(Q_T)$ be a viscosity subsolution of* (3.1)*attaining a nonnegative maximum at $P_0 = (x_0,t_0) \in Q_T$. Then, u is constant in $C(P_0)$.*

Proof. We prove the result via a classical construction for parabolic problems (see e.g. [117, 100]). Suppose by contradiction that there exists a point $P_1 = (x_1,t_0)$ such that $u(P_1) < u(P_0)$. Then, by the fact that $u(P_1) < u(P_0)$ and the upper semicontinuity of u, we can find a ball centered at P_1 such that $u(x,t) < u(P_0)$ for every $(x,t) \in B(P_1,\epsilon)$, where $\epsilon < \mathrm{dist}(P_0,P_1) < \mathrm{dist}(P_1,\partial\Omega)$. As a next step, we construct a family of ellipsoids whose length of the vertical axes is $\frac{\epsilon}{2}$ and the horizontal one is λ. Again by the upper semicontinuity of u and eventually increasing λ, we find a $\bar{\lambda}$ and an ellipsoid

$$\mathcal{E}_{\bar{\lambda},\varepsilon} := \{(x,t) : |x-x_1|^2 + \bar{\lambda}|t-t_0|^2 \leq (\epsilon/2)^2\bar{\lambda}\}$$

such that $u(x,t) < u(P_0)$ for every $(x,t) \in \mathrm{Int}\mathcal{E}_{\bar{\lambda},\varepsilon}$ and $u(x^*,t^*) = M$ at some point $(x^*,t^*) \in \partial\mathcal{E}_{\bar{\lambda},R}$. Let $P^* := (x^*,t^*)$ and note that $x^* \neq x_1$. Let then $B := B((\tilde{x},\tilde{t}),r)$ with r small enough such that $\tilde{x} \neq x^*$. Then we observe that $\mu := (\tilde{x}-x^*,\tilde{t}-t^*)\perp\mathrm{Prop}((x^*,t^*))$ at P^*. Since the vector fields $Z_1,...,Z_m$ fulfill the Hörmander condition, one can find d vector fields $W_j \in \mathcal{L}(Z_1,...,Z_m)$, $j = 1,...,d$, which are linearly independent and satisfy $W_j \cdot \mu = 0$ in view of Proposition 3.5. However, since $(W_j)_{d+1} = 0$, the previous orthogonality condition would imply that the first d components of μ are 0, in contradiction with the fact that $\tilde{x} \neq x^*$.

□

3.3 Vertical Propagation

We are now ready to study the vertical propagation of maxima. In particular, we will prove first that if $u \in \mathrm{USC}(\overline{Q_T})$ is a viscosity subsolution of (3.1) that attains a nonnegative maximum at $P_0 = (x_0,t_0) \in Q_T$, then such a maximum propagates locally in $\Omega \times (0,t_0)$. More precisely, let us consider the following rectangle

$$\mathcal{R} := \{(x,t) : x_0^i - a^i \leq x^i \leq x_0^i + a^i,\ t_0 - a_0 \leq t \leq t_0\}$$

with a^i, a_0 small enough, and denote by $\mathcal{R}_0$ the rectangle $\mathcal{R}$ minus the top face $t = t_0$. We prove that for any rectangle $\mathcal{R} \subseteq \Omega \times [0, t_0]$, $\mathcal{R}_0$ contains a point $P \neq P_0$ such that $u(P) = u(P_0)$.

Proposition 3.7. *Let $u \in \mathrm{USC}(\overline{Q}_T)$ be a viscosity subsolution of* (3.1) *that attains a nonnegative maximum at some point $P_0 = (x_0, t_0) \in Q_T$. Assume that F satisfies (i)-(ii) and $F(x_0, t_0, 0, 0, 0) \geq 0$. Then, for any rectangle $\mathcal{R} \subseteq \Omega \times [0, t_0]$, the set $\mathcal{R}_0$ contains a point $\overline{P} \neq P_0$ such that $u(\overline{P}) = u(P_0)$.*

Proof. Suppose by contradiction that there exists a rectangle $\mathcal{R} \subseteq \Omega \times [0, t_0)$ in which $u < u(P_0)$. Consider in $\mathcal{R}$ the auxiliary function

$$h(x, t) = \epsilon(t - t_0) \ .$$

Then, choose $\epsilon > 0$ sufficiently small such that

$$u(x, t) - u(x_0, t_0) \leq h(x, t)$$

Moreover $u(x_0, t_0) - h(x_0, t_0) = u(x_0, t_0)$. Hence the function $\Phi(x, t) := u(x, t) - h(x, t)$ has a maximum at (x_0, t_0). On one hand, since h is smooth and u is a viscosity subsolution of (3.1) we get

$$\partial_t h(x_0, t_0) + F(x_0, t_0, u(x_0, t_0), Dh(x_0, t_0), D^2 h(x_0, t_0)) \leq 0 \ .$$

On the other hand, we have

$$\partial_t h(x_0, t_0) + F(x_0, t_0, u(x_0, t_0), Dh(x_0, t_0), D^2 h(x_0, t_0)) = \ .$$

$$= \epsilon + F(x_0, t_0, u(x_0, t_0), 0, 0) \geq \epsilon + F(x_0, t_0, 0, 0, 0) > 0 \ ,$$

where the first inequality follows by (i) together with $u(x_0, t_0) \geq 0$ and the second one by the hypothesis $F(x_0, t_0, 0, 0, 0) \geq 0$. This contradiction completes the proof. $\square$

Remark 3.8. As in the elliptic case (cf Remark 2.17) the scaling property and the existence of subunit vector fields imply that $F(x_0, t_0, 0, 0, 0) \geq 0$.

The next result shows the local vertical propagation of maxima. This is an extension of [117, Lemma 2.1.4 pag.36] to the viscosity solutions' framework (see also [100, Corollary 2.3]).

Corollary 3.9. *Let $u \in \mathrm{USC}(\overline{Q}_T)$ be a viscosity subsolution of* (3.1) *attaining a nonnegative maximum at $P_0 = (x_0, t_0) \in Q_T$. Assume that F satisfies (i)-(ii) and $F(x_0, t_0, 0, 0, 0) \geq 0$. Moreover, suppose also that there exist SV fields for F. Then u is constant in any rectangle $\mathcal{R} = \{(x, t) : x_0^i - a^i \leq x^i \leq x_0^i + a^i,\ t_0 - a_0 \leq t \leq t_0\} \subseteq \overline{\Omega} \times [0, t_0]$.*

Proof. Let $\mathcal{R} := \{(x, t) : x_0^i - a^i \leq x^i \leq x_0^i + a^i,\ t_0 - a_0 \leq t \leq t_0\}$ be a rectangle contained in $\overline{\Omega} \times [0, t_0]$ and suppose by contradiction that there is $Q \in \mathcal{R}$ such that $u(Q) < u(P_0)$.
Since $u < u(P_0)$ also in a neighborhood of Q, we may also assume that Q does not lie on $t = t_0$. On the straight segment γ connecting Q to P_0 there exists a point P_1

such that $u(P_1) = u(P_0)$ and $u < u(P_1)$ for all points γ connecting Q and P_1. So we may assume $P_1 = P_0$ and Q lying on $t = t_0 - a_0$, otherwise we can restrict ourselves to a smaller rectangle.
Since for every point $Q' \in \mathcal{R}_0$, $C(Q')$ contains some point of γ and $u < u(P_0)$ in γ, the horizontal propagation implies $u(Q') < u(P_0)$. Hence we may assume $u < u(P_0)$ in $\mathcal{R}$ and proceed as in Proposition 3.7 to get a contradiction. □

The next proof is the same as in [117, Theorem 2.2.1], whose arguments are only based on geometric constructions.

Corollary 3.10 (Strong Maximum Principle)**.** *Let* $u \in \mathrm{USC}(\overline{Q}_T)$ *be a viscosity subsolution of* (3.1) *attaining a nonnegative maximum at* $P_0 = (x_0, t_0) \in Q_T$. *Assume that* $F(x_0, t_0, 0, 0, 0) \geq 0$ *and satisfies (i)-(ii) and that there exist SV* $Z_1, ..., Z_m$ *for* F *at* (x_0, t_0) *satisfying the Hörmander condition. Then* u *is constant in* $S(P_0)$.

Proof. The proof of this result is based only on geometric arguments. Suppose that $u \neq u(P_0)$ in $S(P_0)$. Then there exists a point $Q \in S(P_0)$ such that $u(Q) < u(P_0)$. Connect Q to P_0 by a simple continuous curve γ lying in $S(P_0)$ such that the t-coordinate is nondecreasing from Q to P_0 (it can be done since Ω is connected). As before, on γ there exists a point P_1 such that $u(P_1) = u(P_0)$ and $u(\overline{P}) < u(P_1)$ for all $\overline{P}$ on γ lying between Q and P_1. Denote by γ_0 the segment between Q and P_1. Construct a rectangle whose top face is centered at P_1, i.e.

$$\mathcal{R}' := \{(x, t) : x_1^i - a^i \leq x^i \leq x_1^i + a^i,\ t_1 - a \leq t \leq t_1\}\ ,$$

where a is sufficiently small so that the rectangle lies in Q_T. Applying the local vertical propagation of maxima in Corollary 3.9 we get $u \equiv u(P_1)$ in this rectangle. As a consequence $u \equiv u(P_1)$ in γ_0 (which in fact lies in the rectangle) and this contradicts the definition of P_1.

□

Remark 3.11. Results for the propagation of minima can be deduced from the previous analysis simply by observing that $v \in \mathrm{LSC}(\Omega \times (0, T))$ is a supersolution to (3.1) if and only if $u = -v$ is a subsolution to

$$\partial_t u + F(x, -u, -Du, -D^2 u) = 0 \text{ in } \Omega \times (0, T)\ .$$

Denote by $F^-(x, r, p, X) = -F(x, -r, -p, -X)$. Then, the assumptions needed to get the strong minimum principle are the following

(i') F is upper semicontinuous and proper.

(ii') (*Scaling*) For some $\phi > 0$ the operator satisfies

$$\xi s + F(x, t, \xi r, \xi p, \xi X) \leq\ \phi(\xi)(s + F(x, t, r, p, X))$$

for all $\xi \in (0, 1]$ and $r \in [-1, 0]$ and for every $(x, t) \in \Omega \times (0, T)$, $p \in \mathbb{R}^d \backslash \{0\}$ and $X \in \mathcal{S}_d$.

Finally, $Z \in \mathbb{R}^{d+1}$ is subunit for $\partial_t + F^-$ at $(x, t) \in \mathcal{O}$ if

$$\inf_{\gamma > 0} \{p_t + F(x, t, 0, p_t, p, \gamma p \otimes p - I)\} > 0\ ,$$

for every $(p, p_t) \in \mathbb{R}^{d+1}$ such that $(p, p_t) \cdot Z \neq 0$.

3.4 Some applications

3.4.1 Fully nonlinear degenerate parabolic equations

As in the elliptic case, the main application concerns fully nonlinear evolutive subelliptic equations. In this framework one is given a family $\mathcal{X} = (X_1, ..., X_m)$ of $C^{1,1}$ vector fields defined in $\overline{\Omega}$. The intrinsic gradient and intrinsic Hessian are defined as $D_{\mathcal{X}}u = (X_1u, ..., X_mu)$ and $(D^2_{\mathcal{X}}u)_{ij} = X_i(X_ju)$. After choosing a basis in the Euclidean space, we write $X_j = \sigma^j \cdot D$, with $\sigma^j : \overline{\Omega} \to \mathbb{R}^d$, and $\sigma = \sigma(x) = [\sigma^1(x), ..., \sigma^m(x)] \in \mathbb{R}^{d\times m}$. Then

$$D_{\mathcal{X}}u = \sigma^T Du = (\sigma^1 \cdot Du, ..., \sigma^m \cdot Du)$$

and

$$X_i(X_ju) = (\sigma^T D^2u\ \sigma)_{ij} + (D\sigma^j\ \sigma^i) \cdot Du\ .$$

Denote by Y^* the symmetrized matrix of Y. By the chain rule (see, e.g., [34, Lemma 3]) one can obtain that for $u \in C^2$

$$(D^2_{\mathcal{X}}u)^* = \sigma^T D^2u\sigma + g(x, Du)\ ,$$

where the correction term g is

$$(g(x, P))_{ij} = \frac{1}{2}[(D\sigma^j\ \sigma^i) \cdot p + (D\sigma^i\ \sigma^j) \cdot p]\ .$$

Then, we focus on the parabolic equation (3.3) that can be written as

$$\partial_t u + G(x, t, u, \sigma^T(x)Du, \sigma^T(x)D^2u\sigma(x) + g(x, Du)) = 0\ , \tag{3.8}$$

which is of the form (3.1) if we define

$$p_t + F(x, t, r, p, X) := p_t + G(x, t, r, \sigma^T(x)p, \sigma^T(x)X\sigma(x) + g(x, p))\ . \tag{3.9}$$

Lemma 3.12. *If $p_t + G$ satisfies properties* (i), (ii) *and* (3.5)*, then $p_t + F$ satisfies properties* (i) *and* (ii) *and the vector fields $(\sigma^i, 0)$ are subunit for $\partial_t + F$ in the sense of Definition 3.1.*

Proof. (i) holds because $X \le Y$ implies $\sigma^T(x)X\sigma(x) \le \sigma^T(x)Y\sigma(x)$, so F is proper.

(ii) holds for F if it does for G because $g(x, p)$ is positively 1-homogeneous in the variable p.

To prove that any X_i is SV for F we use property (3.5) with $q = \sigma^T(x)p$, $X = \sigma^T\sigma + g$ to get

$$\begin{aligned} &p_t + F(x, t, 0, p, I - \gamma p \otimes p) \\ &\quad = p_t + G(x, t, 0, \sigma(x)^T p, \sigma^T(x)I\sigma(x) - \gamma(\sigma^T(x)p) \otimes (\sigma^T(x)p) + g(x, p)) > 0 \end{aligned}$$

for some $\gamma > 0$ if $\sigma^i(x) \cdot p \ne 0$. □

Combining this Lemma and Corollary 3.10 we have the following SMP for evolutive subelliptic equations. As largely discussed in Chapter 2, this result applies for instance to PDEs on Carnot groups and over Grushin vector fields.

Corollary 3.13. *Assume that G verifies* (i), (ii)*, $G(x_0, t_0, 0, 0, 0) \geq 0$ and* (3.5)*, and let u be a subsolution of* (2.13)*, attaining a nonnegative maximum at $(x_0, t_0) \in Q_T$. Then we have both horizontal and vertical propagation of maxima.*

The model equation to apply our results is

$$\partial_t u + c(x,t)|u|^{k-1}u + a(x,t)E(D_{\mathcal{X}}u, (D^2_{\mathcal{X}}u)^*) = 0 \ , \tag{3.10}$$

where we assume $E : \mathbb{R}^m \backslash \{0\} \times \mathbb{R}^{m \times m}$ is positively homogeneous of degree $\alpha \geq 0$, a is continuous and

$$c \geq 0 \ , \ a > 0 \ , \quad \text{and} \quad \text{either } c = 0 \text{ or } \alpha \leq k \ , k > 0 \ .$$

Example 3.14. The *Pucci's extremal operators* $\mathcal{M}^+(M)$ and $\mathcal{M}^-(M)$ on symmetric matrices $M \in \mathcal{S}_m$ are 1-homogeneous and satisfy (3.5) because

$$\mathcal{M}^+_{\lambda,\Lambda}(X - \gamma q \otimes q) \geq \mathcal{M}^-_{\lambda,\Lambda}(X - \gamma q \otimes q) \geq \mathcal{M}^-_{\lambda,\Lambda}(X) + \lambda\gamma|q|^2.$$

then the equation (3.10) with $E(D^2_{\mathcal{X}}u)^*) = \mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*)$ and $k = 1$ (which ensures the validity of (ii)) satisfy the SMP and the Strong Minimum principle, and the same holds if $\mathcal{M}^+$ is replaced by $\mathcal{M}^-$. The first result for this evolutive operators appeared in [100] in the Euclidean framework. The first analysis of such parabolic extremal operators goes back to [13].

Example 3.15. Another example of (3.10) are quasi-linear parabolic subelliptic equations of the form

$$\partial_t u - \Delta_{\mathcal{X}}u + b(x,t)|D_{\mathcal{X}}u(x,t)|^m + c(x,t)|u|^{k-1}u = 0 \ .$$

Setting $G(q, X) = -\mathrm{Tr}(X) + b(x,t)|q|^m$, we observe that (3.5) is clearly satisfied since

$$G(q, X - \gamma q \otimes q) = -\mathrm{Tr}(X) + b(x,t)|q|^m + \gamma|q|^2 \ .$$

Moreover, the scaling property is satisfied if either

$$b \geq 0 \text{ and } m \leq 1 \text{ or } b \leq 0 \text{ and } m \geq 1 \ ,$$

and either

$$c \equiv 0 \text{ or } k = 1 \ .$$

Therefore, under these assumptions, we have both horizontal and vertical propagation of maxima for these subelliptic quasi-linear parabolic equations.

Example 3.16. Consider the equation

$$\partial_t u - |D_{\mathcal{X}}u|^{h-3}\Delta_{\mathcal{X},\infty}u = 0$$

for $h \geq 1$, where $\mathcal{X}$ is a system of vector fields inducing a Carnot group. The operator is homogeneous of degree h in space and has been studied in [144] when $h = 1$, [196] for $1 < h < 3$ in the Euclidean case, and lately revisited in the context of Carnot groups in [43]. The results of [100] give both the horizontal and vertical propagation of maxima for the Euclidean counterpart of such evolutive operator. Our analysis provide the SMP for the particular case $h = 1$, which ensures the validity of the scaling assumption (ii) (see also Example 2.28 for the elliptic counterpart). See in particular [144, 43] for the definition of viscosity solution in this context. Note that the above proofs can be accommodated to handle these singular PDEs by using upper and lower semicontinuous envelopes of G as in the hypothesis given in [100].

Remark 3.17. We remark that, unlike the elliptic setting, the parabolic p-Laplacian equation with $p \neq 2$ is not covered by our results, since the scaling property (ii) for the evolutive operator fails (see e.g. [100, Example 2.6]). However, for such parabolic equations the SMP may fail when $p > 2$ due to the results in [146] for the parabolic equation $\partial_t u - \Delta_p u = 0$ in the Euclidean setting. We are not aware of similar results in the subelliptic framework.

3.4.2 Hamilton-Jacobi-Bellman Equations

We are given a family of linear degenerate operators

$$L^\alpha u := -\mathrm{Tr}(A^\alpha(x,t)D^2u) - b^\alpha(x,t)\cdot Du + c^\alpha(x,t)u \tag{3.11}$$

where the parameter α takes values in a given set, $A^\alpha(x,t) \geq 0$, $A_\alpha \in \mathcal{S}_d$ and $c^\alpha(x,t) \geq 0$ for all (x,t) and α. Here, we consider the equations

$$\partial_t u+F_s(x,t,u,Du,D^2u) := \partial_t u+\sup_\alpha L^\alpha u\,, \partial_t u+F_i(x,t,u,Du,D^2u) := \partial_t u+\inf_\alpha L^\alpha u \tag{3.12}$$

and we assume that $F_s(x,t,r,p,X), F_i(x,t,r,p,X)$ are finite and continuous for all entries $(x,t,r,p,X) \in \overline{\mathcal{O}} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d$. They are clearly proper and positively 1-homogeneous.

Remark 3.18. It is straightforward to see that if $Z \in \mathbb{R}^d$ is subunit for F, then the vector field $(Z,0) \in \mathbb{R}^{d+1}$ is SV for the parabolic operator $\partial_t + F$ in the sense of Definition 3.1.

We can characterize the subunit vectors of these operators as follows

Lemma 3.19. *Let $Z \in \mathbb{R}^d$ and $(x,t) \in \mathcal{O}$.*
i) $(Z,0)$ is SV for $\partial_t + F_i$ at (x,t) if and only if Z is subunit for all the matrices $A^\alpha(x,t)$, i.e., $A^\alpha(x,t) \geq Z \otimes Z$ for all α;
ii) $(Z,0)$ is SV for $\partial_t + F_s$ at (x,t) if there exists $\bar\alpha$ such that Z is subunit for the matrix $A^{\bar\alpha}(x,t)$.

Proof. This is a consequence of Lemma 2.31 and Remark 3.18.

□

The results of the previous sections combined with Lemma 3.19 give informations on the sets of propagation of maxima and minima of sub- and supersolutions.

Corollary 3.20. *Assume $Z_i : \overline{\mathcal{O}} \to \mathbb{R}^d$, $i = 1,...,m$, are locally Lipschitz vector fields such that*

$$A^\alpha(x,t) \geq Z_i(x,t) \otimes Z_i(x,t) \quad \textit{for all } \alpha,\ i,\ \textit{and } x,t\,,$$

and satisfying the Hörmander condition. Then
i) any subsolution of $\partial_t u + \inf_\alpha L^\alpha u = 0$ attaining a nonnegative maximum at $P_0 \in Q_T$ is constant in $S(P_0)$,
ii) any supersolution of $\partial_t u + \sup_\alpha L^\alpha u = 0$ attaining a nonpositive minimum at $P_0 \in Q_T$ is constant in $S(P_0)$.

3.4.3 Hamilton-Jacobi-Isaacs Equations

Now we are given a two-parameter family of linear degenerate elliptic operators

$$L^{\alpha,\beta}u := -\mathrm{Tr}(A^{\alpha,\beta}(x,t)D_x^2 u) - b^{\alpha,\beta}(x,t)\cdot Du + c^{\alpha,\beta}(x,t)u$$

where the parameters α,β take values in two given sets, $A^{\alpha,\beta}(x,t) \geq 0$, $A^{\alpha,\beta} \in \mathcal{S}_d$ and $c^{\alpha,\beta}(x,t) \geq 0$ for all x, t, α, β. The Hamilton-Jacobi-Isaacs (briefly, H-J-I) operators are

$$\partial_t u + F_-(x,t,u,Du,D^2u) := \partial_t u + \sup_\beta \inf_\alpha L^{\alpha,\beta}u$$

and

$$\partial_t u + F_+(x,t,u,Du,D^2u) := \partial_t u + \inf_\alpha \sup_\beta L^{\alpha,\beta}u\ .$$

and we assume that $F_-(x,r,p,X), F_+(x,r,p,X)$ are finite and continuous for all entries $(x,t,r,p,X) \in \overline{\mathcal{O}} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d$. They are clearly proper and positively 1-homogeneous. We can find subunit vectors of these operators following the arguments of Corollary 3.19.

Lemma 3.21. *Let $Z \in \mathbb{R}^d$ and $(x,t) \in \mathcal{O}$.*
i) $(Z,0)$ is SV for $\partial_t + F_-$ at (x,t) if there exists $\bar\beta$ such that $A^{\alpha,\bar\beta}(x,t) \geq Z \otimes Z$ for all α;
ii) $(Z,0)$ is SV for $\partial_t + F_+$ at (x,t) if for all α there exists $\beta(\alpha)$ such that $A^{\alpha,\beta(\alpha)}(x,t) \geq Z \otimes Z$.

Then we get the following SMP for the H-J-I equations.

Corollary 3.22. *Assume $Z_i : \overline{\mathcal{O}} \to \mathbb{R}^{d+1}$, $i = 1,...,m$, are locally Lipschitz vector fields satisfying the Hörmander condition. Then*
i) if there exists $\bar\beta$ such that

$$A^{\alpha,\bar\beta}(x,t) \geq Z_i(x,t) \otimes Z_i(x,t) \quad \textit{for all } \alpha,\ i,\ \textit{and } x,t\,,$$

then any subsolution of $\partial_t u + \sup_\beta \inf_\alpha L^{\alpha,\beta}u = 0$ attaining a nonnegative maximum in Q_T is constant in $S(P_0)$;
ii) if for all α there exists $\beta(\alpha)$ such that

$$A^{\alpha,\beta(\alpha)}(x,t) \geq Z_i(x,t) \otimes Z_i(x,t) \quad \textit{for all } i \textit{ and } x,t\,,$$

then any subsolution of $\partial_t u + \inf_\alpha \sup_\beta L^{\alpha,\beta}u = 0$ attaining a nonnegative maximum in Q_T is constant in $S(P_0)$.

Sufficient conditions for the Strong Minimum Principle can be easily found in the same way owing to Remark 3.11, as follows.

Corollary 3.23. *Assume $Z_i : \overline{\mathcal{O}} \to \mathbb{R}^d$, $i = 1,...,m$, are locally Lipschitz vector fields satisfying the Hörmander condition. Then*
i) if for all β there exists $\alpha(\beta)$ such that

$$A^{\alpha(\beta),\beta}(x,t) \geq Z_i(x,t) \otimes Z_i(x,t) \quad \textit{for all } i \textit{ and } x,t\,,$$

then any supersolution of $\partial_t u + \sup_\beta \inf_\alpha L^{\alpha,\beta} u = 0$ *attaining a nonpositive minimum in* Q_T *is constant in* $S(P_0)$*;*
ii) if there exists $\bar{\alpha}$ *such that*

$$A^{\bar{\alpha},\beta}(x,t) \geq Z_i(x,t) \otimes Z_i(x,t) \quad \textit{for all } \beta,\ i,\ \textit{and } x,t\,,$$

then any supersolution of $\partial_t u + \inf_\alpha \sup_\beta L^{\alpha,\beta} u = 0$ *attaining a nonpositive minimum in* Q_T *is constant in* $S(P_0)$*.*

Example 3.24. If $\mathcal{X} = (X_1, ..., X_m)$ are $C^{1,1}$ vector fields on $\overline{\mathcal{O}}$ satisfying the Hörmander condition, $a, b \in C(\overline{\mathcal{O}})$ are nonnegative, and $\mathcal{M}^+, \mathcal{M}^-$ denote the Pucci's extremal operators, then the evolutive equation

$$\partial_t u + a(x,t)\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathcal{X}} u)^*) + b(x,t)\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}} u)^*) = 0$$

is of H-J-I form and satisfies both the SMP and the Strong Minimum Principle.

Chapter 4

Liouville properties for fully nonlinear subelliptic problems

In this chapter we are interested in Liouville theorems for fully nonlinear second order subelliptic equations of the form

$$G(x, u, D_{\mathcal{X}}u, (D^2_{\mathcal{X}}u)^*) = 0 \text{ in } \mathbb{R}^d \ , \tag{4.1}$$

where $D_{\mathcal{X}}u \in \mathbb{R}^m$ and $(D^2_{\mathcal{X}}u)^* \in \mathcal{S}_m$, $m \leq d$, stand for the horizontal gradient and the symmetrized horizontal Hessian respectively, as defined in Subsection 1.0.5. The typical approach to handle such PDEs is to write the equation in Euclidean coordinates by defining the (degenerate) operator

$$F(x, r, Du, D^2u) := G(x, r, \sigma^T(x)p, \sigma^T(x)D^2u\sigma(x) + g(x, Du)) \ .$$

(see Remark 4.11 below). More precisely, we investigate sufficient conditions for the validity of Liouville properties such as:

Any subsolution (resp. supersolution) of (4.1) bounded from above (resp. below) is a constant.

In particular, we first give an abstract result for general equations of the form (4.1) modeled on Hörmander vector fields that are neither convex nor concave. We then apply the results to PDEs driven by Pucci's operators perturbed by first order terms in the subelliptic setting in order to achieve the results for fully nonlinear uniformly subelliptic equations. We provide several examples among which PDEs on the Heisenberg group $\mathbb{H}^d$ and free step 2 Carnot groups with r-generators (see Subsection 1.0.3). We conclude our study by providing also some examples for fully nonlinear equations on the Grushin plane (Subsection 1.0.4), where no group structure is available but the Hörmander condition is still in force.

4.1 A glimpse on the method of proof for linear equations

Before showing our main results, we prefer to present the proof of the Liouville theorem for classical C^2 subsolutions to linear uniformly elliptic equations in the

Euclidean framework, which serves as a guideline for our proof in the nonlinear and subelliptic setting. It is based on classical tools such as strong maximum and comparison principles.

Theorem 4.1. *Assume that the operator* $Lu := -\mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du$ *is uniformly elliptic, with* $a : \mathbb{R}^d \to \mathcal{S}_d$, $b : \mathbb{R}^d \to \mathbb{R}^d$ *bounded and continuous. Suppose also that there exists a classical supersolution* w *to* $Lu = 0$ *which blows up at infinity. More precisely, we assume that there exist* $R > 0$ *and* $w \in C^2(\mathbb{R}^d \backslash \{0\})$ *such that*

(i) $Lw \geq 0$ *for* $|x| > R$

(ii) $\lim_{|x| \to +\infty} w = +\infty$.

Let also $u \in C^2(\mathbb{R}^d)$ *be such that* $Lu \leq 0$ *in* $\mathbb{R}^d$*and* $u(x) \leq C$ *in* $\mathbb{R}^d$. *Then,* u *is constant.*

Remark *4.2.* This result is a special case of [16, Theorem 2.1] and applies to the case of the Laplacian (i.e. $a_{ij} = \delta_{ij}$ and $b = 0$) when $d \leq 2$; therefore it gives a different proof of the Liouville theorem [197, Theorem 2.29]. Indeed, the function $w := \log |x|$ fulfills the above assumptions, giving thus that every bounded from above subharmonic function is constant. However, as pointed out in the introduction, the Liouville property fails as soon as $d \geq 3$, where indeed w is no longer a classical supersolution of $-\Delta u = 0$. It also applies to subsolutions of $-\Delta u + b(x) \cdot Du = 0$ in any space dimension under assumptions on the drift b implying the existence of a Lyapunov-like function w, see e.g. [16, Section 1].

Proof. Let u be a classical subsolution to $Lu = 0$ in $\mathbb{R}^d$ and let $\zeta > 0$. Set

$$v_\zeta(x) = u(x) - \zeta w(x) \text{ for } |x| \geq \bar{R}$$

for some $\bar{R} > R > 0$. Clearly, $v_\zeta \in C^2(\Omega_{\bar{R}})$, where we set $\Omega_{\bar{R}} := \{x \in \mathbb{R}^d : |x| \geq \bar{R}\}$. Moreover, we have

$$\lim_{|x| \to +\infty} v_\zeta(x) = -\infty \text{ and } Lv_\zeta = Lu - \zeta Lw \leq 0 \text{ for every } x \text{ such that } |x| > \bar{R} \ .$$

Define $C_\zeta := \max_{\{|x| = \bar{R}\}} v_\zeta(x)$. Since

$$\lim_{|x| \to +\infty} v_\zeta(x) = -\infty \ ,$$

there exists $K_\zeta > \bar{R}$ such that $v_\zeta < C_\eta$ for every x such that $|x| \geq K_\zeta$. By the weak maximum principle (see [122]) applied on the set $\{x \in \mathbb{R}^d : \bar{R} < |x| < K_\zeta\}$ we have

$$\max_{\{x \in \mathbb{R}^d : \bar{R} < |x| < K_\zeta\}} v_\zeta(x) = \max_{\{x \in \mathbb{R}^d : |x| = \bar{R} \text{ or } |x| = K_\zeta\}} v_\zeta(x)$$

Since $v_\zeta(x) < C_\zeta$ for every x such that $|x| \geq K_\zeta$, we conclude that for every y such that $|y| \geq \bar{R}$

$$v_\zeta(y) = u(y) - \zeta w(y) \leq \max_{\{x \in \mathbb{R}^d : |x| = \bar{R}\}} v_\eta \leq \max_{\{x \in \mathbb{R}^d : |x| = \bar{R}\}} u - \zeta \min_{\{x \in \mathbb{R}^d : |x| = \bar{R}\}} w \ .$$

On one hand, letting $\zeta \to 0$ we conclude

$$u(y) \le \max_{\{x\in\mathbb{R}^d : |x|=\bar{R}\}} u \text{ for all } |y| > \bar{R} \, .$$

On the other hand, owing to the weak maximum principle in the set $B(0,\bar{R})$ we obtain

$$u(y) \le \max_{\{x\in\mathbb{R}^d : |x|=\bar{R}\}} u \text{ for all } |y| < \bar{R} \, .$$

Combining the above inequalities one concludes

$$u(y) \le \max_{\{x\in\mathbb{R}^d : |x|=\bar{R}\}} u \text{ for all } y \in \mathbb{R}^d \, .$$

Hence, u attains its maximum over $\partial B(0,\bar{R})$ and then the conclusion follows by the strong maximum principle for classical linear uniformly elliptic equations [122]. □

Remark 4.3. The same result remains true if L is replaced by a degenerate elliptic operator $L_{\mathcal{X}}u = -\sum_{i,j} X_iX_ju + b(x)\cdot D_{\mathcal{X}}u$, provided the vector fields $\mathcal{X}$ satisfy the Hörmander condition and $b : \mathbb{R}^d \to \mathbb{R}^m$, $m \le d$, bounded and continuous, the proof being exactly the same owing to Bony's strong maximum principle for subelliptic equations. An example of such result is [181, Proposition 3.1].

4.2 Liouville theorem: the general case

4.2.1 Abstract result

In this Section we consider a general equation of the form

$$F(x,u,Du,D^2u) = 0 \text{ in } \mathbb{R}^d. \tag{4.2}$$

We will denote $F[u] := F(x,u,Du,D^2u)$ and make the following assumptions

(i) F is continuous, proper, satisfies

$$F[\varphi - \psi] \le F[\varphi] - F[\psi] \text{ for all } \varphi,\psi \in C^2(\mathbb{R}^d) \tag{S1}$$

and $F(x,r,0,0) \ge 0$ for every $x \in \Omega$ and $r \ge 0$.

(ii) F satisfies the comparison principle in any bounded open set Ω, namely, if u and v are respectively a viscosity subsolution and a viscosity supersolution of (4.1) such that $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.

(iii) There exists $R_o \ge 0$ and $w \in \mathrm{LSC}(\mathbb{R}^d)$ viscosity supersolution of (4.2) for $|x| > R_o$ and satisfying $\lim_{|x|\to\infty} w(x) = +\infty$.

(iv) F satisfies the strong maximum principle, namely, any viscosity subsolution of (4.2) that attains an nonnegative maximum must be constant.

The prototype examples of operators satisfying (S1) are Pucci's minimal operators, or, more in general, Bellman operators defined as infimum of linear operators.
To prove the analogous results for viscosity supersolutions, we need to replace (i)-(ii) and (iv) above by

(i') F is continuous, proper, satisfies

$$F[\varphi - \psi] \geq F[\varphi] - F[\psi] \text{ for all } \varphi, \psi \in C^2(\mathbb{R}^d) \tag{S2}$$

and $F(x, r, 0, 0) \leq 0$ for every $x \in \Omega$ and $r \leq 0$.

(ii') There exists $R_o \geq 0$ and $W \in \mathrm{USC}(\mathbb{R}^d)$ viscosity subsolution to (4.2) for $|x| > R_o$ and satisfying $\lim_{|x|\to\infty} W(x) = -\infty$.

(iv') F satisfies the strong minimum principle.

We now adapt the procedure outlined in Theorem 4.1 for linear equations in the fully nonlinear degenerate setting. Similar arguments were used in [16, 17] for fully nonlinear uniformly elliptic equations and quasi-linear equations with Hörmander diffusion and in [181, Proposition 3.1] for linear PDEs in the first Heisenberg group $\mathbb{H}^1$.

Proposition 4.4. *Assume* (i)-(iv). *Let* $u \in \mathrm{USC}(\mathbb{R}^d)$ *be a viscosity subsolution to* (4.2) *satisfying*

$$\limsup_{|x|\to\infty} \frac{u(x)}{w(x)} \leq 0 \ . \tag{4.3}$$

for w *as in* (iii). *If* $u \geq 0$*, then* u *is constant.*

Proof. We divide the proof in four steps.

Step 1. Define $u_\zeta(x) := u(x) - \zeta w(x)$ for $\zeta > 0$. Possibly increasing R_o, we can assume that u is not constant in $\overline{B}(0, R_o) := \{x \in \mathbb{R}^d : |x| \leq R_0\}$, otherwise we are done. Set

$$C_\zeta := \max_{|x|\leq R_o} u_\zeta(x) \ . \tag{4.4}$$

First of all, note that under the standing assumptions $F[C_\zeta] \geq 0$ for every ζ sufficiently small. Since $u \geq 0$ by the standing assumptions we assume that $C_\zeta > 0$ for ζ sufficiently small. In fact, if this were not the case, we could conclude letting $\zeta \to 0$ that $u(x) = 0$ for every x such that $|x| \leq R_o$, in contradiction with the initial assumption that u is not constant in the ball $\overline{B}(0, R_o)$.

Step 2. The growth condition (4.3) implies

$$\limsup_{|x|\to\infty} \frac{u_\zeta(x)}{w(x)} \leq -\zeta < 0 \ \forall \zeta > 0 \ .$$

As a consequence we have

$$\lim_{|x|\to+\infty} u_\zeta(x) = -\infty \ . \tag{4.5}$$

Then, for all $\zeta > 0$ there exists $R_\zeta > R_o$ such that

$$u_\zeta(x) \leq C_\zeta \text{ for all } |x| \geq R_\zeta \ . \tag{4.6}$$

Step 3. We prove that u_ζ is a viscosity subsolution of $F[u] = 0$ in $\{x \in \mathbb{R}^d : |x| > R_o\}$. Fix $\bar{x}$ such that $|\bar{x}| > R_o$ and a smooth function φ such that $(u_\zeta - \varphi)(\bar{x}) = 0$

and $u_\zeta - \varphi$ has a strict maximum at $\bar{x}$. Assume by contradiction that $F[\varphi(\bar{x})] > 0$. By the continuity of F there exists $\delta > 0$ such that $F[\varphi(\bar{x}) - \delta] > 0$. Hence, using again the continuity of F and the regularity of φ we can conclude that there exists $0 < r < |\bar{x}| - R_o$ such that

$$F[\varphi - \delta] > 0 \text{ in } B(\bar{x}, r) \ .$$

Since $u_\zeta - \varphi$ has a strict maximum at $\bar{x}$, there exists $0 < k < \delta$ such that $u_\zeta - \varphi \leq -k < 0$ on $\partial B(\bar{x}, r)$ Moreover, we claim that $\zeta w + \varphi - k$ satisfies $F[\zeta w + \varphi - k] \geq 0$ in $B(\bar{x}, r)$. Indeed take $\tilde{x} \in B(\bar{x}, r)$ and ψ smooth such that

$$\zeta w + \varphi - k - \psi \text{ has a minimum at } \tilde{x}.$$

Using that w is a viscosity supersolution to (4.1), the fact that G is proper and (S1) we get

$$0 \leq F[\psi(\tilde{x}) - \varphi(\tilde{x}) + k] \leq F[\psi(\tilde{x}) - \varphi(\tilde{x}) + \delta] \leq F[\psi(\tilde{x})] - F[\varphi(\tilde{x}) - \delta] < F[\psi(\tilde{x})] \ ,$$

where in the last inequality we used that $F[\varphi(\tilde{x}) - \delta] > 0$. Therefore $F[\psi(\tilde{x})] \geq 0$, which in turn implies that $\zeta w + \varphi - k$ is a viscosity supersolution to $F[u] = 0$ in $B(\bar{x}, r)$. Since $u \leq \zeta w + \varphi - k$ on $\partial B(\bar{x}, r)$, we can now apply the comparison principle and get

$$u \leq \zeta w + \varphi - k \text{ in } B(\bar{x}, r) \ ,$$

in contradiction with the fact that $u(\bar{x}) = \zeta w(\bar{x}) + \varphi(\bar{x})$.

Step 4. We use the comparison principle in $\Omega = \{x : R_o < |x| < R_\zeta\}$. Since $F[C_\zeta] \geq 0$ by Step 1, we get $u_\zeta \leq C_\zeta$ in Ω by Step 2 (precisely by (4.6)) and Step 3. Therefore we have

$$u_\zeta(x) \leq C_\zeta \text{ for all } |x| \geq R_o \ .$$

By letting $\zeta \to 0^+$ we obtain

$$u(x) \leq \max_{|y| \leq R_o} u(y)$$

and hence u attains its maximum $\bar{x}$ over $\mathbb{R}^d$. If now $u \geq 0$, the SMP holds and we have the desired conclusion. □

The next result emphasizes that the assumption $u \geq 0$ can be dropped provided $r \longmapsto F(x, r, p, X)$ is constant: this will be the case of HJB operators we discuss in the next sections.

Corollary 4.5. *Assume* (i)-(iv). *Let* $u \in \mathrm{USC}(\mathbb{R}^d)$ *be a viscosity subsolution to* (4.2) *such that* (4.3) *holds for* w *as in* (iii). *Assume* $r \mapsto F(x, r, p, X)$ *is constant for all* x, p, X *and* $F(x, r, 0, 0) = 0$ *for every* $x \in \Omega$. *Then,* u *is constant.*

Proof. The proof goes along the same lines as Proposition 4.4. It is sufficient to note that under the standing assumptions on F, $u + |u(\bar{x})|$, $\bar{x}$ standing for the maximum point in Proposition 4.4, is again a subsolution to (4.2), since $r \mapsto F(x, r, p, X)$ is constant for all x, p, X, and one concludes. □

Similar result holds for the case of supersolutions to (4.2).

Proposition 4.6. *Assume* (i'),(ii),(iii') and (iv'). *Let* $v \in \mathrm{USC}(\mathbb{R}^d)$ *be a viscosity supersolution to* (4.2) *satisfying*

$$\liminf_{|x|\to\infty} \frac{v(x)}{W(x)} \le 0 \ . \tag{4.7}$$

for W *as in* (iii'). *If* $v \le 0$, *then* v *is constant.*

Proof. We proceed as in the previous theorem. We consider the function $v_\zeta := v(x) - \zeta W(x)$. As in Step 1 we get $F[c_\zeta] \le 0$ for ζ sufficiently small, where

$$c_\zeta := \min_{|x|\le R_o} v_\zeta(x) \ .$$

Following Step 2, by (4.7) and $W(x) < 0$ for $|x|$ large, we get $\lim_{|x|\to+\infty} v_\zeta(x) = +\infty$ for every $\zeta > 0$. Then for all $\zeta > 0$ sufficiently small there exists $R_\zeta > R_o$ such that

$$v_\zeta(x) \ge c_\zeta \ \forall |x| \ge R_\zeta \ .$$

Moreover, arguing as in Step 3 of the same proof, one can show that $F[v_\eta] \ge 0$ for $|x| \ge R_o$ under assumptions (i') and exploiting (S2). As in Step 4 we use the comparison principle to conclude that $v_\zeta(y) \ge c_\zeta$ for $|y| \ge R_o$. Letting $\zeta \to 0$ we get $v(y) \ge \min_{|x|\le R_o} v(x)$ for $|y| \ge R_o$ meaning that v attains its minimum at a point $\bar{x}$ over $\mathbb{R}^d$. As before, the strong minimum principle gives the conclusion. □

Corollary 4.7. *Assume* (i'),(ii'),(iii) and (iv'). *Let* $v \in \mathrm{USC}(\mathbb{R}^d)$ *be a viscosity supersolution to* (4.2) *satisfying* (4.7) *for* W *as in* (iii'). *Assume* $r \mapsto F(x, r, p, X)$ *is constant for all* x, p, X *and* $F(x, r, 0, 0) = 0$ *for every* $x \in \Omega$, *then* v *is constant.*

We can now state our main result for subsolutions. To do this, we first recall a crucial scaling assumption for the validity of the strong maximum principle for fully nonlinear subelliptic equations together with the concept of generalized subunit vector field. For the definition of Hörmander vector fields and further details we refer to Chapter 2. We assume

(SC) For some $\phi : (0, 1] \to (0, +\infty]$, F satisfies

$$F(x, \xi s, \xi p, \xi X) \ge \ \phi(\xi) F(x, s, p, X)$$

for all $\xi \in (0, 1]$, $s \in [-1, 0]$, $x \in \Omega$, $p \in \mathbb{R}^d \backslash \{0\}$, and $X \in \mathcal{S}_d$;

We briefly recall the definition of generalized subunit vector field introduced in Chapter 2.

Definition 4.8. $Z \in \mathbb{R}^d$ *is a generalized* subunit vector *(briefly, SV) for* $F = F(x, r, p, X)$ *at* $x \in \Omega$ *if*

$$\sup_{\gamma>0} F(x, 0, p, I - \gamma p \otimes p) > 0 \quad \forall p \in \mathbb{R}^d \ \text{ such that } \ Z \cdot p \ne 0;$$

$Z : \Omega \to \mathbb{R}^d$ *is a* subunit vector field *(briefly, SVF) if* $Z(x)$ *is SV for* F *at* x *for every* $x \in \Omega$.

Theorem 4.9. *Let F be such that* (i),(ii), (iii), and (SC) *hold. Furthermore assume that F admits $Z_1, ..., Z_m$ generalized subunit vector fields satisfying the Hörmander condition. Let $u \in \mathrm{USC}(\mathbb{R}^d)$ be a viscosity subsolution to* (4.2) *satisfying* (4.3) *for w as in* (iii)*. Assume either $u \geq 0$, or $r \mapsto F(x,r,p,X)$ is constant for all x, p, X and $F(x,r,0,0) = 0$ for every $x \in \Omega$. Then u is constant.*

Proof. The proof is a consequence of Proposition 4.4 and Corollary 4.5 by recalling that under (1), (SC) and the existence of subunit vector fields for F the strong maximum principle holds (cf Chapter 2). □

Similarly, in the case of supersolutions we have the following result by replacing (SC) with

(SC') For some $\phi : (0,1] \to (0,+\infty]$, F satisfies

$$F(x, \xi s, \xi p, \xi X) \leq \ \phi(\xi) F(x,s,p,X)$$

for all $\xi \in (0,1]$, $s \in [-1,0]$, $x \in \Omega$, $p \in \mathbb{R}^d \backslash \{0\}$, and $X \in \mathcal{S}_d$;

and the condition in Definition 2.3 is replaced by

$$\inf_{\gamma>0} F(x,0,p,\gamma p \otimes p - I) > 0 \quad \forall p \in \mathbb{R}^d \ \text{ such that } \ Z \cdot p \neq 0;$$

Theorem 4.10. *Let F be such that* (i'),(ii'), (iii), and (SC') *hold. Furthermore assume that F admits $Z_1, ..., Z_m$ generalized subunit vector fields satisfying the Hörmander condition. Let $v \in \mathrm{USC}(\mathbb{R}^d)$ be a viscosity supersolution to* (4.2) *satisfying* (4.7) *for W as in* (iii')*. Assume either $v \leq 0$, or $r \mapsto F(x,r,p,X)$ is constant for all x, p, X and $F(x,r,0,0) = 0$ for every $x \in \Omega$. Then u is constant.*

Proof. The proof is a consequence of Proposition 4.6 and Corollary 4.7 by recalling that under (1'), (SC') and the existence of subunit vector fields for F the strong minimum principle holds (cf Chapter 2). □

Remark 4.11. In the subelliptic context, i.e. when dealing with PDEs of the form (4.1), if one assumes that G is *elliptic for any x and p fixed* in the following sense:

$$\sup_{\gamma>0} G(x,0,q,X - \gamma q \otimes q) > 0 \quad \forall\, x \in \Omega,\ q \in \mathbb{R}^m,\ q \neq 0,\ X \in \mathcal{S}_m,$$

then, by rewriting the equation in Euclidean coordinates, i.e.

$$F(x,r,p,X) = G(x,r,\sigma^T(x)p, \sigma^T(x)X\sigma(x) + g(x,Du)) \ ,$$

one finds an equivalent equation of the form (4.2) with F having σ^i as subunit vector fields (cf Lemma 2.23). As Lemma 2.23 shows, if G satisfies (SC) and (SC'), also F does. Therefore, Theorem 4.9 and Theorem 4.10 apply respectively to viscosity sub- and supersolutions to (4.1).

4.2.2 Equations driven by Pucci's subelliptic operators

Important examples of fully nonlinear second order subelliptic operators are the Pucci's extremal operators over horizontal Hessians. They are the simplest examples of degenerate HJB operators and they differ from those defined in the Euclidean setting since horizontal Hessians carry an additional x-dependence through the matrices $\sigma(x)$ (Section 1.0.5). More precisely, the minimal operator $\mathcal{M}^-_{\lambda,\Lambda}$ defined in Example 1.7 enjoys the property (S1) simply as a consequence of the property of duality (i.e. $\mathcal{M}^-_{\lambda,\Lambda}(M) = -\mathcal{M}^+_{\lambda,\Lambda}(-M)$ for every $M \in \mathcal{S}_d$) and the reverse inequalities (i.e. $\mathcal{M}^-_{\lambda,\Lambda}(M+N) \leq \mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N)$ for every $M, N \in \mathcal{S}_d$, see [62, Lemma 2.10 properties (3) and (6)]). Similarly, (S2) holds for the maximal operator $\mathcal{M}^+_{\lambda,\Lambda}$.

We now prove the Liouville property for subsolutions of the equation

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*) + H_i(x, u, (D_{\mathcal{X}}u) = 0 \text{ in } \mathbb{R}^d \ , \tag{4.8}$$

where

$$H_i(x, r, p) := \inf_{\alpha \in A}\{c^\alpha(x)r - b^\alpha(x) \cdot p\} \tag{4.9}$$

and for supersolutions of

$$\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*) + H_s(x, u, D_{\mathcal{X}}u) = 0 \text{ in } \mathbb{R}^d \ , \tag{4.10}$$

where

$$H_s(x, r, p) := \sup_{\alpha \in A}\{c^\alpha(x)r - b^\alpha(x) \cdot p\} \ . \tag{4.11}$$

Note that H_i and H_s satisfy (S1) and (S2) by the properties of infimum and supremum. We assume that $b^\alpha(x)$ is locally Lipschitz in x uniformly in α, namely for all $R > 0$ there exists $K_R > 0$ such that

$$\sup_{|x|,|y| \leq R, \alpha \in A} |b^\alpha(x) - b^\alpha(y)| \leq K_R|x-y| \tag{4.12}$$

and

$$c^\alpha(x) \geq 0 \text{ and continuous in } |x| \leq R \text{ uniformly in } \alpha. \tag{4.13}$$

We recall that $(D^2_{\mathcal{X}}u)^* = \sigma^T(x)D^2u\sigma(x) + g(x, Du)$, where $g = g(x,p)$ is linear and 1-homogeneous in the second entry. Typically $g \equiv 0$ in many interesting cases such as Carnot groups, as outlined in Chapter 2.

Corollary 4.12. *Under the previous conditions on H_i, let $u \in \mathrm{USC}(\mathbb{R}^d)$ be a viscosity subsolution to* (4.8) *satisfying* (4.3) *for w as in* (iii). *If either $u \geq 0$ or $c^\alpha(x) \equiv 0$, then u is constant.*

Proof. The proof is a consequence of Theorem 4.9. First, note that $\mathcal{M}^-$ enjoys property (S1) and the scaling (SC) by the well-known properties [62, Lemma 2.10-(3)-(4) and (6)]; this allows to run the arguments in Step 3 of Proposition 4.4. Moreover, the comparison principle (ii) for $\mathcal{M}^- + H_i = 0$ holds in view of Example 2.45. Finally, observe that when $c^\alpha \equiv 0$, then $G(x, r, 0, 0) = 0$ for every $x \in \Omega$, $r \in \mathbb{R}$, and $r \mapsto G(x, r, p, X)$ is constant for every x, p, X. □

Similar procedures yield the following generalization for supersolutions.

Corollary 4.13. *Under the previous conditions on H_s, let $v \in \mathrm{LSC}(\mathbb{R}^d)$ be a viscosity supersolution to* (4.10) *satisfying* (4.7) *for W as in* (iii'). *If either $v \le 0$ or $c^\alpha(x) \equiv 0$, then v is constant.*

Proof. The result is consequence of Theorem 4.10. First, observe that $\mathcal{M}^+$ enjoys property (S2) and the scaling (SC') by using [62, Lemma 2.10-(3)-(4) and (5)]; this allows to run the arguments in Step 3 of Proposition 4.6. Moreover, the comparison principle for $\mathcal{M}^+ + H_s = 0$ holds in view of Example 2.45. In particular, when $c^\alpha \equiv 0$, one notices that $G(x, r, 0, 0) = 0$ for every $x \in \Omega$, $r \in \mathbb{R}$, and $r \mapsto G(x, r, p, X)$ is constant for every x, p, X. □

4.2.3 Fully nonlinear uniformly subelliptic equations

In this section, we consider the Liouville property for viscosity sub- and supersolutions to the prototype fully nonlinear second order subelliptic equation

$$G(x, u, D_{\mathcal{X}}u, (D^2_{\mathcal{X}}u)^*) = 0 \text{ in } \mathbb{R}^d \,. \tag{4.14}$$

We assume that G satisfies a properly rescaled uniform ellipticity condition, namely we consider those operators fulfilling the following inequalities

$$\mathcal{M}^-_{\lambda,\Lambda}(M - N) \le G(x, r, p, M) - G(x, r, p, N) \le \mathcal{M}^+_{\lambda,\Lambda}(M - N) \,. \tag{4.15}$$

for every $(x, r, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and $M, N \in \mathcal{S}_n$ with $N \ge 0$. By taking $N = 0$ we get

$$\mathcal{M}^-_{\lambda,\Lambda}(M) \le G(x, r, p, M) - G(x, r, p, 0) \le \mathcal{M}^+_{\lambda,\Lambda}(M)$$

(cf Section 1.0.5), and, as a consequence, by setting $H(x, r, p) := G(x, r, p, 0)$, one can infer Liouville results for viscosity subsolutions and supersolutions to (4.14) by studying the corresponding properties to equations driven by Pucci's extremal operators $\mathcal{M}^\pm$ composed with the (symmetrized) horizontal Hessian perturbed by the gradient term $H(x, r, p)$. We recall that this is indeed the main idea behind classical works on qualitative and quantitative properties for second order fully nonlinear uniformly elliptic PDEs, see [62] and references therein. In particular, we will focus on the case in which the first order term is concave or convex and hence can be written as infimum or supremum of linear operators. We further assume that

$$G(x, t, p, 0) \ge H_i(x, t, p) \tag{4.16}$$

for some concave Hamiltonian of the form (4.9). Then Corollary 4.12 gives immediately the conclusion

Corollary 4.14. *Assume* (4.15) *and* (4.16) *with b, c satisfying* (4.12) *and* (4.13). *Let $u \in \mathrm{USC}(\mathbb{R}^d)$ be a viscosity subsolution to* (4.14) *satisfying* (4.3) *for w as in* (iii). *If either $u \ge 0$ or $c^\alpha(x) \equiv 0$, then u is constant.*

Proof. It is sufficient to observe that u satisfies the differential inequality

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*) + H(x, u, D_{\mathcal{X}}u) \le 0 \text{ in } \mathbb{R}^d$$

and apply Corollary 4.12. □

As for supersolutions, instead of (4.16) we impose

$$G(x,t,p,0) \leq H_s(x,t,p) \tag{4.17}$$

for some convex H_s as in (4.11).

Corollary 4.15. *Assume* (4.15) *and* (4.17) *with* b, c *satisfying* (4.12) *and* (4.13). *Let* $v \in \mathrm{LSC}(\mathbb{R}^d)$ *be a viscosity subsolution to* (4.14) *satisfying* (4.7) *for* W *as in* (iii'). *If either* $v \leq 0$ *or* $c^\alpha(x) \equiv 0$*, then* v *is constant.*

Proof. It is sufficient to observe that v satisfies the differential inequality

$$\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*) + H(x,u,D_{\mathcal{X}}u) \leq 0 \text{ in } \mathbb{R}^d$$

and apply Corollary 4.13. □

We conclude this section with an application of the previous results to parabolic problems. This would be a first step to the prove the large-time stabilization with respect to the space variable for parabolic problems (cf [16, Section 5]). Let $u : \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$ and consider the general parabolic equation

$$\partial_t u + G(x,u,D_{\mathcal{X}}u,(D^2_{\mathcal{X}}u)^*) = 0 \text{ in } \mathbb{R}^d \times (0,+\infty)\ .$$

We have the following

Corollary 4.16. *Assume that* G *satisfies* (4.15), (4.16), *together with* (i)-(iv). *If* $u \in \mathrm{USC}(\mathbb{R}^d \times [0,+\infty))$ *satisfies*

$$\partial_t u + G(x,u,D_{\mathcal{X}}u,(D^2_{\mathcal{X}}u)^*) \leq 0 \textit{ in } \mathbb{R}^d \times (0,+\infty)\ ,$$

and

$$\limsup_{|x|\to+\infty} \frac{u(x,t)}{w(x)} \leq 0 \textit{ uniformly in } t \in [0,+\infty)\ ,$$

and either $c = 0$ *or* $u \geq 0$*, then*

$$\limsup_{t\to+\infty, y\to x} u(y,t) = \bar{u}(x)$$

is constant with respect to x.

Proof. As above, we exploit the fact that if u is a subsolution to

$$\partial_t u + G(x,u,D_{\mathcal{X}}u,(D^2_{\mathcal{X}}u)^*) \leq 0 \text{ in } \mathbb{R}^d \times (0,+\infty)\ ,$$

then

$$\partial_t u + \mathcal{M}^-_{\lambda,\Lambda}(D^2_{\mathcal{X}}u)^*) + H_i(x,u,D_{\mathcal{X}}u) \leq 0 \text{ in } \mathbb{R}^d \times (0,+\infty)\ .$$

Then one argues as in [16, Corollary 5.1] to conclude the assertion. □

Remark 4.17. One can immediately prove the counterpart of the above result for supersolutions. Indeed, if G satisfies (4.15),(4.17) and (i')-(ii)-(iii') and (iv'), the result reads as follows: let v be a LSC supersolution to $\partial_t u + G(x,u,D_{\mathcal{X}}u,(D^2_{\mathcal{X}}u)^*) = 0$ such that $\liminf_{|x|\to+\infty} \frac{v(x,t)}{W(x)} \leq 0$ uniformly in t. Assume also that either $c \equiv 0$ or $v \leq 0$, then $\liminf_{t\to+\infty,y\to x} v(y,t) = \bar{v}(x)$ is a constant.

4.3 Example 1: the Heisenberg group

Aim of this section is to specialize the previous results to viscosity subsolutions of (4.14) over the Heisenberg vector fields. In the next theorem we provide sufficient conditions for the validity of the Liouville property for viscosity subsolutions to the fully nonlinear equation (4.8), confirming the behavior that can be observed for sub- and super-solutions of the Heisenberg Laplacian, see Section 4.3.1 below for further details. Here $2d+1$ is the linear dimension of the Heisenberg group, and $m = 2d$ stands for the dimension of the horizontal layer. Here and in the next examples we exploit a classical chain rule to compute the horizontal gradient and Hessian of a "radial" function with respect to the homogeneous norm ρ. Indeed, for a sufficiently smooth radial function $f = f(\rho)$ and given a system of vector fields $\mathcal{X} = \{X_1, ..., X_m\}$, we have

$$D_{\mathcal{X}} f(\rho) = f'(\rho) D_{\mathcal{X}} \rho$$

and

$$D^2_{\mathcal{X}} f(\rho) = f'(\rho) D^2_{\mathcal{X}} \rho + f''(\rho) D_{\mathcal{X}} \rho \otimes D_{\mathcal{X}} \rho \ .$$

In this section we denote the Heisenberg horizontal gradient and symmetrized Hessian by $D_{\mathbb{H}^d}$ and $(D^2_{\mathbb{H}^d})^*$.

Theorem 4.18. *Let $\mathcal{X} = \{X_1, ..., X_{2d}, X_{2d+1}\}$ be the system of vector fields generating the Heisenberg group $\mathbb{H}^d$. Assume that* (4.12) *and* (4.13) *hold and that*

$$\sup_{\alpha \in A} \{ b^\alpha(x) \cdot \frac{\eta}{|x_H|^2} - c^\alpha(x) \frac{\rho^4}{|x_H|^2} \log \rho \} \leq \lambda - \Lambda(Q-1) \tag{4.18}$$

for ρ sufficiently large, where $Q = 2d+2$ stands for the homogeneous dimension of $\mathbb{H}^d$, $b^\alpha(x)$ takes values in $\mathbb{R}^{2d}$ and $\eta = (\eta_i, \eta_{i+d})$ is defined by

$$\eta_i = x_i |x_H|^2 + x_{i+d} x_{2d+1} \ ,$$

$$\eta_{i+d} = x_i |x_H|^2 - x_i x_{2d+1}$$

for $i = 1, ..., d$ and $x_H = (x_1, ..., x_{2d})$.

(a) Let $u \in \mathrm{USC}(\mathbb{R}^{2d+1})$ be a viscosity subsolution of (4.8) *such that*

$$\limsup_{|x| \to \infty} \frac{u(x)}{\log \rho(x)} \leq 0 \ .$$

Assume that either $c^\alpha(x) \equiv 0$ or $u \geq 0$, then u is a constant.

(b) Let $v \in \mathrm{LSC}(\mathbb{R}^{2d+1})$ be a viscosity supersolution of (4.10) *such that*

$$\liminf_{|x| \to \infty} \frac{v(x)}{\log \rho(x)} \geq 0 \ .$$

Assume that either $c^\alpha(x) \equiv 0$ or $v \leq 0$, then v is a constant.

Remark 4.19. We highlight that the above result does not hold for the Heisenberg sub-Laplacian, corresponding to the case $b \equiv c \equiv 0$ and $\lambda = \Lambda = 1$. In fact, condition (4.18) is not satisfied because in the Heisenberg group the homogeneous dimension $Q \geq 4$. This confirms the failure of the Liouville property observed at the beginning of the section in the linear case.

Proof. We only have to check property (iii), namely the existence of the Lyapunov function. Set $w(\rho) = \log \rho$ and note that $\lim_{|x|\to\infty} w(\rho(x)) = \infty$ because $\rho \to \infty$ whenever $|x| \to \infty$. Straightforward computations yield

$$X_i\rho = x_i \frac{|D_{\mathbb{H}^d}\rho|^2}{\rho} + \frac{x_{i+d}x_{2d+1}}{\rho^3} = \frac{\eta_i}{\rho^3}\ ,$$

$$X_{i+d}\rho = x_{i+d} \frac{|D_{\mathbb{H}^d}\rho|^2}{\rho} - \frac{x_i x_{2d+1}}{\rho^3} = \frac{\eta_{i+d}}{\rho^3},$$

and $|D_{\mathbb{H}^d}\rho|^2 = |x_H|^2/\rho^2$ (see e.g. [99, Lemma 3.1]). Then, we recall that for a radial function w (with respect to the homogeneous norm ρ) the eigenvalues of $(D^2_{\mathbb{H}^d} w)^*$ are

$$-\frac{|x_H|^2}{\rho^4}\ , 3\frac{|x_H|^2}{\rho^4} \text{ which are simple },$$

and

$$\frac{|x_H|^2}{\rho^4} \text{ with multiplicity } 2d-2$$

(cf. [99, Lemma 3.2]). Hence we are able to compute the Pucci's minimal operator as in [99, Corollary 3.1]

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathbb{H}^d} w)^*) = \{-\Lambda(2d+1) + \lambda\}\frac{|x_H|^2}{\rho^4}\ .$$

Thus, w is a supersolution at all points where

$$\{-\Lambda(2d+1) + \lambda\}\frac{|x_H|^2}{\rho^4} + \inf_{\alpha\in A}\{c^\alpha(x)\log\rho - b^\alpha(x)\cdot \frac{\eta}{\rho^4}\} \geq 0\ ,$$

where η is defined as in the statement. In particular, this inequality holds when ρ is sufficiently large under condition (4.18) by recalling that $Q = 2d + 2$. Similarly one can check that (4.18) implies that the function $W(\rho) = -\log \rho$ is a subsolution to (4.10) for $|x|$ sufficiently large. Therefore Corollary 4.12 and Corollary 4.13 give the conclusion. □

Remark 4.20. Condition (4.18) is comparable to that obtained in [16, eq. (2.17)], but here typical quantities of Carnot groups appear. The ratio

$$\frac{\rho^4}{|x_H|^2} = \frac{|x_H|^4 + |x_V|^2}{|x_H|^2}$$

plays the same role as $|x|^2$ in [16, condition (2.17)], while the dimension d of the Euclidean setting is replaced by its subelliptic counterpart Q, as expected.

Remark 4.21. A simple condition that implies (4.18) and therefore the Liouville property is

$$\limsup_{|x|\to\infty} \sup_{\alpha\in A} \{b^\alpha(x) \cdot \frac{\eta}{|x_H|^2}\} < \lambda - \Lambda(Q-1) \ ,$$

since $c \geq 0$. Compare the above condition to that in [16, Remark 2.4]: Q replaces the dimension d of the Euclidean case and $x \in \mathbb{R}^d$ is replaced by the vector $\eta/|x_H|^2 \in \mathbb{R}^{2d}$.

We thus have the following

Corollary 4.22. *Assume that the operator G in* (4.14)*, where $\mathcal{X} = \{X_1, ..., X_{2d+1}\}$ are the Heisenberg vector fields, satisfies* (4.15) *and* (4.16)*. Assume also that* (4.12)*,* (4.13) *and* (4.18) *are satisfied. Let $u \in \mathrm{USC}(\mathbb{R}^{2d+1})$ be a subsolution of* (4.14) *satisfying* (4.3) *with $w(x) = \log\rho(x)$. Assume either $c^\alpha(x) \equiv 0$ or $u \geq 0$, then u is constant.*

Proof. It is enough to exploit that u is a subsolution to (4.8) over the Heisenberg vector fields and then apply Theorem 4.18-(i). □

Corollary 4.23. *Assume that the operator G in* (4.14)*, where $\mathcal{X} = \{X_1, ..., X_{2d+1}\}$ are the Heisenberg vector fields, satisfies* (4.15) *and* (4.17)*. Assume also that* (4.12)*,* (4.13) *and* (4.18) *are satisfied. Let $v \in \mathrm{LSC}(\mathbb{R}^{2d+1})$ be a supersolution of* (4.14) *satisfying* (4.7) *with $W(x) = -\log\rho(x)$. Assume either $c^\alpha(x) \equiv 0$ or $v \leq 0$, then v is constant.*

Proof. It is enough to exploit that u is a supersolution to (4.10) over Heisenberg vector fields and then apply Theorem 4.18-(ii). □

We specialize the last corollaries to a class of examples in order to compare with those in [16]. Consider again the general PDE (4.14) satisfying the structure condition (4.15) and assume that

$$G(x, r, p, 0) \geq -\bar{b}(x) \cdot p - g(x)|p| + \bar{c}(x)r \ , \tag{4.19}$$

where $\bar{b} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ and $g : \mathbb{R}^{2d+1} \to \mathbb{R}$ is continuous and $g \geq 0$ and $\bar{c} \geq 0$. We have the following

Corollary 4.24. *Assume that the operator G in* (4.14) *satisfies* (4.15)*, and* (4.12)*-*(4.13) *are in force. Moreover, suppose that* (4.19) *holds and*

$$\bar{b}(x) \cdot \frac{\eta}{|x_H|^2} + g(x)\frac{|\eta|}{|x_H|^2} \leq \bar{c}\frac{\rho^4}{|x_H|^2} \log\rho + \lambda - \Lambda(Q-1) \ . \tag{4.20}$$

where η is defined in Theorem 4.18. Let $u \in \mathrm{USC}(\mathbb{R}^{2d+1})$ be a viscosity subsolution to (4.14) *such that*

$$\limsup_{|x|\to\infty} \frac{u(x)}{\log\rho(x)} \leq 0 \ .$$

Assume that either $c^\alpha(x) \equiv 0$ or $u \geq 0$, then u is a constant.

Proof. Observe that $-|D_{\mathbb{H}^d}u| = -|\sigma^T Du| = \min_{|\alpha|=1}\{-\alpha\cdot\sigma^T Du\}$. Hence we can write the right-hand side of the inequality (4.19) as

$$\inf_{\alpha\in A}\{\bar{c}u - (\bar{b}+g\alpha)\cdot\sigma^T Du\}\ ,$$

where $A = \{\alpha\in\mathbb{R}^{2d} : |\alpha| = 1\}$. Moreover, the Heisenberg gradient can be written as $D_{\mathbb{H}^d}u = \frac{1}{\rho^4}\eta$, where η is defined in Corollary 4.18. Then condition (4.18) becomes (4.20) and the conclusion follows by Corollary 4.14. □

Arguing in a similar manner one gets the result for supersolutions using the conclusions of Corollary 4.15.

Corollary 4.25. *Assume that the operator G in* (4.14) *satisfies* (4.15) *and* (4.12)-(4.13) *are in force. If*

$$G(x,r,p,0)\le -\bar{b}(x)\cdot p + g(x)|p| + \bar{c}(x)r \tag{4.21}$$

holds with $\bar{b}$, g and $\bar{c}$ as above. Let $v\in \mathrm{LSC}(\mathbb{R}^{2d+1})$ be a viscosity supersolution to (4.14) *in $\mathbb{R}^{2d+1}$ and assume that*

$$\liminf_{|x|\to\infty}\frac{v(x)}{\log\rho(x)}\ge 0\ . \tag{4.22}$$

If either $v\le 0$ or $c^\alpha(x)\equiv 0$, then v is constant.

Similar results can be achieved for equations driven by the Pucci's extremal operators $\mathcal{P}^{\pm}$ introduced in Example 1.7. Consider thus the following counterpart of (4.8) and (4.10), namely

$$\mathcal{P}^-_\lambda((D^2_{\mathcal{X}}u)^*) + H_i(x,u,D_{\mathcal{X}}u) = 0 \text{ in } \mathbb{R}^{2d+1}\ , \tag{4.23}$$

and

$$\mathcal{P}^+_\lambda((D^2_{\mathcal{X}}u)^*) + H_s(x,u,D_{\mathcal{X}}u) = 0 \text{ in } \mathbb{R}^{2d+1}\ . \tag{4.24}$$

Sufficient conditions can be directly obtained by comparing the extremal operators $\mathcal{P}^{\pm}$ with $\mathcal{M}^{\pm}$. Indeed, straightforward computations give

$$\mathcal{P}^+_\lambda(M)\le\mathcal{M}^+_{\lambda,\lambda+(1-d\lambda)}(M)$$

and

$$\mathcal{P}^-_\lambda(M)\ge\mathcal{M}^-_{\lambda,\lambda+(1-d\lambda)}(M)$$

for every $M\in\mathcal{S}_{2d}$. However, one can exploit representation formulae for $\mathcal{P}^{\pm}$ (see Example (1.7)) to get optimal sufficient conditions. We have the following

Corollary 4.26. *Let $\mathcal{X} = \{X_1,....,X_{2d},X_{2d+1}\}$ be the system of vector fields generating the Heisenberg group $\mathbb{H}^d$. Assume that* (4.12) *and* (4.13) *are in force and*

$$\sup_{\alpha\in A}\{b^\alpha(x)\cdot\frac{\eta}{|x_H|^2} - c^\alpha(x)\frac{\rho^4}{|x_H|^2}\log\rho\}\le -3+4d\lambda \tag{4.25}$$

for ρ sufficiently large, where $Q = 2d+2$ stands for the homogeneous dimension of $\mathbb{H}^d$, $b^\alpha(x)$ takes values in $\mathbb{R}^{2d}$ and $\eta = (\eta_i, \eta_{i+d})$ is defined by

$$\eta_i = x_i|x_H|^2 + x_{i+d}x_{2d+1} \ ,$$

$$\eta_{i+d} = x_i|x_H|^2 - x_i x_{2d+1}$$

for $i = 1, ..., d$ and $x_H = (x_1, ..., x_{2d})$.

(c) Let $u \in \mathrm{USC}(\mathbb{R}^{2d+1})$ be a viscosity subsolution of (4.23) such that

$$\limsup_{|x|\to\infty} \frac{u(x)}{\log \rho(x)} \leq 0 \ .$$

Assume that either $c^\alpha(x) \equiv 0$ or $u \geq 0$, then u is a constant.

(d) Let $v \in \mathrm{LSC}(\mathbb{R}^{2d+1})$ be a viscosity supersolution of (4.24) such that

$$\liminf_{|x|\to\infty} \frac{v(x)}{\log \rho(x)} \geq 0 \ .$$

Assume that either $c^\alpha(x) \equiv 0$ or $v \leq 0$, then v is a constant.

Proof. The proof is exactly the same as Theorem 4.18 using the Lyapunov function $w(\rho) = \log \rho$ and the representation formulas for $\mathcal{P}^+_\lambda(M) = -\lambda \mathrm{Tr}(M) - (1-2d\lambda)e_1$ and $\mathcal{P}^-_\lambda(M) = -\lambda \mathrm{Tr}(M) - (1-2d\lambda)e_{2d}$ for any $M \in \mathcal{S}_{2d}$. Using the expression of the eigenvalues of $(D^2_{\mathbb{H}^d} w)^*$ one finds

$$\mathcal{P}^-_\lambda((D^2_{\mathbb{H}^d} w)^*) = (4d\lambda - 3)\frac{|x_H|^2}{\rho^4} \ .$$

Similarly, one uses $W = -\log \rho$ as Lyapunov function for the maximal operator $\mathcal{P}^+_\lambda$. □

Remark 4.27. We observe that this condition is better than (4.18) with $\Lambda = \lambda + (1-2d\lambda)$ and $\lambda < \frac{1}{2d}$, since

$$-2d\lambda - (1-2d\lambda)(2d+1) < -3 + 4d\lambda \ .$$

4.3.1 Comparison with the literature and sharpness of the conditions

In this section we make a comparison with the results in the literature, showing the sharpness of our conditions and those of [16] via several counterexamples. We first observe that [16, Corollary 2.4] with $b \equiv c \equiv 0$ states a Liouville-type result in the Euclidean case either for viscosity subsolutions bounded from above to $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = 0$ in $\mathbb{R}^d$ or viscosity supersolutions bounded from below to $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = 0$ in $\mathbb{R}^d$ whenever $d \leq \frac{\lambda}{\Lambda} + 1$. This is consistent with the well-known Liouville property for the Laplace equation in the case $d \leq 2$ (i.e. $\lambda = \Lambda$, cf Remark 4.2) and with some counterexamples when $d > \frac{\lambda}{\Lambda} + 1$ that we show next.

Counterexample 4.28. For $d = 2$ and $\Lambda = 2\lambda$ one can verify that the function

$$u_1(x) = \begin{cases} \frac{1}{8}[10 - |x|^2 - |x|^4] & \text{if } |x| < 1 \ , \\ \frac{1}{|x|} & \text{if } |x| \geq 1 \ . \end{cases}$$

is bounded, satisfies $\mathcal{M}^+_{\lambda,2\lambda}(D^2 u_1) \geq 0$ in $\mathbb{R}^2$ and it is not constant. Set $u_1(x) = f_1(|x|)$. When $|x| < 1$ we have $f_1', f_1'' < 0$ and hence the eigenvalues of the Hessian $D^2 u_1$ are negative, which shows that $\mathcal{M}^+_{\lambda,2\lambda}(D^2 u_1) > 0$. When $|x| \geq 1$ we have $f_1'(|x|) = -1/|x|^2$ and $f_1''(|x|) = 2/|x|^3$ and hence $\mathcal{M}^+_{\lambda,2\lambda}(D^2 u_1) = 0$ because $\Lambda = 2\lambda$. Finally, it is a viscosity supersolution of the Pucci's maximal equation simply by observing that the subjet is empty at point where $|x| = 1$. Similarly, the function $v_1 = -u_1$ gives a counterexample for subsolutions to $\mathcal{M}^-_{\lambda,2\lambda}(D^2 v_1) \leq 0$). More in general, when $d \geq 2$, one can prove that when

$$\beta := \frac{\Lambda}{\lambda}(d-1) + 1 > 2 \ ,$$

the function

$$u_2(x) = \begin{cases} \frac{1}{8}[\beta(\beta-2)|x|^4 - 2(\beta^2 - 4)|x|^2 + \beta(\beta+2)] & \text{if } |x| < 1 \ , \\ \frac{1}{|x|^{\beta-2}} & \text{if } |x| \geq 1 \ . \end{cases}$$

is a bounded classical solution to $\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_2) \geq 0$ in $\mathbb{R}^d$ (cf [98, Remark 3.3, eq. (3.19)]); therefore the Liouville property fails to be true whenever $d > \lambda/\Lambda + 1$. Similarly, $v_2 = -u_2$ gives a counterexample for solutions to $\mathcal{M}^-_{\lambda,\Lambda}(D^2 v_2) \leq 0$ in $\mathbb{R}^d$. Since u_2 is a radial function, the eigenvalues of the Hessian matrix can be immediately computed thanks to [98, Lemma 3.1].

Some remarks are in order to compare [16] with the results obtained in [98]. In the latter work, the authors (cf [98, Theorem 3.2]) provided the Liouville property either for viscosity supersolutions to $\mathcal{M}^-_{\lambda,\Lambda}(D^2 u) = 0$ in $\mathbb{R}^d$ or for viscosity subsolutions to $\mathcal{M}^+_{\lambda,\Lambda}(D^2 u) = 0$ in $\mathbb{R}^d$ under the condition $d \leq \frac{\Lambda}{\lambda} + 1$, therefore providing less restrictive conditions for the validity of the Liouville property in terms of the ratio $\Lambda/\lambda > 1$ compared to that of the Laplace equation.

The next example (cf [98, Remark 3.2]) shows that the assumption in [98, Theorem 3.2] is optimal.

Counterexample 4.29. The condition

$$\alpha := \frac{\lambda}{\Lambda}(d-1) + 1 > 2$$

found in [98, Theorem 3.2] in the Euclidean case is sharp. In fact, one can prove that

$$u_3(x) = \begin{cases} -\frac{1}{8}[\alpha(\alpha-2)|x|^4 - 2(\alpha^2 - 4)|x|^2 + \alpha(\alpha+2)] & \text{if } |x| < 1 \ , \\ -\frac{1}{|x|^{\alpha-2}} & \text{if } |x| \geq 1 \ , \end{cases}$$

is a nonconstant classical solution to $\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_3) \leq 0$ in $\mathbb{R}^d$ which is bounded if $\alpha \geq 2$. Similarly, $v_3 = -u_3$ yields a counterexample for the corresponding property for the minimal operator.

Note that, although the condition for Liouville in [98, Theorem 3.2] is less demanding than the one in [16], such Theorem cannot be applied to general uniformly elliptic operators via the inequalities (4.15), as it is done in [16].
We now turn to consider the case of PDEs over the Heisenberg vector fields. Liouville's theorem for classical harmonic functions on the Heisenberg group is a consequence of the Harnack inequality (see [51, Theorem 8.5.1]). However, Liouville's theorem for (classical) subsolutions (supersolutions) bounded from above (below) to

$$-\Delta_{\mathbb{H}^d} u = 0 \text{ in } \mathbb{R}^{2d+1}$$

is false. Indeed, one can check that the function

$$u_4(x) = \begin{cases} \frac{1}{8}[Q(Q-2)\rho^4 - 2(Q^2-4)\rho^2 + Q(Q+2)] & \text{if } \rho \leq 1 \ , \\ \frac{1}{\rho} & \text{if } \rho \geq 1 \ , \end{cases}$$

$Q = 2d + 2$ standing for the homogeneous dimension of $\mathbb{H}^d$, is a bounded classical supersolution to $-\Delta_{\mathbb{H}^d} u = 0$ in $\mathbb{R}^{2d+1}$ and, similarly, $v_4 = -u_4$ gives a bounded subsolution $-\Delta_{\mathbb{H}^d} u = 0$ in $\mathbb{R}^{2d+1}$. It was observed in [181, Lemma 2.2] that the Liouville property for sub- or supersolutions of linear equations on the first Heisenberg group can be recovered by adding first order terms multiplied by a vector field pointing away from infinity.

The counterpart of [98, Theorem 3.2] within the context of the Heisenberg group is found in [99, Theorem 5.2], where the authors proved the Liouville property either for viscosity supersolutions bounded from below to $\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathbb{H}^d} u)^*) = 0$ in $\mathbb{R}^{2d+1}$ or for viscosity subsolutions bounded from above to $\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathbb{H}^d} u)^*) = 0$ in $\mathbb{R}^{2d+1}$ provided that $Q \leq \frac{\Lambda}{\lambda} + 1$, where Q stands for the homogeneous dimension of the Heisenberg group. Also in this case their result cannot be applied to infer Liouville results for general uniformly subelliptic operators, as it happens in our case. Here, we would like to show via a counterexample that the condition $Q \leq \frac{\Lambda}{\lambda} + 1$ found in [99, Theorem 5.2] is sharp. The proof of the optimality of the condition seems to be omitted in [99] and is new to our knowledge.

Counterexample 4.30. Set $\tilde{\alpha} := \frac{\lambda}{\Lambda}(Q-1) + 1$. One can prove that for $\tilde{\alpha} > 2$

$$u_5(x) = \begin{cases} -\frac{1}{8}[\tilde{\alpha}(\tilde{\alpha}-2)\rho^4 - 2(\tilde{\alpha}^2-4)\rho^2 + \tilde{\alpha}(\tilde{\alpha}+2)] & \text{if } \rho < 1 \ , \\ -\frac{1}{\rho^{\tilde{\alpha}-2}} & \text{if } \rho \geq 1 \ , \end{cases}$$

is a bounded from above classical solution to $\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathbb{H}^d} u_5)^*) \leq 0$ in $\mathbb{R}^{2d+1}$ and it is not constant. Indeed, denote by $u_5(x) = f_5(\rho)$. For $\rho < 1$ we have

$$f_5'(\rho) = -\frac{\tilde{\alpha}-2}{2}\rho[\tilde{\alpha}\rho^2 - (\tilde{\alpha}+2)] \ ,$$

and

$$f_5''(\rho) = -\frac{\tilde{\alpha}-2}{2}[3\rho^2\tilde{\alpha} - (\tilde{\alpha}+2)]$$

Recalling that $|D_{\mathbb{H}^d}\rho|^2 = |x_H|^2/\rho^2$, the eigenvalues of the radial function $f_5(\rho)$ are (cf [99, Lemma 3.2])

$$e_1 = |D_{\mathbb{H}^d}\rho|^2 f_4''(\rho) = -\frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2[3\rho^2\tilde\alpha - (\tilde\alpha+2)]$$

$$e_2 = 3|D_{\mathbb{H}^d}\rho|^2 \frac{f_4'(\rho)}{\rho} = -3\frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2[\tilde\alpha\rho^2 - (\tilde\alpha+2)]$$

which are both simple, and

$$e_3 = |D_{\mathbb{H}^d}\rho|^2 \frac{f_4'(\rho)}{\rho} = -\frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2[\tilde\alpha\rho^2 - (\tilde\alpha+2)]$$

which has multiplicity $2d-2$. Thus we observe that when $\rho < 1$ and $\tilde\alpha > 2$, the eigenvalues e_2, e_3 are always positive. Therefore, when $\rho^2 \le \frac{\tilde\alpha+2}{3\tilde\alpha} < 1$, even e_1 is positive and hence $\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathbb{H}^d}u_5)^*) \le 0$. When $1 > \rho^2 > \frac{\tilde\alpha+2}{3\tilde\alpha}$, $e_1 < 0$, and hence

$$\begin{aligned}
\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathbb{H}^d}u_4)^*) &= \Lambda\frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2[3\rho^2\tilde\alpha - (\tilde\alpha+2)] \\
&\quad + \lambda\left\{\frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2[\tilde\alpha\rho^2 - (\tilde\alpha+2)](2d-2) + 3\frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2[\tilde\alpha\rho^2 - (\tilde\alpha+2)]\right\} \\
&= \frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2\left\{\lambda[\tilde\alpha\rho^2 - (\tilde\alpha+2)](2d-2) + 3\tilde\alpha\rho^2 - 3(\tilde\alpha+2)] + \Lambda[3\rho^2\tilde\alpha - (\tilde\alpha+2)]\right\} \\
&= \frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2\left\{\tilde\alpha\rho^2[(2d+1)\lambda + 3\Lambda] - \lambda(2d+1)(\tilde\alpha+2) - \Lambda(\tilde\alpha+2)\right\} \\
&= \frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2\left\{\tilde\alpha\rho^2[(Q-1)\lambda + 3\Lambda] - [\lambda(Q-1) + \Lambda](\tilde\alpha+2)\right\} \\
&= \frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2\left\{[\lambda(Q-1) + \Lambda](-\tilde\alpha - 2 + \tilde\alpha\rho^2) + 2\Lambda\tilde\alpha\rho^2\right\} \\
&\le \frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2\left\{-2[\lambda(Q-1) + \Lambda] + 2\Lambda\tilde\alpha\right\} \\
&= \frac{\tilde\alpha - 2}{2\rho^2}|x_H|^2\left\{-2\lambda(Q-1) + 2\Lambda(\tilde\alpha - 1)\right\} = 0\ ,
\end{aligned}$$

where the last equality is true in view of $\tilde\alpha - 1 = \frac{\lambda}{\Lambda}(Q-1)$. When $\rho > 1$ we have

$$f_5'(\rho) = -(2-\tilde\alpha)\rho^{1-\tilde\alpha}$$

$$f_5''(\rho) = -(2-\tilde\alpha)(1-\tilde\alpha)\rho^{-\tilde\alpha}$$

and the eigenvalues are

$$e_4 = |D_{\mathbb{H}^d}\rho|^2 f_4''(\rho) = -\frac{|x_H|^2(2-\tilde\alpha)(1-\tilde\alpha)}{\rho^{\tilde\alpha+2}}$$

$$e_5 = 3|D_{\mathbb{H}^d}\rho|^2 \frac{f_4'(\rho)}{\rho} = -3\frac{|x_H|^2(2-\tilde\alpha)}{\rho^{\tilde\alpha+2}}$$

and

$$e_6 = |D_{\mathbb{H}^d}\rho|^2 \frac{f_4'(\rho)}{\rho} = -\frac{|x_H|^2(2-\tilde{\alpha})}{\rho^{\tilde{\alpha}+2}}$$

with multiplicity $2d-2$. Therefore, for $\rho \geq 1$, we have

$$\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathbb{H}^d}u_4)^*) = \frac{|x_H|^2(2-\tilde{\alpha})}{\rho^{\tilde{\alpha}+2}}[\Lambda(1-\tilde{\alpha}) + \lambda(Q-1)] = 0 \ ,$$

Similarly, $v_5 = -u_5$ yields a counterexample for the corresponding property of the miminal operator.

Furthermore, we emphasize that the sufficient condition obtained in Theorem 4.18 is consistent with the behavior observed in the case of the Heisenberg Laplacian (i.e. $b \equiv c \equiv 0$ and $\lambda = \Lambda$), for which the Liouville property for subharmonic (superharmonic) functions bounded from above (below) functions is false (see the function u_4 above). In particular, (4.18) with $b \equiv c \equiv 0$ suggests that one-side Liouville properties for viscosity subsolutions (supersolutions) to the minimal (maximal) Pucci's equations do not hold. Set

$$\tilde{\beta} := \frac{\Lambda}{\lambda}(Q-1) + 1 > 2 \text{ , namely } Q > \frac{\lambda}{\Lambda} + 1 \ .$$

The next counterexample shows the Liouville property for supersolutions bounded from below to $\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathbb{H}^d}u)^*) = 0$ and subsolutions bounded from above to the minimal Pucci's equation $\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathbb{H}^d}u)^*) = 0$ fails (recall that for the Heisenberg group $Q \geq 4$).

Counterexample 4.31. In the same way as in Counterexample 4.30, one can verify that the function

$$u_6(x) = \begin{cases} \frac{1}{8}[\tilde{\beta}(\tilde{\beta}-2)\rho^4 - 2(\tilde{\beta}^2-4)\rho^2 + \tilde{\beta}(\tilde{\beta}+2)] & \text{if } \rho < 1 \ , \\ \frac{1}{\rho^{\tilde{\beta}-2}} & \text{if } \rho \geq 1 \ . \end{cases}$$

is a bounded classical supersolution to $\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathbb{H}^d}u_6)^*) = 0$, which is not constant.

Similarly, $v_6 = -u_6$ gives the counterexample for subsolutions to the minimal operator $\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathbb{H}^d}v_6)^*) = 0$.

Therefore, we conclude that the presence of the gradient terms in Theorem 4.18 are fundamental to prove the Liouville property. The same phenomena occurs for the Liouville properties in the Euclidean case for semilinear equations with fully nonlinear second order terms, where the presence of the semilinear part plays a crucial role (see e.g. [98, Theorem 4.1]).

4.4 Example 2: Free step-2 Carnot groups

This section is devoted to collect some Liouville results for nonlinear PDEs modeled on free step 2 Carnot groups with r generators introduced in Subsection 1.0.3. Here we denote the horizontal gradient and Hessian as $D_{\mathbb{G}_r}$ and $D^2_{\mathbb{G}_r}$ respectively. The homogeneous norm we are going to use here to build the Lyapunov function has the same form of the one used for the Heisenberg group, namely

$$\rho(x) = \left(|x_H|^4 + |x_V|^2\right)^{\frac{1}{4}} = \left((x_1^2 + ... + x_r^2)^2 + t_{2,1}^2 + ... + t_{r,r-1}^2\right)^{\frac{1}{4}} \ .$$

Theorem 4.32. *Assume that* (4.12) *and* (4.13) *hold true and*

$$\sup_{\alpha\in A}\{b^\alpha(x)\cdot\frac{\bar\eta}{|x_H|^2}-c^\alpha(x)\frac{\rho^4}{|x_H|^2}\log\rho\}\le 4\lambda\frac{\rho^2}{|x_H|^2}|D_{\mathbb{G}_r}\rho|^2-3r\Lambda\ , \tag{4.26}$$

for ρ large enough and $\bar\eta=\rho^3 D_{\mathbb{G}_r}\rho$.

(a) Let $u\in \mathrm{USC}(\mathbb{R}^d)$ be a viscosity subsolution of (4.8) *where the vector fields $X_1,...,X_m$ are the generators of a free step-2 Carnot group defined in Section 1.0.3 such that*

$$\limsup_{\rho(x)\to\infty}\frac{u(x)}{\log\rho(x)}\le 0\ .$$

Assume that either $c^\alpha(x)\equiv 0$ or $u\ge 0$, then u is a constant.

(b) Let $v\in \mathrm{LSC}(\mathbb{R}^d)$ be a viscosity supersolution of (4.10) *where the vector fields $X_1,...,X_m$ are the generators of a free step-2 Carnot group defined in Section 1.0.3 such that*

$$\liminf_{\rho(x)\to\infty}\frac{v(x)}{\log\rho(x)}\ge 0\ .$$

Assume that either $c^\alpha(x)\equiv 0$ or $v\le 0$, then v is a constant.

Proof. We have to check property (iii), namely the existence of the Lyapunov function $w:=\log\rho$. By the chain rule applied to $f(\rho)=\rho$, we first need to compute the horizontal Hessian of the homogeneous norm ρ. We have

$$X_k\rho=\frac{1}{\rho^3}\left[x_k\sum_{j=1}^r x_j^2+\left(\sum_{j>k}x_jt_{jk}-\sum_{j<k}x_jt_{kj}\right)\right]\ .$$

Thus, we obtain

$$|D_{\mathbb{G}_r}\rho|^2=\frac{|x_H|^6}{\rho^6}+\frac{1}{\rho^6}\sum_k\left(\sum_{j>k}x_jt_{jk}-\sum_{j<k}x_jt_{kj}\right)^2$$

Moreover, we can compute

$$X_k(X_k\rho)=\frac{1}{\rho^3}\left[\sum_{j=1}^r x_j^2+2x_k^2+2\sum_{j=1,j\neq k}^r x_j^2\right]-\frac{3}{\rho}X_k\rho X_k\rho=\frac{3|x_H|^2}{\rho^3}I_r-\frac{3}{\rho}X_k\rho X_k\rho$$

and

$$X_i(X_k\rho)=\frac{1}{\rho^3}\left[2x_ix_k-t_{ki}-2x_ix_k\right]-\frac{3}{\rho}X_i\rho X_k\rho\text{ for }i<k;$$

$$X_i(X_k\rho)=\frac{1}{\rho^3}\left[2x_ix_k+t_{ik}-2x_ix_k\right]-\frac{3}{\rho}X_i\rho X_k\rho\text{ for }i>k\ .$$

Therefore, the horizontal hessian $D^2_{\mathbb{G}_r}\rho\in\mathbb{R}^{r\times r}$ is given by

$$D^2_{\mathbb{G}_r}\rho=\frac{1}{\rho^3}\left[T+3|x_H|^2I_r\right]-\frac{3}{\rho}D_{\mathbb{G}_r}\rho\otimes D_{\mathbb{G}_r}\rho\ ,$$

where $x_H = (x_1, ..., x_r)$ and T is the skew-symmetric matrix

$$T := \begin{pmatrix} 0 & -t_{21} & \dots & -t_{r1} \\ t_{21} & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ t_{r1} & \dots & \dots & 0 \end{pmatrix}$$

We then compute

$$D^2_{\mathbb{G}_r} \log(\rho) = \frac{1}{\rho^4} \left[T + 3|x_H|^2 I_r\right] - \frac{4}{\rho^2} D_{\mathbb{G}_r}\rho \otimes D_{\mathbb{G}_r}\rho \ .$$

Therefore the symmetrized matrix is given by

$$(D^2_{\mathbb{G}_r} \log(\rho))^* = \frac{3|x_H|^2}{\rho^4} I_r - \frac{4}{\rho^2} D_{\mathbb{G}_r}\rho \otimes D_{\mathbb{G}_r}\rho := N + M$$

Note that the eigenvalues of M are $-\frac{4}{\rho^2}|D_{\mathbb{G}_r}\rho|^2$, which is simple, and 0 with multiplicity $r-1$, while the eigenvalue of N is $3|x_H|^2/\rho^4$ with multiplicity r. We thus compute

$$\begin{aligned} \mathcal{M}^-_{\lambda,\Lambda}(D^2_{\mathbb{G}_r} w) + H_i(x, w, D_{\mathbb{G}_r} w) &\geq \mathcal{M}^-_{\lambda,\Lambda}\left(-\frac{4}{\rho^2} D_{\mathbb{G}_r}\rho \otimes D_{\mathbb{G}_r}\rho\right) + \mathcal{M}^-_{\lambda,\Lambda}\left(\frac{1}{\rho^4}(3|x_H|^2 I_r)\right) \\ &+ \inf_{\alpha\in A}\left\{c^\alpha(x)\log\rho - b^\alpha(x)\cdot\frac{D_{\mathbb{G}_r}\rho}{\rho}\right\} = \frac{4\lambda}{\rho^2}|D_{\mathbb{G}_r}\rho|^2 - \Lambda\frac{3r}{\rho^4}|x_H|^2 \\ &+ \inf_{\alpha\in A}\left\{c^\alpha(x)\log\rho - b^\alpha(x)\cdot\frac{D_{\mathbb{G}_r}\rho}{\rho}\right\} \ . \end{aligned}$$

Hence w is a supersolution of (4.8) if

$$\frac{4\lambda}{\rho^2}|D_{\mathbb{G}_r}\rho|^2 - \Lambda\frac{3r}{\rho^4}|x_H|^2 + \inf_{\alpha\in A}\left\{c^\alpha(x)\log\rho - b^\alpha(x)\cdot\frac{D_{\mathbb{G}_r}\rho}{\rho}\right\} \geq 0$$

Then, one can see that this inequality holds when $|x|$ is sufficiently large under condition (4.26).
Similar computations holds for $W = -\log\rho$ thanks to the superadditivity inequalities of the maximal operator, and noting that

$$(D^2_{\mathbb{G}_r} W)^* = -\frac{1}{\rho^4}\left[3|x_H|^2 I_r\right] + \frac{4}{\rho^2} D_{\mathbb{G}_r}\rho \otimes D_{\mathbb{G}_r}\rho$$

Finally, Corollary 4.12 and Corollary 4.13 give the conclusion. □

Remark 4.33. Note that (4.32) is satisfied for instance when either b is bounded and $c^\alpha(x) > 0$ or $b^\alpha(x) \cdot D_{\mathbb{G}_r}\rho < 0$ and $c^\alpha(x) \geq 0$.

Remark 4.34. Recall that in Remark 1.19 we pointed out that the Heisenberg group $\mathbb{H}^d$ is a free step two Carnot groups if and only if $d = 1$ (and $r = 2$). We compare the above condition with the one obtained in the previous section. Since $\bar{\eta} = \rho^3 D_{\mathbb{G}_r}\rho$ (4.26) reads

$$\sup_{\alpha\in A}\{b^\alpha(x)\cdot\bar{\eta} - c^\alpha(x)\rho^4\log\rho\} \le 4\lambda\rho^2|D_{\mathbb{G}_2}\rho|^2 - 6\Lambda|x_H|^2$$

As pointed out in [99, Lemma 3.1], one immediately sees that $|D_{\mathbb{G}_2}\rho|^2 = \frac{|x_H|^2}{\rho^2}$ and hence (4.26) becomes

$$\sup_{\alpha\in A}\{b^\alpha(x)\cdot\bar{\eta} - c^\alpha(x)\rho^4\log\rho\} \le (4\lambda - 6\Lambda)\,|x_H|^2\ ,$$

and the sufficient condition (4.18) was

$$\sup_{\alpha\in A}\{b^\alpha(x)\cdot\bar{\eta} - c^\alpha(x)\rho^4\log\rho\} \le (\lambda - 3\Lambda)|x_H|^2\ .$$

We observe that $(4\lambda - 6\Lambda) \le (\lambda - 3\Lambda)$ with strict inequality when the constants λ, Λ are such that $0 < \lambda < \Lambda$. As expected, condition (4.26) is more restrictive than (4.18) due to the fact that in Corollary 4.32 sub- and superadditivity inequalities of the extremal operators are used.

We remark that the main trouble in computing a sufficient condition in Corollary 4.32 relies on determining the sign of the eigenvalues of the horizontal Hessian of the radial Lyapunov function. However, within these geometries the extremal operators $\mathcal{P}^\pm$ behave better than $\mathcal{M}^\pm$, since formulae (1.6) and (1.7) requires to know the trace and an extremal eigenvalue of the Hessian only, without necessarily knowing their sign. We have the following

Corollary 4.35. *Let $\mathcal{X} = \{X_1, ..., X_m\}$ be the system of vector fields generating $\mathbb{G}_r$. Assume that* (4.12) *and* (4.13) *are in force and*

$$\sup_{\alpha\in A}\{b^\alpha(x)\cdot\bar{\eta} - c^\alpha(x)\rho^4\log\rho\} \le 3|x_H|^2 - 4\lambda(r-1)\rho^2|D_{\mathbb{G}_r}\rho|^2\ , \tag{4.27}$$

for ρ sufficiently large, $b^\alpha(x)$ takes values in $\mathbb{R}^{2d}$ and $\bar{\eta} = \rho^3 D_{\mathbb{G}_r}\rho$.

(c) Let $u \in \mathrm{USC}(\mathbb{R}^d)$ be a viscosity subsolution of (4.23) *such that*

$$\limsup_{|x|\to\infty}\frac{u(x)}{\log\rho(x)} \le 0\ .$$

Assume that either $c^\alpha(x) \equiv 0$ or $u \ge 0$, then u is a constant.

(d) Let $v \in \mathrm{LSC}(\mathbb{R}^d)$ be a viscosity supersolution of (4.24) *such that*

$$\liminf_{|x|\to\infty}\frac{v(x)}{\log\rho(x)} \ge 0\ .$$

Assume that either $c^\alpha(x) \equiv 0$ or $v \le 0$, then v is a constant.

Remark 4.36. Wee emphasize that when $r = 2$ and $d = 1$ (i.e. $\mathbb{G}_r \simeq \mathbb{H}^d$) we find the same condition as in Corollary 4.26, as expected.

4.5 Example 3: the Grushin plane

We now provide sufficient conditions for the validity of Liouville-type results in the Grushin plane for equations (4.8) and (4.10) and hence, as a byproduct, for general fully nonlinear equation (4.1) satisfying (4.15) (that we will not write for the sake of brevity, being the same to Corollary 4.22 and Corollary 4.23). Recall that the Grushin plane (cf Section 1.0.4) is the sub-Riemannian geometry induced on $\mathbb{R}^2$ by the two-dimensiuonal vector fields

$$X = \partial_x \quad ; Y = x\partial_y$$

for $p = (x, y) \in \mathbb{R}^2$. Here we use the following homogeneous norm

$$\rho(x, y) = (x^4 + 4y^2)^{\frac{1}{4}} \ .$$

We first underline that the Liouville property for the (classical) subsolutions (supersolutions) bounded from above (below) of the Grushin sub-Laplacian does not hold. Indeed, the function

$$\bar{u}(x) = \begin{cases} \frac{1}{8}[15 - 10\rho^2 + 3\rho^4] & \text{if } \rho \leq 1 \ , \\ \frac{1}{\rho} & \text{if } \rho \geq 1 \ . \end{cases}$$

is a nonconstant classical supersolution bounded to $-\Delta_{\mathcal{X}} u = -\partial_{xx} u - x^2 \partial_{yy} u = 0$ in $\mathbb{R}^2$. Similarly, $v = -u$ shows the failure of the Liouville property for subsolutions. This example underlines that as soon as the (classical) ellipticity is not in force, then the Liouville property fails even in the case $d = 2$. We recall that one-side Liouville results for sub- and supersolutions to the Laplace equation are true in $\mathbb{R}^2$, see Remark 4.2.

We denote as usual by $D_{\mathcal{X}} u$ and $(D^2_{\mathcal{X}} u)^*$ the horizontal gradient and symmetrized horizontal Hessian over Grushin vector fields respectively. The same example shows also that one-side Liouville properties do not hold even for sub- and supersolutions to Pucci's extremal equations. In fact, for $0 < \lambda < \Lambda$, in view of the inequalities

$$\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathcal{X}} u)^*) = -\lambda \sum_{e_i > 0} e_i - \Lambda \sum_{e_i < 0} e_i \geq -\Lambda \Delta_{\mathcal{X}} u$$

and

$$-\lambda \Delta_{\mathcal{X}} u \geq -\Lambda \sum_{e_i > 0} -\lambda \sum_{e_i < 0} e_i = \mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}} u)^*) \ ,$$

we can conclude that $\bar{u}$ is a nontrivial bounded classical supersolution to the maximal Pucci's equation $\mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathcal{X}} u)^*) = 0$ and, similarly, $\bar{v} = -\bar{u}$ is a bounded classical subsolution to $\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}} u)^*) = 0$, providing a counterexample for the Liouville property even for the extremal operators over the Grushin horizontal Hessian on $\mathbb{R}^2$.

Corollary 4.37. *Let $\mathcal{X} = \{X, Y\}$ be the system of vector fields generating the Grushin plane. Assume also that* (4.12) *and* (4.13) *are in force and*

$$2 \sup_{\alpha \in A} \{ b^\alpha(x) \cdot \tilde{\eta} - c^\alpha(x) \rho^4 \log \rho \} \leq (-\Lambda - \lambda) x^2 + (-\Lambda + \lambda) \sqrt{9x^4 + 4y^2} \ , \tag{4.28}$$

for $|x|, |y|$ sufficiently large, where $\tilde{\eta} := (x^3, 2xy) \in \mathbb{R}^2$.

(a) Let $u \in \mathrm{USC}(\mathbb{R}^2)$ be a viscosity subsolution of (4.8) *such that*

$$\limsup_{|(x,y)|\to\infty} \frac{u(x)}{\log \rho(x)} \leq 0 \ .$$

Assume that either $c^\alpha(x) \equiv 0$ or $u \geq 0$, then u is a constant.

(b) Let $v \in \mathrm{LSC}(\mathbb{R}^2)$ be a viscosity supersolution of (4.10) *such that*

$$\liminf_{|(x,y)|\to\infty} \frac{v(x)}{\log \rho(x)} \geq 0 \ .$$

Assume that either $c^\alpha(x) \equiv 0$ or $v \leq 0$, then v is a constant.

Proof. Similarly to Corollary 4.18 and Corollary 4.32, we compute the symmetrized horizontal Hessian of the Lyapunov function $w = \log \rho$ having in mind (4.8) over the Grushin vector fields defined above. We first note that $w(\rho(x))$ explodes as $|x| \to \infty$. We have

$$X\rho = \frac{x^3}{\rho^3} = \frac{\tilde{\eta}_1}{\rho^3} \ , Y\rho = \frac{2xy}{\rho^3} = \frac{\tilde{\eta}_2}{\rho^3} \ , D_{\mathcal{X}}\rho = \frac{1}{\rho^3}\left(x^3, 2xy\right)$$

which also gives

$$|D_{\mathcal{X}}\rho|^2 = \frac{x^2}{\rho^2} \ .$$

Moreover, the entries of the intrinsic Hessian $D^2_{\mathcal{X}}\rho$ are given by

$$X(X\rho) = \frac{3x^2}{\rho^3} - \frac{3}{\rho}X\rho X\rho \ ,$$

$$X(Y\rho) = \frac{2y}{\rho^3} - \frac{3}{\rho}X\rho Y\rho \ ,$$

$$Y(X\rho) = -\frac{3}{\rho}Y\rho X\rho \ ,$$

$$Y(Y\rho) = \frac{2x^2}{\rho^3} - \frac{3}{\rho}Y\rho Y\rho \ .$$

Therefore, the matrix $D^2_{\mathcal{X}}\rho$ can be written as

$$D^2_{\mathcal{X}}\rho = \frac{1}{\rho^3}\begin{pmatrix} 3x^2 & 2y \\ 0 & 2x^2 \end{pmatrix} - \frac{3}{\rho}D_{\mathcal{X}}\rho \otimes D_{\mathcal{X}}\rho \ .$$

Setting $w := \log \rho$, by the chain rule we have

$$D^2_{\mathcal{X}}w = \frac{1}{\rho^4}\begin{pmatrix} 3x^2 & 2y \\ 0 & 2x^2 \end{pmatrix} - \frac{4}{\rho^2}D_{\mathcal{X}}\rho \otimes D_{\mathcal{X}}\rho \ .$$

Then, the symmetrized horizontal Hessian takes the form

$$(D^2_{\mathcal{X}}w)^* = \frac{1}{\rho^4}\begin{pmatrix} 3x^2 & y \\ y & 2x^2 \end{pmatrix} - \frac{4}{\rho^2}D_{\mathcal{X}}\rho \otimes D_{\mathcal{X}}\rho \ .$$

We claim that the eigenvalues are

$$\left\{ \frac{x^2+\sqrt{9x^4+4y^2}}{2\rho^4}, \frac{x^2-\sqrt{9x^4+4y^2}}{2\rho^4} \right\} . \tag{4.29}$$

Indeed the (symmetrized) horizontal Hessian is given by

$$(D^2_{\mathcal{X}} w)^* = \begin{pmatrix} \frac{3x^2}{\rho^4} - \frac{4x^6}{\rho^8} & \frac{y}{\rho^4} - \frac{8x^4 y}{\rho^8} \\ \frac{y}{\rho^4} - \frac{8x^4 y}{\rho^8} & \frac{2x^2}{\rho^4} - \frac{16x^2 y^2}{\rho^8} \end{pmatrix} .$$

Then one computes

$$\mathrm{Tr}((D^2_{\mathcal{X}} w)^*) = \frac{5x^2}{\rho^4} - \frac{4x^2}{\rho^8}(x^4+4y^2) = \frac{x^2}{\rho^4} ,$$

and, by recalling the expression of ρ, we also get

$$\det((D^2_{\mathcal{X}} w)^*) = \frac{6x^4}{\rho^8} - \frac{48x^4y^2}{\rho^{12}} - \frac{8x^8}{\rho^{12}} + \frac{64x^8y^2}{\rho^{16}} - \frac{y^2}{\rho^8} - \frac{64x^8y^2}{\rho^{16}} + \frac{16x^4y^2}{\rho^{12}} =$$

$$= \frac{(6x^4-y^2)}{\rho^8} - \frac{32x^4y^2}{\rho^{12}} - \frac{8x^8}{\rho^{12}} .$$

Then the eigenvalues are given by the formulae

$$\lambda_1 := \frac{\mathrm{Tr}((D^2_{\mathcal{X}} w)^*) - \sqrt{\mathrm{Tr}((D^2_{\mathcal{X}} w)^*)^2 - 4\det((D^2_{\mathcal{X}} w)^*)}}{2}, \tag{4.30}$$

and

$$\lambda_2 := \frac{\mathrm{Tr}((D^2_{\mathcal{X}} w)^*) + \sqrt{\mathrm{Tr}((D^2_{\mathcal{X}} w)^*)^2 - 4\det((D^2_{\mathcal{X}} w)^*)}}{2} .$$

Note that

$$\sqrt{\mathrm{Tr}((D^2_{\mathcal{X}} w)^*)^2 - 4\det((D^2_{\mathcal{X}} w)^*)} = \sqrt{\frac{x^4\rho^8}{\rho^{16}} - \frac{4}{\rho^{16}}\left[(6x^4-y^2)\rho^8 - 32x^4y^2\rho^4 - 8x^8\rho^4\right]} =$$

$$= \sqrt{\frac{x^4\rho^8 - 4[(6x^4-y^2)\rho^8 - 8\rho^8 x^4]}{\rho^{16}}} = \frac{1}{\rho^4}\sqrt{9x^4+4y^2} .$$

Then, we get the eigenvalues (4.30). In particular, we immediately observe that λ_1 is positive and λ_2 is negative and this fact allows to compute Pucci's extremal operators over $(D^2_{\mathcal{X}} w)^*$. We have

$$\begin{aligned} &\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}} w)^*) + \inf_{\alpha\in A}\{c^\alpha(x)\log\rho - b^\alpha(x)\cdot\frac{\tilde{\eta}}{\rho^4}\} \\ &= -\Lambda\frac{x^2+\sqrt{9x^4+4y^2}}{2\rho^4} - \lambda\frac{x^2-\sqrt{9x^4+4y^2}}{2\rho^4} + \\ &+ \inf_{\alpha\in A}\{c^\alpha(x)\log\rho - b^\alpha(x)\cdot\frac{\tilde{\eta}}{\rho^4}\} \geq 0 \end{aligned}$$

if condition (4.28) is satisfied. One can obtain the same sufficient condition for equations of the form (4.10) using the Lyapunov function $W(\rho) = -\log\rho$. $\square$

Part II

Fractional Mean Field Games

Chapter 5

Fractional MFGs

In this chapter, we deal with the well-posedness of the evolutive fractional MFG system

$$\begin{cases} -\partial_t u + (-\Delta)^s u + H(x, Du) = F[m(t)](x) & \text{in } Q_T \\ \partial_t m + (-\Delta)^s m - \operatorname{div}(m D_p H(x, Du)) = 0 & \text{in } Q_T \\ m(x,0) = m_0(x),\ u(x,T) = u_T(x) & \text{in } \mathbb{T}^d \ , \end{cases} \tag{5.1}$$

where $Q_T := \mathbb{T}^d \times (0,T)$, $H = H(x, Du)$ is a superlinear Hamiltonian in Du, $(-\Delta)^s u$ is the fractional Laplacian of order $s \in (0,1)$, F is a regularizing coupling and m_0, u_T are given functions. In particular, note that the first equation is a backward-in-time fractional Hamilton-Jacobi equation, while the second equation is a forward fractional Fokker-Planck equation.

As announced, the main contribution of this part of the manuscript, other than providing the well-posedness to the aforementioned system, is to provide the functional setting to handle (nonlinear) nonlocal problems in the L^p setting as perturbation of fractional nonlocal equations, which will be discussed in the next Section 5.3. Then, we will analyze separately both equations in the subsequent sections and, finally, we will prove our main results that we state in detail below for reader convenience.

5.1 Assumptions and main results

We suppose that H is $C^2(\mathbb{T}^d \times \mathbb{R}^d)$, $H(x,p) \geq H(x,0) = 0$, convex in p, and there exist constants $\gamma > 1$ and $c_H, C_H, \tilde{C}_H > 0$ such that

$$D_p H(x,p) \cdot p - H(x,p) \geq C_H |p|^\gamma - c_H \ , \tag{H1F}$$

$$|D_p H(x,p)| \leq C_H |p|^{\gamma-1} + \tilde{C}_H \tag{H2F}$$

$$|D_{xx} H(x,p)| \leq C_H |p|^\gamma + \tilde{C}_H \ , \tag{H3F}$$

$$|D^2_{px} H(x,p)| \leq C_H |p|^{\gamma-1} + \tilde{C}_H \ , \tag{H4F}$$

$$D^2_{pp} H(x,p)\xi \cdot \xi \geq C_H |p|^{\gamma-2} |\xi|^2 - \tilde{C}_H \tag{H5F}$$

for every $x \in \mathbb{T}^d$, $p \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$. We provide some examples of Hamiltonians fulfilling the above listed assumptions in Section 5.2 below. The following are the standing assumptions on the regularizing coupling F: there exist a constant $C_F > 0$

such that

$$F : \mathcal{P}(\mathbb{T}^d) \to C^{2+\alpha}(\mathbb{T}^d) \text{ is continuous,} \tag{F1}$$
$$\|F[m_1] - F[m_2]\|_\infty \leq C_F \mathbf{d}_1(m_1, m_2) \ \forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^d), \tag{F2}$$
$$\|F(\cdot, m)\|_{C^{2+\alpha}(\mathbb{T}^d)} \leq C_F \text{ for every } m \in \mathcal{P}(\mathbb{T}^d). \tag{F3}$$

Finally, we suppose that

$$u_T \in C^{4+\alpha}(\mathbb{T}^d) \text{ and } \|u_T\|_{C^{4+\alpha}(\mathbb{T}^d)} \leq C' \tag{I1}$$
$$m_0 \in C^{4+\alpha}(\mathbb{T}^d) \text{ with } \|m_0\|_{C^{4+\alpha}(\mathbb{T}^d)} \leq C'' \text{ and non-negative, } \int_{\mathbb{T}^d} m_0(x)dx = 1. \tag{I2}$$

We denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$ endowed with the Monge-Kantorovich distance $\mathbf{d}_1$, defined as

$$\mathbf{d}_1(\mu, \nu) := \sup_{\varphi} \int_{\mathbb{T}^d} \varphi d(\mu - \nu) \ ,$$

where the supremum is taken over the 1-Lipschitz maps $\varphi : \mathbb{T}^d \to \mathbb{R}$. Here, we address the existence and uniqueness of solutions to (5.1) through the vanishing viscosity method, namely solutions of (5.1) are obtained as limits (in some sense to be specified below) of solutions u_σ of the approximating viscous coupled system of PDEs

$$\begin{cases} -\partial_t u - \sigma\Delta u + (-\Delta)^s u + H(x, Du) = F[m(t)](x) & \text{in } Q_T \\ \partial_t m - \sigma\Delta m + (-\Delta)^s m - \operatorname{div}(m D_p H(x, Du)) = 0 & \text{in } Q_T \\ m(x, 0) = m_0(x), \ u(x, T) = u_T(x) & \text{in } \mathbb{T}^d \ . \end{cases} \tag{5.2}$$

Theorem 5.1. *Let* (I1)-(I2), (H1F)-(H5F) *and* (F1)-(F3) *be in force. Then, for all* $\sigma > 0$ *and* $s \in (0, 1)$, *there exists a classical solution* $(u_\sigma, m_\sigma) \in C^{4+\alpha,2+\alpha/2}(Q_T) \times C^{4+\alpha,2+\alpha/2}(Q_T)$ *to the fractional MFG system* (5.2).

The proof of this result is a rather standard application of Schauder's fixed point theorem. For fixed $\sigma > 0$, we treat $(-\Delta)^s u$, $(-\Delta)^s m$ as perturbation terms in a viscous MFG system. Semiconcavity estimates for the HJB equation with mixed local and nonlocal diffusion term are obtained by means of the adjoint method, that ensure existence of u. Note that these estimates are stable as $\sigma \to 0$. This limiting procedure is then described by the next main result:

Theorem 5.2. *Let the assumptions of Theorem 5.1 be satisfied. Let* (u_σ, m_σ) *be a solution to* (5.2). *Then, as* $\sigma \to 0$ *and up to subsequences,* u_σ *converges uniformly to* u, Du_σ *converges strongly to* Du, *and* m_σ *converges weakly to* m. *In particular*

- *If* $s \in (0, 1/2]$, *then* (u, m) *is a weak energy solution to* (5.1) *(in the sense of Definition 5.41 and Definition 5.38 respectively);*

- *If* $s \in (1/2, 1)$, *then* $\partial_t u, \partial_t m, (-\Delta)^s u, (-\Delta)^s m$ *belong to some* $\mathcal{C}^{\bar\alpha, \frac{\bar\alpha}{2s}}(Q_T)$, $\bar\alpha \in (0, 1)$, *and* (u, m) *is a classical solution to* (5.1) .

Our uniqueness theorem can be states as follows. For its proof, see Theorems 5.52, 5.54.

Theorem 5.3. *Suppose that* (I1)-(I2), (H1F)-(H5F) *and* (F1)-(F3) *hold. Then* (5.1) *admits a unique solution in the following cases:*

(a) The monotone case. If H is convex and the following monotonicity condition holds

$$\int_{\mathbb{T}^d}(F[m_1](x)-F[m_2](x))d(m_1-m_2)(x)>0\ ,\forall m_1,m_2\in\mathcal{P}(\mathbb{T}^d)\ ,m_1\neq m_2\ ,$$

then (5.1) *admits a unique solution.*

(b) Small-time uniqueness. For $s\in(\frac{1}{2},1)$ and $H\in C^3(\mathbb{T}^d\times\mathbb{R}^d)$, there exists $T^*>0$, depending on d,s,H,F,m_0,u_T such that for all $T\in(0,T^*]$, (5.1) *has at most a solution.*

The chapter is organized as follows: Section 5.3 is devoted to some preliminary tools on the functional spaces used in the following sections. We prove the Sobolev embedding theorem for parabolic spaces in Subsection 5.3.3. Section 5.5 is completely designed to the separate analysis of the viscous fractional Fokker-Planck and HJB equations. In particular, the existence result for the latter is given in Subsection 5.5.2. In Section 5.6 we prove both Theorem 5.1 and Theorem 5.2, postponing the uniqueness to Section 5.7, where Theorem 5.3 is proven. As announced, in the appendices we gather regularity results in Sobolev and Hölder spaces for non-homogeneous fractional heat-type equations together with fractional Leibniz and composition rules on the torus.

5.2 Model Hamiltonians

In this section we list some model Hamiltonians fulfilling (H1F)-(H5F).

Example 5.4. A first example fulfilling the above assumptions is

$$H_1(x,p)=h(x)(1+|p|^2)^{\frac{\gamma}{2}}+b(x)\ ,\gamma>1\ ,$$

where $p\in\mathbb{R}^d$, $h,b\in C^2(\mathbb{T}^d)$ and $h(x)\geq h_0>0$. Here we have for $x\in\mathbb{T}^d$, $p\in\mathbb{R}^d$

$$\begin{aligned}D_pH_1(x,p)\cdot p-H_1(x,p)&=D_pH_1(x,p)\cdot p-\gamma H_1(x,p)+(\gamma-1)H_1(x,p)\\&\geq(\gamma-1)H_1(x,p)-h(x)\gamma\geq h_0(\gamma-1)|p|^\gamma-(\|h\|_\infty\gamma+\|b\|_\infty)\end{aligned}$$

ensuring the validity of (H1F). Moreover, $|D_pH|=h(x)\gamma(1+|p|^2)^{\frac{\gamma}{2}-1}|p|$, so it is immediate to check the validity of (H2F)-(H4F). In addition, we have

$$D^2_{pp}H(x,p)=h(x)\gamma[(\gamma-2)(1+|p|^2)^{\frac{\gamma-4}{2}}p\otimes p+(1+|p|^2)^{\frac{\gamma}{2}-1}I_d]\ .$$

Therefore, H is strictly convex since for every $\xi\in\mathbb{R}^d$ it holds

$$D^2_{pp}H(x,p)\xi\cdot\xi=h(x)\gamma(1+|p|^2)^{\frac{\gamma-4}{2}}|\xi|^2[(\gamma-1)|p|^2+1]\geq h_0\gamma|\xi|^2\ .$$

The same computations shows the validity of (H5F).

Example 5.5. Another example is represented by Hamiltonians behaving like $O(|p|^\gamma)$ for $|p| \to \infty$, namely

$$H_2(x,p) = h(x)|p|^\gamma + b(x) \cdot p \ , \gamma \geq 2 \ ,$$

where $p \in \mathbb{R}^d$, $h, b \in C^2(\mathbb{T}^d)$ and $h(x) \geq h_0 > 0$. Here, simple computations and Young's inequality allows to conclude

$$D_p H_2(x,p) \cdot p - H_2(x,p) \geq |p|^\gamma \left[h(x)(\gamma-1) - \frac{1}{\gamma} \right] + b(x) - \frac{\|b\|_\infty^{\gamma'}}{\gamma'}$$

which gives (H1F) by setting e.g. $C_H := h_0(\gamma-1) - \frac{1}{\gamma}$, $h_0 := \frac{1}{\gamma(\gamma-1)} + \delta$,$\delta > 0$ and $c_H = 2\frac{\|b\|_\infty^{\gamma'}}{\gamma'} - \frac{1}{\gamma}$. It is straightforward to verify (H3F)-(H5F). Moreover, H is convex since it can be written as supremum of linear operators (see e.g. [212]).

Remark 5.6. We remark that if one requires to satisfy $H_3(p) \sim |p|^\gamma$ in the subquadratic case $\gamma < 2$ in Example 5.5, then H fails to be C^2 in a neighborhood of $p = 0$, since $H_3 = H_3(p) \in C^{1,\gamma-1}(\mathbb{R}^d)$.

5.3 Fractional parabolic spaces

5.3.1 Hölder spaces

We first recall the definition of Hölder spaces on the torus and then define the natural parabolic Hölder spaces associated to the heat and fractional heat equation. Let $\alpha \in (0,1]$ and k be a non-negative integer. A real-valued function u defined on $\mathbb{T}^d$ belongs to $C^{k+\alpha}(\mathbb{T}^d)$ if $u \in C^k(\mathbb{T}^d)$ and

$$[D^r u]_{C^\alpha(\mathbb{T}^d)} := \sup_{x \neq y \in \mathbb{T}^d} \frac{|D^r u(x) - D^r u(y)|}{\operatorname{dist}(x,y)^\alpha} < \infty$$

for each multi-index r such that $|r| = k$, where $\operatorname{dist}(x,y)$ is the geodesic distance from x to y on $\mathbb{T}^d$. Note that in the definition of the previous (and following) seminorm, since u can be seen as a periodic function on $\mathbb{R}^d$, $\operatorname{dist}(x,y)$ can be replaced by the euclidean distance $|x-y|$, and the supremum be taken in $\mathbb{R}^d$. We will denote by $\|\cdot\|_{\infty;\Omega}$ the sup-norm on Ω (and eventually drop Ω in the subscript if it is clear from the context).

Let now $I \subseteq [0,T]$ and $Q = \mathbb{T}^d \times I$. First define

$$[u]_{C^\alpha_x(Q)} := \sup_{t \in [0,T]} [u(\cdot,t)]_{C^\alpha(\mathbb{T}^d)}$$

and

$$[u]_{C^\beta_t(Q)} := \sup_{x \in \mathbb{T}^d} [u(x,\cdot)]_{C^\beta(I)} \ .$$

For any integer k we denote by $C^{2k,k}(Q)$ the set of functions $u = u(x,t) : Q \to \mathbb{R}$ which are continuous in Q together with all derivatives of the form $\partial_t^r D_x^\beta u$ for $2r + |\beta| \leq 2k$. Moreover, let $C^{2k+\alpha,k+\alpha/2}(Q)$ be functions of $C^{2k,k}(Q)$ such that the derivatives $\partial_t^r D_x^\beta u$, with $2r+|\beta|=2k$, are α-Hölder in x and $\alpha/2$-Hölder in t, with norm

$$\|u\|_{C^{2k+\alpha,k+\alpha/2}(Q)} = \sum_{2r+|\beta|\le 2k} \|\partial_t^r D_x^\beta u\|_{\infty;Q} + \sum_{2r+|\beta|=2k} [\partial_t^r D_x^\beta u]_{C_x^\alpha(Q)} + [\partial_t^r D_x^\beta u]_{C_t^{\alpha/2}(Q)}.$$

For these classical parabolic Hölder spaces, we refer the interested reader to [119, 154, 159] for a more comprehensive discussion.

We now consider some vector-valued Hölder classes. Let X be a Banach space and $\beta \in (0,1)$. Denote by $C^\beta(I;X)$ the space of functions $u : I \to X$ such that the norm defined as

$$\|u\|_{C^\beta(I;X)} := \sup_{t\in I} \|u(t)\|_X + \sup_{t\neq\tau} \frac{\|u(t)-u(\tau)\|_X}{|t-\tau|^\beta}$$

is finite. Hence, specializing to $X = C^\alpha(\mathbb{T}^d)$, $\alpha \in (0,1)$, we have that $C^\beta(I;C^\alpha(\mathbb{T}^d))$ is the set of functions $u : I \to C^\alpha(\mathbb{T}^d)$ with finite norm

$$\|u\|_{C^\beta(I;C^\alpha(\mathbb{T}^d))} := \|u\|_{\infty;Q} + \sup_{t\in I}[u(\cdot,t)]_{C^\alpha(\mathbb{T}^d)} + [u]_{C^\beta(I;C^\alpha(\mathbb{T}^d))} \ ,$$

where the last seminorm is defined as

$$[u]_{C^\beta(I;C^\alpha(\mathbb{T}^d))} := \sup_{t\neq\tau\in I} \frac{\|u(\cdot,t)-u(\cdot,\tau)\|_{C^\alpha(\mathbb{T}^d)}}{|t-\tau|^\beta} \ .$$

When dealing with regularity of parabolic equations driven by fractional diffusion, we also need the following Hölder spaces with different regularity in time and space. Following the lines of [59] and [114], we define $\mathcal{C}^{\alpha,\beta}(Q)$ as the space of continuous functions u such that the following Hölder parabolic seminorm is finite

$$[u]_{\mathcal{C}^{\alpha,\beta}(Q)} := [u]_{C_x^\alpha(Q)} + [u]_{C_t^\beta(Q)}. \tag{5.3}$$

The norm in the space $\mathcal{C}^{\alpha,\beta}(Q)$ is defined naturally as

$$\|u\|_{\mathcal{C}^{\alpha,\beta}(Q)} := \|u\|_{\infty;Q} + [u]_{\mathcal{C}^{\alpha,\beta}(Q)} \ .$$

Note that if $\beta = \alpha/2$, the space $\mathcal{C}^{\alpha,\beta}(Q)$ coincides with $C^{\alpha,\alpha/2}(Q)$. As pointed out in [114], the following equivalence between seminorms holds

$$[u]_{\mathcal{C}^{\alpha,\beta}(Q)} \sim \sup_{x,y\in\mathbb{T}^d, t,\tau\in[0,T]} \frac{|u(x,t)-u(y,\tau)|}{\mathrm{dist}(x,y)^\alpha + |t-\tau|^\beta} \ .$$

All the spaces above can be defined analogously on $\mathbb{R}^d$ and $Q = \mathbb{R}^d \times I$. Moreover, if u is a periodic function in the x-variable, norms on $\mathbb{T}^d$ and $\mathbb{R}^d$ coincide, e.g. $\|u\|_{C^\alpha(\mathbb{T}^d)} = \|u\|_{C^\alpha(\mathbb{R}^d)}$, ...

Remark 5.7. It is worth noticing that we have to distinguish the vector-valued Hölder spaces $C^\beta([0,T];C^\alpha(\mathbb{T}^d))$ and $\mathcal{C}^{\alpha,\beta}(Q)$, since it results

$$C^\beta([0,T];C^\alpha(\mathbb{T}^d)) \subsetneq \mathcal{C}^{\alpha,\beta}(Q_T) \ .$$

It can be easily seen by taking $\beta = \alpha$ and a periodic function in the x-variable that behaves like $(x+t)^\alpha$ in a neighborhood of $(0,0)$ (see in particular [204, Section 4]). We provide the explicit computations for reader's convenience. On one hand, we have

$$\begin{aligned}
\|u\|_{\mathcal{C}^{\alpha,\beta}(Q_T)} &\sim |u|_{0;Q_T} + \sup_{x\in\mathbb{R}^d}[u(x,\cdot)]_{C^\beta([0,T])} + \sup_{t\in[0,T]}[u(\cdot,t)]_{C^\alpha(\mathbb{T}^d)} \\
&\leq |u|_{0;Q_T} + \sup_{x\in\mathbb{R}^d}[u(x,\cdot)]_{C^\beta([0,T])} + \sup_{t\in[0,T]}[u(\cdot,t)]_{C^\alpha(\mathbb{T}^d)} + [u]_{C^\beta([0,T];C^\alpha(\mathbb{T}^d))} \\
&= \|u\|_{C^\beta([0,T];C^\alpha(\mathbb{R}^d))} \ .
\end{aligned}$$

On the other hand, to see that the converse inclusion does not hold, we consider the function $\bar{u}(x,t) = (x+t)^\alpha$ in $I^2 = [0,T]\times[0,T]$. We first prove that $\bar{u} \in \mathcal{C}^{\alpha,\alpha}(I^2)$ by proving that the Hölder seminorm below

$$[\bar{u}]_{\mathcal{C}^{\alpha,\alpha}(I^2)} := \sup_{t\in I}\sup_{x,x'\in I}\frac{|\bar{u}(x,t)-\bar{u}(x',t)|}{|x-x'|^\alpha} + \sup_{x\in I}\sup_{t,t'\in I}\frac{|\bar{u}(x,t)-\bar{u}(x,t')|}{|t-t'|^\alpha}$$

is finite. We first recall the following simple algebraic lemma

Lemma 5.8. *Let $\alpha\in(0,1)$ and $a>b>0$. Then we have*

$$a^\alpha - b^\alpha \leq 2^{1-\alpha}(a-b)^\alpha \ . \tag{5.4}$$

Proof. The inequality is proven in [103, Lemma I.4.4] and with sharp constant in [53, Lemma A.2].

□

The above result immediately implies that the seminorms

$$\sup_{t\in I]}[\bar{u}(\cdot,t)]_{C^\alpha(I)} \ , \sup_{x\in I}[\bar{u}(\cdot,t)]_{C^\alpha(I)} < \infty \ .$$

Indeed, supposing without loss of generality that $x > x'$ (and hence $x+t > x'+t$ for every $t\in I$), in view of the inequality in (5.4) we conclude for every $t\in I$ the bound

$$\frac{(x+t)^\alpha-(x'+t)^\alpha}{(x-x')^\alpha} \leq 2^{1-\alpha} \ .$$

We now show that the seminorm $[u]_{C^\alpha(I;C^\alpha(I))}$ blows up in a neighborhood of $(0,0)\in I$, showing indeed that u does not belong to $C^\alpha(I;C^\alpha(I))$. We have

$$[\bar{u}]_{C^\alpha(I;C^\alpha(I))} \geq \frac{|\bar{u}(x,t)-\bar{u}(x,0)-\bar{u}(0,t)+\bar{u}(0,0)|}{x^\alpha t^\alpha} \ ,$$

and hence when (x,t) approaches to $(0,0)$ the inequality on the right-hand side behaves like

$$\frac{|\bar{u}(x,t)|}{x^\alpha t^\alpha} = \left(\frac{x+t}{xt}\right)^\alpha \ ,$$

which blows up as (x,t) approaches to $(0,0)$, and so does the seminorm $[u]_{C^\alpha(I;C^\alpha(I))}$.

5.3.2 Fractional Sobolev and Bessel potential spaces

Recall that $L^p(\mathbb{T}^d)$ is the space of all measurable and periodic functions belonging to $L^p_{\text{loc}}(\mathbb{R}^d)$ with norm $\|\cdot\|_p = \|\cdot\|_{L^p((0,1)^d)}$. If $f : \mathbb{T}^d \to \mathbb{R}^d$, for brevity we write $f \in L^p(\mathbb{T}^d)$ instead of $f \in (L^p(\mathbb{T}^d))^d$. If k is a non-negative integer, $W^{k,p}(\mathbb{T}^d)$ consists of $L^p(\mathbb{T}^d)$ functions with (distributional) derivatives in $L^p(\mathbb{T}^d)$ up to order k. For $\mu \in \mathbb{R}$ and $p \in (1,\infty)$, we can directly define the Bessel potential space $H^\mu_p(\mathbb{T}^d)$ as the space of all distributions u such that $(I-\Delta)^{\frac{\mu}{2}}u \in L^p(\mathbb{T}^d)$, where $(I-\Delta)^{\frac{\mu}{2}}u$ is the operator defined in terms of Fourier series

$$(I-\Delta)^{\frac{\mu}{2}}u(x) = \sum_{k\in\mathbb{Z}^d}(1+4\pi^2|k|^2)^{\frac{\mu}{2}}\hat{u}(k)e^{2\pi ik\cdot x} \ ,$$

where

$$\hat{u}(k) = \int_{\mathbb{T}^d} u(x)e^{-2\pi ik\cdot x}dx \ .$$

The norm in $H^\mu_p(\mathbb{T}^d)$ will be denoted by

$$\|u\|_{\mu,p} := \left\|(I-\Delta)^{\frac{\mu}{2}}u\right\|_p .$$

Note that $H^k_p(\mathbb{T}^d)$ coincides with $W^{k,p}(\mathbb{T}^d)$ when k is a non-negative integer and $p \in (1,\infty)$, by standard arguments in Fourier series (see Remark 5.14 below). Moreover, $C^\infty(\mathbb{T}^d)$ is dense in $H^\mu_p(\mathbb{T}^d)$, by a convolution procedure: this fact will be useful to prove several properties of Bessel spaces, as it is sufficient to argue in the smooth setting to get general results.

Bessel potential spaces can be also constructed via complex interpolation. We will briefly present such a construction, that will be helpful to derive some useful properties of $H^\mu_p(\mathbb{T}^d)$. For additional details, we refer to [178, Chapter 2], [36, Chapter 4] and [231, Section 1.9]. We first recall the following

Definition 5.9. *Let X be a Banach space and $\Omega \subset \mathbb{C}$ be an open set. A function $f : \Omega \to X$ is holomorphic if*

$$f'(z_0) := \lim_{h\to 0\ ,h\in\mathbb{C}\setminus\{0\}} \frac{f(z_0+h)-f(z_0)}{h}$$

exists for all $z_0 \in \Omega$. We say that f is weakly holomorphic in Ω if it is continuous in Ω and the complex-valued function $z \longmapsto \langle f(x), x'\rangle_{X',X}$ is holomorphic in Ω for every $x' \in X'$.

In particular, it is straightforward to see that every vector-valued holomorphic function is weakly holomorphic. Moreover, the converse implication is also true [7, Appendix A].

In general, in complex interpolation theory one considers two Banach spaces X, Y, that are continuously embedded in a Hausdorff topological vector space Z. Let S be the set

$$S := \{z \in \mathbb{C} : 0 < \text{Re}z < 1\} \ .$$

We define

$$\mathcal{H}_{X,Y}(S) := \{u(\theta)\,|\,u(\theta) : \overline{S} \to X+Y \text{ bounded and continuous,} \\ \text{holomorphic on } S, \|u(it)\|_X, \|u(1+it)\|_Y \text{ bounded for } t \in \mathbb{R}\}$$

and we equip it with the norm

$$\|u\|_{\mathcal{H}_{X,Y}(S)} = \max\{\sup_{t\in\mathbb{R}} \|u(it)\|_X, \sup_{t\in\mathbb{R}} \|u(1+it)\|_Y\}.$$

For every $\theta \in [0,1]$ we define the complex interpolation space with respect to (X,Y) as

$$[X,Y]_\theta = \{u(\theta) : u \in \mathcal{H}_{X,Y}(S)\}$$

endowed with the norm

$$\|f\|_{[X,Y]_\theta} := \inf_{u\in\mathcal{H}_{X,Y}(S), u(\theta)=f} \|u\|_{\mathcal{H}_{X,Y}(S)}\ .$$

Then, one has that $H^\mu_p(\mathbb{T}^d)$ can be obtained by complex interpolation between $L^p(\mathbb{T}^d)$ and $W^{k,p}(\mathbb{T}^d)$, see, e.g., [213, Section 3] or [36, Theorem 6.4.5 and p. 170], that is

$$H^\mu_p(\mathbb{T}^d) \simeq [L^p(\mathbb{T}^d), W^{k,p}(\mathbb{T}^d)]_\theta, \qquad \text{where } \mu = k\theta.$$

We briefly describe also some tools to construct real interpolation spaces, namely the so-called K-method and the trace method, referring, among others, to [177, Chapter 1] or [178, Chapter 1] for additional details. In general, real interpolation between $L^p(\mathbb{T}^d)$ and $W^{k,p}(\mathbb{T}^d)$ leads to spaces that do not coincide with Bessel potential spaces. Still, we will make use of this other class of fractional spaces to prove useful properties of $(-\Delta)^s$. Let X, Y be Banach spaces with $Y \subset X$, $\theta \in [0,1]$ and $p \in [1,\infty]$. For every $x \in X$ and $t > 0$, define

$$K(t,x,X,Y) = \inf_{x=a+b, a\in X, b\in Y} \|a\|_X + t\|b\|_Y\ .$$

Sometimes we will use the shorter notation $K(t,x)$ to denote the K-functional. If $I \subset (0,\infty)$, we denote by $L^p_*(I)$ the Lebesgue space $L^p(I, \frac{dt}{t})$ and $L^\infty_*(I) = L^\infty(I)$. We define the real interpolation space $(X,Y)_{\theta,p}$ between the Banach spaces X, Y as

$$(X,Y)_{\theta,p} = \{x \in X+Y : t \mapsto t^{-\theta}K(t,x,X,Y) \in L^p_*(0,+\infty)\}$$

endowed with the norm

$$\|x\|_{\theta,p} = \|t^{-\theta}K(t,x,X,Y)\|_{L^p_*(0,+\infty)}\ .$$

It can be proved that this is a Banach space. We remark that such a construction turns out to be useful to prove Hölder regularity of the solution of the fractional heat equation in Theorem B.1. Another frequent characterization of real interpolation spaces is given by means of the trace method (see [231, Section 1.8.1], [177, Section 1.2.2] and [172]). Let X, Y be Banach spaces as above. For $\alpha, p \in \mathbb{R}$ with $p \in (1,+\infty)$ satisfying $0 < \alpha + \frac{1}{p} < 1$, we define the space

$$W(p,\alpha,Y,X) = \{f : \mathbb{R}^+ \to X : t^\alpha f(t) \in L^p(0,+\infty;Y) \text{ and } t^\alpha f'(t) \in L^p(0,+\infty;X)\}\ .$$

It is a Banach space endowed with the norm

$$\|f\|_{W(p,\alpha,Y,X)} := \max\{\|t^\alpha f(t)\|_{L^p(0,+\infty;Y)},\ \|t^\alpha f'(t)\|_{L^p(0,+\infty;X)}\}\ .$$

We then identify with $T(p,\alpha,Y,X)$ the space of traces u of those functions $f(t) \in W(p,\alpha,Y,X)$, equipped with the norm

$$\|u\|_{T(p,\alpha,Y,X)} = \inf_{u=f(0)} \|f\|_{W(p,\alpha,Y,X)}$$

By [177, Proposition 1.2.10], this provides a characterization for the real interpolation space $(X,Y)_{\theta,p}$ as a trace space. For $p \in (1,\infty)$, $\theta \in (0,1)$ and $\theta = \frac{1}{p} + \alpha$, we define fractional Sobolev spaces $W^{1-\theta,p}(\mathbb{T}^d)$ by

$$W^{1-\theta,p}(\mathbb{T}^d) = T(p,\alpha,W^{1,p}(\mathbb{T}^d),L^p(\mathbb{T}^d)).$$

For $\mu > 1$, $W^{\mu,p}(\mathbb{T}^d)$ is defined as the space of functions in $W^{\lfloor\mu\rfloor,p}(\mathbb{T}^d)$ with derivatives of order $\lfloor\mu\rfloor$ in $W^{\mu-\lfloor\mu\rfloor,p}(\mathbb{T}^d)$, while for $\mu < 0$ it is defined by duality. Note that $T(p,\alpha,Y,X) = T(p',-\alpha,X',Y')$ by [172, Theorem 1.2]. We finally mention that spaces $W^{\mu,p}(\mathbb{T}^d)$ defined above can be characterized using the Gagliardo seminorm on $\mathbb{T}^d$ by transposing classical arguments on $\mathbb{R}^d$ (see, e.g., [178]).
Finally, we need to introduce the Besov spaces $B^\mu_{p,q}(\mathbb{T}^d)$, where $\mu \in \mathbb{R}$ stands for the order of differentiability and $1 < p,q \le \infty$ for the orders of integrability. If μ is not an integer, we denote by $[\mu]$ and $\{\mu\}$ be the integral and fractional parts of μ. For $p,q < \infty$ we define

$$B^\mu_{p,q}(\mathbb{T}^d) := \{u \in W^{[\mu],p}(\mathbb{T}^d) : [u]_{B^\mu_{p,q}(\mathbb{T}^d)} < \infty\}$$

where

$$[u]_{B^\mu_{p,q}(\mathbb{T}^d)} := \sum_{|\alpha|=\mu} \left(\int_{\mathbb{T}^d} \frac{dh}{|h|^{d+\{\mu\}q}} \left(\int_{\mathbb{T}^d} |D^\alpha u(x+h) - D^\alpha u(x)|^p\, dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} ,$$

and we set as usual $W^{0,p}(\mathbb{T}^d) = L^p(\mathbb{T}^d)$. When $p = q$ it results $B^\mu_{p,p}(\mathbb{T}^d) = W^{\mu,p}(\mathbb{T}^d)$. These spaces in the intermediate cases $1 < p,q < \infty$ are crucial to characterize the initial traces for parabolic Sobolev spaces we shall define in the next sections. For $p,q = \infty$ the L^p norms are replaced by sup norms and $B^\mu_{\infty,\infty}(\mathbb{T}^d) = C^\mu(\mathbb{T}^d)$ (see e.g. [178, p. 13]). Similarly to the case of fractional Sobolev spaces $W^{\mu,p}$, Besov spaces can be characterized via real interpolation. One can show the following characterizations (cf [178, Example 1.10]):

- For $\theta \in (0,1)$, $m \in \mathbb{N}$ we have

$$(C(\mathbb{T}^d), C^k(\mathbb{T}^d))_{\theta,\infty} = B^{m\theta}_{\infty,\infty}(\mathbb{T}^d)\ .$$

- In particular, if $m\theta$ is not an integer,

$$(C(\mathbb{T}^d), C^k(\mathbb{T}^d))_{\theta,\infty} = C^{m\theta}(\mathbb{T}^d)\ .$$

- For $1 \leq p, q < \infty$, $m \in \mathbb{N}$ we have

$$(L^p(\mathbb{T}^d), W^{m,p}(\mathbb{T}^d))_{\theta,q} = B^{m\theta}_{p,q}(\mathbb{T}^d) \ .$$

We conclude this introductory part with one of the main useful tool in interpolation theory, namely the so-called Reiteration Theorem. We first recall the following definition

Definition 5.10. *Let $\theta \in [0,1]$ and E be a Banach space such that $X \cap Y \subset E \subset X + Y$.*

(i) E belongs to the class J_θ between X and Y if there exists a constant $C_1 > 0$ such that

$$\|x\|_E \leq C_1 \|x\|_X^{1-\theta} \|x\|_Y^{\theta} \ , \forall x \in X \cap Y \ .$$

(ii) E belongs to the class K_θ between X and Y if there exists a constant $C_2 > 0$ such that

$$K(t,x) \leq C_2 t^\theta \|x\|_E \ , \forall x \in E, \ t > 0 \ .$$

Note that when $Y \subset X$ we have $K(t,x) \leq \|x\|_X$ (see [178, (c) pag. 2]). We quote the result from [178, Theorem 1.23].

Theorem 5.11 (Reiteration Theorem)**.** *Let $0 \leq \theta_0 < \theta_1 \leq 1$. Set $\theta \in (0,1)$ and $\omega = (1-\theta)\theta_0 + \theta\theta_1$. The following hold true.*

(a) If E_i belong to the class K_{θ_i}, $i = 0,1$ between X and Y, then

$$(E_0, E_1)_{\theta,p} \subset (X,Y)_{\omega,p} \ , \forall p \in [1,\infty] \ , (E_0,E_1)_\theta \subset (X,Y)_\omega \ .$$

(b) If E_i belong to the lass J_{θ_i}, $i = 0,1$ between X and Y, then

$$(X,Y)_{\omega,p} \subset (E_0, E_1)_{\theta,p} \ , \forall p \in [1,\infty] \ , (X,Y)_\omega \subset (E_0,E_1)_\theta \ .$$

As a consequence, if E_i belong to $K_{\theta_i} \cap J_{\theta_i}$, then

$$(E_0, E_1)_{\theta,p} = (X,Y)_{\omega,p} \ , \forall p \in [1,\infty] \ , (E_0,E_1)_\theta = (X,Y)_\omega$$

with equivalence of their respective norms.

Parabolic spaces. We proceed with the definitions of some functional spaces involving time and space weak derivatives. Let $Q = \mathbb{T}^d \times I$ be as before. For any integer k and $p \geq 1$, we denote by $W^{2k,k}_p(Q)$ the space of functions u such that $\partial_t^r D_x^\beta u \in L^p(Q)$ for any multi-index β and r such that $|\beta| + 2r \leq 2k$ endowed with the norm

$$\|u\|_{W^{2k,k}_p(Q)} = \left(\iint_Q \sum_{|\beta|+2r \leq 2k} |\partial_t^r D_x^\beta u|^p dxdt \right)^{\frac{1}{p}} .$$

We now define the fractional generalization of the above spaces. Let again $\mu \in \mathbb{R}$ and $p \in (1, \infty)$. We denote by $\mathbb{H}_p^\mu(Q) := L^p(0,T; H_p^\mu(\mathbb{T}^d))$ the space of measurable functions $u : (0,T) \to H_p^\mu(\mathbb{T}^d)$ endowed with the norm

$$\|u\|_{\mathbb{H}_p^\mu(Q)} := \left(\int_0^T \|u(\cdot,t)\|_{H_p^\mu(\mathbb{T}^d)}^p \, dt \right)^{\frac{1}{p}} .$$

We define the space $\mathcal{H}_p^\mu(Q) = \mathcal{H}_p^{\mu;s}(Q)$ as the space of functions $u \in \mathbb{H}_p^\mu(Q)$ with $\partial_t u \in (\mathbb{H}_{p'}^{2s-\mu}(Q))'$ equipped with the norm

$$\|u\|_{\mathcal{H}_p^\mu(Q)} := \|u\|_{\mathbb{H}_p^\mu(Q)} + \|\partial_t u\|_{(\mathbb{H}_{p'}^{2s-\mu}(Q))'} .$$

We refer the reader to [84]. Note that the above definitions make sense also when $s = 1$, see e.g. Chapter 6 (here we will usually drop the superscript s for brevity). Those are natural spaces in the standard parabolic setting: see [155] and [92], [50, Chapter 6] for properties in the case $s = 1$. Note that $(\mathbb{H}_{p'}^{2s-\mu}(Q))'$ coincides with $\mathbb{H}_p^{\mu-2s}(Q)$ when $p > 1$. From here on for a function $f : Q_T \to \mathbb{R}$ we will write for brevity $\|Df\|_{L^q(Q_T)}$ meaning the norm $\|Df\|_{L^q(Q_T;\mathbb{R}^d)}$.

Moreover, all the aforementioned spaces can be defined analogously on $\mathbb{R}^d$ and $\mathbb{R}^d \times I$, mutatis mutandis. In particular, one has to consider $(I - \Delta)^{\frac{\mu}{2}} u$ as the operator acting on tempered distributions in terms of the Fourier transform $\mathcal{F}$:

$$\mathcal{F}[(I - \Delta)^{\frac{\mu}{2}} u](\xi) = (1 + 4\pi^2 |\xi|^2)^{\frac{\mu}{2}} \, \mathcal{F}u(\xi), \qquad \forall \xi \in \mathbb{R}^d.$$

5.3.3 The fractional Laplacian on the torus

In this section we recall the definition of the fractional Laplacian on the flat torus. Let $u : \mathbb{T}^d \to \mathbb{R}$. The fractional Laplacian on the torus can be defined via the multiple Fourier series

$$(-\Delta_{\mathbb{T}^d})^\mu u(x) = (2\pi)^{2\mu} \sum_{k \in \mathbb{Z}^d} |k|^{2\mu} \hat{u}(k) e^{2\pi i k \cdot x} , \qquad \mu > 0 .$$

With a slight abuse of notation, we will denote this operator by $(-\Delta)^\mu$. Indeed, generally speaking $(-\Delta_{\mathbb{T}^d})^\mu$ coincides with the standard fractional laplacian on $\mathbb{R}^d$ acting on periodic functions. We refer the reader to [206, 62] for additional details, and to [205] for transference properties from the torus to the Euclidean space. Note that in our analysis of this chapter we never make use of the integral representation formula for the fractional Laplacian on the torus. However, it can be immediately obtained by properties of Fourier transform, as stated in the next

Proposition 5.12. *Let $0 < s < 1$, $x \in \mathbb{T}^d$, $u \in C^\infty(\mathbb{T}^d)$, then*

$$(-\Delta)^s u(x) = c_{d,s} \sum_{k \in \mathbb{Z}^d} \text{P.V.} \int_{\mathbb{T}^d} \frac{u(x) - u(y)}{|x - y - k|^{d+2s}} dy .$$

Proof. This fact has been proved in the 2-dimensional case in [95, Proposition 2.2] and can be easily adapted to the general d-dimensional case.
We further mention that in the case of the whole space the fractional Laplacian can be also defined via the extension problem (see e.g. the seminal paper [61], [56] and the references therein).

$\square$

We present two standard results that will be useful in the sequel

Lemma 5.13. *For every smooth f, g, the following identity holds true for any $s \in (0,1)$*

$$\int_{\mathbb{T}^d} (-\Delta)^s f g dx = \int_{\mathbb{T}^d} (-\Delta)^{s/2} f \,(-\Delta)^{s/2} g dx = \int_{\mathbb{T}^d} f(-\Delta)^s g dx \ .$$

Proof. The functions f and g can be written by multiple Fourier series expansion

$$f(x) = \sum_{\nu\in\mathbb{Z}^d} \hat{f}(\nu) e^{2\pi i\nu\cdot x} \text{ and } g(x) = \sum_{\mu\in\mathbb{Z}^d} \hat{g}(\mu) e^{2\pi i\mu\cdot x}.$$

Then

$$\begin{aligned}
\int_{\mathbb{T}^d} (-\Delta)^s f g dx &= (2\pi)^{2s} \int_{\mathbb{T}^d} \sum_{\nu,\mu\in\mathbb{Z}^d} |\nu|^{2s} \hat{f}(\nu) e^{2\pi i\nu\cdot x} \hat{g}(\mu) e^{2\pi i\mu\cdot x} dx \\
&= (2\pi)^{2s} \sum_{\nu,\mu\in\mathbb{Z}^d} |\nu|^{2s} \hat{f}(\nu)\hat{g}(\mu) \int_{\mathbb{T}^d} e^{2\pi i(\nu+\mu)\cdot x} dx \\
&= (2\pi)^{2s} \sum_{\nu+\mu=0} |\mu|^s|\nu|^s \hat{f}(\nu)\hat{g}(\mu) \int_{\mathbb{T}^d} e^{2\pi i(\nu+\mu)\cdot x} dx \\
&= (2\pi)^{2s} \sum_{\nu,\mu\in\mathbb{Z}^d} |\mu|^s|\nu|^s \hat{f}(\nu)\hat{g}(\mu) \int_{\mathbb{T}^d} e^{2\pi i(\nu+\mu)\cdot x} dx \\
&= (2\pi)^{2s} \int_{\mathbb{T}^d} \sum_{\nu,\mu\in\mathbb{Z}^d} |\nu|^s \hat{f}(\nu) e^{i\nu\cdot x} |\mu|^s \hat{g}(\mu) e^{2\pi i\mu\cdot x} dx = \int_{\mathbb{T}^d} (-\Delta)^{\frac{s}{2}} f (-\Delta)^{\frac{s}{2}} g dx \ ,
\end{aligned}$$

where we used that $\int_{\mathbb{T}^d} e^{2\pi i(\nu+\mu)\cdot x} dx = 0$ if and only if $\mu + \nu \neq 0$ and the fact that the Fourier series defining f and g converge absolutely. $\square$

Remark 5.14. We point out that the operator $(I-\Delta)^{\frac{\mu}{2}}$ maps isometrically $H_p^{\eta+\mu}(\mathbb{T}^d)$ to $H_p^{\eta}(\mathbb{T}^d)$ (and therefore spaces $\mathcal{H}_p^{\eta+\mu}$ to $\mathcal{H}_p^{\eta}$) for any $\eta, \mu \in \mathbb{R}$, This fact extends also to Besov spaces $B_{pp}^{\mu}(\mathbb{T}^d)$. Moreover, for $\mu > 0$ the operator $(-\Delta)^{\frac{\mu}{2}}$ is bounded from $H_p^{\eta+\mu}(\mathbb{T}^d)$ to $H_p^{\eta}(\mathbb{T}^d)$. Indeed, $T^\mu := [(-\Delta)^{\frac{\mu}{2}}(I-\Delta)^{-\frac{\mu}{2}}]$, $\mu > 0$ is bounded in $L^p(\mathbb{R}^d)$ (see [222, p. 133]), so

$$\|(-\Delta)^{\frac{\mu}{2}} u\|_{L^p(\mathbb{R}^d)} \leq C(s,p)\|u\|_{H_p^{\mu}(\mathbb{R}^d)}. \tag{5.5}$$

In other words, $(2\pi)^\mu|\xi|^\mu(1+4\pi^2|\xi|^2)^{-\frac{\mu}{2}}$ defines a Fourier multiplier on $L^p(\mathbb{R}^d)$. Then, by the transference result [223, Theorem VIII.3.8], the periodized operator given by

$$\tilde{T}^\mu u := \sum_{k\in\mathbb{Z}^d} (2\pi)^\mu |k|^\mu (1+4\pi^2|k|^2)^{-\frac{\mu}{2}} \hat{u}(k) e^{2\pi ik\cdot x}$$

is in turn bounded in $L^p(\mathbb{T}^d)$. It then follows

$$\|(-\Delta)^{\frac{\mu}{2}}u\|_{L^p(\mathbb{T}^d)} = \|\tilde{T}^\mu(I-\Delta)^{\frac{\mu}{2}}u\|_{L^p(\mathbb{T}^d)} \leq C\|(I-\Delta)^{\frac{\mu}{2}}u\|_{L^p(\mathbb{T}^d)} = C\|u\|_{H_p^\mu(\mathbb{T}^d)}\ , \quad (5.6)$$

so $(-\Delta)^{\frac{\mu}{2}}$ is bounded from $H_p^\mu(\mathbb{T}^d)$ to $L^p(\mathbb{T}^d)$. The general case follows by using the isometry $(I-\Delta)^{\frac{\eta}{2}}$.

Similarly, $(1+(2\pi)^\mu|\xi|^\mu)/(1+4\pi^2|\xi|^2)^{\frac{\mu}{2}}$ and $(1+4\pi^2|\xi|^2)^{\frac{\mu}{2}}/(1+(2\pi)^\mu|\xi|^\mu)$ define Fourier multipliers on $L^p(\mathbb{R}^d)$ for $1<p<\infty$, and by continuity they transfer to $L^p(\mathbb{T}^d)$. This proves the equivalence of norms $\|\cdot\|_{\mu,p}$ and $\|\cdot\|_p + \|(-\Delta)^{\frac{\mu}{2}}\cdot\|_p$.

The following interpolation estimates hold.

Lemma 5.15. *Let $u \in L^p(\mathbb{T}^d)$, $p \in (1,\infty)$.*

(i) If $s \in (0,\frac{1}{2})$ and $Du \in L^p(\mathbb{T}^d)$, then for every $\delta>0$ there exists $C(\delta)>0$ depending on δ, d, s, p such that

$$\|(-\Delta)^s u\|_p \leq \delta\,\|Du\|_p + C(\delta)\,\|u\|_p\,.$$

(ii) If $s \in [\frac{1}{2},1)$ and $D^2u \in L^p(\mathbb{T}^d)$, then for every $\delta>0$ there exists $C(\delta)>0$ depending on δ, d, s, p such that

$$\|(-\Delta)^s u\|_p \leq \delta\,\left\|D^2u\right\|_p + C(\delta)\,\|u\|_p\,.$$

Proof. The proof follows by interpolation arguments. We prove only the case (i), the other being similar. Since $H_p^{2s}(\mathbb{T}^d) \simeq [L^p(\mathbb{T}^d), W^{1,p}(\mathbb{T}^d)]_\theta$, $\theta = 2s$, by (5.6) and Young's inequality we have

$$\|(-\Delta)^s u\|_p \leq \|u\|_{2s,p} \leq C\,\|u\|_p^{1-\theta}\,\|u\|_{1,p}^\theta \leq (1-2s)\left(\frac{C}{\varepsilon}\right)^{\frac{1}{1-2s}}\|u\|_p + 2s\epsilon^{\frac{1}{2s}}\,\|u\|_{1,p}$$

where $C = C(d,s,p)$. We then conclude (i) by setting $\delta := 2s\epsilon^{\frac{1}{2s}}$ and $C(\delta) := (1-2s)\left(\frac{C}{\varepsilon}\right)^{\frac{1}{1-2s}} + 2s\epsilon^{\frac{1}{2s}}$.

□

Embedding Theorems for H_p^μ, $W^{\mu,p}$ and B_{pp}^μ

We recall some classical embeddings for (stationary) Bessel potential spaces $H_p^\mu(\mathbb{T}^d)$.

Lemma 5.16. *(i) Let $\nu,\mu \in \mathbb{R}$ with $\nu \leq \mu$, then $H_p^\mu(\mathbb{T}^d) \subset H_p^\nu(\mathbb{T}^d)$.*

(ii) If $p\mu > d$ and $\mu - d/p$ is not an integer, then $H_p^\mu(\mathbb{T}^d) \subset C^{\mu-d/p}(\mathbb{T}^d)$.

(iii) Let $\nu,\mu \in \mathbb{R}$ with $\nu \leq \mu$, $p,q \in (1,\infty)$ and

$$\mu - \frac{d}{p} = \nu - \frac{d}{q}\ ,$$

then $H_p^\mu(\mathbb{T}^d) \subset H_q^\nu(\mathbb{T}^d)$. In particular, for $\nu=0$ this gives the continuous embedding of $H_p^\mu(\mathbb{T}^d)$ onto $L^{\frac{dp}{d-\mu p}}$ and hence onto $L^q(\mathbb{T}^d)$ for $1<q\leq\frac{dp}{d-\mu p}$ for $\mu p < d$

(iv) In particular, for $\mu > 0$ such that $\mu p < d$ and $1 < q < \frac{dp}{d-\mu p}$ the embedding of $H_p^\mu(\mathbb{T}^d)$ onto $L^q(\mathbb{T}^d)$ is compact.

Proof. Item (i)-(iii) are proven in [158, Corollary 13.3.9], [158, Theorem 13.8.1] and [158, Theorem 13.8.7] respectively for the whole space case. The transference to the periodic setting can be obtained as follows. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be a cutoff function such that $\chi \equiv 1$ on the unit cube $[0,1]^d$ and $0 \le \chi \le 1$.

Let now u be smooth function on $\mathbb{T}^d$, namely a smooth periodic function on $\mathbb{R}^d$. Then, it is easy to check that the extension operator

$$u \mapsto \tilde{u} = \chi u \qquad \text{on } \mathbb{R}^d \tag{5.7}$$

extends to a linear continuous operator $W^{k,p}(\mathbb{T}^d) \to W^{k,p}(\mathbb{R}^d)$, for all non-negative integers k and $p \ge 1$. The spaces $H_p^\mu(\mathbb{R}^d)$ and $H_p^\mu(\mathbb{T}^d)$ can be both obtained via complex interpolation, that is for some $\theta \in (0,1)$ and $k \ge \mu \ge 0$, $H_p^\mu(\mathbb{T}^d) \simeq [L^p(\mathbb{T}^d), W^{k,p}(\mathbb{T}^d)]_\theta$ and $H_p^\mu(\mathbb{R}^d) \simeq [L^p(\mathbb{R}^d), W^{k,p}(\mathbb{R}^d)]_\theta$. Moreover, they coincide with $H_p^\mu(\mathbb{T}^d)$ and $H_p^\mu(\mathbb{R}^d)$ respectively when μ is a non-negative integer. Therefore, the extension operator (5.7) is also bounded on $H_p^\mu(\mathbb{T}^d) \to H_p^\mu(\mathbb{R}^d)$ by interpolation (see [231, Theorem (a), p. 59] and [178, Chapter 2]).Thus, for all $\mu \ge 0$,

$$\|u\|_{L^p(\mathbb{T}^d)} \le \|\tilde{u}\|_{L^p(\mathbb{R}^d)} \le C\|\tilde{u}\|_{H_p^\mu(\mathbb{R}^d)} \le C_1\|u\|_{H_p^\mu(\mathbb{T}^d)}, \tag{5.8}$$

that implies (i) in the case $\nu = 0$ (note that for the first inequality in (5.8) to be true, it is crucial to work in L^p, so that the restriction operator $L^p(\mathbb{R}^d) \to L^p(\mathbb{T}^d)$ is continuous). The general case $\nu \neq 0$ follows by applying to (5.8) the isometry $(I - \Delta)^{\nu/2}$. Items (ii) and (iii) are obtained analogously.
To prove (iv) one argues by interpolation. We restrict without loss of generality to the case $\mu \in (0,1)$. One observes then that $H_p^\mu(\mathbb{T}^d) = [L^p(\mathbb{T}^d), W^{1,p}(\mathbb{T}^d)]_\theta$, $W^{1,p}(\mathbb{T}^d)$ is compactly embedded onto $L^r(\mathbb{T}^d)$ for $1 < r < \frac{dp}{d-p}$ by Rellich-Kondrachov Theorem and hence the identity map $T : W^{1,p}(\mathbb{T}^d) \to L^p(\mathbb{T}^d)$, $T(u) = u$ is compact. Moreover, T is also continuous from $L^p(\mathbb{T}^d)$ onto itself. Therefore, by classical compactness results in interpolation theory (see e.g. [173]), we have the compact embedding of $H_p^\mu(\mathbb{T}^d)$ onto $L^p(\mathbb{T}^d)$. We now take a bounded sequence u_n in $H_p^\mu(\mathbb{T}^d)$. Then one can extract a subsequence u_{n_k} converging strongly in $L^p(\mathbb{T}^d)$. By interpolation, for every $p < q < \frac{dp}{d-\mu p}$, there exists $\theta \in (0,1)$ such that

$$\|u_{n_k} - u_{n_j}\|_q \le \|u_{n_k} - u_{n_j}\|_p^{1-\theta}\|u_{n_k} - u_{n_j}\|_{\frac{dp}{d-\mu p}}^\theta \to 0$$

as $j,k \to \infty$ since u_{n_k} is bounded in $H_p^\mu(\mathbb{T}^d)$ which is in turn continuously embedded onto $L^{\frac{dp}{d-\mu p}}(\mathbb{T}^d)$ by (iii). Then we have the strong convergence also in L^q with q as above.

□

Lemma 5.17. *(i) Let $\nu, \mu \in \mathbb{R}$ with $\nu \le \mu$, then $W^{\mu,p}(\mathbb{T}^d) \subset W^{\nu,p}(\mathbb{T}^d)$.*

(ii) If $p\mu > d$ and $\mu - d/p$ is not an integer, then $W^{\mu,p}(\mathbb{T}^d) \subset C^{\mu-d/p}(\mathbb{T}^d)$.

(iii) Let $\nu, \mu \in \mathbb{R}$ with $\nu \leq \mu$, $p, q \in (1, \infty)$ and

$$\mu - \frac{d}{p} = \nu - \frac{d}{q} \ ,$$

then $W^{\mu,p}(\mathbb{T}^d) \subset W^{\nu,q}(\mathbb{T}^d)$.

Proof. For (i) and (ii) see [213, Section 3.5.5]. To prove (iii), we use the trace method (see also [213, Section 3.5.5] and references therein for a different proof). By the definition of the fractional Sobolev space $W^{\mu,p}$ it is sufficient to restrict to prove the result when $\mu \in (0,1)$. By using the critical embedding $W^{1,p}(\mathbb{T}^d) \subset L^q(\mathbb{T}^d)$ when $\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$ one concludes that the identity map is linear and continuous from $T(p, \alpha, W^{1,p}(\mathbb{T}^d), W^{0,p}(\mathbb{T}^d)) = W^{1-\theta,p}(\mathbb{T}^d)$ to $T(p, \alpha, L^q(\mathbb{T}^d), L^p(\mathbb{T}^d))$. It is well-known (see e.g. [171]) that $T(p, \alpha, L^q(\mathbb{T}^d), L^p(\mathbb{T}^d)) \subset L^{q(\theta)}$, where

$$\frac{1}{q(\theta)} = \frac{1}{p} - \frac{1-\theta}{d} \ , \theta = \frac{1}{p} + \alpha \ .$$

giving thus the embedding $W^{1-\theta,p}(\mathbb{T}^d)$ onto $L^{q(\theta)}(\mathbb{T}^d)$ for $q(\theta)$ as above. $\square$

Let us remark that if either $\mu > 0$ noninteger or $\mu \in \mathbb{N}$ and $p = 2$ we have $W^{\mu,p} \simeq B^{\mu}_{p,p}$. When μ is an integer and $p \neq 2$ we have $W^{\mu,p} \neq B^{\mu}_{p,p}$. This motivates the next lemma, where we collect some results that link Bessel, Besov and Sobolev-Slobodeckii spaces under different ranges of the integrability exponent p.

Lemma 5.18. *We have the following inclusions for $\mu \in \mathbb{R}$.*

(i) $W^{\mu,p}(\mathbb{T}^d) \subset H^{\mu}_p(\mathbb{T}^d) \subset B^{\mu}_{p,2}(\mathbb{T}^d)$ for $1 < p \leq 2$.

(ii) $B^{\mu}_{p,2}(\mathbb{T}^d) \subset H^{\mu}_p(\mathbb{T}^d) \subseteq W^{\mu,p}(\mathbb{T}^d)$ for $2 \leq p < \infty$.

Proof. The result on $\mathbb{R}^d$ is proven in [231, Section 2.3.3] (see also [36, Theorem 6.4.4]. Recalling that H^{μ}_p is isomorphic to a Triebel-Lizkorin scale (see [213, Theorem 3.5.4-(v)] and the same chapter for the definition of this space), one uses [213, Remark 3.5.1.4-(20)] to show (i) and (ii).

$\square$

Embedding Theorems for parabolic spaces $\mathcal{H}^{\mu}_p$

We now prove continuous embedding theorems for the spaces $\mathcal{H}^{\mu}_p(Q_T) = \mathcal{H}^{\mu;s}_p(Q_T)$, where $Q_T = \mathbb{T}^d \times (0,T)$. As usual, we will denote continuous embeddings of Banach spaces by the symbol $X \hookrightarrow Y$. All the results of this section are valid for $s \in (0,1]$. We will basically follow the strategy of [155, Theorem 7.2], where analogous results are proven for (stochastic) spaces associated to heat-type equations (that is, for $s = 1$) on $\mathbb{R}^d \times (0,T)$ (see also [157]). In addition, we refer to [50, Theorem 6.2.2], [92, Proposition 2.2] and [185, Theorem A.3] for the case $s, \mu = 1$. We first state the main result of this section and, at the end, we will deduce some useful corollaries.

Theorem 5.19. *Let $\mu \in \mathbb{R}$, $p > 1$, $u \in \mathcal{H}_p^\mu(Q_T)$ and $u(0) \in W^{\mu-2s/p,p}(\mathbb{T}^d)$. If β is such that*

$$\frac{s}{p} < \beta < s,$$

then $u \in C^{\frac{\beta}{s}-\frac{1}{p}}([0,T]; H_p^{\mu-2\beta}(\mathbb{T}^d))$. In particular, there exists $C > 0$ depending on d, p, β, s, T, such that

$$\|u(\cdot,t) - u(\cdot,\tau)\|_{\mu-2\beta,p}^p \le C|t-\tau|^{\frac{\beta}{s}p-1}(\|u\|_{\mathcal{H}_p^\mu(Q_T)} + \|u(0)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)})^p$$

for $0 \le t, \tau \le T$. Hence,

$$\|u\|_{C^{\frac{\beta}{s}-\frac{1}{p}}([0,T];H_p^{\mu-2\beta}(\mathbb{T}^d))} \le C(\|u\|_{\mathcal{H}_p^\mu(Q_T)} + \|u(0)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}) \, . \tag{5.9}$$

Note that the constant C remains bounded for bounded values of T.

We first state the following trace result on the hyperplane $t = 0$ for functions belonging to $\mathcal{H}_p^\mu$. The flexibility of this (functional) characterization of initial traces give also a way to prove the result claimed in [159, Lemma II.3.4] for classical spaces associated to heat PDEs, were a proof is not presented.

Lemma 5.20. *If $u \in \mathcal{H}_p^\mu(Q_T)$, $\mu \in \mathbb{R}$ and $p > 1$ satisfying $\mu - 2s/p > 0$, then $u(0) \in W^{\mu-2s/p,p}(\mathbb{T}^d)$.*

Proof. We restrict to the case $\mu = 2s$. This is proven in [178, Corollary 1.14] in an abstract framework and the result is a consequence of embedding properties for the domain of the fractional Laplacian $D(-(-\Delta)^s)$ and the Reiteration Theorem. Indeed, since $s \in (0,1)$, by applying [178, Proposition 4.7] one obtains that $D(-(-\Delta)^s)$ belongs to $J_s(L^p(\mathbb{T}^d), D(-\Delta)) \cap K_s(L^p(\mathbb{T}^d), D(-\Delta))$. We now apply the Reiteration Theorem (see Theorem 5.11) with $\theta_0 = 0$, $\theta_1 = s$, $\theta = 1-1/p$ (giving thus $\omega = s-s/p$) $X = L^p(\mathbb{T}^d)$, $Y = W^{2,p}(\mathbb{T}^d)$, $E_0 = L^p(\mathbb{T}^d)$, $E_1 = D(-(-\Delta)^s)$ to get that

$$(L^p(\mathbb{T}^d), D(-(-\Delta)^s))_{1-1/p,p} = (L^p(\mathbb{T}^d), W^{2,p}(\mathbb{T}^d))_{s-s/p,p} \simeq W^{2s-2s/p,p}(\mathbb{T}^d)$$

Recall that $W^{2s-2s/p,p}(\mathbb{T}^d) \simeq B_{p,p}^{2s-2s/p}(\mathbb{T}^d)$. □

Remark 5.21. We remark that in the context of mixed Lebesgue spaces of the form

$$\mathcal{H}_p^{\mu,q}(Q_T) := W^{1,q}(0,T; H_p^{\mu-2s}(\mathbb{T}^d)) \cap L^q(0,T; H_p^\mu)$$

the initial trace turns out to belong to a Besov space with different orders of summability. When e.g. $\mu = 2s$ we have $u(0) \in B_{p,q}^{2s-2s/p}(\mathbb{T}^d)$ (see e.g. [199]).

We first need some estimates in the spaces of Bessel potentials for the semigroup $\mathcal{T}_t$ associated to the fractional Laplacian. Recall that for a given smooth u, $\mathcal{T}_t u := v(t)$, where v solves

$$\begin{cases} \partial_t v + (-\Delta)^s v = 0 & \text{in } Q_T, \\ v(0) = u & \text{in } \mathbb{T}^d. \end{cases}$$

Then we have the following standard representation formula that can be obtained via Fourier transform

$$\mathcal{T}_t u(x) = \int_{\mathbb{R}^d} p_t(x-y)u(y)dy = p_t \star_{\mathbb{R}^d} u(x), \tag{5.10}$$

where $p_t(x) := \mathcal{F}^{-1}(e^{-t|\xi|^{2s}})(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-t|\xi|^{2s}} d\xi$. In the periodic case, namely if $u : \mathbb{T}^d \to \mathbb{R}$, defining $\hat{p}_t(x) := \sum_{z\in\mathbb{Z}^d} p_t(x+z) = \sum_{z\in\mathbb{Z}^d} e^{-t|z|^{2s}} e^{2\pi i z\cdot x}$, we note that

$$\begin{aligned}\mathcal{T}_t u(x) &= \int_{\mathbb{R}^d} p_t(x-y)u(y)dy = \sum_{z\in\mathbb{Z}^d} \int_{(0,1)^d+z} p_t(x-y)u(y+z)dy \\ &= \int_{\mathbb{T}^d} \sum_{z\in\mathbb{Z}^d} p_t(x-y-z)u(y)dy = \int_{\mathbb{T}^d} \hat{p}_t(x-y)u(y)dy = \hat{p}_t \star_{\mathbb{T}^d} u(x). \end{aligned}\tag{5.11}$$

This shows that some properties of the fractional heat semigroup on the whole space $\mathbb{R}^d$ can be directly transferred to the periodic case. First, $\|p_t\|_{L^1(\mathbb{R}^d)} = 1$, so $\|\hat{p}_t\|_{L^1(\mathbb{T}^d)} = 1$, readily yielding

$$\|\mathcal{T}_t f\|_p \le \|f\|_p \qquad \forall p \in [1,\infty]\ , \tag{5.12}$$

by Young's inequality for convolutions. Moreover, $p_t(x) = t^{-d/2s} p_1(t^{-1/2s}x)$ by rescaling, hence for a multiindex β we have

$$\left\|D^\beta \hat{p}_t\right\|_{L^1(\mathbb{T}^d)} \le \left\|D^\beta p_t\right\|_{L^1(\mathbb{R}^d)} \le t^{-|\beta|/2s} \left\|D^\beta p_1\right\|_{L^1(\mathbb{R}^d)} \le C t^{-|\beta|/2s}, \tag{5.13}$$

by boundedness of $\left\|D^\beta p_1\right\|_{L^1(\mathbb{R}^d)}$ (see, e.g., [236, Lemma 2.4]).

Remark 5.22. Representation formula (5.11) and decay estimates (5.12) imply that for any $f \in C^\infty(\mathbb{T}^d)$ and multiindices $k, m \in \mathbb{N}$,

$$\|D^{k+m}\mathcal{T}_t f\|_p \le C t^{-\frac{k}{2s}} \|D^m f\|_p \qquad \forall p \in [1,\infty]\ . \tag{5.14}$$

On the one hand, this shows that for $t > 0$, $\mathcal{T}_t$ maps $C^m(\mathbb{T}^d)$ onto $C^{k+m}(\mathbb{T}^d)$. On the other hand, exploiting the density of $C^\infty(\mathbb{T}^d)$ in $H_p^\mu(\mathbb{T}^d)$, one obtains that $\mathcal{T}_t$ is bounded from $W^{m,p}(\mathbb{T}^d)$ to $W^{k+m,p}(\mathbb{T}^d)$.

In addition, note that, for $\mu \in \mathbb{R}$, it results

$$\mathcal{T}_t (I-\Delta)^{\frac{\mu}{2}} u = (I-\Delta)^{\frac{\mu}{2}} \mathcal{T}_t u. \tag{5.15}$$

The equality can be verified by taking its Fourier transform.

Lemma 5.23. *(i) For any $p > 1$ and $\nu \in \mathbb{R}, \gamma \ge 0$, we have for all $f \in H_p^\nu(\mathbb{T}^d)$*

$$\|\mathcal{T}_t f\|_{\nu+\gamma,p} \le C t^{-\gamma/2s} \|f\|_{\nu,p}\ ,$$

where $C = C(\nu,\gamma,d,s,p)$.

(ii) For any $\theta \in [0,s]$ and $p > 1$, there exists a constant $C = C(d,s,p,\theta)$ such that, for all $f \in H_p^{2\theta}(\mathbb{T}^d)$, it holds

$$\|\mathcal{T}_t f - f\|_p \le C t^{\theta/s} \|f\|_{2\theta,p}\,. \tag{5.16}$$

(iii) For any $p \geq q > 1$, we have for all $f \in L^q(\mathbb{R}^d)$

$$\|\mathcal{T}_t f\|_p \leq C t^{-\frac{d}{2s}(\frac{1}{q}-\frac{1}{p})} \|f\|_q \ .$$

where $C = C(d, p, q, s)$.

(iv) For any $p \geq q > 1$ and $\mu, \nu \in \mathbb{R}$ we have for all $f \in H^\mu_q(\mathbb{R}^d)$

$$\|\mathcal{T}_t f\|_{\nu,p} \leq C t^{-\frac{d}{2s}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2s}(\nu-\mu)} \|f\|_{\mu,q}$$

where $C = C(d, p, q, \mu, \nu, s)$.

Proof. To prove (i) one can restrict without loss of generality to $\nu = 0$, since the general case will follow by replacing f by $(I - \Delta)^{-\nu} f$. The proof is a consequence of (complex) interpolation between inequalities (5.12) and (5.14), see e.g. [231, Theorem (a) p. 59].

We prove (ii), and follow the strategy of [155, Lemma 7.3]. First, by (i) with $\nu = 2\theta$ and $\gamma = 2s - 2\theta \geq 0$, we get

$$\|\mathcal{T}_t f\|_{2s,p} \leq C t^{\frac{\theta}{s}-1} \|f\|_{2\theta,p} \ , \tag{5.17}$$

where $C = C(d, p, \theta, s)$. Note that $(\mathcal{T}_t f)' = -(-\Delta)^s \mathcal{T}_t f$. Hence, we have

$$\|(\mathcal{T}_t - 1)f\|_p \leq \int_0^t \left\|[(-\Delta)^s (I-\Delta)^{-s}](I-\Delta)^s \mathcal{T}_\tau f\right\|_p d\tau$$

$$\leq C \int_0^t \|\mathcal{T}_\tau f\|_{2s,p} d\tau \leq C \|f\|_{2\theta,p} \int_0^t \tau^{\frac{\theta}{s}-1} d\tau = C t^{\frac{\theta}{s}} \|f\|_{2\theta,p}$$

where we used (5.17) and the fact that $[(-\Delta)^s (I - \Delta)^{-s}]$ is bounded in $L^p(\mathbb{T}^d)$ (see Remark 5.14).

(iii) is a consequence of Young's inequality for convolutions. We have

$$\|\mathcal{T}_t f\|_p = \|\hat{p}_t \star_{\mathbb{T}^d} f\|_p \leq \|p_t\|_r \|f\|_q$$

where $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and p_t stands for the fractional heat kernel. Standard estimates for the fractional heat kernel yields

$$\|\hat{p}_t\|_r \leq C t^{-\frac{d}{2s}(1-\frac{1}{r})}$$

This can be obtained by transference arguments in the periodic setting. Then one immediately yields

$$\|\mathcal{T}_t f\|_p \leq C t^{-\frac{d}{2s}(\frac{1}{p}-\frac{1}{q})} \|f\|_p \ .$$

The last item (iv) is a consequence of (i) and (iii). $\square$

Remark 5.24. We observe that $-(-\Delta)^s$ generates an analytic semigroup $\mathcal{T}_t$ on $L^p(\mathbb{T}^d)$ for all $p > 1$, since the following inequality

$$\|-(-\Delta)^s \mathcal{T}_t f\|_p \leq C t^{-1} \|f\|_p$$

holds (then, argue via [192, Theorem 2.5.2] for example). The above estimate is in turn a straightforward consequence of Lemma 5.23-(i) with $\nu = 0$ and $\gamma = 2s$.

We recall the following useful lemma, and refer to [155, Lemma 7.4] (and references therein) for its proof.

Lemma 5.25. *Let $p \geq 1$ and $\alpha p > 1$. Then, for any continuous L^p-valued function $h(\cdot)$ and $\tau \leq t$ we have*

$$\|h(t) - h(\tau)\|_p^p \leq C(\alpha, p)(t-\tau)^{\alpha p - 1} \int_\tau^t \int_\tau^t \mathbf{1}_{r_2 > r_1} \frac{\|h(r_2) - h(r_1)\|_p^p}{|r_2 - r_1|^{1+\alpha p}} dr_1 dr_2 \tag{5.18}$$

$$= C(\alpha, p)(t-\tau)^{\alpha p - 1} \int_0^{t-\tau} \frac{d\gamma}{\gamma^{1+\alpha p}} \int_\tau^{t-\gamma} \|h(r+\gamma) - h(r)\|_p^p \, dr \ .$$

As a consequence one has

$$\sup_{0 \leq \tau < t \leq T} \frac{\|h(t) - h(\tau)\|_p^p}{(t-\tau)^{\alpha p - 1}} \leq C(\alpha, p) \int_0^T \int_0^T \mathbf{1}_{r_2 > r_1} \frac{\|h(r_2) - h(r_1)\|_p^p}{|r_2 - r_1|^{1+\alpha p}} dr_1 dr_2 \ , \tag{5.19}$$

where $\mathbf{1}_A$ denotes the indicator function of a given set A.

We now proceed with the proof of the embeddings of $\mathcal{H}_p^\mu$.

Proof of Theorem 5.19. Note first that since the operator $(I - \Delta)^{\frac{\eta}{2}}$ maps isometrically $\mathcal{H}_p^\mu(Q_T)$ onto $\mathcal{H}_p^{\mu-\eta}(Q_T)$ for any η, μ (see Remark 5.14), we just consider the case $2\beta = \mu$. We then have to prove that

$$\|u(t) - u(\tau)\|_p^p \leq C|t-\tau|^{\frac{\beta}{s}p - 1}(\|u\|_{\mathcal{H}_p^{2\beta}(Q_T)} + \|u(0)\|_{W^{2\beta - 2s/p, p}(\mathbb{T}^d)}),$$

for $0 \leq t, \tau \leq T$.

Define

$$f := \partial_t u + (-\Delta)^s u, \tag{5.20}$$

and by the variation of constants formula (it is well-defined in view of [177, Definition 4.1.4]) we have

$$u(t) = \mathcal{T}_t u(0) + \int_0^t \mathcal{T}_{t-\tau} f(\tau) d\tau \ ,$$

where $\mathcal{T}_t$ is defined at the beginning of this section. We claim that

$$u(r+\gamma) - u(r) = (\mathcal{T}_\gamma - 1)u(r) + \int_0^\gamma \mathcal{T}_{\gamma - \rho} f(r+\rho) d\rho \ .$$

Indeed we have

$$\begin{aligned}
&\mathcal{T}_\gamma u(r) - u(r) + \int_0^\gamma \mathcal{T}_{\gamma-\rho} f(r+\rho) d\rho \\
&\qquad = \mathcal{T}_{r+\gamma} u(0) + \int_0^r \mathcal{T}_{r+\gamma-\tau} f(\tau) d\tau - u(r) + \int_0^\gamma \mathcal{T}_{\gamma-\rho} f(r+\rho) d\rho \\
&\qquad\qquad = \mathcal{T}_{r+\gamma} u(0) + \int_0^{r+\gamma} \mathcal{T}_{r+\gamma-\tau} f(\tau) d\tau - u(r) = u(r+\gamma) - u(r) \ .
\end{aligned}$$

Therefore,

$$\|u(r+\gamma) - u(r)\|_p^p \leq C(A(r,\gamma) + B(r,\gamma)) \ ,$$

where

$$A(r,\gamma) = \|(\mathcal{T}_\gamma - 1)\, u(r)\|_p^p$$

and

$$B(r,\gamma) = \left\| \int_0^\gamma \mathcal{T}_{\gamma-\rho} f(r+\rho) d\rho \right\|_p^p = \left\| \int_0^\gamma \mathcal{T}_\omega f(r+\gamma-\omega) d\omega \right\|_p^p .$$

Choose α so that $\frac{1}{p} < \alpha < \frac{\beta}{s}$. By Lemma 5.25 we have

$$\|u(t) - u(\tau)\|_p^p \le C(\alpha,p)(t-\tau)^{\alpha p - 1}(I(t,\tau) + J(t,\tau)) \ , \tag{5.21}$$

where

$$I(t,\tau) = \int_0^{t-\tau} \frac{d\gamma}{\gamma^{1+\alpha p}} \int_\tau^{t-\gamma} A(r,\gamma) dr$$

and

$$J(t,\tau) = \int_0^{t-\tau} \frac{d\gamma}{\gamma^{1+\alpha p}} \int_\tau^{t-\gamma} B(r,\gamma) dr \ .$$

To estimate B, we use Hölder's inequality and Lemma 5.23-(i) (with $\nu = 0$ and $\gamma = 2s - 2\beta \in (0,1)$. We have

$$\begin{aligned}
B(r,\gamma) &= \int_{\mathbb{T}^d} \left| \int_0^\gamma \omega^{\frac{\beta}{s}-1} \omega^{1-\frac{\beta}{s}} \mathcal{T}_\omega f(r+\gamma-\omega) d\omega \right|^p dx \\
&\le \left(\int_0^\gamma \omega^{(\frac{\beta}{s}-1)q} d\omega \right)^{\frac{p}{q}} \int_0^\gamma \omega^{(1-\frac{\beta}{s})p} \int_{\mathbb{T}^d} |\mathcal{T}_\omega f(r+\gamma-\omega)|^p dx\, d\omega \\
&\le C(d,p,\beta,s) \gamma^{\frac{\beta}{s}p-1} \int_0^\gamma \|f(r+\gamma-\omega)\|_{2\beta-2s,p}^p \, d\omega \\
&= C(d,p,\beta,s) \gamma^{\frac{\beta}{s}p-1} \int_0^\gamma \|f(r+\rho)\|_{2\beta-2s,p}^p \, d\rho.
\end{aligned}$$

This and the inequality $\alpha < \frac{\beta}{s}$ give

$$\begin{aligned}
J(t,\tau) &\le C(d,p,\alpha,\beta,s) \int_0^{t-\tau} \frac{d\gamma}{\gamma^{2+(\alpha-\frac{\beta}{s})p}} \int_0^\gamma d\rho \int_\tau^{t-\gamma} \|f(r+\rho)\|_{2\beta-2s,p}^p \, dr \\
&\le C(d,p,\alpha,\beta,s) \int_0^{t-\tau} \frac{d\gamma}{\gamma^{2+(\alpha-\frac{\beta}{s})p}} \int_0^\gamma d\rho \int_0^t \|f(r)\|_{2\beta-2s,p}^p \, dr \\
&= C(d,p,\alpha,\beta,s)(t-\tau)^{(-\alpha+\frac{\beta}{s})p} \int_0^t \|f(r)\|_{2\beta-2s,p}^p \, dr.
\end{aligned} \tag{5.22}$$

Recalling that $f = \partial_t u + (-\Delta)^s u$, by (5.5)

$$\begin{aligned}
J(t,\tau) &\le C(d,p,\alpha,\beta,s)(t-\tau)^{(-\alpha+\frac{\beta}{s})p} \int_0^t \left(\|\partial_t u(r)\|_{2\beta-2s,p}^p + \|(-\Delta)^s u(r)\|_{2\beta-2s,p}^p \right) dr \\
&\le C(d,p,\alpha,\beta,s)(t-\tau)^{(-\alpha+\frac{\beta}{s})p} \int_0^t \left(\|\partial_t u(r)\|_{2\beta-2s,p}^p + \|u(r)\|_{2\beta,p}^p \right) dr \\
&= C(d,p,\alpha,\beta,s)(t-\tau)^{(-\alpha+\frac{\beta}{s})p} \left(\|\partial_t u\|_{\mathbb{H}_p^{2\beta-2s}(Q_T)}^p + \|u\|_{\mathbb{H}_p^{2\beta}(Q_T)}^p \right) \\
&= C(d,p,\alpha,\beta,s)(t-\tau)^{(-\alpha+\frac{\beta}{s})p} \|u\|_{\mathcal{H}_p^{2\beta}(Q_T)}^p .
\end{aligned}$$

To estimate I, we apply Lemma 5.23-(ii) with $\theta = \beta \in (0, s)$ and Theorem B.4 to get

$$\int_0^t A(r,\gamma)dr \le C(d,p,\beta,s)\gamma^{\frac{\beta}{s}p} \int_0^t \|u(r)\|_{2\beta,p}^p \, dr$$
$$\le C\gamma^{\frac{\beta}{s}p}\|u\|^p_{\mathbb{H}_p^{2\beta}(Q_T)} \le C_1(d,p,\alpha,\beta,s,T)\gamma^{\frac{\beta}{s}p}(\|f\|^p_{\mathbb{H}_p^{2\beta-2s}(Q_T)} + \|u(0)\|_{W^{2\beta-2s/p,p}(\mathbb{T}^d)}).$$

Thus,

$$I(t,\tau) \le \int_0^{t-\tau} \frac{d\gamma}{\gamma^{1+\alpha p}} \int_0^t A(r,\gamma)dr$$
$$\le C(d,p,\alpha,\beta,s,T)(t-\tau)^{(\frac{\beta}{s}-\alpha)p}(\|\partial_t u + (-\Delta)^s u\|^p_{\mathbb{H}_p^{2\beta-2s}(Q_T)} + \|u(0)\|^p_{W^{2\beta-2s/p,p}(\mathbb{T}^d)})$$
$$\le C(d,p,\alpha,\beta,s,T)(t-\tau)^{(-\alpha+\frac{\beta}{s})p}(\|u\|^p_{\mathcal{H}_p^{2\beta}(Q_T)} + \|u(0)\|^p_{W^{2\beta-2s/p,p}(\mathbb{T}^d)}).$$

Finally, combining the last inequality with (5.21) and (5.22), we proved that

$$\|u(t)-u(\tau)\|_p^p \le C(d,p,\alpha,\beta,s,T)|t-\tau|^{\frac{\beta}{s}p-1}(\|u\|^p_{\mathcal{H}_p^{2\beta}(Q_T)} + \|u(0)\|^p_{W^{2\beta-2s/p,p}(\mathbb{T}^d)}) \ . \tag{5.23}$$

To obtain (5.9), in the special case $\mu = 2\beta$, it remains to show that

$$\sup_{t\le T} \|u(t)\|_p \le C(\|u\|_{\mathcal{H}_p^{2\beta}(Q_T)} + \|u(0)\|_{W^{2\beta-2s/p,p}(\mathbb{T}^d)}) \ , \tag{5.24}$$

This is a consequence of (5.23) and the continuous embedding of $W^{2\beta-2s/p,p}(\mathbb{T}^d)$ into $L^p(\mathbb{T}^d)$, as $\beta > 2s/p$. Indeed,

$$\|u(t)\|_p^p \le C(\beta,s,p,d)\,\|u(0)\|^p_{W^{2\beta-2s/p,p}(\mathbb{T}^d)} + CT^{\frac{\beta}{s}p-1}(\|u\|^p_{\mathcal{H}_p^{2\beta}(Q_T)} + \|u(0)\|^p_{W^{2\beta-2s/p,p}(\mathbb{T}^d)}).$$

□

Remark 5.26. An alternative way of proving the embedding in Theorem 5.19 is provided by the so-called mixed derivative theorem (see [221] and [199, Corollary 4.5.10]). Let us focus on the case $\mu = 2s$ for simplicity and $\mathcal{H}_p^{2s}(\mathbb{R}^d \times \mathbb{R}) = W^{1,p}(\mathbb{R}; L^p(\mathbb{T}^d)) \cap L^p(\mathbb{R}; H_p^{2s}(\mathbb{T}^d))$. Here it suffices to take the operators $A = (I-\Delta)^{\mu/2}$ and $B = (I-\partial_t^2)^{1/2}$ and $X = L^p(\mathbb{T}^d)$. Since the hypothesis of the mixed derivative theorem are fulfilled (see e.g. [186]), then we can apply it to the pair of operator (A,B) acting on $Y = L^p(\mathbb{R}; X)$ obtaining

$$\|A^\xi B^{1-\xi}\|_Y \le C\|Ay + By\|_Y \text{ for all } y \in D(A) \cap D(B)$$

Since $D(B) = H_p^1(\mathbb{R}; X)$, by the above estimate one obtains

$$\mathcal{H}_p^{2s}(Q_T) \hookrightarrow H_p^\xi(0,T; H_p^{2s-2s\xi}(\mathbb{T}^d))$$

for all $\xi \in [0,1]$.

By taking $\xi = \beta/s \in (0,1)$ we have the embedding onto $H_p^{\beta/s}(H_p^{2s-2\beta})$. Using Lemma 5.25 together with the fact that inclusions of Bessel and fractional Sobolev classes (see e.g. Lemma 5.32 below), one gets the desired inclusion onto $C^{\frac{\beta}{s}-\frac{1}{p}}(0,T; H_p^{2s-2\beta}(\mathbb{T}^d))$.

We now present some continuous embedding results that stem from Thereom 5.19.

Proposition 5.27. *Let $q \geq p > 1$, $0 \leq \theta \leq 1$ and $\mu, \eta \in \mathbb{R}$ be such that*

$$\eta < \mu + \frac{d}{q} - \frac{d + 2s(1-\theta)}{p}. \tag{5.25}$$

Then, for any $u \in \mathcal{H}_p^\mu(Q_T)$,

$$\left(\int_0^T \|u(\cdot,t)\|_{\eta,q}^{\frac{p}{\theta}}\,dt\right)^\theta \leq C(\|u\|^p_{\mathcal{H}_p^\mu(Q_T)} + \|u(0)\|^p_{W^{2\beta-2s/p,p}(\mathbb{T}^d)}) \ .$$

In particular, if $\mu > 0$, $1 < p < \frac{d+2s}{\mu}$ and $\frac{1}{q} > \frac{1}{p} - \frac{\mu}{d+2s}$,

$$\|u\|_{L^q(Q_T)} \leq C(\|u\|_{\mathcal{H}_p^\mu(Q_T)} + \|u(0)\|_{W^{2\beta-2s/p,p}(\mathbb{T}^d)})$$

Here, C depends on $d, p, q, \mu, \eta, \theta, T, s$, but remains bounded for bounded values of T.

Proof. Let $0 < \beta < s$ to be chosen. Recall that, for any $\theta \in [0,1]$, if $\nu = \nu(\beta) = (\mu-2\beta)(1-\theta)+\theta\mu$, then H_p^ν can be obtained by (complex) interpolation between H_p^μ and $H_p^{\mu-2\beta}$ (see, e.g., [36, Theorem 6.4.5]). Moreover, H_p^ν is continuously embedded in $H_q^{\nu+d/q-d/p}$ in view of Lemma 5.16. Hence, for a.e. t,

$$c(d,p,s,\beta)\,\|u(t)\|_{\nu-\frac{d}{p}+\frac{d}{q},q} \leq \|u(t)\|_{\nu,p} \leq \|u(t)\|^{1-\theta}_{\mu-2\beta,p}\,\|u(t)\|^{\theta}_{\mu,p}\,.$$

By (5.25), we can choose $2\beta > \frac{2s}{p}$ so that $\eta \leq \nu(\beta) - \frac{d}{p} + \frac{d}{q} < \mu + \frac{d}{q} - \frac{d+2s(1-\theta)}{p}$, and therefore

$$\begin{aligned}
\left(\int_0^T \|u(t)\|_{\eta,q}^{\frac{p}{\theta}}\,dt\right)^\theta &\leq C\left(\int_0^T \|u(t)\|^{(1-\theta)\frac{p}{\theta}}_{\mu-2\beta,p}\,\|u(t)\|^p_{\mu,p}\,dt\right)^\theta \\
&\leq C\sup_{t\leq T}\|u(t)\|^{(1-\theta)p}_{\mu-2\beta,p}\left(\int_0^T \|u(t)\|^p_{\mu,p}\,dt\right)^\theta \\
&\leq C(\|u\|_{\mathcal{H}_p^\mu(Q_T)} + \|u(0)\|_{W^{2\beta-2s/p,p}(\mathbb{T}^d)})^{(1-\theta)p}\,\|u\|^{\theta p}_{\mathbb{H}_p^\mu(Q_T)} \\
&\leq C(\|u\|_{\mathcal{H}_p^\mu(Q_T)} + \|u(0)\|_{W^{2\beta-2s/p,p}(\mathbb{T}^d)})^p
\end{aligned}$$

where, in the last inequality, we used Theorem 5.19 and Young's inequality.

The last statement follows by choosing $\eta = 0$ and $\theta = p/q$. □

Remark 5.28. We point out that the previous result does not allow to obtain the critical embedding in L^q for $\frac{1}{q} = \frac{1}{p} - \frac{\mu}{d+2s}$. In order to do this, a possible way is to slightly modify the above proof using the embeddings e.g. of $\mathcal{H}_p^{2s}$ onto $C(W^{2s-2s/p,p})$ (see Proposition C.3 for further details). An alternative proof of this fact in the case $\mu = 2$ and $s = 1$ can be found in [129, 14]. One can also exploit the mixed derivative theorem introduced in Remark 5.26.

Proposition 5.29. *Let* $\frac{1}{2} < s < 1$ *and* $p > \frac{d+2s}{2s-1}$. *Then for all* $u \in \mathcal{H}_p^{2s-1}(Q_T)$ *the following inequality holds*

$$\|u\|_{\mathcal{C}^{\gamma,\frac{\gamma}{2s}}(Q_T)} \leq C(\|u\|_{\mathcal{H}_p^{2s-1}(Q_T)} + \|u(0)\|_{W^{2\beta-2s/p,p}(\mathbb{T}^d)}),$$

where

$$\gamma = s - \frac{s}{p} - \frac{d}{2p} - \frac{1}{2},$$

and C *depends on* d, s, p, T.

Proof. First apply Theorem 5.19 with $\mu = 2s - 1$ to get

$$\mathcal{H}_p^{\mu}(Q_T) \hookrightarrow C^{\frac{\beta}{s}-\frac{1}{p}}([0,T]; H_p^{\mu-2\beta}(\mathbb{T}^d)) \ .$$

Then, exploit the embedding $H_p^{\mu-2\beta}(\mathbb{T}^d) \hookrightarrow C^{\mu-2\beta-\frac{d}{p}}(\mathbb{T}^d)$ of Lemma 5.16. By choosing β so that $\frac{\beta}{s} - \frac{1}{p} = \frac{\gamma}{2s}$ and γ as in the statement, then $\mu - 2\beta - \frac{d}{p} = \gamma$, and one concludes by the inclusion of $C^{\frac{\gamma}{2s}}(C^{\gamma})$ into $\mathcal{C}^{\gamma,\frac{\gamma}{2s}}$ (see Remark 5.7). □

Remark 5.30. We point out that all the results obtained in this section can be proven exactly in the same manner for the whole space case $\mathbb{R}^d$. Indeed, the arguments turn around decay estimates for the fractional heat operator and fractional heat parabolic regularity that hold to the same extent on $\mathbb{R}^d$ and $\mathbb{T}^d$. One can also prove Sobolev embedding theorems even for the stochastic version of the parabolic spaces introduced in [84] (see also [155] and references therein for a deeper discussion on such spaces). Finally, the flexibility of such approach allows to deal with problems driven by hypoelliptic diffusion, for which the adaptation of this parabolic construction is almost straightforward replacing H_p^{μ} with its horizontal counterpart (see [115]).

Using classical arguments like Aubin-Lions-Simon lemma one can even obtain the compactness of the aforementioned embedding onto Lebesgue classes, as stated in the next proposition.

Proposition 5.31. *If* $1 < r < \frac{d+2s}{\mu}$, *then* $\mathcal{H}_r^{\mu}(Q_T)$, $\mu \in \mathbb{R}$ *is compactly embedded in* $L^q(Q_T)$ *for* $1 \leq q < \frac{(d+2s)r}{d+2s-\mu r}$

Proof. To show the compactness, we restrict to consider the case $\mu \in (0, 2s]$, the general case being consequence of the isometry property of the Bessel potential operator (see Remark 5.14). The idea is to exploit the so-called Aubin-Lions-Simon Lemma. Let $\mu \in \mathbb{R}$ and $0 < \mu \leq 2s$ with p satisfying $1 < p < \frac{d+2s}{\mu}$. Note first that $H_{p'}^{\mu}(\mathbb{T}^d)$ is reflexive and separable. Therefore the space $L^p(0,T; (H_{p'}^{\mu}(\mathbb{T}^d))')$ is isomorphic to $(L^{p'}(0,T; H_{p'}^{\mu}(\mathbb{T}^d)))' \equiv (\mathbb{H}_{p'}^{\mu}(Q_T))'$. One can easily see that by definition $\mathcal{H}_p^{\mu}(Q_T)$ is isomorphic to

$$E := \{u \in L^p(0,T; H_p^{\mu}(\mathbb{T}^d)), \partial_t u \in L^p(0,T; (H_{p'}^{2s-\mu}(\mathbb{T}^d))'\}$$

Note also that $H_p^{\mu}(\mathbb{T}^d)$ is compactly embedded into $L^p(\mathbb{T}^d)$ by Lemma 5.16-(iv) and $L^p(\mathbb{T}^d)$ is continuously embedded in $(H_{p'}^{2s-\mu}(\mathbb{T}^d))'$ since $\mu \leq 2s$. Then, Aubin-Lions-Simon Lemma (see [219] and [215, Proposition III.1.3]) implies that E is compactly embedded into $L^p(Q_T)$. Hence $\mathcal{H}_p^\mu(Q_T)$ is compactly embedded in $L^q(Q_T)$ for any $1 \le q \le p$. Let u_n be a bounded sequence in $\mathcal{H}_p^\mu(Q_T)$. By the previous discussion we may extract a subsequence u_{n_k} converging to u strongly in $L^p(Q_T)$. For any $p < q < \frac{(d+2s)p}{d+2s-\mu p}$, arguing by interpolation, we may assert the existence of $0 < \theta < 1$ such that

$$\left\| u_{n_k} - u_{n_j} \right\|_{L^q(Q_T)} \le \left\| u_{n_k} - u_{n_j} \right\|_{L^p(Q_T)}^{\theta} \left\| u_{n_k} - u_{n_j} \right\|_{L^{\frac{(d+2s)p}{d+2s-\mu p}}}^{1-\theta} \to 0$$

as $j, k \to +\infty$, since u_{n_k} belongs to $\mathcal{H}_r^\mu(Q_T)$, which is in turn continuously embedded onto $L^{\frac{(d+2s)p}{d+2s-\mu p}}$ in view of Proposition C.3, so u_{n_k} converges strongly also in $L^q(Q_T)$. $\square$

5.3.4 Relation between H_p^μ and $W^{\mu,p}$

We prove the embeddings between $W^{\mu,p}$ and H_p^μ via the trace method. Without going into the details, we mention that when $p = 2$ the space $W^{\mu,2}$ coincides with H_2^μ by properties of Fourier transform (or by the fact that complex and real interpolation couples do agree on Hilbert spaces [178]). For general $p \neq 2$, we follow the lines of [172, Theorem 3.1], shortening their proof by using the decay estimates obtained in Lemma 5.23.

Lemma 5.32. *For every $\varepsilon > 0$, $\mu \in \mathbb{R}$ and $1 < p < \infty$ we have*

$$H_p^{\mu+\varepsilon}(\mathbb{T}^d) \hookrightarrow W^{\mu,p}(\mathbb{T}^d) \hookrightarrow H_p^{\mu-\varepsilon}(\mathbb{T}^d) \ .$$

Proof. Step 1. We first prove that $H_p^{1-\theta+\varepsilon}(\mathbb{T}^d) \hookrightarrow W^{1-\theta,p}(\mathbb{T}^d)$ for every $\varepsilon > 0$ and $\theta \in (0,1)$. To show this, it is sufficient to confine ourselves to the case $\varepsilon < \theta$ since $H_p^\nu(\mathbb{T}^d) \hookrightarrow H_p^\eta(\mathbb{T}^d)$ for every $\nu, \eta \in \mathbb{R}$ such that $\nu > \eta$. Set $\lambda := 1 - \theta + \varepsilon$ and take $u \in H_p^\lambda(\mathbb{T}^d)$. We need to show the existence of $f(t)$ such that

$$t^\alpha f(t) \in L^p(0,1; W^{1,p}(\mathbb{T}^d))$$

$$t^\alpha f'(t) \in L^p(0,1; L^p(\mathbb{T}^d))$$

and

$$f(0) = u$$

are fulfilled, for $\alpha = \theta - 1/p$. Once one finds such $f(t)$, it is sufficient to multiply it by a continuously differentiable function $\zeta(t)$ for $t \in [0, +\infty)$, which vanishes for $t \ge 1$ and it is identically 1 for $t \in [0, 1/2]$ and then set $g(t) = \zeta(t)f(t)$ for $t \in [0,1]$ and $g(t) = 0$ for $t > 1$. As a consequence, it follows that $t^\alpha g(t) \in L^p(0, +\infty; W^{1,p}(\mathbb{T}^d))$, $t^\alpha g'(t) \in L^p(0, +\infty; L^p(\mathbb{T}^d))$ and $g(0) = f(0) = u \in W^{1-\theta,p}(\mathbb{T}^d)$. To reach our goal, we use the solution of the fractional heat equation with $s = 1/2$ and initial data equal to u, that is

$$f(t) := \mathcal{T}_t u \ ,$$

where here $\mathcal{T}_t$ is the semigroup associated to the half-laplacian. It is clear that $f(0) = u$. We show only that $t^\alpha f(t) \in L^p(0,1; W^{1,p}(\mathbb{T}^d))$, the other case being similar. By Lemma 5.23-(i) with $\nu = \lambda$ and $\gamma = \theta - \epsilon > 0$ we have

$$\left(\int_0^1 \|t^\alpha \mathcal{T}_t u\|_{1,p}^p dt\right)^{\frac{1}{p}} \le C_1 \left(\int_0^1 t^{\alpha p} t^{-(\theta-\varepsilon)p} \|u\|_{\lambda,p}^p dt\right)^{\frac{1}{p}}$$
$$\le C_2 \left(\int_0^1 t^{(\alpha-\theta+\varepsilon)p} dt\right)^{\frac{1}{p}} \|u\|_{\lambda,p} \le C_3.$$

Step 2. We claim that for every $\varepsilon > 0$ it results $W^{1-\theta,p}(\mathbb{T}^d) \hookrightarrow H_p^{1-\theta-\varepsilon}(\mathbb{T}^d)$. By isometry (see Remark 5.14), the operator $(I-\Delta)^{\frac{1}{2}}$ maps $W^{1,p}(\mathbb{T}^d)$ onto $L^p(\mathbb{T}^d)$ and $L^p(\mathbb{T}^d)$ onto $W^{-1,p}(\mathbb{T}^d)$. In addition, it also maps $H^{1-\theta+\epsilon}(\mathbb{T}^d)$ onto $H^{-\theta+\epsilon}(\mathbb{T}^d)$. By definition we have that it is also an isometry between

$$T(p, \alpha, W^{1,p}(\mathbb{T}^d), L^p(\mathbb{T}^d)) = W^{1-\theta,p}(\mathbb{T}^d)$$

and

$$T(p, \alpha, L^p(\mathbb{T}^d), W^{-1,p}(\mathbb{T}^d)) = (T(p', -\alpha, W^{1,p'}(\mathbb{T}^d), L^{p'}(\mathbb{T}^d)))'$$
$$= (W^{1-(1/p'-\alpha),p'}(\mathbb{T}^d))' = W^{-\theta,p}(\mathbb{T}^d).$$

By Step 1 we obtain

$$H_p^{-\theta+\varepsilon}(\mathbb{T}^d) \hookrightarrow W^{-\theta,p}(\mathbb{T}^d) \ ,$$

which turns out to hold for every $\varepsilon > 0$. By duality we also conclude $W^{\theta,p'}(\mathbb{T}^d) \hookrightarrow H_{p'}^{\theta-\varepsilon}(\mathbb{T}^d)$ and hence the validity of the claim after replacing θ by $1-\theta$.

Step 3. Suppose $\mu \ge 0$. We first prove the left inclusion $H_p^{\mu+\epsilon}(\mathbb{T}^d) \hookrightarrow W^{\mu,p}(\mathbb{T}^d)$. Let $u \in H_p^{\mu+\epsilon}(\mathbb{T}^d)$. Then $D^k u \in L^p(\mathbb{T}^d)$ for all $|k| \le [\mu]$, where $[\cdot]$ stands for the integer part. On the other hand, $D^k u \in H_p^{\mu+\varepsilon-[\mu]}(\mathbb{T}^d)$ for $k = [\mu]$, which gives by Step 1 $D^k u \in W^{\mu-[\mu],p}(\mathbb{T}^d)$. Then $u \in W^{\mu,p}(\mathbb{T}^d)$. Conversely, if $u \in W^{\mu,p}(\mathbb{T}^d)$, it means that $u \in H_p^{[\mu]}(\mathbb{T}^d)$. Thus in view of Step 2 we obtain $D^k u \in H_p^{\mu-[\mu]-\varepsilon}(\mathbb{T}^d)$, namely $u \in H_p^{\mu-\varepsilon}(\mathbb{T}^d)$ which in turn implies $W^{\mu,p}(\mathbb{T}^d) \hookrightarrow H_p^{\mu-\varepsilon}(\mathbb{T}^d)$. The case $\mu < 0$ follows by the previous one arguing by duality. $\square$

5.4 Linear viscous partial integro-differential equations

In this section we collect some results on second order viscous integro-differential equations the form

$$\mathcal{A}u := (\mathcal{L} + (-\Delta)^s)u = -\sum_{i,j=1}^d a_{ij}\partial_{ij}u + \sum_{i=1}^d b_i\partial_i u + cu + (-\Delta)^s u$$

All the results of this section can be adapted to more general integro-differential operators and for this classical matter we refer to [119], but for our purposes we restrict ourselves to the particular case of the fractional Laplacian. We first begin with two auxiliary results, whose proof is standard and based on the transformation $u(x,t) = v(x,t)e^{\lambda t}$ and the fact that at a maximum point $(-\Delta)^s u(x_0,t_0) \geq 0$. We refer the reader to [159, Theorem I.2.5] for similar arguments used for the case of classical diffusion and [119, Theorem II.2.11] for the adaptation to the nonlocal case.

Proposition 5.33. *Let u be a classical solution of the Cauchy problem*

$$\mathcal{A}u = f \text{ in } Q_T \text{ and } u(x,0) = u_0(x) \text{ in } \mathbb{T}^d \tag{5.26}$$

where the local operator $\mathcal{L}$ is a uniformly parabolic operator with bounded continuous coefficients a_{ij}, b_i and c. Then

$$u(x,\tau) \leq \inf_{\lambda > c_0} \max\left\{0, u_0(x)e^{\lambda(\tau-t)}, \frac{\max_{Q_\tau} f(x,t)e^{\lambda(\tau-t)}}{\lambda - c_0}\right\}$$

for every $\tau \in [0,T]$. Similarly, it holds

$$u(x,\tau) \geq \sup_{\lambda > c_0} \min\left\{0, u_0(x)e^{\lambda(\tau-t)}, \frac{\min_{Q_\tau} f(x,t)e^{\lambda(\tau-t)}}{\lambda - c_0}\right\}$$

Proof. We show only the first inequality, the second one being similar. We use the transformation $v(x,t) = e^{-\lambda t}u(x,t)$ with λ to be specified. Then v solves the parabolic problem

$$\begin{cases} (\mathcal{A}+\lambda)v = fe^{-\lambda t} & \text{in } Q_T \\ v(x,0) = u_0(x) & \text{in } \mathbb{T}^d. \end{cases} \tag{5.27}$$

Let $\tau \in (0,T]$. There are three possibilities:

(i) v is non-positive in $\overline{Q}_\tau$ and hence $\max_{Q_\tau} v \leq 0$.

(ii) v is non-negative and the maximum value is attained at $\mathbb{T}^d \times \{0\}$, giving

$$v(x,t) \leq \max_{\mathbb{T}^d \times \{0\}} v(x,0) = \|u_0\|_\infty$$

(iii) v is non-negative and its greatest value is assumed at some point $(x_0,t_0) \in \mathbb{T}^d \times (0,\tau]$, that is

$$0 \leq \max_{Q_\tau} v(x,t) \leq v(x_0,t_0)$$

In particular, we have

$$\partial_t v(x_0,t_0) \geq 0$$

since, if this were not the case, the inequality $\partial_t v(x_0,t_0) < 0$ would contradict the maximality of (x_0,t_0). Moreover

$$\partial_{x_i} v(x_0,t_0) = 0 \ , -\sigma\Delta u(x_0,t_0) \geq 0 \ , (-\Delta)^s u(x_0,t_0) \geq 0$$

Since v solves (5.27) we have

$$(\lambda - c_0)v(x_0,t_0) \leq f(x_0,t_0)e^{-\lambda t_0} \ ,$$

which gives in turn

$$v(x,\tau) \leq \frac{\max_{Q_\tau} f(x,t)e^{-\lambda t}}{\lambda - c_0}$$

Combining all the assertions and using the definition of v we obtain the desired inequality. $\square$

Corollary 5.34. *Let u be a classical solution of the Cauchy problem*

$$\mathcal{A}u = f \text{ in } Q_T \text{ and } u(x,0) = u_0(x) \text{ in } \mathbb{T}^d \tag{5.28}$$

where the local operator $\mathcal{L}$ is a uniformly parabolic operator with bounded continuous coefficients a_{ij}, b_i and c. Assume that $\|u\|_{\infty;Q_T} \leq M$ for some $M > 0$. Then for $\tau \in [0,T]$

$$\|u\|_{\infty;Q_T} \leq (\|u_0\|_\infty + \tau\|f\|_{\infty;Q_T})e^{c_0\tau}$$

Proof. This is a consequence of Proposition 5.34. $\square$

We have the following comparison principle, whose proof relies on classical arguments used for linear local equations [159].

Lemma 5.35. *Let* (H) *be in force, $\sigma > 0$ and u, v be (classical) sub- and supersolutions to the Cauchy problem*

$$\partial_t u - \sigma\Delta u + (-\Delta)^s u + H(x, Du) = V \text{ in } Q_T \text{ and } u(x,0) = u_0(x) \text{ in } \mathbb{T}^d \tag{5.29}$$

with initial condition $u(x,0)$ and $v(x,0)$ respectively. Then, $w := u - v$ satisfies the estimate

$$\|w\|_{\infty;Q_\tau} \leq C\|w(x,0)\|_\infty e^{C\tau}$$

for some $C > 0$.

Proof. By the regularity assumptions on H, w solves a linear equation of the form

$$\partial_t w - \sigma\Delta w + (-\Delta)^s w + b \cdot Dw = 0$$

equipped with the terminal data $w(x,\tau) = u(x,\tau) - v(x,\tau)$, where the coefficients $b^i := \int_0^1 D_p H(x, Du_\eta)d\eta$, $u_\eta := \eta u + (1-\eta)v$, are bounded. Therefore, by Proposition 5.34 we have the desired estimate. $\square$

Here we list some Schauder type theorems for such equations

Theorem 5.36. *Let $a_{ij}, b_i, c \in C^{\alpha,\alpha/2}(Q_T)$. Then, for any $f \in C^{\alpha,\alpha/2}(Q_T)$ and $u_0 \in C^{2+\alpha}(\mathbb{T}^d)$ there exists a unique solution $u \in C^{2+\alpha,1+\alpha/2}(Q_T)$ such that*

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leq C(\|f\|_{C^{\alpha,\alpha/2}(Q_T)} + \|u_0\|_{C^{2+\alpha}(\mathbb{T}^d)}) \ .$$

Proof. This result is standard and can be found in [119, Theorem II.3.1]. However, we present here the idea on how to get the estimate by using tools from interpolation theory in the simpler case $a_{ij} = \delta_{ij}$ and $b_i = c = 0$, since it constitutes the basis for the next Appendix B. As it will be pointed out in the next sections the realization of the full operator $\Delta - (-\Delta)^s$ is given by the composition of the semigroups associated to the operators (see [232]), namely $\bar{\mathcal{T}}_t := e^{-t(-\Delta)^s}(e^{t\Delta})$ is the semigroup associated to the sum $\Delta - (-\Delta)^s$. The crucial point here is to get the decay estimates

$$\|\bar{\mathcal{T}}_t f\|_{C^{\theta_2}(\mathbb{T}^d)} \leq Ct^{-(\theta_2-\theta_1)/2s}\|f\|_{C^{\theta_1}(\mathbb{T}^d)} \ .$$

for every $0 \leq \theta_1 < \theta_2$, $\theta_1, \theta_2 \in \mathbb{R}$ and $C = C(\theta_1, \theta_2)$. Then, the proof uses the representation via Duhamel's formula and the K-method, see e.g [178]. $\square$

Theorem 5.37. *Let $a_{ij} \in C(Q_T)$, $b_i, c \in L^\infty(Q_T)$. Then, for every $f \in L^p(Q_T)$ and $u_0 \in W^{2-2/p,p}(\mathbb{T}^d)$ there exists a unique solution $u \in W_p^{2,1}(Q_T)$ such that*

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C(\|f\|_{L^p(Q_T)} + \|u_0\|_{W^{2-2/p,p}(\mathbb{T}^d)})$$

where C is a positive constant. The same result is true on the whole space $\mathbb{R}^d$.

Proof. This maximal regularity result can be found in [119]. However, in the simpler case $b_i = c = 0$, results can also be obtained via the abstract approach in [160, 202, 134], since the semigroup $\mathcal{T}_t$ generated by $\Delta - (-\Delta)^s$, satisfying the estimate

$$\|\bar{\mathcal{T}}_t f\|_{2s,p} \leq Ct^{-1}\|f\|_p \ ,$$

is analytic (see [192]). □

5.5 Fractional Fokker-Planck and HJB equations

5.5.1 On the fractional Fokker-Planck equation

In this section we gather some results on fractional Fokker-Planck equations in the periodic setting of the form

$$\begin{cases} \partial_t m - \sigma\Delta m + (-\Delta)^s m + \mathrm{div}(bm) = 0 & \text{in } \mathbb{T}^d \times (0,T) \ , \\ m(x,0) = m_0(x) & \text{in } \mathbb{T}^d \ , \end{cases} \tag{5.30}$$

with $\sigma \geq 0$ and $m_0 \in L^\infty(\mathbb{T}^d)$. When $\sigma = 0$, we expect low regularity of solutions, in particular when $0 < s < 1/2$. In this case we will adopt the usual notion of weak solution, with the following integrability requirements.

Definition 5.38. *Let $b \in L^\infty(Q_T)$. A function*

$$m \in L^2(0,T; H_2^s(\mathbb{T}^d)) = \mathbb{H}_2^s(Q_T) \qquad \text{with} \quad \partial_t m \in L^2(0,T; H_2^{-1}(\mathbb{T}^d)) = \mathbb{H}_2^{-1}(Q_T) \tag{5.31}$$

is a weak solution to (5.30) if, for every $\varphi \in C^\infty(\mathbb{T}^d \times [0,T))$, one has

$$\iint_{Q_T} -m\partial_t\varphi - bm \cdot D\varphi + (-\Delta)^{s-\frac{1}{2}} m(-\Delta)^{\frac{1}{2}}\varphi \, dxdt = \int_{\mathbb{T}^d} \varphi(x,0)m_0(x)\, dx \ .$$

Remark 5.39. It can be verified that (5.31) implies $m \in C([0,T]; H_2^{(s-1)/2}(\mathbb{T}^d))$, see e.g. [101, p. 480]. This suggests, by a density argument, that test functions φ in the previous formulation can be chosen so that $\varphi \in L^2(0,T; H_2^1(\mathbb{T}^d))$ with $\partial_t\varphi \in L^2(0,T; H_2^{-s}(\mathbb{T}^d))$, therefore satisfying $\varphi \in C([0,T]; H_2^{(1-s)/2}(\mathbb{T}^d))$. In this case the integration by parts in time formula holds (with an abuse of notation, integration in space is hiding duality pairings here):

$$\iint_{Q_T} \varphi\partial_t m + m\partial_t\varphi \, dxdt = \int_{\mathbb{T}^d} \varphi(x,T)m(x,T)\, dx - \int_{\mathbb{T}^d} \varphi(x,0)m(x,0)\, dx.$$

Uniqueness of solutions in the subcritical regime $s \in (1/2, 1)$ can be deduced via duality, i.e. exploiting the existence for the adjoint equation, see in particular the next analysis in Chapter 7 where the drift actually belongs to mixed Lebesgue spaces fulfilling a suitable interpolated condition. By linearity, it is sufficient to restrict to the case $m(0) = 0$. Then, by using the solution v to the adjoint equation

$$-\partial_t v + (-\Delta)^s v + b(x,t) \cdot Dv = 0 \text{ in } Q_\tau$$

with $v(x,\tau) = v_\tau(x) > 0$ as a test function in the weak formulation above we get

$$0 = \int_{\mathbb{T}^d} m(0)v(0)\, dx = \int_{\mathbb{T}^d} m(\tau)v(\tau)\, dx \ .$$

This fact will be crucial to run the bootstrap argument in Theorem 5.2-Step 3.

We will need the following estimates independent of σ, for classical solutions of the viscous problem, under the assumption[1] $[\operatorname{div} b]^- \in L^\infty(Q_T)$.

Proposition 5.40. *Let $\sigma \geq 0$, $m_0 \in C(\mathbb{T}^d)$ and $b \in C^1_x(Q_T)$ with $[\operatorname{div} b]^- \in L^\infty(Q_T)$ such that*

$$\|m_0\|_\infty + \|b\|_\infty + \|[\operatorname{div} b]^-\|_\infty \leq K.$$

Then, there exists $C = C(K)$ such that for every classical solution m to (5.30) it holds

$$\|m\|_{\infty;Q_T} \leq C, \tag{5.32}$$

$$\sigma \iint_{Q_T} |Dm|^2\, dxdt + \iint_{Q_T} [(-\Delta)^{s/2} m]^2\, dxdt \leq C, \tag{5.33}$$

$$\|\partial_t m\|_{\mathbb{H}_2^{-1}(Q_T)} \leq C. \tag{5.34}$$

Proof. By standard comparison arguments involving the function

$$w(x,t) := m(x,t)e^{-(K+\varepsilon)t} - \|m_0\|_\infty$$

with $\varepsilon \to 0$ (see e.g. [119, Section II.2] and Proposition 5.34), one concludes

$$\|m\|_{\infty;Q_T} \leq \|m_0\|_\infty e^{KT}.$$

Multiply the equation in (5.30) by m and integrate over Q_T to get

$$\frac{1}{2}\int_0^T \frac{d}{dt}\|m\|^2_{L^2(\mathbb{T}^d)} - \sigma \iint_{Q_T} m\Delta m\, dxdt + \iint_{Q_T} m(-\Delta)^s m\, dxdt = -\iint_{Q_T} m \operatorname{div}(bm) dxdt$$

Using Lemma 5.13 and integrating by parts we have

$$\begin{aligned}\frac{1}{2}\int_0^T \frac{d}{dt}\|m(\cdot,t)\|^2_{L^2(\mathbb{T}^d)} + \sigma \iint_{Q_T} |Dm|^2 dxdt + \iint_{Q_T} [(-\Delta)^{\frac{s}{2}} m]^2\, dxdt &= \iint_{Q_T} mb \cdot Dm\, dxdt \\ &= -\frac{1}{2}\iint_{Q_T} (\operatorname{div} b) m^2\, dxdt. \end{aligned} \tag{5.35}$$

[1] In what follows, we will denote by $[u]^-$ the negative part of u.

Using that $[\mathrm{div}(b)]^- \le K$ and the L^∞ bound on m (one could also argue via Gronwall's lemma), we obtain

$$\frac{1}{2}\|m(T)\|^2_{L^2(\mathbb{T}^d)} + \sigma \iint_{Q_T} |Dm|^2 dx + \iint_{Q_T} [(-\Delta)^{s/2} m]^2\, dx \le C(K) + \frac{1}{2}\|m(0)\|^2_{L^2(\mathbb{T}^d)}$$

which gives the desired inequality (5.33).

The last estimate follows by observing that, using the equation in (5.30),

$$\begin{aligned}\left|\iint_{Q_T} \partial_t m \varphi\, dxdt\right| &\le \|b\|_{L^\infty(Q_T)}\|m\|_{L^2(Q_T)}\|D\varphi\|_{L^2(Q_T)} + \|(-\Delta)^{\frac{s}{2}} m\|_{L^2(Q_T)}\|\varphi\|_{\mathbb{H}^s_2(Q_T)}\\ &\le C\|\varphi\|_{\mathbb{H}^1_2(Q_T)}.\end{aligned}$$

□

5.5.2 On the HJB equation

Semiconcavity estimates

This subsection is devoted to the analysis of semiconcavity properties of solutions to backward fractional HJB equations

$$\begin{cases} -\partial_t u - \sigma\Delta u + (-\Delta)^s u + H(x, Du) = V(x,t) & \text{in } Q_T\ , \\ u(x,T) = u_T(x) & \text{in } \mathbb{T}^d\ , \end{cases} \tag{5.36}$$

We prove in particular that u is semiconcave, with semiconcavity constant depending on the data and independent of σ. First, we stress that when $\sigma = 0$ we mean that u is a weak (energy) solution according to the following

Definition 5.41. *Let $\sigma = 0$ and V be a continuous function on Q_T. We say that $u \in \mathcal{H}^s_2(Q_T)$ with $Du \in L^\infty(Q_T)$ is a weak solution to* (5.36) *if*

$$\begin{aligned}-\int_{\mathbb{T}^d} \varphi(x,T)u_T(x)dx + \iint_{Q_T} \partial_t\varphi u dxdt + \iint_{Q_T} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}}\varphi\, dxdt\\ + \iint_{Q_T} H(x,Du)\varphi dxdt = \iint_{Q_T} V\varphi\, dxdt\end{aligned}$$

for all $\varphi \in C^\infty(\mathbb{T}^d \times (0,T])$.

Remark 5.42. We make a preliminary observation, which we will use in the sequel. Recall that $u \in \mathcal{H}^s_2(Q_T)$ means $u \in L^2(0,T; H^s_2(\mathbb{T}^d))$ with $\partial_t u \in L^2(0,T; H^{-s}_2(\mathbb{T}^d))$. Note that $\mathcal{H}^s_2(Q_T)$ is continuously embedded into $C(0,T; L^2(\mathbb{T}^d))$ in view of [101, Theorem XVIII.2.1]), so this is equivalent to

$$\iint_{Q_T} [-\partial_t u\varphi + (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}}\varphi + H(x,Du)\varphi] dxdt = \iint_{Q_T} V\varphi\, dxdt$$

for all $\varphi \in \mathcal{H}^s_2(Q_T)$, and $u(T) = u_T$ in the L^2-sense. Uniqueness of solutions in this sense in the subcritical case holds by usual energy arguments using the crucial property $Du \in L^\infty$ and the C^1 regularity of H. In fact, it is sufficient to observe that the difference $w = u - v$ is a subsolution to an equation with fractional diffusion and drift, like

$$-\partial_t w + (-\Delta)^s w + b \cdot Dw = 0 \text{ in } Q_T$$

with $w(x,T) = 0$. Then, using the comparison principle (see e.g. Proposition 7.10) we get uniqueness of solutions in the above parabolic class.

Proposition 5.43. *Assume that* $V \in C^{2+\alpha,1+\alpha/2}(Q_T)$, (H1F) *and* (H3F)-(H5F) *hold, and*

$$\|V\|_{C^2_x(Q_T)} + \|u_T\|_{C^2(\mathbb{T}^d)} \le K$$

for some $K > 0$. *Then every classical solution* u *to* (5.36) *satisfies*

$$D^2 u(x,t) \le C\, I \qquad \text{on } Q_T,$$

where C *depends on* K. *As a consequence, we have the gradient bound*

$$\|Du\|_{L^\infty(Q_T)} \le C\sqrt{d}$$

The proof will be accomplished via the so-called adjoint method, that is, by using information of the dual linearized problem. This procedure is particularly effective when the Hamiltonian lacks uniform convexity. Here, we are inspired by some results in [127], see also references therein and those provided in the introduction to Part II of this manuscript. We stress that we do not require convexity of H, but just assumptions (H1F) and (H3F)-(H5F). Generally, for uniformly convex Hamiltonians similar results can be obtained in a more straightforward way through maximum principle arguments (see e.g. [182, 74]). When dealing with non-convex Hamiltonians, the latter approach fails in general.

For any given $\rho_\tau \in C^\infty(\mathbb{T}^d)$, $\rho_\tau \ge 0$, $\tau \in [0,T)$ and $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$ we consider the adjoint equation

$$\begin{cases} \partial_t \rho - \sigma\Delta\rho + (-\Delta)^s\rho - \operatorname{div}(D_pH(x,Du)\rho) = 0 & \text{in } \mathbb{T}^d \times [\tau, T]\ , \\ \rho(x,\tau) = \rho_\tau(x) & \text{on } \mathbb{T}^d\ . \end{cases} \tag{5.37}$$

We have the following preliminary result

Lemma 5.44. *There exists a classical solution* ρ *to* (5.37). *Moreover,*

$$\int_\tau^T \int_{\mathbb{T}^d} |Du|^\gamma \rho\, dxdt \le C,$$

where C *depends on* K *and not on* ρ_τ *nor* τ.

Proof. The well-posedness of (5.37) is a consequence of [119, Theorem II.3.1] and the regularity assumptions on H and u. By multiplying the fractional HJB equation by ρ and the adjoint equation by u, one easily obtains the following formula

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx = \int_{\mathbb{T}^d} u(x,T)\rho(x,T)dx + \int_\tau^T \int_{\mathbb{T}^d} V\rho\, dxdt \tag{5.38}$$

$$+\int_\tau^T \int_{\mathbb{T}^d} (D_pH(x,Du)\cdot Du - H(x,Du))\rho\, dxdt.$$

Then, by (H1) we get

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx \geq \int_\tau^T \int_{\mathbb{T}^d} V\rho\, dxdt + C_H \int_\tau^T \int_{\mathbb{T}^d} |Du|^\gamma \rho\, dxdt - \\ - c_H \int_\tau^T \int_{\mathbb{T}^d} \rho\, dxdt + \int_{\mathbb{T}^d} \rho(x,T)u(x,T)dx. \quad (5.39)$$

Then, since u is a classical solution to (5.36), a standard linearization argument and the application of the Comparison Principle for linear viscous integro-differential PDE (see, e.g. [119, Section II.2]) yield

$$\|u\|_{\infty;Q} \leq \|u_T\|_{\infty;\mathbb{T}^d} + T\big(\|V\|_{\infty;Q} + \|H(\cdot,0)\|_{\infty;\mathbb{T}^d}\big). \quad (5.40)$$

Finally, plugging (5.40) in (5.39) and using the fact that $\|\rho(t)\|_1 = 1$ for all t, we conclude the desired estimate. □

We now prove the semiconcavity estimate.

Proof of Proposition 5.43. Since $V \in C^{2+\alpha,1+\alpha/2}(Q_T)$, by a bootstrap argument u belongs to $C^{4+\alpha,2+\alpha/2}(Q_T)$ (see Proposition 5.48 below). So, we can differentiate twice the equation in any direction $\xi \in \mathbb{R}^d$, $|\xi| = 1$. Observe that $v = \partial_{\xi\xi}u$ satisfies

$$-\partial_t v - \sigma\Delta v + (-\Delta)^s v + D_pH(x,Du)\cdot Dv + D_\xi H(x,Du) = \partial_\xi V\ , \quad v(x,0) = \partial_\xi u(0)$$

and $w = \partial_{\xi\xi}u$ solves

$$-\partial_t w - \sigma\Delta w + (-\Delta)^s w + Dv\cdot D^2_{pp}H(x,Du)Dv + D_pH(x,Du)\cdot Dw + \quad (5.41)$$

$$+2D^2_{p\xi}H(x,Du)\cdot Dv + D^2_{\xi\xi}H(x,Du) = \partial_{\xi\xi}V\ , \quad w(x,0) = \partial_{\xi\xi}u(0)\ .$$

Then, multiply (5.41) by the adjoint variable ρ satisfying (5.37) and integrate over $\mathbb{T}^d \times [\tau,T]$ to get

$$\int_{\mathbb{T}^d} w(x,\tau)\rho_\tau(x)\,dx + \int_\tau^T \int_{\mathbb{T}^d} Dv\cdot D^2_{pp}H(x,Du)Dv\rho\,dxdt = \int_{\mathbb{T}^d} w(x,T)\rho(x,T)\,dx -$$

$$-2\int_\tau^T \int_{\mathbb{T}^d} D^2_{p\xi}H(x,Du)\cdot Dv\rho\,dxdt - \int_\tau^T \int_{\mathbb{T}^d} D^2_{\xi\xi}H(x,Du)\rho\,dxdt + \int_\tau^T \int_{\mathbb{T}^d} \partial_{\xi\xi}V\rho\,dxdt\ .$$

On one hand, by (H5F) we have

$$\int_\tau^T \int_{\mathbb{T}^d} Dv\cdot D^2_{pp}H(x,Du)Dv\rho\,dxdt \geq C_1 \int_\tau^T \int_{\mathbb{T}^d} |Du|^{\gamma-2}|Dv|^2\rho\,dxdt \\ - \tilde{C}_1 \int_\tau^T \int_{\mathbb{T}^d} \rho\,dxdt$$

and hence, using also (H3F)-(H4F), we conclude

$$\int_{\mathbb{T}^d} w(x,\tau)\rho_\tau(x)dx + C_1\int_\tau^T\int_{\mathbb{T}^d}|Du|^{\gamma-2}|Dv|^2\rho\,dxdt - \tilde{C}_1\int_\tau^T\int_{\mathbb{T}^d}\rho\,dxdt$$
$$\leq \int_{\mathbb{T}^d} w(x,T)\rho(x,T)dx + C_2\int_\tau^T\int_{\mathbb{T}^d}|Du|^{\gamma-1}|Dv|\rho dxdt + C_3\int_\tau^T\int_{\mathbb{T}^d}|Du|^\gamma\rho\,dxdt$$
$$+(\tilde{C}_2+\tilde{C}_3)\int_\tau^T\int_{\mathbb{T}^d}\rho\,dxdt + \int_\tau^T\int_{\mathbb{T}^d}V_{\xi\xi}\rho\,dxdt.$$

Now, we apply Young's inequality to the second term on the right-hand side of the above inequality to get

$$\int_\tau^T\int_{\mathbb{T}^d}|Du|^{\gamma-1}|Dv|\rho\,dxdt \leq \frac{\epsilon^2}{2}\int_\tau^T\int_{\mathbb{T}^d}|Du|^{\gamma-2}|Dv|^2\rho dxdt + \frac{1}{\epsilon^2}\int_\tau^T\int_{\mathbb{T}^d}|Du|^\gamma\rho dxdt.$$

Taking ϵ so that $C_1 = \frac{\epsilon^2}{2}$ we finally obtain the estimate

$$\int_{\mathbb{T}^d} w(x,\tau)\rho_\tau(x)dx \leq \int_{\mathbb{T}^d} w(x,T)\rho(x,T)dx + \left(\frac{1}{2C_1}+C_3\right)\int_\tau^T\int_{\mathbb{T}^d}|Du|^\gamma\rho\,dxdt+$$
$$+\int_\tau^T\int_{\mathbb{T}^d}V_{\xi\xi}\rho\,dxdt + \tilde{C}_4\ .$$

During the above computations $C_i = C_i(C_H)$. By Lemma 5.44 we finally deduce the desired semiconcavity estimate after passing to the supremum over ρ_τ. The gradient bound is a straightforward consequence of the fact that

$$\|Du\|_{L^\infty(Q_T)} \leq \sqrt{d}\sup_{(x,t)\in Q_T\ ,|\xi|\leq 1} D^2u(x,t)\xi\cdot\xi$$

□

Remark 5.45. The viscosity parameter σ does not play any role in the above proof, and hence if u is sufficiently regular to perform a differentiation procedure in the classical sense, the above scheme can be carried out with merely fractional diffusion of any order $s\in(0,1)$.

We now turn to space-time Hölder bounds for (forward) fractional HJB equations with bounded right hand side. These will be useful in the vanishing viscosity limit to have uniform convergence of solutions, and therefore to bring to the limit the viscosity notion.

Proposition 5.46. *Let $f\in L^\infty(Q_T)$ and u be a classical solution to*

$$\begin{cases}\partial_t u - \sigma\Delta u + (-\Delta)^s u = f(x,t) & \text{in } Q_T\\ u(x,0)=u_0(x) & \text{in } \mathbb{T}^d.\end{cases}$$

with $u_0\in C^1(\mathbb{T}^d)$. Then

$$\|u\|_{\mathcal{C}^{\alpha,\beta}(Q_T)} \leq C \tag{5.42}$$

for some $\alpha,\beta\in(0,1)$, where the constant C depends only on $\|f\|_{L^\infty(Q_T)}$, $\|u_0\|_{C^1(\mathbb{T}^d)}$ and is independent of σ.

Remark *5.47.* To prove the above result, we need to show the counterpart of Lemma 5.23 for the semigroup $\bar{\mathcal{T}}_t$ generated by the full operator $\sigma\Delta - (-\Delta)^s$. We point out that the two semigroups $e^{-t(-\Delta)^s}$ and $e^{t\sigma\Delta}$ commute, and therefore

$$\bar{\mathcal{T}}_t = e^{-t(-\Delta)^s}(e^{t\sigma\Delta}).$$

by a well-known result due to Trotter [232].

Proof of Proposition 5.46. We observe that by Lemma 5.23-(i) and (5.12), it is straightforward to see that, for $\nu \in \mathbb{R}$, $p > 1$ and $\gamma \geq 0$ we have

$$\|\bar{\mathcal{T}}_t f\|_{\nu+\gamma,p} \leq Ct^{-\gamma/2s} \|f\|_{\nu,p} \ . \tag{5.43}$$

Note that C does not depend on σ here.

Write u using Duhamel's formula, that is $u(t) = u_1(t) + u_2(t)$, where

$$u_1(t) = \bar{\mathcal{T}}_t u_0, \qquad u_2(t) = \int_0^t \bar{\mathcal{T}}_{t-\tau} f(\tau) d\tau.$$

The estimate of $u_1(t) := \bar{\mathcal{T}}_t u_0$ follows using the same argument as in Theorem B.4 and the estimates in Lemma 5.23. We focus on $u_2(t) = \int_0^t \bar{\mathcal{T}}_{t-\tau} f(\tau) d\tau$. Take $\nu = 0$, $\gamma = \frac{s}{p}$ in (5.43) to get

$$\|\bar{\mathcal{T}}_{t-\tau} f\|_{s,p}^p \leq C(t-\tau)^{-1/2} \|f\|_{L^\infty(Q_T)}^p \ .$$

Therefore

$$\|u_2\|_{\mathbb{H}_p^s(Q_T)} = \left(\int_0^T \|u_2(t)\|_{s,p}^p\right)^{\frac{1}{p}} \leq CT^{\frac{3}{2p}} \|f\|_{L^\infty(Q_T)}$$

Since u_2 solves $\partial_t u_2 + (-\Delta)^s u_2 = f$, one has

$$\int_0^T \|\partial_t u_2(t)\|_{-s,p}^p \, dt \leq C_1 \left(\int_0^T \|(-\Delta)^s u_2\|_{-s,p}^p + \|f\|_{-s,p}^p dt\right) \leq C_2 \|f\|_{L^\infty(Q_T)} \ ,$$

yielding the full estimate

$$\|u\|_{\mathcal{H}_p^s(Q_T)} \leq C(\|f\|_{L^\infty(Q_T)} + \|u_0\|_{s-2s/p+\epsilon})$$

for $\varepsilon < \frac{2s}{p}$. Then, for $p > \frac{d+2s}{s}$, by Sobolev embedding theorems in Proposition 5.19 we conclude

$$\|u\|_{\mathcal{C}^{\alpha,\beta}(Q_T)} \leq C \|u\|_{\mathcal{H}_p^s(Q_T)} \leq C_1 \ .$$

□

Existence of solutions

In this section we prove an existence result for backward integro-differential HJB equations of the form

$$\begin{cases} -\partial_t u - \Delta u + (-\Delta)^s u + H(x, Du) = V(x,t) & \text{on } Q_T \ , \\ u(x,T) = u_T(x) & \text{on } \mathbb{T}^d \ . \end{cases} \tag{5.44}$$

Proposition 5.48. *Let $V \in C^{2+\alpha,1+\alpha/2}(Q_T)$, H satisfying* (H1F)-(H5F) *and $u_T \in C^{4+\alpha}(\mathbb{T}^d)$. Then, there exists a unique solution $u \in C^{4+\alpha,2+\alpha/2}(Q_T)$ to* (5.44)*, and the following estimate holds*

$$\|u\|_{C^{4+\alpha,2+\alpha/2}(Q_T)} \le C(\|V\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + \|u_T\|_{C^{4+\alpha}(\mathbb{T}^d)}) \ . \tag{5.45}$$

The crucial step to obtain this existence result are the semiconcavity estimates of the previous section, that yield a priori gradient bounds of solutions. Then, the construction of a solution follows by standard arguments. Since we were not able to find a similar result in the literature, we detail the proof here for the convenience of the reader.

Proof. Step 1: Local existence on $Q_\tau = \mathbb{T}^d \times (T-\tau, T)$. Let $\tau \le 1$ and

$$\mathcal{S}_a := \left\{ u \in W^{2,1}_p(Q_\tau) : u(T) = u_T \ , \|u\|_{W^{2,1}_p(Q_\tau)} \le a \ , p > d+2 \right\}$$

be the space on which we apply the contraction mapping principle. The parameter a will be chosen large enough. Fix $z \in W^{2,1}_p(Q_\tau)$, $p > d+2$ and let $w = Jz$ be the solution of the problem

$$\begin{cases} -\partial_t w - \Delta w = V - H(x, Dz) - (-\Delta)^s z & \text{in } \mathbb{T}^d \times (T-\tau, T] \ , \\ w(x,T) = u_T(x) & \text{in } \mathbb{T}^d. \end{cases} \tag{5.46}$$

We recall that since $W^{2,1}_p(Q_\tau)$ is continuously embedded onto $C([0,\tau]; W^{2-2/p,p}(\mathbb{T}^d))$ and hence $Dz \in C([0,\tau]; W^{1-2/p,p}(\mathbb{T}^d))$. By Sobolev embedding (see Lemma 5.17) it turns out that $W^{1-2/p,p}(\mathbb{T}^d)$ is continuously embedded onto an Hölder class for $p > d+2$ and hence we have $Dz \in L^\infty(Q_\tau)$ for the range of p chosen in $\mathcal{S}_a$. By standard (local) parabolic regularity theory (see [159, Theorem IV.9.1] or [91]), since the right hand side of the equation in (5.46) is in $L^p(Q_\tau)$, (5.46) admits a unique solution $w \in W^{2,1}_p(Q_\tau)$ satisfying the following estimate

$$\|w\|_{W^{2,1}_p(Q_\tau)} \le C(\|V\|_{L^p(Q_\tau)} + \|H(x,Dz)\|_{L^p(Q_\tau)} + \|(-\Delta)^s z\|_{L^p(Q_\tau)} + \|u_T\|_{W^{2-2/p,p}(\mathbb{T}^d)}).$$

We show that we can choose $\tau \in (0,T]$ sufficiently small so that $\|Jz\|_{W^{2,1}_p(Q_\tau)} \le a$. By [91, Lemma 2.4]

$$\|H(x,Dz)\|_{L^p(Q_\tau)} \le C_1 \tau^{\frac{1}{2p}} \|H(x,Dz)\|_{L^{2p}(Q_\tau)} \le C_2 \tau^{\frac{1}{2p}} \|Dz\|^\gamma_{\infty;Q_\tau}$$

Moreover, by [91, Proposition 2.5] we have

$$\|Dz\|_{\infty;Q_\tau} \le C_3(\|z\|_{W^{2,1}_p(Q_\tau)} + \|u_T\|_{W^{2-2/p,p}(\mathbb{T}^d)}) \ ,$$

which gives

$$\|H(x,Dz)\|_{L^p(Q_\tau)} \le C_4 \tau^{\frac{1}{2p}} (\|z\|^\gamma_{W^{2,1}_p(Q_\tau)} + \|u_T\|^\gamma_{W^{2-2/p,p}(\mathbb{T}^d)}) \ .$$

Concerning the fractional term we observe that if either $s \in (0, \frac{1}{2})$ or $s \in \left[\frac{1}{2}, 1\right)$, then by Lemma 5.15 we get for some $\delta > 0$

$$\|(-\Delta)^s z\|_{L^p(Q_\tau)} \le \delta \|z\|_{W^{2,1}_p(Q_\tau)} + C(\delta) \|z\|_{L^p(Q_\tau)}$$

where $C(\delta) > 0$ grows as δ approaches to 0. Then, note that by writing

$$z(\cdot, s) = u_T(\cdot) - \int_s^T \partial_t z(\cdot, \omega) d\omega \ ,$$

we obtain

$$\|z\|_{L^p(Q_\tau)} \leq \tau^{\frac{1}{p}} \|u_T\|_{L^p(\mathbb{T}^d)} + \tau \|\partial_t z\|_{L^p(Q_\tau)} \ .$$

Then

$$\begin{aligned}
\|w\|_{W_p^{2,1}(Q_\tau)} &\leq C \Big[\max\{\|z\|_{W_p^{2,1}(Q_\tau)}, \|z\|^\gamma_{W_p^{2,1}(Q_\tau)}\}(\tau^{\frac{1}{2p}} + C(\delta)\tau + \delta) \\
&\quad + (\tau^{\frac{1}{p}} + \tau^{\frac{1}{2p}}) \max\{\|u_T\|_{L^p(\mathbb{T}^d)}, \|u_T\|^\gamma_{W^{2-2/p,p}(\mathbb{T}^d)}\} + \|V\|_{L^p(Q_\tau)} \Big] \\
&\leq C \Big[\max\{\|z\|_{W_p^{2,1}(Q_\tau)}, \|z\|^\gamma_{W_p^{2,1}(Q_\tau)}\}(\tau^{\frac{1}{2p}}(1 + C(\delta)) + \delta) \\
&\quad + 2\tau^{\frac{1}{2p}} \max\{\|u_T\|_{L^p(\mathbb{T}^d)}, \|u_T\|^\gamma_{W^{2-2/p,p}(\mathbb{T}^d)}\} + \|V\|_{L^p(Q_\tau)} \Big] \ .
\end{aligned}$$

At this stage, take

$$a \geq C \left(2 \max\{\|u_T\|_{L^p(\mathbb{T}^d)}, \|u_T\|^\gamma_{W^{2-2/p,p}(\mathbb{T}^d)}\} + \|V\|_{L^p(Q_\tau)} \right) + 2$$

to get

$$\|w\|_{W_p^{2,1}(Q_\tau)} \leq C \left\{ \max\{\|z\|_{W_p^{2,1}(Q_\tau)}, \|z\|^\gamma_{W_p^{2,1}(Q_\tau)}\} \left[(1 + C(\delta))\tau^{\frac{1}{2p}} + \delta \right] \right\} + a - 2 \ .$$

Then, choose $\delta \leq \frac{1}{Ca}$ so that

$$\|w\|_{W_p^{2,1}(Q_\tau)} \leq C \max\{\|z\|_{W_p^{2,1}(Q_\tau)}, \|z\|^\gamma_{W_p^{2,1}(Q_\tau)}\}(1 + C(\delta))\tau^{\frac{1}{2p}} + a - 1$$

and finally τ small to conclude

$$\|w\|_{W_p^{2,1}(Q_\tau)} \leq a \ .$$

This shows that J maps $\mathcal{S}_a$ into itself.

To prove that J is a contraction, one has to argue as above, exploiting also the fact that

for bounded $z \in W_p^{2,1}(Q_\tau)$, $p > d + 2$, then Dz is bounded in $L^\infty(Q_\tau)$ and hence, by using (H2F) we have

$$\|H(x, Dz_1) - H(x, Dz_2)\|_{L^p(Q_\tau)} \leq C \, \|D(z_1 - z_2)\|_{L^p(Q_\tau)}$$

for some positive constant C. Therefore, by using interpolation inequalities (see e.g. [119, Proposition I.1.8]) we have

$$\begin{aligned}
\|H(x, Dz_1) - H(x, Dz_2)\|_{L^p(Q_\tau)} &\leq C(\eta \|z_1 - z_2\|_{W_p^{2,1}(Q_\tau)} + \eta^{-1} \|z_1 - z_2\|_{L^p(Q_\tau)}) \\
&\leq C(\eta \|z_1 - z_2\|_{W_p^{2,1}(Q_\tau)} + \eta^{-1} T \|\partial_t (z_1 - z_2)\|_{L^p(Q_\tau)})
\end{aligned}$$

A similar procedure allows to handle the fractional term as

$$\|(-\Delta)^s z_1 - (-\Delta)^s z_2\|_{L^p(Q_\tau)} \leq \delta \|z_1 - z_2\|_{W_p^{2,1}(Q_\tau)} + C(\delta)T\|\partial_t(z_1 - z_2)\|_{L^p(Q_\tau)})$$

and by choosing δ, η and τ small enough one concludes

$$\|Jz_1 - Jz_2\|_{W_p^{2,1}(Q_{\bar{T}})} \leq \frac{1}{2}\|z_1 - z_2\|_{W_p^{2,1}(Q_{\bar{T}})} ,$$

which ensures the existence of a unique fixed point, $z = Jz$, i.e. a solution z of the HJB equation in the interval $(T - \tau, T]$.
Now note that by Sobolev embedding, if $p > d + 2$, then $u \in C^{1+\alpha, \frac{1+\alpha}{2}}(Q_\tau)$. Then a bootstrap argument allows to conclude $u \in C^{4+\alpha,2+\alpha/2}(Q_\tau)$, since $V \in C^{2+\alpha,1+\alpha/2}(Q_\tau)$.

Step 2. Define

$$T^* := \inf\{\tau \in [0,T] : (5.44) \text{ admits a solution } C^{4+\alpha,2+\alpha/2}(Q_\tau)\}$$

In view of Step 1 we claim that the above set is nonempty. We want to show that $T^* \leq 0$. To this aim, take a sequence $\{(\tau_k, u_k)\}$ in $(T^*, T) \times W_p^{2,1}(\overline{Q}_{\tau_k})$, where τ_k converges decreasingly to T^* and u_k solves (5.44) in $\overline{Q}_{\tau_k}$. Since, by Sobolev Embedding, $u_k \in C^{4+\alpha,2+\alpha/2}(Q_{\tau_k})$, we have that u_k is semiconcave independently on k. Being also bounded by the Comparison Principle for classical solutions of integro-differential uniformly parabolic equations (see [119, Corollary II.2.18], there exists $C > 0$ such that

$$\|Du_k\|_{L^\infty(Q_\tau)} \leq C \qquad \forall k \in \mathbb{N}$$

(see [66, Remark 2.1.8]). Arguing as in Step 1, by [159, Theorem IV.9.1] we claim that u_k satisfies

$$\|u_k\|_{W_p^{2,1}(Q_\tau)} \leq C. \tag{5.47}$$

In particular the solution turns out to be classical by bootstrapping and [119, Theorem II.3.1]. Again by the Comparison Principle, we also have

$$u_k = u_h \text{ on } \overline{Q}_{\tau_h} \text{ for every } k \geq h \; . \tag{5.48}$$

We define a function $u : \mathbb{T}^d \times [T^*, T] \to \mathbb{R}$ by setting $u = u_k$ on $\overline{Q}_{\tau_k}$ for every $k \in \mathbb{N}$ and then by taking its continuous extension to $\mathbb{T}^d \times [T^*, T]$. Moreover, it solves the Cauchy problem on $\mathbb{T}^d \times [T^*, T]$ by continuity of $u, \partial_t u, Du, D^2 u$ (using the results for parabolic Hölder spaces, since, as claimed above, at the end u has classical regularity). If, by contradiction, $T^* > 0$, one argues as in Step 1 to find $w \in W_p^{2,1}(Q_\tau)$ which solves

$$-\partial_t w - \Delta w + (-\Delta)^s w + H(x, Dw) = V \text{ on } Q_\tau, \qquad w(\cdot, T) = u(\cdot, T^*) \text{ on } \mathbb{T}^d$$

(basically one applies the local existence to the backward equation with datum in T^*) which at the end will have $C^{4+\alpha,2+\alpha/2}$ regularity. One can check that

$$u^*(x,t) = \begin{cases} u(x,t) & \text{if } (x,t) \in \mathbb{T}^d \times [T^*, T] \; , \\ w(x, T + t - T^*) & \text{if } (x,t) \in \mathbb{T}^d \times [T^* - \tau, T^*] \end{cases}$$

belongs to $C^{4+\alpha,2+\alpha/2}(\mathbb{T}^d \times [T^* - \tau, T])$ and solves the problem on $\mathbb{T}^d \times [T^* - \tau, T]$, contradicting the minimality of T^*.

□

5.6 Existence for the MFG system

This section is devoted to the proofs of existence for systems (5.1) and (5.2). We begin by the viscous case, then proceed with the vanishing viscosity procedure.

5.6.1 The viscous case

Proof of Theorem 5.1. The statement is a consequence of the Schauder fixed point theorem (see [122, Corollary 11.2]). Let

$$\mathcal{X} = C^{1+\alpha/2}([0,T];\mathcal{P}(\mathbb{T}^d))$$

and

$$\mathcal{C} = \{m \in \mathcal{X} : \|m\|_{C^{1+\alpha/2}([0,T];\mathcal{P}(\mathbb{T}^d))} \leq \overline{C}\}.$$

It is straightforward to see that $\mathcal{C}$ is closed and convex. We construct a map $T : \mathcal{C} \to \mathcal{C}$ in the following way: given $\mu \in \mathcal{C}$, let u be the unique solution to

$$\begin{cases} -\partial_t u - \sigma\Delta u + (-\Delta)^s u + H(x, Du) = F[\mu](x) & \text{in } \mathbb{T}^d \times (0,T) , \\ u(x,T) = u_T(x) & \text{in } \mathbb{T}^d . \end{cases} \tag{5.49}$$

Then, we define $m = T(\mu)$ as the solution to the fractional Fokker-Planck equation

$$\begin{cases} \partial_t m - \sigma\Delta m + (-\Delta)^s m - \operatorname{div}(mD_pH(x, Du)) = 0 & \text{in } \mathbb{T}^d \times (0,T) , \\ m(x,0) = m_0(x) & \text{in } \mathbb{T}^d . \end{cases} \tag{5.50}$$

We divide the proof in three steps.

Step 1. T is well-defined. To show that the map T is well-defined, first note that, since $\mu \in C_t^{1+\alpha/2}(Q_T)$, by the assumptions on F we have $F[\mu] \in C^{2+\alpha,1+\alpha/2}(Q_T)$; in particular, $F[\mu]$ is bounded in $C^{2+\alpha,1+\alpha/2}(Q_T)$ independently with respect to μ. By Proposition 5.48, problem (5.49) has a unique classical solution belonging to $C^{4+\alpha,2+\alpha/2}(Q_T)$, and satisfies the a priori estimate

$$\|u\|_{C^{4+\alpha,2+\alpha/2}(Q_T)} \leq C_1$$

where C_1 in particular depends on $\|u_T\|_{C^{4+\alpha}(\mathbb{T}^d)}$, but does not depend on μ. Then, we can expand the divergence term of the viscous fractional Fokker-Planck equation as

$$\partial_t m - \sigma\Delta m + (-\Delta)^s m - D_pH(x, Du) \cdot Dm - m \operatorname{div}(D_pH(x, Du)) = 0 ,$$

which turns out to be a linear equation with coefficients belonging to $C^{2+\alpha,1+\alpha/2}(Q_T)$, uniformly with respect to μ. Indeed $\operatorname{div}(D_pH(x, Du)) \in C^{2+\alpha,1+\alpha/2}(Q_T)$ owing to [154, Remark 8.8.7]. This gives that

$$\|m\|_{C^{4+\alpha,2+\alpha/2}(Q_T)} \leq C_2 \tag{5.51}$$

by [119, Theorem II.3.1]. In particular, the map T is well-defined from $\mathcal{C}$ into itself by choosing $\overline{C}$ above large enough.

Step 2. T is continuous. To this aim, let $\mu_n \in \mathcal{C}$ converging to some μ. Let $(u_n, m_n), (u, m)$ be the corresponding solutions. By the continuity assumption (F1) we conclude that the map $(x,t) \longmapsto F[\mu_n(t)](x)$ uniformly converge to $(x,t) \longmapsto F[\mu(t)](x)$. We can then consider the equation

$$-\partial_t u_n - \sigma\Delta u_n + (-\Delta)^s u_n + H(x, Du_n) = F[\mu_n(t)](x)$$

whose right-hand side $F[\mu_n(t)](x)$ is uniformly bounded in $C^{2+\alpha,1+\alpha/2}(Q_T)$. Then the sequence $\{u_n\}$ is uniformly bounded in $C^{4+\alpha,2+\alpha/2}(Q_T)$ in view of Proposition 5.48 and thus converges in $C^{4,2}$ to the unique solution u of the HJB equation. As before, the m_n are solutions of a linear equation with Hölder continuous coefficients, providing uniform estimates in $C^{4+\alpha,2+\alpha/2}(Q_T)$ for $\{m_n\}$. Therefore $\{m_n\}$ converges in $C^{4,2}$ to the unique solution m of the Fokker-Planck equation. Note that the convergence holds also in $\mathcal{C}$.

Step 3. $\overline{T(\mathcal{C})}$ is compact. By bounds (5.51), one proves that for every $\mu_n \in \mathcal{C}$, the sequence $m_n = T(\mu_n)$ has a convergent subsequence. □

5.6.2 The vanishing viscosity limit

We emphasize in passing that in the limiting procedure $\sigma \to 0$ for the HJ equation, one passes from classical parabolic $W_p^{2,1}$ regularity to fractional parabolic $\mathcal{H}_p^{2s}(Q_T)$ regularity. Similar phenomena occurs in the case of the Fokker-Planck equation. The strategy will thus be to pass to the limit in some suitable weak sense, and then recover maximal regularity by means of Theorem B.4.

Proof of Theorem 5.2. Let (u_σ, m_σ) be a solution of (5.2). For $\sigma > 0$ we know that a solution exists in view of Theorem 5.1. Collecting the results in Proposition 5.40, Proposition 5.43 and Proposition 5.46, we are able to construct a sequence $\sigma = \{\sigma_n\} \to 0$ such that, if (u_σ, m_σ) is the corresponding solution, we have

(i) u_σ converges to u in $C(Q_T)$ as a consequence of the estimate (5.42) and Ascoli-Arzelá Theorem. Moreover, one easily has bounds for u_σ in $\mathcal{H}_2^s$, so $u_\sigma \to u$ weakly in $\mathcal{H}_2^s$.

(ii) The semiconcavity estimates in Proposition 5.43 yield $Du_\sigma \to Du$ a.e. in Q_T in view of [66, Theorem 3.3.3]. In addition, by [66, Remark 2.1.8] they also imply uniform bounds for Du_σ in $L^\infty(Q_T)$, so $Du_\sigma \to Du$ in the L^∞-weak-$*$ sense. Finally, u is semiconcave with the same semiconcavity bounds.

(iii) By (ii) and dominated convergence theorem $Du_\sigma \to Du$ in $L^p(Q_T)$ for every finite $p \geq 1$.

(iv) As a consequence of the semiconcavity estimates, we have $[\mathrm{div}(b)]^- \leq C$, where $b = -D_pH(x, Du_\sigma)$. Indeed

$$\mathrm{div}(-D_pH(x, Du_\sigma)) = -\sum_{i,j} D^2_{p_i x_j}H - \sum_{i,j} D^2_{p_i p_j}H\partial_{x_i x_j}u_\sigma \geq -\overline{C}.$$

The first term can be controlled by (ii) and (H4). Since $0 \leq D^2_{pp}H(x, Du) \leq C_1 I_d$ and $D^2 u_\sigma \leq CI_d$, we have a control on the second term by a constant independent of σ.

(v) In view of the estimate (5.32), m_σ converges to $m \in L^\infty(Q_T)$, weakly-$*$ in L^∞.

(vi) Proposition 5.40 ensures that m_σ, $\partial_t m_\sigma$ are bounded uniformly with respect to σ in $\mathbb{H}_2^s(Q_T)$ and $\mathbb{H}_2^{-1}(Q_T)$ respectively, so they weakly converge.

In addition, note that $(x,t) \longmapsto F[m_\sigma(t)](x)$ uniformly converges to the map $(x,t) \longmapsto F[m(t)](x)$. We now pass to the limit in the weak formulation of both equations.

Step 1. Fokker-Planck Equation. Multiplying the Fokker-Planck equation by a test function $\varphi \in C^\infty(\mathbb{T}^d \times [0,T))$ and integrating over Q_T we get

$$-\int_{\mathbb{T}^d} m_\sigma(x,0)\varphi(x,0)dx - \iint_{Q_T} m_\sigma \partial_t \varphi dxdt - \sigma \iint_{Q_T} m_\sigma \Delta\varphi dxdt$$
$$+ \iint_{Q_T} (-\Delta)^{s/2} m_\sigma (-\Delta)^{s/2}\varphi dxdt + \iint_{Q_T} m_\sigma D_p H(x, Du_\sigma)\cdot D\varphi dxdt = 0 \quad (5.52)$$

We then let $\sigma \to 0$ to conclude

$$-\int_{\mathbb{T}^d} m(x,0)\varphi(x,0)dx - \iint_{Q_T} m\partial_t\varphi dxdt + \iint_{Q_T} (-\Delta)^{s/2} m(-\Delta)^{s/2}\varphi dxdt +$$
$$+ \lim_{\sigma\to 0} \iint_{Q_T} m_\sigma D_p H(x, Du_\sigma)\cdot D\varphi dxdt = 0 \ ,$$

by the convergence of m_σ stated in (v)-(vi). It remains to prove

$$\iint_{Q_T} m_\sigma D_p H(x, Du_\sigma)\cdot D\varphi dxdt \to \iint_{Q_T} m D_p H(x, Du)\cdot D\varphi dxdt \ .$$

We write

$$\left|\iint_{Q_T} (m_\sigma D_p H(x,Du_\sigma) - m D_p H(x,Du))\cdot D\varphi \, dxdt\right| \le$$
$$\le \iint_{Q_T} |m_\sigma D_p H(x,Du_\sigma) - m_\sigma D_p H(x,Du)|\,|D\varphi|\, dxdt$$
$$+ \iint_{Q_T} |m_\sigma D_p H(x,Du) - m D_p H(x,Du)|\,|D\varphi|\, dxdt \ .$$

The first term on the right-hand side of the above inequality can be handled using (iii)-(v)

$$\iint_{Q_T} |m_\sigma (D_p H(x,Du_\sigma) - D_p H(x,Du))|\,|D\varphi| dxdt$$
$$\le C\,\|m_\sigma\|_{L^\infty(Q_T)}\,\|D_p H(x,Du_\sigma) - D_p H(x,Du)\|_{L^1(Q_T)} \ ,$$

Now observe that one can use the regularity of H together with the fact that

$$D_p H(x,Du_\sigma) - D_p H(x,Du) = \int_0^1 D^2_{pp} H(x, Du + \theta(Du_\sigma - Du))(Du_\sigma - Du)d\theta$$

to get, using also Hölder's inequality with exponents (p, q),

$$\| |D_pH(x, Du_\sigma) - D_pH(x, Du)| \|_{L^1(Q_T)} \le C \|Du_\sigma - Du\|_{L^q(Q_T)}$$

and concluding exploiting the convergence of Du_σ to Du in L^q for every finite $q \ge 1$. Finally,

$$\iint_{Q_T} (m_\sigma - m) D_pH(x, Du) \cdot D\varphi dxdt \to 0$$

in view of the L^∞ weak-$*$ convergence m_σ to m and the fact that

$$\|D_pH(x, Du)\|_{L^1(Q_T)} \le C \|Du\|^{\gamma-1}_{L^{\gamma-1}(Q_T)} < \infty \ .$$

Step 2. The HJB equation. We now pass to the limit in the fractional HJB equation. Multiplying the equation satisfied by u_σ by a test function $\varphi \in C^\infty(\mathbb{T}^d \times (0, T])$ we get

$$\begin{aligned} -\iint_{Q_T} \partial_t u_\sigma \varphi dxdt - \sigma \iint_{Q_T} \Delta u_\sigma \varphi dxdt + \iint_{Q_T} (-\Delta)^s u_\sigma \varphi dxdt \\ + \iint_{Q_T} H(x, Du_\sigma)\varphi dxdt = \iint_{Q_T} F[m_\sigma(t)]\varphi dxdt \end{aligned}$$

We now integrate by parts using Lemma 5.13 to obtain

$$\begin{aligned} &-\int_{\mathbb{T}^d} u_\sigma(x, T)\varphi(x, T)dx + \iint_{Q_T} u_\sigma \partial_t \varphi dxdt + \sigma \iint_{Q_T} Du_\sigma \cdot D\varphi dxdt \\ &+ \iint_{Q_T} (-\Delta)^{\frac{s}{2}} u_\sigma (-\Delta)^{\frac{s}{2}} \varphi dxdt + \iint_{Q_T} H(x, Du_\sigma)\varphi dxdt = \iint_{Q_T} F[m_\sigma(t)]\varphi dxdt. \end{aligned}$$

Now note that (iii) together with Lemma 5.15 implies also that $(-\Delta)^{\frac{s}{2}} u_\sigma \to (-\Delta)^{\frac{s}{2}} u$ in $L^p(Q_T)$. By the regularity assumptions of the coupling F, the term on the right-hand side converges to $\iint_{Q_T} F[m(t)]\varphi dxdt$ as $\sigma \to 0$. We only need to prove that

$$\iint_{Q_T} H(x, Du_\sigma)\varphi dxdt \to \iint_{Q_T} H(x, Du)\varphi dxdt$$

as $\sigma \to 0$. To this aim we argue as above using the assumptions on H and the convergence of Du_σ to Du in L^p for every finite $p \ge 1$.

Step 3. Recall that the energy solution u belongs to the parabolic class $\mathcal{H}^s_2(Q_T)$. Moreover, when $s > 1/2$ weak solutions of fractional Hamilton-Jacobi equations are unique in view of Remark 5.42. Therefore, since $Du \in L^\infty$, one can regard the equation as a perturbation of a fractional heat equation to get $u \in \mathcal{H}^{2s}_p(Q_T)$ for every $p > 1$ via Theorem B.4. As for the solution of the Fokker-Planck equation, we note that in the regime $s \in (0, 1/2]$ we have that $m \in \mathbb{H}^s_2(Q_T)$ with $\partial_t m \in \mathbb{H}^{-1}_2(Q_T)$. However, in the subcritical case $s \in (1/2, 1)$, m belongs also to $\mathcal{H}^{2s-1}_2(Q_T)$ and uniqueness within this class holds by duality (see Remark 5.39). Therefore, since a posteriori m is also bounded, we have $m \in \mathcal{H}^{2s-1}_p(Q_T)$ for every $p > 1$.

Step 4. Finally, if $s > 1/2$ one can set up a bootstrap procedure to obtain classical regularity. This will be proven in the following Theorem 5.50.

□

Remark 5.49. By uniform convergence of u_σ and $F[m_\sigma]$ on Q_T we can also conclude that the limit u solves the HJB equation in (5.1) in the viscosity sense.

5.6.3 Classical regularity in the subcritical case $s > 1/2$

In what follows, we will assume that

$$\frac{1}{2} < s < 1.$$

We aim at proving that (u, m) previously found in Theorem 5.2 solves the MFG system in the classical sense. We stress that for a (linear) bootstrap procedure to be performed, s must be greater than $1/2$, because the Hamiltonian and divergence terms deteriorate the regularity of the unknowns up to one derivative, while the gain realized by the fractional Laplacian is of order $2s$.

Theorem 5.50. *Let $s \in (\frac{1}{2}, 1)$ and (u, m) be a solution to* (5.2) *(in the sense of Definitions 5.38 and 5.41). Then u, m both satisfy* (B.2) *for some $0 < \bar{\alpha} < 1$, and in particular solve* (5.2) *in the classical sense. Moreover, there exists a constant $C > 0$ depending on the data and remaining bounded for bounded values of T such that*

$$\|m\|_\infty + \|Du\|_\infty \leq C.$$

Proof of Theorem 5.50. We first observe that since $m \in \mathcal{H}_p^{2s-1}(Q_T)$ for all $p > 1$, by Proposition 5.29 we have that m is bounded in $\mathcal{C}^{\bar{\alpha}, \frac{\bar{\alpha}}{2s}}(Q_T)$ for some $0 < \bar{\alpha} < 1$, by choosing p large enough. Therefore, in view of (F3), $F[m] \in C^{\bar{\alpha}/2s}([0, T]; C^{2+\alpha}(\mathbb{T}^d))$, that is in turn embedded in $\mathbb{H}_p^2$ for all $p > 1$.

Note that u solves the following equation

$$-\partial_t u + (-\Delta)^s u = G(x, t), \qquad u(x, T) = u_T(x),$$

where $G(x, t) := F[m(t)](x) - H(x, Du(x, t))$, and $Du \in L^\infty$. Then, at first glance, $G \in L^p(Q_T)$ for all p. This yields $u \in \mathcal{H}_p^{2s}(Q_T)$ by applying Theorem B.4, and in particular $Du \in \mathbb{H}_p^{2s-1}(Q_T)$. Then $H(x, Du) \in \mathbb{H}_p^{2s-1-\varepsilon}(Q_T)$ by the fractional chain rule in Lemma A.2, so $G \in \mathbb{H}_p^{2s-1-\varepsilon}(Q_T)$. Using that $s > \frac{1}{2}$ and taking ε small, we can iterate this procedure until, in a finite number of steps, $G \in \mathbb{H}_p^2(Q_T)$, that is the maximal regularity allowed by $F[m] \in \mathbb{H}_p^2(Q_T)$. Another iteration yields $u \in \mathcal{H}_p^{2+2s}(Q_T)$ for all $p > 1$. Since $2+2s > 3$, we can apply Theorem 5.19 with p large and β close to zero to obtain $u \in C^{\alpha_1}([0, T]; C^{3+\alpha_2}(\mathbb{T}^d))$, for some $0 < \alpha_1, \alpha_2 < 1$, thus $H(x, Du) \in C^{\alpha_1}([0, T]; C^{2+\alpha_2}(\mathbb{T}^d))$. As a consequence, $G \in \mathcal{C}^{\bar{\alpha}, \frac{\bar{\alpha}}{2s}}(Q_T)$, possibly for a smaller $\bar{\alpha}$ than the one appeared at the beginning of the proof. So, Theorem B.1 applies, providing the desired regularity for u.

Let us now focus on the Fokker-Planck equation. By similar arguments we have that $D_pH(x, Du) \in \mathbb{H}_p^{1+2s-\varepsilon} \cap L^\infty(Q_T)$. Moreover, $m \in \mathcal{H}_p^{2s-1} \cap L^\infty(Q_T)$, so by Lemma A.1 we obtain that $\mathrm{div}(mD_pH(x, Du)) \in \mathbb{H}_p^{2s-2}(Q_T)$. An application of fractional parabolic regularity stated in Theorem B.4 provides $m \in \mathcal{H}_p^{4s-2}(Q_T)$. We may iterate this procedure until we get $m \in \mathcal{H}_p^{2s+1} \cap L^\infty(Q_T)$, and another time to conclude $m \in \mathcal{H}_p^{4s-\varepsilon}(Q_T)$ for all $p > 1$. Since $4s > 2$, we can use Theorem 5.19 with p large and β small to get $m \in C^{\alpha_3}([0, T]; C^{1+\alpha_4}(\mathbb{T}^d))$, for some $0 < \alpha_3, \alpha_4 < 1$. Since we previously obtained $D_pH(x, Du) \in C^{\alpha_1}([0, T]; C^{2+\alpha_2}(\mathbb{T}^d))$, we finally have $\mathrm{div}(mD_pH(x, Du)) \in \mathcal{C}^{\bar{\alpha}, \frac{\bar{\alpha}}{2s}}(Q_T)$, reducing eventually the value of $\bar{\alpha}$ previously chosen. We deduce the stated regularity for m again from Theorem B.1.

Last, the estimate on the sup-norm of Du on Q_T follows by comparison and semi-concavity bounds. Note that Proposition 5.43 applies in view of $C^{\alpha_1}([0,T];C^{3+\alpha_2}(\mathbb{T}^d))$ regularity of u, see in particular Remark 5.45. Analogous bounds for m are then a direct consequence of Theorems B.4 and 5.19.

□

Remark 5.51. We mention that if u_T, m_0, H and F are smoother, an additional bootstrap procedure yields further regularity of u, m, up to C^∞. We will not detail here this procedure for brevity.

5.7 Uniqueness

Here, we prove some uniqueness results in the case $\sigma = 0$, that is for system (5.1). We assume that equations are satisfied in the sense of Definitions 5.38 and 5.41. The case $\sigma > 0$ is easier, since solutions enjoy classical regularity, and the following arguments apply similarly.

5.7.1 Uniqueness in the monotone case

Theorem 5.52. *Assume that H is convex and the following monotonicity condition holds*

$$\int_{\mathbb{T}^d}(F[m_1](x) - F[m_2](x))d(m_1 - m_2)(x) > 0\ , \quad \forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^d)\ , m_1 \neq m_2\ .$$

Then, the solution to (5.1) *is unique.*

Proof. Uniqueness in the monotone case follows from the usual ideas by Lasry-Lions [163]. One has to be careful that (u, m) is regular enough to run the argument. Let (u_1, m_1) and (u_2, m_2) be two solutions of the MFG system (5.1). Set $v = u_1 - u_2$ and $\mu = m_1 - m_2$. Then v and μ satisfy respectively the equations

$$-\partial_t v + (-\Delta)^s v + H(x, Du_1) - H(x, Du_2) = F[m_1(t)](x) - F[m_2(t)](x)\ , v(x,T) = 0$$

and

$$\partial_t \mu + (-\Delta)^s \mu - \operatorname{div}\big(m_1 D_p H(x, Du_1) - m_2 D_p H(x, Du_2)\big) = 0\ , \mu(x,0) = 0\ .$$

We distinguish between the supercritical-critical (namely $s \in (0, 1/2)$ and $s = 1/2$) case and the subcritical ($s \in (1/2, 1)$) one.

Case 1. The supercritical-critical case. Recall that $u_i, Du_i, m_i \in L^\infty(Q_T)$, so $v, Dv, \mu \in L^\infty(Q_T)$. Moreover, $v \in \mathcal{H}_2^s(Q_T)$. Hence, using $\mu \in \mathbb{H}_2^s(Q_T) \cap L^\infty(Q_T)$ as a test function in the weak formulation of Definition 5.41, we get

$$\iint_{Q_T} -\mu\partial_t v + \mu\big(H(x, Du_1) - H(x, Du_2)\big) - \mu\big(F[m_1(t)](x) - F[m_2(t)](x)\big)dxdt + \\ + \iint_{Q_T} (-\Delta)^{\frac{s}{2}}\mu(-\Delta)^{\frac{s}{2}} v\, dxdt = 0\ . \quad (5.53)$$

Then, we use $v \in \mathcal{H}_2^s(Q_T) \cap L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d))$ as a test function in the weak formulation of the equation satisfied by μ, recalling also that $\partial_t \mu \in \mathbb{H}_2^{-1}(Q_T)$, to conclude

$$0 = \iint_{Q_T} -\mu \partial_t v\, dxdt + (-\Delta)^{\frac{s}{2}}\mu (-\Delta)^{\frac{s}{2}} v\, dxdt + \\ + \iint_{Q_T} Dv \cdot (m_1 D_p H(x,Du_1) - m_2 D_p H(x,Du_2))\, dxdt\ , \quad (5.54)$$

Subtracting (5.54) from (5.53) we obtain

$$0 = \iint_{Q_T} -\mu\big((F[m_1(t)](x) - F[m_2(t)](x)\big) + \mu(H(x,Du_1) - H(x,Du_2))dxdt - \\ - \iint_{Q_T} Dv \cdot \big(m_1 D_p H(x,Du_1) - m_2 D_p H(x,Du_2)\big) dxdt\ . \quad (5.55)$$

The following inequality holds true

$$\iint_{Q_T} \mu(H(x,Du_1) - H(x,Du_2)) - Dv \cdot (m_1 D_p H(x,Du_1) - m_2 D_p H(x,Du_2))dxdt \le 0\ ,$$

by convexity of H. Using (5.55) we can conclude that

$$\iint_{Q_T} (m_1 - m_2)\big(F[m_1(t)] - F[m_2(t)]\big) dtdx \le 0\ ,$$

In view of the monotonicity condition we get $m_1 = m_2$ a.e..

Finally, by the fact that u_1 and u_2 solve the same equation with the same final datum (see Remark 5.53 below), they must concide.

Case 2. The subcritical case. The proof of the case $s \in (\frac{1}{2}, 1)$ is simpler and it can be carried out as in Step 1, observing that (u,m) is a classical solution. □

Remark 5.53. The above uniqueness proof under monotonicity conditions on the cost F in the case $s \in (0,1/2]$ uses in turn an underlying uniqueness result for weak energy solutions to fractional HJ equations. We show this fact using a similar argument that we will also exploit in Part III, where actually rough data are considered. Let u_1, u_2 be two weak energy solutions, in the sense of Definition 5.41, of the (forward) fractional HJ equation,

$$\partial_t u + (-\Delta)^s u + H(x,Du) = f(x,t) \text{ on } Q_T$$

equipped with $u(x,0) = u_0$, being f, u_0 smooth, $H = H(x,p)$ convex in p, as in the assumptions of Theorem 5.52. Take their difference $v := u_1 - u_2$ on $\overline{Q}_T$ with $Dv \in L^\infty$ (as in the requirement of Definition 5.41), which satisfies

$$\partial_t v + (-\Delta)^s v + H(x,Du_1) - H(x,Du_2) = 0 \text{ on } Q_T$$

in weak sense, equipped with zero initial data. Let $\tau \in (0,T]$. By convexity of $H(x,\cdot)$, v solves

$$\int_\omega^\tau \langle \partial_t v(t), \varphi(t)\rangle dt + \iint_{\mathbb{T}^d \times (\omega,\tau)} (-\Delta)^{\frac{s}{2}} v\, (-\Delta)^{\frac{s}{2}}\varphi + D_p H(x,Du_2)\cdot Dv\, \varphi\, dxdt \le 0$$

for all $\omega \in (0,\tau)$, and $v(\cdot,0) = 0$. Let now ρ be the adjoint variable with respect to u_2, namely ρ be the weak solution to

$$\begin{cases} -\partial_t \rho(x,t) + (-\Delta)^s \rho(x,t) - \operatorname{div}\big(D_p H(x, Du_2(x,t))\, \rho(x,t)\big) = 0 & \text{in } Q_\tau\ , \\ \rho(x,\tau) = \rho_\tau(x) & \text{on } \mathbb{T}^d\ , \end{cases}$$

for some non-negative and smooth probability density ρ_τ. Then, by duality we get

$$\int_{\mathbb{T}^d} v(x,\tau)\rho_\tau(x)dx \le \int_{\mathbb{T}^d} v(x,\omega)\rho(x,\omega)dx.$$

Since $Dv \in L^\infty$, we assert that v enjoys better regularity, more precisely $v \in \mathcal{H}_p^{2s}$ for all $p > \frac{d+2s}{2s}$. By the fractional Sobolev embedding developed in Section 5.3.2, one can argue similarly to Proposition 5.29 to conclude $w \in C(\overline{Q}_T)$. Hence w is uniformly continuous on $\overline{Q}_T$, giving $w(\cdot,t) \to w(\cdot,0) \equiv 0$ uniformly in $\mathbb{T}^d$. Moreover, $\int_{\mathbb{T}^d} v(x,\omega)\rho(x,\omega)dx = \int_{\mathbb{T}^d} [v(x,\omega) - v(x,0)]\rho(x,\omega)dx$. Thus, by Hölder's inequality and $\|\rho(s)\|_{L^1(\mathbb{T}^d)} = 1$, $\int_{\mathbb{T}^d} v(\omega)\rho(\omega) \to 0$, yielding

$$\int_{\mathbb{T}^d} v(x,\tau)\rho_\tau(x)dx \le 0$$

for arbitrary ρ_τ. As ρ_τ varies, $u_1(\tau) \le u_2(\tau)$ follows, and by exchanging the role of u_1 and u_2 and varying τ, we eventually obtain $u_1 \equiv u_2$.

5.7.2 Small-time uniqueness

The result of this section is the following

Theorem 5.54. *For $s \in (\frac{1}{2},1)$ and $H \in C^3(\mathbb{T}^d \times \mathbb{R}^d)$, there exists $T^* > 0$, depending on d, s, H, F, m_0, u_T such that for all $T \in (0,T^*]$ system* (5.1) *has at most one solution (u,m).*

Rewriting (5.1) as a forward-forward system for v, m setting $v(\cdot,t) := u(\cdot, T-t)$ for all $t \in [0,T]$, then

$$\begin{cases} v(x,t) = \mathcal{T}_t u_T(x) - \int_0^t \mathcal{T}_{t-\tau} \Phi^v[v,m](\tau)(x)d\tau\ , \\ m(x,t) = \mathcal{T}_t m_0(x) + \int_0^t \mathcal{T}_{t-\tau} \Phi^m[v,m](\tau)(x)d\tau\ , \end{cases} \tag{5.56}$$

where

$$\Phi^v[v,m](\tau)(\cdot) = F[m(T-\tau)](\cdot) - H(\cdot, Dv(\cdot,\tau))\ ,$$

$$\Phi^m[v,m](\tau)(\cdot) = \operatorname{div}(D_p H(\cdot, Dv(\cdot, T-\tau))m(\tau))$$

for $\tau \in [0,T]$. We will exploit the decay properties of $\mathcal{T}_t$.

Proof of Theorem 5.54. For $p > 1$ and $\mu \ge 0$, let us denote by

$$X_p^\mu := C([0,T]; H_p^\mu(\mathbb{T}^d)).$$

First, observe that any solution is classical by Theorem 5.50, and therefore it belongs to $X_p^{2s} \times X_p^{2s-1}$. Moreover, every solution of (5.1) can be seen as a fixed point of the map $\Psi : (v, m) \longmapsto (\hat{v}, \hat{m})$, where

$$\begin{cases} \hat{v}(t) = \mathcal{T}_t u_T(x) - \int_0^t \mathcal{T}_{t-\tau} \Phi^v[v, m](\tau)(x) d\tau \ , \\ \hat{m}(t) = \mathcal{T}_t m_0(x) + \int_0^t \mathcal{T}_{t-\tau} \Phi^m[v, m](\tau)(x) d\tau \ . \end{cases} \tag{5.57}$$

We remark that such representation holds in view of the fact that the solutions are classical. We prove that the fixed point of Ψ defined in (5.57) is unique by the contraction properties of Ψ itself that are valid for small T. Let (v_1, m_1) and (v_2, m_2) be two fixed points of Ψ. Set $\epsilon = d\left(\frac{1}{p} - \frac{1}{\bar{p}}\right) < 2s - 1$ with $\bar{p} > p$. This choice yields

$$\|m(\tau)\|_{2s-1-\epsilon, \bar{p}} \leq C \|m(\tau)\|_{2s-1,p}$$

for some $C > 0$ in view of Lemma 5.16. We apply Lemma 5.23-(i) (with $\nu = 2s-1-\varepsilon$ and $\gamma = 1 + \varepsilon$) and the assumptions on F and H to get

$$\begin{aligned}
&\left\| \int_0^t \mathcal{T}_{t-\tau}(\Phi^v[v_1, m_1](\tau)(x) - \Phi^v[v_2, m_2](\tau)(x)) d\tau \right\|_{2s,p} \leq \\
&\qquad \leq \int_0^t \|\mathcal{T}_{t-\tau}(\Phi^v[v_1, m_1](\tau)(x) - \Phi^v[v_2, m_2](\tau)(x))\|_{2s,p} d\tau \\
&\leq C_1 \left(\int_0^t (t-\tau)^{-\frac{1+\varepsilon}{2s}} \|F[m_1(T-\tau)](\cdot) - F[m_2(T-\tau)](\cdot)\|_{2s-1-\varepsilon,p} d\tau + \right. \\
&\left. + \int_0^t (t-\tau)^{-\frac{1+\varepsilon}{2s}} \|H(\cdot, Dv_1(\cdot, T-\tau)) - H(\cdot, Dv_2(\cdot, T-\tau))\|_{2s-1-\varepsilon,p} d\tau \right) \\
&\qquad \leq C_2 \left(\int_0^t (t-\tau)^{-\frac{1+\varepsilon}{2s}} \|m_1(\cdot, T-\tau) - m_2(\cdot, T-\tau)\|_{2s-1,p} d\tau + \right. \\
&\qquad \left. + \int_0^t (t-\tau)^{-\frac{1+\varepsilon}{2s}} \|Dv_1(\cdot, T-\tau) - Dv_2(\cdot, T-\tau)\|_{2s-1,p} d\tau \right) \\
&\qquad\qquad \leq C_3 T^{\frac{2s-1-\varepsilon}{2s}} \left(\|m_1 - m_2\|_{X_p^{2s-1}} + \|v_1 - v_2\|_{X_p^{2s}} \right) \ ,
\end{aligned}$$

by taking T small enough.
We now consider the term related to the Fokker-Planck equation. We apply Lemma

5.23-(i) with $\nu = 2s-2-\varepsilon$ and $\gamma = 1+\varepsilon$ to obtain

$$
\begin{aligned}
&\| \int_0^t \mathcal{T}_{t-\tau}(\Phi^m[v_1,m_1](\tau)(x) - \Phi^m[v_2,m_2](\tau)(x))d\tau\|_{2s-1,p} \leq \\
&\qquad \leq \int_0^t \|\mathcal{T}_{t-\tau}(\Phi^m[v_1,m_1](\tau)(x) - \Phi^m[v_2,m_2](\tau)(x))\|_{2s-1,p}d\tau \\
&\qquad\qquad \int_0^t (t-\tau)^{-\frac{1+\varepsilon}{2s}} \| \operatorname{div}\big(D_pH(\cdot,Dv_1(\cdot,T-\tau))m_1(\tau) \\
&\qquad\qquad\qquad - D_pH(\cdot,Dv_2(\cdot,T-\tau))m_2(\tau)\big)\|_{2s-2-\epsilon,p} \\
&\leq C_1\Big(\int_0^t (t-\tau)^{-\frac{1+\varepsilon}{2s}} \| \operatorname{div}(D_pH(\cdot,Dv_1(\cdot,T-\tau))(m_1(\tau)-m_2(\tau)))\|_{2s-2-\epsilon,p}d\tau + \\
&\qquad + \int_0^t (t-\tau)^{-\frac{1+\varepsilon}{2s}} \| \operatorname{div}(m_2(\tau)(D_pH(\cdot,Dv_1(\cdot,T-\tau)) \\
&\qquad\qquad -D_pH(\cdot,Dv_2(\cdot,T-\tau))))\|_{2s-2-\epsilon,p}d\tau) \\
&\leq C_2\Big(\int_0^t (t-\tau)^{-\frac{1+\varepsilon}{2s}} \|D_pH(\cdot,Dv_1(\cdot,T-\tau))(m_1(\tau)-m_2(\tau))\|_{2s-1-\epsilon,p}d\tau + \\
&\qquad + \int_0^t (t-\tau)^{-\frac{1+\varepsilon}{2s}} \|m_2(\tau)(D_pH(\cdot,Dv_1(\cdot,T-\tau)) \\
&\qquad\qquad\qquad -D_pH(\cdot,Dv_2(\cdot,T-\tau)))\|_{2s-1-\epsilon,p}d\tau))
\end{aligned}
$$

Then one has to observe that

$$
\begin{aligned}
&\|D_pH(Dv_1(\cdot,T-\tau))(m_1(\tau)-m_2(\tau))\|_{2s-1-\epsilon,p} \\
&\qquad \leq C_3(\|D_pH\|_{\bar q}\|m_1-m_2\|_{2s-1-\epsilon,\bar p} + \|D_pH\|_{2s-1-\epsilon,\bar q}\|m_1-m_2\|_{\bar p}) \\
&\qquad\qquad \leq C_4\|m_1-m_2\|_{2s-1-\epsilon,\bar p} \leq C_5\|m_1-m_2\|_{2s-1,p}\ ,
\end{aligned}
$$

where we applied Lemma A.1 to the second inequality, Lemma 5.16-(iii) to the last one, the fact that $\|D_pH\|_{2s-1-\epsilon,\bar q}$ is bounded independently of T by the regularity assumption on H and the L^∞ bound on Du and m.

Similarly,

$$
\begin{aligned}
&\|m_2(\tau)(D_pH(\cdot,Dv_1(\cdot,T-\tau)) - D_pH(\cdot,Dv_2(\cdot,T-\tau)))\|_{2s-1-\epsilon,p} \\
&\qquad \leq C_1\Big(\|m_2\|_{\bar q}\|D_pH(\cdot,Dv_1)-D_pH(\cdot,Dv_2)\|_{2s-1-\epsilon,\bar p} + \\
&\qquad\qquad + \|m_2\|_{2s-1-\epsilon,\bar q}\|D_pH(\cdot,Dv_1)-D_pH(\cdot,Dv_2)\|_{\bar p}\Big) \\
&\quad \leq C_2\|D_pH(\cdot,Dv_1)-D_pH(\cdot,Dv_2)\|_{2s-1-\epsilon,\bar p} \leq C_3\|D(v_1-v_2)\|_{2s-1-\epsilon,\bar p} \\
&\qquad\qquad \leq C_4\|D(v_1-v_2)\|_{2s-1,p} \leq C_5\|v_1-v_2\|_{2s,p}\,,
\end{aligned}
$$

where $C_i = C_i(d,s,\epsilon,p,\bar p,\bar q)$. This gives

$$
\begin{aligned}
&\left\|\int_0^t \mathcal{T}_{t-\tau}(\Phi^m[v_1,m_1](\tau)(x) - \Phi^m[v_2,m_2](\tau)(x))d\tau\right\|_{2s-1,p} \\
&\qquad\qquad \leq C_4T^{\frac{2s-1-\varepsilon}{2s}}(\|v_1-v_2\|_{X_p^{2s}} + \|m_1-m_2\|_{X_p^{2s-1}})
\end{aligned}
$$

by eventually taking T small enough. At the end we get

$$\begin{aligned}\|v_1 - v_2\|_{X_p^{2s}} + \|m_1 - m_2\|_{X_p^{2s-1}} &= \|\Psi(v_1, m_1) - \Psi(v_2, m_2)\|_{X_p^{2s}\times X_p^{2s-1}} \\ &\leq \frac{1}{2}(\|v_1 - v_2\|_{X_p^{2s}} + \|m_1 - m_2\|_{X_p^{2s-1}}) ,\end{aligned}$$

which allows to conclude $(v_1, m_1) = (v_2, m_2)$ for T sufficiently small. $\square$

Part III

Lipschitz regularity to Hamilton-Jacobi equations with rough data

Chapter 6

Lipschitz regularity to time-dependent viscous Hamilton-Jacobi equations with L^p terms

The aim of this chapter is to develop a duality method to deduce Lipschitz regularity of suitable weak energy solutions to time-dependent viscous Hamilton-Jacobi equations with unbounded right-hand side of the form[1]

$$
\begin{cases}
\partial_t u(x,t) - \sum_{i,j} a_{ij}(x,t)\partial_{ij}u(x,t) + H(x,Du(x,t)) = f(x,t) & \text{in } Q_T = \mathbb{T}^d \times (0,T),\\
u(x,0) = u_0(x) & \text{in } \mathbb{T}^d.
\end{cases}
\tag{6.1}
$$

We recall that the main peculiarity of our approach is to exploit both the diffusion and the coercivity of the Hamiltonian, while usual treatments for regularity estimates of solutions to these equations regard the Laplacian as a perturbation. We refer to Section 6.3 for additional details, references and comparison with the known techniques on gradient bounds for such nonlinear PDEs.

6.1 Assumptions and main results

We now state our two main results we are going to present throughout this chapter, the first one for weak energy solutions and the second one dealing with classical solutions. Assume that $d \geq 2$, and $A = (a_{ij}) : Q_T \to \mathrm{Sym}_d$, where Sym_d is the set of symmetric $d \times d$ real matrices, $a_{ij} \in C(0,T;W^{2,\infty}(\mathbb{T}^d))$ and

$$
\text{for some } \lambda > 0, \quad \lambda|\xi|^2 \leq a_{ij}(x,t)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d \text{ and a. e. } (x,t) \in Q_T. \tag{A}
$$

We perform again our analysis on the flat torus $\mathbb{T}^d$ in order to avoid boundary phenomena. We suppose that $H(x,p)$ is $C^1(\mathbb{T}^d \times \mathbb{R}^d)$, convex in the second variable,

[1]From now on the summation over repeated indices is understood.

and without loss of generality $H \geq 0$ (if not, one may compensate by adding a positive constant to f). Moreover,

$$\begin{aligned}
&\text{there exist constants } \gamma > 1 \text{ and } C_H > 0 \text{ such that}\\
&\quad C_H^{-1}|p|^\gamma - C_H \leq H(x,p) \leq C_H(|p|^\gamma + 1) \ ,\\
&\quad D_pH(x,p)\cdot p - H(x,p) \geq C_H^{-1}|p|^\gamma - C_H \ ,\\
&\quad |D_xH(x,p)| \leq C_H(|p|^\gamma + 1) \ ,\\
&\quad C_H^{-1}|p|^{\gamma-1} - C_H \leq |D_pH(x,p)| \leq C_H|p|^{\gamma-1} + C_H \ ,
\end{aligned} \tag{H}$$

for every $x \in \mathbb{T}^d$, $p \in \mathbb{R}^d$. Moreover, one can add an explicit dependence with respect to the time variable t to H provided that it respects the growth properties stated above in (H).

The first result concerns the regularizing effect of the equation, namely Lipschitz regularity of *weak* solutions u for positive times. Below $\gamma' = \gamma/(\gamma-1)$ is the conjugate exponent of γ.

Theorem 6.1. *Suppose that*

- $a_{ij} \in C(0,T;W^{2,\infty}(\mathbb{T}^d))$ *and satisfies* (A),
- $H \in C^1(\mathbb{T}^d \times \mathbb{R}^d)$, *it is convex in the second variable, and satisfies* (H),
- $f \in L^q(Q_T)$, *for some* $q > d+2$ *and* $q \geq \frac{d+2}{\gamma'-1}$,
- $u_0 \in L^\infty(\mathbb{T}^d)$.

(a) Let u be a local weak solution to (6.1) *(in the sense of Definition 6.5). Then, $u(\cdot,\tau) \in W^{1,\infty}(\mathbb{T}^d)$ for all $\tau \in (0,T]$. In particular, for all $t_1 \in (0,T)$ there exists a positive constant C_1 depending on t_1, λ, $\|a\|_{C(W^{2,\infty})}$, C_H, $\|u\|_{L^\infty(Q_T)}$, $\|f\|_{L^q(Q_T)}$, q, d, T such that*

$$\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C_1 \qquad \text{for all} \quad \tau \in [t_1,T]. \tag{6.2}$$

(b) If, in addition, u is a global weak solution with $u_0 \in W^{1,\infty}(\mathbb{T}^d)$, then there exists a positive constant C_2 depending on λ, $\|a\|_{C(W^{2,\infty})}$, C_H, $\|u_0\|_{W^{1,\infty}(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, q, d, T such that

$$\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C_2 \qquad \text{for all} \quad \tau \in [0,T]. \tag{6.3}$$

Moreover, the same conclusions hold if u is a weak solution to (6.1) *with $\mathcal{P} \neq \mathcal{Q}$ in* (6.13) *whenever $a_{ij}(x,t) = A_{ij}$ on Q_T for some $A_{ij} \in \mathrm{Sym}(\mathbb{R}^d)$ satisfying* (A).

Note that if $\gamma \leq 2$ (i.e. the subquadratic/quadratic regime) f is required to be in $L^q(Q_T)$ for some $q > d+2$, while in the superquadratic case $\gamma > 2$ conditions on f are more strict.
We are then able to show the existence and uniqueness of weak solutions

Theorem 6.2. *Suppose that the assumptions on a, f, H of Theorem 6.1 are in force, If $u_0 \in C(\mathbb{T}^d)$, then there exists a unique local weak solution to* (6.1)*. If $u_0 \in W^{1,\infty}(\mathbb{T}^d)$, then such a solution is a global weak solution.*

If we assume in addition that u is a *classical* solution to (6.1), we will show the following a priori regularity results. Note that, with respect to the previous Theorem 6.1, Lipschitz bounds will depend on weaker properties of the data a, f.

Theorem 6.3. *Suppose that*

- $a_{ij} \in C([0,T]; C^1(\mathbb{T}^d))$ *and satisfies* (A),
- $H \in C^2(\mathbb{T}^d \times \mathbb{R}^d)$ *and satisfies* (H),
- $f \in C([0,T]; C^1(\mathbb{T}^d))$,
- $u_0 \in C^1(\mathbb{T}^d)$.

Let

$$q > \min\left\{d+2, \frac{d+2}{2(\gamma'-1)}\right\}. \tag{6.4}$$

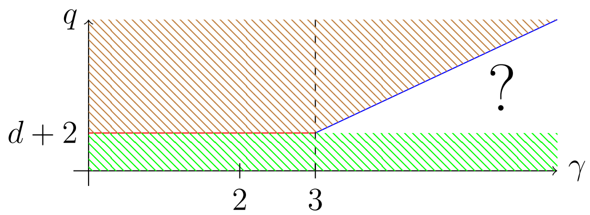
Then, there exists a positive constant C_3 depending on q, d, T, λ, C_H, $\|u_0\|_{W^{1,\infty}(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, $\|a\|_{C(0,T;W^{1,\infty}(\mathbb{T}^d))}$, such that every classical solution to (6.1) *satisfies*

$$\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \le C_3 \qquad \text{for all} \quad \tau \in [0,T]. \tag{6.5}$$

Note that (6.4) reads

$$q > \begin{cases} d+2 & \text{if } 1 < \gamma \le 3, \\ \frac{d+2}{2(\gamma'-1)} & \text{if } \gamma > 3. \end{cases}$$

In particular, we obtain "L^p-maximal regularity" whenever $\gamma \le 3$, that is a control on $\partial_t u, \partial_{ij} u$ and $H(Du)$ in L^q with respect to the the L^q norm of f for any $q > d+2$ by exploiting classical Caldéron-Zygmund results for linear equations. Still, the results obtained for $\gamma > 3$ are new, since, as far as we know, Lipschitz estimates in this regime are not available in the literature of parabolic viscous HJ equations. Anyhow, we remark that Lipschitz bounds in the regime $\gamma > 3$ and $d+2 < q < \frac{d+2}{2(\gamma'-1)}$ are at this stage an open problem.



In the next Section 6.2 we briefly describe our methods, and comment on crucial hypotheses that appear in Theorems 6.1, 6.3 and in the Definition 6.5 of weak solutions to (6.1). In Section 6.5 we present some preliminary facts and results on the adjoint equation. Sections 6.7 and 6.9 will be devoted mainly to the proofs of Theorems 6.1 and 6.3 respectively.

6.2 Heuristic derivation of Lipschitz estimates

The adjoint method implemented here can be heuristically described as follows. Let us assume that u is a smooth solution of the viscous HJ equation

$$\partial_t u(x,t) - \Delta u(x,t) + H(Du(x,t)) = f(x,t) \tag{6.6}$$

with $u(\cdot,0) \in C^1(\mathbb{T}^d)$ and f be C^1 in the space variable. We differentiate the equation to study the regularity of Du, namely, for any direction $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, we consider $v = \partial_\xi u$. Then, v solves the linearized equation

$$\partial_t v - \Delta v + D_p H(Du) \cdot Dv = \partial_\xi f \ . \tag{6.7}$$

For any $\tau \in (0,T)$, $x_0 \in \mathbb{T}^d$, we then look at the adjoint equation which develops a Dirac mass at the terminal time $t = \tau$, namely

$$\begin{cases} -\partial_t \rho - \Delta \rho - \operatorname{div}(D_p H(Du)\rho) = 0 & \text{in } \mathbb{T}^d \times (0,\tau) \ , \\ \rho(\tau) = \delta_{x_0} & \text{on } \mathbb{T}^d \ . \end{cases} \tag{6.8}$$

By duality between (6.7) and (6.8) we immediately get

$$\begin{aligned} \partial_\xi u(x_0,\tau) = \langle v(\tau), \rho(\tau) \rangle = \iint_{\mathbb{T}^d \times (0,\tau)} \partial_\xi f \, \rho + \int_{\mathbb{T}^d} v \rho(0) \\ = - \iint_{\mathbb{T}^d \times (0,\tau)} f \, \partial_\xi \rho + \int_{\mathbb{T}^d} \partial_\xi u \, \rho(0). \end{aligned}$$

Thanks to integration by parts in the previous formula, we realize that our representation of $\partial_\xi u(x_0,\tau)$ roughly depends on $\|f\|_{L^q(Q_T)}$ and $\|D\rho\|_{L^{q'}(Q_T)}$, so, the more we know on the integrability of $D\rho$, the less we can assume on the integrability of the datum f. The difficulty here is that ρ depends on Du itself through the drift in (6.8), and has a final datum that is a Dirac measure. Therefore, even disregarding completely the divergence term in (6.8), and using as final datum an L^1 approximation of δ_{x_0}, the best we can expect is $\|D\rho\|_{L^{q'}(Q_T)}$ for $q' < (d+2)'$. This is actually an integrability limit on $D\rho$ imposed by the heat part of the equation. Therefore, we will always require f to be L^q with $q > d+2$ (which is optimal, see Remark 6.33).

The transport (divergence) term in (6.8) is handled by exploiting a crucial information on the crossed quantity

$$\iint |D_p H(Du)|^{\gamma'} \rho \, dxdt, \tag{6.9}$$

that is obtained using a sort of duality between (6.1) and (6.8), and has a very precise meaning in terms of optimality in stochastic control problems. Such a quantity is actually a weighted $L^{\gamma'}(\rho)$ norm of the drift $-D_p H(Du)$ that appears in the divergence term, and turns out to be enough to derive bounds for $\|D\rho\|_{L^{q'}(Q_T)}$. The presence of this crossed term is standard in the study of the Lebesgue regularity of Fokker-Planck PDEs (see e.g. [50, 185, 184]). This crucial result is stated in Proposition 6.18 and exploits a delicate combination of parabolic regularity, interpolation and embeddings of parabolic spaces. It is worth noting that such an $L^{\gamma'}(\rho)$ integrability deteriorates as γ grows. In particular, we observe that in the subquadratic regime $\gamma \le 2$, this information is strong enough to guarantee $\|D\rho\|_{L^{q'}(Q_T)}$ for $q' < (d+2)'$. We can then regard the div() term in (6.8) as perturbation of a heat equation. On the other hand, in the superquadratic case $\gamma > 2$, we are just able to prove that $\|D\rho\|_{L^{q'}(Q_T)}$ for $q' \le q'_\gamma$, with $q'_\gamma < (d+2)'$, and actually $q'_\gamma \to 1$ as $\gamma \to \infty$. As expected, in the superquadratic case the Hamiltonian term in (6.1) may overcome the regularizing effect of Laplacian. Still, under the additional hypothesis $f \in L^{q_\gamma}$, we obtain Lipschitz regularity results *for every* $\gamma > 1$. This is a major difference with respect to previous works [123, 128], where the techniques involved produce estimates on $D\rho$ only under the assumption that the drift entering into the dual equation is at least $L^2(\rho)$, thus limiting the range of γ.

In the next sections we make precise all the above formal computations, and for more general equations of the form (6.1). In the first part our plan is to obtain Lipschitz regularity of *weak solutions* to (6.1), in a sense specified below (see Definition 6.5). The main issues in this program are the following:

- To exploit duality between (6.1) and (6.8) in a weak framework, one has to understand the right weak setting for both equations. We realize here that a suitable weak notion guaranteeing Lipschitz regularity is basically the usual energy one for both equations (i.e. $u, \rho \in \mathcal{H}_2^1$, see below for the definition). This relies strongly on the additional assumption $D_pH(Du) \in L^{\mathcal{Q}}((0,T); L^{\mathcal{P}})$, which can be considered as a requirement for the adjoint equation (6.8) rather than for the given HJ equation (6.1), but one should always keep in mind the subtle interplay between the two equations. Of course this forces the final datum $\rho(\tau)$ to be in L^2, and therefore introduces an additional approximation step from L^2 to L^1 in our scheme.
  One may argue that, for γ very large, $|Du|^{\gamma-1} \approx D_pH(Du) \in L^{\mathcal{Q}}((0,T); L^{\mathcal{P}})$ is very close to $Du \in L^\infty$. We stress in Section 6.7.5 that to perform this (seemingly) small step, one cannot avoid in general this assumption on Du, and therefore our requirements on weak solutions are optimal to guarantee Lipschitz regularity.

- A weak solution u is not a priori a.e. differentiable, and $f \in L^q$, so no differentiation procedure of (6.1) is justified. This is circumvented by considering difference quotients of u in the x-variable, which are handled via a method that is again based on the optimality of $-D_pH(Du)$ in stochastic optimal control problems (though here PDE methods will be involved only).

- Though they are not our main focus, we have also to be careful with regularity of H and a in the x-variable. Moreover, we are able slightly relax the above assumption $Du(0) \in L^\infty$ by localizing our estimates in time, thus assuming $u(0) \in L^\infty$ only.

The study of regularity, rather than the proof of a priori estimates of smooth solutions to (6.1), is a key difference with respect to works previously mentioned (e.g. [123, 128]). We take this different viewpoint in the final Section 6.9: assuming regularity of the solution, we can improve in some directions the previous procedure.

First, it is possible to enhance (6.9) by absorbing part of the gradient term in the left hand side of the Lipschitz estimate. Second, rather than studying the equation for $\partial_\xi u$, we consider the equation for $|Du|^2$, following a classical idea of Bernstein. This yields a similar "linearized" equation, with additional information on D^2u coming from strict ellipticity of the operator. This allows us to prove *a priori* regularity of smooth solutions u to (6.1) that depend on weaker integrability properties of f and regularity of a_{ij} with respect to x.

6.3 Comparison with the literature

The research on regularity and gradient bounds for (elliptic) PDEs began with the seminal work by S.V. Bernstein [37] and it has been later explored in the context of nonlinear elliptic [122, 174] and parabolic [214, 159] equations. The literature on this subject is too wide to keep track of all the references and hence, without aiming to give a comprehensive bibliography, we rather prefer to focus only on the contributions in the nonlinear parabolic setting, quoting also some results for nonlinear elliptic PDEs which are close to ours and which inspired our analysis. As underlined in the introduction, by performing L^∞ scaling arguments, it is common to distinguish two regimes, namely the subquadratic ($\gamma < 2$) and the superquadratic ($\gamma > 2$) case. We list the works following mainly this classification, describing briefly the assumptions and the techniques used in each work.

The subquadratic/quadratic case ($\gamma \le 2$) If $H(p) \sim |p|^\gamma$ and $1 < \gamma < 2$ the diffusion terms are the prevailing ones at small scales. When the right-hand side $f \in L^\infty$, Lipschitz (and further) regularity of quasi-linear equations of the form (6.1) goes back to classical literature [159, 214]. The most popular approach used to obtain gradient bounds to Hamilton-Jacobi equations like (6.1) with superlinear growth in Du and f continuous is the Ishii-Lions method [140]. This method is in turn based on viscosity solutions' techniques and typically takes advantage of the strict ellipticity of the diffusion, although the assumptions to run such arguments restrict the growth of the Hamiltonian to $\gamma < 2$ [140, assumption (3.2)] (or at most $\gamma < 3$, see e.g. [28, assumption (3.4)]) in the gradient variable. Generalizations in the context of fully nonlinear evolutive PDEs are due to A. Porretta and E. Priola (see e.g. [195]). When the right-hand side is unbounded in space and $\gamma < 2$, i.e. $f \in L^\infty(I; L^q(\mathbb{T}^d))$ and $q > d$, the first Lipschitz estimates for evolutive problems like (6.1) have been obtained by D. Gomes et al [127, Theorem 5.11],[126] for classical solutions with Lipschitz initial data. As for the quadratic regime $\gamma = 2$, Lipschitz estimates are proven in [127, Theorem 8.3] in the context of MFGs, where again the right-hand side $f \in L^\infty(I; L^q(\mathbb{T}^d))$ and $q > d$. These are the first attempts to generalize a very general elliptic result by P.L. Lions [175] (see also [88, Theorem A.3] and the Introduction of this manuscript) to the evolutive framework.

The superquadratic case ($\gamma > 2$) The works treating Lipschitz regularity for time-dependent viscous HJ equations having nonlinearities with general superlinear power growth $\gamma > 1$ in Du, and thus embracing the superquadratic case, when f

is at least bounded, are mainly established via the Bernstein method. This method requires typically to assume a "convexity-type" assumptions, i.e.

$$D_pH(x,p)\cdot p - H(x,p) \geq C_H|p|^\gamma \ ,p\in\mathbb{R}^d \ ,\gamma>1 \ ,$$

which typically appear when differentiating the equation, a usual drawback of the first versions of the Bernstein method.
The first contribution in the parabolic framework goes back to [97], where the authors were able to handle Hamiltonians behaving like either $|p|^\gamma$, $\gamma>1$ with right-hand side $f=f(x)\in C(\mathbb{T}^d)$ or $a(x)|p|^\gamma$, $\gamma\leq 2$, $a\in C(\mathbb{T}^d)$ with $f=f(x)\in C(\mathbb{T}^d)$. Related works in the time-dependent setting are due to G. Barles and P. Souganidis [32] via the weak Bernstein method. Recent results appeared via refinements of the Bernstein method in the viscosity solutions' framework [10], where convexity in the Hamiltonian is not required, and the Ishii-Lions method [168], which also allows to treat degenerate problems [104]. We emphasize that in all of these works both the Hamiltonian and the right-hand side are time-independent, and at least continuous in space. Existence and uniqueness results for viscous HJ equations with superlinear Hamiltonians exploiting the bounds in [10] are given in [102].
As for the case of unbounded right-hand side, the research began with the work by P.-L. Lions in [175] (see also [161]) for the stationary problem, showing Lipschitz regularization effect for $f\in L^q(\mathbb{T}^d)$, $q>d$ and any $\gamma>1$ via an integral variant of the Bernstein method. We also quote the results obtained by M. Bardi and B. Perthame [27], where the authors obtained a "maximal" L^q-regularity result for quasi-linear elliptic equations with natural growth in the gradient (i.e. $\gamma=2$) via a refinement of the same method. This improvement leads to minimal regularity assumptions on the diffusion coefficients , and also permits to treat some degenerated cases, i.e. where $A\geq 0$ only. In this latter case, the aforementioned approach gives an estimate on $|ADu|\in L^q$, implying also Hölder bounds when the Hörmander condition is in force via well-knwon embedding theorems.
After these contributions, we mention the work by P. Souplet [35] (see also [203] and the references therein), which provides an extensive analysis for the Cauchy problem on the whole space with $f\equiv 0$ and initial data belonging to Lebesgue spaces both for absorbing and repulsive gradient terms, and [33] for the corresponding evolutive problem driven by the p-Laplacian. Lately, there has been an increasing interest in the regularization effect for these nonlinear evolutive problem, mainly motivated by the recent research in the context of MFGs. More precisely, P. Cardaliaguet and L. Silvestre [77] proved Hölder's regularization effect for viscous HJ equations with superquadratic growth and unbounded right-hand side, treating also degenerate problems, where the underlying idea is to regard the diffusion as a perturbation of a first order equation and exploit a scaling argument. We mention also the results obtained in [82, 224] via De Giorgi's techniques for the viscous and first order version of the problem, respectively. In the context of MFGs recent results have been obtained by D. Gomes and collaborators [128] via duality methods, showing the Lipschitz regularity for smooth solutions and smooth data with $H(p)\sim|p|^\gamma$ and $\gamma<3$ when $f\in L^\infty(I;L^q(\mathbb{T}^d))$ and $q>d$.

Due to this discussion, we believe that the results we present below improve significantly the knowledge on the subject of parabolic Hamilton-Jacobi equations coercive in the gradient and space-time unbounded terms. More precisely, we improve the aforementioned known results when the growth $\gamma < 3$, treating right-hand sides belonging to space-time Lebesgue spaces in a weaker setting, and we provide the first Lipschitz regularity result when $\gamma \geq 3$ and $f \in L^q(\mathbb{T}^d \times I)$.

6.4 Scaling

In this section, we perform some scaling arguments to guess the critical exponents ensuring the Lipschitz regularization effect. To this aim, let us consider the simpler case $a_{ij} = \delta_{ij}$. As outlined in the introduction, the typical idea is to employ a $W^{1,\infty}$ scaling, meaning to zoom in and look at the function $z(x,t) = \varepsilon^{-1}u(\varepsilon x, \varepsilon^2 t)$, where u solves (6.1). Simple computations yield the following equation satisfied by z

$$\partial_t z - \Delta z + \varepsilon |Dz|^\gamma = \varepsilon f(\varepsilon x, \varepsilon^2 t) =: r_\varepsilon(x,t) \ .$$

It is then straightforward to verify that the $L^q(\mathbb{R}^d \times (0,T))$ norm of $r_\varepsilon(x,t)$ is invariant under the previous scaling precisely when $q = d+2$. Therefore, one expects to obtain Lipschitz regularity of the solution of the HJ equation in the subcritical regime, namely assuming $f \in L^q$ with $q > d+2$. As announced, our arguments involves the estimates of solutions to a dual Fokker-Planck equation of the form

$$\partial_t \rho - \Delta \rho - \operatorname{div}(b\rho) = 0 \ ,$$

where b stands for the drift, in terms of the crossed quantity $\iint |b|^{\gamma'}\rho$. We zoom in again and set $\mu(x,t) = \varepsilon^\alpha \rho(\varepsilon x, \varepsilon^2 t)$ and $v(x,t) = \varepsilon^\beta b(\varepsilon x, \varepsilon^2 t)$ to find that the variable μ with scaled drift v solves the equation

$$\partial_t \mu - \Delta \mu - \operatorname{div}(v\mu) = \varepsilon^{2+\alpha}(\partial_t \rho - \Delta \rho) - \varepsilon^{\alpha+\beta+1}\operatorname{div}(b(x,t)\rho) = 0 \ .$$

Therefore, it is immediate to check that the correct scaling leaving the equation invariant imposes $\beta = 1$. Therefore, the crossed quantity corresponding to the scaled quantities (μ, v) is

$$\iint |v|^{\gamma'}\mu = \varepsilon^{\gamma'+\alpha-d-2} \iint |b|^{\gamma'}\rho \ ,$$

which allows to find the optimal critical exponent

$$\alpha = d + 2 - \gamma' \ .$$

Since our arguments rely on estimates on $D\rho$ in some Lebesgue space $L^{q'}$, where q' is the conjugate of $q > 1$, one finds

$$\iint |D\mu|^{q'} = \varepsilon^{(\alpha+1)q'-d-2} \iint |D\rho|^{q'} \ .$$

Therefore, imposing

$$(\alpha + 1)q' - d - 2 = 0$$

we get

$$q' = \frac{d+2}{d+3-\gamma'}$$

after plugging the previous expression for α. This forces the critical threshold for q to be

$$q = \frac{d+2}{\gamma'-1}\ ,$$

which is in fact the one appearing in Theorem 6.1.

Remark 6.4. Recall that, typically, summability results for solution to parabolic equations as well as embedding theorems can be obtained from their stationary counterpart by the substitution $d \longmapsto d+2$, i.e. taking two more dimensions with respect to the elliptic case. For instance, it is well-known that the first order Sobolev space $W^{1,p}(\mathbb{T}^d)$ is embedded onto $L^{\frac{2d}{d-2}}(\mathbb{T}^d)$ and, in fact, we will prove in Proposition 7.7 that its parabolic analogue $\mathcal{H}_p^1$ is embedded onto $L^{2(d+2)/d}(\mathbb{T}^d)$. This fact can be heuristically inferred to the correspondence "one time derivative-two space derivatives" appearing in the heat equation, which is in fact the diffusion operator appearing in our dynamics.

6.5 Functional spaces, weak solutions and basic properties

First, recall that the Lagrangian $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, $L(x,\xi) := \sup_p\{p \cdot \xi - H(x,p)\}$, namely the Legendre transform of H in the p-variable, is well defined by the superlinear character of $H(x,\cdot)$. Moreover, by convexity of $H(x,\cdot)$,

$$H(x,p) = \sup_{\xi\in\mathbb{R}^d}\{\xi \cdot p - L(x,\xi)\},$$

and

$$H(x,p) = \xi \cdot p - L(x,\xi) \quad \text{if and only if} \quad \xi = D_pH(x,p). \tag{6.10}$$

The following properties of L are standard (see, e.g. [66] and also [89, Proposition 2.1]): for some $C_L > 0$,

$$C_L^{-1}|\xi|^{\gamma'} - C_L \le L(x,\xi) \le C_L|\xi|^{\gamma'} \tag{L1}$$

$$|D_xL(x,\xi)| \le C_L(|\xi|^{\gamma'}+1). \tag{L2}$$

for all $\xi \in \mathbb{R}^d$.

For any time interval $I \subset \mathbb{R}$, let $Q = \mathbb{T}^d \times I$. For any time interval $(t_1,t_2) \subseteq \mathbb{R}$, let $Q_{t_1,t_2} := \mathbb{T}^d \times (t_1,t_2)$. We will also use the notation $Q_{t_2} := \mathbb{T}^d \times (0,t_2)$. For any $p \ge 1$ and $Q = Q_{t_1,t_2}$. Recalling the definition of Lebesgue and Sobolev spaces in the periodic setting given in Subsection 5.3.2, for any $p \ge 1$, we recall that the space $W_p^{2,1}(Q)$ is the space of functions u such that $\partial_t^r D_x^\beta u \in L^p(Q)$ for all multi-indices β and r such that $|\beta| + 2r \le 2$, endowed with the norm

$$\|u\|_{W_p^{2,1}(Q)} = \left(\iint_Q \sum_{|\beta|+2r\le 2} |\partial_t^r D_x^\beta u|^p dxdt\right)^{\frac{1}{p}}.$$

The space $W^{1,0}_p(Q)$ is defined similarly, and is endowed with the norm

$$\|u\|_{W^{1,0}_p(Q)} := \|u\|_{L^p(Q)} + \sum_{|\beta|=1} \left\|D^\beta_x u\right\|_{L^p(Q)} .$$

Similarly to the fractional spaces $\mathcal{H}^{2s-1}_p$ introduced in Part II to study fractional MFG systems, we define the space $\mathcal{H}^1_p(Q)$ (i.e. the local $s=1$ analogue of $\mathcal{H}^{2s-1}_p$) as the space of functions $u \in W^{1,0}_p(Q)$ with $\partial_t u \in (W^{1,0}_{p'}(Q))'$, equipped with the norm

$$\|u\|_{\mathcal{H}^1_p(Q)} := \|u\|_{W^{1,0}_p(Q)} + \|\partial_t u\|_{(W^{1,0}_{p'}(Q))'} .$$

We have the following isomorphisms: $W^{1,0}_2(Q) \simeq L^2(I; W^{1,2}(\mathbb{T}^d))$, and

$$\begin{aligned}\mathcal{H}^1_2(Q) &\simeq \{u \in L^2(I; W^{1,2}(\mathbb{T}^d)),\, \partial_t u \in (\,L^2(I; W^{1,2}(\mathbb{T}^d))\,)'\} \\ &\simeq \big\{u \in L^2(I; W^{1,2}(\mathbb{T}^d)),\, \partial_t u \in L^2\big(I; (W^{1,2}(\mathbb{T}^d))'\big)\big\},\end{aligned}$$

and the latter is known to be continuously embedded into $C(I; L^2(\mathbb{T}^d))$ (see, e.g., [101, Theorem XVIII.2.1]). Sometimes, we will use the compact notation $C(X)$ and $L^q(X)$.

6.5.1 A notion of weak solution to viscous HJ equations

We will say that u is a *weak* solution to (6.1) in the following sense.

Definition 6.5. *We say that*

i) u is a local weak solution to (6.1) *if for all* $0 < s < T$

$$u \in \mathcal{H}^1_2(\mathbb{T}^d \times (s,T)) \cap C(\overline{Q}_T), \quad H(\cdot, Du) \in L^1(s,T; L^\sigma(\mathbb{T}^d)) \text{ for some } \sigma > 1, \tag{6.11}$$

$$\text{and } D_p H(\cdot, Du) \in L^{\mathcal{Q}}(s,T; L^{\mathcal{P}}(\mathbb{T}^d)) \tag{6.12}$$

$$\text{for some } d \le \mathcal{P} \le \infty, \text{ and } 2 \le \mathcal{Q} \le \infty \text{ such that } \frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} \le \frac{1}{2}, \tag{6.13}$$

and for all $0 < s < \tau \le T$, $\varphi \in \mathcal{H}^1_2(\mathbb{T}^d \times (s,\tau)) \cap L^\infty(s,\tau; L^{\sigma'}(\mathbb{T}^d))$

$$\int_s^\tau \langle \partial_t u(t), \varphi(t)\rangle dt + \iint_{\mathbb{T}^d \times (s,\tau)} \partial_i u\, \partial_j(a_{ij}\varphi) + H(x, Du)\varphi\, dxdt = \iint_{\mathbb{T}^d \times (s,\tau)} f\varphi\, dxdt \tag{6.14}$$

(here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(W^{1,2}(\mathbb{T}^d))'$ and $W^{1,2}(\mathbb{T}^d)$).

ii) u is a global weak solution if (6.11)-(6.12)-(6.13) *hold for all* $0 \le s < T$*, that is, on all Q_T (and therefore,* (6.14) *is also satisfied up to $s = 0$).*

Note that when $s = 0$ the parabolic space $\mathcal{H}_2^1(Q_\tau)$ is continuously embedded in $C(0, \tau; L^2(\mathbb{T}^d))$, so the weak formulation above is equivalent to

$$\int_0^\tau \langle \partial_t u(t), \varphi(t) \rangle dt + \iint_{Q_\tau} \partial_i u \, \partial_j (a_{ij}\varphi) + H(x, Du)\varphi \, dxdt = \iint_{Q_\tau} f\varphi \, dxdt$$

for all $\varphi \in C^\infty(Q_\tau)$, and $u(0) = u_0$ in the L^2-sense (here, $\langle \cdot, \cdot \rangle$ is the duality pairing between $(W^{1,2}(\mathbb{T}^d))'$ and $W^{1,2}(\mathbb{T}^d)$). Note that for (6.14) to be meaningful, one could just require $H(x, Du) \in L^1$ (i.e. $u \in L^\gamma(W^{1,\gamma})$); we ask for slightly better integrability since we will use the adjoint variable ρ (see (6.23) below) as test function, that is not necessarily in $L^\infty(Q_T)$ (particularly when (6.13) is satisfied as an equality). In particular, (6.14) holds in general for $\varphi \in \mathcal{H}_2^1(Q_{s,\tau}) \cap L^\infty(s, \tau; L^{\sigma'}(\mathbb{T}^d))$. Anyway, as it will be pointed out in the following remark, *ii)* implies *i)* in many interesting cases. Though condition *ii)* appears to be unrelated to (6.1), it actually guarantees the existence of a weak (energy) solution of the adjoint equation (see Proposition 6.15 below), that will be crucial in our subsequent analysis. In what follows, when talking about local and global weak solutions, we will always assume that they are also distributional solution, as stated in Definition 6.5.

Remark 6.6. Under the growth assumptions (H) on the Hamiltonian, one can easily verify the following implications: if $D_pH(x, Du)$ satisfies *ii)* for some $\mathscr{P} = \mathscr{Q} \geq d+2$, then *i)* holds for sure whenever $\gamma > \frac{d+2}{d+1}$. Or, if $D_pH(x, Du)$ satisfies *ii)* for $\mathscr{Q} = \infty$ and some $\mathscr{P} \geq d$, then *i)* always holds if $\gamma > \frac{d}{d-1}$.

6.5.2 On viscous equations with unbounded drifts

In this section we prove the uniqueness of weak solutions to (6.1) by using classical tools for linear equations with unbounded coefficients. Let us consider the model problem

$$\partial_t u - \sum_{i,j=1}^d a_{ij}(x, t)\partial_{ij} u(x, t) + b(x, t) \cdot Du = 0 \tag{6.15}$$

with A satisfying (A). We first give the following comparison principle for equations with unbounded $L^{\mathscr{Q}}(L^{\mathscr{P}})$ drifts following classical arguments used in [193, 113].

Proposition 6.7. *Let u, v be two weak solutions in $L^\infty(0, T; L^2(\mathbb{T}^d))) \cap L^2(0, T; H^1(\mathbb{T}^d))$ of (6.15) such that $u(x, 0) \leq v(x, 0)$ a.e. in $\mathbb{T}^d$. Then $u \leq v$ a.e. in Q_T under one of the following assumptions on the drift*

(i) $b \in L^{\mathscr{P}}(Q_T)$, $\mathscr{P} \geq d + 2$.

(ii) $b \in L^\infty(0, T; L^d(Q_T))$ and $|B(\cdot, t)|^d$ is uniformly integrable with respect to the time variable.

(iii) $b \in L^{\mathcal{Q}}(L^{\mathcal{P}}(\mathbb{T}^d))$ with $\mathcal{P}, \mathcal{Q}$ fulfilling (6.13) (with strict inequality when $d = 2$).

Proof. We first prove (i). We argue by contradiction assuming $w := u - v > 0$ in $\mathcal{O} \subset Q_T$ with $|\mathcal{O}| > 0$ and denote by

$$w_k := \begin{cases} w^+ - k & \text{if } w^+ > k \\ 0 & \text{otherwise .} \end{cases}$$

for $k \in (0, \sup_{\mathcal{O}} w)$. First, note that $w_k(x,0) = 0$. We use w_k as a test function in the weak formulation to find

$$\int_{\mathbb{T}^d} w w_k(t)\,dx - \int_{\mathbb{T}^d} w(x,0) w_k(x,0)\,dx - \iint_{Q_t} w \partial_t w_k\,dxd\tau + \lambda \iint_{Q_t} |Dw_k|^2\,dxd\tau \\ \leq \iint_{Q_t} |b||Dw_k||w_k|\,dxd\tau \ .$$

One immediately checks that, using the sign of the initial condition, it holds

$$\int_{\mathbb{T}^d} w w_k\,dx - \iint_{Q_t} w \partial_t w_k\,dxd\tau \geq \frac{1}{2}\int_{\mathbb{T}^d} w_k^2(t)\,dx$$

giving thus

$$\frac{1}{2}\int_{\mathbb{T}^d} w_k^2\,dx + \lambda \iint_{Q_t} |Dw_k|^2\,dxd\tau \leq \iint_{Q_t} |b||Dw_k||w_k|\,dxd\tau \ .$$

Passing to the supremum over $t \in [0,T]$ in the left-hand side and applying generalized Hölder's inequality with the triple $(d+2, 2, \frac{2(d+2)}{d})$, we have

$$\frac{1}{2}\text{ess sup}_{t\in(0,T)} \int_{\mathbb{T}^d} w_k^2\,dx + \lambda \iint_{A_k} |Dw_k|^2\,dxd\tau \\ \leq \|b\|_{L^{d+2}(A_k)} \|Dw_k\|_{L^2(Q_T)} \|w_k\|_{L^{\frac{2(d+2)}{d}}(Q_T)} \tag{6.16}$$

where $A_k := \{(x,t) \in Q_T : k < w^+ < \sup_{\mathcal{O}} w\}$. By [103, Proposition I.3.1] we have the embedding

$$\|w_k\|_{L^{\frac{2(d+2)}{d}}(Q_t)} \leq C_1 \left(\text{ess sup}_{t\in(0,T)} \left(\int_{\mathbb{T}^d} w_k^2\,dx \right)^{\frac{1}{2}} + \left(\iint_{A_k} |Dw_k|^2\,dxd\tau \right)^{\frac{1}{2}} \right)$$

where C_1 depends solely on d. By combining all the above inequalities and applying Young's inequality one immediately finds

$$\min\left\{1, \frac{\lambda}{2}\right\} \left(\text{ess sup}_{t\in(0,T)} \int_{\mathbb{T}^d} w_k^2 + \iint_{A_k} |Dw_k|^2\,dxd\tau \right) \\ \leq C_2 \|b\|^2_{L^{d+2}(A_k)} \left(\text{ess sup}_{t\in(0,T)} \int_{\mathbb{T}^d} w_k^2 + \iint_{A_k} |Dw_k|^2\,dxd\tau \right)$$

where $C_2 = C_2(d, \lambda)$. This yields

$$1 \leq \frac{C_2}{\min\left\{1, \frac{\lambda}{2}\right\}} \|b\|^2_{L^{d+2}(A_k)} \ .$$

Then, by letting $k \to \sup_{\mathcal{O}} w$ we note that the right-hand side approaches to 0, giving thus the contradiction.
To prove (ii), we argue as in the elliptic case (see e.g. [122, 193]). We define $A_k^t :=$

$\{x \in \mathbb{T}^d : k < w^+ < \sup_{Q_T} w\}$ for a.e. $t \in [0,T]$ and we use w_k as test function as above. Then, Hölder's and Gagliardo-Nirenberg inequality both imply

$$\frac{1}{2}\text{ess sup}_{t\in(0,T)}\int_{\mathbb{T}^d} w_k^2 + \lambda\int_0^t\int_{A_k}|Dw_k|^2\,dxd\tau \le \int_0^t \|b\|_{L^d(A_k)}\|Dw_k\|_{L^2(\mathbb{T}^d)}\|w_k\|_{L^{\frac{2d}{d-2}}(\mathbb{T}^d)}\,d\tau$$
$$\le C_1\|b\|_{L^\infty(0,t;L^d(A_k))}\|Dw_k\|_{L^2(Q_t)}(\|w_k\|_{L^2(Q_t)} + \|Dw_k\|_{L^2(Q_t)})$$
$$\le C_2\|b\|_{L^\infty(0,t;L^d(A_k))}\|w_k\|^2_{L^2(0,t;W^{1,2}(\mathbb{T}^d))}$$

Moreover

$$\frac{1}{2}\int_{\mathbb{T}^d} w_k^2 + \lambda\int_0^t\int_{A_k}|Dw_k|^2\,dxdt \ge C_3\|w_k\|_{L^2(0,T;W^{1,2}(\mathbb{T}^d))}$$

for some $C_3 > 0$ depending also on λ. Then

$$1 \le \frac{C_2}{C_3}\|b\|_{L^\infty(0,T;L^d(A_k))}$$

and one conclues as above, using the assumption on b.
We finally prove (iii). We have

$$\frac{1}{2}\text{ess sup}_{t\in(0,T)}\int_{\mathbb{T}^d} w_k^2\,dx + \lambda\iint_{A_k}|Dw_k|^2\,dxdt \le \int_0^t\int_{A_k}|b||Dw_k||w_k|\,dxdt$$
$$\le \left(\int_0^t\|bw_k\|^2_{L^2(A_k)}\,dt\right)^{\frac{1}{2}}\left(\int_0^t\|Dw_k\|^2_{L^2(A_k)}\,dt\right)^{\frac{1}{2}}$$
$$\le C(\lambda)\left(\int_0^T\|bw_k\|^2_{L^2(A_k)}\right) + \frac{\lambda}{2}\int_0^T\|Dw_k\|^2_{L^2(A_k)}\,dt$$

We then have

$$\min\left\{1,\frac{\lambda}{2}\right\}\left(\text{ess sup}_{t\in(0,T)}\int_{\mathbb{T}^d} w_k^2 + \iint_{A_k}|Dw_k|^2\,dxdt\right)$$
$$\le C\||b|\|^2_{L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(A_k))}\|w_k\|^2_{L^{2\bar{\mathcal{Q}}'}(0,T;L^{2\bar{\mathcal{P}}'}(A_k))}$$

where $\bar{\mathcal{Q}} = \mathcal{Q}/2$ and $\bar{\mathcal{P}} = \mathcal{P}/2$. First, note that

$$\frac{d}{2\bar{\mathcal{P}}} + \frac{1}{\bar{\mathcal{Q}}} \le 1$$

and then use [12, Lemma 3] to obtain

$$\|w_k\|^2_{L^{2\bar{\mathcal{Q}}'}(0,T;L^{2\bar{\mathcal{P}}'}(A_k))} \le CT^\theta\left(\text{ess sup}_{t\in(0,T)}\int_{\mathbb{T}^d} w_k^2 + \iint_{A_k}|Dw_k|^2\,dxdt\right)$$

with $\theta = 1 - \frac{d}{2\mathcal{P}} - \frac{1}{\mathcal{Q}} > 0$, except for the case $d = 2$, where the quoted [12, Lemma 3] requires the strict inequality. To prove the above fact in dimension 2 when

$$\frac{d}{2\bar{\mathcal{P}}} + \frac{1}{\bar{\mathcal{Q}}} = 1$$

one can use [103, Proposition I.3.3](with $r = 2\bar{\mathcal{Q}}'$, $q = 2\bar{\mathcal{P}}'$, $p = 2$) to obtain

$$1 \le C(\lambda)\||b|\|^2_{L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(A_k^t))}$$

and conclude again as in the previous items. $\square$

Notice that under the assumptions of Definition 6.5, weak solutions of (6.1) must be unique (except for a subtle endpoint case $\mathcal{Q} = \infty$, $\mathcal{P} = d$ where one needs uniform in time integrability as in Proposition 6.7-(ii)). This can be proven via a simple linearization argument:

Theorem 6.8. *Under the standing assumptions* (H)*, every (global) weak solution to* (6.1) *is unique.*

Proof. Let $v(x,t) := u_1(x,t) - u_2(x,t)$ on Q_T, where u_i are two solutions of (6.1) in the sense of Definition 6.5. Then, $v \in \mathcal{H}_2^1(Q_T)$ is a weak (energy) solution to the linear equation

$$\partial_t v - \sum_{i,j=1}^{d} a_{ij}(x,t)\partial_{ij}v(x,t) + B(x,t) \cdot Dv(x,t) = 0,$$

satisfying $v(0) = 0$ in the L^2-sense, where $B(x,t)$ is some measurable vector field such that, in view of (H),

$$|B(x,t)| \le C(|Du_1(x,t)|^{\gamma-1} + |Du_2(x,t)|^{\gamma-1} + 1).$$

Hence, again by (H) and (6.13), $B(x,t) \in L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(\mathbb{T}^d))$ for some $\mathcal{P}$, $\mathcal{Q}$ satisfying $\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} \le \frac{1}{2}$. Hence by Proposition 6.7 we get $v \le 0$. Repeating the same arguments replacing v with $-v$ one gets the conclusion. □

As for the existence, here we argue via a fixed point theorem, since, in general, the operator $\partial_t u - \Delta u + b \cdot Du$ is not coercive, unless $\|b\|_{L^{\mathcal{Q}}(L^{\mathcal{P}})}$ with $\mathcal{Q}, \mathcal{P}$ satisfy the Aronson-Serrin condition is small enough.
We thus prove, for simplicity, the well-posedness in $\mathcal{H}_2^1(Q_\tau)$ of the adjoint problem to (6.23), focusing on the case $\mathcal{P} = \mathcal{Q}$. The full proof for the this result in the general case with unbounded data is well-established and can be found in [159] by means of the Galerkin's approximation method.

Proposition 6.9. *Let* (A) *be in force,* $b \in L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))$ *for some* $\mathcal{P} \ge d$, $\mathcal{Q} \ge 2$ *satisfying* $\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} \le \frac{1}{2}$*. Define the map* $T : \mathcal{H}_2^1(Q_T) \times [0,1] \to \mathcal{H}_2^1(Q_T)$ *such that for any* $v \in \mathcal{H}_2^1(Q_T)$ *we have* $T[v;\sigma] = u$ *if and only if*

$$\partial_t u - \sum_{i,j=1}^{d} a_{ij}(x,t)\partial_{ij}u(x,t) = \sigma b \cdot Dv \text{ in } Q_T\ , u(x,0) = \sigma u_0(x) \text{ in } \mathbb{T}^d\ . \tag{6.17}$$

with $u_0 \in L^2(\mathbb{T}^d)$*. Then*

(i) T *is compact;*

(ii) There exists a constant $M > 0$ *such that*

$$\|u\|_{\mathcal{H}_2^1(Q_T)} \le M$$

for every $u \in \mathcal{H}_2^1(Q_T)$ *and every* $\sigma \in [0,1]$ *such that* $u = T[u;\sigma]$.

Therefore, problem (6.17) *has a unique solution in* $\mathcal{H}_2^1(Q_T)$. *Moreover, if* $u_0 \geq 0$, *then* $u \geq 0$ *a.e. on* Q_T.

Proof. The proof of the existence is a consequence of the Leray-Schauder fixed point theorem (see e.g. [122, Theorem 11.6]) once (i) and (ii) are proven. To prove (i), take a sequence v_n bounded in $\mathcal{H}_2^1(Q_T)$ and let $u_n = T[v_n, \sigma_n]$. We first observe that $|b||Dv_n|$ is bounded in $\mathbb{H}_2^{-1}(Q_T)$. To see this, it is sufficient to note that by the (stationary) Sobolev and Hölder inequalities

$$\begin{aligned}\||b||Dv_n|\|_{L^2(0,T;H^{-1}(\mathbb{T}^d))} &\leq \||b||Dv_n|\|_{L^2(0,T;L^{\frac{2d}{d+2}}(\mathbb{T}^d))} \\ &\leq \||b|\|_{L^2(0,T;L^d(\mathbb{T}^d))}\|Dv_n\|_{L^2(Q_T)} \leq C\||b|\|_{L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(\mathbb{T}^d))}\|Dv_n\|_{L^2(Q_T)} \ .\end{aligned}$$

for $\mathcal{P}, \mathcal{Q}$ satisfying (6.13). By the compactness of the embedding $\mathcal{H}_2^1(Q_T)$ in $L^2(Q_T)$ (see e.g. [92, Proposition 2.2] and also Part II for a similar statement), there exists a subsequence, which we still call u_n, converging strongly to some limiting function u in $L^2(Q_T)$ and such that the sequence of gradients Du_n converges weakly in $L^2(Q_T)$ to Du and $\partial_t u_n$ to $\partial_t u$ in $\mathbb{H}_2^{-1}(Q_T)$. We take $u_n - u$ as a test function to get

$$\begin{aligned}&\int_0^T \frac{1}{2}\frac{d}{dt}\|u_n - u\|^2_{L^2(\mathbb{T}^d)} + \lambda \iint_{Q_T} |D(u_n - u)|^2 \, dxdt \\ &\leq \iint_{Q_T} b \cdot Dv_n(u_n - u)\, dxdt - \lambda \iint_{Q_T} Du \cdot D(u_n - u)\, dxdt + \iint_{Q_T} \partial_t v(v_n - v) \ .\end{aligned}$$

In the case $\mathcal{P} = \mathcal{Q} = d + 2$, the first term on the right-hand side converges to 0 by applying Hölder's inequality with the triple $(d + 2, 2(d + 2)/d, 2)$, and using that $u_n - u$ is bounded in $L^{2(d+2)/d}(Q_T)$ by the parabolic Sobolev embedding $\mathcal{H}_2^1(Q_T) \hookrightarrow L^{2(d+2)/d}(Q_T)$. The general case $\mathcal{P} \neq \mathcal{Q}$ can be handled exploiting the embeddings for mixed summability exponents (cf Remark C.2)

$$\mathcal{H}_{p_1}^{1,p_2}(Q_T) \hookrightarrow L^{q_2}(0, T; L^{q_1}(\mathbb{T}^d))$$

whenever

$$\frac{1}{2} = \frac{d}{2}\left(\frac{1}{p_1} - \frac{1}{q_1}\right) + \frac{1}{p_2} - \frac{1}{q_2} \ .$$

In particular, when $p_1 = p_2 = 2$ we have $\mathcal{H}_2^1(Q_T) \hookrightarrow L^{q_2}(0, T; L^{q_1}(\mathbb{T}^d))$

$$\frac{d}{2q_1} + \frac{1}{q_2} = \frac{d}{4} \ .$$

Therefore, using Hölder inequality in space and time with exponents $(\mathcal{P}, q_1, 2)$ and $(\mathcal{Q}, q_2, 2)$ respectively, with q_1, q_2 as above, we conclude the convergence of the first integral term in the general case.
The convergence of the second term is standard in view of the fact that $Du_n \rightharpoonup Du$ in $L^2(Q_T)$. As for the last term, it is sufficient to exploit the weak convergence of v_n to v in $\mathbb{H}_2^1(Q_T)$ and the fact that $\partial_t v \in \mathbb{H}_2^{-1}(Q_T)$.

This in particular gives the strong convergence of Du_n to Du in $L^2(Q_T)$. Finally, we observe that

$$\left|\iint_{Q_T} \partial_t(u_n - u)\varphi\, dxdt\right| \le C(\lambda)\iint_{Q_T} |D(u_n-u)||D\varphi|\, dxdt$$
$$+ \iint_{Q_T} |b||D(v_n - v)||\varphi|\, dxdt \le\le C(\|u_n - u\|_{\mathbb{H}_2^1(Q_T)} + \|Dv_n - Dv\|_{L^2(Q_T)})$$

for some $C > 0$. Then, the right-hand side approaches to 0 due to the above strong convergences. To prove (ii), we argue by contradiction, assuming that for any n the sequence $u_n \in \mathcal{H}_2^1(Q_T)$, $\sigma_n \in [0,1]$ such that $u_n = T[u_n, \sigma_n]$ and $\|u_n\|_{\mathcal{H}_2^1(Q_T)} \ge n$, namely $\|u_n\|_{\mathcal{H}_2^1(Q_T)} \to \infty$. This implies that

$$\partial_t u_n - a_{ij}(x,t)\partial_{ij}u_n(x,t) = \sigma_n b \cdot Du_n(x,t)\ .$$

Let us set $w_n := \frac{u_n}{\|u_n\|_{\mathcal{H}_2^1(Q_T)}}$ and observe that since $\sigma_n \le 1$

$$\partial_t w_n - a_{ij}(x,t)\partial_{ij}w_n(x,t) \le |b||Dw_n|$$

with $w_n(x,0) = \sigma_n \frac{u_0(x)}{\|u_n\|_{\mathcal{H}_2^1(Q_T)}}$. As in (i), we can deduce the strong convergence of w_n in $\mathcal{H}_2^1(Q_T)$ and hence, by letting $n \to \infty$, we conclude

$$\partial_t w - a_{ij}(x,t)\partial_{ij}w(x,t) \le |b||Dw|$$

with $w(x,0) = 0$, which gives $w \le 0$ by the comparison principle (see Proposition 6.7). The same procedure can be applied mutatis mutandis to $-w$ leading to $w \ge 0$, which allows to conclude $w = 0$ on Q_τ. However, $\|w_n\|_{\mathcal{H}_2^1(Q_T)} = 1$ and hence by the strong convergence we also have $\|w\|_{\mathcal{H}_2^1(Q_T)} = 1$, giving the contradiction. The last conclusion follows readily again by Proposition 6.7. Finally, one notices that $T[u;0] = 0$ by standard results for heat equations. The uniqueness is a consequence of the comparison principle in Proposition 6.7. □

6.5.3 Some auxiliary results

Lemma 6.10. *Let $p > 1$, and suppose that $a_{ij} \in C(Q_T)$ satisfies* (A). *Then, there exists a unique solution in $W_p^{2,1}(Q_T)$ to*

$$\begin{cases} \partial_t u(x,t) - \sum_{i,j=1}^d a_{ij}(x,t)\partial_{ij}u(x,t) = f(x,t) & \text{in } Q_T\ ,\\ u(x,0) = 0 & \text{in } \mathbb{T}^d\ . \end{cases}$$

Moreover, there exists a constant C (depending on λ, p, and the modulus of continuity of a on Q_T) such that

$$\|u\|_{W_p^{2,1}(Q_T)} \le C\|f\|_{L^p(Q_T)}\ . \tag{6.18}$$

For $u(x,0) = u_0 \in W^{2-2/p,p}(\mathbb{T}^d)$, then there exists a constant C (depending on λ, p, and the modulus of continuity of a on Q_T) such that

$$\|u\|_{W_p^{2,1}(Q_T)} \le C(\|f\|_{L^p(Q_T)} + \|u_0\|_{W^{2-2/p,p}(\mathbb{T}^d)})\ . \tag{6.19}$$

Proof. This is a classical maximal L^p regularity statement for uniformly elliptic equations with continuous coefficients, that can be deduced from results contained in [159]; see [91, Appendix A] for additional details on the periodic setting. One can also rely on abstract results on maximal regularity for parabolic equations in [198]. In particular, in the case of non-zero initial trace, one has to use that the sharp space of initial trace is described via the trace method from interpolation theory in Banach spaces (see e.g. [178, Corollary 1.14] and Part II) to conclude that $u(0) \in (L^p, W^{2,p})_{1-1/p,p} \simeq W^{2-2/p,p}$. □

The following continuous embedding result of $\mathcal{H}^1_\sigma(Q_T)$ into $L^p(Q_T)$ is rather known and can be found for example in [92, 185], where, however, the estimates are local in time. Here we need its stability as $T \to 0$ and hence this requires an additional control on the trace at some time (e.g. $t = 0$). We provide a proof here for the reader's convenience, that does not make use of fractional Sobolev spaces.

Proposition 6.11. *If $1 < \sigma < (d+2)/(d+1)$, then $\mathcal{H}^1_\sigma(\mathbb{T}^d \times (\tau_1, \tau))$, $0 < \tau_1 < \tau \leq T$ is continuously embedded into $L^p(\mathbb{T}^d \times (\tau_1, \tau))$ for*

$$\frac{1}{p} = \frac{1}{\sigma} - \frac{1}{d+2}.$$

Moreover, if $u \in \mathcal{H}^1_\sigma(\mathbb{T}^d \times (\tau_1, \tau))$ and $u(\cdot, \tau) \in L^1(\mathbb{T}^d)$, we have

$$\|u\|_{L^p(\mathbb{T}^d \times (\tau_1,\tau))} \leq C\big(\|u\|_{\mathcal{H}^1_\sigma(\mathbb{T}^d \times (\tau_1,\tau))} + \|u(\tau_1)\|_{L^1(\mathbb{T}^d)} \big) , \tag{6.20}$$

where the constant C depends on d, p, σ, T, but remains bounded for bounded values of T.

Proof. Let $f \in L^{p'}(\mathbb{T}^d \times (\tau_1, \tau))$ and φ be the solution to

$$\begin{cases} -\partial_t \varphi(x,t) - \Delta\varphi(x,t) = f(x,t) & \text{in } \mathbb{T}^d \times (\tau_1, \tau) , \\ \varphi(x,\tau) = 0 & \text{in } \mathbb{T}^d . \end{cases}$$

By Lemma 6.10, φ satisfies

$$\|\varphi\|_{W^{2,1}_{p'}(Q_T)} \leq C\|f\|_{L^{p'}(Q_T)} . \tag{6.21}$$

Note that C here may depend on τ, but it is the same for all $\tau \leq 1$ (if $\tau < 1$, it is sufficient to extend trivially f on $\mathbb{T}^d \times (\tau, 1)$ and use (6.19) on $\mathbb{T}^d \times (0, 1)$). Note that $(d+2)/2 < p' < d+2$. Therefore, by the embedding results in [159, Lemma II.3.3],

$$\|\varphi\|_{C(\mathbb{T}^d\times(\tau_1,\tau))} \leq C\|\varphi\|_{W^{2,1}_{p'}(\mathbb{T}^d\times(\tau_1,\tau))}, \qquad \|\varphi\|_{W^{1,0}_{\sigma'}(\mathbb{T}^d\times(\tau_1,\tau))} \leq C\|\varphi\|_{W^{2,1}_{p'}(\mathbb{T}^d\times(\tau_1,\tau))} \tag{6.22}$$

Note that a straightforward application of [159, Lemma II.3.3] yields bounded constants in (6.22) as $\tau \to 0$, plus an additional term on the right-hand sides of the form $C_1\tau^{-1}\|\varphi\|_{L^{p'}(Q_T)}$; this term can be removed using the fact that $\varphi(\tau) = 0$, that guarantees $\|\varphi\|_{L^{p'}(\mathbb{T}^d\times(\tau_1,\tau))} \leq \tau\|\partial_t\varphi\|_{L^{p'}(\mathbb{T}^d\times(\tau_1,\tau))} \leq \|\varphi\|_{W^{2,1}_{p'}(\mathbb{T}^d\times(\tau_1,\tau))}$. Note also that here we can identify norms on $\mathbb{T}^d$ with norms on $\Omega = (0, 1)^d$.

Therefore, integrating by parts in time and using (6.21) and (6.22),

$$
\begin{aligned}
&\left|\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} uf\,dxdt\right| = \left|\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} u(-\partial_t\varphi-\Delta\varphi)\,dxdt\right| \\
&\quad\le \int_{\mathbb{T}^d}|\varphi(x,\tau_1)u(x,\tau_1)|dx + \left|\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d}\partial_t u\,\varphi\,dxdt\right| + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d}|D\varphi|\,|Du|\,dxdt \\
&\quad\le C\Big(\|\varphi(\tau_1)\|_{L^\infty(\mathbb{T}^d)}\|u(\tau_1)\|_{L^1(\mathbb{T}^d)} + \|\partial_t u\|_{\left(W^{1,0}_{\sigma'}(\mathbb{T}^d\times(\tau_1,\tau))\right)'}\|\varphi\|_{W^{1,0}_{\sigma'}(\mathbb{T}^d\times(\tau_1,\tau))} \\
&\qquad\qquad + \|Du\|_{L^\sigma(\mathbb{T}^d\times(\tau_1,\tau))}\|D\varphi\|_{L^{\sigma'}(\mathbb{T}^d\times(\tau_1,\tau))}\Big) \\
&\le C\Big(\|u(\tau_1)\|_{L^1(\mathbb{T}^d)} + \|\partial_t u\|_{\left(W^{1,0}_{\sigma'}(\mathbb{T}^d\times(\tau_1,\tau))\right)'} + \|Du\|_{L^\sigma(\mathbb{T}^d\times(\tau_1,\tau))}\Big)\|f\|_{L^{p'}(\mathbb{T}^d\times(\tau_1,\tau))},
\end{aligned}
$$

yielding the desired result.

□

We need the following generalization of [236, Theorem 2.1.6] for weak derivatives of difference quotients

$$
D^h u := \frac{u(x+h,t)-u(x,t)}{h}\ ,\ h\in\mathbb{R}^d\ .
$$

Lemma 6.12. *Let $1<p<\infty$ and $0<\tau_1<\tau\le T$. Assume $u\in L^p(\tau_1,\tau;W^{1,p}(\mathbb{T}^d))$ and $f\in L^q(\mathbb{T}^d\times(\tau_1,\tau))$, with $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$
\left\|fD^h u\right\|_{L^1(\mathbb{T}^d\times(\tau_1,\tau))} \le \|f\|_{L^q(\mathbb{T}^d\times(\tau_1,\tau))}\,\|Du\|_{L^p(\mathbb{T}^d\times(\tau_1,\tau))}
$$

Proof. Let $u\in C^1(\mathbb{T}^d\times(\tau_1,\tau))\cap L^p(\tau_1,\tau;W^{1,p}(\mathbb{T}^d))$ and for the general case argue by density. We first show that

$$
\|u(\cdot+h,t)-u(\cdot,t)\|_{L^p(\mathbb{T}^d)} \le |h|\,\|Du(\cdot,t)\|_{L^p(\mathbb{T}^d)}
$$

We write

$$
\frac{u(x+h,t)-u(x,t)}{h} = \frac{1}{|h|}\int_0^{|h|}\left|Du\left(x+\theta\frac{h}{|h|},t\right)\right|d\theta
$$

By Jensen's inequality we obtain

$$
\left|\frac{u(x+h,t)-u(x,t)}{h}\right|^p = \frac{1}{|h|}\int_0^{|h|}\left|Du\left(x+\theta\frac{h}{|h|},t\right)\right|^p d\theta
$$

and we then conclude the assertion integrating over Q_T and exchanging the order of integration by Fubini's Theorem. Then

$$
\begin{aligned}
\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} f(x,t)\frac{u(x+h,t)-u(x,t)}{h}dx &\le \|f\|_{L^q(\mathbb{T}^d\times(\tau_1,\tau))}\,\|D^h u\|_{L^p(\mathbb{T}^d\times(\tau_1,\tau))} \\
&\le \|f\|_{L^q(\mathbb{T}^d\times(\tau_1,\tau))}\,\|Du\|_{L^p(\mathbb{T}^d\times(\tau_1,\tau))}
\end{aligned}
$$

□

6.5.4 Well-posedness and regularity of the adjoint equation

This section is devoted to the analysis of the following Fokker-Planck equation

$$\begin{cases} -\partial_t \rho(x,t) - \sum_{i,j=1}^d \partial_{ij}(a_{ij}(x,t)\rho(x,t)) + \operatorname{div}(b(x,t)\,\rho(x,t)) = 0 & \text{in } Q_\tau\ , \\ \rho(x,\tau) = \rho_\tau(x) & \text{in } \mathbb{T}^d\ . \end{cases} \tag{6.23}$$

Note that when the vector field $b(x,t) = -D_pH(x, Du(x,t))$, then (6.23) becomes the adjoint equation of the linearization of (6.1).

Here, $\tau \in (0,T]$ and $Q_\tau := \mathbb{T}^d \times (0,\tau)$.

Definition 6.13. *For $b \in L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(\mathbb{T}^d))$ for some $\mathcal{P} \geq d$, and $\mathcal{Q} \geq 2$ satisfying (6.13), a (weak) solution $\rho \in \mathcal{H}_2^1(Q_\tau)$ is such that $\rho(\tau) = \rho_\tau$ in the L^2-sense, and*

$$-\int_0^\tau \langle \partial_t \rho(t), \varphi(t)\rangle dt + \iint_{Q_\tau} \partial_j(a_{ij}\rho)\partial_i\varphi - b\rho \cdot D\varphi\, dxdt = 0 \tag{6.24}$$

for all $\varphi \in \mathcal{H}_2^1(Q_\tau)$.

Remark 6.14. Note that the integral term

$$I = \iint_{Q_\tau} \rho b \cdot D\varphi\, dxdt$$

is well-posed. Indeed, we have

$$|I| \leq \|D\varphi\|_{L^2(Q_T)} \||b|\|_{L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))} \|\rho\|_{L^{\frac{2\mathcal{Q}}{\mathcal{Q}-2}}(0,\tau;L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(\mathbb{T}^d))}$$

where $\mathcal{Q}, \mathcal{P}$ fulfills

$$\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} = \frac{1}{2}$$

In particular, such condition implies also that

$$\frac{\mathcal{Q}-2}{2\mathcal{Q}} + \frac{d(\mathcal{P}-2)}{4\mathcal{P}} = \frac{d}{4}$$

and this allows to apply [103, Proposition I.3.3] to exploit the embedding of $V_2(Q_\tau) = L^\infty(0,\tau;L^2(\mathbb{T}^d)) \cap L^2(0,\tau;W^{1,2}(\mathbb{T}^d))$ onto $L^{\frac{2\mathcal{Q}}{\mathcal{Q}-2}}(0,\tau;L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(\mathbb{T}^d))$. Then

$$|I| \leq C_1\|D\varphi\|_{L^2(Q_T)}\||b|\|_{L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))}\|\rho\|_{V_2(Q_\tau)} \leq C_2\|D\varphi\|_{L^2(Q_T)}\||b|\|_{L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))}\|\rho\|_{\mathcal{H}_2^1(Q_\tau)}$$

by finally using the fact that $\mathcal{H}_2^1 \hookrightarrow C(L^2)$.

Throughout this section we will assume that

$$\rho_\tau \in C^\infty(\mathbb{T}^d), \quad \rho_\tau \geq 0, \quad \text{and} \quad \int_{\mathbb{T}^d} \rho_\tau(x)\, dx = 1. \tag{6.25}$$

Note that $\rho \in C([0,\tau];L^2(\mathbb{T}^d))$, so $\rho \in C([0,\tau];L^1(\mathbb{T}^d))$, and

$$\int_{\mathbb{T}^d} \rho(x,t)\, dx = 1 \qquad \text{for all } t \in [0,\tau]. \tag{6.26}$$

This can be easily verified using $\varphi \equiv 1$ as a test function in (7.5). Evolutive equations with divergence type terms and discontinuous coefficients were analyzed in [48, 39], while we refer to [47] and references therein for the elliptic counterpart.

Proposition 6.15. *Let* (A) *be in force,* $b \in L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))$ *for some* $\mathcal{P} \geq d$, $\mathcal{Q} \geq 2$ *satisfying* (6.13)*, and* $\rho_\tau \in L^2(\mathbb{T}^d)$*. Then, there exists a unique weak solution* $\rho \in \mathcal{H}_2^1(Q_\tau)$ *to* (6.23) *satisfying the estimate*

$$\|\rho\|_{\mathcal{H}_2^1(Q_\tau)} \leq C$$

If in addition $\rho_\tau \in L^m(\mathbb{T}^d)$ *for* $m \in (1,\infty)$*, then* ρ *is bounded in* $L^\infty(0,\tau;L^m(\mathbb{T}^d)) \cap L^\eta(0,\tau;W^{1,\eta}(\mathbb{T}^d))$*, with* $\eta = 2$ *if* $m \geq 2$ *and* $\eta = \frac{m(d+2)}{m+d}$ *if* $m \in (1,2)$*. Finally,* ρ *is a.e. non-negative on* Q_τ*.*

Proof. The existence part when $\rho_\tau \in L^2(\mathbb{T}^d)$, i.e. the PDE is driven by the Laplacian, can be proven as in Theorem 7.12 for the case of the fractional Laplacian via Caldéron-Zygmund parabolic regularity (roughly speaking, it is enough to set $s = 1$ ans exploit the local analogue of Theorem B.4) and Leray-Schauder fixed point theorem [122, Theorem 11.6], while the general case is proven in [159] via Galerkin approximation method. The uniqueness can be inferred by duality. As for integrability estimates, we quote the results in [48, Lemma 3.2], which can be achieved in the case $p = 2$ by using $\varphi = \rho$ as a test function, and in the more general case $\rho_\tau \in L^m(\mathbb{T}^d)$ by using a suitable power of ρ. The adaptation in the periodic setting of the arguments used there is straightforward. □

Remark 6.16. We remark that when condition 6.13 is fulfilled as a strict inequality, actually ρ is bounded. This can be seen via an appropriate choice of the test function as in [48, Theorem 2.1]. See also the approach via Duhamel's formula in [39] and the next Proposition 7.12 for the case of subcritical fractional diffusion. We also point out that (6.13) is sharp. In fact, in [39] it is proven in the whole space setting that when

$$\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} > \frac{1}{2}\ ,$$

then ρ does not enjoy any estimates uniform in time better than in $L^1(\mathbb{R}^d)$ for a smooth velocity field and initial datum.

By gathering the previous results we basically get the well-posedness of the Fokker-Planck equation for fixed ρ_τ. The main goal is now to derive estimates on ρ that are stable for any ρ_τ satisfying merely (6.25); one may have in mind that ρ_τ is an item of a sequence approaching a Dirac delta. These estimates will be achieved using some information on the integrability of the vector field b with respect to the solution ρ itself, that is a typical datum in the analysis of Hamilton-Jacobi equations.

The following proposition is a modification of [92, Proposition 2.4], and is a kind of Caldéron-Zygmund parabolic regularity result for equations with divergence-type terms. Similar regularity estimates already appeared in [194, Prop. 3.10-(iii)] when $b \in L^2(\rho\, dxdt)$ via the renormalized formulation, and [185, Section 3]. We stress that here the constraint on the integrability exponent q' is completely determined by the regularity of the initial datum. Both in the context of regularity theory for transport equations and MFGs, our main achievement is to obtain Sobolev regularity whenever $b \in L^k(\rho\, dxdt)$ with $k < 2$ (in the context of MFGs this allows a treatment of the superquadratic case $\gamma > 2$)

We will show in the forthcoming Proposition 6.20 that one can actually reach the threshold $q' = 2$ whenever $\rho_\tau \in L^2(\mathbb{T}^d)$ via the same procedure.

Proposition 6.17. *Let ρ be a (non-negative) weak solution to* (6.23) *and*

$$1 < q' < \frac{d+2}{d+1}.$$

Then, there exists $C > 0$, depending on $\lambda, \|a\|_{C(W^{1,\infty})}, q', d, T$ such that

$$\|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} \le C(\|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}). \tag{6.27}$$

Note that C here does not depend on $\tau \in (0, T]$.

Proof. We assume that the coefficients a_{ij}, b_i are smooth, and therefore ρ is smooth as well on Q_τ. The general case $Da \in L^\infty(Q_\tau)$, $b \in L^{\mathcal{Q}}(0, T; L^{\mathcal{P}}(\mathbb{T}^d))$ follows by an approximation argument as in Proposition 6.15.

Fix $k = 1, ..., d$. For $\delta > 0$, let $\psi = \psi_\delta$ be the classical solution to

$$\begin{cases} \partial_t \psi(x,t) - \sum_{i,j} a_{ij}(x,t)\partial_{ij}\psi(x,t) = (\delta + |\partial_k \rho(x,t)|^2)^{\frac{q'-2}{2}} \partial_k \rho(x,t) & \text{in } Q_\tau \ , \\ \psi(x,0) = 0 & \text{on } \mathbb{T}^d \ . \end{cases} \tag{6.28}$$

Since $q' < 2$, $\delta > 0$ serves as a regularizing perturbation. By standard parabolic regularity (see Lemma 6.10), we have (for a positive constant not depending on $\tau \le T$)

$$\|\psi\|_{W^{2,1}_q(Q_\tau)} \le C \left\| (\delta + |\partial_k \rho|^2)^{\frac{q'-2}{2}} \partial_k \rho \right\|_{L^q(Q_\tau)} \le C \left\| |\partial_k \rho|^{q'-1} \right\|_{L^q(Q_\tau)} = C \, \|\partial_k \rho\|^{q'-1}_{L^{q'}(Q_\tau)} . \tag{6.29}$$

Set $\varphi(x,t) = \partial_{x_k}\psi(x,t)$. Then, φ is a classical solution to

$$\begin{cases} \partial_t \varphi - \sum_{i,j} a_{ij}\partial_{ij}\varphi = \partial_k \left[(\delta + |\partial_k \rho|^2)^{\frac{q'-2}{2}} \partial_k \rho \right] + \sum_{i,j} \partial_k(a_{ij})\partial_{ij}\psi & \text{in } Q_\tau \ , \\ \varphi(x,0) = 0 & \text{on } \mathbb{T}^d \ . \end{cases} \tag{6.30}$$

Using φ as a test function for the equation satisfied by ρ,

$$\iint_{Q_\tau} \rho(\partial_t \varphi - a_{ij}\partial_{ij}\varphi - b \cdot D\varphi) dxdt = -\int_{\mathbb{T}^d} \rho_\tau(x)\varphi(x,\tau) dx \ ,$$

and using the equation in (6.30) satisfied by φ we get, after integration by parts

$$\iint_{Q_\tau} (\delta + |\partial_k \rho|^2)^{\frac{q'-2}{2}} |\partial_k \rho|^2 - \partial_k(a_{ij})\partial_{ij}\psi \, \rho + b\rho \cdot D\varphi \, dxdt = \int_{\mathbb{T}^d} \rho_\tau(x)\varphi(x,\tau) dx \ ,$$

Applying Hölder's inequality,

$$\begin{aligned} \iint_{Q_\tau} (\delta + |\partial_k \rho|^2)^{\frac{q'-2}{2}} |\partial_k \rho|^2 \, dxdt \le \|Da\|_{L^\infty(Q_\tau)} \|\psi\|_{W^{2,1}_q(Q_\tau)} \|\rho\|_{L^{q'}(Q_\tau)} \\ + \|b\rho\|_{L^{q'}(Q_\tau)} \|D\varphi\|_{L^q(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)} \|\varphi(\cdot,\tau)\|_\infty . \end{aligned}$$

Since $q > d+2$, by [159, Lemma II.3.3], the parabolic space $W^{2,1}_q(Q_\tau)$ is continuously embedded into $C([0,\tau]; C^1(\mathbb{T}^d))$, therefore $\|\varphi(\cdot,\tau)\|_\infty \le \|\psi(\cdot,\tau)\|_{C^1(\mathbb{T}^d)} \le$

$C\|\psi\|_{W^{2,1}_q(Q_\tau)}$ (to be sure that C does not explode as $\tau \to 0$, one has to exploit that $\psi(0) = 0$, and argue as in the proof of Proposition 6.11). Hence, since $\varphi = \partial_{x_k}\psi$,

$$\iint_{Q_\tau} (\delta + |\partial_k \rho|^2)^{\frac{q'-2}{2}} |\partial_k \rho|^2 \, dxdt \le C(\|\rho\|_{L^{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)})\|\psi\|_{W^{2,1}_q(Q_\tau)}.$$

By (6.29) and letting $\delta \to 0$,

$$\iint_{Q_\tau} |\partial_k \rho|^{q'} \, dxdt \le C(\|\rho\|_{L^{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)})\|\partial_k \rho\|^{q'-1}_{L^{q'}(Q_\tau)}.$$

Summarizing, we conclude

$$\|D\rho\|_{L^{q'}(Q_\tau)} \le C(\|\rho\|_{L^{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}) \ . \tag{6.31}$$

By Poincaré-Wirtinger inequality and (6.31), together with the fact that $\int_{\mathbb{T}^d} \rho(x,t)dx = 1$ for all $t \in [0,\tau]$, we obtain

$$\|\rho\|^{q'}_{L^{q'}(Q_\tau)} \le C(\|D\rho\|^{q'}_{L^{q'}(Q_\tau)} + \tau\|\rho_\tau\|^{q'}_{L^1(\mathbb{T}^d)}) \ ,$$

yielding, together with (6.31)

$$\|\rho\|_{W^{1,0}_{q'}(Q_\tau)} \le C(\|\rho\|_{L^{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}).$$

Finally, for any smooth test function φ (which may not vanish at the terminal time τ), again by Hölder's inequality

$$\begin{aligned}\left|\int_0^\tau \langle \partial_t \rho(t), \varphi(t)\rangle dt\right| &\le \iint_{Q_\tau} |\partial_j(a_{ij}\rho)\partial_i \varphi| + |b\rho|\,|D\varphi| \, dxdt \\ &\le \left[(\|a\|_{L^\infty(Q_\tau)} + \|Da\|_{L^\infty(Q_\tau)})\, \|\rho\|_{W^{1,0}_{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)}\right]\|D\varphi\|_{L^q(Q_\tau)}.\end{aligned}$$

Thus,

$$\|\partial_t \rho\|_{(W^{1,q}(Q_\tau))'} \le C(\|\rho\|_{L^{q'}(Q_\tau)} + \|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}) \ .$$

□

Proposition 6.18. *Let ρ be the (non-negative) weak solution to* (6.23) *and*

$$1 < q' < \frac{d+2}{d+1}.$$

Then, there exists $C > 0$, depending on $\lambda, \|a\|_{C(W^{1,\infty})}, T, q', d$ such that

$$\|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} \le C\left(\iint_{Q_\tau} |b(x,t)|^{r'}\rho(x,t)\, dxdt + 1\right), \tag{6.32}$$

where

$$r' = 1 + \frac{d+2}{q}. \tag{6.33}$$

Proof. Inequality (6.27), (6.25) and the generalized Hölder's inequality yield

$$\begin{aligned}\|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} &\le C(\|b\rho^{1/r'}\rho^{1/r}\|_{L^{q'}(Q_\tau)} + \|\rho\|_{L^{q'}(Q_\tau)} + 1) \\ &\le C\left(\left(\iint_{Q_\tau} |b|^{r'}\rho\,dxdt\right)^{1/r'} \|\rho\|^{1/r}_{L^p(Q_\tau)} + \|\rho\|_{L^{q'}(Q_\tau)} + 1\right), \quad (6.34)\end{aligned}$$

for $p > q'$ satisfying

$$\frac{1}{q'} = \frac{1}{r'} + \frac{1}{rp}. \tag{6.35}$$

Then, by Young's inequality, for all $\varepsilon > 0$

$$\|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} \le C\left(\frac{1}{\varepsilon}\iint_{Q_\tau} |b|^{r'}\rho\,dxdt + \varepsilon\|\rho\|_{L^p(Q_\tau)} + \|\rho\|_{L^{q'}(Q_\tau)} + 1\right), \tag{6.36}$$

Since $\|\rho\|_{L^1(Q_\tau)} = \tau$, by interpolation between $L^1(Q_\tau)$ and $L^p(Q_\tau)$ we have $\|\rho\|_{L^{q'}(Q_\tau)} \le \tau^{1/r'}\|\rho\|^{1/r}_{L^p(Q_\tau)}$, and again by Young's inequality

$$\|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} \le \widetilde{C}\left(\frac{1}{\varepsilon}\iint_{Q_\tau} |b|^{r'}\rho\,dxdt + \varepsilon\|\rho\|_{L^p(Q_\tau)} + 1\right), \tag{6.37}$$

One can verify that (6.33) and (6.35) yield

$$\frac{1}{p} = \frac{1}{q'} - \frac{1}{d+2}.$$

The continuous embedding of $\mathcal{H}^1_{q'}(Q_\tau)$ in $L^p(Q_\tau)$ stated in Proposition 6.11 then implies

$$\|\rho\|_{L^p(Q_\tau)} \le C_1\big(\|\rho\|_{\mathcal{H}^1_{q'}(Q_\tau)} + \tau\big).$$

Hence, the term $\varepsilon\|\rho\|_{L^p(Q_\tau)}$ can be absorbed by the left hand side of (6.37) by choosing $\varepsilon = (2\widetilde{C}C_1)^{-1}$, thus providing the assertion. □

6.6 Additional regularity results

We remark that the above results, especially those obtained in Proposition 6.17, are based on (parabolic) Caldéron-Zygmund regularity and duality arguments. These parabolic regularity results allows to get Sobolev regularity of the adjoint equation when $b \in L^k(\rho)$, i.e. in terms of the crossed quantity

$$\iint |b|^k\rho < \infty\ ,$$

for some $k > 1$. We conclude this section by providing a Sobolev regularity result for solutions of the forward equation driven by the Laplacian

$$\begin{cases}\partial_t\rho - \Delta\rho - \operatorname{div}(b(x,t)\rho) = 0 & \text{in } \mathbb{T}^d \times (0,\tau)\ , \\ \rho(0) = \rho_0(x) & \text{on } \mathbb{T}^d\ .\end{cases} \tag{6.38}$$

in terms of $\iint |b|^k\rho$, without using maximal L^p-regularity results and duality arguments, which, however, we will not use in the sequel. This is inspired by the approach used in the stationary setting in [184] (see also [185, Proposition 3.3] for similar results).

Proposition 6.19. *Let $b \in L^k(\rho)$, i.e. $\iint_{Q_T} |b|^k\rho < \infty$ for $2 < k < 2 + d/2$, $\beta := \frac{k-2}{d+2-k}$ and $\rho_0 \in L^{\beta+1}(\mathbb{T}^d)$. Then every non-negative weak solution $\rho \in \mathcal{H}_2^1(Q_\tau) \cap L^\infty(Q_\tau)$ to (6.38) satisfies, for k as above, the estimate*

$$\|\rho\|_{L^{\frac{d+2}{d+2-k}}(Q_\tau)} \leq C\left(\iint_{Q_\tau} |b|^k\rho\, dxdt + \|\rho_0\|_{L^{\beta+1}(\mathbb{T}^d)}\right)$$

where C is a positive constant depending on d, k and $q = \frac{d+2}{k-2}$. Finally, as a consequence, we have

$$\|\rho\|_{\mathcal{H}^1_{\frac{d+2}{d+3-k}}(Q_\tau)} \leq C\left(\iint_{Q_\tau} |b|^k\rho\, dxdt + \|\rho_0\|_{L^{\beta+1}(\mathbb{T}^d)}\right)$$

Proof. As in the elliptic case [184] the strategy is to use $\varphi = \rho^\beta$ with $\beta := \frac{k-2}{d+2-k}$ (or better $\varphi = (\rho+\varepsilon)^\beta$ and then let $\varepsilon \to 0$). By Young's inequality we have

$$\begin{aligned}\frac{1}{\beta+1}\int_{\mathbb{T}^d} |\rho(x,\tau)|^{\beta+1}\, dx + \beta\iint_{Q_\tau} \rho^{\beta-1}|D\rho|^2\, dxdt \\ \leq \iint_{Q_\tau} |b|\rho^\beta|D\rho|\, dxdt + \frac{1}{\beta+1}\int_{\mathbb{T}^d} |\rho(x,0)|^{\beta+1}\, dx \\ \leq C_\beta \iint_{Q_\tau} |b|^2\rho^{\beta+1}\, dxdt + \frac{\beta}{4}\iint_{Q_\tau} \rho^{\beta-1}|D\rho|^2\, dxdt + \frac{1}{\beta+1}\int_{\mathbb{T}^d} |\rho(x,0)|^{\beta+1}\, dx\ .\end{aligned}$$

Denote by

$$J = \text{ess sup}_{t\in(0,T)} \int_{\mathbb{T}^d} |\rho(x,\tau)|^{\beta+1}\, dx + \beta\iint_{Q_\tau} \rho^{\beta-1}|D\rho|^2\, dxdt\ .$$

and by first passing to the supremum over $t \in (0,\tau)$ and then applying the Sobolev embedding in [103, Proposition I.3.1]

$$\begin{aligned}\frac{J}{\beta+1} = \frac{1}{\beta+1}\left(\text{ess sup}_{t\in(0,T)} \int_{\mathbb{T}^d} (|\rho(x,\tau)|^{\frac{\beta+1}{2}})^2\, dx + \beta\iint_{Q_\tau} |D\rho^{\frac{\beta+1}{2}}|^2\, dxdt\right) \\ \geq C\left(\iint_{Q_\tau} \rho^{(\beta+1)\frac{d+2}{d}}\, dxdt\right)^{1-\frac{2}{d+2}}\ .\end{aligned}$$

We then have

$$\begin{aligned}\left(\iint_{Q_\tau} \rho^{(\beta+1)\frac{d+2}{d}}\, dxdt\right)^{1-\frac{2}{d+2}} + \frac{\beta}{4}\iint_{Q_\tau} \rho^{\beta-1}|D\rho|^2\, dxdt \\ \leq C_1\left(\iint_{Q_\tau} |b|^k\rho\, dxdt\right)^{\frac{2}{k}}\left(\iint_{Q_\tau} \rho^{\beta\frac{k}{k-2}+1}\right)^{1-2/k} + C_2\|\rho_0\|^{\beta+1}_{L^{\beta+1}(Q_\tau)}.\end{aligned}$$

We then apply Hölder's and Young's inequalities to the first term of the right-hand side of the above inequality to get

$$
\begin{aligned}
C_1 \left(\iint_{Q_\tau} |b|^k \rho \, dxdt \right)^{\frac{2}{k}} \left(\iint_{Q_\tau} \rho^{\beta \frac{k}{k-2}+1} \right)^{1-2/k} \\
\le C_2 \left(\iint_{Q_\tau} |b|^k \rho \, dxdt \right)^{\frac{d}{d+2-k}} + \frac{1}{2} \left(\iint_{Q_\tau} \rho^{(\beta+1)\frac{d+2}{d}} \, dxdt \right)^{1-\frac{2}{d+2}} .
\end{aligned}
$$

We note then that

$$
\beta q = (\beta+1)\frac{d+2}{d} = \beta \frac{k}{k-2} + 1 = \frac{d+2}{d+2-k}
$$

giving thus

$$
\begin{aligned}
\left(\iint_{Q_\tau} \rho^{\frac{d+2}{d+2-k}} \right)^{\frac{d}{d+2}} + \frac{\beta}{4} \iint_{Q_\tau} \rho^{\beta-1} |D\rho|^2 \, dxdt \\
\le C \left[\left(\iint_{Q_\tau} |b|^k \rho \, dxdt \right)^{\frac{d}{d+2-k}} + \|\rho_0\|^{\beta+1}_{L^{\beta+1}(Q_\tau)} + 1 \right] ,
\end{aligned}
$$

and finally, noting that $\beta + 1 = \frac{d}{d+2-k}$ we have

$$
\left(\iint_{Q_\tau} \rho^{\frac{d+2}{d+2-k}} \right)^{\frac{d+2-k}{d+2}} \le C \left(\iint_{Q_\tau} |b|^k \rho \, dxdt + \|\rho_0\|_{L^{\beta+1}(\mathbb{T}^d)} \right) .
$$

The second estimate can be obtained via [185, Lemma 3.2]. However, it can be proven simply by observing that the above computations gives a bound on $\iint_{Q_\tau} \rho^{\beta-1}|D\rho|^2 \, dxdt$. Therefore, we can use it to estimate

$$
\begin{aligned}
\|D\rho\|_{L^{\frac{d+2}{d+3-k}}(Q_\tau)} &\le \|\rho^{(\beta-1)/2} D\rho\|_{L^2(Q_\tau)} \|\rho^{(1-\beta)/2}\|_{L^{\frac{2(d+2)}{d+4-2k}}(Q_\tau)} \\
&= \left(\iint_{Q_\tau} \rho^{\beta-1} |D\rho|^2 \, dxdt \right)^{\frac{1}{2}} \left(\iint_{Q_\tau} \rho^{\frac{d+2}{d+2-k}} \right)^{\frac{d+4-2k}{2(d+2)}} .
\end{aligned}
$$

The estimate for $\partial_t \rho$ follows by duality (see [185]). □

Proposition 6.20. *Let ρ be a (non-negative) weak solution to* (6.23)*. Then, there exists $C > 0$, depending on $\lambda, \|a\|_{C(W^{1,\infty})}, d, T$ such that*

$$
\|\rho\|_{\mathcal{H}^1_2(Q_\tau)} \le C(\|b\rho\|_{L^2(Q_\tau)} + \|\rho\|_{L^2(Q_\tau)} + \|\rho_\tau\|_{L^2(\mathbb{T}^d)}). \tag{6.39}
$$

Proof. We assume that the coefficients a_{ij}, b_i are smooth, and therefore ρ is smooth as well on Q_τ. The general case $Da \in L^\infty(Q_\tau)$, $b \in L^{\mathcal{Q}}(0,T; L^{\mathcal{P}}(\mathbb{T}^d))$ follows by an approximation argument.

Fix $k = 1, ..., d$. Let ψ be the classical solution to

$$
\begin{cases}
\partial_t \psi(x,t) - \sum_{i,j} a_{ij}(x,t) \partial_{ij} \psi(x,t) = \partial_k \rho(x,t) & \text{in } Q_\tau \ , \\
\psi(x,0) = 0 & \text{on } \mathbb{T}^d \ .
\end{cases} \tag{6.40}
$$

By standard parabolic regularity (see Lemma 6.10), we have (for a positive constant not depending on $\tau \le T$)

$$\|\psi\|_{W_2^{2,1}(Q_\tau)} \le C\|\partial_k \rho\|_{L^2(Q_\tau)}. \tag{6.41}$$

Set $\varphi(x,t) = \partial_{x_k}\psi(x,t)$. Then, φ is a classical solution to

$$\begin{cases} \partial_t \varphi - \sum_{i,j} a_{ij}\partial_{ij}\varphi = \partial^2_{kk}\rho + \sum_{i,j}\partial_k(a_{ij})\partial_{ij}\psi & \text{in } Q_\tau \ , \\ \varphi(x,0) = 0 & \text{on } \mathbb{T}^d \ . \end{cases} \tag{6.42}$$

Using φ as a test function for the equation satisfied by ρ,

$$\iint_{Q_\tau} \rho(\partial_t\varphi - a_{ij}\partial_{ij}\varphi - b\cdot D\varphi)dxdt = -\int_{\mathbb{T}^d}\rho_\tau(x)\varphi(x,\tau)dx \ ,$$

and using the equation in (6.42) satisfied by φ we get, after integration by parts

$$\iint_{Q_\tau} |\partial_k\rho|^2 - \partial_k(a_{ij})\partial_{ij}\psi\,\rho + b\rho\cdot D\varphi\,dxdt = \int_{\mathbb{T}^d}\rho_\tau(x)\varphi(x,\tau)dx \ .$$

Applying Hölder's inequality,

$$\begin{aligned}\iint_{Q_\tau} |\partial_k\rho|^2\,dxdt \le{}& \|Da\|_{L^\infty(Q_\tau)}\|\psi\|_{W_2^{2,1}(Q_\tau)}\|\rho\|_{L^2(Q_\tau)} \\ &+ \|b\rho\|_{L^2(Q_\tau)}\|D\varphi\|_{L^2(Q_\tau)} + \|\rho_\tau\|_{L^2(\mathbb{T}^d)}\|\varphi(\tau)\|_{L^2(\mathbb{T}^d)}.\end{aligned}$$

Hence, since $\varphi = \partial_{x_k}\psi$,

$$\iint_{Q_\tau} |\partial_k\rho|^2\,dxdt \le C(\|\rho\|_{L^2(Q_\tau)} + \|b\rho\|_{L^2(Q_\tau)} + \|\rho_\tau\|_{L^2(\mathbb{T}^d)})\|\psi\|_{W_2^{2,1}(Q_\tau)}.$$

By (6.41) we have

$$\iint_{Q_\tau} |\partial_k\rho|^2\,dxdt \le C(\|\rho\|_{L^2(Q_\tau)} + \|b\rho\|_{L^2(Q_\tau)} + \|\rho_\tau\|_{L^2(\mathbb{T}^d)})\|\partial_k\rho\|_{L^2(Q_\tau)}.$$

Summarizing, we conclude

$$\|D\rho\|_{L^2(Q_\tau)} \le C(\|\rho\|_{L^2(Q_\tau)} + \|b\rho\|_{L^2(Q_\tau)} + \|\rho_\tau\|_{L^2(\mathbb{T}^d)}) \ . \tag{6.43}$$

By Poincaré-Wirtinger inequality and (6.43), together with the fact that $\int_{\mathbb{T}^d}\rho(x,t)dx = 1$ for all $t \in [0,\tau]$, we obtain

$$\|\rho\|^2_{L^2(Q_\tau)} \le C(\|D\rho\|^2_{L^2(Q_\tau)} + \tau\|\rho_\tau\|^2_{L^2(\mathbb{T}^d)}) \ ,$$

yielding, together with (6.43)

$$\|\rho\|_{W_2^{1,0}(Q_\tau)} \le C(\|\rho\|_{L^2(Q_\tau)} + \|b\rho\|_{L^2(Q_\tau)} + \|\rho_\tau\|_{L^2(\mathbb{T}^d)}).$$

Finally, for any smooth test function φ (which may not vanish at the terminal time τ), again by Hölder's inequality

$$\begin{aligned}\left|\int_0^\tau \langle\partial_t\rho(t),\varphi(t)\rangle dt\right| &\le \iint_{Q_\tau} |\partial_j(a_{ij}\rho)\partial_i\varphi| + |b\rho|\,|D\varphi|\,dxdt \\ &\le \left[(\|a\|_{L^\infty(Q_\tau)} + \|Da\|_{L^\infty(Q_\tau)})\,\|\rho\|_{W_2^{1,0}(Q_\tau)} + \|b\rho\|_{L^2(Q_\tau)}\right]\|D\varphi\|_{L^2(Q_\tau)}.\end{aligned}$$

Thus,

$$\|\partial_t\rho\|_{(W^{1,2}(Q_\tau))'} \le C(\|\rho\|_{L^2(Q_\tau)} + \|b\rho\|_{L^2(Q_\tau)} + \|\rho_\tau\|_{L^2(\mathbb{T}^d)}) \ .$$

□

6.7 Lipschitz regularity

This section is devoted to the proof of Lipschitz regularity of u, stated in Theorem 6.1. We will assume that the assumptions of Theorem 6.1 are in force: $a_{ij} \in C(0,T;W^{2,\infty}(\mathbb{T}^d))$ and satisfies (A), $H \in C^1(\mathbb{T}^d \times \mathbb{R}^d)$, it is convex in the second variable, and satisfies (H) and $u_0 \in L^\infty(\mathbb{T}^d)$. Moreover, $f \in L^q(Q_T)$ for some $q > d+2$. At a certain stage we will require $q \geq \frac{d+2}{\gamma'-1}$ also.

The result will be obtained using regularity properties of the adjoint variable ρ, i.e. the solution to

$$\begin{cases} -\partial_t \rho(x,t) - \sum_{i,j=1}^{d} \partial_{ij}(a_{ij}(x,t)\rho(x,t)) - \operatorname{div}\big(D_pH(x,Du(x,t))\,\rho(x,t)\big) = 0 & \text{in } Q_\tau\ , \\ \rho(x,\tau) = \rho_\tau(x) & \text{on } \mathbb{T}^d\ , \end{cases} \tag{6.44}$$

for $\tau \in (0,T)$, $\rho_\tau \in C^\infty(\mathbb{T}^d)$, $\rho_\tau \geq 0$ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$. Recall that u is a weak solution to the viscous Hamilton-Jacobi equation (6.1). By the integrability assumptions on D_pH, the adjoint state $\rho \in \mathcal{H}_2^1(Q_\tau)$ is, for any ρ_τ, well-defined, non-negative and bounded in $L^\infty(0,\tau;L^{\sigma'}(\mathbb{T}^d))$ for all $\sigma' > 1$, by a straightforward application of Proposition 6.15.

In what follows, we establish bounds on ρ that are independent on the choice of τ and ρ_τ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$.

6.7.1 Estimates on the adjoint variable

Let us point out first that from now on we will denote by $C, C_1, \ldots$ positive constants that may depend on λ, C_H, $\|u_0\|_{C(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, $\|a\|_{C(W^{1,\infty})}$, $\|D^2a\|_{L^\infty(Q_\tau)}, T, q, d$, but do not depend on τ, ρ_τ.

Lemma 6.21. *Let u be a local weak solution to* (6.1). *Assume that ρ is a weak solution to* (6.44). *Then, for all τ_1, τ_2 such that $0 < \tau_1 < \tau_2 \leq T$ we have*

$$\int_{\mathbb{T}^d} u(x,\tau_2)\rho(x,\tau_2)dx = \int_{\mathbb{T}^d} u(x,\tau_1)\rho(x,\tau_1) + \int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^d} L(x, D_pH(x,Du))\rho dxdt + \int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^d} f\rho\, dxdt. \tag{6.45}$$

Moreover, if u is a global weak solution, the previous identity holds up to $s = 0$.

Proof. Using $-\rho \in \mathcal{H}_2^1(\mathbb{T}^d \times (\tau_1,\tau_2)) \cap L^\infty(\tau_1,\tau_2;L^{\sigma'}(\mathbb{T}^d))$ as a test function in the weak formulation of problem (6.1), $u \in \mathcal{H}_2^1(\mathbb{T}^d \times (\tau_1,\tau_2))$ as a test function for the corresponding adjoint equation (6.44) and summing both expressions, one obtains

$$-\int_{\tau_1}^{\tau_2} \langle \partial_t u(t), \rho(t)\rangle dt - \int_{\tau_1}^{\tau_2} \langle \partial_t \rho(t), u(t)\rangle dt + \int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^d} (D_pH(x,Du)\cdot Du - H(x,Du))\rho dxdt + \int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^d} f\rho\, dxdt = 0\ .$$

The desired equality follows after integrating by parts in time and using property (7.28) of L. Note that since $H(x, Du) \in L^1(\tau_1, \tau_2; L^\sigma(\mathbb{T}^d))$, and then $L(x, D_pH(Du)) \in L^1(\tau_1, \tau_2; L^\sigma(\mathbb{T}^d))$ by (L1) and (H), so all the terms in (6.45) make sense. $\square$

We are now ready to prove a crucial estimate on the the integrability of D_pH with respect to ρ, that depends in particular on the sup norm $\|u\|_{C(\overline{Q}_T)}$. Note that this estimate is obtained on the whole parabolic cylinder.

Proposition 6.22. *Let u be a local weak solution to* (6.1) *and ρ be a weak solution to* (6.44). *Then, there exist positive constants C (depending on λ, $\|a\|_{C(W^{1,\infty})}$, $\|u\|_{C(\overline{Q}_T)}$, C_H, $\|f\|_{L^q(Q_T)}$, q, d, T) such that*

$$\iint_{Q_\tau} |D_pH(x, Du(x,t))|^{\gamma'} \rho(x,t)\, dxdt \le C. \tag{6.46}$$

Remark 6.23. Note that as a straightforward consequence of (6.46), one has

$$\iint_{Q_\tau} |Du(x,t)|^{\beta} \rho(x,t)\, dxdt \le C_\beta \qquad \text{for all } 1 \le \beta \le \gamma. \tag{6.47}$$

Indeed, by (H), $\iint_{Q_\tau} |Du(x,t)|^\gamma \rho(x,t)\, dxdt \le C$, which yields (6.47) for $\beta = \gamma$. For $\beta < \gamma$ it is sufficient to use Young's inequality and (6.26).

Proof. Rearrange the representation formula (6.45) to get, for $s \in (0, \tau)$,

$$\iint_{Q_{s,\tau}} L(x, D_pH(x, Du))\rho\, dxdt = \int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx - \int_{\mathbb{T}^d} u(x,s)\rho(x,s)dx - \iint_{Q_{s,\tau}} f\rho\, dxdt. \tag{6.48}$$

Fix some η such that $(d+2)/\gamma' < \eta < d+2$ $(< q)$. Use now bounds on the Lagrangian (L1), and Hölder's inequality to obtain

$$\begin{aligned} C_L^{-1} \iint_{Q_{s,\tau}} |D_pH(x, Du)|^{\gamma'} \rho\, dxdt &\le \iint_{Q_{s,\tau}} L(x, D_pH(x, Du))\rho\, dxdt \\ &\le 2\|u\|_{C(\overline{Q}_T)} + \|f\|_{L^\eta(Q_{s,\tau})} \|\rho\|_{L^{\eta'}(Q_{s,\tau})}. \end{aligned} \tag{6.49}$$

Let now $\bar{q}$ be such that

$$\frac{1}{\eta'} = \frac{1}{\bar{q}'} - \frac{1}{d+2}$$

By Proposition 6.11, $\mathcal{H}^1_{\bar{q}'}(Q_{s,\tau})$ is continuously embedded in $L^{\eta'}(Q_{s,\tau})$. Moreover, choosing $\eta > (d+2)/2$ guarantees $\bar{q}' < (d+2)/(d+1)$, so by inequality (6.32) (with q replaced by $\bar{q}$),

$$\|\rho\|_{L^{\eta'}(Q_{s,\tau})} \le C(\|\rho\|_{\mathcal{H}^1_{\bar{q}'}(Q_{s,\tau})} + 1) \le C_1 \left(\iint_{Q_{s,\tau}} |D_pH(x, Du)|^{r'} \rho(x,t)\, dxdt + 1 \right), \tag{6.50}$$

where $r' = 1 + \frac{d+2}{\bar{q}}$. Plugging this inequality into (6.49), we obtain

$$C_L^{-1} \iint_{Q_{s,\tau}} |D_pH(x,Du)|^{\gamma'} \rho \, dxdt$$
$$\leq 2\|u\|_{C(\overline{Q}_T)} + C_1 \|f\|_{L^\eta(Q_{s,\tau})} \left(\iint_{Q_{s,\tau}} |D_pH(x,Du)|^{r'} \rho(x,t) \, dxdt + 1 \right)$$

Finally, the right hand side can be absorbed in the left hand side whenever $r' < \gamma'$ by Young's inequality: this is assured by

$$r' = 1 + \frac{d+2}{\bar{q}} = \frac{d+2}{\eta} < \gamma'.$$

One then gets (6.46) by taking the limit $s \to 0$ (constants here remain bounded for $s \in (0, \tau)$).

□

Integrability of D_pH with respect to ρ provides finally $L^{q'}$ regularity of $D\rho$. From now on, we will suppose that $q > d+2$ and $q \geq \frac{d+2}{\gamma'-1}$.

Corollary 6.24. *Let u be a local weak solution to* (6.1) *and ρ be a weak solution to* (6.44)*. Let $\bar{q}$ be such that*

$$\bar{q} > d+2 \quad \textit{and} \quad \bar{q} \geq \frac{d+2}{\gamma'-1}.$$

Then, there exists a positive constant C such that

$$\|\rho\|_{\mathcal{H}^1_{\bar{q}'}(Q_\tau)} \leq C \ ,$$

where C depends in particular on $\lambda, \|a\|_{C(W^{1,\infty})}, C_H$, $\|f\|_{L^q(Q_\tau)}$, $\bar{q}, d, T$ (but not on τ, ρ_τ), $\|u_0\|_{L^\infty(\mathbb{T}^d)}$ if u is a local weak solution and $\|u\|_{L^\infty(Q_T)}$ if u is a local weak solution.

Proof. Since $\bar{q}' < \frac{d+2}{d+1}$, (6.32) applies (with $q = \bar{q}$), yielding

$$\|\rho\|_{\mathcal{H}^1_{\bar{q}'}(Q_\tau)} \leq C \left(\iint_{Q_\tau} |D_pH(x,Du(x,t))|^{r'} \rho(x,t) \, dxdt + 1 \right),$$

with

$$r' = 1 + \frac{d+2}{\bar{q}} \leq \gamma'.$$

If $r' = \gamma'$, use Proposition 6.22 to conclude. Otherwise, if $r' < \gamma'$ then use Young's inequality first to control $\iint |D_pH(x,Du(x,t))|^{r'} \rho \, dxdt$ with $\iint |D_pH(x,Du)|^{\gamma'} \rho \, dxdt + \tau$. □

Remark 6.25. It is worth noting that in the sub-quadratic regime $\gamma \leq 2$, the information $b \in L^{\gamma'}(\rho)$ is strong enough to guarantee $\|D\rho\|_{L^{q'}(Q_T)}$ for all $q' < (d+2)/(d+1)$, that is expected for distributional solutions to heat equations with L^1

data (see e.g. [194]). We can then regard the div() term in (6.8) as perturbation of a heat equation. On the other hand, in the super-quadratic case $\gamma > 2$, we are just able to prove the weaker regularity $\|D\rho\|_{L^{q'}(Q_T)}$ for $q' \le q'_\gamma$, with $q'_\gamma < (d+2)/(d+1)$, where actually $q'_\gamma \to 1$ as $\gamma \to \infty$. As expected, in the super-quadratic case the Hamiltonian term in (6.1) may overcome the regularizing effect of Laplacian.

Finally, if one thinks $\rho(t)$ as a flow of probability measures, then ρ enjoys also some Hölder regularity in time.

Corollary 6.26. *Let u be a local weak solution to* (6.1) *and ρ be a weak solution to* (6.44)*. Then, there exists a positive constant C such that*

$$\mathbf{d_1}(\rho(t), \rho(t')) \le C|t-t'|^{\frac{1}{2}\wedge\frac{1}{\gamma}} \qquad \forall t, t' \in [0, \tau],$$

where C depends in particular on $\lambda, \|a\|_{C(W^{1,\infty})}$, C_H, $\|u\|_{C(\overline{Q}_T)}$, $\|f\|_{L^q(Q_\tau)}$, d, T (but not on τ, ρ_τ).

Proof. Since ρ solves the Fokker-Planck equation (6.44) with drift $D_pH(x, Du(x,t))$, given the L^1 bound (6.46) on $|D_pH(\cdot, Du)|^{\gamma'}\rho$, the result is a straightforward application of [73, Lemma 4.1]. □

6.7.2 Further bounds for global weak solutions

If u is a global weak solution, i.e. an energy solution up to initial time, it is possible to control its sup norm in terms of $\|u_0\|_{C(\mathbb{T}^d)}$. This will be done in the next proposition.

Proposition 6.27. *There exists $C > 0$ (depending on $\lambda, \|a\|_{C(W^{1,\infty})}, T, d$) such that any global weak solution u to* (6.1) *satisfies*

$$\|u(\cdot,\tau)\|_{C(\mathbb{T}^d)} \le C \qquad \text{for all } \tau \in [0, T]. \tag{6.51}$$

Proof. First, we prove a bound from above for u:

$$u(x,\tau) \le \|u_0\|_{C(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)} \tag{6.52}$$

for all $\tau \in (0, T)$ and $x \in \mathbb{T}^d$. Consider indeed the (strong) non-negative solution of the following backward problem

$$\begin{cases} -\partial_t\mu(x,t) - \sum_{i,j} \partial_{ij}(a_{ij}(x,t)\mu(x,t)) = 0 & \text{on } Q_\tau\ , \\ \mu(x,\tau) = \mu_\tau(x) & \text{on } \mathbb{T}^d\ . \end{cases}$$

with $\mu_\tau \in C^\infty(\mathbb{T}^d)$, $\mu_\tau \ge 0$ and $\|\mu_\tau\|_{L^1(\mathbb{T}^d)} = 1$. Note that μ is a solution of a Fokker-Planck equation of the form (6.23) with drift $b \equiv 0$. Then, since $q' < (d+2)/(d+1)$, by Proposition 6.18 there exists a positive constant C (not depending on τ, μ_τ) such that $\|\mu\|_{\mathcal{H}^1_{q'}(Q_\tau)} \le C$.

Use μ as a test function in the weak formulation of the Hamilton-Jacobi equation (6.1) to get

$$\int_{\mathbb{T}^d} u(x,\tau)\mu_\tau(x)dx = \int_{\mathbb{T}^d} u_0(x)\mu(x,0)dx + \iint_{Q_\tau} f\mu dxdt - \iint_{Q_\tau} H(x, Du)\mu dxdt\ .$$

Applying Hölder's inequality to the second term of the right-hand side of the above inequality and the fact that $\|\mu(t)\|_{L^1(\mathbb{T}^d)} = 1$ for all $t \in [0, \tau]$, we get

$$\int_{\mathbb{T}^d} u(x,0)\mu(x,0)dx + \int_0^\tau \int_{\mathbb{T}^d} f\mu dxdt \le \|u_0\|_{C(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)} \ ,$$

By the assumption $H(x,p) \ge 0$, we then conclude

$$\int_{\mathbb{T}^d} u(x,\tau)\mu_\tau(x)dx \le \|u_0\|_{C(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)} \ .$$

Finally, by passing to the supremum over $\mu_\tau \ge 0$, $\|\mu_\tau\|_{L^1(\mathbb{T}^d)} = 1$, one deduces the estimate (6.52) by duality.

To prove the bound from below of u, one can argue exactly as in the proof of Proposition 6.22, starting from the representation formula (6.48) with $s = 0$. Using the additional upper bound (6.52),

$$\iint_{Q_\tau} |D_pH(x, Du(x,t))|^{\gamma'} \rho(x,t)\, dxdt \le 2\|u_0\|_{C(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)} + \|f\|_{L^\eta(Q_\tau)}\|\rho\|_{L^{\eta'}(Q_\tau)}.$$

This provides as before a control on $\iint_{Q_\tau} |D_pH(x, Du))|^{\gamma'} \rho\, dxdt$ and thus on $\|\rho\|_{L^{\eta'}(Q_\tau)}$, which depends on $\|u_0\|_{C(\mathbb{T}^d)}$ instead of the full sup norm $\|u\|_{C(\overline{Q}_T)}$. Going back to (6.45),

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx \ge \int_{\mathbb{T}^d} u(x,0)\rho(x,0) - C_L \iint_{Q_\tau} \rho(x,t)dxdt + \iint_{Q_\tau} f\rho\, dxdt.$$

Since $\iint f\rho$ can be bounded (from below) by Hölder's inequality,

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx \ge -\|u(\cdot,0)\|_{C(\mathbb{T}^d)} - C_L\tau - C.$$

Since ρ_τ can be arbitrarily chosen so that $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$, we have the desired result. □

6.7.3 Proof of Theorem 6.1

Theorem 6.28. *Let u be a distributional solution to* (7.3)*.*

(i) Let u be a local weak solution to (7.3) *with $\mathcal{P} = \mathcal{Q}$ and $\eta = \eta(t) \in C_0^\infty((0,T])$ be a positive smooth function satisfying $\eta(t) \le 1$ for all t. Then, $(\eta u)(\cdot,\tau) \in W^{1,\infty}(\mathbb{T}^d)$ for all $\tau \in (0,T)$, and there exists $C > 0$ depending on C_H, $\|u\|_{C(\overline{Q}_T)}$, $\|f\|_{L^q(Q_T)}$, q, d, T such that*

$$\eta(\tau)\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \le C\Big(\|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{(d+2)(\gamma-1)}(Q_\tau)} + \sup_{(0,T)}|\eta'(t)| + 1\Big)$$

for all $\tau \in (0,T]$. Without requiring $\mathcal{P} = \mathcal{Q}$ in (6.13)*, but assuming in addition that $Da \equiv 0$ on Q_T, we have the same assertion, and in particular*

$$\eta(\tau)\|Du(\cdot,\tau)\|_{L^\infty(\mathbb{T}^d)} \le C\Big(\sup_{(0,T)}|\eta'(t)| + 1\Big)$$

for all $\tau \in [0,T]$.

Proof. **Step 1.** Since H is convex and superlinear we can write for a.e. $(x,t) \in Q_T$

$$H(x, Du(x,t)) = \sup_{\nu \in \mathbb{R}^d} \{\nu \cdot Du(x,t) - L(x,\nu)\}.$$

Hence we get, for $0 < s < \tau$,

$$\int_s^\tau \langle \partial_t u(t), \varphi(t)\rangle dt + \iint_{Q_{s,\tau}} \partial_i u(x,t)\, \partial_j (a_{ij}(x,t)\varphi(x,t)) + [\Xi(x,t)\cdot Du(x,t) - L(x,\Xi(x,t))]\varphi \, dxdt \leq \iint_{Q_{s,\tau}} f(x,t)\varphi(x,t)\, dxdt \quad (6.53)$$

for all test functions $\varphi \in \mathcal{H}_2^1(Q_{s,\tau}) \cap L^\infty(s,\tau; L^{\sigma'}(\mathbb{T}^d))$ and measurable $\Xi : Q_{s,\tau} \to \mathbb{R}^d$ such that $L(\cdot, \Xi(\cdot,\cdot)) \in L^1(s,\tau; L^\sigma(\mathbb{T}^d))$ and $\Xi \cdot Du \in L^1(s,\tau; L^\sigma(\mathbb{T}^d))$. Note that the previous inequality becomes an equality if $\Xi(x,t) = D_pH(x, Du(x,t))$ in $Q_{s,\tau}$.

We fix ρ_τ as in (6.25). Set

$$w(x,t) = \eta(t) u(x,t).$$

Use now (6.53) with $\Xi(x,t) = D_pH(x,Du(x,t))$ and $\varphi = \eta\rho \in \mathcal{H}_2^1(Q_\tau) \cap L^\infty(s,\tau; L^{\sigma'}(\mathbb{T}^d))$ for all $1 < \sigma' < \infty$, where ρ is the adjoint variable (i.e. the weak solution to (6.44)) to find

$$\int_s^\tau \langle \partial_t w(t), \rho(t)\rangle dt + \iint_{Q_{s,\tau}} \partial_i w\, \partial_j(a_{ij}\rho) + D_pH(x,Du)\cdot Dw\rho - L(x, D_pH(x,Du))\eta\rho\, dxdt = \iint_{Q_{s,\tau}} f\eta\rho\, dxdt + \iint_{Q_{s,\tau}} u\eta'\rho\, dxdt. \quad (6.54)$$

Then, use $w \in \mathcal{H}_2^1(Q_T)$ as a test function in the weak formulation of the equation satisfied by ρ to get

$$-\int_s^\tau \langle \partial_t \rho(t), w(t)\rangle dt + \iint_{Q_{s,\tau}} \partial_j(a_{ij}\rho)\partial_i w + D_pH(x,Du)\rho \cdot Dw\, dxdt = 0. \quad (6.55)$$

We now fix s small so that $\eta(s) = 0$. We then obtain, subtracting the previous equality to (6.54), and integrating by parts in time

$$\int_{\mathbb{T}^d} w(x,\tau)\rho_\tau(x) dx = \iint_{Q_{s,\tau}} \eta(t) f(x,t)\rho(x,t) dxdt + \iint_{Q_{s,\tau}} \eta(t) L\big(x, D_pH(x,Du(x,t))\big)\rho(x,t) dxdt + \iint_{Q_{s,\tau}} \eta'(t) u(x,t)\rho(x,t) dxdt. \quad (6.56)$$

For $h > 0$ and $\xi \in \mathbb{R}^d$, $|\xi| = 1$, define $\hat\rho(x,t) := \rho(x - h\xi, t)$. After a change of variables in (6.44), it can be seen that $\hat\rho$ satisfies, using w as a test function,

$$-\int_s^\tau \langle \partial_t \hat\rho(t), w(t)\rangle dt + \iint_{Q_{s,\tau}} \partial_j\big(a_{ij}(x-h\xi,t)\hat\rho(x,t)\big)\partial_i w + D_pH(x-h\xi, Du(x-h\xi,t))\hat\rho(x,t)\cdot Dw(x,t)\, dxdt = 0. \quad (6.57)$$

As before, plugging $\Xi(x,t) = D_pH(x-h\xi, Du(x-h\xi,t))$ and $\varphi = \eta\hat\rho$ in (6.53) yields

$$\int_s^\tau \langle \partial_t w(t), \hat\rho(t)\rangle dt + \iint_{Q_{s,\tau}} \partial_i w\, \partial_j(a_{ij}\hat\rho) + D_pH(x-h\xi, Du(x-h\xi,t))\cdot Dw\hat\rho - L(x, D_pH(x-h\xi, Du(x-h\xi,t)))\eta\hat\rho\, dxdt \leq \iint_{Q_{s,\tau}} f\eta\hat\rho\, dxdt + \iint_{Q_{s,\tau}} u\eta'\hat\rho\, dxdt.$$

Hence, subtracting (6.57) to the previous inequality,

$$\int_{\mathbb{T}^d} w(x,\tau)\hat\rho_\tau(x)dx \leq \iint_{Q_{s,\tau}} \partial_j\Big(\big(a_{ij}(x-h\xi,t) - a_{ij}(x,t)\big)\hat\rho(x,t)\Big)\partial_i w\, dxdt + \iint_{Q_{s,\tau}} L(x, D_pH(x-h\xi, Du(x-h\xi,t)))\eta\hat\rho\, dxdt + \iint_{Q_{s,\tau}} f\eta\hat\rho\, dxdt + \iint_{Q_{s,\tau}} u\eta'\hat\rho\, dxdt,$$

which, after the change of variables $x \mapsto x + h\xi$, becomes

$$\begin{aligned}\int_{\mathbb{T}^d} w(x+h\xi,\tau)\rho_\tau(x)dx \leq \iint_{Q_{s,\tau}} &\partial_j\Big(\big(a_{ij}(x-h\xi,t) - a_{ij}(x,t)\big)\rho(x,t)\Big)\partial_i w\, dxdt \\ &+ \iint_{Q_{s,\tau}} \eta(t)L(x+h\xi, D_pH(x, Du(x,t)))\rho(x,t)\, dxdt \\ &+ \iint_{Q_{s,\tau}} f\eta\hat\rho\, dxdt + \iint_{Q_{s,\tau}} u\eta'\hat\rho\, dxdt, \end{aligned} \tag{6.58}$$

Taking the difference between (6.58) and (6.56) we obtain

$$\begin{aligned}\int_{\mathbb{T}^d}(w(x+h\xi,\tau) - w(x,\tau))\rho_\tau(x)dx \leq &\iint_{Q_{s,\tau}} \partial_j\Big(\big(a_{ij}(x-h\xi,t) - a_{ij}(x,t)\big)\rho(x,t)\Big)\partial_i w\, dxdt \\ &+ \iint_{Q_{s,\tau}} \eta(t)\Big(L(x+h\xi, D_pH(x, Du(x,t))) - L(x, D_pH(x, Du(x,t)))\Big)\rho(x,t)\, dxdt \\ &+ \iint_{Q_{s,\tau}} \eta(t)f(x,t)\big(\rho(x-h\xi,t) - \rho(x,t)\big)\, dxdt \\ &+ \iint_{Q_{s,\tau}} \eta'(t)u(x,t)\big(\rho(x-h\xi,t) - \rho(x,t)\big)\, dxdt. \end{aligned} \tag{6.59}$$

Step 2. We now estimate all the right hand side terms of (6.59). We stress that constants $C, C_1, \ldots$ are not going to depend on τ, ρ_τ, h, ξ.

Regarding the first term, assuming that $\mathcal{P} = \mathcal{Q}$ holds in (6.13), we have by the growth assumptions (H) on D_pH that $\eta Du \in L^{(d+2)(\gamma-1)}(Q_\tau)$. Note that this fact will be used in the next chain of inequalities only. By Young's and Holder's inequality

$$\begin{aligned}
&\left|\iint_{Q_{s,\tau}} \partial_j\Big(\big(a_{ij}(x-h\xi,t)-a_{ij}(x,t)\big)\rho(x,t)\Big)\partial_i w\,dxdt\right| = \\
&\qquad\left|\iint_{Q_{s,\tau}} \big(\partial_j a_{ij}(x-h\xi,t)-\partial_j a_{ij}(x,t)\big)\rho\,\partial_i w\,dxdt\right. \\
&\qquad\left. + \iint_{Q_{s,\tau}} (a_{ij}(x-h\xi,t)-a_{ij}(x,t))\partial_j\rho\,\partial_i w\,dxdt\right| \\
&\le \|D^2 a\|_{L^\infty(Q_{s,\tau})}|h|\iint_{Q_{s,\tau}} |Du|\rho\,dxdt + \|Da\|_{L^\infty(Q_{s,\tau})}|h|\iint_{Q_{s,\tau}} |\eta Du|\,|D\rho|\,dxdt \\
&\qquad \le C|h|\left(\iint_{Q_{s,\tau}} |Du|^\gamma\rho\,dxdt + \tau\right. \\
&\qquad \left. +\|Da\|_{L^\infty(Q_{s,\tau})}\|\eta Du\|_{L^{(d+2)(\gamma-1)}(Q_{s,\tau})}\|D\rho\|_{L^{((d+2)(\gamma-1))'}(Q_{s,\tau})}\right) \\
&\qquad \le C_1|h|\left(\|Da\|_{L^\infty(Q_{s,\tau})}\|\eta Du\|_{L^{(d+2)(\gamma-1)}(Q_{s,\tau})}+1\right), \qquad (6.60)
\end{aligned}$$

where in the last inequality we used (6.47) and Corollary 6.24 (with $\bar q = (d+2)(\gamma-1) = (d+2)/(\gamma'-1)$).

Next, using first the mean value theorem (that yields a function $\zeta : \mathbb{T}^d \to \mathbb{T}^d$), then property (L2) of $D_x L$ and (6.46),

$$\begin{aligned}
&\left|\iint_{Q_{s,\tau}} \eta(t)\Big(L(x+h\xi, D_pH(x,Du(x,t))) - L(x, D_pH(x,Du(x,t)))\Big)\rho(x,t)\,dxdt\right| \\
&\qquad \le |h|\iint_{Q_{s,\tau}} \big|D_x L\big(\zeta(x), D_pH(x,Du(x,t))\big)\big|\rho(x,t)\,dxdt \\
&\qquad\qquad \le C_L|h|\iint_{Q_{s,\tau}} \big(|D_pH(x,Du(x,t))|^{\gamma'}+1\big)\rho(x,t)\,dxdt \le C|h|.
\end{aligned}$$

Denote by $D^h\rho(x,t) := |h|^{-1}(\rho(x+h\xi,t)-\rho(x,t))$. Then, for the term involving f we use again Corollary 6.24, with $\bar q = q$, and control the $L^{q'}$ norm of difference quotient $D^h\rho$ via $D\rho$ (as in, e.g. Lemma 6.12), to get

$$\begin{aligned}
&\left|\iint_{Q_{s,\tau}} \eta(t) f(x,t)\big(\rho(x-h\xi,t)-\rho(x,t)\big)\,dxdt\right| \\
&\qquad \le |h|\iint_{Q_{s,\tau}} |f(x,t)|\,|D^h\rho(x,t)|\,dxdt \le |h|\|f\|_{L^q(Q_{s,\tau})}\|D\rho\|_{L^{q'}(Q_{s,\tau})} \le C|h|.
\end{aligned}$$

Finally, by boundedness of u stated in (6.51) and again Corollary 6.24

$$\left|\iint_{Q_{s,\tau}} \eta'(t)u(x,t)\big(\rho(x-h\xi,t)-\rho(x,t)\big)\,dxdt\right| \\ \leq |h|\big(\sup_{(0,T)}|\eta'(t)|\big)\|u\|_{L^\infty(Q_{s,\tau})}\|D\rho\|_{L^1(Q_{s,\tau})} \\ \leq C|h|\sup_{(0,T)}|\eta'(t)|.$$

Plugging all the estimates in (6.59) we obtain

$$\int_{\mathbb{T}^d}(w(x+h\xi,\tau)-w(x,\tau))\rho_\tau(x)dx \leq C|h|\Big(\|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{(d+2)(\gamma-1)}(Q_\tau)}+\sup_{(0,T)}|\eta'(t)|+1\Big). \tag{6.61}$$

Step 3. Since (6.61) holds for all smooth $\rho_\tau \geq 0$ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)}=1$, we get

$$\eta(\tau)[u(x+h\xi,\tau)-u(x,\tau)] \leq C|h|\Big(\|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{(d+2)(\gamma-1)}(Q_\tau)}+\sup_{(0,T)}|\eta'(t)|+1\Big)$$

for all $x\in\mathbb{T}^d$, $\xi\in\mathbb{R}^d$, $h>0$. Thus, $u(\cdot,\tau)$ is Lipschitz continuous, and

$$\eta(\tau)\|Du(\cdot,\tau)\|_{L^\infty(\mathbb{T}^d)} \leq C\Big(\|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{(d+2)(\gamma-1)}(Q_\tau)}+\sup_{(0,T)}|\eta'(t)|+1\Big).$$

Since C does not depend on $\tau\in(0,T)$, we have proved the theorem.

Finally, for the special case $Da\equiv 0$ on Q_T, one may follow the very same lines, with the difference that there is no need to control the term appearing in (6.60) (which is identically zero). Therefore, there is no need to keep track of $\|\eta Du\|_{L^{(d+2)(\gamma-1)}(Q_\tau)}$, and therefore the theorem is proven without assuming the constraint $\mathcal{P}=\mathcal{Q}$ in (6.13). □

The following lemma shows that $\|Du\|_{L^\gamma(Q_T)}$ can be bounded by a constant depending on the data only.

Lemma 6.29. *Let u be a local weak solution. Then, there exists a constant C depending on C_H, $\|u\|_{C(\overline{Q}_T)}$, $\|f\|_{L^q(Q_T)}$, $\|Da\|_{L^\infty(Q_\tau)}$, q,d,T such that*

$$\|Du\|_{L^\gamma(Q_T)} \leq C.$$

Proof. Plugging $\varphi\equiv 1$ as a test function in the weak formulation of (6.1) we obtain, for $s>0$,

$$\int_{\mathbb{T}^d}u(x,T)\,dx-\int_{\mathbb{T}^d}u(x,s)\,dx+\iint_{Q_{s,T}}\partial_i u\,\partial_j(a_{ij})+H(x,Du)\,dxdt=\iint_{Q_{s,T}}f\,dxdt$$

Hence, using (H), and Young's inequality we get

$$C_H\iint_{Q_{s,T}}|Du|^\gamma\,dxdt \leq \|u(\cdot,T)\|_{C(\mathbb{T}^d)}+\|u(\cdot,s)\|_{C(\mathbb{T}^d)}+\frac{C_H}{2}\iint_{Q_{s,T}}|Du|^\gamma\,dxdt \\ +CT\|Da_{ij}\|^{\gamma'}_{L^\infty(Q_{s,T})}+\iint_{Q_{s,T}}|f|^q dxdt+T+C_H^{-1}T.$$

Therefore, we conclude by passing to the limit $s\to 0$.

□

We are now ready to prove the main theorem on Lipschitz regularity stated in the introduction.

Proof of Theorem 6.1. For $t_1 \in (0,T)$, let $\eta = \eta(t)$ be a non negative smooth function on $[0,T]$ satisfying $\eta(t) \leq 1$ for all t, $\eta(t) \equiv 1$ on $[t_1, T]$ and vanishing on $[0, t_1/2]$. Then, Theorem 6.28 yields $u(\cdot,\tau) \in W^{1,\infty}(\mathbb{T}^d)$ for all $\tau \in (0,T)$, and the existence of $C > 0$ (depending on the data and η, so t_1 itself) such that

$$\eta(\tau)\|Du(\cdot,\tau)\|_{L^\infty(\mathbb{T}^d)} \leq C\big(\|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^{(d+2)(\gamma-1)}(Q_\tau)} + 1\big)$$

for all $\tau \in [0,T]$. If $(d+2)(\gamma-1) \leq \gamma$, we immediately conclude (6.5) using Lemma 6.29. Otherwise, by interpolation of $L^{(d+2)(\gamma-1)}(Q_\tau)$ between $L^\gamma(Q_\tau)$ and $L^\infty(Q_\tau)$ we get

$$\eta(\tau)\|Du(\cdot,\tau)\|_{L^\infty(\mathbb{T}^d)} \leq C\left(\|Da\|_{L^\infty(Q_\tau)}\|\eta Du\|_{L^\infty(Q_\tau)}^{1-\frac{\gamma}{(d+2)(\gamma-1)}}\|\eta Du\|_{L^\gamma(Q_\tau)}^{\frac{\gamma}{(d+2)(\gamma-1)}} + 1\right),$$

that implies (6.5) after passing to the supremum with respect to $\tau \in (0,T)$, and again using Lemma 6.29 to control $\|\eta Du\|_{L^\gamma(Q_\tau)}$.

To prove the global in time bound (6.3) one may follow the same lines, using $\eta \equiv 1$ on $[0,T]$ instead. Being the solution global, $s = 0$ can indeed be chosen throughout the proof of Theorem 6.28, and norms $\|u\|_{C(\overline{Q}_T)}$ can be replaced by $\|u_0\|_{C(\mathbb{T}^d)}$ in view of Proposition 6.27. Note that an additional term $\int_{\mathbb{T}^d}(u(x+h,0)-u(x,0))\rho(x,0)dx$ pops up in (6.59): this can be easily bounded by $\|Du_0\|_{L^\infty(\mathbb{T}^d)}$.

Finally, if $a_{ij}(x,t) = A_{ij}$ on Q_T for some A_{ij} satisfying (A), then $Da \equiv 0$ on Q_T, and we obtain the same conclusion, exploiting the fact that Theorem 6.28 does not require anymore $\mathcal{P} = \mathcal{Q}$.

□

6.7.4 Some consequences of Lipschitz regularity

Once Lipschitz regularity is established, one can deduce further properties of weak solutions. Indeed, the viscous HJ equation (6.1) can be treated in terms of regularity as a linear equation, being the $H(x,Du)$ term (locally in time) bounded in L^∞. Thus, the classical Calderón-Zygmund parabolic theory applies, and the so-called maximal regularity for u follows, i.e.: $\partial_t u, \partial_{ij}u, H(x,Du) \in L^q$.

Corollary 6.30. *Under the assumptions of Theorem 6.1, any local weak solution u of* (6.1) *is a strong solution belonging to $W^{2,1}_q(\mathbb{T}^d \times (t_1,T))$ for all $t_1 \in (0,T)$, namely it solves* (6.1) *almost everywhere in Q_T.*

Proof. For any $t_1 > 0$, Theorem 6.1 yields $H(x,Du(x,t)) \in L^\infty(\mathbb{T}^d \times (t_1/2,T))$. Therefore, since $f \in L^q(\mathbb{T}^d \times (t_1/2,T))$ and $q > d+2$, there exists a weak (energy) solution v to the linear equation

$$\partial_t v(x,t) - \sum_{i,j=1}^d a_{ij}(x,t)\partial_{ij}v(x,t) = -H(x,Du(x,t)) + f(x,t) \qquad \in L^q(\mathbb{T}^d \times (t_1/2,T)), \tag{6.62}$$

that satisfies $v(t_1/2) = u(t_1/2)$ in the L^2-sense, and enjoys the additional strong regularity property $W^{2,1}_q(\mathbb{T}^d \times (t_1, T))$. This can be proven using, e.g., local estimates in [159, Theorem IV.10.1]. Since weak solutions to (6.62) are unique, u coincides a.e. with v on $\mathbb{T}^d \times (t_1, T)$, and we obtain the assertion. □

6.7.5 Some remarks on the exponents $\mathcal{P}$, $\mathcal{Q}$, q

In the following remarks, we stress the importance of the condition $D_pH(x, Du) \in L^{\mathcal{Q}}(0, T; L^{\mathcal{P}}(\mathbb{T}^d))$ with $\mathcal{P}$, $\mathcal{Q}$ satisfying

$$\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} \le \frac{1}{2}. \tag{6.63}$$

Not only it guarantees Lipschitz regularity of u, but is also related to uniqueness of solutions in the distributional sense. In the following examples it is indeed possible to observe multiple solutions; among them, there is one that satisfies (6.63) and is Lipschitz continuous, while the other(s) are not, showing therefore that Lipschitz regularity for positive times stated in Theorem 6.1 fails in general without extra integrability properties of $D_pH(x, Du)$.
We will also comment on the condition $f \in L^q(Q_T)$ for some $q > d + 2$.

Remark 6.31. We consider first the super-quadratic regime $\gamma > 2$. For $\mathcal{Q} = \infty$, (6.63) reads

$$D_pH(x, Du) \in L^\infty(L^{\mathcal{P}}(\mathbb{T}^d)) \qquad \text{for some} \quad \mathcal{P} \ge d.$$

Let $a_{ij} = \delta_{ij}$ and $H(x, p) = |p|^\gamma$, $\gamma > 2$. For $c, \alpha > 0$, we consider the (time-independent) function

$$u_1(x, t) = c\psi(x)|x|^\alpha \qquad \text{on } Q_T,$$

where ψ is a smooth function having support on $B_{1/2}(0)$ and is identically one in $B_{1/4}(0)$. Note that ψ has the role of a localizing term only, so that $u_1(x, t)$ is a representative on $[-1/2, 1/2]^d$ of a periodic function on $\mathbb{R}^d$. If we let

$$\alpha = \frac{\gamma - 2}{\gamma - 1}, \qquad c = \frac{(d + \alpha - 2)^{\frac{1}{\gamma-1}}}{\alpha}$$

then u_1 solves, for some $f_1 \in L^\infty(\mathbb{T}^d)$ (that vanishes on $B_{1/4}(0)$)

$$\begin{cases} \partial_t u - \Delta u(x, t) + |Du(x, t)|^\gamma = f_1(x) \\ u(x, 0) = c\psi(x)|x|^\alpha, \end{cases} \tag{6.64}$$

in the sense that it satisfies all the requirements in Definition 6.5, except the Aronson-Serrin condition (6.12)-(6.13). More precisely,

$$(\gamma - 1)|Du|^{\gamma-1} = |D_pH(x, Du)| \in L^\infty(0, T; L^{\mathcal{P}}(\mathbb{T}^d)) \qquad \text{if and only if} \quad \mathcal{P} < d.$$

Moreover, $u_1(\cdot, \tau)$ is clearly not Lipschitz continuous for any $\tau \in [0, T]$.

Note that $u(x, 0) \in C(\mathbb{T}^d)$ and $f_1 \in L^\infty(Q_T)$, so by Theorem 6.8 there exists a unique solution to (6.64) in the sense of Definition 6.5. Thus, (6.64) admits two distinct strong solutions, but only the one satisfying fully the Definition 6.5, in particular the crucial integrability condition on $D_pH(x, Du)$, enjoys Lipschitz regularity.

Remark 6.32. In the sub-quadratic regime $1 + 2/(d+2) < \gamma < 2$, for $a_{ij} = \delta_{ij}$ and $H(x,p) = |p|^\gamma$, we can produce an energy solution to (6.1) such that $D_pH(x,Du) \in L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(\mathbb{T}^d))$ if and only if

$$\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} > \frac{1}{2},$$

that is not Lipschitz continuous, and not even bounded in L^∞ uniformly on $\overline{Q}_T$. It then satisfies all requirements of Definition 6.5 except the Aronson-Serrin condition (6.12)-(6.13) and the continuity up to $t = 0$: the initial datum is assumed in the L^2-sense only.

The construction of such a u is based on the existence, for $k > 0$ small, of $U \in C^2(0,\infty) \cap C^1[0,\infty)$ to the Cauchy problem

$$\begin{cases} U''(y) + \left(\frac{d-1}{y} + \frac{y}{2}\right) U'(y) + U(y) + |U'(y)|^\gamma = 0 & \text{for } 0 < y < \infty \\ U'(0) = 0 \\ U(0) = \alpha_0, \end{cases}$$

for some $\alpha_0 > 0$, that satisfies for some positive c

$$|U(y)| + |U'(y)| + |U''(y)| \le ce^{-y} \qquad \text{as } y \to \infty.$$

The existence of such a U is proven in [35, Section 3], see in particular Theorem 3.5, Proposition 3.11, Proposition 3.14 and Remark 3.8 (see also [166]). As in our Remark 6.31, we need a smooth localization term ψ having support on $(-1/2, 1/2)$ and identically one in $[-1/4, 1/4]$. Let then

$$u_2(x,t) = -t^{-\sigma} U(|x|\, t^{-1/2})\psi(|x|), \qquad \sigma = \frac{2-\gamma}{2(\gamma-1)}.$$

We have that u_2 is a classical solution to

$$\partial_t u(x,t) - \Delta u(x,t) + |Du(x,t)|^\gamma = f_2(x,t), \tag{6.65}$$

where $u_2(0) = 0$ in the L^2-sense (since $\gamma > 1 + 2/(d+2)$). Moreover,

$$\begin{aligned} f_2(x,t) = -t^{-\sigma-1} \Bigg\{ \Bigg[U''(|x|\, t^{-1/2}) &+ \left(\frac{d-1}{|x|\, t^{-1/2}} + \frac{|x|\, t^{-1/2}}{2} \right) U'(|x|\, t^{-1/2}) \\ &+ kU(|x|\, t^{-1/2}) \Bigg] \psi(|x|) \\ &+ \left| U'(|x|\, t^{-1/2})\psi(|x|) + t^{1/2} U(|x|\, t^{-1/2})\psi'(|x|) \right|^\gamma \\ + 2t^{1/2} U'(|x|\, t^{-1/2})\psi'(|x|) &+ tU(|x|\, t^{-1/2})\psi''(|x|) + \frac{d-1}{|x|} tU(|x|\, t^{-1/2})\psi'(|x|) \Bigg\}. \end{aligned}$$

Note that $f_2(x,t)$ is identically zero on $|x| \le 1/4$ and $|x| \ge 1/2$; otherwise, it is bounded in L^∞, since $|U(|x|\, t^{-1/2})| + |U'(|x|\, t^{-1/2})| + |U''(|x|\, t^{-1/2})| \le ce^{-t^{-1/2}/4}$. Therefore, one should expect the existence of a weak solution to the HJ equation (6.65) with zero initial datum that is Lipschitz continuous on the whole Q_T (by Theorem 6.8), but such a solution cannot be u_2, since $u_2(t)$ becomes unbounded as $t \to 0$.

Remark 6.33. To have Lipschitz bounds for solutions to (6.1), one cannot avoid in general the condition

$$f \in L^q(Q_T) \quad \text{for some} \quad q > d+2. \tag{6.66}$$

This constraint is actually imposed by the linear (heat) part of (6.1). Consider indeed $a_{ij} = \delta_{ij}$ and $H(x,p) = |p|^\gamma$, $\gamma > 1$. For $T > 0$, let $\chi \in C_0^\infty(\mathbb{R}^d)$, $\Gamma(x,t)$ be fundamental solution of the heat equation in $\mathbb{R}^d$, $f_3(x,t) := \chi(x/\sqrt{T-t})[\sqrt{T-t}\,\log(T-t)]^{-1}$ and u_3 be the function

$$u_3(x,t) := \iint_{\mathbb{R}^d\times(0,t)} f_3(y,s)\Gamma(x-y,t-s)\,dyds \qquad \text{on } Q_T$$

Clearly, u_3 is a classical solution to

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) + |Du(x,t)|^\gamma = f_3(x,t) + |Du_3(x,t)|^\gamma \\ u(x,0) = 0, \end{cases}$$

$f_3 \in L^q(Q_T)$ for all $q \le d+2$ and $|Du_3|^\gamma \in L^\infty(0,T;L^\beta(\mathbb{T}^d))$ for all $\beta < \infty$. In turn, we have that $\|Du_3(\cdot,t)\|_\infty \to \infty$ as $t \to T$. Note that this example can be recast into the periodic setting by multiplying u_3 by a cut-off function ψ, as in the previous remarks.

Therefore, with respect to integrability requirements on f, Theorem 6.3 is optimal, at least when $\gamma < 3$, namely when $d+2 \ge \frac{d+2}{2(\gamma'-1)}$. We do not know whether (6.66) is enough also when $\gamma \ge 3$.

6.8 Existence and uniqueness of solutions

This section is devoted to the proof of existence and uniqueness of solutions to the HJ equation (6.1). We start with the simpler case of regular initial datum.

Proof of Theorem 6.8. **Existence.** We start with a sequence of classical solutions u_n to regularized problems

$$\begin{cases} \partial_t u_n(x,t) - \displaystyle\sum_{i,j=1}^d a_{ij}(x,t)\partial_{ij}u_n(x,t) + H(x,Du_n(x,t)) = f_n(x,t) & \text{in } Q_T, \\ u_n(x,0) = u_{n,0}(x) & \text{in } \mathbb{T}^d, \end{cases} \tag{6.67}$$

where $f_n, u_{n,0}$ are smooth functions converging to f, u_0 in $L^q(Q_T), C(\mathbb{T}^d)$ respectively. The existence of solutions to the regularized equations can be proven using standard methods as in Proposition 5.48. We sketch the proof for reader's convenience.
Let $\tau \in (0,T]$ and $\alpha \in (0,1)$ be the exponent in [102, Proposition 2.1]. We set

$$\mathcal{S}_a := \{u \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_\tau) : u(0) = u_0 \text{ and } \|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(Q_\tau)} \le a\}$$

We define J mapping $\mathcal{S}_a$ into itself by $u = Jv$, where $u_n = u$ solves

$$\begin{cases} \partial_t u(x,t) - \sum_{i,j} a_{ij}(x,t)\partial_{ij}u(x,t) + H_n(x, Dv(x,t)) = f_n(x,t) & \text{in } Q_T = \mathbb{T}^d \times (0,T), \\ u(x,0) = u_0^n(x) & \text{in } \mathbb{T}^d. \end{cases}$$

By [159, Theorem IV.5.1] (see also [91, Proposition 2.6] for the periodic setting) the above problem has a unique solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_\tau)$ satisfying the estimate

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_\tau)} \le C(\|H_n(x,Dv)\|_{C^{\alpha,\frac{\alpha}{2}}(Q_\tau)} + \|u_0^n\|_{C^{2+\alpha}(\mathbb{T}^d)}) \ ,$$

where C is a constant independent of τ, u_0 and v. By using classical interpolation arguments in Hölder spaces (see e.g. [102, Proposition 2.2] and the same strategy applied in Proposition 5.48 in the L^p setting), one can choose τ small enough to get

$$\|Jv\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(Q_\tau)} \le a$$

to show that J maps $\mathcal{S}_a$ into itself for such τ. This allows to apply the contraction mapping principle to show the existence and uniqueness of a fixed point in Q_{τ^*}, $\tau^* \in (0,T]$.
Step 2. One uses the same arguments of [102, Proposition 2.2] and Proposition 5.48 to show the continuation in time by using the gradient bound (6.3) and get a solution on the whole Q_T.

The global bound on $\|u_n\|_{C(\overline{Q}_T)}$ depending on $\|u_0\|_{C(\mathbb{T}^d)}$ (see Proposition 6.27) and the local in time Lipschitz estimate (6.5) hold, namely, for any fixed $t_1 > 0$,

$$\|Du_n(\cdot,t)\|_{L^\infty(\mathbb{T}^d)} \le C_{t_1} \qquad \text{for all} \quad t \in [t_1, T].$$

Hence, since f_n is equibounded in $L^q(Q_T)$, u_n is equibounded in $W_q^{2,1}(Q_{t_1,T})$ by standard maximal parabolic regularity (e.g. [159, Theorem IV.10.1]). Then, weak limits $\partial_t u, D^2 u$ exist (up to subsequences), and are in L^q locally in time. Moreover, since $q > d+2$, parabolic embeddings of $W_q^{2,1}$ (see e.g. [159]) guarantee that u_n and Du_n are equibounded and equicontinuous in $[t_1, T]$ for all $t_1 > 0$. Therefore, Ascoli theorem and a further diagonalization argument imply that, again up to subsequences, u_n converges uniformly on $[t_1, T]$ for all $t_1 > 0$ to some limit u, and the same convergence holds for Du_n. Note that the desired limit equation is locally satisfied in the strong sense, namely a.e. on Q_T.

To prove that u is a local weak solution, it just remains to show that it is continuous up to $t = 0$. This is a delicate step since the control on Du deteriorates as $t \to 0$. We start with the l.s.c. inquality

$$u_0(x_0) \le \liminf_{\substack{x \to x_0 \\ t \to 0}} u(x,t) \qquad \forall x_0.$$

The following fact will be crucial: for all $(\bar{x}, \bar{t}) \in Q_T$, there exists $\rho = \rho_{\bar{x},\bar{t}} \in C^{\frac{1}{2}\wedge\frac{1}{\gamma}}([0,\bar{t}], \mathcal{P}(\mathbb{T}^d)) \cap \mathcal{H}_{q'}^1(Q_{\bar{t}})$ such that $\rho_{\bar{x},\bar{t}}(\bar{t}) = \delta_{\bar{x}}$ and

$$u(\bar{x},\bar{t}) \ge \int_{\mathbb{T}^d} u_0(x)\rho_{\bar{x},\bar{t}}(0,dx) + \iint_{Q_{\bar{t}}} f(x,t)\rho_{\bar{x},\bar{t}}(x,t)dxdt - C_L\bar{t}, \tag{6.68}$$

and $\rho_{\bar{x},\bar{t}}$ is bounded in $C^{\frac{1}{2}\wedge\frac{1}{\gamma}}([0,\bar{t}],\mathcal{P}(\mathbb{T}^d))\cap\mathcal{H}^1_{q'}(Q_{\bar{t}})$ uniformly in $(\bar{x},\bar{t})$. Indeed, let u_n be as in the previous part of the proof, and ρ_n be the corresponding adjoint variable solving (6.23), where $\rho_n(\bar{t})$ is any sequence converging to $\delta_{\bar{x}}$ in the sense of measures. By duality (see Lemma 6.21) we get

$$\int_{\mathbb{T}^d} u_n(x,\bar{t})\rho_n(x,\bar{t}) = \int_{\mathbb{T}^d} u_{n,0}(x)\rho_n(x,0)dx \\ + \iint_{Q_{\bar{t}}} \Big(L(x, D_pH(x,Du_n))\rho_n dxdt + f_n\rho_n\Big)\, dxdt.$$

Moreover, ρ_n is bounded in $C^{\frac{1}{2}\wedge\frac{1}{\gamma}}([0,\bar{t}],\mathcal{P}(\mathbb{T}^d))\cap\mathcal{H}^1_{q'}(Q_{\bar{t}})$ by means of Corollaries 6.24 and 6.26, and these bounds do not depend on $\rho_n(\bar{t})$ nor on $(\bar{x},\bar{t})$. By (L1), $L\geq -C_L$. Moreover, $u_{n,0}(\cdot)$ and $u_n(\cdot,\bar{t})$ converge uniformly in $\mathbb{T}^d$, $\rho_n(t)$ converges in the sense of measures, f_n converges strongly to f in $L^q(Q_{\bar{t}})$ while ρ_n enjoys the same convergence in the weak $L^{q'}$ sense, eventually up to subsequences (actually it could be made strong convergence by compact embeddings of parabolic spaces). Hence we obtain (6.68) by passing to the limit $n\to\infty$.

Fix now $x_0\in\mathbb{T}^d$, and let $(\bar{x}_m,\bar{t}_m)$ be any sequence such that $(\bar{x}_m,\bar{t}_m)\to(x_0,0)$. By adding $u_0(x_0)$ to both sides of (6.68), rearranging the terms and using Hölder's inequality, we have

$$u_0(x_0)\leq u(\bar{x}_m,\bar{t}_m) \\ + \|f\|_{L^q(\mathbb{T}^d\times(0,\bar{t}_m))}\|\rho_{\bar{x}_m,\bar{t}_m}\|_{L^{q'}(\mathbb{T}^d\times(0,\bar{t}_m))} + C_L\bar{t}_m + \int_{\mathbb{T}^d} u_0(x)\big(\delta_{x_0}-\rho_{\bar{x}_m,\bar{t}_m}(0)\big)(dx).$$

On one hand, $\|f\|_{L^q(\mathbb{T}^d\times(0,\bar{t}_m))}\to 0$ as $\bar{t}_m\to 0$, while $\|\rho_{\bar{x}_m,\bar{t}_m}\|_{L^{q'}}$ is equibounded. On the other hand, as $\bar{x}_m\to x_0$,

$$\int_{\mathbb{T}^d} u_0(x)\big(\delta_{x_0}-\rho_{\bar{x}_m,\bar{t}_m}(0)\big)(dx) = u_0(x_0)-u_0(\bar{x}_m) \\ + \int_{\mathbb{T}^d} u_0(x)\big(\rho_{\bar{x}_m,\bar{t}_m}(\bar{t}_m)-\rho_{\bar{x}_m,\bar{t}_m}(0)\big)(dx)\to 0,$$

by continuity of u_0, and the fact that $\mathbf{d_1}(\rho_{\bar{x}_m,\bar{t}_m}(\bar{t}_m),\rho_{\bar{x}_m,\bar{t}_m}(0))\leq C|\bar{t}_m|^{\frac{1}{2}\wedge\frac{1}{\gamma}}\to 0$ implies the convergence of $\rho_{\bar{x}_m,\bar{t}_m}(\bar{t}_m)$ to $\rho_{\bar{x}_m,\bar{t}_m}(0)$ in the weak sense of measures. We then get the claimed lower semicontinuity of u on $\overline{Q}_T$.

The reverse inequality

$$u_0(x_0)\geq \limsup_{\substack{x\to x_0\\ t\to 0}} u(x,t) \qquad \forall x_0$$

can be obtained following analogous lines: instead of testing the approximating equation for u_n by solutions ρ_n to the adjoint Fokker-Planck equation, it is sufficient to use

$$-\partial_t\mu_n(x,t)-\sum_{i,j}\partial_{ij}\big(a_{ij}(x,t)\mu_n(x,t)\big)=0 \qquad \text{on } Q_{\bar{t}}\ ,$$

i.e. a solution of a Fokker-Planck equation of the form (6.23) with drift $b \equiv 0$, such that $\mu_n(\bar t)$ converges to $\delta_{\bar x}$ in the sense of measures. By duality with u_n and $H \geq -C_H$, it holds

$$\int_{\mathbb{T}^d} u_n(x,\bar t)\mu_n(x,\bar t)dx \leq \int_{\mathbb{T}^d} u_0(x)\mu_n(x,0)dx + \iint_{Q_{\bar t}} f_n\mu_n dxdt + C_H\bar t,$$

and by taking limits

$$u(\bar x,\bar t) \leq \int_{\mathbb{T}^d} u_0(x)\mu_{\bar x,\bar t}(0,dx) + \iint_{Q_{\bar t}} f\mu dxdt + C_H\bar t,$$

so it is possible to proceed as before.

Uniqueness. Consider two solutions u_1, u_2 of the HJ equation, and take their difference $w := u_1 - u_2$ on $\overline{Q}_T$. Let $\tau \in (0,T]$. By convexity of $H(x,\cdot)$, w solves

$$\int_s^\tau \langle \partial_t w(t), \varphi(t)\rangle dt + \iint_{\mathbb{T}^d\times(s,\tau)} \partial_i w\,\partial_j(a_{ij}\varphi) + D_pH(x,Du_2)\cdot Dw\,\varphi\,dxdt \leq 0$$

for all $s \in (0,\tau)$, and $w(\cdot,0)=0$. Let now ρ be the adjoint variable with respect to u_2, namely ρ be the weak solution to

$$\begin{cases} -\partial_t\rho(x,t) - \displaystyle\sum_{i,j=1}^d \partial_{ij}(a_{ij}(x,t)\rho(x,t)) - \mathrm{div}\left(D_pH(x,Du_2(x,t))\,\rho(x,t)\right) = 0 & \text{in } Q_\tau\ , \\ \rho(x,\tau) = \rho_\tau(x) & \text{on } \mathbb{T}^d\ , \end{cases} \tag{6.69}$$

for some non-negative and smooth probability density ρ_τ. Then, by duality we get

$$\int_{\mathbb{T}^d} w(x,\tau)\rho_\tau(x)dx \leq \int_{\mathbb{T}^d} w(x,s)\rho(x,s)dx.$$

Since $w \in C(\overline{Q}_T)$, it is uniformly continuous on $\overline{Q}_T$, so $w(\cdot,t) \to w(\cdot,0) \equiv 0$ uniformly in $\mathbb{T}^d$. Moreover, $\int_{\mathbb{T}^d} w(x,s)\rho(x,s)dx = \int_{\mathbb{T}^d}[w(x,s)-w(x,0)]\rho(x,s)dx$. Thus, by Hölder's inequality and $\|\rho(s)\|_{L^1(\mathbb{T}^d)} = 1$, $\int_{\mathbb{T}^d} w(s)\rho(s) \to 0$, yielding

$$\int_{\mathbb{T}^d} w(x,\tau)\rho_\tau(x)dx \leq 0$$

for arbitrary ρ_τ. As ρ_τ varies, $u_1(\tau) \leq u_2(\tau)$ follows, and by exchanging the role of u_1 and u_2 and varying τ, we eventually obtain $u_1 \equiv u_2$.

Additional regularity. When $u_0 \in W^{1,\infty}$, using global Lipschitz bounds (6.3) one can bring Lipschitz (and further) regularity of u_n to the limit solution u on the whole time interval $[0,T]$.

□

Remark 6.34. Note that the uniqueness proof works in the sub-quadratic case $\gamma \leq 2$ if one requires $u_0 \in L^\infty(\mathbb{T}^d)$ and $u_i(s) \overset{*}{\rightharpoonup} u_0$ in L^∞ only. This follows by the fact that ρ in (6.69) can be proven (as in Proposition 6.22) to satisfy $\int_0^T \int |D_pH(x,Du_2)|^{\gamma'}\rho <$

∞. When $\gamma' \geq 2$, then $\rho \in C([0,T], L^1(\mathbb{T}^d))$ by [194, Theorem 3.6]. Strong convergence of $\rho(s)$ in L^1 and weak-* convergence of $u_1(s) - u_2(s)$ is then enough to have $\int_{\mathbb{T}^d} w(s_n)\rho(s_n) \to 0$ along some sequence $s_n \to 0$. We believe that existence and Lipschitz regularity of solutions could be addressed in this weaker framework, but this is a bit beyond the scopes of this analysis. Nevertheless, these considerations are consistent with the principle that in the super-quadratic case $\gamma > 2$, the HJ equation "sees points" [77], and thus requires u_0 to be continuous in order to be well-posed, while for $\gamma \leq 2$ it may be enough to have informations a.e. at initial time.

6.9 A priori estimates: Bernstein's and the adjoint methods

This section is devoted to the proof of Theorem 6.3, and complements regularity results of the previous section. Here, u is a classical solution to (6.1). This will allow to perform the Bernstein's method, namely to analyse the equation satisfied by $|Du|^2$. The adjoint of such an equation is basically (6.44). As before we will exploit the interplay between the equation itself and its adjoint.

We will assume that $a_{ij} \in C([0,T]; C^1(\mathbb{T}^d))$ and satisfies (A), $H \in C^2(\mathbb{T}^d \times \mathbb{R}^d)$ and satisfies (H), $f \in C([0,T]; C^1(\mathbb{T}^d))$, $u_0 \in C^1(\mathbb{T}^d)$ and

$$q > \min\left\{d+2, \frac{d+2}{2(\gamma'-1)}\right\}.$$

As before, for any fixed $\tau \in (0,T)$, $\rho_\tau \in C^\infty(\mathbb{T}^d)$, $\rho_\tau \geq 0$ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$, let ρ be the (classical) solution to (6.44). Note that Proposition 6.27, Lemma 6.21 and Proposition 6.22 apply. We start with a revised version of Corollary 6.24.

Corollary 6.35. *Let u and ρ be solutions to* (6.1) *and* (6.44) *respectively. Let $\bar{q}$ be such that*

$$\bar{q} > \frac{d+2}{2(\gamma'-1)}.$$

Then, there exist constants $C > 0$ and $0 < \delta < 1$ such that

$$\|\rho\|_{\mathcal{H}^1_{\bar{q}'}(Q_\tau)} \leq C\big(\|Du\|^{1-\delta}_{L^\infty(Q_\tau)} + 1\big)\ ,$$

where C depends in particular on $\lambda, \|a\|_{C(W^{1,\infty})}$, C_H, $\|u_0\|_{L^\infty(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, $\bar{q}, d, T$ (but not on τ, ρ_τ).

A straightforward consequence of the corollary is that

$$\|\rho\|_{L^{\bar{p}'}(Q_\tau)} \leq C\big(\|Du\|^{1-\delta}_{L^\infty(Q_\tau)} + 1\big), \qquad \text{for all} \quad \bar{p} > \frac{d+2}{2(\gamma'-1)+1}. \tag{6.70}$$

Indeed, since $\bar{q}' < \frac{d+2}{d+1}$, Proposition 6.11 gives the result.

Proof. Since $\bar{q}' < \frac{d+2}{d+1}$, (6.32) applies (with $q = \bar{q}$), yielding by (H)

$$
\begin{aligned}
\|\rho\|_{\mathcal{H}^1_{\bar{q}'}(Q_\tau)} \le C \left(\iint_{Q_\tau} |D_p H(x, Du)|^{r'} \rho \, dxdt + 1 \right) \\
\le C_1 \left(\iint_{Q_\tau} |Du|^{(\gamma-1)r'} \rho \, dxdt + 1 \right) \\
\le C_1 \left(\|Du\|^{1-\delta}_{L^\infty(Q_\tau)} \iint_{Q_\tau} |Du|^{(\gamma-1)r'-1+\delta} \rho \, dxdt + 1 \right),
\end{aligned}
$$

with $r' = 1+(d+2)\bar{q}^{-1}$. Note that $\delta > 0$ can be chosen small so that $(\gamma-1)r'-1+\delta \le \gamma$. One then uses the estimate (6.47) on $\iint |Du|^\gamma \rho$ to conclude. $\square$

We are now ready to prove our main a priori Lipschitz regularity result.

Proof of Theorem 6.3. **Step 1.** Set $z(x,t) := \frac{|Du(x,t)|^2}{2}$ on Q_T. Straightforward computations yield

$$
\partial_i z = Du \cdot D\partial_i u \ , \quad \partial_{ij} z = D\partial_j u \cdot D\partial_i u + Du \cdot D\partial_{ij} u \ , \quad \partial_t z = Du \cdot D(\partial_t u) \ ,
$$

which give

$$
\mathrm{Tr}(AD^2 z) = \sum_{k=1}^d AD\partial_k u \cdot D\partial_k u + Du \cdot D\{\mathrm{Tr}(AD^2 u)\} - \sum_{k=1}^d \partial_k u \mathrm{Tr}(\partial_k A D^2 u) \ . \quad (6.71)
$$

Then, differentiating the HJ equation (6.1) with respect to x_k, multiplying the resulting equation by $\partial_k u$, and summing for $k = 1, \dots, d$, one finds

$$
Du \cdot D(\partial_t u) - Du \cdot D\{\mathrm{Tr}(AD^2 u)\} + D_p H \cdot Dz + D_x H \cdot Du = Df \cdot Du \ .
$$

Therefore, by plugging (6.71) into the previous equality we obtain the following equation satisfied by z

$$
\begin{aligned}
\partial_t z - \mathrm{Tr}(AD^2 z) + \sum_{k=1}^d AD\partial_k u \cdot D\partial_k u + D_p H \cdot Dz = \sum_{k=1}^d \partial_k u \mathrm{Tr}(\partial_k A D^2 u) \\
- D_x H \cdot Du + Df \cdot Du \ . \quad (6.72)
\end{aligned}
$$

Using the uniform ellipticity condition (A) we estimate the third term on the left-hand side by

$$
\sum_{k=1}^d AD\partial_k u \cdot D\partial_k u \ge \lambda \mathrm{Tr}((D^2 u)^2).
$$

Multiply (6.72) by the adjoint variable ρ and integrate by parts in space-time to get

$$
\begin{aligned}
\int_{\mathbb{T}^d} z(x,\tau)\rho_\tau(x) \, dx + \lambda \iint_{Q_\tau} \mathrm{Tr}((D^2 u)^2)\rho \, dxdt \le \int_{\mathbb{T}^d} z(x,0)\rho(x,0) \, dxdt + \\
\iint_{Q_\tau} |D_x H||Du|\rho \, dxdt + \iint_{Q_\tau} Df \cdot Du\rho \, dxdt + \iint_{Q_\tau} \partial_k u \mathrm{Tr}(\partial_k A D^2 u)\rho \, dxdt.
\end{aligned}
\quad (6.73)
$$

Step 2. We proceed by estimating the four terms on the right hand side of (6.73). First,

$$\int_{\mathbb{T}^d} z(x,0)\rho(x,0)\,dxdt \le \frac{1}{2}\|Du(\cdot,0)\|^2_{L^\infty(\mathbb{T}^d)}. \tag{6.74}$$

Second, thanks to (H), Proposition 6.22 and Young's inequality,

$$\iint_{Q_\tau} |D_xH||Du|\rho \le \|Du\|_{L^\infty(Q_\tau)}\left[C_H\iint_{Q_\tau}|Du|^\gamma\rho\,dxdt + C_H\tau\right] \le C_2+\frac{1}{8}\|Du\|^2_{L^\infty(Q_\tau)}. \tag{6.75}$$

We then consider $\iint Df\cdot Du\rho$. Integrating by parts,

$$\begin{aligned}\left|\iint_{Q_\tau} Df\cdot Du\rho\,dxdt\right| &= \left|\iint_{Q_\tau} f\mathrm{div}(Du\rho)\,dxdt\right|\\ &\le \left|\iint_{Q_\tau} fDu\cdot D\rho\,dxdt\right| + \left|\iint_{Q_\tau} f\mathrm{Tr}(D^2u)\rho\,dxdt\right| =: I_1+I_2\end{aligned}$$

The term I_1 can be controlled by means of Hölder's and Young's inequalities, and the control on $\|\rho\|_{\mathcal{H}^1_{\bar{q}'}}$ stated in Corollary 6.35:

$$\begin{aligned}I_1 \le \|Du\|_{L^\infty(Q_\tau)}\|f\|_{L^{\bar{q}}(Q_\tau)}\|D\rho\|_{L^{\bar{q}'}(Q_\tau)} &\le C\|Du\|_{L^\infty(Q_\tau)}\|f\|_{L^{\bar{q}}(Q_\tau)}\big(\|Du\|^{1-\delta}_{L^\infty(Q_\tau)}+1\big)\\ &\le C_3+\frac{1}{16}\|Du\|^2_{L^\infty(Q_\tau)}.\end{aligned} \tag{6.76}$$

We apply to I_2 also Hölder's and Young's inequalities to get, for a $\bar{p}>1$ to be chosen,

$$\begin{aligned}I_2 &\le \frac{1}{2\lambda}\iint_{Q_\tau} f^2\rho\,dxdt + \frac{\lambda}{2}\iint_{Q_\tau}\mathrm{Tr}((D^2u)^2)\rho\,dxdt\\ &\le \frac{1}{2\lambda}\|f\|^2_{L^{2\bar{p}}(Q_\tau)}\|\rho\|_{L^{\bar{p}'}(Q_\tau)} + \frac{\lambda}{2}\iint_{Q_\tau}\mathrm{Tr}((D^2u)^2)\rho\,dxdt.\end{aligned}$$

Let us focus on the first term of the right-hand side of the above inequality: it can be bounded by (6.70) and $\|f\|_{L^q(Q_T)}$ whenever there exists $\bar{p}$ such that

$$\frac{2(d+2)}{2(\gamma'-1)+1} < 2\bar{p} \le q.$$

Such a $\bar{p}$ indeed exists, since $q > \min\left\{d+2, \frac{d+2}{2(\gamma'-1)}\right\}$. Therefore,

$$I_2 \le C_3 + \frac{1}{16}\|Du\|^2_{L^\infty(Q_\tau)} + \frac{\lambda}{2}\iint_{Q_\tau}\mathrm{Tr}((D^2u)^2)\rho\,dxdt. \tag{6.77}$$

For the last term $\iint u_{x_k}\mathrm{Tr}(A_{x_k}D^2u)\rho$, Cauchy-Schwartz and Young's inequalities yield

$$\iint_{Q_\tau} u_{x_k}\mathrm{Tr}(A_{x_k}D^2u)\rho\,dxdt \le C\|Da\|^2_\infty\iint_{Q_\tau}|Du|^2\rho\,dxdt + \frac{\lambda}{2}\iint_{Q_\tau}\mathrm{Tr}((D^2u)^2)\rho\,dxdt$$

We distinguish two cases: if $\gamma \geq 2$, we have by (6.47) (with $\beta = 2$) that $\iint_{Q_\tau} |Du|^2\rho \leq C$. Otherwise, if $1 < \gamma < 2$,

$$\iint_{Q_\tau} |Du|^2\rho \leq \|Du\|^{2-\gamma}_{L^\infty(Q_\tau)} \iint_{Q_\tau} |Du|^\gamma \rho \, dxdt \leq C\|Du\|^{2-\gamma}_{L^\infty(Q_\tau)}.$$

In both cases we end up with

$$\iint_{Q_\tau} \partial_k u \mathrm{Tr}(\partial_k A D^2 u)\rho \, dxdt \leq C_4 + \frac{1}{8}\|Du\|^2_{L^\infty(Q_\tau)} + \frac{\lambda}{2}\iint_{Q_\tau} \mathrm{Tr}((D^2u)^2)\rho \, dxdt. \tag{6.78}$$

Step 3. Plugging (6.74), (6.75), (6.76), (6.77) and (6.78) into (6.73) we get

$$\frac{1}{2}\int_{\mathbb{T}^d} |Du(x,\tau)|^2\rho_\tau(x)\,dx = \int_{\mathbb{T}^d} z(x,\tau)\rho_\tau(x)\,dx \leq \frac{1}{2}\|Du(\cdot,0)\|^2_{L^\infty(\mathbb{T}^d)} + C + \frac{3}{8}\|Du\|^2_{L^\infty(Q_\tau)}.$$

Since this inequality holds for all smooth $\rho_\tau \geq 0$ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$, we obtain

$$\frac{1}{2}\|Du(\cdot,\tau)\|^2_{L^\infty(\mathbb{T}^d)} \leq \frac{1}{2}\|Du(\cdot,0)\|^2_{L^\infty(\mathbb{T}^d)} + C + \frac{3}{8}\|Du\|^2_{L^\infty(Q_\tau)},$$

and we conclude by passing to the supremum with respect to $\tau \in (0,T)$. □

Chapter 7

Transport equations with subcritical fractional diffusion and applications to Hamilton-Jacobi equations

This chapter is concerned with the study of the regularity properties of solutions to evolutive transport equations with subcritical fractional viscosity and unbounded coefficients. Our final aim is to apply this analysis to study the regularization effect of fractional Hamilton-Jacobi equations with superlinear growth in the gradient and possibly unbounded data in the spirit of Chapter 6.
More precisely, we begin focusing on the backward problem

$$\begin{cases} -\partial_t \rho(x,t) + (-\Delta)^s \rho(x,t) + \operatorname{div}(b(x,t)\,\rho(x,t)) = 0 & \text{in } \mathbb{T}^d \times (0,\tau)\ , \\ \rho(x,\tau) = \rho_\tau(x) & \text{in } \mathbb{T}^d\ , \end{cases} \tag{7.1}$$

with $\tau \in (0,T)$, where the diffusion term is a fractional Laplacian operator $(-\Delta)^s$ of order $s \in (1/2,1)$ and the velocity field $b = b(x,t)$ belongs to some suitable space-time Lebesgue space. We remind the reader that when the above problem is recasted as a PDE on the whole space $\mathbb{R}^d$ with non-periodic data, the above advection-diffusion equation is invariant under the (parabolic) scaling $\rho_\lambda(x,t) = \rho(\lambda x, \lambda^{2s} t)$ and $b_\lambda(x,t) = \lambda^{2s-1} b(\lambda x, \lambda^{2s} t)$ and, in particular, the $L^{\mathcal{Q}}(L^{\mathcal{P}})$ norm of the velocity field is invariant under the previous scaling of b when $d/(2s\mathcal{P}) + 1/\mathcal{Q} = 1 - 1/2s$. In the classical viscous case $s = 1$, it is well-known that this is the critical threshold in terms of the integrability of the drift ensuring well-posedness of the Fokker-Planck equation (see Chapter 6). Therefore, we will work under the following assumption for the drift b:

$$b(x,t) \in L^{\mathcal{Q}}(0,\tau; L^{\mathcal{P}}(\mathbb{T}^d)) \text{ with } \frac{d}{2s\mathcal{P}} + \frac{1}{\mathcal{Q}} \leq \frac{2s-1}{2s} \tag{7.2}$$

where

$$\mathcal{P} \geq d/(2s-1) \text{ and } \mathcal{Q} \geq 2s/(2s-1)\ .$$

This can be regarded as a fractional analogue of the interpolated condition (6.13) presented in the previous chapter. The analysis on these transport-type equations with unbounded coefficients and fractional diffusion can be tracked back to the literature of Surface Quasi-Geostrophic equations (see e.g. [93, 235, 63]), where, however, information on the divergence of the velocity field is available. We are not able to find a treatment on nonlocal diffusive transport equations without information on the divergence (typically the incompressibility condition $\mathrm{div}(b) = 0$). As for (7.1), unlike the discussion developed in Chapter 6, we assume that (7.2) is satisfied with a strict inequality (see Remark 7.7 for further details). In particular, we prove the following

Theorem 7.1. *Let $b \in L^{\mathcal{Q}}(0, \tau; L^{\mathcal{P}}(\mathbb{T}^d))$ with $\mathcal{P} \geq d/(2s-1)$ and $\mathcal{Q} \geq 2s/(2s-1)$ satisfying*

$$\frac{d}{2s\mathcal{P}} + \frac{1}{\mathcal{Q}} < \frac{2s-1}{2s} \ ,$$

and $\rho_\tau \in H^{s-1}(\mathbb{T}^d)$. Then, there exists a unique weak solution $\rho \in \mathcal{H}_2^{2s-1}(Q_\tau)$ to (7.1)*. If, in addition, $\rho_\tau \in L^p(\mathbb{T}^d)$, $p \in (1, \infty]$, then $\rho \in L^\infty(0, \tau; L^p(\mathbb{T}^d))$. Moreover, if $\rho_\tau \geq 0$, then $\rho \geq 0$ a.e. on Q_τ.*

Our main goal is to apply the above results to study the regularization effect of the fractional (forward) Hamilton-Jacobi equation

$$\begin{cases} \partial_t u(x,t) + (-\Delta)^s u(x,t) + H(x, Du(x,t)) = f(x,t) & \text{in } Q_T = \mathbb{T}^d \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}^d, \end{cases} \tag{7.3}$$

where the right-hand side f belongs to some vector-valued Lebesgue space $L^q(0, T; X)$, $q > 1$, where X will be a suitable Bessel potential space of positive order of differentiability (to be specified later) and H behaving like $|Du|^\gamma$ in the second entry, exactly under the assumptions (H) in force throughout the previous chapter. As announced, we work in the subcritical regime $s > \frac{1}{2}$ for the fractional diffusion operator $(-\Delta)^s$ due to the fact that under this condition the diffusion components are the dominating terms at small scales. In particular, by means of the duality procedure implemented in Chapter 6, we seek to prove that weak solutions (in a suitable sense) with bounded initial data u_0 become immediately Lipschitz continuous at positive times. In particular, we prove the following

Theorem 7.2. *Suppose that $s \in (1/2, 1)$ and*

- *$H \in C^1(\mathbb{T}^d \times \mathbb{R}^d)$, it is convex in the second variable, and satisfies* (H)*,*
- *$f \in L^q(0, T; H_q^{2-2s}(\mathbb{T}^d))$ with $q > d + 2s$ and $q > \frac{d+2s}{(\gamma'-1)(2s-1)}$*
- *$u_0 \in L^\infty(\mathbb{T}^d)$.*

Let u be a distributional solution to (7.3) *(in the sense of Definition 7.17)*

(1) If u is a local weak solution to (7.3)*, then $u(\cdot, \tau) \in W^{1,\infty}(\mathbb{T}^d)$ for all $\tau \in (0, T]$. In particular, for all $t_1 \in (0, T)$ there exists a constant $C_1 > 0$ depending on t_1, C_H, $\|u\|_{L^\infty(Q_T)}$, $\|f\|_{L^q(0,T;H_q^{2-2s}(\mathbb{T}^d))}$, q, d, T, s such that*

$$\|u(\cdot, \tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C_1$$

for all $\tau \in [t_1, T]$.

(2) If, in addition, u is a global weak solution to (7.3) with $u_0 \in W^{1,\infty}(\mathbb{T}^d)$, then there exists $C_2 > 0$ depending on C_H, $\|u_0\|_{W^{1,\infty}(\mathbb{T}^d)}$, $\|f\|_{L^q(0,T;H_q^{2-2s}(\mathbb{T}^d))}$, q, d, T, s such that

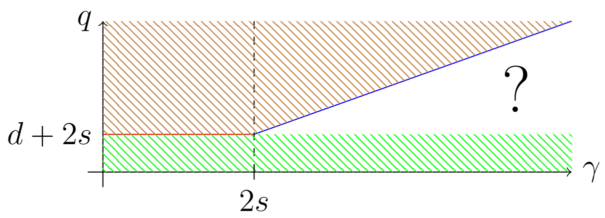
$$\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C_2$$

for all $\tau \in [0,T]$.

We would like to highlight that when $s = 1$ our analysis is consistent with the results obtained in Chapter 6 (cf Theorem 6.1). However, as it is clear from the context of PDEs with fractional diffusion, we are not able to directly reproduce the gradient estimate in Theorem 6.3 via the interplay between the Bernstein and adjoint method developed in Section 6.9. Anyhow, this latter Lipschitz regularity result seems to be the first one for Hamilton-Jacobi equations driven by fractional diffusion dealing with coercive Hamiltonians with general superlinear growth in the gradient and unbounded coefficients on the right-hand side source term f (at least in time). This would be also the first step towards maximal L^p-regularity for such quasi-linear integro-differential PDEs, which is still open nowadays even in the viscous case. We remark that as far as the integrability of $f \in L^q(H_q^{2-2s})$ is concerned, the results are somehow optimal (see Section 7.5). We emphasize that the integrability condition on q assumed in Theorem 7.2 can be rewritten as

$$q > \begin{cases} d + 2s & \text{if } 1 < \gamma \leq 2s, \\ \frac{d+2s}{(2s-1)(\gamma'-1)} & \text{if } \gamma > 2s, \end{cases}$$

and thus, even under these assumptions, the regularization effect for the range $d + 2s < q < \frac{d+2s}{(2s-1)(\gamma'-1)}$ remains open, as summarized in the next picture.



7.1 Some known results

We recall that (Hölder's) regularization effect of degenerate HJ equations with super-quadratic growth in the gradient and unbounded right-hand side has been discussed in [77] (see also [82] for a different approach). We also mention the Bernstein-type argument developed for quasi-linear equations with quadratic growth and degenerate diffusion in [27] in the stationary setting. The intermediate case in which the dynamics is driven by a jump process has been widely investigated during the last decade in the context of viscosity solutions' theory for integro-differential PDEs. Regularity results for HJ equations of the form

$$\partial_t u + (-\Delta)^s u + H(Du) = 0$$

with H locally Lipschitz are well-known. The conservation of Lipschitz regularity (i.e. starting with $u(0) \in W^{1,\infty}$) for every $s \in (0,1)$ goes back to [105, Theorem 5] (see also [139, 147]). Lipschitz regularity was then investigated in the case of critical diffusion $s = 1/2$ by L. Silvestre in [217]. Hölder's regularity of nonlocal HJ equations with superquadratic growth has been first analyzed in [76]. Hölder's regularization effect of (viscosity) solutions to fractional HJ equations in the "fractional" quadratic regime $\gamma = 2s$ (i.e. when $H(p) \sim \delta|p|^{2s}$) starting from a bounded initial data has been observed by L. Silvestre in [216]. There, the author obtains also a Hölder's regularity result in the fractional superquadratic regime $\gamma > 2s$ under a smallness condition on $\|u\|_{L^\infty}$. More recently, the regularization effect in Besov spaces when $s = 1/2$ is investigated in [141] under a smallness condition on the initial data. Lipschitz regularity for viscosity solutions of coercive Hamilton-Jacobi equations has been widely analyzed using revisited techniques coming from classical viscosity solutions' theory. In [30] the authors study Lipschitz regularity of solutions via Ishii-Lions method when f is bounded (which unfortunately requires to restrict the growth to the fractional subquadratic regime $\gamma < 2s$, as it happens for the classical viscous case $s = 1$) and via a weak version of the Bernstein method in the periodic setting [31], where $f \in W^{1,\infty}$ in the space variable and $\gamma > 1$, even for more general integro-differential operators than fractional powers of the Laplacian. We finally mention that fractional HJ-type PDEs with coercive Hamiltonians have been also recently investigated in the framework of periodic homogenization [18].

The plan of this chapter is the following: after some preliminary results on embedding theorems for fractional parabolic spaces, in Section 7.3 we present some crucial results on transport equations with subcritical fractional diffusion. Section 7.4 will be mainly devoted to the proof of Theorem 7.2.

7.2 Preliminaries

7.2.1 Sobolev embedding theorems for parabolic Bessel potential spaces

This section is devoted to present a Sobolev embedding theorem for the parabolic Bessel potential class $\mathcal{H}_2^{2s-1}$. We borrow from Part II of this manuscript all functional tools and definitions, which we will not recall here to avoid a cumbersome discussion.

More precisely, we show a trace result for parabolic Bessel potential spaces on the hyperplane $t = 0$ that can be regarded as the fractional counterpart of Proposition 6.11.

Proposition 7.3. *Let $s \in (\frac{1}{2}, 1)$. If $1 < \sigma < (d+2s)/(d+2s-1)$, then $\mathcal{H}_\sigma^{2s-1}(Q_T)$ is continuously embedded onto $L^p(Q_T)$ for*

$$\frac{1}{p} = \frac{1}{\sigma} - \frac{2s-1}{d+2s}.$$

Moreover, if $u \in \mathcal{H}_\sigma^{2s-1}(Q_T)$ and $u(\cdot,0) \in L^1(\mathbb{T}^d)$, we have

$$\|u\|_{L^p(Q_T)} \leq C\big(\|u\|_{\mathcal{H}_\sigma^{2s-1}(Q_T)} + \|u(0)\|_{L^1(\mathbb{T}^d)}\big)\ , \tag{7.4}$$

where the constant C depends on d, p, σ, T, but remains bounded for bounded values of T.

Proof. Let $f \in L^{p'}(Q_T)$ and let φ be the unique strong solution to the backward problem

$$\begin{cases} -\partial_t\varphi(x,t) + (-\Delta)^s\varphi(x,t) = f(x,t) & \text{in } Q_T \ , \\ \varphi(x,T) = 0 & \text{in } \mathbb{T}^d \ . \end{cases}$$

By maximal-L^p regularity results for fractional evolution equations, see Theorem B.4, we write

$$\|\varphi\|_{\mathcal{H}^{2s}_{p'}(Q_T)} \leq C\|f\|_{L^{p'}(Q_T)} \ .$$

By the embedding result in Proposition 5.27 we get

$$\|\varphi\|_{\mathbb{H}^1_{\sigma'}(Q_T)} \leq C\|\varphi\|_{\mathcal{H}^{2s}_{p'}(Q_T)} \ , \tag{7.5}$$

owing to the fact that φ has null terminal trace and $D\varphi \in \mathcal{H}^{2s-1}_{p'}(Q_T)$. Moreover, by the embeddings in Hölder's spaces developed in Part II (see Proposition 5.42) we have

$$\|\varphi\|_{C(Q_T)} \leq C\|\varphi\|_{\mathcal{H}^{2s}_{p'}(Q_T)} \ . \tag{7.6}$$

for

$$\frac{d+2s}{2s} < p' < \frac{d+2s}{2s-1} \ .$$

Therefore, integrating by parts in time and using (7.5) and (7.6) we have

$$\begin{aligned}
&\left|\iint_{Q_T} uf\,dxdt\right| = \left|\iint_{Q_T} u(-\partial_t\varphi + (-\Delta)^s\varphi)\,dxdt\right| \\
&\leq \int_{\mathbb{T}^d} |\varphi(x,0)u(x,0)|dx + \left|\iint_{Q_T} \partial_t u\,\varphi\,dxdt\right| + \left|\iint_{Q_T} (-\Delta)^{\frac{1}{2}}\varphi\,(-\Delta)^{s-\frac{1}{2}}u\,dxdt\right| \\
&\leq C_1\Big(\|\varphi(0)\|_{L^\infty(\mathbb{T}^d)}\|u(0)\|_{L^1(\mathbb{T}^d)} + \|\partial_t u\|_{\left(\mathbb{H}^1_{\sigma'}(Q_T)\right)'}\|\varphi\|_{\mathbb{H}^1_{\sigma'}(Q_T)} + \|u\|_{\mathbb{H}^{2s-1}_{\sigma}(Q_T)}\|D\varphi\|_{L^{\sigma'}(Q_T)}\Big) \\
&\leq C_2\Big(\|u(0)\|_{L^1(\mathbb{T}^d)} + \|\partial_t u\|_{\left(\mathbb{H}^1_{\sigma'}(Q_T)\right)'} + \|u\|_{\mathbb{H}^{2s-1}_{\sigma}(Q_T)}\Big)\|f\|_{L^{p'}(Q_T)} \ .
\end{aligned} \tag{7.7}$$

yielding the desired result.

□

***Remark* 7.4.** As in Proposition 7.7, one can prove the result for a generic time-intervals $I = (\tau_1, \tau)$ with obvious modifications.

7.3 Fractional diffusion equations with unbounded drifts

7.3.1 Scaling

In this section we perform different scalings to guess the critical integrability exponents ensuring the well-posedness and integrability estimates of solutions to

$$\partial_t\rho + (-\Delta)^s\rho - \mathrm{div}(b(x,t)\rho) = 0$$

and the regularity of solutions under rough assumptions on the right-hand side $f(x,t)$ of the fractional HJ equation (7.3). Let ρ be a solution to the above equation with drift $b=b(x,t)$, $\mu(x,t)=\varepsilon^{\alpha}\rho(\varepsilon x,\varepsilon^{2s}t)$ and $v(x,t)=\varepsilon^{\beta}b(\varepsilon x,\varepsilon^{2s}t)$ for some $\alpha,\beta\in\mathbb{R}$ to be later determined. Simple computations yield that the variable μ with drift v satisfies the equality

$$\partial_t\mu+(-\Delta)^s\mu-\operatorname{div}(v(x,t)\mu)=\varepsilon^{2s+\alpha}(\partial_t\rho+(-\Delta)^s\rho)+\varepsilon^{\alpha+\beta+1}\operatorname{div}(b(x,t)\rho)\ .$$

We look for possible scalings under which the above fractional Fokker-Planck equation is invariant, i.e. we impose

$$2s+\alpha=\alpha+\beta+1\ ,$$

giving $\beta=2s-1>0$ since $s\in(1/2,1)$.
As announced, our arguments will exploit a bound on the crossed quantity $\iint|b|^{\gamma'}\rho$. In particular, we will use this bound to extract regularity informations for ρ to handle a crucial term of the form $\iint f|D\rho|$. Therefore, we observe that

$$\iint|v|^{\gamma'}\mu=\varepsilon^{(2s-1)\gamma'+\alpha-d-2s}\iint|b|^{\gamma'}\rho\ ,$$

to find the optimal critical exponent

$$\alpha=d+2s-(2s-1)\gamma'\ ,$$

which leaves the crossed quantity invariant under the scaling. First, since the equation is in divergence form, by parabolic Caldéron-Zygmund theory (see e.g. Theorem B.4 with $\mu=2s-1$) we expect at most to control $(-\Delta)^{s-1/2}\rho$ in some Lebesgue space $L^{q'}$, where q' is the conjugate of $q>1$ to be later determined. Hence

$$\iint|(-\Delta)^{s-1/2}\mu|^{q'}=\varepsilon^{(\alpha+2s-1)q'-d-2s}\iint|(-\Delta)^{s-1/2}m|^{q'}\ .$$

Therefore, we impose

$$(\alpha+2s-1)q'-d-2s=0$$

giving

$$q'=\frac{d+2s}{d+2s-(2s-1)(\gamma'-1)}$$

after plugging the previous expression for α. This forces q to be

$$q=\frac{d+2s}{(2s-1)(\gamma'-1)}\ ,$$

which is the threshold appearing in Theorem 7.2. In view of this fact, in order to control $\iint f|D\rho|$ we need to require some additional space regularity on the right-hand side of (7.3), that is we impose $f\in L^q(H_q^{2-2s})$, see (7.25). By performing a $W^{1,\infty}$-scaling argument, if u solves (7.3), then one immediately notices that $w_\varepsilon(x,t)=\varepsilon^{-1}u(\varepsilon x,\varepsilon^{2s}t)$ is a solution to

$$\partial_t w_\varepsilon+(-\Delta)^s w_\varepsilon+\varepsilon^{2s-1}|Dw_\varepsilon|^{\gamma}=\varepsilon^{2s-1}f(\varepsilon x,\varepsilon^{2s}t)=r_\varepsilon(x,t)\ .$$

In particular, we observe that $r_\varepsilon\in H_q^{2-2s}$ if and only if $\|r_\varepsilon\|_q+\|(-\Delta)^{1-s}r_\varepsilon\|_q<\infty$, and the last norm is invariant under the scaling precisely when $q=d+2s$, which is the other threshold appearing in Theorem 7.2. We remark that when $s=1$ the above arguments are consistent with the integrability assumptions of Chapter 6.

7.3.2 Weak solutions for the fractional Fokker-Planck equation and its dual

This part is devoted to study the following advection equation with fractional diffusion

$$\begin{cases} -\partial_t\rho(x,t) + (-\Delta)^s\rho(x,t) + \mathrm{div}(b(x,t)\,\rho(x,t)) = 0 & \text{in } Q_\tau\ , \\ \rho(x,\tau) = \rho_\tau(x) & \text{in } \mathbb{T}^d\ . \end{cases} \tag{7.8}$$

Note that when the vector field $b(x,t) = -D_pH(x,Du(x,t))$, then (7.8) becomes the adjoint equation of the linearization of (6.1). Here, $\tau \in (0,T]$ and $Q_\tau := \mathbb{T}^d \times (0,\tau)$. From now on, unless otherwise specified, we will focus on $d > 2$. We will consider the following notion of weak solution

Definition 7.5. *Let $b \in L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(\mathbb{T}^d))$ with $\mathcal{P} \geq d/(2s-1)$ and $\mathcal{Q} \geq 2s/(2s-1)$ be such that*

$$\frac{d}{2s\mathcal{P}} + \frac{1}{\mathcal{Q}} < \frac{2s-1}{2s} \tag{7.9}$$

and $\rho_\tau \in H^{s-1}(\mathbb{T}^d)$. A (weak) solution ρ belongs to $\mathcal{H}_2^{2s-1}(Q_\tau)$ and satisfies

$$\int_0^\tau \int_{\mathbb{T}^d} \partial_t\varphi\rho\,dxdt + \iint_{Q_\tau} (-\Delta)^{s-\frac{1}{2}}\rho\,(-\Delta)^{\frac{1}{2}}\varphi - b\rho\cdot D\varphi\,dxdt = \int_{\mathbb{T}^d} \rho_\tau(x)\varphi(x,\tau)\,dx \tag{7.10}$$

for all $\varphi \in C^\infty(\mathbb{T}^d \times (0,\tau])$.

In particular, the above formulation holds even when test functions are chosen to belong to the class $\mathcal{H}_2^{1;s} := \{\varphi \in L^2(0,\tau;H^1(\mathbb{T}^d))\ ,\partial_t\varphi \in L^2(0,\tau;H^{-2s+1}(\mathbb{T}^d))\}$. We stress out that when $s = 1$ the above setting falls within the classical matter described in [48, 159, 39] under the interpolated condition (6.13) on the velocity field. We remark in passing that $\rho \in \mathcal{H}_2^{2s-1}(Q_\tau) \hookrightarrow C([0,T];(H^{2s-1}(\mathbb{T}^d),H^{-1}(\mathbb{T}^d))_{1/2,2}) \simeq C([0,T];H^{s-1}(\mathbb{T}^d))$ in view of the classical abstract trace result [101, Section XVIII.3 eq. (1.61)]. We finally point out that time-integration by parts holds by using that $C^\infty([0,T];H^{2s-1}(\mathbb{T}^d))$ is dense in $\mathcal{H}_2^{2s-1}(Q_T)$, see [101, p. 480].

Remark 7.6. Some observations are in order to compare the various notions of weak solutions met throughout the thesis. Note that the functional framework above described is different from the one developed in Section 5.5.1 (when $\sigma = 0$): in fact, here ρ belong to the smaller parabolic space $\mathcal{H}_2^{2s-1}$ instead of

$$\{\rho \in L^2(0,\tau;H^s(\mathbb{T}^d))\ ,\partial_t\rho \in L^2(0,\tau;H^{-1}(\mathbb{T}^d))\}\ .$$

This is due to the additional assumption on the negative part of the divergence (see Definition 5.38). Here, the parabolic space $\mathcal{H}_2^{2s-1}$ is the suited functional class to develop the fractional counterpart of the results in [159] under the rough assumptions on the drift stated in (7.9).

Remark 7.7. Here, as announced in the introduction, we will obtain our existence and integrability results under the assumption that the exponents $\mathcal{P},\mathcal{Q}$ meet

$$\frac{d}{2s\mathcal{P}} + \frac{1}{\mathcal{Q}} < \frac{2s-1}{2s}\ .$$

For instance, this is required when dealing with the variation of constants formula for abstract evolution equations in Step 2 of Theorem 7.12. Some details about the critical case

$$\frac{d}{2s\mathcal{P}} + \frac{1}{\mathcal{Q}} = \frac{2s-1}{2s}$$

will be provided in Remark 7.14.

Throughout this section we will assume that

$$\rho_\tau \in H^{s-1}(\mathbb{T}^d), \quad \rho_\tau \geq 0, \quad \text{and} \quad \langle \rho_\tau, 1 \rangle = 1 \ . \tag{7.11}$$

We further observe that $\rho \in \mathcal{H}_2^{2s-1}$ and $\mathcal{H}_2^{2s-1} \hookrightarrow L^2 \hookrightarrow L^1$ and hence $\rho(t) \in L^1(\mathbb{T}^d)$ for a.e. t. Therefore, by using suitable test functions one obtains $\int_{\mathbb{T}^d} \rho(t) = 1$ for fixed t.
Classical Fokker-Planck equations with low regularity on the drift have been studied in [194, 185, 50] and references therein. A key role to understand properties both for HJ and Fokker-Planck equations is played by the dual equation to (7.8) with subcritical fractional diffusion, namely

$$\begin{cases} \partial_t v + (-\Delta)^s v - b \cdot Dv = 0 & \text{in } Q_\omega := \mathbb{T}^d \times (\omega, \tau) \ , \\ v(x,\omega) = v_\omega(x) & \text{in } \mathbb{T}^d \ . \end{cases} \tag{7.12}$$

with $\omega \in [0,\tau)$, where b satisfies (7.9) (see e.g. Chapter 6 and [194, Section 3.1] for the viscous case, $s=1$). We consider the following notion of weak solution for (7.12)

Definition 7.8. [1] *For $b \in L^{\mathcal{Q}}(0,T;L^{\mathcal{P}}(\mathbb{T}^d))$ for $\mathcal{P}, \mathcal{Q}$ satisfying (7.9), a (weak) solution $v \in \mathcal{H}_2^{1;s}(Q_\omega) := \{v \in L^2(\omega,\tau;H^1(\mathbb{T}^d)),\ \partial_t v \in L^2(\omega,\tau;H^{1-2s}(\mathbb{T}^d))\}$ to (7.12) is such that*

$$-\iint_{Q_\omega} v \partial_t \varphi \, dt + \iint_{Q_\omega} (-\Delta)^{\frac{1}{2}} v \, (-\Delta)^{s-1/2} \varphi + b \cdot Dv \, \varphi \, dxdt = \int_{\mathbb{T}^d} v_\omega(x) \varphi(x,\omega) \, dx \tag{7.13}$$

for all $\varphi \in C^\infty(\mathbb{T}^d \times [\omega,\tau))$.

Here test functions can be chosen to belong to $\mathcal{H}_2^{2s-1}(Q_\tau)$. Note that by well-known abstract results for parabolic spaces we have that

$$v \in \mathcal{H}_2^{1;s}(Q_\omega) \hookrightarrow C([\omega,\tau];(H^1(\mathbb{T}^d), H^{s-1}(\mathbb{T}^d))_{\frac{1}{2},2}) \simeq C([\omega,\tau];H^{1-s}(\mathbb{T}^d))$$

and hence the trace $v(\omega)$ makes sense in $H^{1-s}(\mathbb{T}^d)$.

We recall that the comparison principle ensures uniqueness of suitably defined weak solutions to Hamilton-Jacobi equations by a simple linearization argument (see Remark 7.18 below and the discussion in Chapter 6). We prove the following comparison principle, which is a simple consequence of the parabolic Kato's inequality stated in the next lemma (see e.g. [167, Theorem 34] for the proof).

[1] Here we use the superscript $s \in (0,1)$ to emphasize the difference with the classical space $\mathcal{H}_2^1$ used in the case of classical diffusion throughout Chapter 6.

Lemma 7.9 (parabolic Kato's inequality). *Let Ω be a bounded domain in $\mathbb{R}^d$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing bounded continuous function (except at most in a finite number of points). Then, for any weak solution to the fractional heat equation*

$$\partial_t u + (-\Delta)^s u = f(x,t) \text{ in } \Omega \times (0,T)$$

with $f \in L^1(\Omega \times (0,T))$ we have that, in the weak sense, it holds

$$\partial_t \Phi(u) + (-\Delta)^s \Phi(u) \leq (\partial_t u + (-\Delta)^s u)\Phi(u) \ ,$$

where $\Phi(r) = \int_0^r \varphi(\sigma)\, d\sigma$.

Proposition 7.10. *Let $b \in L^{\mathcal{Q}}(\omega, \tau; L^{\mathcal{P}}(\mathbb{T}^d))$ with $\mathcal{P}, \mathcal{Q}$ satisfying (7.9) and let $u_1, u_2 \in \mathcal{H}_2^{1;s}(Q_\omega)$ be a weak sub- and a supersolution with $u(\omega) \leq v(\omega)$, $\omega \in [0,\tau)$, a.e. on $\mathbb{T}^d$. Then $u \leq v$ on Q_ω. Moreover, if v is a solution to (7.12) with $v(\omega) \geq 0$, then $v \geq 0$ a.e. in Q_ω.*

Proof. We have just to prove that if u_1 and u_2 are two solutions of (7.12), then $v := u_1 - u_2$ and $v(\omega) \leq 0$ satisfying

$$\partial_t v + (-\Delta)^s v = b \cdot Dv$$

fulfills $v \leq 0$.

To see this, let us restrict to the case $v^+ := \max\{v, 0\}$. First, note that since $v \in \mathcal{H}_2^{1;s}(Q_\omega)$, we have $v \in \mathbb{H}_q^1(Q_\omega)$ for $1 < q < \frac{d+2s}{d+1}$ and this implies that $|b||Dv|$ belong at least to $L^1(Q_\omega)$. Therefore, by the parabolic Kato's inequality in Lemma 7.9, we can write

$$\partial_t v^+ + (-\Delta)^s v^+ \leq |b||Dv^+| \ . \tag{7.14}$$

Let φ be the unique solution in $\mathcal{H}_2^{2s-1}(Q_\omega)$ to the problem

$$-\partial_t \varphi + (-\Delta)^s \varphi + \operatorname{div}(\varphi \mathcal{G}) = 1 \tag{7.15}$$

with zero terminal trace $\varphi(\tau) = 0$, where

$$\mathcal{G} := \begin{cases} |b|\frac{Dv^+}{|Dv^+|} & \text{if } |Dv^+| \neq 0 \\ 0 & \text{otherwise} \end{cases} .$$

First, notice that this equation is the adjoint of (7.12). Note that such a solution exists in view of the next Proposition 7.12 and, moreover, it enjoys $\varphi \in L^\infty(Q_\omega)$ and $\varphi \geq 0$ a.e. We use $\varphi \in \mathcal{H}_2^{2s-1}(Q_\omega)$ as admissible test function in the weak formulation to equation (7.14) and v^+ as a test function for (7.15) to get

$$-\int_{\mathbb{T}^d} v^+(x,\omega)\varphi(x,\omega) + \iint_{Q_\omega} \varphi \mathcal{G} \cdot Dv^+ \, dxdt + \iint_{Q_\omega} v^+ \, dxdt \leq \iint_{Q_\omega} |Dv^+||b|\varphi \, dxdt$$

which gives in particular $\iint_{Q_\tau} v^+ \leq 0$, i.e. $v^+ \leq 0$, owing to the fact that $\varphi(x,\tau) = 0$, $\varphi(x,\omega) \geq 0$ and $v^+(x,\omega) \leq 0$. $\square$

We have the following existence result obtained by similar arguments used in Proposition 6.9, covering also the equality in (7.9).

Proposition 7.11. *Let $v_\omega \in H^{1-s}(\mathbb{T}^d)$ and $b \in L^{\mathcal{Q}}(\omega,\tau;L^{\mathcal{P}}(\mathbb{T}^d))$ with $\mathcal{P} \geq d/(2s-1)$ and $\mathcal{Q} \geq 2s/(2s-1)$ such that $d/2s\mathcal{P} + 1/\mathcal{Q} \leq 1 - 1/2s$. Then, there exists a unique solution $v \in \mathcal{H}_2^{1;s}(Q_\omega)$ to (7.12).*

Proof. Let $\mathcal{Y} = \mathcal{H}_2^{1;s}(Q_\omega)$ and define the map $\Psi : \mathcal{Y} \times [0,1] \to \mathcal{Y}$ defined as $z \longmapsto \Psi[z;\sigma] = v$ with v solving the parametrized PDE

$$\partial_t v + (-\Delta)^s v = \sigma b \cdot Dz \text{ in } Q_\omega \ , v(x,\omega) = \sigma v_\omega \text{ in } \mathbb{T}^d \ ,$$

for the parameter $\sigma \in [0,1]$. We consider the case $\mathcal{P} = \mathcal{Q}$ only, so that $\mathcal{P} \geq \frac{d+2s}{2s-1}$, the general case being handled as in Proposition 6.9 exploiting the (critical) embeddings in mixed Lebesgue spaces, see also Appendix C. First, observe that $\Psi[z;0] = 0$ by standard results for fractional heat equations. We also note that $b \cdot Dz \in \mathbb{H}_2^{1-2s}(Q_\omega)$ by a straightforward consequence of Sobolev inequality in Lemma 5.16-(iii) (with $\mu = 2s-1$) and Hölder's inequality. Indeed, we have

$$\begin{aligned}\||b||Dz|\|_{L^2(\omega,\tau;H^{1-2s}(\mathbb{T}^d))} &\leq \||b||Dz|\|_{L^2(\omega,\tau;L^{\frac{2d}{d-2(2s-1)}}(\mathbb{T}^d))} \\ &\leq \||b|\|_{L^2(\omega,\tau;L^{\frac{d}{2s-1}}(\mathbb{T}^d))}\|Dz\|_{L^2(Q_\omega)} \leq C\||b|\|_{L^{\mathcal{Q}}(\omega,\tau;L^{\mathcal{P}}(\mathbb{T}^d))}\|Dz\|_{L^2(Q_\omega)} \ .\end{aligned}$$

Then, in view of Theorem B.4, we can infer that $v \in \mathcal{H}_2^{1;s}(Q_\omega)$ to (7.12) and the following estimate holds

$$\|v\|_{\mathcal{H}_2^{1;s}(Q_\omega)} \leq C(\||b||Dz|\|_{\mathbb{H}_2^{1-2s}(Q_\omega)} + \|v_\omega\|_{H^{1-s}(\mathbb{T}^d)})$$

since $\sigma \in [0,1]$. This shows that the map Ψ is well-defined. The same procedure by contradiction implemented in Proposition 6.9-(ii), which works exploiting the comparison principle stated in Proposition 7.10, actually gives the a priori estimate for every fixed point $g \in \mathcal{Y}$ of the map Ψ, that is satisfying $g = \Psi[g;\sigma]$.
We now prove the compactness of the map. Let z_n be a bounded sequence in $\mathcal{H}_2^{1;s}(Q_\omega)$ and $v_n = \Psi[z_n,\sigma]$. Arguing as above, we exploit the compactness of $\mathcal{H}_2^{1;s}(Q_\omega)$ onto $L^2(Q_\omega)$ (see Proposition 5.31 with $\mu = 1$), so as to have the strong convergence of v_n to v in $L^2(Q_\omega)$ and the weak convergence of $(-\Delta)^{\frac{1}{2}}v_n$ to $(-\Delta)^{\frac{1}{2}}v$ in $L^2(Q_\omega)$ along a subsequence. The compactness of Ψ follows exactly as in Proposition 6.9 of Chapter 6, by using now $\varphi := (-\Delta)^{1-s}(v_n - v)$ that satisfies $\varphi \in L^2(H^{2s-1})$ with $\partial_t\varphi \in L^2(H^{-1})$, and so it is an admissible test function. We write

$$\begin{aligned}&\int_\omega^\tau \frac{1}{2}\frac{d}{dt}\|(-\Delta)^{\frac{1-s}{2}}(v_n - v)\|^2_{L^2(\mathbb{T}^d)} + \iint_{Q_\omega} |(-\Delta)^{\frac{1}{2}}(v_n - v)|^2\,dxdt \\ &\quad\leq \iint_{Q_\omega} b \cdot Dz_n(-\Delta)^{1-s}(v_n - v)\,dxdt - \iint_{Q_\omega} (-\Delta)^{\frac{1}{2}}v \cdot (-\Delta)^{\frac{1}{2}}(v_n - v)\,dxdt \\ &\qquad\qquad - \iint_{Q_\omega} \partial_t v(-\Delta)^{1-s}(v_n - v)\,dxdt \ ,\end{aligned}$$

which shows the strong convergence of $(-\Delta)^{\frac{1}{2}}v_n$ to $(-\Delta)^{\frac{1}{2}}v$ in $L^2(Q_\omega)$ by using the aforementioned strong convergence of v_n to v in $L^2(Q_\omega)$, the weak convergence of $(-\Delta)^{\frac{1}{2}}v_n$ to $(-\Delta)^{\frac{1}{2}}v$ in $L^2(Q_\omega)$ and the bound of $|b||Dz| \in \mathbb{H}_2^{1-2s}(Q_\omega)$. By duality we finally get the strong convergence of the time-derivative in $\mathbb{H}_2^{1-2s}(Q_\omega)$ arguing exactly as in Proposition 6.9. We conclude the existence of solutions by the Leray-Schauder fixed point theorem [122, Theorem 11.6]. The uniqueness follows by Proposition 7.10.

□

7.3.3 Existence and integrability estimates to fractional Fokker-Planck equations

We present the main result of this section

Theorem 7.12. *Let $b \in L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))$ with $\mathcal{P},\mathcal{Q}$ satisfying (7.9). Then, there exists a unique solution $\rho \in \mathcal{H}_2^{2s-1}(Q_\tau)$ to (7.8) satisfying the estimate*

$$\|\rho\|_{\mathcal{H}_2^{2s-1}(Q_\tau)} \le C_1$$

for some positive $C_1 > 0$ depending in particular on $\|b\|_{L^{\mathcal{Q}}(L^{\mathcal{P}})}$ and $\|\rho_\tau\|_{H^{s-1}(\mathbb{T}^d)}$. In particular, if $\rho_\tau \ge 0$, then $\rho \ge 0$ a.e. in Q_τ.
Let now $\rho_\tau \in L^p(\mathbb{T}^d)$, $p \in (1,\infty]$. Then, $\rho \in L^\infty(0,\tau;L^p(\mathbb{T}^d))$ and we have

$$\|\rho(\cdot,t)\|_{L^p(\mathbb{T}^d))} \le C_2\|\rho_\tau\|_{L^p(\mathbb{T}^d)} \text{ for a.e. } t \in [0,\tau]$$

for some $C_2 > 0$ where $C_2 = C_2(d,p,\mathcal{P},\mathcal{Q},s,\|b\|_{L^{\mathcal{Q}}(L^{\mathcal{P}})})$.

Remark 7.13. We remark that our existence proof in Step 1 is based on a fixed point argument and exploits Caldéron-Zygmund regularity for fractional PDEs in divergence form (cf Theorem B.4). This is a different procedure than the one implemented in [48], where the authors argue by regularization and truncation due to the low regularity of the diffusion term (it is typically the approach when the data are only measurable and thus classical maximal L^p regularity approach is not reasonable) and also to [159], where a refinement of the Galerkin method for the case of unbounded coefficients is used.

Proof. Step 1. Existence and uniqueness in the energy space $\mathcal{H}_2^{2s-1}(Q_\tau)$. We apply Leray-Schauder fixed point theorem for the existence (see [122, Theorem 11.6] on the space

$$\mathcal{X} = \mathcal{H}_2^{2s-1}(Q_\tau)\ .$$

Consider the map $G : \mathcal{X} \times [0,1] \to \mathcal{X}$ defined by $m \longmapsto \rho = G[m;\sigma]$ given by solving the following parametrized PDE

$$-\partial_t\rho + (-\Delta)^s\rho = \sigma\mathrm{div}(b(x,t)m) \text{ in } Q_\tau\ , \rho(x,\tau) = \sigma\rho_\tau(x) \text{ in } \mathbb{T}^d\ .$$

Note that $G[m;0] = 0$ by standard results for fractional heat equations. We first show that $G : \mathcal{X} \times [0,1] \to \mathcal{X}$ is well-defined. We first consider the case $\mathcal{P} = \mathcal{Q}$ for simplicity (whence condition (7.9) becomes $\mathcal{P} > d+2$). By parabolic Caldéron-Zygmund regularity theory (cf Theorem B.4) we have

$$\begin{aligned}\|\rho\|_{\mathcal{X}} = \|\rho\|_{\mathcal{H}_2^{2s-1}(Q_\tau)} &\le \tilde{C}(T)(\sigma\|bm\|_{L^2(Q_\tau)} + \sigma\|\rho_\tau\|_{H^{s-1}(\mathbb{T}^d)})\\ &\le \tilde{C}(T)(\|b\|_{L^{\mathcal{P}}(Q_\tau)}\|m\|_{L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(Q_\tau)} + \|\rho_\tau\|_{H^{s-1}(\mathbb{T}^d)})\\ &\le C_2\|b\|_{L^{\mathcal{P}}(Q_\tau)} + \frac{1}{2}\|m\|_{\mathcal{H}_2^{2s-1}(Q_\tau)} + \tilde{C}(T)\|\rho_\tau\|_{H^{s-1}(\mathbb{T}^d)} \quad (7.16)\end{aligned}$$

The last inequality is due to the fact that we have

$$1 < \frac{2\mathcal{P}}{\mathcal{P}-2} < \frac{2(d+2s)}{d+2-2s} \ ,$$

which allows to argue by interpolation and exploit the embedding of $\mathcal{H}_2^{2s-1}(Q_\tau) \hookrightarrow L^{\frac{2(d+2s)}{d+2-2s}}(Q_\tau)$ (see Remark 5.28 and Proposition C.3) to show

$$\|m\|_{L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(Q_\tau)} \le C_1 \|m\|_{L^1(Q_\tau)}^{\theta} \|m\|^{1-\theta}_{L^{\frac{2(d+2s)}{d+2-2s}}(Q_\tau)} = C_1 T^{\theta} \|m\|^{1-\theta}_{L^{\frac{2(d+2s)}{d+2-2s}}(Q_\tau)} \le C_2 + \frac{1}{2\tilde{C}} \|m\|_{\mathcal{H}_2^{2s-1}(Q_\tau)}$$

for some $\theta \in (0,1)$. This shows that G is well-defined from $\mathcal{X}$ into itself since $m \in \mathcal{X}$. Moreover, if $\rho \in \mathcal{X}$ and $\sigma \in [0,1]$ is such that $\rho = G[\rho;\sigma]$ we have that $\rho \in \mathcal{X}$ is a solution of (7.8) and the a priori estimate (7.16) carry through uniformly on $\sigma \in [0,1]$. Thus we obtain the existence of a constant $M > 0$ depending only on the data (namely $\|b\|_{L^{\mathcal{P}}(Q_\tau)}$, ρ_τ, T) such that

$$\|\rho\|_{\mathcal{X}} \le M \ .$$

We finally show that the map T is compact using similar arguments to that of Proposition 6.9-(ii). Let m_n be a bounded sequence in $\mathcal{H}_2^{2s-1}(Q_\tau)$ and let $\rho_n = G[m_n;\sigma]$. Since $|b|m_n \in L^2(Q_\tau)$ we have that $\operatorname{div}(bm_n) \in \mathbb{H}_2^{-1}(Q_\tau)$ and hence by Theorem B.4 we have $\rho_n \in \mathcal{H}_2^{2s-1}(Q_\tau)$. By the compactness of $\mathcal{H}_2^{2s-1}$ onto $L^2(Q_\tau)$ (cf Proposition 5.31), which is ensured by the restriction $s > 1/2$, we have that, along a subsequence, ρ_n converges strongly in $L^2(Q_\tau)$ to ρ and $(-\Delta)^{s-1/2}\rho_n$ converges weakly to $(-\Delta)^{s-1/2}\rho$ in $L^2(Q_\tau)$. We use $(-\Delta)^{s-1}(\rho_n - \rho) \in \mathcal{H}_2^{1;s}(Q_\tau)$ as admissible test function to the weak formulation of the equation satisfied by ρ_n.

$$\begin{aligned}
&\iint_{Q_\tau} \partial_t(\rho_n - \rho)(-\Delta)^{s-1}(\rho_n-\rho)\,dxdt + \iint_{Q_\tau} |(-\Delta)^{s-\frac12}(\rho_n-\rho)|^2\,dxdt \\
&\quad \le C \iint_{Q_\tau} |b|m_n||(-\Delta)^{s-\frac12}(\rho_n-\rho)|\,dxdt - \iint_{Q_\tau} (-\Delta)^s\rho(-\Delta)^{s-1}(\rho_n-\rho)\,dxdt \\
&\qquad - \iint_{Q_\tau} \partial_t\rho(-\Delta)^{s-1}(\rho_n-\rho)\,dxdt
\end{aligned}$$

Since $|b|m_n \in L^2(Q_\tau)$ and $(-\Delta)^{s-1/2}\rho_n$ converges weakly to $(-\Delta)^{s-1/2}\rho$ in $L^2(Q_\tau)$ the first term on the right-hand side of the above inequality converges to 0. Similarly, since $\partial_t\rho \in \mathbb{H}_2^{-1}(Q_\tau)$ and exploiting again the weak convergence of $(-\Delta)^{s-1/2}\rho_n$ in $L^2(Q_\tau)$ the third term goes to 0. Similar motivations provide the convergence of the second term. This shows that $(-\Delta)^{s-1/2}\rho_n$ converges strongly to $(-\Delta)^{s-1/2}\rho$ in $L^2(Q_\tau)$. Finally, to show the strong convergence of $\partial_t\rho_n$ to $\partial_t\rho$ in $\mathbb{H}_2^{-1}(Q_\tau)$ we argue by duality. For every $\varphi \in \mathbb{H}_2^1(Q_\tau)$ we have

$$\begin{aligned}
\left|\iint_{Q_\tau} \partial_t(\rho_n-\rho)\varphi\,dxdt\right| &\le \left|\iint_{Q_\tau} (-\Delta)^s(\rho_n-\rho)\varphi\,dxdt\right| + \left|\iint_{Q_\tau} \operatorname{div}(b(\rho_n-\rho))\varphi\,dxdt\right| \\
&\le C \iint_{Q_\tau} |(-\Delta)^{s-\frac12}(\rho_n-\rho)||D\varphi|\,dxdt + \iint_{Q_\tau} |\rho_n-\rho||b||D\varphi|\,dxdt
\end{aligned}$$

which yields the strong convergence of $\partial_t\rho_n$ to $\partial_t\rho$ in $\mathbb{H}_2^{-1}(Q_\tau)$ in view of the previous claims. The general case $\mathcal{P}\neq\mathcal{Q}$ can be treated similarly, observing that

$$1<\frac{2\mathcal{P}}{\mathcal{P}-2}<\frac{2d}{d-2(2s-1)}$$

which yields by interpolation

$$\|m(\cdot,t)\|_{L^{\frac{2\mathcal{P}}{\mathcal{P}-2}}(\mathbb{T}^d)}\leq C\|m(\cdot,t)\|^{\theta}_{L^1(\mathbb{T}^d)}\|m(\cdot,t)\|^{1-\theta}_{L^{\frac{2d}{d-2(2s-1)}}(\mathbb{T}^d)}$$

for a.e. $t\in(0,\tau)$. Then one integrates in time the above inequality and applies first Hölder's inequality and then exploits the Sobolev embedding $H_2^{2s-1}(\mathbb{T}^d)$ onto $L^{\frac{2d}{d-2(2s-1)}}(\mathbb{T}^d)$ and argue exactly as above. The uniqueness of solutions can be obtained by duality, see Step 3 below.

Step 2. A priori estimate via Duhamel's formula. We claim that there exists $t^*\in(0,\tau]$ independently of $\rho_\tau\in L^p(\mathbb{T}^d)$ such that

$$\|\rho(\cdot,t)\|_{L^p(\mathbb{T}^d)}\leq C_2\|\rho_\tau\|_{L^p(\mathbb{T}^d)}\text{ for all }t\in[t^*,\tau]$$

for some $C_2>0$. Set $\tilde\rho(\cdot,t):=\rho(\cdot,\tau-t)$ for all $t\in[0,\tau]$. We use Duhamel's formula to represent the solution as

$$\tilde\rho(t)=\mathcal{T}_t\rho_\tau-\int_0^t\mathcal{T}_{t-\omega}\mathrm{div}(b\tilde\rho)(\cdot,\omega)d\omega\ .$$

Though this is a formal computation, it can be made rigorous by approximation. We have

$$\begin{aligned}\|\tilde\rho(t)\|_{L^p(\mathbb{T}^d)}&\leq\|\mathcal{T}_t\rho_\tau\|_{L^p(\mathbb{T}^d)}+\left\|\int_0^t\mathcal{T}_{t-\omega}\mathrm{div}(b\tilde\rho)(\cdot,\omega)d\omega\right\|_{L^p(\mathbb{T}^d)}\\&\leq\|\rho_\tau\|_{L^p(\mathbb{T}^d)}+\int_0^t(t-\omega)^{-\frac{d}{2s}(\frac{1}{b}-\frac{1}{p})-\frac{1}{2s}}\|\mathrm{div}(b\tilde\rho)(\cdot,\omega)\|_{H_b^{-1}(\mathbb{T}^d)}d\omega\\&\leq\|\rho_\tau\|_{L^p(\mathbb{T}^d)}+\int_0^t(t-\omega)^{-\frac{d}{2s}(\frac{1}{b}-\frac{1}{p})-\frac{1}{2s}}\|b\tilde\rho(\cdot,\omega)\|_{L^b(\mathbb{T}^d)}d\omega\ ,\end{aligned}$$

where we applied the decay estimates in Lemma 5.23-(iv). We then use Hölder's inequality to bound the right-hand side of the last inequality with

$$\begin{aligned}&\|\tilde\rho\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))}\int_0^\tau(t-\omega)^{-\frac{d}{2s}(\frac{1}{b}-\frac{1}{p})-\frac{1}{2s}}\|b(\cdot,\omega)\|_{L^{\mathcal{P}}(\mathbb{T}^d)}d\omega\\&\qquad\leq\left(\int_0^t(t-\omega)^{[-\frac{d}{2s}(\frac{1}{b}-\frac{1}{p})-\frac{1}{2s}]\mathcal{Q}'}\right)^{\frac{1}{\mathcal{Q}'}}\|b\|_{L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))}\|\tilde\rho\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))}\end{aligned}$$

where

$$\frac{1}{b}=\frac{1}{\mathcal{P}}+\frac{1}{p}\ .$$

In particular, the above integral term is well-posed provided that

$$\alpha:=\left(-\frac{d}{2s\mathcal{P}}-\frac{1}{2s}\right)\mathcal{Q}'>-1\ ,$$

which is indeed satisfied precisely when

$$\frac{d}{2s\mathcal{P}}+\frac{1}{\mathcal{Q}}<\frac{2s-1}{2s}\ .$$

Hence

$$\|\tilde{\rho}\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))}\leq\|\rho_\tau\|_{L^p(\mathbb{T}^d)}+C\|b\|_{L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))}t^{\frac{\alpha+1}{\mathcal{Q}'}}\|\tilde{\rho}\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))}\ ,$$

which gives

$$\|\tilde{\rho}\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))}\leq 2\|\rho_\tau\|_{L^p(\mathbb{T}^d)}$$

by taking

$$t\geq t^*:=\left(\frac{1}{2C\|b\|_{L^{\mathcal{Q}}(0,\tau;L^{\mathcal{P}}(\mathbb{T}^d))}}\right)^{\frac{\mathcal{Q}'}{\alpha+1}}$$

and hence the validity of the estimate on $[0,t^*]$ with $C_2=2$. Note that t^* does not depend on $\|\rho_\tau\|_{L^p(\mathbb{T}^d)}$ and hence one can iterate the argument in the following way. Let n be the integer part of $\frac{1}{t^*}$. Then one applies the above scheme in the intervals $[0,t^*]$, $[t^*,2t^*]$,....,$[nt^*,1]$ getting the estimate with $C_2=2^{n+1}$.

Step 3. Positivity. This is a straightforward consequence of the comparison principle in Proposition 7.10, whose proof can be done e.g. by duality. We take $\varphi=v$ as a test function in the weak formulation of (7.8), where v solves (7.12) in $Q_\omega=\mathbb{T}^d\times(\omega,\tau)$ with $v(\omega)=v_\omega\geq 0$ and ρ as a test function to (7.12). By summing the expressions one obtains

$$\int_{\mathbb{T}^d}v(\omega)\rho(\omega)\,dx=\int_{\mathbb{T}^d}\rho(\tau)v(\tau)\,dx$$

and since the right-hand side is nonnegative, the left-hand side is so since $v_\omega\geq 0$.

□

Remark 7.14. In Step 1 we can actually reach the threshold

$$\frac{d}{2s\mathcal{P}}+\frac{1}{\mathcal{Q}}=\frac{2s-1}{2s}$$

by assuming a smallness condition on $\|b\|_{L^{\mathcal{Q}}(L^{\mathcal{P}})}$, since interpolation inequalities are no longer available. Furtheremore, the approach used to get $L^\infty(L^2)$ estimate in Step 2 (i.e. with $p=2$) can be modified by accommodating the above endpoint case by implementing a L^2 version of the adjoint method (see e.g. [125, Section 4]), i.e. by testing the equation against the solution $v\in\mathcal{H}_2^{1;s}$ to (7.12) with initial data $v_\omega\in L^2(\mathbb{T}^d)$ with $\|v_\omega\|_{L^2(\mathbb{T}^d)}=1$, following the lines described in Step 3. However, we prefer to keep the strict inequality for the sake of exposition.

Estimates on parabolic Sobolev spaces $\mathcal{H}_{q'}^{2s-1}(Q_\tau)$

We finally describe further regularity results that rely on the information $b\in L^k(\rho)$ for some $k>1$, that will be used in the next section.

Proposition 7.15. *Let ρ be a (non-negative) weak solution to* (6.23) *and*

$$1 < q' < \frac{d+2s}{d+2s-1} \ . \tag{7.17}$$

Then, there exists $C > 0$, depending on q', d, T, s such that

$$\|\rho\|_{\mathcal{H}_{q'}^{2s-1}(Q_\tau)} \leq C(\|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}). \tag{7.18}$$

Note that C here does not depend on $\tau \in (0, T]$.

Proof. Let us rewrite the equation (7.8) as a perturbation of the fractional heat equation

$$-\partial_t \rho + (-\Delta)^s \rho = \operatorname{div}(b(x,t)\rho) \text{ on } Q_\tau$$

with terminal data $\rho(x,\tau) := \rho_\tau(x)$ on $\mathbb{T}^d$. We observe that $\rho \in \mathcal{H}_2^{2s-1}$ readily implies $\rho \in \mathcal{H}_{q'}^{2s-1}$ for every $1 < q' < 2$. By parabolic regularity theory (see Theorem B.4) ρ enjoys the estimate

$$\|\rho\|_{\mathcal{H}_{q'}^{2s-1}(Q_\tau)} \leq C(\|b\rho\|_{L^{q'}(Q_\tau)} + \|\rho_\tau\|_{W^{2s-1-2s/q',q'}(\mathbb{T}^d)}) \ .$$

By exploiting Sobolev embedding for fractional Sobolev spaces in Lemma 5.17, one immediately obtains that

$$\|\rho_\tau\|_{W^{2s-1-2s/q',q'}(\mathbb{T}^d)} \leq C\|\rho_\tau\|_{L^1(\mathbb{T}^d)}$$

whenever $1 < q' < \frac{d+2s}{d+2s-1}$. Indeed

$$\begin{aligned}\|\rho_\tau\|_{W^{2s-1-2s/q',q'}(\mathbb{T}^d)} &= \sup_{\varphi \in W^{2s/q'-2s+1,q}(\mathbb{T}^d) \ ,\|\varphi\|_{W^{2s/q'-2s+1,q}(\mathbb{T}^d)}=1} \left| \int_{\mathbb{T}^d} \rho_\tau \varphi \, dx \right| \\ &\leq \|\varphi\|_\infty \|\rho_\tau\|_{L^1(\mathbb{T}^d)} \leq C\|\varphi\|_{W^{2s/q'-2s+1,q}(\mathbb{T}^d)} \|\rho_\tau\|_{L^1(\mathbb{T}^d)} \leq C\|\rho_\tau\|_{L^1(\mathbb{T}^d)}\end{aligned}$$

where the last inequality is a consequence of the embedding $W^{2s/q'-2s+1,q}(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$ (cf Lemma 5.17-(ii)) when

$$(2s/q' - 2s + 1)q > d \ ,$$

that is $q > d + 2s$ or, in other words, when q' satisfies (7.17). □

Proposition 7.16. *Let ρ be the (non-negative) weak solution to* (7.8) *and*

$$1 < q' < \frac{d+2s}{d+2s-1}.$$

Then, there exists $C > 0$, depending on T, q', d, s such that

$$\|\rho\|_{\mathcal{H}_{q'}^{2s-1}(Q_\tau)} \leq C \left(\iint_{Q_\tau} |b(x,t)|^{r'} \rho(x,t) \, dxdt + 1 \right), \tag{7.19}$$

where

$$r' = 1 + \frac{d+2s}{q(2s-1)}. \tag{7.20}$$

Proof. Inequality (7.18), (7.11) and the generalized Hölder's inequality yield

$$
\begin{aligned}
\|\rho\|_{\mathcal{H}^{2s-1}_{q'}(Q_\tau)} &\le C(\|b\rho^{1/r'}\rho^{1/r}\|_{L^{q'}(Q_\tau)}+1)\\
&\le C\left(\left(\iint_{Q_\tau}|b|^{r'}\rho\,dxdt\right)^{1/r'}\|\rho\|^{1/r}_{L^p(Q_\tau)}+1\right), \quad (7.21)
\end{aligned}
$$

for $p>q'$ satisfying

$$
\frac{1}{q'}=\frac{1}{r'}+\frac{1}{rp}. \quad (7.22)
$$

Then, by Young's inequality, for all $\varepsilon>0$

$$
\|\rho\|_{\mathcal{H}^{2s-1}_{q'}(Q_\tau)}\le C\left(\frac{1}{\varepsilon}\iint_{Q_\tau}|b|^{r'}\rho\,dxdt+\varepsilon\|\rho\|_{L^p(Q_\tau)}+1\right). \quad (7.23)
$$

One can verify that (7.20) and (7.22) yield

$$
\frac{1}{p}=\frac{1}{q'}-\frac{2s-1}{d+2s}.
$$

Indeed, (7.22) gives

$$
\frac{1}{p}=\frac{r}{q'}-\frac{r}{r'}=\frac{1}{q'}-\frac{r-1}{q}
$$

and then the definition of r' in (7.20) gives the conclusion. The continuous embedding of $\mathcal{H}^{2s-1}_{q'}(Q_\tau)$ in $L^p(Q_\tau)$ stated in Proposition 7.3 then implies

$$
\|\rho\|_{L^p(Q_\tau)}\le C_1\big(\|\rho\|_{\mathcal{H}^{2s-1}_{q'}(Q_\tau)}+\tau\big),
$$

finally giving

$$
\|\rho\|_{L^p(Q_\tau)}\le CC_1\left(\frac{1}{\varepsilon}\iint_{Q_\tau}|b|^{r'}\rho\,dxdt+\varepsilon\|\rho\|_{L^p(Q_\tau)}+1\right), \quad (7.24)
$$

Hence, the term $\varepsilon\|\rho\|_{L^p(Q_\tau)}$ can be absorbed by the left hand side of (7.24) by choosing $\varepsilon=(2CC_1)^{-1}$, thus providing the assertion. □

7.4 Lipschitz regularity to Hamilton-Jacobi equations with subcritical fractional diffusion

This last part is concerned with the proof of Lipschitz regularity of u, stated in Theorem 7.2. The adjoint method implemented here follows the very same lines of Chapter 6 and one can repeat the same heuristic idea described in Section 6.2 to obtain the gradient bound. However, unlike the classical case in which $D\rho$ can be controlled in some Lebesgue norm, here, due to the subtle gap between the (subcritical) fractional diffusion operator and the divergence term, we expect to control $\|(-\Delta)^{s-\frac{1}{2}}\rho\|_{L^q(Q_\tau)}$ for some $q>1$, as we observed in the previous sections. This forces to assume some (spatial) fractional differentiability on the right-hand side of the fractional HJ equation, just as a simple consequence of vector-valued Hölder's inequality. More precisely

$$
\begin{aligned}
\left|\iint_{\mathbb{T}^d\times(0,\tau)} \partial_\xi f\,\rho\right| &\le \|\partial_\xi f\|_{L^q(0,T;H^{1-2s}_{q'}(\mathbb{T}^d))}\|\rho\|_{L^q(0,T;H^{2s-1}_q(\mathbb{T}^d))} \\
&\lesssim \|(-\Delta)^{\frac12} f\|_{L^{q'}(0,T;H^{1-2s}_{q'}(\mathbb{T}^d))}\|\rho\|_{L^q(0,T;H^{2s-1}_q(\mathbb{T}^d))} \\
&\lesssim \|f\|_{L^{q'}(0,T;H^{2-2s}_{q'}(\mathbb{T}^d))}\|\rho\|_{L^q(0,T;H^{2s-1}_q(\mathbb{T}^d))}\ . \qquad (7.25)
\end{aligned}
$$

This is the major difference with respect to the Lipschitz regularity analyzed in Chapter 6.

We will suppose that the assumptions of Theorem 7.2 stated in the Introduction are in force: $H \in C^1(\mathbb{T}^d \times \mathbb{T}^d)$, it is convex in the second variable, satisfies (H) and $u_0 \in L^\infty(\mathbb{T}^d)$. Moreover, $f \in L^q(0,T;H^{2-2s}_q(\mathbb{T}^d))$ for some $q > d+2s$. At some stage we will require $q \ge \frac{d+2s}{(\gamma'-1)(2s-1)}$ also. The result will be accomplished using regularity properties of the adjoint variable ρ, i.e. the solution to

$$
\begin{cases}
-\partial_t\rho(x,t) + (-\Delta)^s\rho(x,t) - \operatorname{div}\big(D_pH(x,Du(x,t))\,\rho(x,t)\big) = 0 & \text{in } Q_\tau\ , \\
\rho(x,\tau) = \rho_\tau(x) & \text{on } \mathbb{T}^d\ ,
\end{cases} \qquad (7.26)
$$

for $\tau \in (0,T)$, $\rho_\tau \in C^\infty(\mathbb{T}^d)$, $\rho_\tau \ge 0$, and $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$ and u is a weak solution to (6.1) (see the next section). By the integrability assumptions on D_pH, the adjoint state $\rho \in \mathcal{H}^{2s-1}_2(Q_\tau)$ is, for any ρ_τ, well-defined, non-negative and bounded in $L^\infty(Q_\tau)$. In what follows, we establish bounds on ρ that are independent on the choice of τ and ρ_τ.
We will say that u is a *weak* solution to (7.3) in the following sense. In what follows we denote by $Q_{(t_1,t_2)} = \mathbb{T}^d \times (t_1,t_2)$ for $0 \le t_1 \le t_2 \le T$.

Definition 7.17. *A function $u \in L^\gamma(0,T;W^{1,\gamma}(\mathbb{T}^d)) \cap L^\infty(Q_T)$ is a distributional solution to* (7.3) *if*

$$
-\int_{\mathbb{T}^d} u_0\varphi(0)dx + \iint_{Q_T} -u\,\partial_t\varphi + (-\Delta)^{\frac12}u\,(-\Delta)^{s-\frac12}\varphi + H(x,Du)\varphi\,dxdt = \iint_{Q_T} f\varphi\,dxdt \qquad (7.27)
$$

for all $\varphi \in C^\infty(\mathbb{T}^d \times [0,T))$.

(1) We say that u is a local weak solution if (7.27) *holds and, in addition,*

i) $u \in \mathcal{H}^{1;s}_2(Q_{(t,T)}) = \{u \in L^2(t,T;H^1(\mathbb{T}^d))\ , \partial_t u \in L^2(t,T;H^{1-2s}(\mathbb{T}^d))\}$ for all $t \in (0,T)$,

ii) $D_pH(x,Du) \in L^{\mathcal{Q}}(t,T;L^{\mathcal{P}}(\mathbb{T}^d))$ for $t \in (0,T)$, $\frac{d}{2s-1} \le \mathcal{P} \le \infty$ and $\frac{2s}{2s-1} \le \mathcal{Q} \le \infty$ such that (7.9) *holds ,*

(2) We say that u is a global weak solution if (i)-(ii) in (1) *holds on $Q_T = \mathbb{T}^d \times (0,T)$.*

In what follows, when talking about local and global weak solutions, we will always assume that they are also distributional solutions as in Definition 7.17-(1) and so identity (7.27) holds. In particular, in the case (2) of global weak solution, (7.27) holds in general for $\varphi \in \mathcal{H}_2^{2s-1}(Q_T) \cap L^\infty(Q_T)$. In view of the results of the previous section it turns out that *ii)* actually guarantees the well-posedness of the adjoint equation by the results of the previous section. Specifically, in the case of local weak solution we will consider $\varphi = \rho$ defined on (t, τ) for all $t > 0$, while in the case of global weak solutions ρ is defined on $(0, \tau)$. This condition, as in Chapter 6, will be crucial in our analysis in order to achieve Lipschitz regularity.

Remark 7.18. Notice that under the assumptions of Definition 7.17, global weak solutions of (7.3) must be unique, as it happens in the viscous case, as pointed out in Proposition 6.8: this can be seen again via a simple linearization argument.

Let us point out first that from now on we will denote by $C, C_1, ...$ positive constants that may depend on C_H, $\|u_0\|_{L^\infty(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, T, q, d, but do not depend on τ, ρ_τ.

We first bound from above the solution of the Hamilton-Jacobi equation (6.1), using a duality argument that involves solutions of a backward heat equation. First, recall that the Lagrangian $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, $L(x, \xi) := \sup_p\{p \cdot \xi - H(x, p)\}$, namely the Legendre transform of H in the p-variable, is well defined by the superlinearity of $H(x, \cdot)$ in the gradient variable. Moreover, by convexity of $H(x, \cdot)$,

$$H(x, p) = \sup_{\xi \in \mathbb{R}^d}\{\xi \cdot p - L(x, \xi)\},$$

and

$$H(x, p) = \xi \cdot p - L(x, \xi) \quad \text{if and only if} \quad \xi = D_pH(x, p). \tag{7.28}$$

We further recall the following properties of L are standard: for some $C_L > 0$,

$$C_L^{-1}|\xi|^{\gamma'} - C_L \le L(x, \xi) \le C_L|\xi|^{\gamma'} \tag{L1}$$

$$|D_xL(x, \xi)| \le C_L(|\xi|^{\gamma'} + 1). \tag{L2}$$

for all $\xi \in \mathbb{T}^d$.

Proposition 7.19. *There exists $C > 0$ (depending on T, q', d) such that any global weak solution u to* (7.3) *satisfies*

$$u(x, \tau) \le \|u_0\|_{L^\infty(\mathbb{T}^d)} + C\|f\|_{L^q(Q_T)} \tag{7.29}$$

for all $\tau \in (0, T)$ and a.e. $x \in \mathbb{T}^d$.

Proof. Let $\tau \in (0, T)$. Consider the nonnegative solution of the following backward fractional heat equation

$$\begin{cases} -\partial_t\mu(x, t) + (-\Delta)^s\mu(x, t) = 0 & \text{on } Q_\tau\ , \\ \mu(x, \tau) = \mu_\tau(x) & \text{on } \mathbb{T}^d\ . \end{cases}$$

with $\mu_\tau \in C^\infty(\mathbb{T}^d)$, $\mu_\tau \ge 0$ and $\|\mu_\tau\|_{L^1(\mathbb{T}^d)} = 1$. Note that μ can be seen as a solution of a Fokker-Planck equation of the form (7.8) with drift $b \equiv 0$. Then, since

$q' < (d+2s)/(d+2s-1)$, by Proposition 7.16 there exists a positive constant C (not depending on μ_τ) such that $\|\mu\|_{\mathcal{H}^{2s-1}_{q'}(Q_\tau)} \le C$.

Use μ as a test function in the weak formulation of the HJ equation (7.3) and recalling that for $u \in \mathcal{H}^{1,s}_2(Q_\tau)$ time-integration by parts holds, one gets

$$\int_{\mathbb{T}^d} u(x,\tau)\mu_\tau(x)dx = \int_{\mathbb{T}^d} u_0(x)\mu(x,0)dx + \iint_{Q_\tau} f\mu dxdt - \iint_{Q_\tau} H(x,Du)\mu dxdt \ .$$

Applying Hölder's inequality to the second term of the right-hand side of the above inequality, the estimate on $\mu \in L^{q'}(Q_\tau)$, and the fact that $\|\mu(t)\|_{L^1(\mathbb{T}^d)} = 1$ for all $t \in (0,\tau)$, we get

$$\begin{aligned}\int_{\mathbb{T}^d} u(x,\tau)\mu_\tau(x)dx + \int_0^\tau \int_{\mathbb{T}^d} f\mu dxdt &\le \|u_0\|_{L^\infty(\mathbb{T}^d)} + \|\mu\|_{L^{q'}(Q_\tau)}\|f\|_{L^q(Q_\tau)} \\ &\le \|u_0\|_{L^\infty(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)} \ .\end{aligned}$$

By the assumption $H(x,Du) \ge 0$, we then conclude

$$\int_{\mathbb{T}^d} u(x,\tau)\mu_\tau(x)dx \le \|u_0\|_{L^\infty(\mathbb{T}^d)} + C\|f\|_{L^q(Q_\tau)} \ .$$

Finally, by passing to the supremum over $\mu_\tau \ge 0$, $\|\mu_\tau\|_{L^1(\mathbb{T}^d)} = 1$, one deduces the estimate (6.52) by duality. □

Remark 7.20. We remark that actually gradient of solutions to

$$\begin{cases} -\partial_t\mu(x,t) + (-\Delta)^s\mu(x,t) = 0 & \text{on } Q_\tau \ , \\ \mu(x,\tau) = \mu_\tau(x) & \text{on } \mathbb{T}^d \ . \end{cases}$$

with $\mu_\tau \in L^1$ enjoys better regularity. This can be immediately seen by noting that Caldéron-Zygmund theory applies on the space $\mathcal{H}^{1;s}_{q'}$ (cf Theorem B.4) yielding

$$\|\mu\|_{\mathcal{H}^{1;s}_{q'}(Q_\tau)} \le C\|\mu_\tau\|_{W^{1-\frac{2s}{q'},q'}(\mathbb{T}^d)} \ .$$

Then, one has the estimate

$$\|\mu_\tau\|_{W^{1-\frac{2s}{q'},q'}(\mathbb{T}^d)} \le C_1\|\mu_\tau\|_{L^1(\mathbb{T}^d)}$$

when $q' < \frac{d+2s}{d+1}$ by arguing as in Proposition 7.15, giving thus a little gain of integrability of the test function μ.

Lemma 7.21. *Let u be a local weak solution to* (7.3)*. Assume that ρ is a weak solution to* (7.26)*. Then, for all $0 < \tau_1 < \tau_2 \le T$*

$$\begin{aligned}\int_{\mathbb{T}^d} u(x,\tau_2)\rho_\tau(x)dx = \int_{\mathbb{T}^d} u(x,\tau_1)\rho(x,\tau_1)\,dx + \int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^d} L(x, D_pH(x,Du))\rho dxdt \\ + \int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^d} f\rho\, dxdt. \quad (7.30)\end{aligned}$$

Moreover, if u is a global weak solution, (7.30) *holds also with $\tau_1 = 0$.*

Proof. Using $-\rho \in \mathcal{H}_2^{2s-1}(Q_{(\tau_1,\tau_2)}) \cap L^\infty(Q_{(\tau_1,\tau_2)})$ as a test function in the weak formulation of problem (7.3), $u \in \mathcal{H}_2^{1,s}(Q_{(\tau_1,\tau_2)})$ as a test function for the corresponding adjoint equation (7.26) and summing both expressions, one obtains for all $0 < \tau_1 < \tau_2 \leq T$

$$
\begin{aligned}
-\int_{\tau_1}^{\tau_2} \langle \partial_t u(t), \rho(t)\rangle dt - \int_{\tau_1}^{\tau_2} \langle \partial_t \rho(t), u(t)\rangle dt \\
+ \int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^d} (D_pH(x,Du)\cdot Du - H(x,Du))\rho dxdt + \int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^d} f\rho\, dxdt = 0\ .
\end{aligned}
$$

The desired equality follows after integrating by parts in time and using property (7.28) of L. Note that since $u \in \mathbb{H}_1^\gamma(Q_{(\tau_1,\tau_2)})$ is a distributional solution, then $H(x,Du) \in L^1(Q_{(\tau_1,\tau_2)})$, and also $L(x, D_pH(x,Du)) \in L^1(Q_{(\tau_1,\tau_2)})$ by (L1) and (H), so all the terms in (7.30) make sense. $\square$

We now prove the crossed integrability bound on D_pH with respect to ρ.

Proposition 7.22. *Let u be a local weak solution to* (7.3) *and ρ be a weak solution to* (7.26). *Then, there exist a positive constant C (depending on C_H, $\|u\|_{L^\infty(Q_T)}$,$\|u_0\|_{L^\infty(\mathbb{T}^d)}$, $\|f\|_{L^q(Q_T)}$, q, d, T, s) such that for all $\tau \in (0,T)$*

$$\iint_{Q_\tau} |D_pH(x,Du(x,t))|^{\gamma'}\rho(x,t)\, dxdt \leq C\ , \tag{7.31}$$

and if u is a global weak solution, then

$$\|u(\cdot,\tau)\|_{L^\infty(\mathbb{T}^d)} \leq C_1 \qquad \text{for all } \tau \in [0,T]. \tag{7.32}$$

Remark 7.23. An immediate consequence of (7.31) is the bound

$$\iint_{Q_\tau} |Du(x,t)|^\beta \rho(x,t)\, dxdt \leq C_\beta \qquad \text{for all } 1 \leq \beta \leq \gamma. \tag{7.33}$$

Indeed, by (H) and $\int_{\mathbb{T}^d}\rho(t) = 1$ for a.e. t, $\iint_{Q_\tau}|Du(x,t)|^\gamma\rho(x,t)\,dxdt \leq C$, which yields (7.33) for $\beta = \gamma$. For $\beta < \gamma$ it is sufficient to use Young's inequality and $\|\rho(t)\|_{L^1(\mathbb{T}^d)} = 1$.

Proof. Rearrange the representation formula (7.30) to get, for $0 < \tau_1 < \tau < T$,

$$
\begin{aligned}
\iint_{Q_{(\tau_1,\tau)}} L(x, D_pH(x,Du))\rho\, dxdt = \int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx - \int_{\mathbb{T}^d} u(x,\tau_1)\rho(x,\tau_1) \\
- \iint_{Q_{(\tau_1,\tau)}} f\rho\, dxdt.
\end{aligned} \tag{7.34}
$$

Use the bounds on the Lagrangian (L1), (7.29) and Hölder's inequality with the exponent $q = \bar{q}$ such that

$$\bar{q} > d + 2s \text{ and } \bar{q} > \frac{d+2s}{(\gamma'-1)(2s-1)}$$

and its conjugate $\bar{q}'$ to obtain

$$C_L^{-1} \iint_{Q_{(\tau_1,\tau)}} |D_p H(x, Du)|^{\gamma'} \rho \, dxdt \leq \iint_{Q_{(\tau_1,\tau)}} L(x, D_p H(x, Du)) \rho \, dxdt$$
$$\leq 2\|u\|_{L^\infty(Q_T)} + C\|f\|_{L^{\bar{q}}(\tau_1,\tau;H_{\bar{q}}^{2-2s}(\mathbb{T}^d))} \|\rho\|_{L^{\bar{q}'}(\tau_1,\tau;H_{\bar{q}'}^{2s-2}(\mathbb{T}^d))} \ , \quad (7.35)$$

where we exploit the fact that for a.e. $t \in (0, T)$

$$\|u(t)\|_{L^\infty(\mathbb{T}^d)} \leq \|u\|_{L^\infty(Q_T)} \ .$$

Since

$$\bar{q}' < \frac{d+2s}{d+2s-1} \ ,$$

owing to (7.19) (with q replaced by $\bar{q}$), one finds that inequality (7.35) is less than or equal to

$$C(\|\rho\|_{\mathcal{H}_{\bar{q}'}^{2s-1}(Q_{(\tau_1,\tau)})} + 1) \leq C_1 \left(\int_{\tau_1}^{\tau} \int_{\mathbb{T}^d} |D_p H(x, Du)|^{r'} \rho(x,t) \, dxdt + 1 \right), \qquad (7.36)$$

where $r' = 1 + \frac{d+2s}{\bar{q}(2s-1)}$. Finally, the right hand side of (7.36) can be absorbed in the left hand side of (7.35) whenever $r' < \gamma'$ by Young's inequality. This is in fact guaranteed by

$$r' = 1 + \frac{d+2s}{\bar{q}(2s-1)} < \gamma'.$$

As a byproduct, one obtains (7.31) after letting $\tau_1 \to 0$.

Regarding (7.32), in view of Proposition 7.19 we have that $u(\cdot, \tau)$ is essentially bounded from above. To prove the bound from below, use formula (7.30) and the bounds from below for the Lagrangian (L1) to get

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx \geq \int_{\mathbb{T}^d} u(x,0)\rho(x,0) - C_L \iint_{Q_\tau} \rho(x,t)dxdt + \iint_{Q_\tau} f\rho \, dxdt.$$

Since $\iint f\rho$ can be bounded from below using as before Hölder's inequality and (7.36),

$$\int_{\mathbb{T}^d} u(x,\tau)\rho_\tau(x)dx \geq -\|u(\cdot,0)\|_{L^\infty(\mathbb{T}^d)} - C_L\tau - C,$$

that holds for any smooth ρ_τ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)} = 1$, implying the desired result.

□

The crossed integrability of $D_p H$ against the adjoint variable ρ finally provides the $L^{q'}$ regularity of $(-\Delta)^{s-1/2}\rho$. From now on, we will suppose that $q > d + 2s$ and $q \geq \frac{d+2s}{(\gamma'-1)(2s-1)}$.

Corollary 7.24. *Let u be a weak solution to* (7.3) *and ρ be a weak solution to* (7.26)*. Let $\bar{q}$ be such that*

$$\bar{q} > d + 2s \quad \textit{and} \quad \bar{q} \geq \frac{d+2s}{(\gamma'-1)(2s-1)}.$$

Then, there exists a positive constant C such that

$$\|\rho\|_{\mathcal{H}^{2s-1}_{\bar{q}'}(Q_\tau)} \leq C \ ,$$

where C depends in particular on C_H, $\|f\|_{L^q(0,\tau;H^{2-2s}_q(\mathbb{T}^d))}$, $\bar{q}, d, T, s$ (but not on τ, ρ_τ) and either on $\|u_0\|_{L^\infty(\mathbb{T}^d)}$ if u is a global weak solution or $\|u\|_{L^\infty(Q_T)}$ if u is a local weak solution.

Proof. Since $\bar{q}' < \frac{d+2s}{d+2s-1}$, (7.19) applies (with $q = \bar{q}$), yielding

$$\|\rho\|_{\mathcal{H}^{2s-1}_{\bar{q}'}(Q_\tau)} \leq C \left(\iint_{Q_\tau} |D_p H(x, Du(x,t))|^{r'} \rho(x,t)\, dxdt + 1 \right),$$

with

$$r' = 1 + \frac{d+2s}{\bar{q}(2s-1)} \leq \gamma'.$$

If $r' = \gamma'$, use Proposition 7.22 to conclude, otherwise. If $r' < \gamma'$ use Young's inequality first to control $\iint |D_p H(x, Du(x,t))|^{r'} \rho\, dxdt$ with $\iint |D_p H(x, Du)|^{\gamma'}\, dxdt + \tau$. □

Remark 7.25. We remark that the assumption

$$\bar{q} > d + 2s \quad \text{and} \quad \bar{q} \geq \frac{d+2s}{(\gamma'-1)(2s-1)}.$$

reduces to $\bar{q} > d + 2s$ when $\gamma \leq 2s$. Anyhow, we provide the above estimates for the solution ρ of the fractional Fokker-Planck equation even for the (fractional) superquadratic regime $\gamma > 2s$, even though the above threshold for $\bar{q}$ deteriorates as γ increases.

7.4.1 Proof of the main results

We now prove our main result.

Theorem 7.26. *Let u be a distributional solution to* (7.3).

(i) Let u be a local weak solution to (7.3)*. Then, there exists $\eta = \eta(t) \in C_0^\infty((0,T])$ positive smooth function satisfying $\eta(t) \leq 1$ for all t such that $(\eta u)(\cdot, \tau) \in W^{1,\infty}(\mathbb{T}^d)$ for all $\tau \in (0,T)$, and there exists $C > 0$ depending on C_H, $\|u\|_{L^\infty(Q_T)}$, $\|f\|_{L^q(0,T;H^{2-2s}_q(\mathbb{T}^d))}$, q, d, T, s such that*

$$\eta(\tau)\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C$$

for all $\tau \in (0,T]$.

(ii) Let u be a global weak solution to (7.3) *and $\eta = \eta(t) \in C^\infty([0,T])$ be a positive smooth function satisfying $\eta(t) \leq 1$ for all t. Then, $(\eta u)(\cdot,\tau) \in W^{1,\infty}(\mathbb{T}^d)$ for all $\tau \in (0,T)$, and there exists $C > 0$ depending on C_H, $\|u_0\|_{W^{1,\infty}(\mathbb{T}^d)}$, $\|f\|_{L^q(0,T;H^{2-2s}_q(\mathbb{T}^d))}$, q, d, T, s such that*

$$\eta(\tau)\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C\Big(\eta(0)\|Du_0\|_{L^\infty(Q_T)} + \sup_{(0,T)} |\eta'(t)| + 1\Big)$$

for all $\tau \in (0,T]$.

Proof. **Step 1.** Since H is convex and superlinear we can write for a.e. $(x,t) \in Q_T$

$$H(x, Du(x,t)) = \sup_{\xi \in \mathbb{R}^d} \{\xi \cdot Du(x,t) - L(x,\xi)\}.$$

Let $0 < \tau_1 < \tau < T$. We then obtain

$$\begin{aligned}\int_{\tau_1}^{\tau} \langle \partial_t u(t), \varphi(t)\rangle dt + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} (-\Delta)^{\frac{1}{2}} u(x,t)\,(-\Delta)^{s-\frac{1}{2}}\varphi(x,t) \\ + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} [\Xi(x,t)\cdot Du(x,t) - L(x,\Xi(x,t))]\varphi\,dxdt \le \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} f(x,t)\varphi(x,t)\,dxdt\end{aligned} \tag{7.37}$$

for all test functions $\varphi \in \mathcal{H}_2^{2s-1}(Q_{(\tau_1,\tau)}) \cap L^\infty(Q_{(\tau_1,\tau)})$ and measurable $\Xi : Q_{(\tau_1,\tau)} \to \mathbb{T}^d$ such that $L(\cdot,\Xi(\cdot,\cdot)) \in L^1(Q_{(\tau_1,\tau)})$ and $\Xi \cdot Du \in L^1(Q_{(\tau_1,\tau)})$. Note that the previous inequality becomes an equality if $\Xi(x,t) = D_pH(x, Du(x,t))$ in $Q_{(\tau_1,\tau)}$. Let η be as in (i) satisfying the additional requirement

$$\frac{\eta'}{\eta^\theta} \in L^q(0,T)\ . \tag{7.38}$$

We first fix $\tau \in (0,T)$, $\tau_1 > 0$ outside $\mathrm{supp}(\eta)$, ρ_τ as in (7.11) and $0 \neq h \in \mathbb{R}^d$. Set

$$w(x,t) = \eta(t)u(x,t).$$

Use now (7.37) with $\Xi(x,t) = D_pH(x, Du(x,t))$ and $\varphi = \eta\rho \in \mathcal{H}_2^{2s-1}(Q_{(\tau_1,\tau)}) \cap L^\infty(Q_{(\tau_1,\tau)})$, where ρ is the adjoint variable (i.e. the weak solution to (7.26)) to find

$$\begin{aligned}\int_{\tau_1}^{\tau} \langle \partial_t w(t), \rho(t)\rangle dt + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} (-\Delta)^{\frac{1}{2}} w(x,t)\,(-\Delta)^{s-\frac{1}{2}}\rho(x,t) \\ + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} D_pH(x,Du)\cdot Dw\rho - L(x, D_pH(x,Du))\eta\rho\,dxdt \\ = \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} f\eta\rho\,dxdt + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} u\eta'\rho\,dxdt.\end{aligned} \tag{7.39}$$

Then, use $w \in \mathcal{H}_2^1(Q_{(\tau_1,\tau)})$ as a test function in the weak formulation of the equation satisfied by ρ to get

$$-\int_{\tau_1}^{\tau} \langle \partial_t \rho(t), w(t)\rangle dt + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} (-\Delta)^{\frac{1}{2}} w(x,t)\,(-\Delta)^{s-\frac{1}{2}}\rho(x,t) + D_pH(x,Du)\rho\cdot Dw\,dxdt = 0\,. \tag{7.40}$$

We obtain, subtracting the previous equality to (7.39), and integrating by parts in time

$$\begin{aligned}\int_{\mathbb{T}^d} w(x,\tau)\rho_\tau(x)dx = \int_{\mathbb{T}^d} w(x,\tau_1)\rho(x,\tau_1)dx + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \eta(t)f(x,t)\rho(x,t)dxdt \\ + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \eta(t)L\big(x, D_pH(x,Du(x,t))\big)\rho(x,t)dxdt + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \eta'(t)u(x,t)\rho(x,t)dxdt.\end{aligned} \tag{7.41}$$

For $h>0$ and $h \in \mathbb{R}^d$, $|h|=1$ define $\hat\rho(x,t) := \rho(x-h,t)$. After a change of variables in (7.26), it can be seen that $\hat\rho$ satisfies, using w as a test function,

$$-\int_{\tau_1}^{\tau} \langle \partial_t \hat\rho(t), w(t)\rangle dt$$
$$+\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} (-\Delta)^{s-\frac12}\hat\rho(x,t)(-\Delta)^{\frac12}w + D_pH(x-h, Du(x-h,t))\hat\rho(x,t)\cdot Dw(x,t)\,dxdt = 0. \tag{7.42}$$

As before, plugging $\Xi(x,t) = D_pH(x-h, Du(x-h,t))$ and $\varphi = \eta\hat\rho$ in (7.37) yields

$$\int_{\tau_1}^{\tau}\langle \partial_t w(t), \hat\rho(t)\rangle dt + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} (-\Delta)^{\frac12}w\,(-\Delta)^{s-\frac12}\hat\rho$$
$$+\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} D_pH(x-h, Du(x-h,t))\cdot Dw\hat\rho - L(x, D_pH(x-h, Du(x-h,t)))\eta\hat\rho\,dxdt$$
$$\le \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} f\eta\hat\rho\,dxdt + \iint_{Q_\tau} u\eta'\hat\rho\,dxdt.$$

Hence, subtracting (7.42) to the previous inequality,

$$\int_{\mathbb{T}^d} w(x,\tau)\hat\rho_\tau(x)dx - \int_{\mathbb{T}^d} w(x,\tau_1)\hat\rho(x,\tau_1)dx \le$$
$$\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} L(x, D_pH(x-h, Du(x-h,t)))\eta\hat\rho\,dxdt + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} f\eta\hat\rho\,dxdt + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} u\eta'\hat\rho\,dxdt,$$

which, after the change of variables $x \mapsto x+h$, becomes

$$\int_{\mathbb{T}^d} w(x+h,\tau)\rho_\tau(x)dx - \int_{\mathbb{T}^d} w(x+h,\tau_1)\rho(x,\tau_1)dx$$
$$\le \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \eta(t)L(x+h, D_pH(x, Du(x,t)))\rho(x,t)\,dxdt$$
$$+\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} f\eta\hat\rho\,dxdt + \int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} u\eta'\hat\rho\,dxdt, \tag{7.43}$$

Taking the difference between (7.43) and (7.41) we obtain

$$\int_{\mathbb{T}^d} (w(x+h,\tau) - w(x,\tau))\rho_\tau(x)dx \le \int_{\mathbb{T}^d} (w(x+h,\tau_1) - w(x,\tau_1))\rho(x,\tau_1)dx$$
$$+\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \eta(t)\Big(L(x+h, D_pH(x, Du(x,t))) - L(x, D_pH(x, Du(x,t)))\Big)\rho(x,t)\,dxdt$$
$$+\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \eta(t)f(x,t)\big(\rho(x-h,t) - \rho(x,t)\big)\,dxdt$$
$$+\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \eta'(t)u(x,t)\big(\rho(x-h,t) - \rho(x,t)\big)\,dxdt. \tag{7.44}$$

Step 2. We now estimate all the right hand side terms of (7.44). We remark that constants $C, C_1, \ldots$ are not going to depend on τ, ρ_τ, h. First, since $\|\rho(x,\tau_1)\|_{L^1(\mathbb{T}^d)} = 1$,

$$\left|\int_{\mathbb{T}^d} (w(x+h,\tau_1) - w(x,\tau_1))\rho(x,\tau_1)dx\right| \leq \eta(\tau_1)\|Du(\tau_1)\|_{L^\infty(\mathbb{T}^d)}|h|. \tag{7.45}$$

Next, using (7.31) and property (L2) of D_xL

$$\begin{aligned}
&\left|\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \eta(t)\Big(L(x+h, D_pH(x,Du(x,t))) - L(x, D_pH(x,Du(x,t)))\Big)\rho(x,t)\,dxdt\right| \\
&\qquad \leq |h|\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \|D_xL(\cdot, D_pH(x,Du(x,t)))\|_{L^\infty(\mathbb{T}^d)}\rho(x,t)\,dxdt \\
&\qquad\qquad \leq |h|\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \big(|D_pH(x,Du(x,t))|^{\gamma'}+1\big)\rho(x,t)\,dxdt \leq C|h|.
\end{aligned}$$

Denote by $D^h\rho(x,t) := |h|^{-1}(\rho(x+h,t)-\rho(x,t))$. Then, for the term involving f we use again Corollary 7.24, with $\bar{q} = q$, and control the difference quotient $D^h\rho(x,t)$ via the norm of ρ in the space of Bessel potentials $\mathbb{H}^{2s-1}_{q'}(Q_T)$, to get

$$\begin{aligned}
&\left|\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \eta(t)f(x,t)\big(\rho(x-h,t)-\rho(x,t)\big)\,dxdt\right| \\
&\qquad = \left|\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} \eta(t)f(x,t)\big(\rho(x-h,t)-\rho(x,t)\big)\,dxdt\right| \\
&\qquad \leq |h|\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d} |f(x,t)|\,|D^h(I-\Delta)^{2-2s}z(x,t)|\,dxdt \\
&\leq C_1|h|\|f\|_{L^q(\tau_1,\tau;H^{2-2s}_q(\mathbb{T}^d))}\|D\rho\|_{\mathbb{H}^{2s-2}_{q'}(Q_{(\tau_1,\tau)})} \leq C_2\|\rho\|_{\mathbb{H}^{2s-1}_{q'}(Q_{(\tau_1,\tau)})}|h| \leq C_3|h|\ .
\end{aligned}$$

Here, we exploited the Bessel potential representation of ρ by means of $z \in L^{q'}$ in the first inequality (see e.g. [3, Section 1.2.6]) and we applied Lemma 6.12 since $(I-\Delta)^{2s-2}\rho(x,t) \in L^{q'}(0,\tau; W^{1,q'}(\mathbb{T}^d))$. Arguing by interpolation, we exploit the following abstract result

$$[L^q(0,T;X_1), L^q(0,T;X_2)]_\theta \simeq L^q(0,T;[X_1,X_2]_\theta)$$

where $(X_1,X_2) = (H^{s_1}_p, H^{s_2}_p)$ with $s_1 = 0$, $s_2 = 1$ and $\theta = 2s-1$ so that $[X_1,X_2]_\theta = H^{2-2s}_q$ (see [231, Section 1.18.4]) in order to interpolate the $L^q(H^{2-2s}_q)$ norm in the following way:

$$\|u\|_{L^q(\tau_1,\tau;H^{2-2s}_q(\mathbb{T}^d))} \leq C\|u\|^{1-\theta}_{L^q(Q_{(\tau_1,\tau)})}\|u\|^{\theta}_{\mathbb{H}^1_q(Q_{(\tau_1,\tau)})}\ .$$

Then, using the above inequality and Corollary 7.24 we deduce

$$\left|\int_{\tau_1}^{\tau}\int_{\mathbb{T}^d}\eta'(t)u(x,t)\big(\rho(x-h,t)-\rho(x,t)\big)\,dxdt\right|$$
$$\leq C_1|h|\left(\int_{\tau_1}^{\tau}(\eta'(t))^q\|u(t)\|^q_{H_q^{2-2s}(\mathbb{T}^d)}\,dt\right)^{\frac{1}{q}}\left(\int_{\tau_1}^{\tau}\|\rho(t)\|^{q'}_{H_{q'}^{2s-1}(\mathbb{T}^d)}\,dt\right)^{\frac{1}{q'}}$$
$$\leq C_2|h|\left(\int_{\tau_1}^{\tau}(\eta')^q\|u(t)\|^{(1-\theta)q}_{L^\infty(\mathbb{T}^d)}\|Du(t)\|^{\theta q}_{L^\infty(\mathbb{T}^d)}\,dt\right)^{\frac{1}{q}}$$
$$\leq C_2|h|\left(\int_{\tau_1}^{\tau}\left(\frac{\eta'}{\eta^\theta}\right)^q\right)^{\frac{1}{q}}\|u\|^{1-\theta}_{L^\infty(Q_{(\tau_1,\tau)})}\|\eta Du\|^{\theta}_{L^\infty(Q_{(\tau_1,\tau)})}\leq C+\frac{1}{2}|h|\|\eta Du\|_{L^\infty(Q_{(\tau_1,\tau)})}$$

for $\theta=2s-1\in(0,1)$ using (7.38). Plugging all the estimates in (7.44) and recalling the choice of $\eta\in C_c^\infty((0,T])$ with $\tau_1\notin\text{supp}(\eta)$, we obtain

$$\int_{\mathbb{T}^d}(w(x+h,\tau)-w(x,\tau))\rho_\tau(x)dx\leq C|h|+\frac{1}{2}|h|\|\eta Du\|_{L^\infty(Q_{(\tau_1,\tau)})}\tag{7.46}$$

Since (7.46) holds for all smooth $\rho_\tau\geq 0$ with $\|\rho_\tau\|_{L^1(\mathbb{T}^d)}=1$, we conclude

$$\eta(\tau)(u(x+h,\tau)-u(x,\tau))\leq C|h|+\frac{1}{2}|h|\|\eta Du\|_{L^\infty(Q_{(\tau_1,\tau)})}\ .$$

Observe that the previous inequality holds for any $h\neq 0\in\mathbb{R}^d$. Therefore, one may select a continuous representative of $u(\cdot,\tau)$ such that the above inequality is fulfilled for all $x\in\mathbb{T}^d$ and $h\in\mathbb{R}^d$. To this aim, one could take the uniform limit as $\delta\to 0$ of $u\star\delta^{-d}\chi(\cdot/\delta)$, where χ is a smooth mollifier). Thus, $u(\cdot,\tau)$ has a Lipschitz continuous representative and

$$\eta(\tau)\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)}\leq C\ .\tag{7.47}$$

The proof of (ii) in the context of global weak solution follows goes through the very same steps . For $\eta\in C^\infty([0,T])$ as in (ii), it is enough to let $\tau_1\to 0$ and use the bound (7.45) to deduce the estimate

$$\eta(\tau)(u(x+h,\tau)-u(x,\tau))\leq C|h|(1+\eta(0)\|Du_0\|_{L^\infty(Q_T)})+\frac{1}{2}|h|\|\eta Du\|_{L^\infty(Q_T)}\ .$$

and conclude as above. □

Proof of Theorem 7.2. The first part of Theorem 7.2 is a straightforward consequence of Theorem 7.26-(i), while the second part follows by Theorem 7.26-(ii). □

7.5 Final remarks on the integrability exponent of the right-hand side

As outlined in the introduction, by performing a $W^{1,\infty}$ scaling, i.e. by zooming in and looking at $z(x,t)=\varepsilon^{-1}u(\varepsilon x,\varepsilon^{2s}t)$, one finds the following equation satisfied by z

$$\partial_t z+(-\Delta)^s z+\varepsilon^{2s-1}|Dz|^\gamma=\varepsilon^{2s-1}f(\varepsilon x,\varepsilon^{2s}t)=:r_\varepsilon(x,t)\ .$$

If one wants to observe a (Lipschitz) regularization effect under space-time Lebesgue integrability assumptions on the right-hand side, it is straightforward to verify that the $L^q(\mathbb{R}^d \times (0,T))$ norm of $r_\varepsilon(x,t)$ is invariant under the previous scaling when $q = (d+2s)/(2s-1)$. Therefore, following the analysis of Chapter 6, one expects to obtain the Lipschitz regularization effect of the solution of the HJ equation assuming $f \in L^q$ with $q > \frac{d+2s}{2s-1}$ only (note that throughout this chapter we have $f \in L^q(H_q^{2-2s})$ with at least $q > d+2s$). To run all the arguments in Chapter 6, one needs to control $D\rho$ in some Lebesgue space and this regularity of the gradient of the solution to (7.8) is not a priori expected, and, at this stage, seems to be still unknown. However, arguing as in Section 7.3.1, setting $\mu(x,t) = \varepsilon^\alpha \rho(\varepsilon x, \varepsilon^{2s} t)$ and $v(x,t) = \varepsilon^\beta b(\varepsilon x, \varepsilon^{2s} t)$, we find that (μ, v) solves the equation

$$\varepsilon^{2s+\alpha}(\partial_t \rho + (-\Delta)^s \rho) + \varepsilon^{\alpha+\beta+1} \mathrm{div}(b(x,t)\rho) = 0 \ .$$

The equation is invariant under the scaling when $\beta = 2s-1 > 0$ since $s \in (1/2, 1)$. Therefore, we observe that

$$\iint |v|^{\gamma'} \mu = \varepsilon^{(2s-1)\gamma' + \alpha - d - 2s} \iint |b|^{\gamma'} \rho$$

and thus find the optimal exponent

$$\alpha = d + 2s - (2s-1)\gamma' \ .$$

Then, if one wants the estimate of $D\rho$ in some Lebesgue space $L^{q'}$, where q' is the conjugate of $q > 1$, we find

$$\iint |D\mu|^{q'} = \varepsilon^{(\alpha+1)q' - d - 2s} \iint |D\rho|^{q'} \ .$$

Therefore, we impose

$$(\alpha + 1)q' - d - 2s = 0 \ .$$

giving

$$q' = \frac{d+2s}{d+2s-(2s-1)\gamma'+1}$$

after plugging the previous expression for α. In particular, note that when $\gamma = 2s$ we have

$$q' = \frac{d+2s}{d+1} \ ,$$

which is the threshold for the maximal L^p-regularity for the gradient of the fractional heat equation. Anyhow, this forces q to be

$$q = \frac{d+2s}{(2s-1)\gamma' - 1} \ .$$

The additional regularity of $D\rho$ is an interesting open problem which, apparently, cannot be achieved via parabolic Caldéron-Zygmund regularity.
Anyhow, assuming our additional regularity $f \in L^q(H_q^{2-2s})$ and using similar arguments to Remark 6.33, namely exploiting the parabolic regularity of the fractional

heat kernel, we can show that our assumption $q > d+2s$ is minimal to have the Lipschitz regularization effect. To this aim, consider $H(x,p) = |p|^\gamma$, $\gamma > 1$. For $T > 0$, let $\chi \in C_0^\infty(\mathbb{R}^d)$, $\Gamma(x,t)$ be fundamental solution of the fractional heat equation in $\mathbb{R}^d$,

$$\bar{f}(x,t) := \frac{\chi\left(\frac{x}{(T-t)^{1/2s}}\right)}{(T-t)^{1/2s}\log(T-t)}$$

and $\bar{u}$ be the function

$$\bar{u}(x,t) := \iint_{\mathbb{R}^d\times(0,t)} \bar{f}(y,s)\Gamma(x-y,t-s)\,dyds \qquad \text{on } Q_T$$

Clearly, $\bar{u}$ is a classical solution to

$$\begin{cases}\partial_t u(x,t) + (-\Delta)^s u(x,t) + |Du(x,t)|^\gamma = \bar{f}(x,t) + |Du(x,t)|^\gamma \\ u(x,0) = 0,\end{cases}$$

$\bar{f} \in L^q(0,T;H_q^{2-2s}(\mathbb{T}^d))$ if and only if $q \le d+2s$ and $|D\bar{u}|^\gamma \in L^\infty(0,T;L^\beta(\mathbb{T}^d))$ for all $\beta < \infty$. In turn, we have that $\|D\bar{u}(\cdot,t)\|_\infty \to \infty$ as $t \to T$. Note that this example can be recast into the periodic setting multiplying $\bar{u}$ by a suitable cut-off function, as in Chapter 6.

Therefore, with respect to integrability requirements on f, Theorem 7.2 is optimal, at least when $\gamma \le 2s$, namely when $d+2s \ge \frac{d+2s}{(2s-1)(\gamma'-1)}$.

Appendix A

Fractional product and chain rules on the torus

We first present a version of the Kato-Ponce inequality on Bessel potential spaces on the torus. We refer the reader to the classical results in [150, 131] (and references therein) for more recent developments, all stated in the Euclidean case (see also [227, eq. (3.1.59)], [228] and [165, Proposition 2] for the periodic setting).

Lemma A.1. *Let $\mu \in (0,1)$ and $1 < p, p_1, q_1, p_2, q_2 < \infty$ and such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Then,*

$$\|fg\|_{H^\mu_p(\mathbb{T}^d)} \leq C(\|f\|_{L^{p_1}(\mathbb{T}^d)} \|g\|_{H^\mu_{q_1}(\mathbb{T}^d)} + \|f\|_{H^\mu_{p_2}(\mathbb{T}^d)} \|g\|_{L^{q_2}(\mathbb{T}^d)})$$

for some $C > 0$.

We recall that the inequality can be proven in the Euclidean case as follows, see e.g. [130]. First, a bilinear multiplier operator with symbol m acting on $f, g \in \mathcal{S}(\mathbb{R}^d)$ is defined as

$$T_m(f,g)(x) := \iint_{\mathbb{R}^{2d}} m(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i(\xi+\eta)\cdot x} d\xi d\eta \ . \tag{A.1}$$

We are interested in the symbol $|\xi+\eta|^\mu$, since

$$(-\Delta)^{\mu/2}(fg)(x) = \iint_{\mathbb{R}^{2d}} |\xi+\eta|^\mu \mathcal{F}f(\xi)\mathcal{F}g(\eta) d\xi d\eta \ .$$

Then one performs the partition $m(\xi,\eta) = \sigma_1(\xi,\eta)|\xi|^\mu + \sigma_2(\xi,\eta)|\eta|^\mu$, where

$$\sigma_1(\xi,\eta) := \frac{|\xi+\eta|^\mu}{|\xi|^\mu}\left(1-\phi\left(\frac{|\xi|}{|\eta|}\right)\right), \qquad \sigma_2(\xi,\eta) := \frac{|\xi+\eta|^\mu}{|\eta|^\mu}\phi\left(\frac{|\xi|}{|\eta|}\right)$$

and ϕ is a suitable C^∞_0 cut-off function; we are then reduced to prove the boundedness of the operators T_{σ_i} on $L^{p_i}(\mathbb{R}^d) \times L^{q_i}(\mathbb{R}^d)$. Indeed, this would yield

$$\begin{aligned}\left\|(-\Delta)^{\mu/2}(fg)\right\|_{L^p(\mathbb{R}^d)} \leq C \Big(&\left\|(-\Delta)^{\mu/2}f\right\|_{L^{p_1}(\mathbb{R}^d)} \|g\|_{L^{q_1}(\mathbb{R}^d)} \\ &+ \|f\|_{L^{p_2}(\mathbb{R}^d)} \left\|(-\Delta)^{\mu/2}g\right\|_{L^{q_2}(\mathbb{R}^d)}\Big) ,\end{aligned}$$

and the desired estimate with H^μ_p norms would follow by equivalence of $\|\cdot\|_{\mu,p}$ with $\|\cdot\|_p + \|(-\Delta)^{\frac{\mu}{2}}\cdot\|_p$. The key result for boundedness of T_{σ_i} is the Coifman-Meyer multiplier theorem (see [131, Theorem A] and references therein). Note that the assumptions of such theorem are fulfilled, since the multipliers σ_i are homogeneous of degree zero.

Proof of Lemma A.1. One may argue as in the Euclidean case. We start by observing that bilinear operators T_{σ_i} have a periodic counterpart defined on the torus, that is

$$B_{\sigma_i}(f,g)(x) := \sum_{\mu\in\mathbb{Z}^d}\sum_{\nu\in\mathbb{Z}^d} \sigma_i(\mu,\nu)\hat{f}(\mu)\hat{g}(\nu)e^{2\pi i(\mu+\nu)\cdot x} \tag{A.2}$$

By the transference results on multilinear multipliers in [110, Theorem 3], since σ_i are bilinear Coifman-Meyer multipliers on $\mathbb{R}^d\times\mathbb{R}^d$, then they are so also on $\mathbb{T}^d\times\mathbb{T}^d$. One has just to be careful since σ_i are discontinuous at $(0,0)$, but it is sufficient to have them defined in $(0,0)$ so that $(0,0)$ is a Lebesgue point for both σ_i. $\square$

We also present a chain rule for fractional Sobolev spaces.

Lemma A.2. *Let $\mu > 0$, and $\Psi : \mathbb{T}^d\times\mathbb{R}^d \to \mathbb{R}$ be of class $C^{\lceil\mu\rceil}(\mathbb{T}^d\times\mathbb{R}^d)$ with bounded derivatives on $\mathbb{T}^d\times\mathbb{R}^d$ up to order $\lceil\mu\rceil$. Let $u\in W^{\mu,p}(\mathbb{T}^d)\cap H^\mu_p(\mathbb{T}^d)$. Then*

$$\|\Psi(\cdot,u(\cdot))\|_{W^{\mu,p}(\mathbb{T}^d)} \le C(\|u\|_{W^{\mu,p}(\mathbb{T}^d)}+1)\ ,$$

and, for all $\varepsilon>0$,

$$\|\Psi(\cdot,u(\cdot))\|_{H^{\mu-\varepsilon}_p(\mathbb{T}^d)} \le C(\|u\|_{H^\mu_p(\mathbb{T}^d)}+1)\ .$$

Remark A.3. As far as the fractional composition rules are concerned, we stress out that such results continue to hold even for $\epsilon = 0$, as in the seminal paper by [86] (see also [165, Proposition 3] for the periodic setting)

Proof. We just consider the case $0<\mu<1$, the general case being treated similarly. We start with the inequality in $W^{\mu,p}$ spaces, using their construction through the trace method. It is sufficient to recall that

$$\|u\|_{W^{1-\mu,p}(\mathbb{T}^d)} = \inf_{u=f(0)} \max\{\|t^{\mu-1/p}f(t)\|_{L^p(0,\infty;W^{1,p}(\mathbb{T}^d))}; \|t^{\mu-1/p}f'(t)\|_{L^p(\mathbb{T}^d\times(0,\infty))}\},$$

and observe that

$$\|\Psi(x,f(x))\|_{W^{1,p}(\mathbb{T}^d)} \le C(1+\|f\|_{W^{1,p}(\mathbb{T}^d)}),$$

where the constant C depends on global bounds on the derivatives of Ψ. Then, one uses $\Psi(x,f(x))$ to estimate $\|\Psi(\cdot,u(\cdot))\|_{W^{1-\mu,p}(\mathbb{T}^d)}$, where f is close to the infimum in the definition of $\|u\|_{W^{1-\mu,p}(\mathbb{T}^d)}$. The analogous inequality in H^μ_p spaces is then a consequence of Lemma 5.32. $\square$

Appendix B

Regularity in parabolic fractional Hölder and Bessel spaces

We consider the problem

$$\begin{cases} \partial_t u + (-\Delta)^s u = f(x,t) & \text{in } Q_T \ , \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}^d \ . \end{cases} \tag{B.1}$$

The purpose of this section is to present a fractional analogue of classical parabolic Hölder and Sobolev regularity. We point out that related results for this problem on the Euclidean space appeared in [59, Appendix A] and [84], see also references therein. We stress that transference of these results to the periodic setting is delicate, in particular concerning regularity in Sobolev spaces, and to our knowledge they are not explicitly stated in the literature. We present some proofs that make use of interpolation methods and results for abstract parabolic equations, with some details for the reader's convenience.

As for regularity in Hölder spaces, we follow the approach of [176, Chapter 3-4] (see also [177, Chapter 5]).

Theorem B.1. *Let $\alpha \in (0,1)$ so that $2s+\alpha$ is not an integer, $f \in \mathcal{C}^{\alpha,\frac{\alpha}{2s}}(Q_T)$ and $u_0 \in C^{2s+\alpha}(\mathbb{T}^d)$. Then problem* (B.1) *has a unique classical solution u, and there exists a positive constant C depending on d, T, α, s (which remains bounded for bounded values of T) such that*

$$\|\partial_t u\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2s}}(Q_T)} + \|(-\Delta)^s u\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2s}}(Q_T)} \leq C(\|u_0\|_{C^{2s+\alpha}(\mathbb{T}^d)} + \|f\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2s}}(Q_T)}) \ . \tag{B.2}$$

We begin with some preliminary decay estimates for the fractional heat semigroup $\mathcal{T}_t$ in Hölder spaces.

Lemma B.2. *For every $0 \leq \theta_1 < \theta_2$, $\theta_1, \theta_2 \in \mathbb{R}$, there exists $C = C(\theta_1,\theta_2)$ such that for all $f \in C^{\theta_1}(\mathbb{T}^d)$*

$$\|\mathcal{T}_t f\|_{C^{\theta_2}(\mathbb{T}^d)} \leq C t^{-(\theta_2-\theta_1)/2s} \|f\|_{C^{\theta_1}(\mathbb{T}^d)} \ .$$

Proof. Computations of Remark 5.22 (in particular the representation formula for $\mathcal{T}_t$ and Young's inequality for convolution) show that for every $k > h$, $k, h \in \mathbb{N} \cup \{0\}$ there exists $C = C(h,k)$

$$\|\mathcal{T}_t f\|_{C^{k+h}(\mathbb{T}^d)} \leq C t^{-\frac{k}{2s}} \|f\|_{C^h(\mathbb{T}^d)} \ .$$

This implies that $\mathcal{T}_t f : C^h(\mathbb{T}^d) \to C^{k+h}(\mathbb{T}^d)$ is bounded for $t > 0$. Recall that, as a consequence of Theorem 5.11 one easily gets

$$(C^h(\mathbb{T}^d), C^{k+h}(\mathbb{T}^d))_{\alpha,\infty} = C^{h+\alpha}(\mathbb{T}^d)$$

(See e.g. [176, Example 1.1.7], where the proofs can be readily adapted to the periodic setting). In addition, one also has $\mathcal{T}_t f : L^\infty(\mathbb{T}^d) \to L^\infty(\mathbb{T}^d)$. By interpolation (see [177, Proposition 1.2.6]), $\mathcal{T}_t$ maps $C^{\theta_1}(\mathbb{T}^d)$ onto $C^{\theta_2}(\mathbb{T}^d)$ with the desired estimate. □

Proof of Theorem B.1. Step 1. We first prove the existence of a constant $C > 0$ such that

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{C^{2s+\alpha}(\mathbb{T}^d)} \le C\big(\sup_{t\in[0,T]} \|f(\cdot,t)\|_{C^\alpha(\mathbb{T}^d)} + \|u_0\|_{C^{2s+\alpha}(\mathbb{T}^d)}\big) \ .$$

We first observe that for $s, \alpha \in (0,1)$ such that $2s+\alpha$ is not an integer we have

$$C^{2s+\alpha}(\mathbb{T}^d) = (C^{\alpha+\delta}(\mathbb{T}^d), C^{2s+\alpha+\delta}(\mathbb{T}^d))_{1-\delta/2s,\infty} \ , \quad 0 < \delta < 2s \ .$$

We show that $u(\cdot,t)$ is bounded with values in $C^{2s+\alpha}(\mathbb{T}^d)$. Fix $t \in [0,T]$. Then, for every $\xi > 0$ we split $u(t)$ as $u(t) = a(\xi) + b(\xi) + c(\xi)$ using Duhamel's formula, that is

$$\begin{aligned}
a(\xi) &= \int_0^{\min\{\xi,t\}} \mathcal{T}_\tau f(t-\tau)(x)d\tau, \\
b(\xi) &= \int_{\min\{\xi,t\}}^t \mathcal{T}_\tau f(t-\tau)(x)d\tau, \\
c(\xi) &= \mathcal{T}_{t-\min\{\xi,t\}}\mathcal{T}_{\min\{\xi,t\}}u_0.
\end{aligned}$$

Then $a(\xi) \in C^{\alpha+\delta}(\mathbb{T}^d)$, $b(\xi), c(t) \in C^{2s+\alpha+\delta}(\mathbb{T}^d)$ for each $\delta \in (0,2s)$. Indeed,

$$\begin{aligned}
\|a(\xi)\|_{C^{\alpha+\delta}(\mathbb{T}^d)} &\le \int_0^{\min\{\xi,t\}} \frac{C}{\tau^{\delta/2s}} d\tau \sup_{\tau\in[0,T]} \|f(\tau)\|_{C^\alpha(\mathbb{T}^d)} \\
&\le \frac{C}{1-\delta/2s}\xi^{1-\delta/2s} \sup_{\tau\in[0,T]} \|f(\tau)\|_{C^\alpha(\mathbb{T}^d)} \ .
\end{aligned}$$

In addition

$$\begin{aligned}
\|b(\xi)\|_{C^{2s+\alpha+\delta}(\mathbb{T}^d)} &\le \int_{\min\{\xi,t\}}^t \frac{C}{\tau^{1+\delta/2s}} d\tau \sup_{\tau\in[0,T]} \|f(\tau)\|_{C^\alpha(\mathbb{T}^d)} \\
&\le \frac{C}{\delta/2s}\xi^{-\delta/2s} \sup_{\tau\in[0,T]} \|f(\tau)\|_{C^\alpha(\mathbb{T}^d)} \ .
\end{aligned}$$

Similarly to the above computations we have

$$\|c(\xi)\|_{C^{2s+\alpha+\delta}(\mathbb{T}^d)} \le \|\mathcal{T}_{\min\{\xi,t\}}u_0\|_{C^{2s+\alpha+\delta}(\mathbb{T}^d)} \le C\xi^{-\delta/2s}\|u_0\|_{C^{2s+\alpha}(\mathbb{T}^d)}.$$

Therefore, by the definition of K in Section 5.3.2 we have

$$\begin{aligned}\xi^{-(1-\delta/2s)}&K(\xi, u(t), C^{\alpha+\delta}(\mathbb{T}^d), C^{2s+\alpha+\delta}(\mathbb{T}^d))\\ &\leq \xi^{-(1-\delta/2s)}(\|a(\xi)\|_{C^{\alpha+\delta}(\mathbb{T}^d)} + \xi\|b(\xi)+c(\xi)\|_{C^{2s+\alpha+\delta}(\mathbb{T}^d)})\\ &\qquad\leq C(\sup_{\tau\in[0,T]}\|f(\tau)\|_{C^{\alpha}(\mathbb{T}^d)} + \|u_0\|_{C^{2s+\alpha}(\mathbb{T}^d)})\ .\end{aligned}$$

This shows in particular that $u(t) \in C^{2s+\alpha}(\mathbb{T}^d) = (C^{\alpha+\delta}(\mathbb{T}^d), C^{2s+\alpha}(\mathbb{T}^d))_{1-\delta/2s,\infty}$ and

$$\|u(t)\|_{C^{2s+\alpha}(\mathbb{T}^d)} \leq C(\|f\|_{C^{\alpha}_x(Q_T)} + \|u_0\|_{C^{2s+\alpha}(\mathbb{T}^d)})\ .$$

for all $t \in [0,T]$. Since $\partial_t u = -(-\Delta)^s u + f$ and $\|(-\Delta)^s u(t)\|_{C^{\alpha}(\mathbb{T}^d)}$ is controlled by $\|u(t)\|_{C^{2s+\alpha}(\mathbb{T}^d)}$ (see, e.g. [206, Theorem 1.4]), we obtain the bound on $\|\partial_t u\|_{C^{\alpha}(\mathbb{T}^d)} + \|(-\Delta)^s u(t)\|_{C^{\alpha}(\mathbb{T}^d)}$.

Step 2. We need to show that $\partial_t u$ and $(-\Delta)^s u$ are both $\alpha/2s$-Hölder continuous in time. In view of the regularity of f, it is sufficient to estimate the term $(-\Delta)^s u$. Our setting falls into the broader treatment for abstract parabolic equations of [220], [176, Theorem 4.0.15] and [177, Theorem 4.3.1]. Anyhow, one can proceed adapting the arguments in [176, Theorem 4.0.14] to the fractional framework, and essentially use estimates of Lemma B.2. We thus provide the proof for reader convenience to have a self-contained discussion. We use Lemma B.2 with $\theta_1 = 0$ and $\theta_2 = 0, 2s, 4s$ to obtain

$$\|\mathcal{T}_t\|_{\mathcal{L}(C(\mathbb{T}^d))} \leq C_0;\ \| - (-\Delta)^s\mathcal{T}_t\|_{\mathcal{L}(C(\mathbb{T}^d))} \leq \frac{C_1}{t};\ \|(-(-\Delta)^s)^2\mathcal{T}_t\|_{\mathcal{L}(C(\mathbb{T}^d))} \leq \frac{C_2}{t^2}$$

respectively and with $\theta_1 = \alpha, \theta_2 = 2s$ to conclude

$$\| - (-\Delta)^s\mathcal{T}_t u\|_{\mathcal{L}(C^{\alpha}(\mathbb{T}^d), C(\mathbb{T}^d))} \leq \frac{C_{1,\alpha,s}}{t^{1-\alpha/2s}}$$

Recall that such decay estimates can be also obtained exploiting the one for the Laplacian obtained in [178] and arguing via Bochner-Pollard subordination identity. We then split $u = u_1 + u_2$ as

$$u_1(\cdot,t) = \int_0^t \mathcal{T}_{t-\sigma}(f(\cdot,\sigma) - f(\cdot,t))d\sigma\ ,$$

and

$$u_2(\cdot,t) = \mathcal{T}_t u_0 + \int_0^t \mathcal{T}_{t-\sigma} f(\cdot,t)d\sigma$$

for $t \in [0,T]$. Direct computations gives

$$-(-\Delta)^s u_1(t) = \int_0^t -(-\Delta)^s\mathcal{T}_{t-\sigma}(f(\sigma) - f(t))d\sigma$$

and

$$-(-\Delta)^s u_2(t) = -(-\Delta)^s\mathcal{T}_t u_0 + (\mathcal{T}_t - 1)f(t)d\sigma\ ,$$

which in turn yields for $0 \le \tau \le t \le T$

$$
\begin{aligned}
-(-\Delta)^s u_1(t) - [-(-\Delta)^s u_1(\tau)] \le \int_0^\tau -(-\Delta)^s(\mathcal{T}_{t-\sigma} - \mathcal{T}_{\tau-\sigma})(f(\sigma) - f(\tau))d\sigma + \\
+ (\mathcal{T}_t - \mathcal{T}_{t-\tau})(f(\tau) - f(\cdot,t)) + \int_\tau^t (-\Delta)^s \mathcal{T}_{t-\sigma}(f(\sigma) - f(\tau))d\sigma
\end{aligned}
$$

Since $-(-\Delta)^s(\mathcal{T}_{t-\sigma} - \mathcal{T}_{\tau-\sigma}) = \int_{\tau-\sigma}^{t-\sigma}(-(-\Delta)^s)^2 \mathcal{T}_\omega d\omega$, we have

$$
\begin{aligned}
&|-(-\Delta)^s u_1(\cdot,t) - [-(-\Delta)^s u_1(\cdot,\tau)]| \\
&\qquad \le C_2 \int_0^\tau (\tau-\sigma)^{\frac{\alpha}{2s}} \int_{\tau-\sigma}^{t-\sigma} \omega^{-2} d\omega d\sigma \sup_{x\in\mathbb{R}^d} [f(x,\cdot)]_{C^{\frac{\alpha}{2s}}([0,T])} \\
&+ 2C_0(t-\tau)^{\frac{\alpha}{2s}} \sup_{x\in\mathbb{R}^d} [f(x,\cdot)]_{C^{\frac{\alpha}{2s}}([0,T])} + C_1 \int_\tau^t (\tau-\sigma)^{\frac{\alpha}{2s}-1} d\sigma \sup_{x\in\mathbb{R}^d} [f(x,\cdot)]_{C^{\frac{\alpha}{2s}}([0,T])} \\
&\qquad \le C_2 \int_0^\tau d\sigma \int_{\tau-\sigma}^{t-\sigma} \omega^{\frac{\alpha}{2s}-2} d\omega \sup_{x\in\mathbb{R}^d} [f(x,\cdot)]_{C^{\frac{\alpha}{2s}}([0,T])} \\
&\qquad + \left(2C_0 + \frac{2sC_1}{\alpha}\right)(t-\tau)^{\frac{\alpha}{2s}} \sup_{x\in\mathbb{R}^d} [f(x,\cdot)]_{C^{\frac{\alpha}{2s}}([0,T])} \\
&\qquad \le \frac{C_2}{(\alpha/2s)(1-\alpha/2s)} (t-\tau)^{\frac{\alpha}{2s}} \sup_{x\in\mathbb{R}^d} [f(x,\cdot)]_{C^{\frac{\alpha}{2s}}([0,T])} \\
&\qquad\qquad + \left(2C_0 + \frac{2sC_1}{\alpha}\right)(t-\tau)^{\frac{\alpha}{2s}} \sup_{x\in\mathbb{R}^d} [f(x,\cdot)]_{C^{\frac{\alpha}{2s}}([0,T])} .
\end{aligned}
$$

where we used (5.4) within the last inequality. Concerning u_2, we add and subtract $(\mathcal{T}_t - \mathcal{T}_\tau) f(\cdot,0)$ to obtain

$$
\begin{aligned}
&|-(-\Delta)^s u_2(\cdot,t) - [-(-\Delta)^s u_2](\cdot,\tau)| \le |(\mathcal{T}_t - \mathcal{T}_\tau)(-(-\Delta)^s u_0 + f(\cdot,0)| \\
&\qquad + |(\mathcal{T}_t - \mathcal{T}_\tau) f(\cdot,0)| + |(\mathcal{T}_t - I)(f(\cdot,t) - f(\cdot,\tau))| \\
&\le \int_\tau^t \|-(-\Delta)^s \mathcal{T}_\sigma\|_{\mathcal{L}(C^\alpha(\mathbb{T}^d),C(\mathbb{T}^d))} d\sigma \| -(-\Delta)^s u_0 + f(\cdot,0)\|_{C^\alpha(\mathbb{T}^d)} + \\
&\qquad + \tau^{\alpha/2s} \| \int_\tau^t -(-\Delta)^s \mathcal{T}_\sigma d\sigma \|_{\mathcal{L}(C^\alpha(\mathbb{T}^d),C(\mathbb{T}^d))} \sup_{x\in\mathbb{R}^d} [f(x,\cdot)]_{C^{\frac{\alpha}{2s}}([0,T])} \\
&\qquad + (C_0+1)(t-\tau)^{\frac{\alpha}{2s}} \sup_{x\in\mathbb{R}^d} [f(x,\cdot)]_{C^{\frac{\alpha}{2s}}([0,T])} \\
&\le \frac{2sC_{1,\alpha,s}}{\alpha} \|(-\Delta)^s u_0 + f(\cdot,0)\|_{C^\alpha(\mathbb{T}^d)} (t-\tau)^{\frac{\alpha}{2s}} + (C_0+1)(t-\tau)^{\frac{\alpha}{2s}} \sup_{x\in\mathbb{R}^d} [f(x,\cdot)]_{C^{\frac{\alpha}{2s}}([0,T])} .
\end{aligned}
$$

showing that that $-(-\Delta)^s$ is $\alpha/2s$-Hölder continuous in time with the following estimate in force

$$
\sup_{x\in\mathbb{T}^d} [\partial_t u(\cdot,t)]_{C^{\frac{\alpha}{2s}}([0,T])} + \sup_{x\in\mathbb{R}^d} [-(-\Delta)^s u(x,\cdot)]_{C^{\frac{\alpha}{2s}}([0,T])} \le C(\|f\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2s}}(Q_T)} + \|u_0\|_{C^{2s+\alpha}(\mathbb{T}^d)})
$$

We also observe that the first partial derivative $\partial_i u$ are $\frac{2s-1+\alpha}{2s}$-time Hölder continuous noting that $C^1(\mathbb{T}^d)$ belongs to the class $J_{1/2s-\alpha/2s}$ between $C^\alpha(\mathbb{T}^d)$ and

$C^{2s+\alpha}(\mathbb{T}^d)$ (more generally, it holds that for $k \in \mathbb{N}$ and $\alpha < k < \beta$, the space $C^k(\mathbb{T}^d)$ belongs to the class $J_{(k-\alpha)/(\beta-\alpha)}$ between $C^\alpha(\mathbb{T}^d)$ and $C^\beta(\mathbb{T}^d)$, see [177, Proposition 1.1.3]). Then, using that $\|\partial_t u\|_{C^\alpha(\mathbb{T}^d)}$ is bounded in $[0,T]$, then the map $t \longmapsto u(\cdot,t)$ is Lipschitz continuous with values in $C^\alpha(\mathbb{T}^d)$ and Lipschitz constant $\sup_{\sigma\in[0,T]} \|\partial_t u(\cdot,\sigma)\|_{C^\alpha(\mathbb{T}^d)}$.

$$
\begin{aligned}
\|u(\cdot,t)-u(\cdot,\tau)\|_{C^1(\mathbb{T}^d)} &\le C\|u(\cdot,t)-u(\cdot,\tau)\|_{C^\alpha(\mathbb{T}^d)}^{1-\frac{1}{2s}+\frac{\alpha}{2s}}\|u(\cdot,t)-u(\cdot,\tau)\|_{C^{2s+\alpha}(\mathbb{T}^d)}^{\frac{1}{2s}-\frac{\alpha}{2s}}\\
&\le C((t-\tau)\sup_{\sigma\in[0,T]}\|\partial_t u(\cdot,\sigma)\|_{C^\alpha(\mathbb{T}^d)})^{1-\frac{1}{2s}+\frac{\alpha}{2s}}(2\sup_{\sigma\in[0,T]}\|u(\cdot,\sigma)\|_{C^{2s+\alpha}(\mathbb{T}^d)})^{\frac{1}{2s}-\frac{\alpha}{2s}}\\
&\le C'(t-\tau)^{1-\frac{1}{2s}+\frac{\alpha}{2s}}
\end{aligned}
$$

□

Remark B.3. This result is actually consequence of the optimal regularity results in Hölder spaces appeared in [220] (see also [177, Theorem 1 and Theorem 2]).

We now turn to the case of strong solutions. Recall that (B.1) can be written as an abstract evolution equation with diffusion semigroup $A = (-\Delta)^s$ as

$$
u'(t) + Au(t) = f(t)\ , t \in [0,T]\ . \tag{B.3}
$$

and $f \in L^p(Q_T)$. Concerning strong solutions, we have the following maximal regularity result in L^p classes.

Theorem B.4. *Let $p > 1$. Suppose that $u \in \mathcal{H}_p^\mu(Q_T)$ solves* (B.1)*. Then there exists a unique strong solution to* (B.1) *and there exists a constant $C > 0$, that depends on d, T, p, s (but remains bounded for bounded values of T) such that*

$$
\|u\|_{\mathcal{H}_p^\mu(Q_T)} \le C(\|f\|_{\mathbb{H}_p^{\mu-2s}(Q_T)} + \|u_0\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}).
$$

Proof. Recall that $(-\Delta)^s$ generates the analytic semigroup $\mathcal{T}_t$ on $L^p(\mathbb{T}^d)$ in view of Remark 5.24. Without loss of generality we can restrict ourselves to consider the case $\mu = 2s$, the general case being consequence of the isometry property of the operator $(I-\Delta)^{\frac{\mu}{2}}$. This observation allows to apply the abstract regularity result [160, Theorem 1] (see also the more recent works [134, Section 3.2.E] and [198], both in the framework of evolution problems in Banach spaces and [84, Theorem 1.4] for the stochastic counterpart of these spaces). Note in particular that the initial trace belongs to the real interpolation space $(H_p^{2s}(\mathbb{T}^d), L^p(\mathbb{T}^d))_{1/p,p} \simeq W^{2s-2s/p,p}(\mathbb{T}^d) \simeq B_{pp}^{2s-2s/p,p}(\mathbb{T}^d)$. However, if one works in the larger space of initial traces $H_p^{\mu-2s/p+\varepsilon}$, the estimate of the term involving the initial datum is simpler than the one involving fractional Sobolev (or equivalently Besov) spaces (cf [84, Lemma 3.2]). Indeed, the estimate

$$
\|\mathcal{T}_t u_0\|_{\mathcal{H}_p^\mu(Q_T)} \le C\|u_0\|_{\mu-2s/p+\varepsilon,p}
$$

can be directly managed using decay estimates of $\mathcal{T}_t$ (see e.g. [156]). We assume without loss of generality $\varepsilon < \frac{2s}{p}$. By Lemma 5.23-(i) we have

$$
\|u_1(t)\|_{\mu,p} = \|\mathcal{T}_t u_0\|_{\mu,p} \le Ct^{-\frac{1}{p}+\frac{\epsilon}{2s}}\|u_0\|_{\mu-2s/p+\epsilon,p}\ .
$$

Note that C here does not depend on T. Integrating between 0 and T we have

$$\|u_1\|^p_{\mathbb{H}^\mu_p(Q_T)} = \int_0^T \|u_1(\cdot,t)\|^p_{\mu,p}\,dt \le CT^{\frac{p\epsilon}{2s}}\,\|u_0\|^p_{\mu-2s/p+\epsilon,p}$$

Since u_1 solves $\partial_t u_1 + (-\Delta)^s u_1 = 0$ we get

$$\begin{aligned}\|\partial_t u_1\|^p_{\mathbb{H}^{\mu-2s}_p(Q_T)} &= \int_0^T \|\partial_t u_1(\cdot,t)\|^p_{\mu-2s,p} = \int_0^T \|(-\Delta)^s u_1(\cdot,t)\|^p_{\mu-2s,p}\,dt \\ &\le C\int_0^T \|(I-\Delta)^s u_1(\cdot,t)\|^p_{\mu-2s,p}\,dt = C\int_0^T \|u_1(\cdot,t)\|^p_{\mu,p}\,dt\ ,\end{aligned}$$

that allows to conclude. □

Remark B.5. As pointed out in [160, 134, 202], the result holds even for f belonging to spaces with different order of summability in space and time. More precisely, let $p,q>1$ and suppose that $u\in\mathcal{H}^{\mu,q}_p(Q_T)$ solves (B.1). Then there exists $C>0$, that depends on d,T,p,q,s (but remains bounded for bounded values of T) such that

$$\|u\|_{\mathcal{H}^{\mu,q}_p(Q_T)} \le C(\|f\|_{\mathbb{H}^{\mu-2s,q}_p(Q_T))} + \|u_0\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}).$$

where

$$\mathcal{H}^{\mu,q}_p(Q_T) = \{u\in L^q(0,T;H^\mu_p(\mathbb{T}^d));\partial_t u\in L^q(0,T;H^{\mu-2s}_p(\mathbb{T}^d))\}\ .$$

and $\mathbb{H}^{\mu,q}_p(Q_T) = L^q(0,T;H^\mu_p(\mathbb{T}^d))$

Appendix C

Some other facts on embedding theorems for (parabolic) Sobolev spaces

The purpose of this section is to first complete the results presented in [155, 157] for the deterministic case and $s = 1$, where the critical embedding onto Lebesgue classes is not discussed. Recall that for $s = 1$ the space $\mathcal{H}_p^{2k}$ is isomorphic to $W_p^{2k,k}$, $k \in \mathbb{N}$. The results we present below provide a different proof of the critical embedding compared to those existing in literature, which are well-established at least when $\mu = 2k$, $k \in \mathbb{N}$, namely for the spaces $W_p^{2k,k}(Q_T)$ (see e.g. [159, Lemma II.3.3] and references therein, [14, Theorem 1.7], [129, Theorem 5.1]). As for the space $\mathcal{H}_p^1(Q_T)$, which is the natural one for equations with divergence type terms as analyzed throughout Chapter 6, a careful analysis seems to be available only within [185, Appendix A] (see also [92] for the periodic setting), which however does not cover the trace on the hyperplane $t = 0$ since the estimates are local in time. Aim of Proposition C.1 is to give the embedding onto Lebesgue classes in the local case $s = 1$ (cf [159, Lemma II.3.3]). For the sake of simplicity we only sketch the result in the case $\mu = 2$, the procedure being similar for the general case exploiting the isometry properties described in Remark 5.14. Then, Proposition C.3 covers the case of the fractional Bessel potential spaces introduced in Part II. We stress out that such scheme easily extends to the whole space case $\mathbb{R}^d \times (0, T)$, for which the result is not written down anywhere in the literature to our knowledge.

Proposition C.1. *Let $1 < p < \frac{d+2}{2}$. Then $\mathcal{H}_p^2(Q_T) \simeq W_p^{2,1}(Q_T)$, is continuously embedded onto $L^{q^*}(Q_T)$, where $\frac{1}{q^*} = \frac{1}{p} - \frac{2}{d+2}$ and*

$$\|u\|_{L^{q^*}(Q_T)} \leq C(\|u\|_{\mathcal{H}_p^2(Q_T)} + \|u(0)\|_{W^{2-2/p,p}(\mathbb{T}^d)}) .$$

Proof. Let $\nu = \nu(\beta) = (2 - 2/p)(1 - \theta) + 2\theta$. We now use the interpolation in the Sobolev-Slobodeckij scale to observe that $W^{\nu,p}$ can be obtained by interpolation between $W^{2,p}$ and $W^{2-2/p,p}$ (see [231, Theorem 2.4.2 p.186 and eq. (16)]. We recall that for $W_p^{2,1}$ the sharp space of initial trace is $W^{2-2/p,p}$ (see e.g. [178, Corollary 1.14]). Moreover, $W^{\nu,p}$ is continuously embedded in $W^{\nu+d/q-d/p,q}$ in view of Lemma 5.17. Hence, for a.e. t,

$$c(d, p, s, \beta)\, \|u(t)\|_{W^{\nu-\frac{d}{p}+\frac{d}{q},q}(\mathbb{T}^d)} \leq \|u(t)\|_{W^{\nu,p}(\mathbb{T}^d)} \leq \|u(t)\|^{1-\theta}_{W^{2-2/p,p}(\mathbb{T}^d)}\, \|u(t)\|^{\theta}_{W^{2,p}(\mathbb{T}^d)} .$$

Then, for all $\eta \le \nu - \frac{d}{p} + \frac{d}{q} = \mu + \frac{d}{q} - \frac{d+2(1-\theta)}{p}$ we have

$$\left(\int_0^T \|u(t)\|_{\eta,q}^{\frac{p}{\theta}} \, dt \right)^{\theta} \le C \left(\int_0^T \|u(t)\|_{W^{2-2/p,p}(\mathbb{T}^d)}^{(1-\theta)\frac{p}{\theta}} \|u(t)\|_{W^{2,p}(\mathbb{T}^d)}^{p} \, dt \right)^{\theta}$$

$$\le C \sup_{t \le T} \|u(t)\|_{W^{2-2/p,p}(\mathbb{T}^d)}^{(1-\theta)p} \left(\int_0^T \|u(t)\|_{W^{2,p}(\mathbb{T}^d)}^{p} \, dt \right)^{\theta}$$

At this stage, one has to use the embedding in [178, Corollary 1.14] to get

$$\mathcal{H}_p^2(Q_T) \hookrightarrow C([0,T]; W^{2-2/p,p}(\mathbb{T}^d))$$

and finally conclude the assertion suing Young's inequality and recalling that $W^{2,p}(\mathbb{T}^d)$ is isomorphic to $H_p^2(\mathbb{T}^d)$. □

Remark C.2. Similarly, when $\mu = 1$ one can prove the embedding of $\mathcal{H}_2^1$ onto L^{q^*} for the range $1/q^* = 1/2 - 1/(d+2)$ (see e.g. [185, Appendix A], [92]). In addition, by a suitable choice of $\theta \in (0,1)$ the same arguments yield the embedding of the space

$$\mathcal{H}_2^1(Q_T) \hookrightarrow L^{q_2}(0,T; L^{q_1}(\mathbb{T}^d))$$

when

$$\frac{1}{2} = \frac{d}{2}\left(\frac{1}{2} - \frac{1}{q_1}\right) + \frac{1}{2} - \frac{1}{q_2}$$

which is consistent with the embeddings found in [129, Theorem 5.1] for $W^{2k,k}$, $k \in \mathbb{N}$.

We now show the critical embedding for the spaces associated to the fractional heat operator, i.e. $\mathcal{H}_p^{\mu,s}$, noting a different behavior depending on the range of p.

We recall that thorughout this manuscript we also provided a proof of the critical embedding in Proposition 6.11 and Proposition 7.3 without using interpolation theory and exploiting duality arguments.

Proposition C.3. *Let $1 < p < \frac{d+2s}{\mu}$, $\mu > 2s/p$, $\mu \in \mathbb{R}$. Then $\mathcal{H}_p^{\mu;s}(Q_T)$ is continuously embedded onto $L^{q^*}(Q_T)$, where $\frac{1}{q^*} = \frac{1}{p} - \frac{\mu}{d+2s}$ and*

$$\|u\|_{L^{q^*}(Q_T)} \le C(\|u\|_{\mathcal{H}_p^{\mu;s}(Q_T)} + \|u(0)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}) \; .$$

Proof. We drop the superscript s and write $\mathcal{H}_p^{\mu;s}(Q_T) = \mathcal{H}_p^{\mu}(Q_T)$ to simplify the notation. Here, we distinguish the cases $1 < p \le 2$ and $2 < p < \infty$ in view of the inclusions stated in Lemma 5.18. To prove the first case $1 < p \le 2$, we note that for any $\theta \in (0,1)$, if $\nu = \nu(\theta) = (\mu - 2s/p)(1-\theta) + \mu\theta$, then H_p^{ν} can be obtained by interpolation between H_p^{μ} and $H_p^{\mu-2s/p}$ (see, e.g., [36, Theorem 6.4.5]). Moreover, H_p^{ν} is continuously embedded in $H_q^{\nu+d/q-d/p}$ in view of Lemma 5.16. Hence, for a.e. t,

$$c(d,p,s,\beta)\, \|u(t)\|_{\nu-\frac{d}{p}+\frac{d}{q},q} \le \|u(t)\|_{\nu,p} \le \|u(t)\|_{\mu-2s/p,p}^{1-\theta} \|u(t)\|_{\mu,p}^{\theta} \, .$$

Therefore, for all $\eta \le \nu - \frac{d}{p} + \frac{d}{q} = \mu + \frac{d}{q} - \frac{d+2s(1-\theta)}{p}$,

$$\left(\int_0^T \|u(t)\|_{\eta,q}^{\frac{p}{\theta}}\, dt\right)^{\theta} \le C_1 \left(\int_0^T \|u(t)\|_{\mu-2s/p,p}^{(1-\theta)\frac{p}{\theta}} \|u(t)\|_{\mu,p}^{p}\, dt\right)^{\theta}$$
$$\le C_2 \left(\int_0^T \|u(t)\|_{W_p^{\mu-2s/p,p}(\mathbb{T}^d)}^{(1-\theta)\frac{p}{\theta}} \|u(t)\|_{\mu,p}^{p}\, dt\right)^{\theta}$$

where we use that for $1 < p \le 2$, $W^{\mu-2s/p,p}$ is embedded onto $H_p^{\mu-2s/p}$ (cf Lemma 5.18-(i)). Then, the last inequality is less than or equal to

$$C \sup_{t\le T} \|u(t)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}^{(1-\theta)p} \left(\int_0^T \|u(t)\|_{\mu,p}^{p}\, dt\right)^{\theta}$$
$$\le C(\|u\|_{\mathcal{H}_p^{\mu}(Q_T)} + \|u(0)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)})^{(1-\theta)p} \|u(t)\|_{\mathbb{H}_p^{\mu}(Q_T)}^{\theta p}$$
$$\le C(\|u\|_{\mathcal{H}_p^{\mu}(Q_T)} + \|u(0)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)})^{p}$$

where, in the second inequality we used again the embedding

$$\mathcal{H}_p^{\mu}(Q_T) \hookrightarrow C([0,T]; W^{\mu-2s/p,p}(\mathbb{T}^d))$$

(see [4, Theorem III.4.10.2] and [199]), while, in the last one, Young's inequality.
As for the case $2 < p < \infty$, we need to restrict ourselves to the case $\eta \in \mathbb{Z}$ in order to exploit that $H_q^{\eta} \simeq W^{\eta,q}$ and interpolate in the Sobolev-Slobodeckij scale. In particular, one uses that $W^{\nu,p}$ can be obtained by interpolation between $W^{\mu,p}$ and $W^{\mu-2s/p,p}$. Moreover, $W^{\nu,p}$ is continuously embedded in $W^{\nu+d/q-d/p,q}$ in view of Lemma 5.17-(iii). Hence, for a.e. t,

$$c(d,p,s,\beta)\, \|u(t)\|_{W^{\nu-\frac{d}{p}+\frac{d}{q},q}(\mathbb{T}^d)} \le \|u(t)\|_{W^{\nu,p}(\mathbb{T}^d)} \le \|u(t)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}^{1-\theta} \|u(t)\|_{W^{\mu,p}(\mathbb{T}^d)}^{\theta}.$$

Then, for all $\eta \in \mathbb{Z}$ such that $\eta \le \nu - \frac{d}{p} + \frac{d}{q} \le \mu + \frac{d}{q} - \frac{d+2s(1-\theta)}{p}$ we have

$$\left(\int_0^T \|u(t)\|_{\eta,q}^{\frac{p}{\theta}}\, dt\right)^{\theta} = \left(\int_0^T \|u(t)\|_{W^{\eta,q}(\mathbb{T}^d)}^{\frac{p}{\theta}}\, dt\right)^{\theta} \le C_1 \left(\int_0^T \|u(t)\|_{W^{\nu-\frac{d}{p}+\frac{d}{q},q}(\mathbb{T}^d)}^{\frac{p}{\theta}}\, dt\right)^{\theta}$$
$$\le C_2 \left(\int_0^T \|u(t)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}^{(1-\theta)p} \|u(t)\|_{W^{\mu,p}(\mathbb{T}^d)}^{p}\, dt\right)^{\theta}$$
$$\le C_3 \sup_{t\le T} \|u(t)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}^{(1-\theta)p} \left(\int_0^T \|u(t)\|_{\mu,p}^{p}\, dt\right)^{\theta}$$

where we used that H_p^{μ} is embedded onto $W^{\mu,p}$ when $p > 2$ (see Lemma 5.18). At this stage, one has to use the maximal regularity result (see [4, Theorem III.4.10.2]) to get

$$\mathcal{H}_p^{\mu}(Q_T) \hookrightarrow C([0,T]; W^{\mu-2s/p,p}(\mathbb{T}^d))$$

and finally conclude the assertion setting $\eta = 0$ to get

$$\left(\int_0^T \|u(t)\|_q^q\, dt\right)^{\frac{p}{q}} \le C(\|u\|_{\mathcal{H}_p^{\mu}(Q_T)} + \|u(0)\|_{W^{\mu-2s/p,p}(\mathbb{T}^d)}^{p})$$

□

Remark C.4. By a suitable choice of θ one can even obtain the critical parabolic embeddings of $\mathcal{H}_p^{\mu;s}$ onto $L^{\frac{p}{\theta}}(0,T;L^q(\mathbb{T}^d))$ under the fractional counterpart of the conditions listed in Remark C.2.

Bibliography

[1] Y. Achdou, G. Barles, H. Ishii, and G. L. Litvinov. *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, volume 2074 of *Lecture Notes in Mathematics*. Springer, Heidelberg; Fondazione C.I.M.E., Florence, 2013. Lecture Notes from the CIME Summer School held in Cetraro, August 29–September 3, 2011, Edited by Paola Loreti and Nicoletta Anna Tchou, Fondazione CIME/CIME Foundation Subseries.

[2] T. Adamowicz and B. Warhurst. Three-spheres theorems for subelliptic quasilinear equations in Carnot groups of Heisenberg-type. *Proc. Amer. Math. Soc.*, 144(10):4291–4302, 2016.

[3] D. R. Adams and L. I. Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996.

[4] H. Amann. *Linear and quasilinear parabolic problems. Vol. I*, volume 89 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1995. Abstract linear theory.

[5] D. Applebaum. Lévy processes—from probability to finance and quantum groups. *Notices Amer. Math. Soc.*, 51(11):1336–1347, 2004.

[6] D. Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.

[7] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2011.

[8] S. N. Armstrong and B. Sirakov. Sharp Liouville results for fully nonlinear equations with power-growth nonlinearities. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 10(3):711–728, 2011.

[9] S. N. Armstrong, B. Sirakov, and C. K. Smart. Fundamental solutions of homogeneous fully nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 64(6):737–777, 2011.

[10] S.N. Armstrong and H. V. Tran. Viscosity solutions of general viscous Hamilton-Jacobi equations. *Math. Ann.*, 361(3-4):647–687, 2015.

[11] D. G. Aronson and J. Serrin. Local behavior of solutions of quasilinear parabolic equations. *Arch. Rational Mech. Anal.*, 25:81–122, 1967.

[12] N. Aronszajn and K. T. Smith. Theory of Bessel potentials. I. *Ann. Inst. Fourier (Grenoble)*, 11:385–475, 1961.

[13] E. Astesiano. Operatori parabolici estremanti. *Boll. Un. Mat. Ital. (4)*, 1:403–421, 1968.

[14] R. J. Bagby. Lebesgue spaces of parabolic potentials. *Illinois J. Math.*, 15:610–634, 1971.

[15] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.* Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1997.

[16] M. Bardi and A. Cesaroni. Liouville properties and critical value of fully nonlinear elliptic operators. *J. Differential Equations*, 261(7):3775–3799, 2016.

[17] M. Bardi, A. Cesaroni, and L. Manca. Convergence by viscosity methods in multiscale financial models with stochastic volatility. *SIAM J. Financial Math.*, 1(1):230–265, 2010.

[18] M. Bardi, A. Cesaroni, and E. Topp. Cauchy problem and periodic homogenization of nonlocal Hamilton-Jacobi equations with coercive gradient terms. arXiv:1806.05930, to appear on Proc. Roy. Soc. Edinburgh Sect. A, 2019.

[19] M. Bardi and M. Cirant. Uniqueness of solutions in mean field games with several populations and Neumann conditions. In *PDE models for multi-agent phenomena*, volume 28 of *Springer INdAM Ser.*, pages 1–20. Springer, Cham, 2018.

[20] M. Bardi, M. G. Crandall, L. C. Evans, H. M. Soner, and P. E. Souganidis. *Viscosity solutions and applications*, volume 1660 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin; Centro Internazionale Matematico Estivo (C.I.M.E.), Florence, 1997. Lectures given at the 2nd C.I.M.E. Session held in Montecatini Terme, June 12–20, 1995, Edited by I. Capuzzo Dolcetta and P. L. Lions, Fondazione CIME/CIME Foundation Subseries.

[21] M. Bardi and F. Da Lio. On the strong maximum principle for fully nonlinear degenerate elliptic equations. *Arch. Math. (Basel)*, 73(4):276–285, 1999.

[22] M. Bardi and F. Da Lio. Propagation of maxima and strong maximum principle for viscosity solutions of degenerate elliptic equations. I. Convex operators. *Nonlinear Anal.*, 44(8, Ser. A: Theory Methods):991–1006, 2001.

[23] M. Bardi and F. Da Lio. Propagation of maxima and strong maximum principle for viscosity solutions of degenerate elliptic equations. II. Concave operators. *Indiana Univ. Math. J.*, 52(3):607–627, 2003.

[24] M. Bardi and M. Fischer. On non-uniqueness and uniqueness of solutions in finite-horizon Mean Field Games. *ESAIM Control Optim. Calc. Var.*, 25:Art. 44, 33, 2019.

[25] M. Bardi and P. Mannucci. On the Dirichlet problem for non-totally degenerate fully nonlinear elliptic equations. *Commun. Pure Appl. Anal.*, 5(4):709–731, 2006.

[26] M. Bardi and P. Mannucci. Comparison principles and Dirichlet problem for fully nonlinear degenerate equations of Monge-Ampère type. *Forum Math.*, 25(6):1291–1330, 2013.

[27] M. Bardi and B. Perthame. Uniform estimates for some degenerating quasilinear elliptic equations and a bound on the Harnack constant for linear equations. *Asymptotic Anal.*, 4(1):1–16, 1991.

[28] G. Barles. Interior gradient bounds for the mean curvature equation by viscosity solutions methods. *Differential Integral Equations*, 4(2):263–275, 1991.

[29] G. Barles and J. Busca. Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term. *Comm. Partial Differential Equations*, 26(11-12):2323–2337, 2001.

[30] G. Barles, E. Chasseigne, A. Ciomaga, and C. Imbert. Lipschitz regularity of solutions for mixed integro-differential equations. *J. Differential Equations*, 252(11):6012–6060, 2012.

[31] G. Barles, O. Ley, and E. Topp. Lipschitz regularity for integro-differential equations with coercive Hamiltonians and application to large time behavior. *Nonlinearity*, 30(2):703–734, 2017.

[32] G. Barles and P. E. Souganidis. On the large time behavior of solutions of Hamilton-Jacobi equations. *SIAM J. Math. Anal.*, 31(4):925–939, 2000.

[33] J.-P. Bartier and P. Laurençot. Gradient estimates for a degenerate parabolic equation with gradient absorption and applications. *J. Funct. Anal.*, 254(3):851–878, 2008.

[34] F. H. Beatrous, T. J. Bieske, and J. J. Manfredi. The maximum principle for vector fields. In *The p-harmonic equation and recent advances in analysis*, volume 370 of *Contemp. Math.*, pages 1–9. Amer. Math. Soc., Providence, RI, 2005.

[35] M. Ben-Artzi, P. Souplet, and F. B. Weissler. The local theory for viscous Hamilton-Jacobi equations in Lebesgue spaces. *J. Math. Pures Appl. (9)*, 81(4):343–378, 2002.

[36] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[37] S. Bernstein. Sur la généralisation du problème de Dirichlet. *Math. Ann.*, 69(1):82–136, 1910.

[38] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[39] S. Bianchini, M. Colombo, G. Crippa, and L. V. Spinolo. Optimality of integrability estimates for advection-diffusion equations. *NoDEA Nonlinear Differential Equations Appl.*, 24(4):Art. 33, 19, 2017.

[40] T. Bieske. On ∞-harmonic functions on the Heisenberg group. *Comm. Partial Differential Equations*, 27(3-4):727–761, 2002.

[41] T. Bieske. A sub-Riemannian maximum principle and its application to the p-Laplacian in Carnot groups. *Ann. Acad. Sci. Fenn. Math.*, 37(1):119–134, 2012.

[42] T. Bieske and L. Capogna. The Aronsson-Euler equation for absolutely minimizing Lipschitz extensions with respect to Carnot-Carathéodory metrics. *Trans. Amer. Math. Soc.*, 357(2):795–823, 2005.

[43] T. Bieske and E. Martin. The parabolic infinite-Laplace equation in Carnot groups. *Michigan Math. J.*, 65(3):489–509, 2016.

[44] I. Birindelli. Superharmonic functions in the Heisenberg group: estimates and Liouville theorems. *NoDEA Nonlinear Differential Equations Appl.*, 10(2):171–185, 2003.

[45] I. Birindelli, I. Capuzzo Dolcetta, and A. Cutrì. Liouville theorems for semilinear equations on the Heisenberg group. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 14(3):295–308, 1997.

[46] I. Birindelli, G. Galise, and F. Leoni. Liouville theorems for a family of very degenerate elliptic nonlinear operators. *Nonlinear Anal.*, 161:198–211, 2017.

[47] L. Boccardo and G. Croce. *Elliptic partial differential equations*, volume 55 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, 2014. Existence and regularity of distributional solutions.

[48] L. Boccardo, L. Orsina, and A. Porretta. Some noncoercive parabolic equations with lower order terms in divergence form. *J. Evol. Equ.*, 3(3):407–418, 2003.

[49] V. I. Bogachev, G. Da Prato, and M. Röckner. Uniqueness for solutions of Fokker-Planck equations on infinite dimensional spaces. *Comm. Partial Differential Equations*, 36(6):925–939, 2011.

[50] V. I. Bogachev, N. V. Krylov, M. Röckner, and S. V. Shaposhnikov. *Fokker-Planck-Kolmogorov equations*, volume 207 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[51] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[52] J.-M. Bony. Principe du maximum, inégalite de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés. *Ann. Inst. Fourier (Grenoble)*, 19(fasc. 1):277–304 xii, 1969.

[53] P. Bousquet and L. Brasco. C^1 regularity of orthotropic p-harmonic functions in the plane. *Anal. PDE*, 11(4):813–854, 2018.

[54] M. Bramanti. *An invitation to hypoelliptic operators and Hörmander's vector fields*. SpringerBriefs in Mathematics. Springer, Cham, 2014.

[55] R. W. Brockett. Control theory and singular Riemannian geometry. In *New directions in applied mathematics (Cleveland, Ohio, 1980)*, pages 11–27. Springer, New York-Berlin, 1982.

[56] C. Bucur and E. Valdinoci. *Nonlocal diffusion and applications*, volume 20 of *Lecture Notes of the Unione Matematica Italiana*. Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016.

[57] X. Cabré. Elliptic PDE's in probability and geometry: symmetry and regularity of solutions. *Discrete Contin. Dyn. Syst.*, 20(3):425–457, 2008.

[58] X. Cabré and L. A. Caffarelli. Interior $C^{2,\alpha}$ regularity theory for a class of nonconvex fully nonlinear elliptic equations. *J. Math. Pures Appl. (9)*, 82(5):573–612, 2003.

[59] L. Caffarelli and A. Figalli. Regularity of solutions to the parabolic fractional obstacle problem. *J. Reine Angew. Math.*, 680:191–233, 2013.

[60] L. Caffarelli, Y. Li, and L. Nirenberg. Some remarks on singular solutions of nonlinear elliptic equations III: viscosity solutions including parabolic operators. *Comm. Pure Appl. Math.*, 66(1):109–143, 2013.

[61] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[62] L. A. Caffarelli and X. Cabré. *Fully nonlinear elliptic equations*, volume 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.

[63] L. A. Caffarelli and A. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. of Math. (2)*, 171(3):1903–1930, 2010.

[64] F. Cagnetti, D. Gomes, H. Mitake, and H. V. Tran. A new method for large time behavior of degenerate viscous Hamilton-Jacobi equations with convex Hamiltonians. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(1):183–200, 2015.

[65] F. Camilli and R. De Maio. A time-fractional mean field game. *Adv. Differential Equations*, 24(9-10):531–554, 2019.

[66] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, volume 58 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2004.

[67] L. Capogna and X. Zhou. Strong comparison principle for p-harmonic functions in Carnot-Caratheodory spaces. *Proc. Amer. Math. Soc.*, 146(10):4265–4274, 2018.

[68] I. Capuzzo Dolcetta. Liouville theorems and a priori estimates for elliptic semilinear equations. *Rend. Sem. Mat. Fis. Milano*, 68:1–18 (2001), 1998.

[69] I. Capuzzo Dolcetta and A. Cutrì. On the Liouville property for sub-Laplacians. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(1-2):239–256 (1998), 1997. Dedicated to Ennio De Giorgi.

[70] I. Capuzzo Dolcetta and A. Cutrì. Hadamard and Liouville type results for fully nonlinear partial differential inequalities. *Commun. Contemp. Math.*, 5(3):435–448, 2003.

[71] P. Cardaliaguet. Notes on Mean Field Games. Technical report, 2010.

[72] P Cardaliaguet and P. J. Graber. Mean field games systems of first order. *ESAIM Control Optim. Calc. Var.*, 21(3):690–722, 2015.

[73] P. Cardaliaguet, P. J. Graber, A. Porretta, and D. Tonon. Second order mean field games with degenerate diffusion and local coupling. *NoDEA Nonlinear Differential Equations Appl.*, 22(5):1287–1317, 2015.

[74] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, and A. Porretta. Long time average of mean field games with a nonlocal coupling. *SIAM J. Control Optim.*, 51(5):3558–3591, 2013.

[75] P. Cardaliaguet, A. Porretta, and D. Tonon. Sobolev regularity for the first order Hamilton-Jacobi equation. *Calc. Var. Partial Differential Equations*, 54(3):3037–3065, 2015.

[76] P. Cardaliaguet and C. Rainer. Hölder regularity for viscosity solutions of fully nonlinear, local or nonlocal, Hamilton-Jacobi equations with superquadratic growth in the gradient. *SIAM J. Control Optim.*, 49(2):555–573, 2011.

[77] P. Cardaliaguet and L. Silvestre. Hölder continuity to Hamilton-Jacobi equations with superquadratic growth in the gradient and unbounded right-hand side. *Comm. Partial Differential Equations*, 37(9):1668–1688, 2012.

[78] Pierre Cardaliaguet, François Delarue, Jean-Michel Lasry, and Pierre-Louis Lions. *The master equation and the convergence problem in mean field games*, volume 201 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2019.

[79] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. I*, volume 83 of *Probability Theory and Stochastic Modelling.* Springer, Cham, 2018. Mean field FBSDEs, control, and games.

[80] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. II*, volume 84 of *Probability Theory and Stochastic Modelling.* Springer, Cham, 2018. Mean field games with common noise and master equations.

[81] A. Cesaroni, M. Cirant, S. Dipierro, M. Novaga, and E. Valdinoci. On stationary fractional mean field games. *J. Math. Pures Appl. (9)*, 122:1–22, 2019.

[82] C. H. Chan and A. Vasseur. De Giorgi techniques applied to the Hölder regularity of solutions to Hamilton-Jacobi equations. In *From particle systems to partial differential equations*, volume 209 of *Springer Proc. Math. Stat.*, pages 117–137. Springer, Cham, 2017.

[83] P. Chan and R. Sircar. Fracking, renewables, and mean field games. *SIAM Rev.*, 59(3):588–615, 2017.

[84] T. Chang and K. Lee. On a stochastic partial differential equation with a fractional Laplacian operator. *Stochastic Process. Appl.*, 122(9):3288–3311, 2012.

[85] H. Chen and P. Felmer. On Liouville type theorems for fully nonlinear elliptic equations with gradient term. *J. Differential Equations*, 255(8):2167–2195, 2013.

[86] F. M. Christ and M. I. Weinstein. Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. *J. Funct. Anal.*, 100(1):87–109, 1991.

[87] A. Ciomaga. On the strong maximum principle for second-order nonlinear parabolic integro-differential equations. *Adv. Differential Equations*, 17(7-8):635–671, 2012.

[88] M. Cirant. *Nonlinear PDEs in ergodic control, Mean Field Games and prescribed curvature problems.* PhD thesis, Università di Padova, 2013.

[89] M. Cirant. On the solvability of some ergodic control problems in $\mathbb{R}^d$. *SIAM J. Control Optim.*, 52(6):4001–4026, 2014.

[90] M. Cirant. Stationary focusing mean-field games. *Comm. Partial Differential Equations*, 41(8):1324–1346, 2016.

[91] M. Cirant, R. Gianni, and P. Mannucci. Short-time existence for a backward-forward parabolic system arising from mean-field games. arXiv:1806.08138, to appear in Dynamic Games and Applications, 2018.

[92] M. Cirant and D. Tonon. Time-dependent focusing mean-field games: the sub-critical case. *J. Dynam. Differential Equations*, 31(1):49–79, 2019.

[93] P. Constantin and J. Wu. Behavior of solutions of 2D quasi-geostrophic equations. *SIAM J. Math. Anal.*, 30(5):937–948, 1999.

[94] R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[95] A. Córdoba and D. Córdoba. A maximum principle applied to quasi-geostrophic equations. *Comm. Math. Phys.*, 249(3):511–528, 2004.

[96] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[97] M. G. Crandall, P.-L. Lions, and P. E. Souganidis. Maximal solutions and universal bounds for some partial differential equations of evolution. *Arch. Rational Mech. Anal.*, 105(2):163–190, 1989.

[98] A. Cutrì and F. Leoni. On the Liouville property for fully nonlinear equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17(2):219–245, 2000.

[99] A. Cutrì and N. Tchou. Barrier functions for Pucci-Heisenberg operators and applications. *Int. J. Dyn. Syst. Differ. Equ.*, 1(2):117–131, 2007.

[100] F. Da Lio. Remarks on the strong maximum principle for viscosity solutions to fully nonlinear parabolic equations. *Commun. Pure Appl. Anal.*, 3(3):395–415, 2004.

[101] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 5*. Springer-Verlag, Berlin, 1992. Evolution problems. I, With the collaboration of Michel Artola, Michel Cessenat and Hélène Lanchon, Translated from the French by Alan Craig.

[102] A. Davini. Existence and uniqueness of solutions to parabolic equations with superlinear Hamiltonians. *Commun. Contemp. Math.*, 21(1):1750098, 25, 2019.

[103] E. DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.

[104] N. Dirr and V.D. Nguyen. Some new results on Lipschitz regularization for parabolic equations. *J. Evol. Equ.*, 2019.

[105] J. Droniou and C. Imbert. Fractal first-order partial differential equations. *Arch. Ration. Mech. Anal.*, 182(2):299–331, 2006.

[106] L. C. Evans. The 1-Laplacian, the ∞-Laplacian and differential games. In *Perspectives in nonlinear partial differential equations*, volume 446 of *Contemp. Math.*, pages 245–254. Amer. Math. Soc., Providence, RI, 2007.

[107] L. C. Evans. Adjoint and compensated compactness methods for Hamilton-Jacobi PDE. *Arch. Ration. Mech. Anal.*, 197(3):1053–1088, 2010.

[108] L. C. Evans. *A strong maximum principle for reaction-diffusion systems and a weak convergence scheme for reflected stochastic differential equations.* PhD thesis, Massachusetts Institute of Technology, 2010.

[109] L. C. Evans and C. K. Smart. Adjoint methods for the infinity Laplacian partial differential equation. *Arch. Ration. Mech. Anal.*, 201(1):87–113, 2011.

[110] D. Fan and S. Sato. Transference on certain multilinear multiplier operators. *J. Aust. Math. Soc.*, 70(1):37–55, 2001.

[111] C. Fefferman and D. H. Phong. Subelliptic eigenvalue problems. In *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., pages 590–606. Wadsworth, Belmont, CA, 1983.

[112] E. Feleqi and F. Rampazzo. Iterated Lie brackets for nonsmooth vector fields. *NoDEA Nonlinear Differential Equations Appl.*, 24(6):Art. 61, 43, 2017.

[113] F. Feo. A remark on uniqueness of weak solutions for some classes of parabolic problems. *Ric. Mat.*, 63(1, suppl.):S143–S155, 2014.

[114] X. Fernández-Real and X. Ros-Oton. Regularity theory for general stable operators: parabolic equations. *J. Funct. Anal.*, 272(10):4165–4221, 2017.

[115] G. B. Folland. Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.*, 13(2):161–207, 1975.

[116] A. Friedman. Remarks on the maximum principle for parabolic equations and its applications. *Pacific J. Math.*, 8:201–211, 1958.

[117] A. Friedman. *Partial differential equations of parabolic type.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[118] W. Gangbo and A. Swiech. Existence of a solution to an equation arising from the theory of mean field games. *J. Differential Equations*, 259(11):6573–6643, 2015.

[119] M. G. Garroni and J.-L. Menaldi. *Green functions for second order parabolic integro-differential problems*, volume 275 of *Pitman Research Notes in Mathematics Series.* Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1992.

[120] B. Gaveau. Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents. *Acta Math.*, 139(1-2):95–153, 1977.

[121] Y. Giga and M. Ohnuma. On strong comparison principle for semicontinuous viscosity solutions of some nonlinear elliptic equations. *Int. J. Pure Appl. Math.*, 22(2):165–184, 2005.

[122] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[123] D. Gomes and H. Sánchez Morgado. A stochastic Evans-Aronsson problem. *Trans. Amer. Math. Soc.*, 366(2):903–929, 2014.

[124] D. A. Gomes, J. Mohr, and R. R. Souza. Continuous time finite state mean field games. *Appl. Math. Optim.*, 68(1):99–143, 2013.

[125] D. A. Gomes and E. Pimentel. Local regularity for mean-field games in the whole space. *Minimax Theory Appl.*, 1(1):65–82, 2016.

[126] D. A. Gomes, E. A. Pimentel, and H. Sánchez-Morgado. Time-dependent mean-field games in the subquadratic case. *Comm. Partial Differential Equations*, 40(1):40–76, 2015.

[127] D. A. Gomes, E. A. Pimentel, and V. Voskanyan. *Regularity theory for mean-field game systems*. SpringerBriefs in Mathematics. Springer, [Cham], 2016.

[128] D. A. Gomes and H. Pimentel, E.and Sánchez-Morgado. Time-dependent mean-field games in the superquadratic case. *ESAIM Control Optim. Calc. Var.*, 22(2):562–580, 2016.

[129] V. R. Gopala Rao. Parabolic function spaces with mixed norm. *Trans. Amer. Math. Soc.*, 246:451–461, 1978.

[130] L. Grafakos. Multilinear operators in harmonic analysis and partial differential equations. In *Harmonic analysis and nonlinear partial differential equations*, RIMS Kôkyûroku Bessatsu, B33, pages 11–27. Res. Inst. Math. Sci. (RIMS), Kyoto, 2012.

[131] L. Grafakos and S. Oh. The Kato-Ponce inequality. *Comm. Partial Differential Equations*, 39(6):1128–1157, 2014.

[132] G. Gripenberg. On the strong maximum principle for degenerate parabolic equations. *J. Differential Equations*, 242(1):72–85, 2007.

[133] G. Grubb. Regularity in L_p Sobolev spaces of solutions to fractional heat equations. *J. Funct. Anal.*, 274(9):2634–2660, 2018.

[134] M. Hieber and J. Prüss. Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations. *Comm. Partial Differential Equations*, 22(9-10):1647–1669, 1997.

[135] C. D. Hill. A sharp maximum principle for degenerate elliptic-parabolic equations. *Indiana Univ. Math. J.*, 20:213–229, 1970/71.

[136] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[137] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.*, 6(3):221–251, 2006.

[138] Q. Huang. On the regularity of solutions to fully nonlinear elliptic equations via the Liouville property. *Proc. Amer. Math. Soc.*, 130(7):1955–1959, 2002.

[139] C. Imbert. A non-local regularization of first order Hamilton-Jacobi equations. *J. Differential Equations*, 211(1):218–246, 2005.

[140] H. Ishii and P.-L. Lions. Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. *J. Differential Equations*, 83(1):26–78, 1990.

[141] T. Iwabuchi and T. Kawakami. Existence of mild solutions for a Hamilton-Jacobi equation with critical fractional viscosity in the Besov spaces. *J. Math. Pures Appl. (9)*, 107(4):464–489, 2017.

[142] T. Jakubowski and K. Szczypkowski. Time-dependent gradient perturbations of fractional Laplacian. *J. Evol. Equ.*, 10(2):319–339, 2010.

[143] R. Jensen. Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations. *Indiana Univ. Math. J.*, 38(3):629–667, 1989.

[144] P. Juutinen and B. Kawohl. On the evolution governed by the infinity Laplacian. *Math. Ann.*, 335(4):819–851, 2006.

[145] P. Juutinen, P.and Lindqvist and J. J. Manfredi. On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. *SIAM J. Math. Anal.*, 33(3):699–717, 2001.

[146] S. Kamin and J. L. Vázquez. Fundamental solutions and asymptotic behaviour for the p-Laplacian equation. *Rev. Mat. Iberoamericana*, 4(2):339–354, 1988.

[147] G. Karch and W. A. Woyczyński. Fractal Hamilton-Jacobi-KPZ equations. *Trans. Amer. Math. Soc.*, 360(5):2423–2442, 2008.

[148] B. Kawohl and N. Kutev. Strong maximum principle for semicontinuous viscosity solutions of nonlinear partial differential equations. *Arch. Math. (Basel)*, 70(6):470–478, 1998.

[149] B. Kawohl and N. Kutev. Comparison principle for viscosity solutions of fully nonlinear, degenerate elliptic equations. *Comm. Partial Differential Equations*, 32(7-9):1209–1224, 2007.

[150] C. E. Kenig, G. Ponce, and L. Vega. Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Comm. Pure Appl. Math.*, 46(4):527–620, 1993.

[151] A. E. Kogoj and E. Lanconelli. An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations. *Mediterr. J. Math.*, 1(1):51–80, 2004.

[152] A. E. Kogoj and E. Lanconelli. Liouville theorem for X-elliptic operators. *Nonlinear Anal.*, 70(8):2974–2985, 2009.

[153] A. E. Kogoj and E. Lanconelli. L^p-Liouville theorems for invariant partial differential operators in $\mathbb{R}^n$. *Nonlinear Anal.*, 121:188–205, 2015.

[154] N. V. Krylov. *Lectures on elliptic and parabolic equations in Hölder spaces*, volume 12 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.

[155] N. V. Krylov. An analytic approach to SPDEs. In *Stochastic partial differential equations: six perspectives*, volume 64 of *Math. Surveys Monogr.*, pages 185–242. Amer. Math. Soc., Providence, RI, 1999.

[156] N. V. Krylov. The heat equation in $L_q((0,T), L_p)$-spaces with weights. *SIAM J. Math. Anal.*, 32(5):1117–1141, 2001.

[157] N. V. Krylov. Some properties of traces for stochastic and deterministic parabolic weighted Sobolev spaces. *J. Funct. Anal.*, 183(1):1–41, 2001.

[158] N. V. Krylov. *Lectures on elliptic and parabolic equations in Sobolev spaces*, volume 96 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[159] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[160] D. Lamberton. Équations d'évolution linéaires associées à des semi-groupes de contractions dans les espaces L^p. *J. Funct. Anal.*, 72(2):252–262, 1987.

[161] J.-M. Lasry and P.-L. Lions. Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem. *Math. Ann.*, 283(4):583–630, 1989.

[162] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.

[163] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.

[164] E. Le Donne, A. Pinamonti, and G. Speight. Universal differentiability sets and maximal directional derivatives in Carnot groups. *J. Math. Pures Appl. (9)*, 121:83–112, 2019.

[165] G. E. Lee. Local wellposedness for the critical nonlinear Schrödinger equation on $\mathbb{T}^3$. *Discrete Contin. Dyn. Syst.*, 39(5):2763–2783, 2019.

[166] T. Leonori and M. Magliocca. Comparison results for unbounded solutions for a parabolic cauchy-dirichlet problem with superlinear gradient growth. *Commun. Pure Appl. Anal.*, 18(6):2923–2960, 2019.

[167] T. Leonori, I. Peral, A. Primo, and F. Soria. Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations. *Discrete Contin. Dyn. Syst.*, 35(12):6031–6068, 2015.

[168] O. Ley and V.D. Nguyen. Lipschitz regularity results for nonlinear strictly elliptic equations and applications. *J. Differential Equations*, 263(7):4324–4354, 2017.

[169] Y. Li, L. Nguyen, and B. Wang. Towards a Liouville theorem for continuous viscosity solutions to fully nonlinear elliptic equations in conformal geometry. arXiv:1901.03646, 2019.

[170] Y. Li and B. Wang. Strong comparison principles for some nonlinear degenerate elliptic equations. *Acta Math. Sci. Ser. B (Engl. Ed.)*, 38(5):1583–1590, 2018.

[171] J. L. Lions. Sur les espaces d'interpolation; dualité. *Math. Scand.*, 9:147–177, 1961.

[172] J.-L. Lions and E. Magenes. Problemi ai limiti non omogenei. III. *Ann. Scuola Norm. Sup. Pisa (3)*, 15:41–103, 1961.

[173] J.-L. Lions and J. Peetre. Sur une classe d'espaces d'interpolation. *Inst. Hautes Études Sci. Publ. Math.*, (19):5–68, 1964.

[174] P.-L. Lions. Résolution de problèmes elliptiques quasilinéaires. *Arch. Rational Mech. Anal.*, 74(4):335–353, 1980.

[175] P.-L. Lions. Quelques remarques sur les problèmes elliptiques quasilinéaires du second ordre. *J. Analyse Math.*, 45:234–254, 1985.

[176] A. Lunardi. How to use interpolation in pde's. On-line lecture notes, available at http://people.dmi.unipr.it/alessandra.lunardi/.

[177] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems.* Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1995. [2013 reprint of the 1995 original] [MR1329547].

[178] A. Lunardi. *Interpolation theory*, volume 16 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)].* Edizioni della Normale, Pisa, 2018. Third edition.

[179] J. J. Manfredi. Nonlinear subelliptic equations on Carnot groups. Notes of a course given at the third school on Analysis and Geometry in metric spaces available at `http://www.pitt.edu/~manfredi/papers/fullynonlsubtrentofinal.pdf`, 2003.

[180] P. Mannucci. The Dirichlet problem for fully nonlinear elliptic equations non-degenerate in a fixed direction. *Commun. Pure Appl. Anal.*, 13(1):119–133, 2014.

[181] P. Mannucci, C. Marchi, and N. Tchou. The ergodic problem for some subelliptic operators with unbounded coefficients. *NoDEA Nonlinear Differential Equations Appl.*, 23(4):Art. 47, 26, 2016.

[182] M. Masoero. On the long time convergence of potential MFG. *NoDEA Nonlinear Differential Equations Appl.*, 26(2):Art. 15, 45, 2019.

[183] S. Mayorga. Short time solution to the master equation of a first order Mean Field Game system. arXiv:1811.08964, 2018.

[184] G. Metafune, D. Pallara, and A. Rhandi. Global properties of invariant measures. *J. Funct. Anal.*, 223(2):396–424, 2005.

[185] G. Metafune, D. Pallara, and A. Rhandi. Global properties of transition probabilities of singular diffusions. *Teor. Veroyatn. Primen.*, 54(1):116–148, 2009.

[186] M. Meyries and R. Schnaubelt. Interpolation, embeddings and traces of anisotropic fractional Sobolev spaces with temporal weights. *J. Funct. Anal.*, 262(3):1200–1229, 2012.

[187] H. Mitake and H. V. Tran. Dynamical properties of Hamilton-Jacobi equations via the nonlinear adjoint method: large time behavior and discounted approximation. In *Dynamical and geometric aspects of Hamilton-Jacobi and linearized Monge-Ampère equations—VIASM 2016*, volume 2183 of *Lecture Notes in Math.*, pages 125–228. Springer, Cham, 2017.

[188] S. Modena and L. Székelyhidi, Jr. Non-uniqueness for the transport equation with Sobolev vector fields. *Ann. PDE*, 4(2):Art. 18, 38, 2018.

[189] A. Montanari and E. Lanconelli. Pseudoconvex fully nonlinear partial differential operators: strong comparison theorems. *J. Differential Equations*, 202(2):306–331, 2004.

[190] N. Nadirashvili, V. Tkachev, and S. Vlăduţ. *Nonlinear elliptic equations and nonassociative algebras*, volume 200 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014.

[191] L. Nirenberg. A strong maximum principle for parabolic equations. *Comm. Pure Appl. Math.*, 6:167–177, 1953.

[192] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[193] A. Porretta. Elliptic equations with first order terms. Lecture notes, CIMPA school Alexandria, 2009.

[194] A. Porretta. Weak solutions to Fokker-Planck equations and mean field games. *Arch. Ration. Mech. Anal.*, 216(1):1–62, 2015.

[195] A. Porretta and E. Priola. Global Lipschitz regularizing effects for linear and nonlinear parabolic equations. *J. Math. Pures Appl. (9)*, 100(5):633–686, 2013.

[196] M. Portilheiro and J. L. Vázquez. Degenerate homogeneous parabolic equations associated with the infinity-Laplacian. *Calc. Var. Partial Differential Equations*, 46(3-4):705–724, 2013.

[197] M. H. Protter and H. F. Weinberger. *Maximum principles in differential equations*. Springer-Verlag, New York, 1984. Corrected reprint of the 1967 original.

[198] J. Prüss and R. Schnaubelt. Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time. *J. Math. Anal. Appl.*, 256(2):405–430, 2001.

[199] J. Prüss and G. Simonett. *Moving interfaces and quasilinear parabolic evolution equations*, volume 105 of *Monographs in Mathematics*. Birkhäuser/Springer, [Cham], 2016.

[200] C. Pucci. Un problema variazionale per i coefficienti di equazioni differenziali di tipo ellittico. *Ann. Scuola Norm. Sup. Pisa (3)*, 16:159–172, 1962.

[201] C. Pucci. Operatori ellittici estremanti. *Ann. Mat. Pura Appl. (4)*, 72:141–170, 1966.

[202] P. Pucci and J. Serrin. *The maximum principle*, volume 73 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 2007.

[203] P. Quittner and P. Souplet. *Superlinear parabolic problems*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.

[204] P. J. Rabier. Vector-valued Morrey's embedding theorem and Hölder continuity in parabolic problems. *Electron. J. Differential Equations*, pages No. 10, 10, 2011.

[205] L. Roncal and P. R. Stinga. Transference of fractional Laplacian regularity. In *Special functions, partial differential equations, and harmonic analysis*, volume 108 of *Springer Proc. Math. Stat.*, pages 203–212. Springer, Cham, 2014.

[206] L. Roncal and P. R. Stinga. Fractional Laplacian on the torus. *Commun. Contemp. Math.*, 18(3):1550033, 26, 2016.

[207] X. Ros-Oton. *Integro-differential equations: regularity theory and Pohozaev identities*. PhD thesis, Universitat Politècnica de Catalunya, 2014.

[208] X. Ros-Oton. Nonlocal elliptic equations in bounded domains: a survey. *Publ. Mat.*, 60(1):3–26, 2016.

[209] X. Ros-Oton and H. Vivas. Higher-order boundary regularity estimates for nonlocal parabolic equations. *Calc. Var. Partial Differential Equations*, 57(5):Art. 111, 20, 2018.

[210] L. Rossi. Non-existence of positive solutions of fully nonlinear elliptic equations in unbounded domains. *Commun. Pure Appl. Anal.*, 7(1):125–141, 2008.

[211] G. Samorodnitsky and M. S. Taqqu. *Stable non-Gaussian random processes.* Stochastic Modeling. Chapman & Hall, New York, 1994. Stochastic models with infinite variance.

[212] F. Santambrogio. *Optimal transport for applied mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and their Applications.* Birkhäuser/Springer, Cham, 2015. Calculus of variations, PDEs, and modeling.

[213] H.-J. Schmeisser and H. Triebel. *Topics in Fourier analysis and function spaces.* A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1987.

[214] J. Serrin. Gradient estimates for solutions of nonlinear elliptic and parabolic equations. In *Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971)*, pages 565–601. Academic Press, New York, 1971.

[215] R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*, volume 49 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997.

[216] L. Silvestre. Hölder continuity for integro-differential parabolic equations with polynomial growth respect to the gradient. *Discrete Contin. Dyn. Syst.*, 28(3):1069–1081, 2010.

[217] L. Silvestre. On the differentiability of the solution to the Hamilton-Jacobi equation with critical fractional diffusion. *Adv. Math.*, 226(2):2020–2039, 2011.

[218] L. Silvestre, V. Vicol, and A. Zlatoš. On the loss of continuity for super-critical drift-diffusion equations. *Arch. Ration. Mech. Anal.*, 207(3):845–877, 2013.

[219] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[220] E. Sinestrari. On the abstract Cauchy problem of parabolic type in spaces of continuous functions. *J. Math. Anal. Appl.*, 107(1):16–66, 1985.

[221] P. E. Sobolevskiĭ. Fractional powers of coercively positive sums of operators. *Dokl. Akad. Nauk SSSR*, 225(6):1271–1274, 1975.

[222] E. M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[223] E. M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces.* Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[224] L. F. Stokols and A. F. Vasseur. De Giorgi techniques applied to Hamilton-Jacobi equations with unbounded right-hand side. *Commun. Math. Sci.*, 16(6):1465–1487, 2018.

[225] D. W. Stroock and S. R. S. Varadhan. On the support of diffusion processes with applications to the strong maximum principle. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. III: Probability theory*, pages 333–359. Univ. California Press, Berkeley, Calif., 1972.

[226] K. Taira. *Diffusion processes and partial differential equations.* Academic Press, Inc., Boston, MA, 1988.

[227] M. E. Taylor. *Pseudo differential operators.* Lecture Notes in Mathematics, Vol. 416. Springer-Verlag, Berlin-New York, 1974.

[228] M. E. Taylor. *Tools for PDE*, volume 81 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2000. Pseudodifferential operators, paradifferential operators, and layer potentials.

[229] M. E. Taylor. *Partial differential equations III. Nonlinear equations*, volume 117 of *Applied Mathematical Sciences.* Springer, New York, second edition, 2011.

[230] H. V. Tran. Adjoint methods for static Hamilton-Jacobi equations. *Calc. Var. Partial Differential Equations*, 41(3-4):301–319, 2011.

[231] H. Triebel. *Interpolation theory, function spaces, differential operators.* Johann Ambrosius Barth, Heidelberg, second edition, 1995.

[232] H. F. Trotter. On the product of semi-groups of operators. *Proc. Amer. Math. Soc.*, 10:545–551, 1959.

[233] N. S. Trudinger. Comparison principles and pointwise estimates for viscosity solutions of nonlinear elliptic equations. *Rev. Mat. Iberoamericana*, 4(3-4):453–468, 1988.

[234] C. Wang. The Aronsson equation for absolute minimizers of L^∞-functionals associated with vector fields satisfying Hörmander's condition. *Trans. Amer. Math. Soc.*, 359(1):91–113, 2007.

[235] J. Wu. Dissipative quasi-geostrophic equations with L^p data. *Electron. J. Differential Equations*, pages No. 56, 13, 2001.

[236] X. Zhang. Stochastic differential equations with Sobolev drifts and driven by α-stable processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 49(4):1057–1079, 2013.