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QUANTITATIVE STATISTICAL PROPERTIES FOR TWO DIMENSIONAL PARTIALLY HYPERBOLIC SYSTEMS

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Abstract

This thesis focuses on the study of the statistical properties of chaotic dynamical systems, especially in the area of partially hyperbolic systems. The general aim of this field is attempting to predict the behaviour of the system for long times. Ruelle in the 1970s has shown that, instead of considering individual trajectories, it is much more natural to consider the evolution of densities under the study of a linear operator, called Transfer Operator (or Ruelle-Perron-Frobenius). Following this idea, in the last years, an extremely powerful method has been developed: the functional approach. It consists in the study of the spectral properties of the transfer operator on suitable Banach spaces. In this work we apply this approach to partially hyperbolic systems in two dimensions, establishing the germ of a general theory. To illustrate the scope of the theory, the results are used in the case of fast-slow partially hyperbolic systems, pointing out how to pursue the arguments for further progresses.

Notations

$\mathbb{R}^d, \mathbb{R}$	The d-dimensional Euclidean space for $d \geq 2$ and $d = 1$ respectively.
$\mathbb{T}^2, \mathbb{T}$	The d-dimensional flat torus $\mathbb{R}^d/\mathbb{Z}^d$ for$d \geq 2$ and $d = 1$ respectively.
$B(x, r)$	The open ball of radius r centered at x.
$C^0(\Omega, \mathbb{R}^d)$	The space of $\mathbb{R}^d$-valued continuous functions on $\Omega \subseteq \mathbb{T}^2$.
$C^k(\Omega, \mathbb{R}^d)$	The space of $\mathbb{R}^d$-valued continuous functions on $\Omega \subseteq \mathbb{T}^2$ with continuous derivatives of order j, $j = 1, \dots, k$.
$\partial_{x_i} u$, ∇u	Partial derivatives with respect to the x_i-th variable and gradient vector of u.
$\partial_t u$, $\frac{d}{dt} u$, u'	Differentiation with respect to the variable t .
Du	Jacobian matrix of $u \in \mathcal{C}^r$.
$D^2 f$	Hessian matrix of u.
$\mathbb{1}_A(x)$	The characteristic function of A defined by $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$.
$\mathcal{L}(X, Y)$	Banach space of linear continuous operators from the Banach space X to the Banach space Y equipped with the norm topology. When $X = Y$ we only write $\mathcal{L}(X)$.
$X \hookrightarrow Y$	$X \subset Y$ with continuous injection.
$\mathcal{F}$	Fourier transform $\mathcal{F}u(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} u(x) dx$.
$\mathcal{H}^s$	The completion of $\mathcal{C}^\infty(\mathbb{T}^2)$ with respect to the norm induced by the inner product $\langle u, v \rangle_s = \sum_{\xi \in \mathbb{Z}^2} (1 + \|\xi\|^2)^s \mathcal{F}u(\xi) \overline{\mathcal{F}v(\xi)}$.
$\{A, B\}^+$	The maximum between the quantities A and B.

Some useful inequalities

- *Generalized Young's inequality*: For $p \in (1, \infty)$ and $p' = p/(p-1)$ and any positive $\epsilon > 0$ we have

$$ab \leq \epsilon^p \frac{a^p}{p} + \frac{1}{\epsilon^{p'}} \frac{b^{p'}}{p'} \ \forall a, b > 0 \ .$$

- *Logarithmic mean*: For each $x, y > 0, x \neq y$

$$\sqrt{xy} \leq \left(\frac{\sqrt{x} + \sqrt{y}}{2} \right) (xy)^{\frac{1}{4}} \leq \frac{x - y}{\ln(x) - \ln(y)} \leq \left(\frac{\sqrt{x} + \sqrt{y}}{2} \right)^2 \leq \frac{x + y}{2}.$$

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Chapter 1

Introduction

The general purpose of this thesis is to give a contribution in the area of *dynamical systems*, in particular in the branch concerning *partially hyperbolic systems*. In this introductory chapter we will briefly outline the motivations behind the study and the main questions and issues we would like to address, discussing the results we obtained.

1.1 Motivation

A dynamical system can be described by the (discrete or continuous) time evolution of a map over some set. For our discussion it is enough to consider a map $F : M \to M$ which is a diffeomorphism, or a local diffeormorphism, on a compact Riemannian smooth manifold M. It is well known at least since the works of H. Poincaré that studying the topological discrete dynamical system (F, M) to make long time predictions may lead to unsatisfying results, due to either a rather complicated behaviour of the orbits, or the sensibility to the initial conditions. We therefore change the point of view, studying instead the measurable dynamical system $(F_*, \mathcal{M}(M))$, where $\mathcal{M}(M)$ is the space of Borel probability measures on M and

$$F_*\mu(A) = \mu(F^{-1}(A)), \quad \text{for each measurable set } A.$$

The first problem is to select the relevant invariant measures. In this setting it can be shown that the measures belonging to the class of a finite, smooth Riemannian volume on M are globally preserved by F. By Morse theory, any Riemannian volume is locally equal to the Lebesgue measure, up to a smooth change of coordinates. Therefore a good choice is to look at measures which are absolutely continuous with respect to the Lebesgue measure. [1] If F is a conservative system, it is well known that there is a natural invariant measure in the class of the volume measures (for example, in the Hamiltonian case, such a measure is the Liouville one). On the other hand, there is no distinguished invariant measure in the dissipative case. For instance, if F has periodic points, we can take the average along the orbit of each such a point and construct an associated invariant measure, which is of course not included in the volume class.

[1] We will show this fact in a concrete simple example in Section 2.

1.2 Physical measures

It is the work of Sinai, Ruelle and Bowen (1960-1970) on hyperbolic attractors that rigorously established the existence of invariant measures which are "physically observable".[2]

Definition 1.2.1. *A physical measure (or SRB measure) is an F-invariant probability measure μ such that the set*

$$B(\mu) := \left\{ x \in M : \frac{1}{n} \sum_{k=0}^{n-1} \delta_{F^k(x)} \to \mu \quad \textit{weakly as} \quad n \to \infty \right\}$$

has positive Lebesgue measure.

By weak convergence we mean that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi\left(F^k(x)\right) = \int_M \varphi d\mu, \quad \forall \varphi \in \mathcal{C}^0(M, \mathbb{R}).$$

Since then, many progresses have been made concerning the existence, uniqueness and statistical properties of SRB measures in the case of hyperbolic maps, in which the tangent bundle splits into contracting and expanding invariant subspaces. On the other hand, a need to study cases beyond hyperbolicity, such as *non-uniform hyperbolic systems* and *partially hyperbolic systems*, has clearly emerged. This work focuses on the second type of systems, the partially hyperbolic ones, which allow central directions at each point, in which the expansion and contraction is dominated by the behaviour in the hyperbolic directions. In the case of volume preserving diffeomorphisms substantial progresses have been made in the study of ergodicity starting with [44, 50, 57] to the point of establishing very general results, e.g. [15]. For the dissipative case, if the invariant measure is not a priori known, then establishing the existence of SRB measures is a serious problem in itself, see [14, 1, 53] for some important partial results. Moreover, it is well known, at least since the work of Krylov [41], that for many applications ergodicity does not suffice and mixing (usually in the form of effective quantitative estimates on the decay of correlations) is of paramount importance. By mixing we mean that, if μ is an invariant measure for F, then

$$\lim_{n\to\infty} \int_M \varphi_1 \cdot \varphi_2 \circ F^n d\mu = \int_M \varphi_1 d\mu \int_M \varphi_2 d\mu, \tag{1.2.1}$$

for suitable observables $\varphi_1, \varphi_2 : M \to \mathbb{R}$. Of course it is desirable to have information on the speed of convergence in the above limit. If the limit in (1.2.1)

[2]Since in this work we deal only with *partially hyperbolic endomorphisms*, we will follow common practice (see for instance [28, Corollary 2]) and choose to use the terms *physical measure* or *SRB measure* to indicate the same object in Definition 1.2.1 . Nonetheless, the reader must be aware that in many situations SRB measures and physical measures are defined in different ways, as they do not coincide in general (see [59] for an exhaustive discussion).

occurs exponentially fast we talk about *exponential decay of correlations.* Some results in this direction exist in the case of mostly expanding central direction [2], and mostly contracting central direction [25, 18]. For central direction with zero Lyapunov exponents (or close to zero) there exist quantitative results on exponential decay of correlations only for group extensions of Anosov maps and Anosov flows [26, 19, 24, 46, 55], but none of them apply to an open class (with notable exceptions of [17, 56]; also, some form of rapid mixing is known to be typical for large classes of flows [32, 47]). Such results, albeit important, are often not easy to apply since it is very difficult to estimate the central Lyapunov exponent. Hence, the problem of effectively studying the quantitative mixing properties of partially hyperbolic systems is wide open.

1.3 A functional approach

In the last years, starting with [13, 43, 7], an extremely powerful method to investigate the statistical properties of hyperbolic systems has been developed: *the functional approach.* It consists in the study of the spectral properties of the transfer operator on appropriate Banach spaces. Although the basic idea can be traced back, at least, to Von Neumann ergodic theorem, the new ingredient consists in the understanding that non standard functional spaces must be used, and in the insight of how to embed the key geometrical properties of the system in the topology of the Banach space. See [6] for a recent review of this approach.

This point of view has produced many important results, e.g. see [46, 40, 33, 31, 29, 8] just to cite a few. It is then natural to investigate if the functional approach can be extended to partially hyperbolic systems. Some results that hint at this possibility already exist (e.g. [4, 30]), however, a general approach is totally missing. Nonetheless, the idea that some quantitative form of accessibility should play a fundamental role has slowly emerged ([51, 48, 16]).

1.4 Main results

In this work we combine ideas from [4] and [43] to advance the functional analytic point of view to a large class of two dimensional partially hyperbolic endomorphisms. In Theorem 4.3.2 we find checkable conditions that imply the existence of finitely many physical measures for such a class, and we prove that they all belong to some Sobolev space of function. We also show (Theorem 4.5.2) that such conditions are fulfilled for an open set of physically relevant systems, motivated by [27, 20, 21, 22, 23]. Moreover, for such systems, we are able to obtain some quantitative information on the regularity of the eigenvectors of the transfer operator (Theorem 4.5.4), which hopefully should allow further progress. In addition, we show how the results obtained here can be combined with averaging results to provide a very detailed description of the physical measures, see Theorem 4.5.3. We believe that this approach can be further refined and extended to produce results in a much more general class of systems. The attempt to obtain precise quantitative information is responsible for much of the length of the work, as it entails a strenuous effort to keep track of many constants. Indeed, it is customary to think that the constants appearing in Lasota-Yorke type inequalities are largely irrelevant. This is certainly not the case in the context discussed in section 10, as the possibility to consider the class of maps discussed there as a perturbation of a limiting case depends crucially of the size of such constants. It was then essential to push the estimates to their extreme in order to find out if perturbative ideas could be applied or identify the source of a possible obstruction.

1.5 Structure of the thesis

The thesis is structured in Chapters organized in three parts: the first is a brief overview of the background, the second is the core of the work, in which we prove the main theorems in a general setting, and in the third part we apply our results in a concrete case.

- **Part I**
  - **Chapter 2**. We briefly show the necessary background concerning the functional approach, recalling some important abstract results about the spectrum of bounded operators. To present such results in a simple fashion, we apply them to the case of expanding maps on the circle.
  - **Chapter 3**. Using a simple example, we present the concept of transversality between unstable cones introduced by Tsujii, which is central in our discussion.
- **Part II**
  - **Chapter 4**. We describe the systems we consider and we state our results.
  - **Chapter 5**. We introduce the necessary notation and prove several facts needed to define the Banach spaces we are interested in. In particular we provided several preliminary estimates that we will use in the rest of the manuscript.
  - **Chapter 6**. We define the main Banach spaces and we prove a first Lasota-Yorke inequality (Theorem 6.0.1). Unfortunately, the spaces considered in this section do not embed compactly in each other and hence one cannot deduce the quasi-compactness of the operator from such inequalities.
  - **Chapter 7**. In this Chapter we prepare some results needed in the next Chapter: we prove a Lasota-Yorke inequality in the Sobolev spaces $\mathcal{H}^s$ (Lemma 7.1.1), and we give some results on the transversality of unstable cones.
  - **Chapter 8**. This Chapter is the core of the thesis where some inequalities relating the previous geometric norms to the Sobolev norms $\mathcal{H}^s$

are obtained (Theorem 8.0.2), solving the compactness problem mentioned in Chapter 6. The key result to prove the Theorem is contained in Proposition 8.1.3.

  - **Chapter 9**. We collect the work done and we prove Theorem 9.0.1, which will allow us to prove Theorem 4.3.2.

- **Part III**

  - **Chapter 10**. In this final Chapter we show that fast-slow systems satisfy our conditions. We prove that Theorem 9.0.1 applies in this case, which allows us to prove Theorem 4.5.2. We then conclude showing some implications, proving Theorem 4.5.4 and Theorem 4.5.3.

Part I

Background: a functional analytic approach to dynamical systems

Chapter 2

Expanding maps on $\mathbb{T}^1$

2.1 Invariant measures

In this chapter we present the main tools in the simplest possible case,[1] which is a map $F \in \mathcal{C}^r(\mathbb{T}^1, \mathbb{T}^1)$, $r \geq 2$, such that $\inf_{x \in \mathbb{T}^1} F'(x) \geq \lambda > 1$. As explained in the introduction, studying the topological dynamical system $(F, \mathbb{T}^1)$ to make long time predictions may lead to unsatisfying results, due to the sensibility to the initial conditions. We then change the point of view studying instead the measurable dynamical system $(F_*, \mathcal{M}(\mathbb{T}^1))$, where $\mathcal{M}(\mathbb{T}^1)$ is the space of measures on $\mathbb{T}^1$ and

$$F_*\mu(\varphi) = \mu(\varphi \circ F), \quad \forall \varphi \in \mathcal{C}^0(\mathbb{T}^1, \mathbb{R}).$$

We want now to select the initial measures which are pushed forward by the dynamics. As one can imagine, a reasonable choice is to take μ absolutely continuous with respect to Lebesgue. We then let $h \in L^1(\mathbb{T}^1)$ be the density of μ. As we are interested in the invariant measures, we need to select the ones such that

$$\mu(\varphi \circ F) = \mu(\varphi), \quad \forall \varphi \in \mathcal{C}^0(\mathbb{T}^1, \mathbb{R}),$$

namely we look for the fixed points of the linear operator F_*. In the case at hand we know by Riesz-Markov Theorem that $\mathcal{M}(\mathbb{T}^1) = \mathcal{C}^0(\mathbb{T}^1)'$, and each measure μ is characterized by

$$\mu(\varphi) = \int_{\mathbb{T}^1} \varphi(x)\mu(dx).$$

According to Krylov-Bogoliubov Theorem if μ is a probability measure on $\mathcal{M}(\mathbb{T}^1)$, then the sequence

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} F_*^k \mu \right\}_{n \in \mathbb{N}}$$

has accumulation points which are fixed points of F_*. Instead of studying directly the composition operators, it is more convenient to look at its dual version which, as we will see in Section 2.2.1, has some smoothing properties. Let $I_i \subset \mathbb{T}^1$ be an

[1] See [45] for a more detailed exposition.

interval such that $F_{|I_i}$ is invertible. We partition $\mathbb{T}^1 = \bigcup_i I_i$ and we set $\mathfrak{h}_i := F_{|I_i}^{-1}$. Then, if m is the Lebesgue measure on $\mathbb{T}^1$, by changing the variables

$$F_*\mu(\varphi) = \sum_i m\left(h\mathbb{1}_{I_i}\varphi \circ F\right) = \sum_i m\left(\mathbb{1}_{I_i}\left(h \circ \mathfrak{h}_i\varphi\right) \circ F\right)$$
$$= \int_{\mathbb{T}^1} \varphi\left(\sum_{y\in F^{-1}x} \frac{h(y)}{F'(y)}\right) dx.$$

If we then define the operator

$$\mathcal{L}_F h(x) = \sum_{y\in F^{-1}x} \frac{h(y)}{F'(y)}, \tag{2.1.1}$$

it has the fundamental properties that, for each μ such that $d\mu = hdm$

$$\frac{dF_*\mu}{dm} = \mathcal{L}_F h.$$

The operator $\mathcal{L}_F$ is called *Perron-Frobenius-Ruelle* operator, or more simply *transfer operator* and it is the main object of our studies. The first thing that can be noticed is that $\mathcal{L}_F : L^1(\mathbb{T}^1) \to L^1(\mathbb{T}^1)$, and it is a contraction on L^1, indeed

$$\int |\mathcal{L}_F h| dm \leq \int \mathcal{L}_F |h| dm = \int 1 \circ F |h| dm = \int |h| dm. \tag{2.1.2}$$

Moreover if $d\mu = hdm$, then μ is an invariant measure of F if and only if $\mathcal{L}_F h = h$, i.e. the densities of the invariant measures correspond to the eigenfunctions of $\mathcal{L}$ associated to the eigenvalues of modulus one. We then want to study the spectral properties of $\mathcal{L}$, which of course will depend on the functional space on which the operator acts. Unfortunately, it turns out that the spectrum of $\mathcal{L}$ in L^1 is the full unitary disk, which does not give enough information for our purposes.
To continue, in the next section we recall some basic facts about the spectrum of an operator acting on Banch spaces.

2.2 Spectral picture

In this section we give a briefly refresh of the main definitions about the spectral properties of bounded operators, highlighting only the tools and the results needed in this work. We essentially follow [5] and [45].
During the course of this section $\mathcal{L}$ will denote a bounded linear operator acting on a Banach space $\mathcal{B}$ endowed with a norm $\|\cdot\|$. In particular there exists a constant $C > 0$ such that

$$\|\mathcal{L}u\| \leq C\|u\|, \quad \forall u \in \mathcal{B},$$

and we denote by $L(\mathcal{B}, \mathcal{B})$ the set of bounded linear operators from $\mathcal{B}$ to $\mathcal{B}$ endowed with the norm

$$\|\mathcal{L}\|_{op} := \sup_{\substack{u\in\mathcal{B} \\ \|u\|=1}} \|\mathcal{L}u\|.$$

Notice that for the transfer operator of the previous section this corresponds to equation (2.1.2) with $C = 1$.

Definition 2.2.1. *Given $\mathcal{L} \in L(\mathcal{B}, \mathcal{B})$ we define the following quantities*

- *The resolvent set of $\mathcal{L}$ is*
$$\mathcal{R}_{\mathcal{B}}(\mathcal{L}) := \left\{\lambda \in \mathbb{C} : \exists \left(\mathcal{L} - \lambda \operatorname{Id}\right)^{-1} \in L(\mathcal{B}, \mathcal{B})\right\}.$$
- *The set $\sigma_{\mathcal{B}}(\mathcal{L}) := \mathbb{C} \setminus \mathcal{R}(\mathcal{L})$ is the spectrum of $\mathcal{L}$.*[2]
- *$\lambda \in \mathbb{C}$ is called an eigenvalue of $\mathcal{L}$ if the operator $\mathcal{L} - \lambda \operatorname{Id}$ is not injective.*
- *The algebraic multiplicity of $\lambda \in \mathbb{C}$ is $m_a(\lambda) = \dim \operatorname{Ker}(\mathcal{L} - \lambda \operatorname{Id}) \leq \infty$.*
- *The geometric multiplicity of $\lambda \in \mathbb{C}$ is*
$$m_g(\lambda) = \dim\{u \in \mathcal{B} : \exists m \geq 1 : (\mathcal{L} - \lambda \operatorname{Id})^m u = 0\} \leq \infty.$$
- *$\rho(\mathcal{L}) = \sup\{|\lambda| : \lambda \in \sigma_{\mathcal{B}}(\mathcal{L})\}$ is the spectral radius of $\mathcal{L}$.*
- *We call $\rho_{ess} := \rho_{ess}(\mathcal{L})$ the essential spectral radius of $\mathcal{L}$ and it is the smallest number $\rho_{ess} \geq 0$ such that any $\lambda \in \sigma(\mathcal{L})$ with $|\lambda| > \rho_{ess}$ is an isolated eigenvalue with finite multiplicity.*

It is important to remark that, if $\mathcal{L}$ is a compact operator, it is well known that its spectrum is made of countably many eigenvalues which may accumulates in zero. Hence in this case the essential spectral radius would be zero. On the other hand it could also happens that $\rho_{\text{ess}}(\mathcal{L}) = \rho(\mathcal{L})$, which gives poor information in terms of the dynamics. We then come up naturally with the idea of defining a *quasi-compact* operator, that is, roughly speaking, an operator which satisfies $\rho_{\text{ess}}(\mathcal{L}) < \rho(\mathcal{L})$.

Definition 2.2.2. *The operator $\mathcal{L} \in L(\mathcal{B}, \mathcal{B})$ is quasi-compact if there exist $\mathcal{B}_1, \mathcal{B}_2$ closed subsets of $\mathcal{B}$, and $0 < \rho_0 < \rho(\mathcal{L})$ such that*

- *$\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$, with $\mathcal{L}(\mathcal{B}_1) \subset \mathcal{B}_1$ and $\mathcal{L}(\mathcal{B}_2) \subset \mathcal{B}_2$,*
- *$\dim \mathcal{B}_1 < \infty$ and if λ is an eigenvalue of $\mathcal{L}|_{\mathcal{B}_1}$, then $|\lambda| > \rho_0$,*
- *$\rho(\mathcal{L}|_{\mathcal{B}_2}) < \rho_0$.*

Remark 2.2.3. *In terms of the transfer operator $\mathcal{L}$ introduced in the previous section, we gain important information if we are able to prove the quasi-compactness in some space $\mathcal{B} \subset L^1$. In fact it can be proved that the eigenvalues on the peripheral spectrum $\sigma_{ph}(\mathcal{L}) = \{z \in \mathbb{C} : |z| = \rho(\mathcal{L}) = 1\}$ have equal algebraic and geometric multiplicity, which essentially follows from (2.1.2) which implies that there are no Jordan blocks associated to eigenvalues on $\sigma_{ph}(\mathcal{L})$. It follows that, since one is an eigenvalue, the dimension of the eigenspace associated to the eigenvalue one corresponds to the number of SRB measures.*

[2] We will simply use the notation $\mathcal{R}(\mathcal{L})$ as well as $\sigma(\mathcal{L})$ when the Banach space $\mathcal{B}$ is clear from the context.

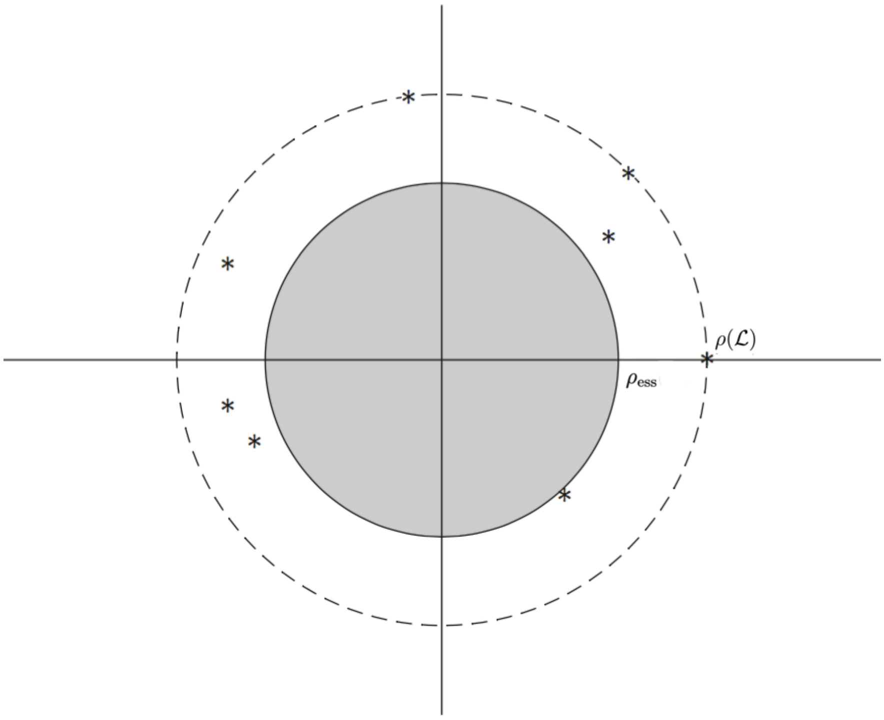

Figure 2.1: Spectral picture.

We thus have a criterion to prove the existence of finitely many physical measures: find a Banach space $\mathcal{B} \subset L^1$ on which the transfer operator $\mathcal{L}$ acts as a quasi-compact operator. In this direction an abstract result in functional analysis is available due to Hennion-Nussbaum ([37]), that we state here in the following form

Theorem 2.2.4. *Let $(\mathcal{B}, \|\cdot\|) \subset (\mathcal{B}_w, \|\cdot\|_w)$ be two Banach spaces and $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ a linear operator such that, for some $M > \eta > 0$, $A, B, C > 0$ and for each $n \in \mathbb{N}$, $u \in \mathcal{B}$*

(1) $\|\mathcal{L}^n u\|_w \leq C M^n \|u\|_w$

(2) $\|\mathcal{L}^n u\| \leq A\eta^n \|u\| + BM^n \|u\|_w$

(3) $\mathcal{B} \hookrightarrow \mathcal{B}_w$ *is compact,*

then $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ is quasi-compact with $\rho(\mathcal{L}) \leq M$ and $\rho_{ess}(\mathcal{L}) \leq \eta$.

Remark 2.2.5. *The first two items are called Lasota-Yorke (or Doeblin-Fortet) inequalities.*

To conclude let us recall also the definition of *spectral gap*, which is stronger than the one of quasi-compactness and provides an abstract criterion to prove the existence and uniqueness of the invariant measure of the system.

Definition 2.2.6. *An operator $\mathcal{L} \in L(\mathcal{B}, \mathcal{B})$ has a spectral gap in $\mathcal{B}$ if $\mathcal{L} = \lambda P + Q$ where*

- $P^2 = P$ *and* $\dim(\text{Range}(P)) = 1$,
- $PQ = QP = 0$,
- $Q \in L(\mathcal{B}, \mathcal{B})$ *such that* $\rho(Q) < |\lambda|$.

It can be proved that if $\mathcal{L}$ has a spectral gap then λ is a simple eigenvalue and there exists $\nu < 1$ such that the spectrum is decomposed into

$$\sigma_{\mathcal{B}}(\mathcal{L}) = \{\lambda\} \cup \mathcal{K},$$

where $\mathcal{K} \subset \{z \in \mathbb{C} : |z| < \nu|\lambda|\}$.

Remark 2.2.7. *Again, in terms of the transfer operator, this results can be extremely helpful in the understanding of the statistical properties of the systems. For example, if there exists a unique SRB measure, then either the map is not mixing (there are other eigenvalues, besides one, on the unit circle) or it mixes exponentially fast (one is the only eigenvalue on the peripheral spectrum and hence the operator has a spectral gap).*

2.2.1 A simple application

As an example we wish to show that the assumptions of Theorem 2.2.4 are satisfied in the case of a one dimensional $\mathcal{C}^2$ expanding map on the circle, to conclude the argument introduced in the previous Section. We have the following

Theorem 2.2.8. *Let* $F : \mathbb{T}^1 \to \mathbb{T}^1$ *be a* $\mathcal{C}^2$ *expanding map with*

$$\inf_{x \in \mathbb{T}^1} F'(x) \geq \lambda > 1, \tag{2.2.1}$$

and $\mathcal{L} := \mathcal{L}_F$ *the transfer operator defined in (2.1.1). Then* $\mathcal{L}$ *satisfies the assumptions of Theorem 2.2.4 with* $\mathcal{B} = L^2(\mathbb{T}^1)$, $\mathcal{B}_w = \mathcal{H}^1(\mathbb{T}^1)$,[3] $M = 1$ *and* $\eta = \lambda^{-1}$.

Remark 2.2.9. *It would be easier to consider* L^1 *as the weak space in the above theorem, nevertheless we wish to prove the quasi-compactness in this setting as it is instructive for our future computations.*

Proof. We need to prove assumptions (1),..,(3) of Theorem 2.2.4. First of all, by the definition of $\mathcal{L}$ in (2.1.1), it is easy to see that, for each $n \in \mathbb{N}$, $x \in \mathbb{T}^1$ and $u \in L^2$,

$$\mathcal{L}^n u(x) = \sum_{y \in F^{-n}(x)} \frac{u(y)}{(F^n)'(y)},$$

where $F^n = \underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}}$ and, by the chain rule,

$$(F^n)'(x) = \prod_{k=0}^{n-1} F'(F^k(x)) =: \Gamma_n(x). \tag{2.2.2}$$

[3] See Appendix C for the precise definition of $\mathcal{H}^1$.

Next

$$
\begin{aligned}
(\mathcal{L}^n u)'(x) &= \sum_{y\in F^{-n}(x)} \left(\frac{u'(y)}{(F^n)'(y)^2} - u(y)\frac{(F^n)''(y)}{(F^n)'(y)^3} \right) \\
&= \mathcal{L}^n \left(\frac{u'}{(F^n)'} \right)(x) - \mathcal{L}^n \left(u \cdot \frac{(F^n)''}{[(F^n)']^2} \right)(x).
\end{aligned} \tag{2.2.3}
$$

We set for each $n \in \mathbb{N}$ and $x \in \mathbb{T}^1$, $D_n(x) := \frac{(F^n)''(x)}{[(F^n)'(x)]^2}$.[4] We then have, by (2.2.2),

$$
(F^n)''(x) = \left(e^{\log \Gamma_n(x)}\right)' = \Gamma_n(x) \sum_{k=0}^{n-1} \left(\frac{F''}{F'} \right) \circ F^k(x) \Gamma_k(x).
$$

Setting $C_F = \sup_{x\in\mathbb{T}^1} \frac{F''(x)}{F(x)}$, equations (2.2.1) and (2.2.3) yield

$$
|D_n(x)| = \left| \frac{\sum_{k=0}^{n-1} \frac{F''}{F} \circ F^k(x)\Gamma_k(x)}{\Gamma_n(x)} \right| \leq C_F \sum_{k=0}^{n-1} \lambda^{k-n} \leq D \quad \forall x \in \mathbb{T}^1, \tag{2.2.4}
$$

where $D = C_F\,(\lambda - 1)^{-1}$. By (2.1.2), (2.2.4) and (2.2.3) it follows that

$$
\begin{aligned}
\|\mathcal{L}^n u\|_{W^{1,1}} &= \|(\mathcal{L}^n u)'\|_{L^1} + \|\mathcal{L}^n u\|_{L^1} \\
&\leq \lambda^{-n}\|u'\|_{L^1} + \|D_n\|_{L^\infty}\|u\|_{L^1} + \|u\|_{L^1} \\
&\leq \lambda^{-n}\|u\|_{W^{1,1}} + (D+1)\|u\|_{L^1},
\end{aligned}
$$

which gives for each $n \in \mathbb{N}$

$$
\|\mathcal{L}^n 1\|_{L^\infty(\mathbb{T}^1)} \leq \|\mathcal{L}^n 1\|_{W^{1,1}} \leq C, \tag{2.2.5}
$$

with $C > 0$ depending only on F. We use the above facts to prove items (1) and (2). First

$$
\begin{aligned}
\|\mathcal{L}^n u\|_{L^2}^2 &= \int_{\mathbb{T}^1} u\,(\mathcal{L}^n u) \circ F^n \leq \|u\|_{L^2} \left(\int_{\mathbb{T}^1} (\mathcal{L}^n u)^2 \circ F^n \right)^{\frac{1}{2}} \\
&= \|u\|_{L^2} \left(\int_{\mathbb{T}^1} (\mathcal{L}^n u)^2\, \mathcal{L}^n 1 \right)^{\frac{1}{2}} \leq C^{\frac{1}{2}} \|u\|_{L^2}\, \|\mathcal{L}^n u\|_{L^2}.
\end{aligned}
$$

Hence, again by (2.2.3)

$$
\|\mathcal{L}^n u\|_{\mathcal{H}^1} = \|(\mathcal{L}^n u)'\|_{L^2} + \|\mathcal{L}^n u\|_{L^2} \leq C^{\frac{1}{2}}\lambda^{-n}\|u\|_{\mathcal{H}^1} + C^{\frac{1}{2}}(D+1)\|u\|_{L^2},
$$

from which we deduce item (2). Finally, by standard Sobolev embedding, the inclusion $\mathcal{H}^1(\mathbb{T}^1) \hookrightarrow L^2(\mathbb{T}^1)$ is compact (this is essentially Rellich-Kondrachov Theorem applied in the case of $\mathbb{T}^1$), hence item (3). $\square$

[4] This term is usually called *distortion* and it measures how much the map deviate from being linear.

Chapter 3

Transversality: a simple example

We present the transversality condition introduced by Tsujii in a simple setting, which is a modification of an example given in [53] (see also [11, Section 11.4]). Let us fix $\varepsilon > 0$ small and set $\mu_\varepsilon = 1 - \varepsilon$. Let $X = \mathbb{T}^1 \times [-1, 1]$, $\lambda \in \mathbb{N}$, $\lambda \geq 2$, setting $Q_i = [\frac{i-1}{\lambda}, \frac{i}{\lambda}] \times [-1, 1]$, for $i \in \{1, \cdots, \lambda\} =: I$, we define the maps

$$F_{i,\varepsilon}(x, z) = (\lambda x, \mu_\varepsilon z + \beta_i x + \gamma_i)$$
$$F_\varepsilon(x, z) = \sum_{i=1}^{\lambda} F_{i,\varepsilon}(x, z) \mathbb{1}_{Q_i}(x, z),$$

where the parameters β_i, γ_i are chosen such that the image of F is well defined. Notice that each $F_{i,\varepsilon}$ is a diffeomorphism and for every $p \in \mathbb{T}^1 \times [-1, 1]$ and $i \in I$,

$$D_p F_{i,\varepsilon} = \begin{pmatrix} \lambda & 0 \\ \beta_i & \mu_\varepsilon \end{pmatrix}.$$

Next we assume that $\lambda\mu_\varepsilon > 1$, which implies $\det DF_\varepsilon > 1$. For $\chi_u > 0$ let us now consider the cone

$$\mathbf{C}_{\chi_u} = \{(\xi, \eta) \in \mathbb{R}^2 : |\eta| \leq \chi_u |\xi|\}.$$

We want to show that there exists χ_u such that $DF_\varepsilon \mathbf{C}_{\chi_u} \subset \mathbf{C}_{\chi_u}$. If we take $(1, u)$ with $|u| \leq \chi_u$, then

$$D_p F_{i,\varepsilon} \begin{pmatrix} 1 \\ u \end{pmatrix} = (\lambda, \beta_i + \mu_\varepsilon u).$$

Hence, if

$$\chi_u \geq \frac{\max_i |\beta_i|}{\lambda - \mu_\varepsilon},$$

then the image of $(1, u)$ belongs to the cone $\mathbf{C}_{\chi_u}$, whereby proving the claim. Henceforth we set $\beta = \max_i |\beta_i|$, $\chi_u = \beta(\lambda - \mu_\varepsilon)^{-1}$ and $\mathbf{C}_u := \mathbf{C}_{\chi_u}$.
We are now ready to introduce the notion of transversality between unstable cones in this context.

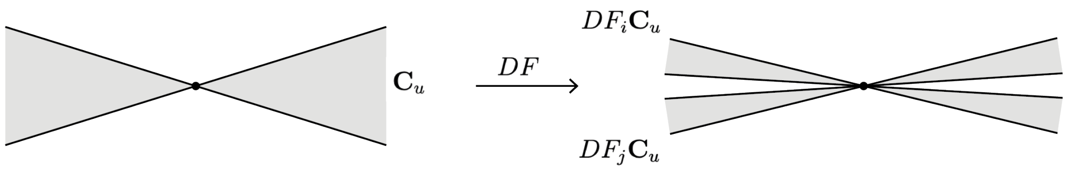

Figure 3.1: Transversality of unstable cones.

Definition 3.0.1. *Given $i, j \in I$ with $i \neq j$, we say that i is transversal to j if*

$$|\beta_i - \beta_j| > 3\mu_\varepsilon \chi_u, \tag{3.0.1}$$

and we write $i \pitchfork j$. Notice in particular that this implies

$$DF_{i,\varepsilon}\mathbf{C}_u \cap DF_{j,\varepsilon}\mathbf{C}_u = \{0\},$$

and we say that the cones are transversal. We write $j \not\pitchfork i$ if condition (3.0.1) *is violated.*

To quantify the number of non-transversal cones we define

$$\mathcal{N}_{F_\varepsilon} := \sup_{i \in I} \sharp\{j : j \not\pitchfork i\}. \tag{3.0.2}$$

3.1 Existence of invariant measures

To study the invariant measures of the system we consider the transfer operator $\mathcal{L}_\varepsilon := \mathcal{L}_{F_\varepsilon}$ associated to the map. In this particular case we can see that, if $u \in L^2(X)$

$$\mathcal{L}u(x) = \sum_{y \in F^{-1}(x)} \frac{1}{\lambda\mu_\varepsilon} u(y) = \sum_{i=1}^{\lambda} \mathcal{L}_{\varepsilon,i} u(y) \tag{3.1.1}$$

where

$$\mathcal{L}_{\varepsilon,i} u(x) = \frac{1}{\lambda\mu_\varepsilon} \left(u \circ F_i^{-1}\right)(x) \mathbb{1}_{F_i(Q_i)}(x).$$

Let us define the following set:

$$\mathcal{B} = \left\{ u \in L^2 : u(x,z) = \sum_{\alpha \in \mathcal{A}} u_\alpha(0, x - \alpha z), \exists u_\alpha \in L^2, \exists \mathcal{A} \text{ finite }, |\alpha| \le \chi_u \right\},$$

namely $u \in \mathcal{B}$ if it is the sum of functions in L^2 which are constants on the line segments of slope α. We are going to prove the following

Lemma 3.1.1. *The space $\mathcal{B}$ is $\mathcal{L}_\varepsilon$-invariant, namely $\mathcal{L}_\varepsilon(\mathcal{B}) \subset \mathcal{B}$. Moreover, if F satisfies the following condition*

$$\nu_\varepsilon := \frac{\mathcal{N}_{F_\varepsilon}}{\lambda \mu_\varepsilon} < 1, \tag{3.1.2}$$

then, for each $n \ge 1$ and $u \in \mathcal{B}$,

$$\|\mathcal{L}_\varepsilon^n u\|_{L^2}^2 \le \nu_\varepsilon^n \|u\|_{L^2}^2 + C_\varepsilon \|u\|_{L^1}, \tag{3.1.3}$$

where $C_\varepsilon = \frac{\lambda^2(\lambda-1)}{\chi_u \mu_\varepsilon (1-\nu_\varepsilon)}$.

We will prove Lemma 3.1.1 in several steps. Notice that if $u = u_\alpha(0, y - \alpha x)$ then

$$\mathcal{L}_{\varepsilon,i} u(x,z) = \frac{1}{\lambda \mu_\varepsilon} u \circ F_i^{-1}(x) \mathbb{1}_{F_i(Q_i)} = u_{\alpha_i}(0, y - \alpha_i x), \tag{3.1.4}$$

where α_i are the slopes of the line segments obtained as images under F_i of the line segments $\{y = \alpha x\}_\alpha$. In other words, by the invariance of $\mathbf{C}_u$, $\mathcal{L}_\varepsilon(\mathcal{B}) \subset \mathcal{B}$. The main characterization of the functions u_α is that, if $u_i = u_{\alpha_i}(0, x - \alpha_i z)$, $i \in \{1,2\}$, for $\alpha_1 \ne \alpha_2$

$$\langle u_1, u_2 \rangle_{L^2} \le \frac{1}{|\alpha_1 - \alpha_2|} \|u_1\|_{L^1} \|u_2\|_{L^1}, \tag{3.1.5}$$

which follows by a simple change of variables in the integrals. Next, by (3.1.1) we write for $u \in \mathcal{B}$

$$\begin{aligned} \|\mathcal{L}_\varepsilon u\|_{L^2}^2 &= \sum_{(i,j) \in I \times I} \langle \mathcal{L}_{\varepsilon,i} u, \mathcal{L}_{\varepsilon,j} u \rangle_{L^2} \\ &= \sum_{(i,j): i \pitchfork j} \langle \mathcal{L}_{\varepsilon,i} u, \mathcal{L}_{\varepsilon,j} u \rangle_{L^2} + \sum_{(i,j): i \not\pitchfork j} \langle \mathcal{L}_{\varepsilon,i} u, \mathcal{L}_{\varepsilon,j} u \rangle_{L^2}. \end{aligned}$$

We are going to estimate the two sums above separately.

$\boxed{i \pitchfork j}$ In this case, if $u = \sum_{\alpha \in \mathcal{A}} u_\alpha$, where u_α is constant along the segments $z = \alpha x$, we have

$$\langle \mathcal{L}_{\varepsilon,i} u, \mathcal{L}_{\varepsilon,j} u \rangle_{L^2} = \sum_{\alpha, \alpha'} \langle \mathcal{L}_{\varepsilon,i} u_\alpha, \mathcal{L}_{\varepsilon,j} u_{\alpha'} \rangle_{L^2}.$$

By (3.1.4) $\mathcal{L}_{\varepsilon,i} u_\alpha$ and $\mathcal{L}_{\varepsilon,j} u_{\alpha'}$ are constant along the segments of slopes $\alpha_i = \lambda^{-1}(\beta_i + \alpha \mu_\varepsilon)$ and $\alpha'_j = \lambda^{-1}(\beta_j + \alpha' \mu_\varepsilon)$. Since $i \pitchfork j$, condition (3.0.1) and the fact that $|\alpha| \le \chi_u$, $|\alpha'| \le \chi_u$ imply

$$|\alpha_i - \alpha'_j| \ge \lambda^{-1} \big| |\beta_i - \beta_j| - |\alpha| \mu_\varepsilon - |\alpha'| \mu_\varepsilon \big| \ge \chi_u \frac{\mu_\varepsilon}{\lambda}. \tag{3.1.6}$$

By the above discussion and by (3.1.5) we conclude that

$$\sum_{\alpha,\alpha'}\langle\mathcal{L}_{\varepsilon,i}u_\alpha,\mathcal{L}_{\varepsilon,j}u_{\alpha'}\rangle_{L^2}\le\sum_{\alpha,\alpha'}\frac{1}{|\alpha_i-\alpha'_j|}\int\mathcal{L}_{\varepsilon,i}u_\alpha\int\mathcal{L}_{\varepsilon,j}u_{\alpha'}$$
$$\le\frac{\lambda}{\chi_u\mu_\varepsilon}\int\sum_\alpha u_\alpha\int\sum_{\alpha'}u_{\alpha'}\le\frac{\lambda}{\chi_u\mu_\varepsilon}\|u\|^2_{L^1}.$$

We then sum over $i \pitchfork j$ and obtain

$$\sum_{(i,j):i\pitchfork j}\langle\mathcal{L}_{\varepsilon,i}u,\mathcal{L}_{\varepsilon,j}u\rangle_{L^2}\le\frac{\lambda^2(\lambda-1)}{\chi_u\mu_\varepsilon}\|u\|^2_{L^1}. \tag{3.1.7}$$

$\boxed{i\not\pitchfork j}$ By the Cauchy-Schwarz inequality

$$\begin{aligned}\sum_{(i,j):i\not\pitchfork j}\langle\mathcal{L}_{\varepsilon,i}u,\mathcal{L}_{\varepsilon,j}u\rangle_{L^2}&\le\sum_{(i,j):i\not\pitchfork j}\frac{\|\mathcal{L}_{\varepsilon,i}u\|^2_{L^2}+\|\mathcal{L}_{\varepsilon,j}u\|^2_{L^2}}{2}\\&\le\mathcal{N}_{F_\varepsilon}\sum_{i\in I}\|\mathcal{L}_{\varepsilon,i}u\|^2_{L^2}.\end{aligned} \tag{3.1.8}$$

It remains to bound the last sum. By the definition of the transfer operator

$$\|\mathcal{L}_{\varepsilon,i}u\|^2_{L^2}=\int_{F_i(Q_i)}\frac{1}{(\lambda\mu_\varepsilon)^2}u^2\circ F_i^{-1}=\int_{Q_i}\frac{1}{\lambda\mu_\varepsilon}u^2,$$

hence

$$\sum_{i=1}^{\lambda}\|\mathcal{L}_{\varepsilon,i}u\|^2_{L^2}=\frac{1}{\lambda\mu_\varepsilon}\sum_{i=1}^{\lambda}\int_{Q_i}u^2=\frac{1}{\lambda\mu_\varepsilon}\|u\|^2_{L^2}. \tag{3.1.9}$$

To sum up, by (3.1.7), (3.1.8) and (3.1.9)

$$\|\mathcal{L}_\varepsilon u\|^2_{L^2}\le\frac{\mathcal{N}_{F_\varepsilon}}{\lambda\mu_\varepsilon}\|u\|^2_{L^2}+\frac{\lambda^2(\lambda-1)}{\chi_u\mu_\varepsilon}\|u\|^2_{L^1}. \tag{3.1.10}$$

We are now ready to prove inequality (3.1.3). Let us set $\nu_\varepsilon=\frac{\mathcal{N}_{F_\varepsilon}}{\lambda\mu_\varepsilon}$ and $A_\varepsilon=\frac{\lambda^2(\lambda-1)}{\chi_u\mu_\varepsilon}$, then iterating inequality (3.1.10)

$$\|\mathcal{L}^n_\varepsilon u\|^2_{L^2}\le\nu^n_\varepsilon\|u\|^2_{L^2}+A_\varepsilon\sum_{k=0}^{n-1}\nu^k_\varepsilon\|u\|^2_{L^1}.$$

As $\nu_\varepsilon<1$ by assumption, we obtain inequality (3.1.3) setting $C_\varepsilon=\frac{A_\varepsilon}{1-\nu_\varepsilon}$. □
An important consequence of Lemma 3.1.1 is the following

Corollary 3.1.2. *There exists an invariant probability measure for F which is absolutely continuous with respect to the Lebesgue measure on X.*

Proof. The proof corresponds essentially to an adaptation of Krylov-Bogoliubov theorem in this setting. Let $dm = dxdz$ be the Lebesgue measure on X, and consider the sequence of measures

$$\mu_n(dm) = \frac{1}{n}\sum_{k=0}^{n-1} F_*^k(dm).$$

Then each μ_n is absolutely continuous with respect to Lebesgue with density

$$h_n = \frac{1}{n}\sum_{k=0}^{n-1}\mathcal{L}^k 1.$$

By Lemma 3.1.1 the sequence $\mathcal{L}^k 1$ is uniformly bounded in L^2, hence according to Banach-Alaoglu theorem, there exists a subsequence h_{n_k} and $h_* \in L^2$ such that

$$\lim_{k\to\infty}\langle h_{n_k}, h\rangle_{L^2} = \langle h_*, h\rangle_{L^2}, \quad \forall h \in L^2.$$

Consequently, the measure $d\mu = h_* dm$ is an accumulation point of μ_{n_k}. We now prove that μ is F-invariant. It is well known that the claim is equivalent to showing that

$$\mu(\phi\circ F) = \mu(\phi), \quad \forall \phi \in \mathcal{C}^0(X).$$

We have

$$\begin{aligned}\mu(\phi\circ F) &= \lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1} F_*^i m(\phi\circ F) = \lim_{k\to\infty}\frac{1}{n_k}\sum_{i=0}^{n_k-1} m\left(\phi\circ F^{i+1}\right)\\ &= \lim_{k\to\infty}\frac{1}{n_k}\left\{\sum_{i=0}^{n_k-1} F_*^i m(\phi) + m\left(\phi\circ F^{n_k}\right) - m(\phi)\right\} = \mu(\phi).\end{aligned}$$

□

The above example is merely explicative, and gives just an idea of the role of the transversality in a simple case. In particular it shows that the quantity $\mathcal{N}_{F_\varepsilon}$ enters directly into inequality (3.1.3) and, if one can check condition (3.1.2) for the given system, we have a quantitative estimate of the essential spectral radius of $\mathcal{L}_\varepsilon$. Furthermore it is important to remark that the Lasota-Yorke inequality in (3.1.3) still holds as $\varepsilon \to 0$. In terms of F this case corresponds to a two dimensional map with two Lyapunov exponents, one is positive (due to the expansion in the horizontal direction) and the other indefinite. In this case F is a simple example of a so called *partially hyperbolic system*. An interesting model related to the above one is given by the following *skew-product*:

$$F_\varepsilon : \mathbb{T}^2 \longrightarrow \mathbb{T}^2 \tag{3.1.11}$$

$$(x,z) \longmapsto F_\varepsilon(x,z) = (f(x), z + \varepsilon\omega(x)) \tag{3.1.12}$$

where f is an expanding map on $\mathbb{T}^1$. The ergodic properties of this kind of model have been extensively studied (varying also the conditions on f and ω), from the qualitative point of view, as in [58], to more quantitative results ([16], [30] [26], [42], [60]), just to mention a few. Especially in [16], [30] and [60] already emerges in this case the key role of quantities similar to (3.0.1). It is then natural wondering if a strategy similar to the one used to prove Lemma 3.1.1 can be developed in the *non-skew product* case. In other words, it is possible to say something about the quantitative statistical properties when f and ω depend on both x and z? Which conditions do they need to satisfy to get information on the spectral properties of the transfer operator associated? These are the kind of questions we are attempting to answer in the rest of this work.

Part II

Vertical Partially Hyperbolic Systems

Chapter 4

The systems and the results

In this section we introduce the class of systems we are interested in, the main assumptions and some definitions necessary to present the results. In this work $\mathbb{T}^2$ and $\mathbb{T}$ represent the quotients $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathbb{R}/\mathbb{Z}$ respectively. For a local diffeomorphism $F : \mathbb{T}^2 \to \mathbb{T}^2$ we define the functions $\mathfrak{m}_F^*, \mathfrak{m}_F : \mathbb{T}^2 \times \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}_+$ as [1]

$$\mathfrak{m}_F(z, v) = \frac{\|D_z F v\|}{\|v\|} \quad ; \quad \mathfrak{m}_F^*(z, v) = \frac{\|(D_z F)^{-1} v\|}{\|v\|}.$$

Notation. *As we would like to apply our results to open sets of maps F, all the constants appearing in the text are really functions of F. We will call a constant* uniform *if it depends continuously only on the $\mathcal{C}^r$ norm of the map F, on $(\lambda_- - \mu_+)^{-1}$, χ_c^{-1}, $(1 - \iota_\star)^{-1}$ and $C_\star$ (see hypothesis (**H1**) for the definition of λ_-, μ_+, χ_c, $\iota_\star$ and $C_\star$) . In order to make the reading more fluid, we will use the notation $f \lesssim g$ to mean that there exists a uniform constant $C_\sharp > 0$, such that $f \leq C_\sharp g$. The values of the constants $C_\sharp$ can change from one occurrence to the next. Moreover, in the following we will use $C_{a,b,\dots}, c_{a,b,\dots}$ to designate constants that depend also on the quantities $a, b, \dots$, but they are uniform when the quantities in the subscripts are fixed, since no confusion can arise we will call such constants uniform as well.*

Note that χ_c, χ_u, which determines the size of the central and unstable cone, respectively, are not uniquely determined by the map. Given our convention, we must keep track of how the constants depend on χ_u^{-1} and we cannot hide such a dependency inside a constant $C_\sharp$. Indeed, in the next sections it will be apparent that it may be convenient to choose χ_u as small as possible while it is convenient to choose χ_c as large as possible.
Finally, to simplify notations, we use $\{a, b, \dots\}^+$ to designate the maximum between the quantities $a, b, \dots$

[1] By $\|\cdot\|$ we mean the Riemannian metric in $\mathbb{T}^2$ induced by the Euclidean norm in $\mathbb{R}^2$.

4.1 Partially hyperbolic systems

Let $r \geq 4$ and $F : \mathbb{T}^2 \mapsto \mathbb{T}^2$ be a surjective $\mathcal{C}^r$ local diffeomorphism. We call F a *partially hyperbolic system* [2] if there exist a continuous splitting, not necessarily invariant, of the tangent bundle into subspaces $\mathcal{T}\mathbb{T}^2 = E^c \oplus E^u$, $\sigma > 1$ and $c > 0$ such that for each $n \in \mathbb{N}$

$$\begin{aligned} &\|DF^n_{|E^u}\| > c\sigma^n \\ &\|DF^n_{|E^c}\| < c\sigma^{-n}\|DF^n_{|E^u}\|. \end{aligned} \tag{4.1.1}$$

Notice that for non-invertible map the unstable direction is not necessarily unique, nor invariant. It is then more convenient to work with cones instead than distributions. Indeed, it is well known (see e.g [36]) that the above conditions are equivalent to the existence of smooth invariant transversal cone fields $\mathbf{C}_u(z), \mathbf{C}_c(z)$, which satisfy conditions equivalent to (4.1.1). To simplify the following arguments we will restrict ourselves to maps without critical points. We can thus assume, without further loss of generality.

(**H0**) for all $p \in \mathbb{T}^2$ we have $\det(D_pF) > 0$.

In addition, to simplify notations, we make the assumption that the cone fields can be chosen constant since this hypothesis applies to all the examples we have in mind. Hence we assume:

(**H1**) There exist $\chi_u, \chi_c \in (0,1)$ and $0 < \mu_- < 1 < \mu_+ < \lambda_- \leq \lambda_+$ such that, setting

$$\begin{aligned} \mathbf{C}_u &:= \{(\xi,\eta) \in \mathcal{T}_z\mathbb{T}^2 : |\eta| \leq \chi_u|\xi|\} \\ \mathbf{C}_c &:= \{(\xi,\eta) \in \mathcal{T}_z\mathbb{T}^2 : |\xi| \leq \chi_c|\eta|\}, \end{aligned} \tag{4.1.2}$$

defining

$$\begin{aligned} \lambda_n^-(z) &:= \inf_{v\in\mathbb{R}^2\setminus\mathbf{C}_c} \mathfrak{m}_{F^n}(z,v) & \lambda_n^+(z) &:= \sup_{v\in\mathbb{R}^2\setminus\mathbf{C}_c} \mathfrak{m}_{F^n}(z,v), \\ \mu_n^-(z) &:= \inf_{v\in\mathbf{C}_c\setminus\{0\}} \mathfrak{m}^*_{F^n}(z,v) & \mu_n^+(z) &:= \sup_{v\in\mathbf{C}_c\setminus\{0\}} \mathfrak{m}^*_{F^n}(z,v), \end{aligned} \tag{4.1.3}$$

and letting $\lambda_n^- = \inf_z \lambda_n^-(z)$ and $\lambda_n^+ = \sup_z \lambda_n^+(z)$ we have the following:
There exist uniform $C_\star \geq 1$ and $\iota_\star \in (0,1)$ such that, for all $z \in \mathbb{T}^2$ and $n \in \mathbb{N}$,[3]

$$D_zF\mathbf{C}_u \subset \{(\xi,\eta) : |\eta| \leq \iota_\star\chi_u|\xi|\} \Subset \mathbf{C}_u; \quad D_zF^{-1}\mathbf{C}_c \Subset \mathbf{C}_c, \tag{4.1.4}$$

$$C_\star^{-1}\mu_-^n \leq \mu_n^-(z) \leq \mu_n^+(z) \leq C_\star\mu_+^n\ ; \quad C_\star^{-1}\lambda_-^n \leq \lambda_n^- \leq \lambda_n^+ \leq C_\star\lambda_+^n \tag{4.1.5}$$

From now on we set $\mu := \{\mu_+, \mu_-^{-1}\}^+ > 1$. Note that the above conditions imply, in particular, $\det(DF) \neq 0$.

[2] In the present case the term *partially expanding* would be more appropriate, as there is only an expanding direction which is dominant.

[3] $A \Subset B$ means $\overline{A} \subset \mathrm{int}(B) \cup \{0\}$.

(**H2**) Let Υ be the family of closed curve $\gamma \in \mathcal{C}^r(\mathbb{T}, \mathbb{T}^2)$ such that [4]

c0) $\gamma' \neq 0$,

c1) γ has homotopy class $(0,1)$,

c2) $\gamma'(t) \in \mathbf{C}_c$, for each $t \in \mathbb{T}$,

then $F^{-1}(\gamma)$ is the disjoint union of closed curves and $\Upsilon \supset F^{-1}(\Upsilon)$.

(**H3**) Let

$$\tilde{\zeta}_r := \frac{1}{3}\left[(r+1)!(6r-1)+1\right]. \tag{4.1.6}$$

Then we say that F satisfies the *pinching* condition if

$$\mu^{\tilde{\zeta}_r} < \lambda_-. \tag{4.1.7}$$

Remark 4.1.1. *Notice that condition* (4.1.7) *implies in particular that* $\mu < \lambda_-$. *The presence of the factorials in* (4.1.6) *is probably not optimal. This is a condition we did not try to optimise since it is irrelevant for our main application in which* μ *is very close to one.*

A partially hyperbolic system satisfying (4.1.7) will be called *strongly dominated.*

Remark 4.1.2. *Note that, since* F *is a local diffeomorphism, then it can be lifted to a diffeomorphism* $\mathbb{F}$ *of* $\mathbb{R}^2$ *with the projection* π *map being* $\mod 1$, *so that* $\pi(0,0) = 0$. *Then we can define* $\mathbb{G}(x,\theta) = \mathbb{F}(x,\theta) - (0,\theta)$ *and write* $F \circ \pi(x,\theta) = \pi(\mathbb{G}(x,\theta) + (0,\theta))$. *Thus in the following, with a slight abuse of notation, we will often confuse the map with his covering and write*

$$F(x,\theta) = (f(x,\theta), \theta + \omega(x,\theta)). \tag{4.1.8}$$

*In addition, note that if the map satisfies condition (*__H2__*) then for each* $x \in \mathbb{R}^2$ *the curve* $\gamma_x(t) = (x,t)$, $t \in \mathbb{T}$ *has a preimage* $\nu \in \Upsilon$ *homotopic to the curve* $\bar{\gamma}_p(t) = p + (0,t)$, $p \in \nu$, $F(p) = (x,0)$. *This implies that* $F(\bar{\gamma}_p(t))$ *is a curve homotopic to* γ_x. *Thus for each* $(x,\theta) \in \mathbb{R}^2$ *the lift has the property* $\mathbb{F}(x,\theta+1) = \mathbb{F}(x,\theta) + (0,1)$, *which implies that* ω *lifts to a periodic function in the second variable.*

In the following we will need some quantitative information on the Lipschitz constant of the graphs associated to "unstable manifolds." To simplify matters, we prove the needed results in Lemma D.0.1. We require then that our maps satisfy the hypotheses of such a Lemma. However, be aware that such hypotheses are not optimal and the following condition is used only in Lemma D.0.1, hence it becomes superfluous if in a given system one can prove Lemma D.0.1 independently.

(**H4**) With the notation (4.1.8) we require, for each $p \in \mathbb{T}^2$,

$$\partial_x f(p) > \{2(1 + \|\partial_x \omega\|_\infty), |\partial_\theta f(p)|\}^+ .$$

[4] As usual we consider equivalent two curves that differ only by a $\mathcal{C}^r$ reparametrization. In the following we will mostly use curves that are parametrized by vertical length.

Definition 4.1.3. *We call a map F a strongly dominated vertical partially hyperbolic system (SVPH for simplicity) if it satisfies assumptions (**H0**),.., (**H4**).*

Remark 4.1.4. *Note that if F satisfies (**H1**) and (**H2**), then so does F^n, $n \in \mathbb{N}$. Thus it may be convenient to consider F^n, instead of F, to check (**H3**) and (**H4**).*

From now on we will write a SVPH in the form (4.1.8) when convenient.

4.2 Transversality of unstable cones

In [51] Tsujii introduces the following notion of transversality.

Definition 4.2.1. *Given $n \in \mathbb{N}$, $y \in \mathbb{T}^2$ and $z_1, z_2 \in F^{-n}(y)$, we say that z_1 is transversal to z_2 (at time n) if $D_{z_1}F^n\mathbf{C}_u \cap D_{z_2}F^n\mathbf{C}_u = \{0\}$, and we write $z_1 \pitchfork z_2$.*

For each $y \in \mathbb{T}^2$ and $z_1 \in F^{-n}(y)$, we define

$$\mathcal{N}_F(n, y, z_1) := \sum_{\substack{z_2 \not\pitchfork z_1 \\ z_2 \in F^{-n}(y)}} |\det D_{z_2}F^n|^{-1} \tag{4.2.1}$$

and set $\mathcal{N}_F(n) = \sup_{y\in\mathbb{T}^2} \sup_{z_1\in F^{-n}(y)} \mathcal{N}_F(n, y, z_1)$.

Remark 4.2.2. *Note that if all the preimages are non-transversal, then the sum in (4.2.1) corresponds to the classical transfer operator applied to one ($\mathcal{L}_F 1$).*

In essence, $\mathcal{L}_F 1 - \mathcal{N}_F(n)$ provides a quantitative version of the notion of accessibility in our systems.

As $\mathcal{N}_F$ is difficult to estimate we also introduce a related quantity, inspired by [51]. Given $y \in \mathbb{T}^2$ and a line L in $\mathbb{R}^2$ passing through the origin, define

$$\widetilde{\mathcal{N}}_F(n, y, L) := \sum_{\substack{z\in F^{-n}(y) \\ DF^n(z)\mathbf{C}_u \supset L}} |\det DF^n(z)|^{-1}. \tag{4.2.2}$$

As before we set $\widetilde{\mathcal{N}}_F(n) = \sup_{y\in\mathbb{T}^2} \sup_L \widetilde{\mathcal{N}}_F(n, y, L)$. Section 7.2 provides the properties of $\widetilde{\mathcal{N}}_F$ and Lemma 7.2.3 explains the relation between $\mathcal{N}_F$ and $\widetilde{\mathcal{N}}_F$.

4.3 Result for SVPH

A *physical measure* is an F-invariant probability measure ν such that the set

$$B(\nu) := \left\{ p \in \mathbb{T}^2 : \frac{1}{n}\sum_{k=0}^{n-1} \delta_{F^k(p)} \to \nu \quad \text{weakly as} \quad n \to \infty \right\}$$

has positive Lebesgue measure. One way to obtain information on the physical measures of the system is to study the spectral properties of the Transfer operator.

Definition 4.3.1. *Given a map $F : \mathbb{T}^2 \to \mathbb{T}^2$, we define $\mathcal{L}_F : L^1(\mathbb{T}^2) \to L^1(\mathbb{T}^2)$, the transfer operator associated to F, as*

$$\mathcal{L}_F u(z) = \sum_{y \in F^{-1}(z)} \frac{u(y)}{|\det(D_y F)|}. \tag{4.3.1}$$

Iterating (4.3.1) yields

$$\mathcal{L}_F^n u(z) = \sum_{y \in F^{-n}(z)} \frac{u(y)}{|\det(D_y F^n)|}, \quad n \in \mathbb{N}. \tag{4.3.2}$$

It is a well known fact that $\|\mathcal{L}_F u\|_{L^1} \leq \|u\|_{L^1}$.
For each integer $1 \leq s \leq r-1$ we define[5]

$$\begin{aligned}
&\alpha = \frac{\log(\lambda_- \mu^{-2})}{\log(\lambda_+)} \\
&\alpha_s := 2(2+s-\alpha) \quad ; \quad \beta_s := 2(s+2) \quad ; \quad \zeta_s := \frac{1}{3} + (s+2)! \left(2s + \frac{5}{6}\right).
\end{aligned} \tag{4.3.3}$$

We are now ready to state the main result for SVPH, whose proof is given in Section 9.

Theorem 4.3.2. *Let $F \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$ be SVPH, and let $\alpha, \alpha_s, \beta_s, \zeta_s$, as in* (4.3.3). *We assume that there exist $n_1 \in \mathbb{N}$ and $\nu_s < 1$ such that, for some $1 \leq s \leq r-3$,*

$$\left\{\mu^{\zeta_s} \lambda_-^{-\frac{1}{2}}, \sqrt{\widetilde{\mathcal{N}}_F(\lceil \alpha n_1 \rceil) \mu^{\alpha_s n_1 + \beta_s \overline{m}_{\chi_u}}}\right\}^+ < \nu_s < 1, \tag{4.3.4}$$

where $\overline{m}_{\chi_u}$ is defined in (8.1.24). *Then there exists Banach spaces $\mathcal{B}_{s,*}$, $\mathcal{C}^{r-1}(\mathbb{T}^2) \subset \mathcal{B}_{s,*} \subset \mathcal{H}^s(\mathbb{T}^2)$ such that $\mathcal{L}_F(\mathcal{B}_{s,*}) \subset \mathcal{B}_{s,*}$. The restriction of $\mathcal{L}_F$ to $\mathcal{B}_{s,*}$ is a bounded quasi-compact operator, with spectral radius one and essential spectral radius smaller than ν_s.*

Remark 4.3.3. *By standard arguments, Theorem 4.3.2 has the following consequences: there exist finitely many physical measures absolutely continuous with densities in the Hilbert space $\mathcal{H}^s$, for some $1 \leq s \leq r-3$. Moreover, for each mixing physical measure we have exponential decay of correlations for Hölder observables.*

Remark 4.3.4. *By the definition of $\widetilde{\mathcal{N}}_F$, it can be noticed that, under the assumption (***H3***), condition* (4.3.4) *is automatically satisfied if $\widetilde{\mathcal{N}}_F$ grows subexponentially for n large. According to [53], this latter fact holds generically for partially hyperbolic systems in two dimensions. In this sense, the result and all the consequences of Theorem 4.3.2 hold generically. For more details see Remark 8.0.1.*

[5]Note that in (4.3.3) $0 < \alpha < 1$, thanks to hypothesis (4.1.6).

In particular, Theorem 4.3.2 implies that the map has finitely many physical measures and that if it is topologically mixing, then it mixes exponentially fast for all Hölder observables. Note that the condition involves only a finite power of the map and it is, at least in principle, checkable for a given map. Of course checking it may be quite laborious and may entail some computer assisted strategy. It is then interesting to consider less general models in which the previous condition can be explicitly verified.

4.4 A general class of models

It is natural to ask when a map of the form (4.1.8) satisfies (**H0**),.., (**H4**). Here we provide checkable conditions implying (**H0**),.., (**H4**).

Lemma 4.4.1. *Let* $\lambda := \inf_{\mathbb{T}^2} \partial_x f, \Lambda := \sup_{\mathbb{T}^2} \partial_x f$ *and suppose that:*

1. $\partial_x f(p) > \{2(1 + \|\partial_x \omega\|_\infty), |\partial_\theta f(p)|\}^+ \quad \forall p \in \mathbb{T}^2$,
2. $\|\partial_x \omega\|_\infty + \|\partial_\theta \omega\|_\infty < 1$,
3. $\|\partial_\theta \omega\|_\infty < \frac{1+\|\partial_x \omega\|_\infty}{\lambda - 1}$
4. $1 + \|\partial_\theta f\|_\infty + \|\partial_\theta \omega\|_\infty + \|\partial_x \omega\|_\infty < \lambda$,
5. $\|\partial_\theta f\|_\infty < \frac{1}{2}\left(-1 + \sqrt{1 + 2\lambda^2 \Lambda^{-1}}\right)$,
6. $\chi_c \|\partial_x \omega\|_\infty + \|\partial_\theta \omega\|_\infty < \frac{\ln \lambda}{2\,\tilde{\zeta}_r}$,

with $\tilde{\zeta}_r$ *as in* (4.1.6). *Then* F *satisfies assumptions (*$\mathbf{H0}$*),..,(*$\mathbf{H4}$*) with* χ_u *given by* (4.4.5), (4.4.13), χ_c *given by* (4.4.7) *and*

$$\mu := \{(1 - \chi_c \|\partial_x \omega\|_\infty - \|\partial_\theta \omega\|_\infty)^{-1}, e^{\chi_c \|\partial_x \omega\|_\infty + \|\partial_\theta \omega\|_\infty)}\}^+. \tag{4.4.1}$$

Proof. To start, note that (1) coincides with **(H4)**, which implies in particular that $\lambda > 2$. We have thus to prove only **(H0)** up to **(H3)**.

We start with **(H0)**. First we show that $\partial_x f(p) > [\partial_\theta f \partial_x \omega - \partial_x f \partial_\theta \omega](p)$, for each $p \in \mathbb{T}^2$. The latter, by (2) and (3), is implied by $\lambda(1 - |\partial_\theta \omega|) > \lambda - 1 - \|\partial_\theta \omega\|_\infty - \|\partial_x \omega\|_\infty$ which, in turn, is implied by (3).
Next we prove (**H1**). Following [20] we start by proving that $D_p F(\mathbf{C}_u) \Subset \mathbf{C}_u$ and $D_p F^{-1}(\mathbf{C}_c) \Subset \mathbf{C}_c$. We consider a vector $(1, u) \in \mathbf{C}_u$ and we write a formula for the unstable slope field

$$D_p F(1, u) = (\partial_x f + u \partial_\theta f)(1, \Xi(p, u)), \qquad \Xi(p, u) = \frac{\partial_x \omega(p) + u \partial_\theta \omega(p) + u}{\partial_x f(p) + u \partial_\theta f(p)}. \tag{4.4.2}$$

Notice that

$$\frac{d}{du}\Xi(p, \cdot) = \frac{\partial_x f + (\partial_\theta \omega \partial_x f - \partial_\theta f \partial_x \omega)}{(\partial_x f + u \partial_\theta f)^2} = \frac{\det DF(x, \theta)}{(\partial_x f + u \partial_\theta f)^2} > 0, \tag{4.4.3}$$

since $\det DF > 0$ by (1). Hence, checking the invariance of $\mathbf{C}_u$ under DF is equivalent to showing that, for each $p \in \mathbb{T}^2$, $|\Xi(p, \pm\chi_u)| \le \chi_u$. That is

$$\|\partial_\theta f\|_\infty \chi_u^2 - (\lambda - \|\partial_\theta \omega\|_\infty - 1)\,\chi_u + \|\partial_x \omega\| \le 0. \tag{4.4.4}$$

Setting $\phi = \lambda - \|\partial_\theta \omega\|_\infty - 1$, inequality (4.4.4) has positive solutions since $\phi > 0$ by (4), which also implies

$$\phi^2 - 4\|\partial_\theta f\|_\infty \|\partial_x \omega\|_\infty \ge (\|\partial_\theta f\|_\infty - \|\partial_x \omega\|_\infty)^2 > 0.$$

Setting $\Phi_\pm = \phi \pm \sqrt{\phi^2 - 4\|\partial_\theta f\|_\infty \|\partial_x \omega\|_\infty}$, we can then choose

$$\chi_u \in \left(\frac{\Phi_-}{2\|\partial_\theta f\|_\infty}, 1 \right). \tag{4.4.5}$$

Note that the interval it is not empty due to (4).
On the other hand, if $(c, 1) \in \mathbf{C}_c$ we consider the center slope field

$$\Xi^-(p, c) = \frac{(1 + \partial_\theta \omega(p))\, c - \partial_\theta f(p)}{\partial_x f(p) - \partial_x \omega(p) c}, \tag{4.4.6}$$

and by an analogous computation we obtain $|\Xi^-(p, \pm\chi_c)| \le \chi_c$ if

$$\chi_c \in \left(\frac{\Phi_-}{2\|\partial_x \omega\|_\infty}, 1 \right). \tag{4.4.7}$$

Again, the interval it is not empty due to (4), we have thus proved (4.1.4).
Next, by the invariance of the cones we can define real quantities λ_n, μ_n, u_n and c_n such that, for each $p \in \mathbb{T}^2$,[6]

$$D_pF^n(1, 0) = \lambda_n(p)\,(1, u_n(p)) \;\; ; \quad D_pF^n\,(c_n(p), 1) = \mu_n(p)(0, 1),$$

with $\|u_n\|_\infty \le \chi_u$, $\|c_n\|_\infty \le \chi_c$. Moreover, by definition

$$D_pF\,(c_n(p), 1) = \frac{\mu_n(p)}{\mu_{n-1}\,(F(p))}\,(c_{n-1}\,(F(p))\,, 1)\,,$$

from which it follows, by (4.1.8),

$$\mu_n(p) = \mu_{n-1}\,(F(p))(1 + \partial_\theta \omega(p) + c_n(p)\partial_x \omega(p)).$$

Since $\|c_n\|_\infty \le \chi_c$, setting $b := \|\partial_\theta \omega\|_\infty + \chi_c \|\partial_x \omega\|_\infty$, we have

$$(1 - b)^n \le \mu_n(p) \le (1 + b)^n. \tag{4.4.8}$$

Note in particular that, by (4.4.8), we can make the choice (4.4.1) which immediately implies (**H3**) by (6). Similarly,

$$\begin{aligned}\lambda_n(p) &= \lambda_{n-1}(F(p))(\partial_x f(p) + \partial_\theta f(p) u_n(p)) \\ &= \prod_{k=0}^{n-1} \partial_x f(F^k p) \left(\partial_x f(F^k p) + \frac{\partial_\theta f(F^k)}{\partial_x f(F^k p)} u_{n-k}(F^k p) \right),\end{aligned}$$

[6] Note that the definition of λ_n differs from the one of $\lambda_n^\pm$ in (4.1.5), since we are considering iteration of vectors inside the unstable cone. Nevertheless, they are related since there exists an integer m such that $F^m(\mathbb{R}^2 \setminus \mathbf{C}_c) \Subset \mathbf{C}_u$.

which, setting $a := \chi_u \| \frac{\partial_\theta f}{\partial_x f} \|_\infty$, implies

$$(1-a)^n \prod_{k=0}^{n-1} \partial_x f(F^k(p)) \le \lambda_n(p) \le (1+a)^n \prod_{k=0}^{n-1} \partial_x f(F^k(p)), \tag{4.4.9}$$

which yields (4.1.5) with $C_\star = 1$,

$$\lambda_+ = (1+a)\lambda \quad \text{and} \quad \lambda_- = (1-a)\lambda, \tag{4.4.10}$$

since, by the definition of χ_u in (4.4.5), we can check that $\lambda_- > 1$. By (4.4.8) and (4.4.9) we have, for each $n \in \mathbb{N}$ and $p \in \mathbb{T}^2$,

$$\frac{\|D_p F^n(c_n, 1)\|}{\|D_p F^n(1,0)\|} = \frac{|\mu_n(p)|}{|\lambda_n(p)|\sqrt{1+u_n^2}} \le \frac{(1+b)^n}{(1-a)^n \lambda^n}. \tag{4.4.11}$$

To conclude, we need to check that $\frac{(1+b)}{(1-a)\lambda} < 1$, form which we deduce **(H1)**. This is implied by

$$1 + \|\partial_\theta \omega\|_\infty + \|\partial_x \omega\|_\infty + \|\partial_\theta f\|_\infty < \lambda$$

which correspond to equation (4).

It remains to prove (**H2**). Since $\lambda > 2$, F has rank at least two at each point, hence it is a covering map and each point has the same number of preimages, says d. Let then $\gamma : [0,1] \to \mathbb{T}^2$ be a smooth closed curve $\gamma(t) = (c(t), t)$ such that $\gamma' \in \mathbf{C}_c$ with homotopy class $(0,1)$. If $p = (x, \theta) \in \gamma(t)$ then $F^{-1}(p) = \{q_1, \cdots, q_d\}$. Note that, by the implicit function theorem, locally $F^{-1}\gamma$ is a curve, also, due to the above discussion, it belongs to the central cone. If we call η the local curve in $F^{-1}\gamma$ such that $\eta(0) = q_i$ we can extend it uniquely to a curve $\nu : [0,1] \to \mathbb{T}^2$. We will prove that $\nu(1) = q_i = \nu(0)$. In turn this implies that $F^{-1}\gamma$ is the union of d closed curves $\nu_1, \cdots, \nu_d$ with $\nu_i' \in \mathbf{C}_c$, each one with homotopy class $(0,1)$, by the lifting property of covering maps (see [34, Proposition 1.30]). We argue by contradiction: assume that $\nu(1) = q_j \ne q_i$. Let $q_k = (x_k, \theta_k)$, $k \in \{1, \ldots d\}$, then

$$\theta_i + \omega(x_i, \theta_i) = \theta_j + \omega(x_j, \theta_j)$$

implies

$$|\theta_i - \theta_j| \le \frac{\|\partial_x \omega\|_\infty}{1 - \|\partial_\theta \omega\|_\infty} |x_i - x_j|. \tag{4.4.12}$$

Hence the segment joining q_i and q_j belong to the unstable cone if

$$\chi_u \ge \frac{\|\partial_x \omega\|_\infty}{1 - \|\partial_\theta \omega\|_\infty} \tag{4.4.13}$$

which is possible since (2) implies that this condition is compatible with (4.4.5). It follows that the image of the segment $\ell = \{tq_i + (1-t)q_j\}$ is an unstable curve and hence it cannot join p to itself without wrapping around the torus. In particular, if $q_i \ne q_j$, then the horizontal length of $F(\ell)$ must be larger than one. Then, setting $\delta = |x_i - x_j|$,

$$1 \le \int_0^1 \left|\langle e_1, D_{\ell(t)} F \ell'(t)\rangle\right| \le \|\partial_x f\|_\infty \left(1 + \chi_u \frac{\|\partial_\theta f\|_\infty}{\|\partial_x f\|_\infty}\right) |x_i - x_j| \le (1+a)\Lambda\delta. \tag{4.4.14}$$

To conclude we must show that ν cannot move horizontally by δ whereby obtaining the wanted contradiction. Let $\nu(t) = (\alpha(t), \beta(t))$, then

$$\begin{pmatrix} c'(t) \\ 1 \end{pmatrix} = \gamma'(t) = DF\nu' = \begin{pmatrix} \alpha'\partial_x f + \beta'\partial_\theta f \\ \alpha'\partial_x\omega + (1+\partial_\theta\omega)\beta' \end{pmatrix}.$$

Since we know that $|c'| \le \chi_c$ and $|\alpha'| \le \chi_c|\beta'|$ we have

$$|\beta'| \le (1 - \chi_c\|\partial_x\omega\|_\infty - \|\partial_\theta\omega\|_\infty)^{-1}$$
$$|\alpha'| \le \frac{\chi_c}{\lambda} + \frac{\|\partial_\theta f\|_\infty}{\lambda(1 - \chi_c\|\partial_x\omega\|_\infty - \|\partial_\theta\omega\|_\infty)}.$$

I follows that it must be

$$\frac{1}{(1+a)\Lambda} \le \delta \le \int_0^1 |\alpha'(t)|dt \le \frac{\chi_c}{\lambda} + \frac{\|\partial_\theta f\|_\infty}{\lambda(1 - \chi_c\|\partial_x\omega\|_\infty - \|\partial_\theta\omega\|_\infty)}.$$

We thus have a contradiction if we can choose χ_c such that

$$\left(1 + \frac{\|\partial_\theta f\|_\infty}{\lambda}\right)\Lambda\left[\frac{\chi_c}{\lambda} + \frac{\|\partial_\theta f\|_\infty}{\lambda(1 - \|\partial_x\omega\|_\infty - \|\partial_\theta\omega\|_\infty)}\right] < 1$$

which, by (4.4.7), is possible only if

$$\frac{\Phi_-}{2\|\partial_x\omega\|_\infty} < \left(1 + \frac{\|\partial_\theta f\|_\infty}{\lambda}\right)^{-1}\frac{\lambda}{\Lambda} - \frac{\|\partial_\theta f\|_\infty}{1 - \|\partial_x\omega\|_\infty - \|\partial_\theta\omega\|_\infty} =: A.$$

Note that if $A \ge 1$, then the inequality is trivially satisfied. We must consider then only the case $A < 1$. A direct computation shows that the above inequality is implied by

$$\|\partial_\theta f\|_\infty < A\left[\phi - A\|\partial_x\omega\|_\infty\right] = A\left[\lambda - \|\partial_\theta\omega\|_\infty - 1 - A\|\partial_x\omega\|_\infty\right] \tag{4.4.15}$$

Let us set for simplicity $\varpi := \|\partial_\theta\omega\|_\infty + \|\partial_x\omega\|_\infty$. Since $A < 1$ the above equation is in turn implied by the following inequality

$$\|\partial_\theta f\|_\infty < \left[\left(1 + \frac{\|\partial_\theta f\|_\infty}{\lambda}\right)^{-1}\frac{\lambda}{\Lambda} - \frac{\|\partial_\theta f\|_\infty}{1 - \varpi}\right](\lambda - (1 + \varpi)). \tag{4.4.16}$$

By elementary algebra (4.4.16) is equivalent to

$$\|\partial_\theta f\|_\infty(\|\partial_\theta f\|_\infty + 1) < \frac{\lambda^2}{\Lambda}\left(1 - \frac{1}{\lambda + \varpi}\right). \tag{4.4.17}$$

Since $\lambda > 2$, (4.4.17) is implied by $\|\partial_\theta f\|_\infty(\|\partial_\theta f\|_\infty + 1) < \frac{1}{2}\lambda^2\Lambda^{-1}$, which is true if
$\|\partial_\theta f\|_\infty < \frac{1}{2}\left(-1 + \sqrt{1 + 2\lambda^2\Lambda^{-1}}\right)$. Hence the conclusion by condition (5). □

We have thus explicit conditions that imply (**H0**),..,(**H4**). It remains to investigate how to check condition (4.3.4), which is, by far, the hardest to verify. One can directly investigate (4.3.4) in any concrete example (possibly via a computer assisted strategy), however to verify it for an explicit open set of maps we further restrict the class of systems under consideration. Note however that the endomorphisms we are going to consider still include a large class of physically relevant systems.

4.5 Fast slow systems

We consider a class of systems given by the following model introduced in [22] (and inspired by the more physically relevant model introduced in [27]). Let $F_0(x, \theta) = (f(x, \theta), \theta)$ be $\mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$, for $r \geq 4$, such that $\inf_{(x,\theta)\in\mathbb{T}^2} \partial_x f(x, \theta) \geq \lambda > 2$. For any $\omega \in \mathcal{C}^r(\mathbb{R}^2, \mathbb{R})$, periodic of period one, and $\varepsilon > 0$, we define

$$F_\varepsilon(x, \theta) = (f(x, \theta), \theta + \varepsilon\omega(x, \theta)). \tag{4.5.1}$$

Before stating our result we need the following definition.

Definition 4.5.1. *The function $\omega \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{R})$ is called x-constant with respect to F_0 if there exist $\theta \in \mathbb{T}$, $\Phi_\theta \in \mathcal{C}^0(\mathbb{T}, \mathbb{R})$ and a constant $c \in \mathbb{R}$ such that, for each $x \in \mathbb{T}$,*

$$\omega(x, \theta) = \Phi_\theta(f(x, \theta)) - \Phi_\theta(x) + c.$$

Note that it is fairly easy to check that a function is not x-constant by looking at the periodic orbits. Hence, the condition that ω is not x-constant is considerably easier to check than (4.3.4).
The following theorem is proven in section 10.3.

Theorem 4.5.2. *Under condition* (5) *of Lemma 4.4.1, there exists ε_* such that the map F_ε is SVPH for any $\varepsilon \in (0, \varepsilon_*)$. In addition, if ω is not x-constant, then, for each $s \geq 1$, there exists $\nu_s \in (0, 1)$ such that, for each $\varepsilon \in (0, \varepsilon_*)$, the transfer operator $\mathcal{L}_{F_\varepsilon}$ is quasi compact on the spaces $\mathcal{B}_{s,*}$, with spectral radius one and essential spectral radius bounded by ν_s.*

The above result imply the following Theorem which is proved in section 10.5.

Theorem 4.5.3. *In the hypothesis of Theorem 4.5.2, the eigenvectors of $\mathcal{L}_{F_\varepsilon}$ associated to the eigenvalue one are the physical measures of F_ε. Moreover, we have the decomposition $\mathcal{L}_{F_\varepsilon} = \Pi_0 + Q$ where $\Pi_0 Q = Q\Pi_0 = 0$, Π_0 is the finite rank projector on the eigenspace associated to the eigenvalue* 1, *and Q has spectral radius strictly smaller than one. Moreover, let $h_*(\cdot, \theta)$ be the unique invariant probability density of $f(\cdot, \theta)$ and consider the operator $P : L^1 \to (\mathcal{C}^1)'$ defined by*

$$\int_{\mathbb{T}^2} \varphi(x, \theta)[Ph](dx, d\theta) = \int_{\mathbb{T}^1} dx \varphi(x, \theta) h_*(x, \theta) \int_{\mathbb{T}^1} dy h(y, \theta), \quad \forall \varphi \in \mathcal{C}^1.$$

Then

$$\|\Pi_0 - P\Pi_0\|_{L^1 \to (\mathcal{C}^1)'} \leq C_\sharp \varepsilon [\ln \varepsilon^{-1}]^2. \tag{4.5.2}$$

Finally, for each $\tau > 0$ let h_ν be the eigenfunction associated to the eigenvalue ν with $|\nu| \geq e^{-\varepsilon^\tau}$. Then we have

$$\left\| h_\nu - h_* \int h_\nu(y, \cdot) dy \right\|_{(\mathcal{C}^1)'} \leq C_\sharp \varepsilon (\ln \varepsilon^{-1})^2. \tag{4.5.3}$$

The above Theorem is much stronger than the results in [53] (where only the existence of the physical measure is discussed and the results hold only generically) or [14, 1] (where the existence of SRB measures is obtained under an additional condition on the contraction or the expansion in the center foliation, even though for more general systems). However, the papers [21, 22] show that, using the standard pair technology and investigating limit theorems, it is possible to obtain considerably more detailed information on the system. Unfortunately, on the one hand the arguments in [21] are rather involved and, on the other hand, the conclusions concerning the physical measure in [20] hold only for mostly contracting systems (contrary to the present ones). It is then very important to investigate if the present strategy can provide further information.

First of all we have an explicit bound on the regularity of the eigenfunctions. The reader can find the proof of the following theorem at the end of section 10.4.

Theorem 4.5.4. *If ω is not $x-$constant, then there exist $c_\star > 0$ such that, for each $\varepsilon > 0$ small enough, and $\mathbb{r} \in (0,1)$, if $\nu \in \sigma_{\mathcal{B}_{1,*}}(\mathcal{L}_{F_\varepsilon}) \cap \{z \in \mathbb{C} : 1-\mathbb{r}c_\star[\ln \varepsilon^{-1}]^{-1} \leq |z|\}$, and u is an eigenvector with eigenvalue ν with $\|u\|_{\mathcal{B}_0} = 1$,*[7] *then for all $\alpha > \frac{11}{2}$,*

$$\|u\|_{\mathcal{H}^1} \leq C_\alpha \varepsilon^{-(1+\mathbb{r})\alpha}.$$

Remark 4.5.5. *It is not clear if the above Theorem is sharp. Certainly some form of blow-up is inevitable. For example: let $f_\theta(\cdot) = f(x,\theta)$ and call $h_*(\cdot,\theta)$ the unique invariant probability density of f_θ. Let $\bar\omega(\theta) = \int_{\mathbb{T}} \omega(x,\theta)h_*(x,\theta)dx$. If $\bar\omega$ has non degenerate zeroes $\{\theta_i\}_{i=1}^N$ such that $\bar\omega'(\theta_i) < 0$, then [23] (see also the discussion below below) implies that there must exist an eigenfunction u essentially concentrated in the $\sqrt{\varepsilon}$ neighborhood of each θ_i. This implies that $\|u\|_{\mathcal{H}^1} \geq C_\sharp \varepsilon^{-\frac{1}{4}}$. However, there is a large gap between such a lower bound and the upper bound provided by Theorem 4.5.4. In particular, much more information on the spectrum could be obtained if one could establish an upper bound of the type $\varepsilon^{-\beta}$ with $\beta < 1$.*

Finally, in the setting of Remark 4.5.5, let $\widehat{P} : L^1 \to (\mathcal{C}^1)'$ be the finite rank operator defined by: for all $\varphi \in \mathcal{C}^1$

$$\int_{\mathbb{T}^2} \varphi(x,\theta)[\widehat{P}h](dx,d\theta) = \sum_j \int_{\mathbb{T}^1} dx\varphi(x,\theta_j)h_*(x,\theta_j) \int_{\mathbb{T}^1\times U_j} dy\, ds h(y,s),$$

where U_j is the basin of attraction of the stable equilibrium point $\{\theta_j\}$ of the averaged dynamics

$$\begin{aligned} \dot{\bar\theta} &= \bar\omega(\bar\theta) \\ \bar\theta(0,\theta) &= \theta. \end{aligned} \tag{4.5.4}$$

Then, an immediate consequence of Theorem 4.5.2 and [23, Proposition 4] is that the eigenfunctions h for the eigenvalue 1 satisfy, for $\gamma \in (0, \frac{1}{4})$,

$$\|h - \widehat{P}h\|_{L^1\to(\mathcal{C}^1)'} \leq \left(C_\sharp \varepsilon^{1/2-2\gamma} + C_\sharp \varepsilon \ln \varepsilon^{-1}\right). \tag{4.5.5}$$

[7] See Section 6 for the definition of the space $\mathcal{B}_0$.

Remark 4.5.6. *Note that the results of [23] are conditional to the existence of the physical measure which has been previously proven only for the generic case [51] (and hence may not apply to the present concrete situation) or in the case in which the central Lyapunov exponent is negative, see [20]. On the contrary here the existence of the physical measures is ensured by Theorems 4.5.2, 4.5.3, and the central Lyapunov exponent might be slightly positive. This leaves open the very exciting possibility to obtain the results in [21] using a simplified argument which relies on some improved version of the present results.*

Chapter 5

Preliminary Estimates

In this Chapter we provide several basic definitions and we prove many estimates that will be extensively used in the following..

5.1 C^r-norm

Since we will need to work with high order derivatives, it is convenient to choose a norm $\|\cdot\|_{\mathcal{C}^r}$ equivalent to the standard one, which ensures our spaces to be Banach Algebras. We thus define the weighted norm in $\mathcal{C}^r(\mathbb{T}^2, \mathcal{M}(m,n))$, where $\mathcal{M}(m,n)$ are the $m \times n$ matrices,[1]

$$\begin{aligned} \|\varphi\|_{\mathcal{C}^0} &= \sup_{x\in\mathbb{T}^2} \sup_{i\in\{1,..,n\}} \sum_{j=i}^{m} |\varphi_{i,j}(x)| \\ \|\varphi\|_{\mathcal{C}^{\rho+1}} &= 2^{\rho+1}\|\varphi\|_{\mathcal{C}^0} + \sup_i \|\partial_{x_i}\varphi\|_{\mathcal{C}^\rho}. \end{aligned} \tag{5.1.1}$$

where, for a multi-index $\alpha = (\alpha_1, ..., \alpha_k)$ with $\alpha_k \in \{1, 2\}$, and we will use the notation $|\alpha| = k$ and $\partial^\alpha = \partial_{x_{\alpha_1}} \cdots \partial_{x_{\alpha_k}}$.[2] The above definition implies

$$\|\varphi\|_{\mathcal{C}^\rho} \le \sum_{k=0}^{\rho} 2^{\rho-k} \sup_{|\alpha|=k} \|\partial^\alpha\varphi\|_{\mathcal{C}^0}. \tag{5.1.2}$$

We will often need to compute the $\mathcal{C}^\rho$ norm of φ along a curve $\nu \in \mathcal{C}^r(\mathbb{T}, \mathbb{T}^2)$. In this case we use the notation $\|\varphi\|_{\mathcal{C}^\rho_\nu} := \|\varphi \circ \nu\|_{\mathcal{C}^\rho}$.
The following Lemma is proven in Appendix A. Note that the estimate in the Lemma are not sharp, however they try to optimize the balance between simplicity and usefulness.[3]

Lemma 5.1.1. *For each $\rho, n, m, s \in \mathbb{N}_0, \psi \in \mathcal{C}^\rho(\mathbb{T}^2, \mathcal{M}(n,m))$, $\varphi \in \mathcal{C}^\rho(\mathbb{T}^2, \mathcal{M}(m,s))$ we have*

$$\|\varphi\psi\|_{\mathcal{C}^\rho} \le \|\varphi\|_{\mathcal{C}^\rho}\|\psi\|_{\mathcal{C}^\rho}.$$

[1] According with the previous notations we set $x_1 = x$ and $x_2 = \theta$.

[2] Notice that this is at odd with the usual multi-index definition in PDE, however we prefer it for homogeneity with the case, treated later, of non-commutative vector fields.

[3] See [10, 35] for precise, but much more cumbersome, formulae.

Moreover, there exists $C_j^\star > 0$ such that, if $\varphi \in \mathcal{C}^\rho(\mathbb{T}^2, \mathcal{M}(n,m))$ and $\psi \in \mathcal{C}^\rho(\mathbb{T}^2, \mathbb{T}^2)$,

$$\|\varphi \circ \psi\|_{\mathcal{C}^\rho} \le C_\rho^\star \sum_{s=0}^{\rho} \|\varphi\|_{\mathcal{C}^s} \sum_{k \in \mathcal{K}_{\rho,s}} \prod_{l \in \mathbb{N}} \|D\psi\|_{\mathcal{C}^{l-1}}^{k_l} \tag{5.1.3}$$

where $\mathcal{K}_{\rho,s} = \{k \in \mathbb{N}_0^{\mathbb{N}} \ : \ \sum_{l=1}^\infty k_l \le s, \sum_{l=1}^\infty l k_l \le \rho\}$.

Using the above Lemma it follows that there exists a constant $\Lambda > 1$ such that

$$\|DF^n\|_{\mathcal{C}^r} + \|(DF^n)^{-1}\|_{\mathcal{C}^r} \le \Lambda^n, \quad \forall n \in \mathbb{N}. \tag{5.1.4}$$

5.2 Admissible curves

In this section we introduce the notion of *admissible curve* in order to define important auxiliary spaces and norms in the next section. We start by fixing some notations and defining exactly what we mean by *inverse branch.*

Lemma 5.2.1. *Let γ be a differentiable closed curve in the homotopy class $(0,1)$ such that $\gamma'(t) \notin \mathbf{C}_u$ for each $t \in \mathbb{T}$ and $F^{-1}\gamma = \bigcup_{k=1}^d \nu_k$, where the ν_k are disjoint closed curves in the homotopy class $(0,1)$. Then, there exist open sets $\Omega_\gamma, \Omega_{\nu_k}$, with $\bar{\Omega}_\gamma = \mathbb{T}^2$, and diffeomorphisms (the inverse branches) $\mathfrak{h}_{\nu_k} : \Omega_\gamma \to \Omega_{\nu_k}$ satisfying,*

- *$F \circ \mathfrak{h}_{\nu_k} = Id|_{\Omega_\gamma}$,*
- *If $\nu_k, \nu_j \in F^{-1}\gamma$, $k \neq j$, then $\Omega_{\nu_k} \cap \Omega_{\nu_j} = \emptyset$,*
- *$\bigcup_{\nu_k \in F^{-1}\gamma} \overline{\Omega}_{\nu_k} = \mathbb{T}^2$.*

Remark 5.2.2. *Note that if $\gamma \in \Upsilon$, then the hypotheses of the Lemma are satisfied thanks to hypothesis* **(H2)***.*

Proof of Lemma 5.2.1. The circle $\mathfrak{q} = \{(a,0)\}_{a \in \mathbb{T}}$ intersects each ν_k in only one point $p_k = \nu_k \cap \mathfrak{q}$. Indeed, by the backward invariance of the complement of $\mathbf{C}_u$, ν_k is locally monotone so it can meet twice $\mathfrak{q}$ only if it wraps around the torus more than once, which cannot happen since ν_k belongs to the homotopy class $(0,1)$. We can then label the ν_k so that the map $k \to p_k$ is orientation preserving $(\bmod\ d)$, let us call it positively oriented.[4] Also, calling $\tilde{\gamma}$ the curve obtained by translating γ by $\frac{1}{2}$ in the horizontal direction, we consider $A := F^{-1}(\tilde{\gamma}) \cap \mathfrak{q}$. Since F is a local diffeomorphism, if $\tilde{p} \in A$, in a neighborhood of $\tilde{p}$ the set $F^{-1}(\tilde{\gamma})$ consists of a curve with derivative outside $\mathbf{C}_u$, hence transversal to $\mathfrak{q}$. Accordingly A is a finite collection of points. Suppose that $\tilde{p}_k \in A$ is between p_k and p_{k+1}, then $\mathbb{T}^2 \setminus \nu_k$ is a cylinder and ν_{k+1} separates the cylinder in two disjoint regions (by Jordan curve theorem), thus $\tilde{p}_k$ belongs to a cylinder defined by the curves ν_k, ν_{k+1}. We can then follow the curve in $F^{-1}\tilde{\gamma}$ starting from $\tilde{p}_k$, such curve cannot exit the cylinder (since γ and $\tilde{\gamma}$ are disjoint). If it intersects again $\mathfrak{q}$ at a point p' then the image, under F, of the segment of $\mathfrak{q}$ between $\tilde{p}_k$ and p' is

[4] This definition is ambiguous if $d = 2$, but in such a case the ambiguity is irrelevant.

an unstable curve that starts and ends at $\tilde{\gamma}$, hence it must cross γ, contrary to the hypothesis. It follows that $p' = \tilde{p}_k$, that is $F^{-1}\tilde{\gamma} = \bigcup_{k=1}^{d} \tilde{\nu}_k$, where the $\tilde{\nu}_k$ are disjoint closed curves, of homotopy type $(0,1)$, and $\tilde{p}_k = \tilde{\nu}_k \cap \mathfrak{q}$. As before, we can label the curves so that the $\tilde{p}_k$ are positively oriented and $\tilde{p}_{k-1}, p_k, \tilde{p}_k$, where the indexes are mod d. Next, for $i \in \{1, \cdots, d\}$ and $q \in \nu_i$, we define the horizontal segment $\{\xi_q(t)\}_{t\in(-\delta_-(q),\delta_+(q))}$ where $\xi_q(t) = q + e_1 t$, $\xi(\delta_+(q)) \in \tilde{\nu}_i$ and $\xi(-\delta_-(q)) \in \tilde{\nu}_{i-1}$. We then define the regions

$$\Omega_{\nu_i} = \bigcup_{q\in\nu_i} \xi_q. \tag{5.2.1}$$

Clearly, $\Omega_{\nu_i} \cap \Omega_{\nu_j} = \emptyset$ if $i \neq j$, and $\bigcup_i \overline{\Omega}_{\nu_i} = \mathbb{T}^2$. Note that $F : \Omega_{\nu_i} \cup \tilde{\nu}_{i-1} \to \mathbb{T}^2$ is a bijection, although the inverse is not continuous. However, if we restrict the map to the set Ω_{ν_i} then it is an diffeomorphism between Ω_{ν_i} and $\Omega_\gamma = \mathbb{T}^2 \setminus \{\tilde{\gamma}\}$. Thus it is well defined the diffeomorphism $\mathfrak{h}_{\nu_i} : \Omega_\gamma \to \Omega_{\nu_i}$ such that $F \circ \mathfrak{h}_{\nu_i} = Id|_{\Omega_\gamma}$. □

We call $\mathfrak{h}_\nu$ the inverse branch of F associated to ν and simply $\mathfrak{h}$ when the curve ν is clear from the context. We denote by $\mathfrak{H}$ the set of inverse branches of F. Likewise, for each $n \in \mathbb{N}$ we denote with $\mathfrak{H}_n$ the set of inverse branches of F^n. As usual, we wish to identify the elements of $\mathfrak{H}_n$ as compositions of elements of $\mathfrak{H}$. Unfortunately, Lemma 5.2.1 tells us that each $\mathfrak{h} \in \mathfrak{H}$ is defined on a domain obtained by removing a curve in Υ from $\mathbb{T}^2$. Therefore the composition of two inverse branches in $\mathfrak{H}$ may not be well defined. We can however consider the following sets: denoting as $\mathcal{D}_\mathfrak{h}$ and $\mathcal{R}_\mathfrak{h}$ the domain and the range of $\mathfrak{h}$ respectively. For a curve $\gamma \in \Upsilon$ and $n \in \mathbb{N}$ we define[5]

$$\begin{aligned} &\mathfrak{H}_{\gamma,n} := \{\mathfrak{h} \in \mathfrak{H}_n : \mathcal{D}_\mathfrak{h} = \mathbb{T}^2 \setminus \{\gamma\}\}, \\ &\mathfrak{H}^n_{*,\gamma} := \left\{\mathfrak{h}_n = (\mathfrak{h}^*_1, \cdots, \mathfrak{h}^*_n) \in \mathfrak{H}^n : \mathcal{D}_{\mathfrak{h}^*_j} \subset \mathcal{R}_{\mathfrak{h}^*_{j-1}}, j \in \{2, .., n\}, \mathcal{D}_{\mathfrak{h}^*_1} \cap \{\gamma\} \neq \emptyset\right\}. \end{aligned} \tag{5.2.2}$$

In $\mathfrak{H}^n_{*,\gamma}$ there exists the obvious equivalence relation $\mathfrak{h}_n \sim \mathfrak{h}'_n$ if $\mathfrak{h}^*_n \circ \cdots \circ \mathfrak{h}^*_1 = \mathfrak{h}'^*_n \circ \cdots \circ \mathfrak{h}'^*_1$ and the quotient of $\mathfrak{H}^n_{*,\gamma}$ is naturally isomorphic to $\mathfrak{H}_{\gamma,n}$. In the following we will use the two notations interchangeably. Finally, we define

$$\mathfrak{H}^\infty_\gamma = \left\{\mathfrak{h} = (\mathfrak{h}^*_1, \cdots) \in \mathfrak{H}^\mathbb{N} \ : \ \mathcal{D}_{\mathfrak{h}^*_{j+1}} \subset \mathcal{R}_{\mathfrak{h}^*_j}, j \in \mathbb{N}\,; \mathcal{D}_{\mathfrak{h}^*_1} \cap \{\gamma\} \neq \emptyset\right\}.$$

For $\mathfrak{h} \in \mathfrak{H}^\infty_\gamma$, the symbol $\mathfrak{h}_n$ will denote the restriction of $\mathfrak{h}$ to $\mathfrak{H}^n_{*,\gamma}$ and we will say that $\mathfrak{h} \sim \mathfrak{h}'$ iff their restrictions are equivalent for each $n \in \mathbb{N}$.[6]

In the following we will often suppress the subsripts γ, ν if it does not create confusion.

[5]Here we are using the notation $\mathfrak{H}^n = \underbrace{\mathfrak{H} \times \cdots \times \mathfrak{H}}_{\text{n-times}}$.

[6] As it is not obvious how to make sense of infinite compositions, we define the equivalence relation indirectly.

Some further notation

For technical reason it is convenient to work with cones which are slightly smaller than $\mathbf{C}_u$ and $\mathbf{C}_c$. Take $\epsilon > 0$ arbitrarily small but fixed[7] and, setting $\epsilon^* = 1 - \epsilon$, let us consider the cone

$$\mathbf{C}_{\epsilon,u} = \{(x, y) \in \mathbb{R}^2 : |y| \leq \chi_u \epsilon^* |x|\}, \tag{5.2.3}$$

which is strictly contained in $\mathbf{C}_u$. Moreover the difference between the angle of $\mathbf{C}_u$ and the angle of $\mathbf{C}_{u,\epsilon}$ is smaller than ϵ. In the same way it is defined $\mathbf{C}_{\epsilon,c}$. For each $p \in \mathbb{T}^2$ let $\mathfrak{H}_p^n := \{\mathfrak{h} \in \mathfrak{H}^n \ : \ p \in \mathcal{D}_\mathfrak{h}\}$. By the expansion of the unstable cone under backward dynamics and the backward invariance of the central cone we can define $m_{\chi_u}(p, \mathfrak{h}) : \mathbb{T}^2 \times \mathfrak{H}_p^\infty \to \mathbb{N}$ and $m_{\chi_u} \in \mathbb{N}$ as

$$\begin{aligned} m_{\chi_u}(p, \mathfrak{h}) &= \min\{n \in \mathbb{N} : D_p\mathfrak{h}_n(\mathbb{R}^2 \setminus \mathbf{C}_u) \subset \mathbf{C}_{\epsilon,c}\} \\ m_{\chi_u}(p) &= \sup_{\mathfrak{h} \in \mathfrak{H}^\infty} m_{\chi_u}(p, \mathfrak{h}) \\ m_{\chi_u} &= \sup_{p \in \mathbb{T}^2} \sup_{\mathfrak{h} \in \mathfrak{H}^\infty} m_{\chi_u}(p, \mathfrak{h}). \end{aligned} \tag{5.2.4}$$

To guarantee that the above quantities are finite, we choose ϵ such that $\mathbf{C}_{\epsilon,c} \supset D_p\mathfrak{h}\mathbf{C}_c$, where $\mathfrak{h} \circ F(p) = p$. Note that the latter condition is possible because of (4.1.4), the continuity of $D_p\mathfrak{h}\mathbf{C}_c$ and the compactness of $\mathbb{T}^2$.
By a direct computation (see Sub-Lemma 5.4.5 for the details) equation (5.2.4) implies

$$\lambda^-_{m_{\chi_u}(p,\mathfrak{h})}(p)^{-1} \mu^{m_{\chi_u}} < \epsilon^* \chi_c \chi_u, \quad \forall p \in \mathbb{T}^2, \mathfrak{h} \in \mathfrak{H}^\infty, \tag{5.2.5}$$

$$m_{\chi_u} < \bar{c}_2 \log \chi_u^{-1}, \tag{5.2.6}$$

for some uniform constant $\bar{c}_2 > 0$ to be chosen later on (see Lemma 5.4.2). Next, consider a vector $v = (1, u_0) \in \mathbf{C}_u$, so that $|u_0| \in [-\chi_u, \chi_u]$. By forward invariance of the unstable cone, there exist continuous functions $\Upsilon_n, \Xi_n : \mathbb{N} \times \mathbb{T}^2 \times [-\chi_u, \chi_u] \to \mathbb{R}$ such that

$$D_pF^n v = \Upsilon_n(p, u_0)(1, \Xi_n(p, u_0)),$$

where $\|\Xi_n\|_\infty \leq \chi_u$. We are interested in the evolution of the slope field Ξ_n. For this purpose it is convenient to introduce the dynamics $\Phi(p, u_0) = (F(p), \Xi(p, u_0))$, for $p \in \mathbb{T}^2$, $u_0 \in [-\chi_u, \chi_u]$ and where we use the notation $\Xi = \Xi_1$. The map Φ will describe how the slopes of the cones change while iterating F. Note that

$$\Phi^n(p, u_0) = (F^n(p), \Xi_n(p, u_0)). \tag{5.2.7}$$

Finally, for $n \in \mathbb{N}$ and $\mathfrak{h} \in \mathfrak{H}^\infty$, let us define the function

$$u_{\mathfrak{h},n}(p, u_0) = \pi_2 \circ \Phi^n(\mathfrak{h}_n(p), u_0) : \mathbb{T}^2 \times [-\chi_u, \chi_u] \to [-\chi_u, \chi_u], \tag{5.2.8}$$

where π_2 is the projection on the second coordinate. By Lemma D.0.1, applied with $u = u' = u_0$ and $\varepsilon_0 = 1$, we see that $u_{\mathfrak{h},n}(p, u_0)$ is Lipschitz and the Lipschitz constant can be computed using (D.0.2).

[7]During the following sections ϵ will have to satisfies different conditions. However, it is important to note that, once the conditions are satisfied, the value of ϵ is fixed once and for all.

Admissible central and unstable curves

In the following $\pi_k : \mathbb{T}^2 \to \mathbb{T}$ will denote the projection on the k^{th} component, for $k = 1, 2$. Also, for $\varphi \in \mathcal{C}^r(\mathbb{T}, \mathbb{C})$ we use the notation $(\varphi)^{(j)}(t) = \frac{d^j}{dt^j}\varphi(t)$ and φ' in the case $j = 1$.

Definition 5.2.3. *Let c be a positive constant, then $\Gamma_j(c)$ is the set of the $\mathcal{C}^r$ closed curves $\gamma : \mathbb{T} \to \mathbb{T}^2$ which are parametrized by vertical length, i.e. $\gamma(t) = (\gamma_1(t), t)$, satisfy conditions c1) and c2) of assumption* **(H2)**, *and:*

c3) for every $2 \leq \ell \leq j$: $|(\pi_1 \circ \gamma(t))^{(\ell)}| \leq c^{(\ell-1)!}$.

Given $\mathbb{c} > 0$ and $j \leq r$ we will call $\gamma \in \Gamma_j(\mathbb{c})$ a $(j, \mathbb{c})$-admissible central curve (or simply admissible curve if the context is clear). We will choose $\mathbb{c}$ in Corollary 5.4.3.
Similarly, a curve $\eta \in \mathcal{C}^r(I, \mathbb{T}^2)$ of length δ defined on a compact interval $I = [0, \delta]$ of $\mathbb{T}$ is called an admissible unstable curve if $\eta'(t) \in \mathbf{C}_u$, it is parametrized by horizontal length and its j-derivative is bounded by $c^{(j-1)!}$.

The basic objects used in the paper are integrals along admissible (or pre-admissible) curves. To estimate precisely such objects are necessary several technical estimates that are developed in the next subsections.

5.3 Preliminary estimates on derivatives

We start with the following simple, but very helpful, propositions.

Proposition 5.3.1. *There exists a uniform constant $C_* \geq 1$ such that, for every $z \in \mathbb{T}^2$, any $n \in \mathbb{N}$, any vectors $v^u \in \mathbf{C}_u$ and $v^c \in \mathbf{C}_c$ such that $(a, b) := D_z F^n v^c \notin \mathbf{C}_u$, we have :*

$$C_*^{-1} \frac{\|D_z F^n v^u\|}{\|v^u\|} \frac{|b|}{\|v^c\|} \leq |\det D_z F^n| \leq C_* \frac{\|D_z F_z^n v^u\|}{\|v^u\|} \frac{|b|}{\|v^c\|}.$$

Proof. Recall that for a matrix $D \in GL(2, \mathbb{R})$ and vectors $v_1, v_2 \in \mathbb{R}^2$ linearly independent

$$|\det D| = \frac{|Dv_1 \wedge Dv_2|}{|v_1 \wedge v_2|} = \frac{\|Dv_1\|}{\|v_1\|} \frac{\|Dv_2\|}{\|v_2\|} \frac{\sin(\measuredangle(Dv_1, Dv_2))}{\sin(\measuredangle(v_1, v_2))}. \tag{5.3.1}$$

Let $\theta = \measuredangle(DF^n v^u, DF^n v^c)$, $\theta_1 = \measuredangle(DF^n v^u, e_1)$, $\theta_2 = \measuredangle(DF^n v^c, e_1)$ and $\theta_u = \arctan \chi_u$. Since $D_z F^n v^u \in DF\mathbf{C}_u$ we have $|\theta_1| \leq c\theta_u$, for some fixed $c \in (0, 1)$. On the other hand, by hypothesis, $|\theta_2| \geq \theta_u$. Thus

$$\frac{|\theta|}{|\theta_2|} = \frac{|\theta_2 - \theta_1|}{|\theta_2|} \leq \frac{|\theta_2| + |\theta_1|}{|\theta_2|} \leq 1 + c$$
$$\frac{|\theta|}{|\theta_2|} \geq \frac{|\theta_2| - |\theta_1|}{|\theta_2|} \geq 1 - c.$$

The Lemma follows since $\|DF^n v^c\| \sin \theta_2 = b$. □

We introduce the following quantities for each $n \in \mathbb{N}, m \le n, p \in \mathbb{T}^2$ and some constant $C_\sharp > 0$:

$$C_{\mu,n} := C_\sharp \frac{1-\mu^{-n}}{\mu - 1} \le C_\sharp \min\{n, (\mu-1)^{-1}\}; \quad C_{\mu,0} = 0, \tag{5.3.2}$$

$$\bar{\varsigma}_{n,m}(p) := \{C_{\mu,n-m}, \lambda_m^+(p)\}^+ \tag{5.3.3}$$

$$\varsigma_{n,m}(p) := \{1, C_{\mu,n} + (\chi_u + \|\omega\|_{\mathcal{C}^2})\bar{\varsigma}_{n,m}(p)\}^+; \quad \varsigma_{n,n} := \varsigma_n, \tag{5.3.4}$$

and we will use the notation $\bar{\varsigma}_{n,m} := \|\bar{\varsigma}_{n,m}\|_\infty$ and $\varsigma_{n,m} = \|\varsigma_{n,m}\|_\infty$.

Remark 5.3.2. *Note that we can always estimate $C_{\mu,n}$ with $(\mu-1)^{-1}$, which is independent on n, and we will do it in the general case (SVPH) if we need estimates uniform in n. However, such a bound will deteriorate when μ approaches one, a case we want to investigate explicitly in Section 10, and for which* (5.3.2) *is more convenient.*

Next, we provide sharper estimates of various quantities relevant in the next sections.

Proposition 5.3.3. *For any $n \in \mathbb{N}$ and $p \in \mathbb{T}^2$, we have:*

$$\begin{aligned} &\lambda_n^+(p) \le C_\sharp \lambda_n^-(p) \\ &\|(DF^n)^{-1}\|_{\mathcal{C}^0(\mathbb{T}^2)} \le C_\sharp \mu^n. \end{aligned} \tag{5.3.5}$$

In addition, for each $\mathbb{c} > 0$, $m \le n$ and $\nu \in \Gamma_2(\mathbb{c})$ such that $DF^{n-m}\nu' \in \mathbf{C}_c$,[8]

$$\begin{aligned} &\|DF^n\|_{\mathcal{C}^0_\nu} \le C_\sharp \lambda_n^+ \\ &\|DF^n\|_{\mathcal{C}^1_\nu} \le C_\sharp \lambda_n^+ \bar{\varsigma}_{n,m}\mu^{n-m} \\ &\|DF^n\|_{\mathcal{C}^2_\nu} \le C_\sharp \lambda_n^+ (\bar{\varsigma}_{n,m}\mu^{n-m})^2 + C_\sharp (\lambda_m^+)^2 \mathbb{c} \\ &\|\frac{d}{dt}(D_{\nu(t)}F^n)^{-1}\| \le C_\sharp \mu^{2n-m} \varsigma_{n,m} \circ \nu(t) \\ &\|\frac{d^2}{dt^2}(D_{\nu(t)}F^n)^{-1}\| \le C_\sharp \mu^n \varsigma_n^2 \circ \nu(t) \lambda_m^+(\nu(t)) + C_\sharp \varsigma_n \circ \nu(t)(\bar{\varsigma}_{n,m}(\nu(t))\mu^{n-m} + \mathbb{c}). \end{aligned} \tag{5.3.6}$$

Proof. Let $v^c \in \mathcal{T}_{F^n(p)}\mathbb{T}^2$ with $v^c \in \mathbf{C}_c$ unitary, and $w_u \in \mathbf{C}_u$. Define

$$\tilde{w}_u = \frac{D_p F^n w_u}{\|D_p F^n w_u\|} \in \mathbf{C}_u.$$

For each $v \in \mathcal{T}_{F^n(p)}\mathbb{T}^2$ we can write $v = \alpha v^c + \beta \tilde{w}_u$, then

$$\|(D_{F^n p}F^n)^{-1} v\| \le |\alpha| \|(D_{F^n p}F^n)^{-1} v^c\| + |\beta| \|(D_{F^n p}F^n)^{-1}\tilde{w}_u\|$$

By (4.1.3) and (4.1.5) we have the following

1. $\|(D_{F^n p}F^n)^{-1}\tilde{w}_u\| \le C_\star \lambda_-^{-n}$,

[8] Recall Section 5.1 for the definition of $\|\cdot\|_{\mathcal{C}^r_\nu}$ and (5.3.4) for the notations used.

2. $\|(D_{F^n p}F^n)^{-1}v^c\| \leq C_\star \mu^n$.

Hence,

$$\|(D_{F^n p}F^n)^{-1}v\| \leq C_\star \mu^n |\alpha| + C_\star \lambda_-^{-n}|\beta|,$$

A direct computation shows

$$\{|\alpha|, |\beta|\}^+ \leq \frac{1 + |\langle v^c, \tilde{w}_u\rangle|}{1 - \langle v^c, \tilde{w}_u\rangle^2}\|v\| \leq \frac{1+\cos\vartheta}{1-(\cos\vartheta)^2}\|v\|$$

where

$$\cos\vartheta := \cos\left[\inf_{v\in\mathbf{C}_u, w\in\mathbf{C}_c}\{|\measuredangle(v,w)|\}\right] \leq \frac{1}{\sqrt{1+\chi_c^2}} < 1.$$

From the above the second statement of (5.3.5) follows. The strategy for proving the first of (5.3.5) is similar. We take $w_1, w_2 \notin \mathbf{C}_c$ unitary and $v^c = (0,1) \in \mathbf{C}_c$, and we set $\tilde{v}_c = \frac{(D_{F^n p}F^n)^{-1}v^c}{\|(D_{F^n p}F^n)^{-1}v^c\|} \in \mathbf{C}_c$. Notice that $\|D_p F^n \tilde{v}^c\| \leq C\mu^n$. Let $w_2 = \alpha w_1 + \beta\tilde{v}^c$. By (4.1.4) it follows that there exists a minimal angle between $w_1 \notin \mathbf{C}_c$ and $\tilde{v}^c \in (DF)^{-1}\mathbf{C}_c$, thus $|\alpha| + |\beta| \leq C$ for some constant $C_\sharp > 0$. Hence,

$$\|D_p F^n w_1 - D_p F^n w_2\| \leq |1-\alpha|\|D_p F^n w_1\| + C_\sharp \mu^n \leq (1 + C_\sharp)\|D_p F^n w_1\| + C_\sharp \mu^n.$$

Since $\|D_p F^n w_1\| \geq C\lambda_n^-(p)$, it follows that

$$\left|1 - \frac{\|D_p F^n w_2\|}{\|D_p F^n w_1\|}\right| \leq \left|\frac{D_p F^n w_1}{\|D_p F^n w_1\|} - \frac{D_p F^n w_2}{\|D_p F^n w_1\|}\right| \leq (1 + C_\sharp) + C_\sharp \frac{\mu^n}{\lambda_n^-(p)}.$$

Equation (5.3.5) follows by the arbitrariness of w_1, w_2 and since $\mu < \lambda_-$. To conclude we must compute the derivatives of $DF^n, (DF^n)^{-1}$. By (4.1.3), we have

$$\|D_x F^k\| \leq C_\sharp \lambda_k^+(x). \tag{5.3.7}$$

Moreover, for each $n, k \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{d}{dt}D_{\nu(t)}F^n &= \sum_{s=1}^{2}\sum_{k=0}^{n-1} D_{F^{k+1}(\nu(t))}F^{n-k-1}\partial_{x_s}(D_{F^k(\nu(t))}F)D_{\nu(t)}F^k(D_{\nu(t)}F^k\nu')_s \\
\frac{d}{dt}(D_{\nu(t)}F^n)^{-1} &= \sum_{s=1}^{2}\sum_{k=0}^{n-1}(D_{\nu(t)}F^k)^{-1}\left[\partial_{x_s}(DF)^{-1}(D_{F(\cdot)}F^{n-k-1})^{-1}\right] \circ F^k(\nu(t)) \\
&\qquad \cdot (D_{\nu(t)}F^k\nu')_s.
\end{aligned} \tag{5.3.8}$$

The above, also differentiating once more, implies that

$$\begin{aligned}
\|\frac{d}{dt}(D_{\nu(t)}F^n)\| &\leq C_\sharp \lambda_n^+ \mu^{n-m}\{C_{\mu,n-m}, \lambda_m^+\}^+ = C_\sharp \lambda_n^+ \bar{\varsigma}_{n,m}\mu^{n-m}, \\
\|\frac{d^2}{dt^2}(D_{\nu(t)}F^n)\| &= \|\sum_{\ell,s}(\partial_{x_\ell}\partial_{x_s}D_x F^n)\nu'_\ell\nu'_s + \sum_s \partial_{x_s}D_x F^n \nu''_s\| \\
&\leq C_\sharp \lambda_n^+(\mu^{n-m}\bar{\varsigma}_{n,m})^2 + C_\sharp(\lambda_m^+)^2\mathbb{c}.
\end{aligned} \tag{5.3.9}$$

To estimate the second of (5.3.8), note that for each $p \in \mathbb{T}^2$, there exists $\xi \in \mathcal{C}^{r-2}(\mathbb{T}^2, \mathbb{R}^2)$, $\|\xi\|_{\mathcal{C}^{r-2}} \leq C_\sharp$, such that, for all $w \in \mathbb{R}^2$, and $|\alpha| \leq r-2$,

$$\left\|\partial_\alpha (DF)^{-1} w - e_1 \langle \partial^\alpha \xi, w\rangle\right\| \leq C_\sharp \|w\| \|\omega\|_{\mathcal{C}^{|\alpha|+2}}. \tag{5.3.10}$$

Thus, setting $\eta_k(p) = D_p F^k e_1 \|D_p F^k e_1\|^{-1}$, we have $\|\eta_k - e_1\| \leq C_\sharp \chi_u$ and, for all $w \in \mathbb{R}^2$,

$$\begin{aligned}
\left\|(D_x F^k)^{-1} \partial_{x_i} (D_p F)^{-1} w\right\| &\leq \left\|(D_x F^k)^{-1} \eta_k(x) \langle \partial_{x_i} \xi, w\rangle\right\| \\
&\quad + \left\|(D_x F^k)^{-1} \partial_{x_i} (D_p F)^{-1} w - (D_x F^k)^{-1} \eta_k(x) \langle \partial_{x_i} \xi, w\rangle\right\| \\
&\leq C_\sharp \frac{\|w\|}{\lambda_k^-(x)} + C_\sharp \mu^k \|w\| (\chi_u + \|\omega\|_{\mathcal{C}^2}).
\end{aligned}$$

For simplicity we set $C_F := \chi_u + \|\omega\|_{\mathcal{C}^r}$. Hence, using the above and (5.3.5),

$$\begin{aligned}
\|\frac{d}{dt}((D_\nu F^n)^{-1})\| &\leq C_\sharp \sum_{k=0}^{n-m-1} \mu^{n-1} \{(\lambda_k^- \circ \nu)^{-1} + C_F \mu^k\} \\
&\quad + C_\sharp \sum_{k=n-m}^{n-1} \mu^{n-k-1} \{(\lambda_k^- \circ \nu)^{-1} + C_F \mu^k\} \mu^{n-m} \lambda_{m-n+k}^+ \circ \nu \\
&\leq \mu^{2n-m} \left[C_{\mu,m} + C_F \{C_{\mu,n-m}, \lambda_m^+ \circ \nu\}^+\right].
\end{aligned}$$

Therefore

$$\|\frac{d}{dt}((D_{\nu(t)} F^n)^{-1})\| \leq C_\sharp \mu^{2n-m} \varsigma_{n,m} \circ \nu(t), \tag{5.3.11}$$

which yields the statement for the first derivative. Next, differentiating once more the second of (5.3.8),

$$\begin{aligned}
\frac{d^2}{dt^2}(D_\nu F^n)^{-1} &= \sum_{s=1}^{2} \sum_{k=0}^{n-1} \left[\frac{d}{dt}(D_{\nu(t)} F^k)^{-1}\right] \left[\partial_{x_s} (DF)^{-1} (D_{F(\cdot)} F^{n-k-1})^{-1}\right] \circ F^k(\nu) \\
&\cdot (D_{\nu(t)} F^k \nu')_s + \sum_{s,\ell=1}^{2} \sum_{k=0}^{n-1} (D_\nu F^k)^{-1} \left\{\partial_{x_\ell} \left[\partial_{x_s} (DF)^{-1} (D_{F(\cdot)} F^{n-k-1})^{-1}\right]\right\} \circ F^k(\nu) \\
&\cdot (D_\nu F^k \nu')_\ell (D_\nu F^k \nu')_s + \sum_{s=1}^{2} \sum_{k=0}^{n-1} (D_\nu F^k)^{-1} \left[\partial_{x_s} (DF)^{-1} (D_{F(\cdot)} F^{n-k-1})^{-1}\right] \circ F^k(\nu) \\
&\cdot \left\{\left[\frac{d}{dt} D_\nu F^k\right] \nu' + D_\nu F^k \nu''\right\}.
\end{aligned}$$

We estimate the three sums above separately. By (5.3.10) and (5.3.11), the first one is bounded by

$$\begin{aligned}
&C_\sharp \sum_{k=0}^{n-m-1} \mu^{2k} \varsigma_{k,0} \circ \nu \mu^{n-k-1} \mu^k + C_\sharp \sum_{k=n-m}^{n-1} \mu^{n-k-1} \mu^k \varsigma_{k,k} \circ \nu \mu^{n-k-1} \mu^{n-m} \lambda_{m-n+k}^+ \circ \nu \\
&\qquad \leq C_\sharp \mu^{2n} C_{\mu,n-m} \varsigma_{n-m,0} \circ \nu + \mu^{2n-m} \varsigma_{n,n} \lambda_m^+ \circ \nu \leq \mu^{2n} \varsigma_{n,n} \circ \nu \lambda_m^+ \circ \nu.
\end{aligned}$$

The second one is equal to

$$\sum_{k=0}^{n-1}(D_\nu F^k)^{-1}\left\{\partial^2_{x_\ell,x_s}(DF)^{-1}\cdot(D_{F(\cdot)}F^{n-k-1})^{-1}\right\}\circ F^k(\nu)\cdot(D_\nu F^k\nu')_\ell(D_\nu F^k\nu')_s$$
$$+(D_\nu F^k)^{-1}\left\{\partial_{x_\ell}(DF)^{-1}\partial_{x_\ell}(D_{F(\cdot)}F^{n-k-1})^{-1}\right\}\circ F^k(\nu)\cdot(D_\nu F^k\nu')_\ell(D_\nu F^k\nu')_s,$$

so we can use again (5.3.10) to get the bound

$$C_\sharp\sum_{k=0}^{n-1}\left[(\lambda_k^+\circ\nu(t))^{-1}+C_F\mu^k\right]\mu^{n-k}\left\{1+\left[C_{\mu,n-k}+C_F\lambda^+_{n-k}\circ\nu\right]\right\}$$
$$\cdot\,\mu^{2\min\{k,n-m\}}(\lambda^+_{\{0,k-n+m\}^+}\circ\nu(t))^2\le C_\sharp\mu^n\varsigma^2_{n,n}\circ\nu\lambda^+_m\circ\nu.$$

For the last term we use the estimates above and, recalling (5.3.9), we obtain the bound

$$\left\{C_{\mu,n}+C_F\lambda^+_n(\nu(t))\right\}(\bar\varsigma_{n,m}\mu^{n-m}+\mathbb{c})\le C_\sharp\varsigma_{n,n}\circ\nu(t)(\bar\varsigma_{n,m}(\nu(t))\mu^{n-m}+\mathbb{c}).$$

Collecting the above estimates, the last of the (5.3.6) readily follows. □

5.4 Iteration of curves

We first check how the above curves behave under iteration. The following is a more quantitative version of [53, Lemma 3.2] adapted to our case.

Lemma 5.4.1. *Let F be SVPH. There exist uniform constants $\bar n\in\mathbb{N}$, $C_\flat>1$ and $\eta<1$ such that, if $\mathbb{c}>\frac{9}{4}C_\flat^3\mu^{3\bar n}$, for each $c_\star>\mathbb{c}/2$, $\gamma\in\Gamma_\ell(c_\star)$, $1\le\ell\le r$, and $n\ge\bar n$, setting $\nu_n\in F^{-n}\gamma$, there exist diffeomorphisms $h_{n,\nu}=:h_n\in\mathcal{C}^r(\mathbb{T})$ such that:*
(a) *The curve $\hat\nu_n=\nu_n\circ h_n$ is in $\Gamma_\ell(\eta^n c_\star+\mathbb{c}/2)$ and*

$$\|h_n\|_{\mathcal{C}^\ell}\le\begin{cases}C_\flat\mu^n & \text{if}\quad \ell=1\\ C_\flat^3c_\star C_{\mu,n}\mu^{2n} & \text{if}\quad \ell=2\\ (C_\flat^2c_\star)^{\ell!}\,C_{\mu,n}^{a_\ell}\mu^{\ell!n} & \text{if}\quad \ell>2,\end{cases}\tag{5.4.1}$$

where $a_\ell=(\ell-1)!\sum_{k=0}^{\ell-1}\frac{1}{k!}$, and $C_{\mu,n}$ as in (5.3.2).

Proof. Fix $\gamma\in\Gamma_j(c_\star)$ and $n\in\mathbb{N}$. Let ν_n be a pre-image of γ under F^n and consider $\mathfrak{h}\in\mathfrak{H}^\infty$ such that $\nu_n=\mathfrak{h}_n\circ\gamma$. Let $h_n:\mathbb{T}\to\mathbb{T}$ be the diffeomorphism such that $\hat\nu_n=\nu_n\circ h_n$ is parametrized by vertical length. We then want to check properties $c1),\dots,c3)$ for $\hat\nu_n$. The first two follow immediately by assumption (**H2**), thus we only have to check property $c3)$. By definition we have

$$F^n\hat\nu_n=\gamma\circ h_n.\tag{5.4.2}$$

Differentiating equation (5.4.2) twice we obtain

$$(\partial_t D_{\hat\nu_n}F^n)\hat\nu_n'+D_{\hat\nu_n}F^n\hat\nu_n''=\gamma''\circ h_n(h_n')^2+\gamma'\circ h_nh_n''.\tag{5.4.3}$$

Similarly, if we differentiate equation (5.4.2) j-th times,

$$R_j(F^n, \hat{\nu}_n) + D_{\hat{\nu}_n} F^n \hat{\nu}_n^{(j)} = \gamma^{(j)} \circ h_n (h_n')^j + Q_j(h_n, \gamma) + \gamma' \circ h_n \cdot h_n^{(j)}, \qquad (5.4.4)$$

where R_j is the sum of monomials, with coefficients depending only of $(\partial^\alpha F^n) \circ \hat{\nu}_n$ with $|\alpha| \leq j$, in the variables $\hat{\nu}_n^{(s)}$, $s \in \{0, \dots j-1\}$, where if k_s is the degree of $\hat{\nu}^{(s)}$ we have $\sum_{s=1}^{j-1} s k_s = j$. Likewise the Q_j are the sum of monomials that are linear in $\gamma^{(\sigma)}$, $\sigma \in \{2, \dots j-1\}$, and of degree p_s in $h_n^{(s)}$, $s \in \{1, \dots j-\sigma+1\}$, such that $\sum_{s=1}^{j-\sigma+1} s\, p_s = j$.[9] In order to obtain an estimate for $\|\hat{\nu}_n^{(j)}\|$ it is convenient to introduce the vectors $\eta_{n,j} = D_{\hat{\nu}_n} F^n \hat{\nu}_n^{(j)}$. We then define the unitary vectors $\eta_{n,j}^\perp, \hat{\eta}_{n,j}$ such that $\langle \eta_{n,j}^\perp, \eta_{n,j} \rangle = 0$ and $\hat{\eta}_{n,j} = \frac{\eta_{n,j}}{\|\eta_{n,j}\|}$. Multiplying equation (5.4.4) by $\eta_{n,j}^\perp$ and $\hat{\eta}_{n,j}$ respectively, we obtain the system of equations

$$\begin{aligned} &\langle \eta_{n,j}^\perp, R_j(F^n, \hat{\nu}_n) \rangle = \left\langle \eta_{n,j}^\perp, \gamma^{(j)} \circ h_n (h_n')^j + Q_j(h_n, \gamma) + \gamma' \circ h_n \cdot h_n^{(j)} \right\rangle \\ &\langle \hat{\eta}_{n,j}, R_j(F^n, \hat{\nu}_n) \rangle + \|\eta_{n,j}\| = \left\langle \hat{\eta}_{n,j}, \gamma^{(j)} \circ h_n (h_n')^j + Q_j(h_n, \gamma) + \gamma' \circ h_n \cdot h_n^{(j)} \right\rangle. \end{aligned} \qquad (5.4.5)$$

Notice that, since $\hat{\nu}_n^{(j)}$, $j > 1$, is a horizontal vector, by the invariance of the unstable cone $\eta_{n,j} \in \mathbf{C}_u$. Moreover $\gamma' \in \mathbf{C}_c$ by assumption and $\|\eta_{n,j}^\perp\| = 1$, thus there exists $\vartheta \in (0,1)$ such that

$$|\langle \eta_{n,j}^\perp, \gamma' \circ h_n \rangle| \geq \vartheta \|\gamma' \circ h_n\| \geq \vartheta. \qquad (5.4.6)$$

Using (5.4.6) and setting $R_{j,n} := \|R_j(F^n, \hat{\nu}_n)\| + \|Q_j(h_n, \gamma)\|$, equation (5.4.5) yields

$$\begin{aligned} |h_n^{(j)}| &\leq \frac{|h_n'|^j \|\gamma^{(j)} \circ h_n\| + R_{j,n}}{\vartheta \|\gamma' \circ h_n\|}, \\ \|\eta_{n,j}\| &\leq \|\gamma^{(j)} \circ h_n\| |h_n'|^j + \|\gamma' \circ h_n\| |h_n^{(j)}| + R_{j,n}. \end{aligned} \qquad (5.4.7)$$

By equation (5.4.2) it follows that

$$\|\hat{\nu}_n'\| = |h_n'| \|(D_{\hat{\nu}_n} F^n)^{-1} \gamma' \circ h_n\|, \qquad (5.4.8)$$

which yields, by (4.1.5) and the fact that $\hat{\nu}_n' = ((\pi_1 \circ \hat{\nu}_n)', 1) \in \mathbf{C}_c$,

$$\frac{\mu^{-n}}{C_\star \sqrt{1+\chi_c^2}} \leq |h_n'| \leq \frac{C_\star \mu^n \|\hat{\nu}_n'\|}{\|\gamma' \circ h_n\|} \leq \sqrt{1+\chi_c^2} C_\star \mu^n =: \bar{C}_\star \mu^n. \qquad (5.4.9)$$

---

[9] The reader can check this by induction (equation (5.4.3) gives the case $j = 2$). E.g., if a term Q in R_j has the form $P = \prod_{s=0}^{j-1} \alpha_s(\hat{\nu}_n^{(s)})$ where $\alpha_s(x)$ is homogeneous of degree k_s in x, then $\partial_t Q$ will be a sum of terms of the same type with homogeneity degrees k_s'. Let us compute such homogeneity degrees: if the derivative does not hit a $\hat{\nu}_n^{(s)}$, $s > 0$, then, by the chain rule, we will get a monomial with $k_1' = k_1 + 1$ while all the other homogeneity degree are unchanged: $k_s' = k_s$ for $s > 0$. Hence, $\sum_{s=0}^{j} k_s' = j + 1$. If the derivative hits one $\hat{\nu}_n^{(i)}$, then it produces a monomial with $k_s' = k_s$ for $s \notin \{i, i+1\}$ while $k_i' = k_i - 1$ and $k_{i+1}' = k_{i+1} + 1$. Then $\sum_{s=0}^{j} k_s' = j - ik_i - (i+1)k_{i+1} + i(k_i - 1) + (i+1)(k_{i+1} + 1) = j + 1$.

Using this in (5.4.7) and observing that $\|\eta_{n,j}\| = \|D_{\hat\nu_n} F^n \hat\nu_n^{(j)}\| \geq \lambda_n^- \|\hat\nu_n^{(j)}\|$, we obtain

$$\|\hat\nu_n^{(j)}\| \leq \|\gamma^{(j)} \circ h_n\| (\lambda_n^-)^{-1} (\bar C_\star \mu^n)^j A + R^\star_{j,n}, \tag{5.4.10}$$

where $A = (1+\vartheta^{-1})$ and $R^\star_{j,n} = (\lambda_n^-)^{-1} A R_{j,n}$. We choose $\bar n$ and $\eta < 1$ such that

$$\begin{aligned} &3^{r!}(1+\vartheta)(\bar C_\star \mu^{\bar n})^r (\lambda_{\bar n}^-)^{-1} < 1, \\ &\eta := \left(3^{r!}(1+\vartheta)(\bar C_\star \mu^{\bar n})^r (\lambda_{\bar n}^-)^{-1}\right)^{\frac{1}{2\bar n r!}}. \end{aligned} \tag{5.4.11}$$

Therefore we have

$$3^{j!} A (\bar C_\star \mu^{\bar n})^j (\lambda_{\bar n}^-)^{-1} \leq 3^{r!} A (\bar C_\star \mu^{\bar n})^r (\lambda_{\bar n}^-)^{-1} \leq \eta^{2\bar n r!} < 1. \tag{5.4.12}$$

Note in particular that, as $\bar C_\star$ and ϑ are uniform constants, so are both $\bar n$ and η. We are ready to conclude. For $j = 1$ the Lemma is trivial since $\|\hat\nu'_{\bar n}\| \leq \sqrt{1+\chi_c^2}$ and $h'_{\bar n}$ can by bounded by (5.4.9), provided $C_\flat \geq \bar C_\star$. Equation (5.4.2) implies that $\|R_2(F^{2\bar n}, \hat\nu_{2\bar n})\| \leq C_\sharp$ and $Q_2 = 0$, thus $R_{2,2\bar n} \leq C_\sharp$. Then the first of (5.4.7), remembering (5.4.3), and (5.4.8), together with equation (5.4.10) imply

$$\begin{aligned} &\|h_n^{(2)}\| \leq C_\sharp \bar C_\star^2 c_\star \mu^{2n} \quad \forall n \leq 2\bar n \\ &\|\hat\nu_n^{(2)}\| \leq A (\lambda_n^-)^{-1} \left\{ \|\gamma^{(2)} \circ h_n\| (\bar C_\star \mu^n)^2 + C_\sharp \right\}. \end{aligned} \tag{5.4.13}$$

Next, we proceed by induction on $j < \ell$ to prove that for each $\bar n \leq n \leq 2\bar n$

$$\begin{aligned} &\|h_n^{(j)}\| \leq C_\sharp c_\star^{(j-1)!} \mu^{j! n} \\ &\|\hat\nu_n^{(j)}\| \leq (\eta^n c_\star + \mathbb{c}/2)^{(j-1)!}. \end{aligned} \tag{5.4.14}$$

By (5.4.13) we have the case $j = 2$, let us assume it for all $s \leq j > 2$. Recalling the structure of R_j, Q_j, see after (5.4.4), and setting $c_n := \eta^n c_\star + 2^{-1}(1-\eta^n)\mathbb{c} \leq c_\star$ we have

$$R_{j+1,n} \leq C_\sharp \left\{ \sum_k c_{\bar n}^{\sum_{s=1}^j (s-1)! k_s} + C_\flat^{j+1} \sum_{\sigma=2}^j \sum_p c_\star^{(\sigma-1)! + \sum_{s=1}^{j+1-\sigma} p_s s!} \mu^{n \sum_{s=0}^{j+2-\sigma} p_s s!} \right\}.$$

Note that $\sum_{s=0}^j (s-1)! k_s \leq (j-2)! \sum_{s=1}^j s k_s = (j-2)!(j+1)$. If $\sigma = j$, then

$$(\sigma - 1)! + \sum_{s=1}^{j+1-\sigma} p_s s! = (j-1)! + j.$$

On the other hand if $\sigma < j$, then we have

$$(\sigma - 1)! + \sum_{s=1}^{j+1-\sigma} p_s s! \leq (j-2)! + (j-\sigma)! j \leq (j-2)!(j+1).$$

Accordingly, since the sums in k and p have at most j^j terms, setting $\tau_j = \{(j-1)! + j, (j-2)!(j+1)\}$,

$$\begin{aligned} &R_{j+1,n} \leq C_\sharp \left\{ j^j c_{\bar n}^{(j-2)!(j+1)} + j^{j+1} C_\flat^{j+1} c_\star^{\tau_j} \mu^{n(j-1)!(j+1)} \right\} \\ &R^\star_{j+1,n} \leq 3^{-j!} \eta^{2\bar n r!} (\bar C_\star \mu^n)^{-j-1} R_{j+1,\bar n}. \end{aligned} \tag{5.4.15}$$

Let us show the first of (5.4.14). Substituting the above in the first of (5.4.7) and using (5.4.9) we have

$$\|h_n^{(j+1)}\| \le \frac{(\bar{C}_\star \mu^n)^{j+1}}{\vartheta} c_\star^{j!} + C_\sharp j^{j+1} \left\{ c_\star^{(j-2)!(j+1)} + C_\flat^{j+1} c_\star^{\tau_j} \mu^{n(j-1)!(j+1)} \right\}.$$

We can finally choose $C_\flat \ge C_\sharp\{2\frac{\bar{C}_\star^r}{\vartheta}, 1\}^+$ and write

$$\|h_n^{(j+1)}\| \le C_\flat c_\star^{j!} \left\{ \frac{1}{2} + j^{j+1} C_\flat^j c_\star^{\tau_j - (j+1)!} \right\} \mu^{n(j+1)!}.$$

Note that for $j = 3$ we have $\tau_3 = 5$, which yields the wanted estimate if $\mathbb{c} \ge 2^5 C_\flat^2$. If $j > 3$, then $\tau_j = (j-2)!(j+1)$ and the first of (5.4.14) follows. Next, we substitute (5.4.15) in (5.4.10) and, using (5.4.12), write

$$\begin{aligned}
\|\hat{\nu}_n^{(j+1)}\| &\le 3^{-j!} \eta^{nj!} \left\{ c_\star^{j!} + C_\sharp r^{r+1} c_{\bar{n}}^{\frac{2}{3}j!} + (\bar{C}_\star \mu^{\bar{n}})^{-j-1} C_\flat^{j+1} c_\star^{\tau_j} \mu^{n\frac{3}{2}j!} \right\} \\
&\le \left\{ \eta^n 3^{-1} \left(c_\star + [C_\sharp r^{r+1}]^{1/j!} c_{\bar{n}}^{\frac{2}{3}} + C_\flat^{2/3} c_\star^{\frac{j+1}{j(j-1)}} \mu^{3\bar{n}} \right) \right\}^{j!}.
\end{aligned}$$

Observing that $c_{\bar{n}}^{2/3} \le (\eta^{\bar{n}} c_\star + 2\mathbb{c})^{2/3} \le \eta^{3\bar{n}/2} c_\star^{\frac{2}{3}} + (2\mathbb{c})^{2/3}$ we have, for each $j > 2$,[10]

$$\|\hat{\nu}_n^{(j+1)}\| \le \left\{ \eta^n \left[c_\star \left(\frac{1}{3} + C_\sharp \eta^{3\bar{n}/2} c_\star^{-\frac{1}{3}} + c_\star^{-\frac{1}{3}} \mu^{3\bar{n}} C_\flat^{2/3} \right) + C_\sharp \sqrt{\mathbb{c}} \right] \right\}^{j!}$$

Hence the second of (5.4.14) will follows if the term in the brace is smaller than $\eta^n c_\star + \mathbb{c}/2$. Choosing $C_\flat$ greater than all the constants $C_\sharp$ appearing in the above equation, it will be enough to check that

$$\begin{aligned}
&\frac{1}{3} + C_\flat \eta^{3\bar{n}/2} c_\star^{-\frac{1}{3}} + c_\star^{-\frac{1}{3}} \mu^{3\bar{n}} C_\flat^{2/3} \le 1; \\
&\eta^n C_\flat^2 \sqrt{\mathbb{c}} \le \mathbb{c}/2.
\end{aligned}$$

Since $c_\star > \mathbb{c}/2$ by assumption, the first equation is satisfied if $\mathbb{c} \ge \frac{9}{4} C_\flat^3 \mu^{3\bar{n}}$ while the second one if $\mathbb{c} \ge 4C_\flat^2$, as $\eta^{2n} < 1$ for each $n \ge \bar{n}$. Collecting all the conditions on $\mathbb{c}$, we see that it must be

$$\mathbb{c} \ge \{2^5 C_\flat, 4C_\flat^2, \frac{9}{4} C_\flat^3 \mu^{3\bar{n}}\}^+ = \frac{9}{4} C_\flat^3 \mu^{3\bar{n}}, \tag{5.4.16}$$

eventually enlarging $C_\flat$. Hence the second of (5.4.14) is satisfied.
In particular $\hat{\nu}_n \in \Gamma_\ell(c_n)$ for each $\ell \le r$ and $\bar{n} \le n \le 2\bar{n}$. Next, let $c_{\star,1} = c_{\bar{n}} \le c_\star$, we have, for each integer $k \ge 2$,

$$\frac{\mathbb{c}}{2} \le c_{\star,k} = \eta^{\bar{n}} c_{\star,k-1} + \frac{\mathbb{c}}{2} \le \eta^{\bar{n}k} c_\star + \frac{\mathbb{c}}{2(1-\eta^{\bar{n}})}.$$

It follows that $\hat{\nu}_{k\bar{n}} \in \Gamma_\ell(c_{\star,k})$ where, for all $m \in \{\bar{n}, \dots 2\bar{n}\}$,[11]

$$\hat{\nu}_{k\bar{n}+m} = \mathfrak{h}_{k\bar{n}+m-1}^* \circ \cdots \circ \mathfrak{h}_{k\bar{n}+1}^* \circ \hat{\nu}_{k\bar{n}} \circ h_{m,k+1}^*,$$

[10] Note that here we are including r^{r+1} into $C_\sharp$ and using that $\frac{j+1}{j(j-1)} \le \frac{2}{3}$

[11] Recall the definition of $\mathfrak{h}_n^*$ in (5.2.2).

$h^*_{\bar n,1} = h_{\bar n}$, and

$$\|h^*_{m,(k+1)}\|_{\mathcal{C}^j} \leq 2C_\flat c_{\star,k}^{(j-1)!}\mu^{j!m}. \tag{5.4.17}$$

Hence, applying iteratively the above argument to $\hat\nu_n$ for $k\bar n \leq n \leq (k+1)\bar n$, we obtain the second of (5.4.14) for each $n \geq \bar n$. It remains to prove the estimate for h_n, $n \geq \bar n$. We write $n = m + k\bar n$, $m \in \{\bar n, \dots, 2\bar n\}$ and

$$h_n = h^*_{m,k+1} \circ h^*_{\bar n,k} \circ \cdots \circ h^*_{\bar n,1} = h^*_{m,k+1} \circ h_{k\bar n}. \tag{5.4.18}$$

Note that (5.4.9) yields $\|h_n\|_{\mathcal{C}^1} \leq C_\flat \mu^n$, provided we choose $C_\flat \geq 3\bar C_\star$. It is then natural to start by investigating the second derivative. In fact, it turns out to be more convenient to study the following ratio

$$\frac{h''_n}{h'_n} = \left(\log[(h^*_{m,k+1})' \circ h_{k\bar n}]\right)' + \frac{h''_{k\bar n}}{h'_{k\bar n}} =: Q_1 + Q_2. \tag{5.4.19}$$

Since (5.3.5) and (5.4.8) imply $|h'_{\bar n,i}| \geq c_0\mu^{-n}$ for each i, for some constant c_0, formula (5.1.3) and (5.4.17) yield $\|\log h^{*\prime}_{m,k}\|_{\mathcal{C}^\ell} \leq C_\flat^{\ell+1} c_{\star,k}^{(\ell-1)!}\mu^{(\ell+1)!m}$, provided $C_\flat$ has been chosen large enough. It then follows immediately that $\|Q_1\|_{\mathcal{C}^0} \leq C_\sharp C_\flat^2 c_{\star,k}\mu^{2m}\mu^{k\bar n} \leq C_\sharp C_\flat^2 c_\star \mu^n$. To estimate $\|Q_2\|_{\mathcal{C}^0}$ we write

$$\frac{h''_{k\bar n}}{h'_{k\bar n}} = \frac{\left(\prod_{i=1}^k h^{*\prime}_{\bar n,i} \circ h_{i\bar n}\right)'}{\prod_{i=1}^k h^{*\prime}_{\bar n,i} \circ h_{i\bar n}} = \left(\log \prod_{i=1}^k h^{*\prime}_{\bar n,i} \circ h_{i\bar n}\right)' = \sum_{i=1}^k \left(\log h^{*\prime}_{\bar n,i} \circ h_{i\bar n}\right)'. \tag{5.4.20}$$

Using formulae (5.4.9), (5.4.13) and (5.4.19) we have, since $\bar n \leq m$,

$$\begin{aligned}\|Q_1\|_{\mathcal{C}^0} &\leq C_\sharp \left\|\sum_{i=1}^k \log h^{*\prime}_{\bar n,i} \circ h_{i\bar n}\right\|_{\mathcal{C}^1} \leq C_\sharp \sum_{i=1}^k \|\log h^{*\prime}_{\bar n,i}\|_{\mathcal{C}^1}\|h'_{i\bar n}\|_{\mathcal{C}^0} \\ &\leq C_\sharp C_\flat^3 c_{\star,1}\mu^{2\bar n}\sum_{i=1}^k \mu^{i\bar n} \leq C_\sharp C_\flat^3 c_{\star,1}\mu^{2\bar n}\frac{1-\mu^{-k\bar n}}{\mu-1}\mu^{k\bar n} \\ &\leq \mu^{2\bar n} C_\flat^3 c_{\star,1} C_{\mu,k\bar n}\mu^{k\bar n} \leq C_\flat^3 c_{\star,1} C_{\mu,n}\mu^n,\end{aligned} \tag{5.4.21}$$

Hence, using the above and (5.4.9), it follows by (5.4.20)

$$\begin{aligned}\|h_n\|_{\mathcal{C}^2} &\leq C_\sharp \left\|\frac{h''_n}{h'_n}\right\|_{\mathcal{C}^0}\|h'_n\|_{\mathcal{C}^0} \leq C_\sharp \mu^n \left\|\frac{h''_n}{h'_n}\right\|_{\mathcal{C}^0} \\ &\leq C_\flat^3 c_\star C_{\mu,n}\mu^{2n} + C_\flat^2 c_\star \mu^{2n} \leq C_\flat^3 c_\star C_{\mu,n}\mu^{2n}.\end{aligned} \tag{5.4.22}$$

This proves the second of (5.4.1). Next we prove the general case by induction on $j \leq \ell$. Assume it true for all $i \leq j$. Using again (5.1.3), by the inductive assumption we have

$$\|Q_1\|_{\mathcal{C}^{j-1}} = \|\log[(h^*_{m,k+1})' \circ h_{k\bar n}]\|_{\mathcal{C}^j} \leq C_\flat^{j+1} c_{\star,1}^{j!}\mu^{m(j+1)!} C_{\mu,n}^{a_j}\mu^{\bar n k j!}. \tag{5.4.23}$$

On the other hand, by formulae (5.4.20), (5.1.3) and the inductive assumption

$$\begin{aligned}\|Q_2\|_{\mathcal{C}^{j-1}} &\leq C_\sharp \sum_{i=1}^k \|\log h^{*\prime}_{\bar n,i}\|_{\mathcal{C}^j}\sum_{q=0}^{j-1}\|h_{i\bar n}\|^q_{\mathcal{C}^j} \\ &\leq C_\sharp C_\flat^{j+1+2(j+1)!j} c_{\star,i}^{j!}\mu^{(j+1)!\bar n}\sum_{i=1}^k\sum_{q=0}^{j-1}(C_{\mu,i\bar n}^{a_j}\mu^{j!i\bar n})^q\end{aligned} \tag{5.4.24}$$

To estimate the last sum, notice that by definition

i $1 \le C_{\mu,\bar n i} \le C_{\mu,\bar n k}, \quad \forall i \le k,$

ii $C_{\mu^a,n} \le C_{\mu,n}, \quad \forall a > 1,$

Hence,

$$\sum_{i=1}^{k}\sum_{q=0}^{j-1}(C_{\mu,\bar n i}^{a_j}\mu^{j!i\bar n})^q \le C_{\mu,\bar n k}^{a_j j}\sum_{i=1}^{k}\mu^{j!(j-1)\bar n i} \le C_{\mu,\bar n k}^{a_j(j-1)+1}\mu^{j!(j-1)\bar n k}.$$

Using this in (5.4.24) we obtain

$$\|Q_2\|_{\mathcal{C}^{j-1}} \le C_\sharp C_\flat^{j+1+2(j+1)!j} c_{\star,i}^{j!}\mu^{(j+1)!\bar n} C_{\mu,\bar n k}^{a_j(j-1)+1}\mu^{j!(j-1)\bar n k}. \tag{5.4.25}$$

Therefore, by the inductive assumption, equations (5.4.23), (5.4.25) and (5.4.19), and provided we choose $C_\flat$ large enough, we finally have[12]

$$\begin{aligned}\|h_n\|_{\mathcal{C}^{j+1}} &\le C_\sharp\|h_n''\|_{\mathcal{C}^{j-1}} \le C_\sharp\left\|\frac{h_n''}{h_n'}\right\|_{\mathcal{C}^{j-1}}\|h_n\|_{\mathcal{C}^j}\\ &\le C_\flat^{2(j+2)!}c_{\star,1}^{(j+1)!}C_{\mu,n}^{a_{j+1}}\mu^{(j+1)!n}.\end{aligned}$$ □

In Section 10 we will need much sharper estimates (but limited to the first derivatives) than the ones provided by Lemma 5.4.1; we prove them next.

Lemma 5.4.2. *Under the hypothesis of Lemma 5.4.1, we choose $\bar c_2$ in (5.2.6), depending on $\bar n$, such that the set $\{\bar n, \dots, \bar c_2 \ln \chi_u^{-1}\}$ is not empty. Then there exist $C_3, C_4, \bar c_1, \bar c_3, c_\flat \ge 1$ uniform such that, for all $n_\star \in \{\bar n, \dots, \bar c_2 \ln \chi_u^{-1}\}$, setting $a_{n_\star} = (\bar c_3)^{n_\star^{-1}}$, $c_{n_\star} = (\bar c_1)^{n_\star^{-1}}$ and*[13]

$$\begin{aligned} b_{n_\star} &:= (C_4 c_\flat \bar c_1^2 \varsigma_{n_\star})^{\frac{1}{n_\star}}\\ \$_{n_\star} &= \{\mu^{2n_\star}\varsigma_{n_\star}^2, C_\flat\varsigma_{n_\star}C_{\mu,n_\star}\mu^{4n_\star}, C_{\mu,n_\star}^2\mu^{6n_\star}\}^+,\end{aligned} \tag{5.4.26}$$

we have, for all $n \ge \bar n$,

$$\begin{aligned}\|\hat\nu_n''(t)\| &\le c_\flat c_{n_\star}^n\mu^{2n}\lambda_n^-(\gamma\circ h_n(t))^{-1}c_\star + C_{\mu,n_\star}\mu^{3n_\star}C_3\\ \|\hat\nu_n'''\| &\le c_\flat a_{n_\star}^n\mu^{3n}(\lambda_n^-(\gamma\circ h_n))^{-1}c_\star^2 + c_\flat b_{n_\star}^n\mu^{3n}(\lambda_n^-(\gamma\circ h_n))^{-1}c_\star + \$_{n_\star}.\end{aligned} \tag{5.4.27}$$

[12] Here we are using the following elementary facts:

- $(j+1)! + 2(j+1)!(j-1) + 2(j+1)! \le 2(j+2)!$
- $j(j+1)! + (j+1)! \le (j+2)!$
- $a_j(j-1) + a_j + 1 = a_{j+1}$.

[13] Recall (5.3.4) for the definition of ς_n.

Proof. To prove the first of (5.4.27) it is convenient to go back to equation (5.4.3) and, recalling (5.3.8), for each $v \in \mathbb{R}^2$, $\|v\| = 1$, we have

$$\begin{aligned}
&\left|\langle v, \hat\nu_n''\rangle - \langle v, \hat\nu_n'\rangle \frac{h_n''}{h_n'}\right| \leq |\langle v, (D_{\hat\nu_n}F^n)^{-1}\gamma'' \circ h_n (h_n')^2\rangle| \\
&+ \sum_{k=0}^{n-1}\sum_{i=1}^{2} \left|\langle v, (D_{\hat\nu_n}F^{k+1})^{-1}\left[\partial_{x_i} D_{F^k(\hat\nu_n)}F\right] D_{\hat\nu_n}F^k \hat\nu_n'\rangle\right| \|(D_{\hat\nu_n}F^k)\hat\nu_n'\| \\
&\leq |\langle v, (D_{\hat\nu_n}F^n)^{-1}\gamma'' \circ h_n (h_n')^2\rangle| + C_\sharp \sum_{k=0}^{n-1} \|(D_{\hat\nu_n}F^{k+1})^{-1}\|\|F\|_{\mathcal{C}^2}\|(D_{\hat\nu_n}F^k)\hat\nu_n'\|^2.
\end{aligned} \tag{5.4.28}$$

Note that, recalling (5.2.6), for each $n \leq n_\star \leq \bar c_2 \log \chi_u^{-1}$ we have $(D_{\hat\nu_n(t)}F^n)^{-1}e_1 \notin \mathbf{C}_c$. Consequently

$$|\langle v, (D_{\hat\nu_n}F^n)^{-1}\gamma'' \circ h_n\rangle| \leq (\lambda_n^-(\hat\nu_n(t)))^{-1}\|\gamma'' \circ h_n(t)\|, \quad \forall n \leq n_\star. \tag{5.4.29}$$

Next, if v is perpendicular to $\hat\nu_n'$, then it must be $|v_2| \leq \chi_c |v_1|$, hence

$$|\langle v, \hat\nu_n''\rangle| = |v_1|\|\hat\nu_n''\| \geq (1+\chi_c^2)^{-\frac{1}{2}}\|\hat\nu_n''\|. \tag{5.4.30}$$

On the other hand, if v is perpendicular to $\hat\nu_n''$, then $v = e_2$ and $|\langle v, \hat\nu_n'\rangle| = 1$. Accordingly, recalling Proposition 5.3.3 and equations (4.1.5), (5.4.9) we have for $n \leq n_\star$

$$\begin{aligned}
&\|\hat\nu_n''(t)\| \leq (1+\chi_c^2)^{\frac{1}{2}}(\lambda_n^-(\hat\nu_n(t)))^{-1}C_\flat^2\mu^{2n}\|\gamma'' \circ h_n(t)\| + \sum_{k=0}^{n-1} C_\star^3\mu^{3k}C_\sharp, \\
&\|h_n''/h_n'\| \leq (\lambda_n^-(\hat\nu_n(t)))^{-1}C_\flat^2\mu^{2n}\|\gamma'' \circ h_n(t)\| + \sum_{k=0}^{n-1} C_\star^3\mu^{3k}C_\sharp.
\end{aligned} \tag{5.4.31}$$

Setting $c_{n_\star} = \left[(1+\chi_c^2)^{\frac{1}{2}}C_\flat^2\right]^{\frac{1}{n_\star}}$ we obtain

$$\|\hat\nu_{n_\star}''(t)\| \leq c_{n_\star}^{n_\star}\mu^{2n_\star}(\lambda_{n_\star}^-(\hat\nu_{n_\star}(t)))^{-1}\|\gamma'' \circ h_{n_\star}(t)\| + \sum_{k=0}^{n_\star-1} C_\star^3\mu^{3k}C_\sharp \tag{5.4.32}$$

We can now proceed by induction since, setting $h_{l,m}^* = h_{ln_\star+m} \circ h_{ln_\star}^{-1}$, if $n = ln_\star + m$, $m \leq n_\star$, then

$$\begin{aligned}
&\|\hat\nu_n''(t)\| \leq c_{n_\star}^m\mu^{2m}(\lambda_m^-(\hat\nu_n(t)))^{-1}\|\hat\nu_{ln_\star}'' \circ h_{l,m}^*(t)\| + \sum_{k=0}^{n_\star-1} C_\star^3\mu^{3k}C_\sharp \\
&\leq c_{n_\star}^n\mu^{2n}(\lambda_m^-(\hat\nu_n(t)))^{-1}(\lambda_{n_\star}^-(\hat\nu_{ln_\star} \circ h_{l,m}^*(t)))^{-1}\dots(\lambda_{n_\star}^-(\gamma \circ h_n(t)))^{-1}c_\star \\
&\quad + \sum_{s=1}^{l} c_{n_\star}^{sn_\star}\mu^{2sn_\star}\lambda_-^{-sn_\star}\sum_{k=0}^{n_\star-1} C_\star^3\mu^{3k}C_\sharp \\
&\leq c_{n_\star}^n\mu^{2n}c_\flat c_\flat^{\frac{n}{n_\star}}(\lambda_n^+(\gamma \circ h_n(t)))^{-1}c_\star + C_{\mu,n_\star}\mu^{3n_\star}
\end{aligned} \tag{5.4.33}$$

where we have called $c_\flat$ the constant in (5.3.5). It remains to bound the third derivative of $\hat\nu_n$. The strategy is basically the same. Recalling that $\hat\nu_n' = (D_{\hat\nu_n}F^n)^{-1}\gamma'\circ h_n h_n'$, we differentiate this expression twice and multiply by a unitary vector v orthogonal to $\hat\nu_n'$:

$$\begin{aligned}\langle \hat\nu_n''', v\rangle = \Big\langle &[(D_{\hat\nu_n}F^n)^{-1}]''\gamma'\circ h_n h_n' + 2[(D_{\hat\nu_n}F^n)^{-1}]'(\gamma''\circ h_n (h_n')^2 + \gamma'\circ h_n h_n'')\\ &+ [(D_{\hat\nu_n}F^n)^{-1}]\left(\gamma'''\circ h_n (h_n')^3 + 3\gamma''\circ h_n h_n' h_n''\right), v\Big\rangle.\end{aligned} \tag{5.4.34}$$

We will estimate the norms of the terms in the first line of the above equation one at a time, for each $n\le n_\star$. First, using (5.3.6) with $m=0$ and $\mathbb{c} = \|\hat\nu_n''\|$ (where the latter is estimated using (5.4.31)), and (5.4.1) we have, for some uniform $A_1>0$

$$\begin{aligned}\|[(D_{\hat\nu_n}F^n)^{-1}]''\gamma'\circ h_n h_n'\| \le\ & \mu^{2n}\varsigma_n^2 + A_1 C_\flat \varsigma_n c_{n_\star}^{n_\star}\mu^{3n}(\lambda_n^-(\hat\nu_n(t)))^{-1}\|\gamma''\circ h_n(t)\|\\ &+ C_\sharp C_\flat\varsigma_n\mu^n C_{\mu,n_\star}\mu^{3n_\star}.\end{aligned}$$

Next, notice that $(D_{\hat\nu_{n_\star}}F^{n_\star})^{-1}\gamma''\notin \mathbf{C}_c$, hence by the second of (5.3.8) and subsequent, there is uniform $A_2>0$ such that

$$\|[(D_{\hat\nu_n}F^n)^{-1}]'\gamma''\circ h_n(h_n')^2\| \le A_2 C_\flat^2\mu^{3n}(\lambda_n^-(\hat\nu_n(t)))^{-1}\varsigma_n\|\gamma''\circ h_n(t)\|. \tag{5.4.35}$$

It is convenient to write the third term as

$$\begin{aligned}[D_{\hat\nu_n}F^n)^{-1}]'\gamma'\circ h_n h_n'' &= \frac{h_n''}{h_n'}[D_{\hat\nu_n}F^n)^{-1}]'\gamma'\circ h_n h_n'\\ &= \frac{h_n''}{h_n'}\left(\hat\nu_n'' - [(D_{\hat\nu_n}F^n)^{-1}]\gamma''\circ h_n(h_n')^2 - \hat\nu_n'\frac{h_n''}{h_n'}\right).\end{aligned}$$

The last term vanishes when we multiplied by v; hence, by (5.4.29) and (5.4.31), we have[14]

$$\begin{aligned}\left|\langle [D_{\hat\nu_n}F^n)^{-1}]'\gamma'\circ h_n h_n'', v\rangle\right| &\le \left|\frac{h_n''}{h_n'}\right|\left\{\|\hat\nu_n''\| + \|[D_{\hat\nu_n}F^n)^{-1}]\gamma''\circ h_n(h_n')^2\|\right\}\\ &\le c_{n_\star}^{n_\star}C_\flat^2\mu^{4n}(\lambda_n^-(\hat\nu_n(t)))^{-2}\|\gamma''\circ h_n(t)\|^2\\ &\quad + c_{n_\star}^{n_\star}C_{\mu,n}\mu^{5n}(\lambda_n^-(\hat\nu_n(t)))^{-1}\|\gamma''\circ h_n(t)\| + C_{\mu,n}^2\mu^{6n}.\end{aligned}$$

For the two terms in the second line of (5.4.34), when the matrix hits γ'' or γ''', we can use (5.4.29) for $n\le n_\star$ and (5.4.1) with $\|\gamma'\circ h_n(t)\|$ instead of $c_\star$. Collecting all the above estimates in (5.4.34) we finally have, recalling also (5.4.30),

$$\begin{aligned}(1+\chi_c^2)^{-\frac12}\|\hat\nu_n'''\| \le\ & C_\sharp\mu^{2n}\varsigma_n^2 + A_1C_\flat\varsigma_n c_{n_\star}^{n_\star}\mu^{3n}(\lambda_n^-(\hat\nu_n(t)))^{-1}\|\gamma''\circ h_n(t)\|\\ &+ C_\flat\varsigma_n\mu^n C_{\mu,n}\mu^{3n}\\ &+ A_2C_\flat^2\mu^{3n}\varsigma_n(\lambda_n^-(\hat\nu_n(t)))^{-1}\|\gamma''\circ h_n(t)\|\\ &+ c_{n_\star}^{n_\star}C_\flat^2\mu^{4n}(\lambda_n^-(\hat\nu_n(t)))^{-2}\|\gamma''\circ h_n(t)\|^2\\ &+ c_{n_\star}^{n_\star}C_{\mu,n}\mu^{5n}(\lambda_n^-(\hat\nu_n(t)))^{-1}\|\gamma''\circ h_n(t)\| + C_{\mu,n}^2\mu^{6n}\\ &+ C_\flat^3\mu^{3n}(\lambda_n^-(\hat\nu_n(t)))^{-1}\|\gamma'''\circ h_n(t)\|\\ &+ (\lambda_n^-(\hat\nu_n(t)))^{-1}C_\flat^2\mu^{3n}C_{\mu,n}\|\gamma'\circ h_n(t)\|.\end{aligned}$$

[14]Recall also the lower bound for $|h_n'|$ in (5.4.9).

Hence, setting $\tilde a_{n_\star} = [(1+\chi_c^2)^{1/2} C_\flat^3]^{1/n_\star}$, $\tilde b_{n_\star} = [\{A_1, A_2\}^+ (1+\chi_c^2)^{1/2} C_\flat \varsigma_{n_\star}]^{1/n_\star} c_{n_\star}$ and recalling the second of (5.4.26) we

$$\begin{aligned}\|\hat\nu_n'''\| &\le \tilde a_{n_\star}^{n_\star} \mu^{3n} (\lambda_n^-(\hat\nu_n(t)))^{-1} \|\gamma''' \circ h_n(t)\| + c_{n_\star}^{2n_\star} \mu^{4n} (\lambda_n^-(\hat\nu_n(t)))^{-2} \|\gamma'' \circ h_n(t)\|^2 \\ &\quad + \tilde b_{n_\star}^{n_\star} (\lambda_n^-(\hat\nu_n))^{-1} \mu^{5n} \|\gamma'' \circ h_n(t)\| + \lambda_n^-(\hat\nu_n(t)))^{-1} c_{n_\star}^{n_\star} \mu^{3n} C_{\mu,n} \|\gamma' \circ h_n(t)\| + \mathbb{s}_{n_\star}.\end{aligned}$$

We can now iterate as in (5.4.33), using the latter to estimate the terms involving γ'' and γ' and, proceeding by induction, we obtain[15]

$$\|\hat\nu_n'''\| \le c_\flat^{\frac{n}{n_\star}+1} \tilde a_{n_\star}^n \mu^{3n} (\lambda_n^-(\gamma \circ h_n))^{-1} c_\star^2 + c_\flat^{\frac{n}{n_\star}+1} \tilde b_{n_\star}^n \mu^{3n} (\lambda_n^-(\gamma \circ h_n))^{-1} c_\star + \mathbb{s}_{n_\star},$$

from which the third of (5.4.27) follows setting $C_4 = \{A_1, A_2\}^+, \bar c_1 = c_\flat (1+\chi_c^2)^{\frac12} C_\flat^2, \bar c_3 = c_\flat (1+\chi_c^2)^{\frac12} C_\flat^3$, and the Lemma is proved. □

Lemmata 5.4.1 and 5.4.2 imply immediately the following important result.

Corollary 5.4.3. *Let $n_\star \in \{\bar n, \dots, \bar c_2 \ln \chi_u^{-1}\}$ and $\ell \in \{2, \cdots, r\}$. We define a constant $\mathbb{c}$ which depend on ℓ and $n_\star$ as follows: if $\ell \in \{2,3\}$ we set*

$$\mathbb{c} = 2\mathbb{s}_{n_\star} = 2\{\mu^{2n_\star} \varsigma_{n_\star}^2, C_\flat \varsigma_{n_\star} C_{\mu,n_\star} \mu^{4n_\star}, C_{\mu,n_\star}^2 \mu^{6n_\star}\}^+, \tag{5.4.36}$$

if $\ell > 3$ we set $\mathbb{c} = \chi_u^{-\varpi_{\bar n,\chi_u}}$, for some $\varpi_{\bar n,\chi_u} \ge 1$. [16] *Then for all $n \ge n_\star$ and such that $\eta^n \le \frac12$, and for each $\ell \in \{2, \cdots, r\}$, we have the inclusion $F^{-n}\Gamma_\ell(\mathbb{c}) \subset \Gamma_\ell(\mathbb{c})$.*

Proof. The case $\ell > 3$ follows directly by Lemma 5.4.1 choosing $\varpi_{\bar n,\chi_u}$ and $c_\star$ such that

$$c_\star = \mathbb{c} = \chi^{-\varpi_{\bar n,\chi_u}} = \frac92 C_\flat^3 \mu^{3\bar n}. \tag{5.4.37}$$

The result then follows since $\eta^n \le \frac12$. For $\ell = 3$, first we note that, recalling (5.3.4), $\mathbb{c}$ given in (5.4.36) is greater than $\frac92 C_\flat^3 \mu^{3\bar n}$, eventually enlarging $\bar n \le n_\star$. On the other hand, recalling (5.4.27) and since $c_\star \ge \mathbb{c}/2$, for each $n_\star \le n \le 2n_\star$ we have

$$\begin{aligned}&c_\flat a_{n_\star}^n \mu^{3n} (\lambda_n^-(\gamma \circ h_n))^{-1} c_\star^2 + c_\flat b_{n_\star}^n \mu^{3n} (\lambda_n^-(\gamma \circ h_n))^{-1} c_\star \\ &\qquad \le c_\star^2 \left\{ c_\flat a_{n_\star}^{2n_\star} \mu^{6n_\star} \lambda_-^{-n_\star} + c_\flat b_{n_\star}^{2n_\star} \mu^{6n_\star} \lambda_-^{-n_\star} \mathbb{s}_{n_\star}^{-1} \right\}.\end{aligned} \tag{5.4.38}$$

Since $\lambda_-^{-1}\mu^6 < 1$ and $a_{n_\star}^{2n_\star} = \bar c_3^2$, the first addend in the brace is strictly smaller than $2^{-(n_\star+1)}$ provided $\bar n$ has been chosen large enough at the beginning. For the other term, recall the definition of $b_{n_\star}$ in (5.4.26). We then have $b_{n_\star}^{2n_\star} = (C_4 c_\flat \bar c_1^2)^2 \varsigma_{n_\star}^2$. Moreover, an easy computation shows that

$$\frac{\varsigma_{n_\star}^2}{\mathbb{s}_{n_\star}} \le \left\{ \mu_{n_\star}^{-2}, \frac{C_{\mu,n_\star}}{C_\flat \sigma_{n_\star}} \right\}^+$$

Hence, the second addend in the brace in (5.4.38) can be made smaller than $2^{-(n_\star+1)}$ as well, for $\bar n$ large enough. Iterating the above argument we obtain, for each $n \ge n_\star$,

$$\|\hat\nu_n'''\| \le \frac12 c_\star^2 + \mathbb{s}_{n_\star},$$

form which we conclude the proof for $\ell = 3$ choosing $c_\star^2 = \mathbb{c} = 2\mathbb{s}_{n_\star}$. The case $\ell = 2$ is made in a similar but easier manner. □

[15] Here we use again that $\mu^r \lambda^{-1} < 1$.

[16] Unfortunately, this yields worst estimates, this is why we make such a choice only for $\ell > 3$.

From now until the end of this section we fix $\bar{n}$ as in Lemma 5.4.1.

The above results tell us that the space of admissible central curves is stable under backward iteration of the map. Arguing as above, but forward in time, we can prove that the space of admissible unstable curves is stable under the iteration of F^n, for n greater than $\bar{n}$. In particular, if $\eta : I \to \mathbb{T}^2$ is an admissible unstable curve, and η_n is the image of η under F^n, then there exists a diffeomorphism $p_{n,\eta} =: p_n$ such that

$$p_n'(t) = \frac{\|D_\eta F^n \cdot \eta'(t)\|}{\|\eta'(t)\|}, \tag{5.4.39}$$

and $\eta_n \circ p_n = F^n \circ \eta$ is an admissible unstable curve. Moreover, as F acts as an expanding map along those curves, we have the following standard distortion estimate for each $n \geq 1$:

$$\frac{p_n'(t)}{p_n'(s)} \lesssim 1, \quad \forall t, s \in I. \tag{5.4.40}$$

In the following we will need to control the evolution also of curves not in the center cone. To this end it is convenient to introduce a further quantity. Given a smooth curve γ such that $\pi_1 \circ \gamma'(t) \neq 0$ for each $t \in \mathbb{T}$, let

$$\begin{aligned} \vartheta_\gamma(t) &= \left\{ \frac{|\pi_2 \circ \gamma'(t)|}{|\pi_1 \circ \gamma'(t)|}, \chi_u \right\}^+ \\ \vartheta_\gamma &= \inf_t \{\vartheta_\gamma(t))\}. \end{aligned} \tag{5.4.41}$$

Lemma 5.4.4. *Let F be a SVPH and $\Delta_\gamma \in L^\infty(\mathbb{T}^1, [1, +\infty])$ and consider any closed curve $\gamma \in \mathcal{C}^r$, homotopic to $(0,1)$, such that $\|\gamma'(t)\| = 1$ and $\|\gamma^{(j+1)}(t)\| \leq \Delta_\gamma(t)^j$,*[17] *for all $j \in \{1, \ldots, r\}$ and $t \in \mathbb{T}$. For $\mathfrak{h} \in \mathfrak{H}^\infty$ let $n_0 \geq 0$ and $m > \{\bar{n}, n_0\}^+$ be the smallest integers such that, for all $t \in \mathbb{T}$,*

$$D_{\gamma(t)} \mathfrak{h}_{n_0} \gamma'(t) \notin \mathbf{C}_u \quad \text{and} \quad D_{\gamma(t)} \mathfrak{h}_m \gamma'(t) \in \operatorname{Int}(\mathbf{C}_c). \tag{5.4.42}$$

Let $\nu_k = \mathfrak{h}_k(\gamma)$, $k \in \mathbb{N}$. Note that for $k \geq n_0$ then there exists a reparametrization h_k such that, setting $\hat{\nu}_k = \nu_k \circ h_k$, $\pi_2 \circ \hat{\nu}_k(t) = t$.

If, for some $\mathfrak{h}$, we have $m < \infty$, then:
***a)** For $\eta < 1$, given in Lemma 5.4.1,*[18] *Λ as in (5.1.4), $\overline{m} = \sigma m$, where*

$$\sigma = \left\lceil \frac{\ln(\mu \|\Delta_\gamma\|_\infty \Lambda)^{-1}}{\ln \eta} \right\rceil, \tag{5.4.43}$$

and $\mathbb{c}$ as in Corollary 5.4.3, we have $\hat{\nu}_{\overline{m}} \in \Gamma_j(\mathbb{c})$ for each $j \geq 3$, and the $\mathcal{C}^j$-norm of $h_{\overline{m}}$ satisfies (5.4.1) with $c_\star = \chi_u^{-1} \|\Delta_\gamma\|_\infty (\mu\Lambda)^m$.
In the case $j \in \{1, 2\}$ we have the following sharper version:
***b)** For each $p \in \gamma$ and $n_\star \in \{\bar{n}, \cdots, \bar{c}_2 \log \chi_u^{-1}\}$, let $\overline{m}(p, \mathfrak{h}, n_\star) \equiv \overline{m}$ be the minimum integer such that*

$$\begin{aligned} \eta_{n_\star}(\overline{m}, m; t) M_{n_0}(m, t) &\leq C_{\mu, n_\star} \mu^{3n_\star}, \\ \bar{\eta}_{n_\star}(\overline{m}, m; t) \overline{M}_{n_0}(m, t) &\leq s_{n_\star}, \end{aligned} \tag{5.4.44}$$

[17] We will apply this Lemma with $\Delta_\gamma(t)$ given by (E.0.1).
[18] See (5.4.11) for a precise definition of η.

where

$$
\begin{aligned}
&\eta_{n_\star}(\overline{m}, m; t) := c_\flat\{b_{n_\star}^{\overline{m}}, c_{n_\star}^{\overline{m}}\}^+\mu^{3\overline{m}}\lambda_{\overline{m}}^+(\hat{\nu}_m \circ h_{\overline{m}-m}(t))^{-1},\\
&\bar{\eta}_{n_\star}(\overline{m}, m; t) := c_\flat a_{n_\star}^{\overline{m}}\mu^{3\overline{m}}\lambda_{\overline{m}}^+(\hat{\nu}_m \circ h_{\overline{m}-m}(t))^{-1},\\
&M_{n_0}(m;t) := \Big\{\Lambda^{2n_0}\mu^m\Delta_\gamma(t), (1+\mu^{2m}\vartheta_{\hat{\nu}_{n_0}}^{-1}\|\omega\|_{\mathcal{C}^2})\lambda_{m-n_0}^+(\hat{\nu}_m(t))\Big\}^+\\
&\overline{M}_{n_0}(m,t) := \Big\{\mu^{4m}\Lambda^{3n_0}\Delta_\gamma^2(t), M_{n_0}(m,t)\vartheta_{\hat{\nu}_{n_0}}^{-1}, \vartheta_{\hat{\nu}_{n_0}}^{-2}, \lambda_m^+(\hat{\nu}_m(t))\vartheta_{\nu_{n_0}}^{-1}\Big\}^+.
\end{aligned}
\tag{5.4.45}
$$

and $a_{n_\star}, b_{n_\star}, c_{n_\star}$, $\mathfrak{s}_{n_\star}$ *are defined in Lemma 5.4.2. Then* $\hat{\nu}_{\overline{m}} \in \Gamma_3(\mathbb{c})$ *and*

$$
\begin{aligned}
&C_\sharp\Lambda^{-n_0}\vartheta_{\hat{\nu}_{n_0}}(t)^{-1}\mu^{-\overline{m}} \le |h_{\overline{m}}'(t)| \le C_\sharp\Lambda^{n_0}\vartheta_{\hat{\nu}_{n_0}}(t)^{-1}\mu^{\overline{m}}\\
&|h_{\overline{m}}''(t)| \le C_\sharp\overline{M}_{n_0}(m,t).
\end{aligned}
\tag{5.4.46}
$$

Proof. Let us start proving item **a**) first. Let $\mathfrak{h} \in \mathfrak{H}^\infty$ such that $\nu_m = \mathfrak{h}_m\gamma$. Recalling (5.1.4), we can apply (5.1.3) and we have for each $j \le r$

$$
\|\nu_m\|_{\mathcal{C}^{j+1}} = \|\mathfrak{h}_m \circ \gamma\|_{\mathcal{C}^{j+1}} \le C_\sharp(\|\Delta_\gamma\|_\infty\Lambda^m)^j. \tag{5.4.47}
$$

We set $\phi(t) := (\pi_2 \circ \nu_m)(t)$. By (5.4.42) there exists $c_{u,\gamma} \ge \chi_u\mu^{-m}$ such that we have $|\phi'| > c_{u,\gamma} > 0$, so it is well defined the diffeomorphism $h_m(t) = \phi^{-1}(t)$, so that $\hat{\nu}_m = \nu_m \circ h_m$ is parametrized by vertical length. We want to estimate the higher order derivatives of h_m using a formula for inverse functions given in [39]. For the reader convenience we write it down here for our case:

$$
h_m^{(j+1)}(t) = \frac{d^{j+1}\phi^{-1}(t)}{dt^{j+1}} = \sum_{k=0}^{j}[\phi'(t)]^{-j-k-1}\sum_{\substack{b_1+\cdots+b_k=j+k\\ b_l\ge 2}} B_{j,k,\{b_l\}_{l=1}^k}\prod_{l=1}^{k}\phi^{(b_l)}(t),
\tag{5.4.48}
$$

where $B_{j,k,\{b_l\}_{l=1}^k} = \frac{(j+k)!}{k!b_1!\cdots b_k!}$. It follows by (5.4.47) and (5.4.48) that for each t

$$
|h_m^{(j+1)}(t)| \le C_\sharp\left(c_{u,\gamma}^{-2}\|\Delta_\gamma\|_\infty\Lambda^m\right)^j. \tag{5.4.49}
$$

By (5.4.47), (5.4.49) and formula (5.1.3) for the composition,

$$
\begin{aligned}
\|\hat{\nu}_m\|_{\mathcal{C}^{j+1}} = \|\nu_m \circ h_m\|_{\mathcal{C}^{j+1}} &\le C_\sharp\sum_{s=0}^{j+1}\|\hat{\nu}_m\|_{\mathcal{C}^s}\sum_{k\in\mathcal{K}_{\rho,s}}\prod_{l\in\mathbb{N}}\|h_m\|_{\mathcal{C}^l}^{k_l}\\
&\le C_\sharp(\bar{c}_{u,\gamma}\|\Delta_\gamma\|_\infty\Lambda^{c_\sharp m})^{2j} \le (\bar{c}_{u,\gamma}\|\Delta_\gamma\|_\infty\Lambda^{c_\sharp m})^{(j+1)!},
\end{aligned}
\tag{5.4.50}
$$

where $\bar{c}_{u,\gamma} = \{c_{u,\gamma}^{-2}, 1\}^+$. Hence, setting $c_\star(m) = \bar{c}_{u,\gamma}\|\Delta_\gamma\|_\infty\Lambda^m$ we have that $\hat{\nu}_m \in \Gamma_j(c_\star(m))$. Since $\overline{m} > m > \bar{n}$ we can apply Lemma 5.4.1 and we have that the curve $\hat{\nu}_{\overline{m}} = \nu_m \circ h_{\overline{m}}$ belongs to $\Gamma_j(\eta^{\overline{m}}c_\star(m) + \frac{\mathbb{c}}{2})$. By definition, $c_\star(m) \le \chi_u^{-2}\|\Delta_\gamma\|_\infty(\mu\Lambda)^m$ and by Corollary 5.4.3, $\mathbb{c} \ge \bar{c}_{u,\gamma}$ (since $j \ge 3$), having chosen ϖ large enough. The statement then follows choosing $\overline{m} = \sigma m$, with σ defined in (5.4.43).

Let us prove item **b**). Let $\nu_n = \mathfrak{h}_n \circ \gamma$ for each $n \in \mathbb{N}$. Then, $C_\sharp\vartheta_\gamma(t)|\pi_1 \circ \nu_{n_0}'(t)| \ge |\pi_2 \circ \nu_{n_0}'(t)| \ge \vartheta_\gamma(t)|\pi_1 \circ \nu_{n_0}'(t)| > 0$, and we can reparametrize ν_n, $n \ge n_0$, by vertical length $\hat{\nu}_n(t) = \nu_n(h_n(t))$. Note that $\|\hat{\nu}_{n_0}'(t)\| \le C_\sharp\vartheta_\gamma(t)^{-1}$.

If $n_0 = 0$, then $C_\sharp \vartheta_{\hat\nu_0}(t)^{-1} = C_\sharp \vartheta_\gamma \circ h_0(t)^{-1} \leq |h_0'(t)| \leq C_\sharp \vartheta_{\hat\nu_0}(t)^{-1}$ and (5.4.28) yields[19]

$$
\begin{aligned}
|h_{n_0}''(t)| &\leq \frac{\|\gamma'' \circ h_{n_0}(t)\| |h_{n_0}'(t)|^3}{|\langle e_1, \hat\nu_{n_0}'(t)\rangle|} \leq C_\sharp \Delta_\gamma \circ h_{n_0}(t) \vartheta_{\hat\nu_{n_0}}(t)^{-2} \\
\|\hat\nu_{n_0}''(t)\| &\leq C_\sharp \Delta_\gamma \circ h_{n_0}(t) \vartheta_{\hat\nu_{n_0}}(t)^{-1}.
\end{aligned}
$$

If $n_0 > 0$, then $C_\sharp \Lambda^{-n_0} \vartheta_{\hat\nu_{n_0}}(t)^{-1} \leq |h_{n_0}'(t)| \leq C_\sharp \Lambda^{n_0} \vartheta_{\hat\nu_{n_0}}(t)^{-1}$ and $\|\nu_{n_0}''(t)\| \leq C_\sharp \Lambda^{2n_0} \|\gamma''(t)\|$. Moreover

$$
\begin{aligned}
|h_{n_0}''(t)| &\leq C_\sharp \Lambda^{3n_0} \Delta_\gamma \circ h_{n_0}(t) \vartheta_{\hat\nu_{n_0}}(t)^{-2} \\
\|\hat\nu_{n_0}''(t)\| &\leq C_\sharp \Lambda^{2n_0} \Delta_\gamma \circ h_{n_0}(t) \vartheta_{\hat\nu_{n_0}}(t)^{-1} \\
\|\hat\nu_{n_0}'''(t)\| &\leq C_\sharp \Lambda^{3n_0} \Delta_\gamma^2 \circ h_{n_0}(t) \vartheta_{\hat\nu_{n_0}}(t)^{-2}.
\end{aligned} \tag{5.4.51}
$$

Remark that

$$
\|(D_{\hat\nu_m(t)} F^k) \hat\nu_m'(t)\| \leq \sqrt{1+\chi_c^2} C_\star \lambda_k^+(\hat\nu_m(t)),
$$

and, setting $F^{m-n_0} \hat\nu_m = \hat\nu_{n_0} \circ \bar h_{m-n_0}$, we have

$$
\begin{aligned}
|\bar h_{m-n_0}'(t)| &= |\langle e_2, D_{\hat\nu_m(t)} F^{m-n_0} \hat\nu_m'(t)\rangle| \leq C_\sharp \lambda_{m-n_0}^+(\hat\nu_m(t)) \vartheta_{\hat\nu_{n_0}}(\bar h_{m-n_0}(t)) \\
|\bar h_{m-n_0}'(t)| &\geq C_\sharp \lambda_{m-n_0}^-(\hat\nu_m(t)) \vartheta_{\hat\nu_{n_0}}(\bar h_{m-n_0}(t)).
\end{aligned} \tag{5.4.52}
$$

Next, we want to use equation (5.4.28), with γ replaced by $\hat\nu_{n_0}$. Note that there exists $\xi_i \in \mathcal{C}^{r-2}$, $\|\xi_i\|_{\mathcal{C}^{r-2}} \leq C_\sharp$, such that, for all $w \in \mathbb{R}^2$,

$$
\left\| \partial_{x_i} (D_{\hat\nu_{n_j}} F) w - e_1 \langle \xi_i, w\rangle \right\| \leq C_\sharp \|w\| \|\omega\|_{\mathcal{C}^2}. \tag{5.4.53}
$$

In addition, it must be $(D_{\hat\nu_m} F^k)^{-1} e_1 \notin \mathbf{C}_c$, for all $k < m - n_0$, otherwise, by the monotonicity of the dynamics in the tangent bundle, it would be that $\hat\nu_{m-1} \in \mathbf{C}_c$ contrary to the hypothesis. Accordingly, recalling (4.1.3), (5.3.6), (5.3.5) and setting $m_0 = m - n_0$,

$$
\begin{aligned}
\|(D_{\hat\nu_m} F^{k+1})^{-1} \left[\partial_{x_i} D_{F^k(\hat\nu_m)} F\right] D_{\hat\nu_m} F^k \hat\nu_m'\| &\leq \frac{\lambda_k^+(\hat\nu_m)}{\lambda_{k+1}^-(\hat\nu_m)} + C_\sharp \mu^{k+1} \|\omega\|_{\mathcal{C}^2} \lambda_k^+(\hat\nu_m) \\
&\leq C_\sharp \left(1 + \mu^{k+1} \lambda_k^+(\hat\nu_m) \|\omega\|_{\mathcal{C}^2}\right).
\end{aligned} \tag{5.4.54}
$$

Arguing as in the proof of Lemma 5.4.1, just after (5.4.28), the above and (5.4.51) yields,

$$
\begin{aligned}
\|\hat\nu_m''(t)\| &\leq C_\sharp \lambda_{m_0}^+(\hat\nu_m(t)) \vartheta_{\hat\nu_{n_0}}(\bar h_{m_0}(t)) \Lambda^{2n_0} \Delta_\gamma \circ h_{n_0}(t) \\
&\quad + \sum_{k=0}^{m_0-1} C_\sharp \left\{1 + \mu^k \lambda_k^+(\hat\nu_m(t)) \|\omega\|_{\mathcal{C}^2}\right\} \lambda_k^+(\hat\nu_m(t)) \\
|\bar h_{m_0}''(t)| &\leq C_\sharp (\lambda_{m_0}^+(\hat\nu_m(t)))^2 \vartheta_{\hat\nu_{n_0}}(\bar h_{m_0}(t))^2 \Lambda^{2n_0} \Delta_\gamma \circ h_{n_0}(t) \\
&\quad + \sum_{k=0}^{m_0-1} C_\sharp \left\{1 + \mu^k \lambda_k^+(\hat\nu_m(t)) \|\omega\|_{\mathcal{C}^2}\right\} \lambda_k^+(\hat\nu_m(t))^2 \vartheta_{\hat\nu_{n_0}}(\bar h_{m_0}(t)).
\end{aligned}
$$

To continue we need the following

[19] Note that (5.4.28) holds also if γ is not parametrized vertically.

Sublemma 5.4.5. *If m_0 is the smallest integer for which $\hat{\nu}'_{m_0}(t) \notin \mathbf{C}_c$ for each t, then*

$$\chi_u \lambda^+_{m_0}(\hat{\nu}_{m_0}(t)) \le C_\sharp \chi_c^{-1} \mu^{m_0}, \quad \forall t \in \mathbb{T}^2. \tag{5.4.55}$$

Proof. If we define w, $\|w\| = 1$, such that $DF^{m_0} w = \|DF^{m_0} w\| e_2$, then $\hat{\nu}'_{m_0} = \alpha e_1 + \beta w$, with $c_\sharp \le |\alpha|, |\beta| \le C_\sharp$. Then, since $D_{\hat{\nu}_{m_0}} F^{m_0} e_1 \in \mathbf{C}_u$, $w \in \mathbf{C}_c$ and using (4.1.3) again,

$$\begin{aligned} &C_\sharp \lambda^+_{m_0} \circ \hat{\nu}_{m_0} \ge |\langle e_1, D_{\hat{\nu}_{m_0}} F^{m_0} \hat{\nu}'_{m_0} \rangle| \ge C_\sharp \lambda^-_{m_0} \circ \hat{\nu}_{m_0} \\ &|\langle e_2, D_{\hat{\nu}_{m_0}} F^{m_0} \hat{\nu}'_{m_0} \rangle| \le C_\sharp (\mu^{m_0} + \lambda^+_{m_0} \circ \hat{\nu}_{m_0} \chi_u). \end{aligned} \tag{5.4.56}$$

Next, let $v \in \mathbb{R}^2$, $\|v\| = 1$ such that $DF^{m_0} v = \|DF^{m_0} v\|(1, \chi_u)$. Note it must be $v \notin \mathbf{C}_c$, otherwise we would have $\hat{\nu}'_{m_0} \in \mathbf{C}_c$, contrary to the hypothesis. We can then write again $v = a e_1 + b w$. Note that $w \in (DF)^{-1}\mathbf{C}_c$, moreover the uniform cone contraction implies that there exists $\vartheta_* \in (0, 1)$ such that, for all $p \in \mathbb{T}^2$, $D_p F \mathbf{C}_u \subset \{(x, y) \in \mathbb{R}^2 \,:\, |y| \le \vartheta_* \chi_u |x|\}$ and $(D_p F)^{-1} \mathbf{C}_c \subset \{(x, y) \in \mathbb{R}^2 \,:\, |x| \le \vartheta_* \chi_c |y|\}$. It follows $|w_1| \le \vartheta_* \chi_c |w_2|$ while $|v_1| \ge \chi_c |v_2|$, thus $v_2 = b w_2$ and

$$|a| \ge \chi_c |v_2| - |b w_1| \ge \chi_c (1 - \vartheta_*) |b| |w_2| \ge \chi_c (1 - \vartheta_*)(1 + \chi_c^2 \vartheta_*^2)^{-\frac{1}{2}} |b|$$

which implies $\frac{|b|}{|a|} \le C_\sharp \chi_c^{-1}$. Finally, by equations (4.1.3) and (5.3.5), we can write

$$\chi_u = \frac{|\langle e_2, DF^{m_0} v \rangle|}{|\langle e_1, DF^{m_0} v \rangle|} \le \frac{|b| \mu^{m_0} + |a| \, |\langle e_2, DF^{m_0} e_1 \rangle|}{|a| \, |\langle e_1, DF^{m_0} e_1 \rangle|} \le C_\sharp \frac{|b|}{|a|} \mu^{m_0} (\lambda^+_{m_0} \circ \hat{\nu}_{m_0})^{-1} + \vartheta_* \chi_u,$$

that is (5.4.55). □

By the above Sub-Lemma it follows that

$$\chi_c \ge \vartheta_{\hat{\nu}_m}(t) \ge \mu^{-m_0} \lambda^+_{m_0} \circ \hat{\nu}_{m_0} \vartheta_{\hat{\nu}_{n_0}}(t). \tag{5.4.57}$$

Thus

$$\begin{aligned} &\|\hat{\nu}''_m(t)\| \le C_\sharp \Lambda^{2n_0} \mu^m \Delta_\gamma \circ h_{n_0}(t) + C_\sharp \left\{ 1 + \mu^{2m} \vartheta^{-1}_{\hat{\nu}_{n_0}} \|\omega\|_{\mathcal{C}^2} \right\} \lambda^+_{m_0}(\hat{\nu}_m(t)), \\ &|\bar{h}''_{m_0}(t)| \le C_\sharp \Lambda^{2n_0} \mu^{2m} \Delta_\gamma \circ h_{n_0}(t) + C_\sharp \left\{ 1 + \mu^{2m} \vartheta^{-1}_{\hat{\nu}_{n_0}} \|\omega\|_{\mathcal{C}^2} \right\} \lambda^+_{m_0}(\hat{\nu}_m(t)) \mu^m \\ &|\bar{h}'_{m_0}(t)| \le C_\sharp \chi_c^{-1} \mu^{m_0}. \end{aligned} \tag{5.4.58}$$

To estimate $\hat{\nu}'''_m$ we use (5.4.34) where $\hat{\nu}_n, \gamma, h_n$ are replaced by $\hat{\nu}_m, \hat{\nu}_{n_0}, \bar{h}_{m_0}$. In this case the curve $\hat{\nu}_{n_0} \notin \mathbf{C}_c$, and so is $\mathfrak{h}_k(\hat{\nu}_{n_0})$ for each $k < m_0$, $\mathfrak{h}_k \in \mathfrak{H}^k$. Therefore, using Proposition 5.3.3, we have the following estimates

$$\begin{aligned} &\| \left[(D_{\hat{\nu}_m} F^{m_0})^{-1} \right]'' \hat{\nu}'_{n_0} \bar{h}'_{m_0} \| \lesssim \{ \mu^m \varsigma_m \lambda^+_m(\hat{\nu}_m(t)) + \varsigma_m \|\hat{\nu}''_m\| \} \|\hat{\nu}'_{n_0}\| |\bar{h}'_{m_0}| \\ &\| \left[(D_{\hat{\nu}_m} F^{m_0})^{-1} \right]' \hat{\nu}'_{n_0} \bar{h}''_{m_0} \| \lesssim \lambda^-_m(\hat{\nu}_m(t))^{-1} \mu^m \varsigma_m |\bar{h}''_{m_0}|, \end{aligned}$$

Additionally, again by Proposition 5.3.3,

$$\begin{aligned} &\| \left[(D_{\hat{\nu}_m} F^{m_0})^{-1} \right]' \hat{\nu}''_{n_0} (\bar{h}'_{m_0})^2 \| \lesssim \varsigma_m \mu^m \vartheta^{-1}_{\hat{\nu}_{n_0}} |\bar{h}'_{m_0}|^2, \\ &\| \left[(D_{\hat{\nu}_m} F^{m_0})^{-1} \right] \hat{\nu}'''_{n_0} (\bar{h}'_{m_0})^3 \| \lesssim \mu^m \|\hat{\nu}'''_{n_0}\| |\bar{h}'_{m_0}|^3 \\ &\| \left[(D_{\hat{\nu}_m} F^{m_0})^{-1} \right] \hat{\nu}''_{n_0} \bar{h}'_{m_0} \bar{h}''_{m_0} \| \lesssim \mu^m \|\hat{\nu}''_{n_0}\| |\bar{h}'_{m_0}| |\bar{h}''_{m_0}|. \end{aligned}$$

Using the above estimates in (5.4.34) and recalling (5.4.52), (5.4.58), and (5.4.51) we conclude

$$\|\hat\nu_m'''\| \leq C_\sharp M_{n_0}(m,t)\vartheta_{\hat\nu_{n_0}}^{-1}[\mu^m\varsigma_m + \mu^{2m}\Lambda^{3n_0}\Delta_\gamma^2\circ\bar h_{m_0}(t)] \\ + C_\sharp\vartheta_{\hat\nu_{n_0}}^{-2}\mu^{4m}\Lambda^{3n_0}\Delta_\gamma^2\circ\bar h_{m_0}(t) + C_\sharp\mu^{2m}\varsigma_m\lambda_m^+(\hat\nu_m(t))\vartheta_{\nu_{n_0}}^{-1} \leq A_0\overline{M}_{n_0}(m,t),$$

for some $A_0 > 0$. Next we set $\overline{m} \equiv \overline{m}(\mathfrak{h}, p)$ and $F^{\overline{m}-m}\hat\nu_{\overline{m}} = \hat\nu_m\circ\bar h_{\overline{m}-m}$. First,

$$\begin{aligned} |\bar h_{\overline{m}-m}'(t)| &= |\langle e_2, D_{\hat\nu_{\overline{m}}(t)}F^{\overline{m}-m}\hat\nu_{\overline{m}}'(t)\rangle| \leq C_\sharp\chi_c^{-1}\mu^{\overline{m}-m} \\ |\bar h_{\overline{m}-m}'(t)| &\geq C_\sharp\chi_c^{-1}\mu^{-\overline{m}+m}. \end{aligned} \tag{5.4.59}$$

We can now apply Lemma 5.4.1, in particular (5.4.27), to $\hat\nu_{\overline{m}}$ and $h_{\overline{m}-m}$ with γ replaced by $\hat\nu_m$, and $c_\star$ and $c_\star^2$ replaced by $M_{n_0}(m,t)$ and $\overline{M}_{n_0}(m,t)$ respectively, defined in (5.4.45). We thus obtain

$$\begin{aligned} &\|\hat\nu_{\overline{m}}''\| \leq c_\flat c_{n_\star}^{\overline{m}}\mu^{2\overline{m}}\lambda_{\overline{m}}^+(\hat\nu_m\circ\bar h_{\overline{m}-m})^{-1}M_{n_0}(m,\cdot) + C_{\mu,n_\star}\mu^{3n_\star} \\ &\|\hat\nu_{\overline{m}}'''\| \leq a_{n_\star}^{\overline{m}}\mu^{3\overline{m}}c_\flat\lambda_{\overline{m}}^+(\hat\nu_m\circ\bar h_{\overline{m}-m})^{-1}\overline{M}_{n_0}(m,\cdot) + b_{n_\star}^{\overline{m}}\mu^{3\overline{m}}c_\flat\lambda_{\overline{m}}^+(\gamma\circ h_{n_\star})^{-1}M_{n_0}(m,\cdot) + \mathbb{s}_{n_\star} \\ &|\bar h_{\overline{m}-m}''| \leq C_\sharp M_{n_0}(m,\cdot)\mu^{2\overline{m}}C_{\mu,\overline{m}}. \end{aligned} \tag{5.4.60}$$

We are ready to conclude. Recalling Corollary 5.4.3, the first two of the above equations plus condition (5.4.44) give $\hat\nu_{\overline{m}} \in \Gamma_3(\mathbb{c})$. Next we set $m_1 = \overline{m} - m$. If $F^{\overline{m}}\hat\nu_{\overline{m}} = \gamma\circ h_{\overline{m}}$, by definition we have

$$h_{\overline{m}} = h_{n_0}\circ\bar h_{m_0}\circ\bar h_{m_1}. \tag{5.4.61}$$

Hence, differentiating (5.4.61) and recalling (5.4.52), (5.4.59) and

$$C_\sharp\Lambda^{-n_0}\vartheta_{\hat\nu_{n_0}}(t)^{-1}|h_{n_0}'(t)| \leq C_\sharp\Lambda^{n_0}\vartheta_{\hat\nu_{n_0}}(t)^{-1},$$

we have the first of (5.4.46). Taking two derivatives of (5.4.61) and using the second lines of (5.4.51), (5.4.58) and the third of (5.4.60), we have[20]

$$\begin{aligned} |h_{\overline{m}}''| &\leq |h_{n_0}''\circ\bar h_{m_0}\circ\bar h_{m_1}\cdot\bar h_{m_0}'\circ h_{m_1}\cdot\bar h_{m_1}'| \\ &\quad + |\bar h_{n_0}'\circ\bar h_{m_0}\cdot\bar h_{m_1}\left(h_{m_0}''\circ\bar h_{m_1}\cdot\bar h_{m_1}' + \bar h_{m_1}''\cdot\bar h_{m_0}'\circ\bar h_{m_1}\right)| \\ &\leq C_\sharp\left(\vartheta_{\nu_0}^{-2}\mu^{\overline{m}} + \vartheta_{\nu_0}^{-1}\mu^{\overline{m}}M_{n_0}(m) + \vartheta_{\nu_0}^{-1}C_{\mu,\overline{m}}\mu^{2\overline{m}}M_{n_0}(m)\right), \end{aligned}$$

form which the second of (5.4.46) follows and the Lemma is proven. □

Remark 5.4.6. *From now on we will use Γ to denote $\Gamma_r(\mathbb{c})$ where $\mathbb{c}$ is defined in Lemma 5.4.1 and has thus the invariance property stated in Corollary 5.4.3.*

5.5 Distortion

We conclude this section with some technical distortion results needed in the following.

[20] Here we drop the dependence on t to ease notations.

Lemma 5.5.1. *For all $n \in \mathbb{N}$, $\nu \in F^{-n}(\Gamma(\mathbb{c}))$ and $x, y \in \nu$, we have*

$$e^{-\mu^n C_{\mu,n}\|x-y\|} \le \frac{\lambda_n^+(x)}{\lambda_n^+(y)} \le e^{\mu^n C_{\mu,n}\|x-y\|}. \tag{5.5.1}$$

Proof. We prove it by induction. To start with, let $x = \nu(t_1), y = \nu(t_2)$ such that $\|x - y\| \le \tau_n$ for some τ_n to be chosen shortly. For $n = 1$ we have, for all unit vector $v \notin \mathbf{C}_c$,

$$\frac{\|D_x F v\|}{\|D_y F v\|} \le e^{\ln\left[1+\frac{\|D_x F v - D_y F v\|}{\|D_y F v\|}\right]} \le e^{\frac{\|D_x F v - D_y F v\|}{\|D_y F v\|}}.$$

$$\|D_x F v - D_y F v\| \le \int_{t_1}^{t_2} \|\frac{d}{ds} D_{\nu(s)} F v\| ds \le C_\sharp |t_2 - t_1| \le C_\sharp \|x - y\|, \tag{5.5.2}$$

the case $n = 1$ follows. Assume it is true for each $k < n$, then, by the triangular inequality

$$\|D_y F^n v - D_x F^n v\| \le \sum_{k=0}^{n-1} \|D_{F^{k+1}y} F^{n-k-1}(D_{F^k y} F - D_{F^k x} F) D_x F^k v\|$$
$$\le C_\sharp \sum_{k=0}^{n-1} \lambda_{n-k-1}^+(F^k y) \lambda_k^+(x) \|D_{F^k y} F - D_{F^k x} F\| \le C_\sharp \sum_{k=0}^{n-1} \lambda_{n-k-1}^+(F^k y) \lambda_k^+(x) \mu^k \|x - y\|.$$

Since $\nu \in F^{-n}(\Gamma(c))$, $\|D_{F^k y} F - D_{F^k x} F\| \le C_\sharp \mu^k \|x - y\|$. Also remark that (5.3.5) and the induction hypothesis imply

$$\lambda_{n-k}^+(F^k y) \lambda_k^+(x) \le e^{\mu^k C_{\mu,k}\|x-y\|} \lambda_{n-k}^+(F^k y) \lambda_k^+(y) \le C_\sharp \lambda_n^-(y),$$

provided we have chosen τ_n small enough. Accordingly, since $\|D_y F^n v\| \ge \lambda_n^-(y)$,

$$\frac{\|D_x F^n v\|}{\|D_y F^n v\|} \le e^{\frac{\|D_x F^n v - D_y F^n v\|}{\|D_y F^n v\|}} \le e^{C_\sharp \sum_{k=0}^{n-1} \mu^k \|x-y\|}.$$

We can now choose v such that $\|D_x F^n v\| = \lambda_n^+(x)$ so

$$\frac{\lambda_n^+(x)}{\lambda_n^+(y)} \le \frac{\|D_x F^n v\|}{\|D_y F^n v\|} \le e^{C_\sharp \sum_{k=0}^{n-1} \mu^k \|x-y\|},$$

which proves the upper bound, for points close enough. Next, for all $x, y \in \nu$ we can consider close intermediate points $\{x_i\}_{i=0}^l$, $x_0 = x$, $x_l = y$, to which the above applies, hence

$$\frac{\lambda_n^+(x)}{\lambda_n^+(y)} \le \frac{\|D_x F^n v\|}{\|D_y F^n v\|} = \prod_{i=0}^{l-1} \frac{\|D_{x_i} F^n v\|}{\|D_{x_{i+1}} F^n v\|} \le e^{C_\sharp \sum_{k=0}^{n-1} \mu^k \sum_{i=0}^{l-1} \|x_{i+1} - x_i\|}.$$

Taking the limit for $l \to \infty$ we have the distance, along the curve, between x and y which is bounded by $C_\sharp \|x - y\|$. This proves the upper bound. The lower bound is proven similarly. □

Next, we prove two more distortion Lemmata, inspired by Lemma 6.2 in [43]. Even though the basic idea of the proof is the same, the presence of the central direction creates some difficulties.

Lemma 5.5.2. *For each $\gamma \in \Gamma(\mathbb{c})$, $n > \bar{n}$ and $0 \le \rho \le r-1$, we have*

$$\begin{aligned} \sum_{\nu_n \in F^{-n}\gamma} \left\| \frac{h_n'}{\det D_{\hat{\nu}_n} F^n} \right\|_{\mathcal{C}^\rho(\mathbb{T})} &\le C_\sharp \mathbb{c}^{a_\rho} C_{\mu,n}^{\tilde{a}_\rho} \mu^{\tilde{b}_\rho n} \\ \sum_{\nu_n \in F^{-n}\gamma} \left\| \frac{1}{\det D_{\hat{\nu}_n} F^n} \right\|_{\mathcal{C}^\rho(\mathbb{T})} &\le C_\sharp \mathbb{c}^{a_\rho} C_{\mu,n}^{\tilde{a}_\rho} \mu^{(\tilde{b}_\rho+1)n} \end{aligned} \tag{5.5.3}$$

where[21] $\tilde{a}_\rho = a_\rho \rho(\rho+1)/2 + 1$ *and* $\tilde{b}_\rho = \rho! \rho(\rho+1)/2 + 1$.

Proof. For every $\nu \in F^{-n}\gamma$ define

$$\Psi_{\nu_n}(t) = \frac{h_n'(t)}{\det D_{\hat{\nu}_n(t)} F^n},$$

and recall that in dimension one holds $\|\Psi_{\nu_{\overline{m}}}\|_{\mathcal{C}^0} \le \|\Psi_{\nu_{\overline{m}}}\|_{L^1} + \|\Psi'_{\nu_{\overline{m}}}\|_{L^1}$. We then first look for a bound of the $W^{1,1}(\mathbb{T})$-norm of $\Psi_{\nu_{\overline{m}}}$. Since $e_1 = (1,0) \in \mathbf{C}_u$, $D_{\hat{\nu}_n} F^n \hat{\nu}_n' \notin \mathbf{C}_u$ and recalling that $F^n \hat{\nu}_n = \gamma \circ h_n$, we have

$$h_n' D_{\hat{\nu}_n} F^n e_1 \wedge \gamma' \circ h_n = D_{\hat{\nu}_n} F^n e_1 \wedge D_{\hat{\nu}_n} F^n \hat{\nu}_n' = \det(D_{\hat{\nu}_n} F^n) e_1 \wedge \hat{\nu}_n'.$$

Thus we have the equation

$$\frac{h_n'(t)}{\det D_{\hat{\nu}_n(t)} F^n} = \frac{e_1 \wedge \hat{\nu}_n'(t)}{D_{\hat{\nu}_n(t)} F^n e_1 \wedge \gamma' \circ h_n(t)} \tag{5.5.4}$$

Arguing as in Proposition 5.3.1 and since $\|\gamma'\| \ge 1$ we have, recalling definition (5.4.41),

$$|D_{\hat{\nu}_n} F^n e_1 \wedge \gamma' \circ h_n| \ge C_\sharp \vartheta_\gamma \circ h_n \|D_{\hat{\nu}_n} F^n e_1\|. \tag{5.5.5}$$

Therefore, since $\|\hat{\nu}_n'\|^2 \le 1 + \chi_c^2$, we have

$$\sum_{\nu_n \in F^{-n}\gamma} \|\Psi_{\nu_n}\|_{L^1} \lesssim \sum_{\nu_n \in F^{-n}\gamma} \left\| \frac{1}{\vartheta_\gamma \circ h_n \|D_{\hat{\nu}_n} F^n \cdot e_1\|} \right\|_{L^1}. \tag{5.5.6}$$

Recall that, by Lemma 5.2.1, for each $\hat{\nu}_n$ we have an inverse branch $\mathfrak{h}_{\hat{\nu}_n} : \Omega_\gamma \to \Omega_{\hat{\nu}_n}$ such that $F^n \circ \mathfrak{h}_{\hat{\nu}_n} = Id_{\Omega_\gamma}$. More precisely, the domain $\Omega_{\hat{\nu}_n} = \bigcup_{t \in \mathbb{T}} \xi_{t,\hat{\nu}_n}$, where $\xi_{t,\hat{\nu}_n}(s) = \hat{\nu}_n(t) + s e_1$ are horizontal segments defined on an interval I_t of length $\delta_{\hat{\nu}_n(t)}$ whose images are unstable curves $\xi_{t,\gamma}^\sharp$ with length$(\xi_{t,\gamma}^\sharp) = \delta_{t,\gamma}^\sharp \ge 1$. Let $p_{n,\xi_{t,\nu_n}}$ be the diffeomorphism associated to ξ_{t,ν_n}, see formula (5.4.39). By equation (5.4.40) $p'_{n,\xi_{t,\nu_n}}(s) \lesssim p'_{n,\xi_{t,\nu_n}}(0) = \|D_{\hat{\nu}_n(t)} F^n e_1\|$. It follows

$$1 \le \delta_{t,\gamma}^\sharp = \int_{I_t} \left\| \frac{d}{ds} F^n(\xi_{t,\hat{\nu}_n}(s)) \right\| ds \le C_\sharp \delta_{\hat{\nu}_n(t)} p'_{n,\xi_{t,\nu_n}}(0),$$

[21] Recall the definition of a_ρ in Lemma 5.4.1

from which

$$\|D_{\hat\nu_n(t)}F^n e_1\| \gtrsim \frac{1}{\delta_{\hat\nu_n(t)}}. \tag{5.5.7}$$

Since by Lemma 5.2.1 the Ω_{ν_n} are all disjoints and the ν_n are parametrized vertically, by (5.5.7) we have[22]

$$\sum_{\nu_{\overline{m}}\in F^{\overline{m}}\gamma}\left\|\frac{1}{\|D_{\hat\nu_{\overline{m}}(t)}F^n e_1\|}\right\|_{L^1} \lesssim \sum_{\hat\nu_{\overline{m}}\in F^{\overline{m}}\gamma}\int_{\mathbb{T}^1}\delta_{\hat\nu_{\overline{m}}(t)} \lesssim \sum_{\nu_{\overline{m}}\in F^{\overline{m}}\gamma} m(\Omega_{\hat\nu_{\overline{m}}}) \lesssim m(\mathbb{T}^2) \lesssim 1.$$

Using this in (5.5.6) yields

$$\sum_{\nu_n\in F^n\gamma}\|\Psi_{\nu_n}\|_{L^1} \le C_\sharp \vartheta_\gamma^{-1} \le C_\sharp, \tag{5.5.8}$$

since $|\pi_1\circ\gamma'(t)|^{-1} \ge \chi_c^{-1} > 1 > \chi_u$ implies $\vartheta_\gamma^{-1} \le 1$. To bound the L^1 norm of the derivative we can notice that:

$$\|\Psi'_{\nu_n}\|_{L^1} \le \left\|\frac{\Psi'_{\nu_n}}{\Psi_{\nu_n}}\right\|_{\mathcal{C}^0}\|\Psi_{\nu_n}\|_{L^1}. \tag{5.5.9}$$

To continue it is useful to see $\hat\nu_n = \nu_n\circ h_n$ as the time evolution of curves parametrized by vertical length. For each $0\le i\le n$, let $\nu_{n-i} = F^i\nu_n$ and h_i the diffeomorphism such that $\hat\nu_i = \nu_i\circ h_i$ is parametrized by vertical length. Define the diffeomorphisms h_i^* by

$$\hat\nu_i = F\circ\hat\nu_{i+1}\circ(h_{i+1}^*)^{-1}, \tag{5.5.10}$$

where $\nu_0=\gamma$ and $h_0^* = h_0$. It is immediate to check that $h_i = h_1^*\circ\cdots\circ h_i^*$. We can then write

$$\begin{aligned}\Psi_{\nu_n}(t) = \frac{\frac{d}{dt}h_n(t)}{\det D_{\hat\nu_n(t)}F^n} &= \frac{\prod_{i=1}^n (h_i^*)'\circ h_{i+1}^*\circ\cdots\circ h_n^*}{\prod_{i=1}^n(\det D_{\hat\nu_i}F)\circ h_{i+1}^*\circ\cdots\circ h_n^*}(t)\\ &= \prod_{i=1}^n(\psi_i\circ h_{i+1}^*\circ\cdots\circ h_n^*)(t),\end{aligned}$$

where $\psi_i(t) = (h_i^*)'(t)\cdot(\det D_{\hat\nu_i(t)}F)^{-1}$. Hence,

$$\left|\frac{\Psi'_{\nu_n}}{\Psi_{\nu_n}}\right| \le \sum_{i=1}^n\left|\left(\frac{\psi_i'}{\psi_i}\circ h_{i+1}^*\circ\cdots\circ h_n^*\right)(h_{i+1}^*\circ\cdots\circ h_n^*)'\right|. \tag{5.5.11}$$

By (5.5.10), since $\hat\nu_n\in\Gamma(\mathbb{c})$, it follows by (5.1.3) that $\|\psi_i\|_{\mathcal{C}^\ell} \le C_\sharp \mathbb{c}^{(\ell-1)!}$ for each $\ell\le\rho$. Thus, setting $b_\ell := \ell!$ and $h_{i,n} = h_{i+1}^*\circ\cdots\circ h_n^*$, by (5.1.3) and (5.4.1) we have

$$\begin{aligned}\left\|\frac{\Psi'_{\nu_n}}{\Psi_{\nu_n}}\right\|_{\mathcal{C}^{\ell-1}} &\lesssim \sum_{i=0}^{n-1}\left\|(\log\psi_i\circ h_{i,n})'\right\|_{\mathcal{C}^{\ell-1}} \lesssim \sum_{i=0}^{n-1}\|\log\psi_i\circ h_{i,n}\|_{\mathcal{C}^\ell}\\ &\lesssim \sum_{i=0}^{n-1}\|\log\psi_i\|_{\mathcal{C}^\ell}\sum_{j=0}^{\ell-1}\|h'_{i,n}\|^j_{\mathcal{C}^{\ell-1}}\\ &\lesssim \mathbb{c}^{(\ell-1)!}\sum_{i=0}^{n-1}\sum_{j=0}^{\ell-1}\|h_{i,n}\|^j_{\mathcal{C}^\ell} \lesssim \mathbb{c}^{(\ell-1)!}C_{\mu,n}^{\ell a_\ell}\mu^{n\ell b_\ell},\end{aligned}$$

[22]Here $m(A)$ is the Lebesgue measure of a set A.

In particular the above estimates in the case $\ell = 1$ and (5.5.9) gives

$$\sum_{\nu_n \in F^n\gamma} \|\Psi'_{\nu_n}\|_{L^1} \le C_\sharp C_{\mu,n}\mu^n \sum_{\nu_n \in F^n\gamma} \|\Psi_{\nu_n}\|_{L^1} \le C_\sharp C_{\mu,n}\mu^n,$$

which gives the result for $\rho = 0$. Once we have the bound of the $\mathcal{C}^0-$norm, we can obtain the general case $\rho \in [1, r-1]$ as follows:

$$\begin{aligned}
\sum_{\nu_n \in F^{-n}\gamma} \|\Psi_{\nu_n}\|_{\mathcal{C}^\rho} &\lesssim \sum_{\nu_n \in F^{-n}\gamma} \|\Psi'_{\nu_n}\|_{\mathcal{C}^{\rho-1}} \lesssim \sum_{\nu_n \in F^{-n}\gamma} \left\|\frac{\Psi'_{\nu_n}}{\Psi_{\nu_n}}\right\|_{\mathcal{C}^{\rho-1}} \|\Psi_{\nu_n}\|_{\mathcal{C}^{\rho-1}} \\
&\lesssim \mathbb{c}^{(\rho-1)!} C_{\mu,n}^{\rho a_\rho} \mu^{n\rho b_\rho} \sum_{\nu_n \in F^{-n}\gamma} \|\Psi_{\nu_n}\|_{\mathcal{C}^{\rho-1}} \\
&\lesssim \mathbb{c}^{a_\rho} C_{\mu,n}^{a_\rho \sum_{k=0}^{\rho} k} \mu^{n b_\rho \sum_{k=0}^{\rho} k} \sum_{\nu_n \in F^{-n}\gamma} \|\Psi_{\nu_n}\|_{\mathcal{C}^0} \\
&\lesssim \mathbb{c}^{a_\rho} C_{\mu,n}^{a_\rho \frac{\rho(\rho+1)}{2}+1} \mu^{(b_\rho \frac{\rho(\rho+1)}{2}+1)n}.
\end{aligned}$$

The procedure to prove the second of (5.5.3) is analogous, with the difference that, by (5.5.4) and (5.4.9), the estimate for $\rho = 0$ gives another $C_\sharp \mu^n$, while the computation for $\rho \ge 1$ is exactly the same, but using $\psi_i = (\det D_{\hat{\nu}_i(t)} F)^{-1}$ instead. □

The next result is a refinement of the previous Lemma in the more general case in which the curve γ is simply not contained in $\mathbf{C}_u$. To state the result it is convenient to define the following quantities

$$\begin{aligned}
\mathbb{J}_{\gamma,n} &= \int_{\mathbb{T}^1} \frac{1}{\{1 - n\|\omega\|_\infty \vartheta_{\hat{\nu}_{n_0}}(s)^{-1}, \chi_u \vartheta_{\hat{\nu}_{n_0}}(s)^{-1}\}^+} ds, \\
\mathbb{I}_{\gamma,n,m} &= \left[\varsigma_m + \chi_u \mu^n \Delta_\gamma \vartheta_\gamma^{-1}\right]
\end{aligned} \tag{5.5.12}$$

Lemma 5.5.3. *In the same hypothesis of Lemma 5.4.4 with $n_0 = 0$, we have*

$$\begin{aligned}
\sum_{\nu_{\overline{m}} \in F^{-\overline{m}}\gamma} \left\|\frac{h'_{\overline{m}}}{\det D_{\hat{\nu}_{\overline{m}}} F^{\overline{m}}}\right\|_{\mathcal{C}^0(\mathbb{T})} &\le C_\sharp \left(\mathbb{c} + \mathbb{I}_{\gamma,n,m}\vartheta_\gamma^{-1}\right) \mu^{\overline{m}} \mathbb{J}_{\gamma,m} \\
\sum_{\nu_{\overline{m}} \in F^{-\overline{m}}\gamma} \left\|\frac{h'_{\overline{m}}}{\det D_{\hat{\nu}_{\overline{m}}} F^{\overline{m}}}\right\|_{\mathcal{C}^1(\mathbb{T})} &\le (C_{\mu,\overline{m}}\mu)^{2\overline{m}} \left(\mathbb{c} + \mathbb{I}_{\gamma,n,m}\vartheta_\gamma^{-1}\right)^2 \mu^{\overline{m}} \mathbb{J}_{\gamma,m} \\
\sum_{\nu_{\overline{m}} \in F^{-\overline{m}}\gamma} \left\|\frac{h'_{\overline{m}}}{\det D_{\hat{\nu}_{\overline{m}}} F^{\overline{m}}}\right\|_{\mathcal{C}^2(\mathbb{T})} &\le O_\star \overline{M}_0(m) \{\vartheta_\gamma^{-2}, \overline{M}_0(m), (\lambda_m^+)^2\}^+ \\
\sum_{\nu_{\overline{m}} \in F^{-\overline{m}}\gamma} \left\|\frac{h'_{\overline{m}}}{\det D_{\hat{\nu}_{\overline{m}}} F^{\overline{m}}}\right\|_{\mathcal{C}^\rho(\mathbb{T})} &\le C_\sharp \Lambda^{c_\sharp \overline{m}}, \quad \rho > 2,
\end{aligned} \tag{5.5.13}$$

where, recalling (5.4.45), $\overline{M}_0(m) = \|\overline{M}_0(m, \cdot)\|_\infty$, *and*

$$\begin{aligned}
O_\star = O_\star(\overline{m}, m) :&= C_{\mu,\overline{m}}^4 \mu^{7\overline{m}} \left(\mathbb{c} + \mathbb{I}_{\gamma,\overline{m},m}\vartheta_\gamma^{-1}\right) \mathbb{J}_{\gamma,m} \cdot \\
&\cdot \left\{(\lambda_m^+ \vartheta_\gamma^{-1})^{-1} \mu^{\overline{m}}, \left[\mathbb{c} + \mathbb{I}_{\gamma,\overline{m},m}\right]^2, \varsigma_m \mu^{3\overline{m}} \Delta_\gamma^2 \{\varsigma_m, (\lambda_m^+ \vartheta_\gamma)^{-1}\}^+\right\}^+.
\end{aligned} \tag{5.5.14}$$

Proof. . We use the same notations of the proof of Lemma 5.5.2. In the case $\rho > 2$ we content ourselves with a rough estimate, so we can proceed exactly as in the proof of the above Lemma and, using (5.4.47) and (5.4.49), the estimate is immediate. In the other cases we need to be more careful in the estimation of (5.5.6). Setting $J^*_k(x) = \det D_x F^k$, we write, recalling (5.4.61) and $m_1 = \overline{m} - m$,

$$\begin{aligned}\sum_{\nu_{\overline{m}}\in F^{-\overline{m}}\gamma}\left\|\frac{h'_{\overline{m}}}{J^*_{\overline{m}}(\hat{\nu}_{\overline{m}})}\right\|_{\mathcal{C}^\rho} &\leq \sum_{\nu_m\in F^{-m}\gamma}\sum_{\nu_{m_1}\in F^{-m_1}\hat{\nu}_m}\left\|\frac{\bar{h}'_m\circ\bar{h}_{m_1}\cdot\bar{h}'_{m_1}}{J^*_{m_1}(\hat{\nu}_{m_1})J^*_m(\hat{\nu}_m\circ\bar{h}_{m_1})}\right\|_{\mathcal{C}^\rho}\\ &= \sum_{\nu_m\in F^{-m}\gamma}\left\|\frac{\bar{h}'_m\circ\bar{h}_{m_1}}{J^*_m(\hat{\nu}_m\circ\bar{h}_{m_1})}\right\|_{\mathcal{C}^\rho}\sum_{\nu_{m_1}\in F^{-m_1}\hat{\nu}_m}\left\|\frac{\bar{h}'_{m_1}}{J^*_{m_1}(\hat{\nu}_{m_1})}\right\|_{\mathcal{C}^\rho}\\ &= \sum_{\nu_m\in F^{-m}\gamma}\left\|\Psi_{\hat{\nu}_m}\circ\bar{h}_{m_1}\right\|_{\mathcal{C}^\rho}\sum_{\nu_{m_1}\in F^{-m_1}\hat{\nu}_m}\left\|\Psi_{\hat{\nu}_{m_1}}\right\|_{\mathcal{C}^\rho}.\end{aligned} \tag{5.5.15}$$

First we are going to estimate the last sum, for $\rho = 2$. By the results of Lemma 5.4.4, $\hat{\nu}_m$ is an admissible central curve and, by equation (5.4.58), $\|\hat{\nu}''_m(t)\| \leq M_0(t,m)$. Therefore we can apply Lemma 5.5.2 with $\mathbb{c}^{a_2}$ replaced by $M_0(t,m)$ and we have

$$\sum_{\nu_{m_1}\in F^{-m_1}\nu_m}\left\|\Psi_{\hat{\nu}_{m_1}}\right\|_{\mathcal{C}^2} \leq \|M_0(\cdot,m)\|_\infty C^{\tilde{a}_2}_{\mu,m_1}\mu^{\tilde{b}_2 m_1}. \tag{5.5.16}$$

Next, arguing as in (5.5.4) we have

$$\frac{\bar{h}'_m\circ\bar{h}_{m_1}}{J^*_m(\hat{\nu}_m\circ\bar{h}_{m_1})} = \frac{e_1\wedge\hat{\nu}'_{\overline{m}}(t)}{D_{\hat{\nu}_{\overline{m}}(t)}F^m e_1\wedge\gamma'\circ h_{\overline{m}}(t)} =: \tilde{\Psi}_{\hat{\nu}_{\overline{m}}} \tag{5.5.17}$$

By (5.5.7) we have

$$\begin{aligned}\sum_{\nu_m\in F^{-m}\gamma}\|\tilde{\Psi}_{\nu_m}\|_{L^1} &\leq C_\sharp\sum_{\nu_m\in F^{-m}\gamma}\int_{\mathbb{T}^1}\frac{\delta_{\hat{\nu}_m(s)}}{\vartheta_\gamma\circ h_m(s)}|\bar{h}'_{m_1}(s)|ds\\ &\leq C_\sharp\mu^{m_1}\sum_{\nu_m\in F^{-m}\gamma}\int_{\mathbb{T}^1}\frac{\delta_{\hat{\nu}_m(s)}}{\vartheta_\gamma\circ h_m(s)}ds.\end{aligned}$$

Since $|\pi_2(F^m(x,\theta)) - \theta| \leq m\|\omega\|_\infty$ it follows that, given $\hat{\nu}_{*,m} \in F^{-m}\gamma$, for each $\hat{\nu}_m \in F^{-m}\gamma$

$$|\pi_2(F^m(\hat{\nu}_{*,m}(t))) - \pi_2(F^m(\hat{\nu}_m(t)))| \leq m\|\omega\|_\infty,$$

accordingly, since $\gamma' \notin \mathbf{C}_u$, we have, calling $h_{\hat{\nu}_m}$ the reparametrizartion associated to $\hat{\nu}_m$,

$$\vartheta_\gamma\circ h_{\hat{\nu}_m}(t) \geq \{\vartheta_\gamma\circ h_{\hat{\nu}_{*,m}}(t) - m\|\omega\|_\infty, \chi_u\}^+.$$

Hence,

$$\begin{aligned}\sum_{\nu_m\in F^{-m}\gamma}\|\tilde{\Psi}_{\nu_m}\|_{L^1} &\leq C_\sharp\int_{\mathbb{T}^1}\frac{\sum_{\nu_m\in F^{-m}\gamma}\delta_{\hat{\nu}_m(t)}}{\{\vartheta_\gamma\circ h_{\hat{\nu}_{*,m}}(t) - m\|\omega\|_\infty, \chi_u\}^+}dt\\ &\leq C_\sharp\int_{\mathbb{T}^1}\frac{1}{|h'_{\hat{\nu}_{*,m}}(h^{-1}_{\hat{\nu}_{*,m}}(s))|\{\vartheta_\gamma(s) - m\|\omega\|_\infty, \chi_u\}^+}dt.\end{aligned}$$

Recalling (5.4.46) we obtain

$$\sum_{\nu_m\in F^{-m}\gamma}\|\tilde\Psi_{\nu_m}\|_{L^1}\leq \mu^{m_1}C_\sharp\int_{\mathbb{T}^1}\frac{\mu^m}{\{1-n\|\omega\|_\infty\vartheta_\gamma(s)^{-1},\chi_u\vartheta_\gamma(s)^{-1}\}^+}dt=C_\sharp\mu^{\overline{m}}\mathbb{J}_{\gamma,m}. \tag{5.5.18}$$

Next, we want to compute, using (5.5.17),

$$\begin{aligned}\frac{\tilde\Psi'_{\nu_{\overline{m}}}}{\tilde\Psi_{\nu_{\overline{m}}}}&=\frac{e_1\wedge\hat\nu''_{\overline{m}}}{e_1\wedge\hat\nu'_{\overline{m}}}-\frac{\partial_t\left(D_{\hat\nu_{\overline{m}}}F^me_1\right)\wedge\gamma'\circ h_{\overline{m}}+D_{\hat\nu_{\overline{m}}}F^me_1\wedge\gamma''\circ h_{\overline{m}}\cdot h'_{\overline{m}}}{D_{\hat\nu_{\overline{m}}}F^me_1\wedge\gamma'\circ h_{\overline{m}}}\\&=-\frac{\left[(D_{\hat\nu_{\overline{m}}}F^m)^{-1}\partial_t\left(D_{\hat\nu_{\overline{m}}}F^m\right)\right]e_1\wedge\hat\nu'_{\overline{m}}+e_1\wedge(D_{\hat\nu_{\overline{m}}}F^m)^{-1}\gamma''\circ h_{\overline{m}}\cdot(h'_{\overline{m}})^2}{e_1\wedge\hat\nu'_{\overline{m}}}\\&\quad+\frac{e_1\wedge\hat\nu''_{\overline{m}}}{e_1\wedge\hat\nu'_{\overline{m}}}\end{aligned} \tag{5.5.19}$$

where we have used equation (5.5.4). Next, note that $\gamma''(s)=\alpha e_1$ wiht $|\alpha|\leq\Delta_\gamma$ and $e_1=a\eta+be_2$ with $|b|\leq\chi_u$ and $(DF^m)^{-1}\eta\wedge e_1=0$. Using (5.3.8), arguing as in (5.4.54), we have

$$\begin{aligned}&\|\left[(D_{\hat\nu_{\overline{m}}}F^m)^{-1}\partial_t\left(D_{\hat\nu_{\overline{m}}}F^m\right)\right]e_1\|\leq\sum_{k=0}^{m-1}\|(D_{\hat\nu_{\overline{m}}}F^{k+1})^{-1}\left[\partial_{x_i}D_{F^k(\hat\nu_{\overline{m}})}F\right]D_{\hat\nu_{\overline{m}}}F^ke_1\|\\&\cdot\|D_{\hat\nu_{\overline{m}}}F^k\hat\nu'_{\overline{m}}\|\leq\sum_{k=0}^{m-1}C_\sharp(1+\mu^k(\|\omega\|_{\mathcal{C}^2}+\chi_u)\lambda_k^+)\|D_{\hat\nu_{\overline{m}}}F^k\hat\nu'_{\overline{m}}\|\\&\leq C_\sharp\varsigma_m|h'_{\overline{m}}|\end{aligned}$$

Thus, using (5.4.46),

$$\left|\frac{\tilde\Psi'_{\nu_{\overline{m}}}}{\tilde\Psi_{\nu_{\overline{m}}}}\circ\bar h_{m_1}\right|\leq C_\sharp\left(\mathbb{c}+\varsigma_m|h'_{\overline{m}}|+\chi_u\mu^{\overline{m}}\Delta_\gamma|h'_{\overline{m}}|^2\right)\leq C_\sharp\left(\mathbb{c}+\left[\varsigma_m+\chi_u\mu^{\overline{m}}\Delta_\gamma\vartheta_\gamma^{-1}\right]\vartheta_\gamma^{-1}\right).$$

The first of (5.5.13) follows by (5.5.9) and (5.5.18). While,

$$\|\tilde\Psi'_{\nu_{\overline{m}}}\|_{\mathcal{C}^0}\leq\left\|\frac{\tilde\Psi'_{\nu_{\overline{m}}}}{\tilde\Psi_{\nu_{\overline{m}}}}\right\|_{\mathcal{C}^0}\|\tilde\Psi_{\nu_{\overline{m}}}\|_{\mathcal{C}^0}. \tag{5.5.20}$$

leads immediately to the second of (5.5.13). To conclude the lemma we must compute $\tilde\Psi''_{\nu_{\overline{m}}}$, which can be obtained by (5.5.19):

$$\begin{aligned}\frac{\tilde\Psi''_{\nu_{\overline{m}}}}{\tilde\Psi_{\nu_{\overline{m}}}}&=\frac{e_1\wedge\hat\nu'''_{\overline{m}}}{e_1\wedge\hat\nu'_{\overline{m}}}-\frac{(e_1\wedge\hat\nu''_{\overline{m}})^2}{(e_1\wedge\hat\nu'_{\overline{m}})^2}+\left[\frac{e_1\wedge\hat\nu''_{\overline{m}}}{e_1\wedge\hat\nu'_{\overline{m}}}-\frac{\tilde\Psi'_{\nu_{\overline{m}}}}{\tilde\Psi_{\nu_{\overline{m}}}}\right]\frac{e_1\wedge\hat\nu''_{\overline{m}}}{e_1\wedge\hat\nu'_{\overline{m}}}\\&-\frac{\partial_t\left[(D_{\hat\nu_{\overline{m}}}F^m)^{-1}\partial_t\left(D_{\hat\nu_{\overline{m}}}F^m\right)\right]e_1\wedge\hat\nu'_{\overline{m}}+\left[(D_{\hat\nu_{\overline{m}}}F^m)^{-1}\partial_t\left(D_{\hat\nu_{\overline{m}}}F^m\right)\right]e_1\wedge\hat\nu''_{\overline{m}}}{e_1\wedge\hat\nu'_{\overline{m}}}\\&-\frac{e_1\wedge\left[\partial_t(D_{\hat\nu_{\overline{m}}}F^m)^{-1}\right]\gamma''\circ h_{\overline{m}}\cdot(h'_{\overline{m}})^2+e_1\wedge(D_{\hat\nu_{\overline{m}}}F^m)^{-1}\gamma'''\circ h_{\overline{m}}\cdot(h'_{\overline{m}})^3}{e_1\wedge\hat\nu'_{\overline{m}}}\\&-\frac{2e_1\wedge(D_{\hat\nu_{\overline{m}}}F^m)^{-1}\gamma''\circ h_{\overline{m}}\cdot h'_{\overline{m}}h''_{\overline{m}}}{e_1\wedge\hat\nu'_{\overline{m}}}+\left[\frac{\tilde\Psi'_{\nu_{\overline{m}}}}{\tilde\Psi_{\nu_{\overline{m}}}}\right]^2.\end{aligned} \tag{5.5.21}$$

We estimate the lines of (5.5.21) one at a time. The first line is bounded by

$$C_\sharp \left\{ \mathbb{c}^2 + \mathbb{c}\left[\varsigma_m + \chi_u \mu^{\overline{m}} \Delta_\gamma \vartheta_\gamma^{-1} \right] \vartheta_\gamma^{-1} \right\} \tag{5.5.22}$$

To estimate the second line we firt note that

$$(D_{\hat{\nu}_{\overline{m}}} F^{\overline{m}})^{-1} \partial_t (D_{\hat{\nu}_{\overline{m}}} F^{\overline{m}}) \leq \sum_{s=1}^{2} \sum_{k=0}^{n-1} (D_{\hat{\nu}_{\overline{m}}} F^{k+1})^{-1} \partial_{x_s} (D_{F^k(\nu(t))} F) D_{\nu(t)} F^k (D_{\nu(t)} F^k \nu')_s.$$

We can thus use the fourth (5.3.6) and (5.4.54) to bound the second line of (5.5.21) with

$$C_\sharp \varsigma_m \circ \hat{\nu}_{\overline{m}} \left[(\lambda_m^+ \circ \hat{\nu}_{\overline{m}})^2 + (\lambda_m^+ \circ \hat{\nu}_{\overline{m}})^2 + \lambda_m^+ \circ \hat{\nu}_{\overline{m}} \mathbb{c} \right] \leq C_\sharp \varsigma_m (\lambda_m^+ \circ \hat{\nu}_{\overline{m}})^2 \mu^{2\overline{m} - m}. \tag{5.5.23}$$

To estimate the third line we use the second line of (5.3.8), arguing as above, and (5.4.46)

$$\begin{aligned} &C_\sharp \left[\varsigma_m^2 \mu^{\overline{m} - m} \Delta_\gamma \left(\vartheta_\gamma(t)^{-1} \mu^{\overline{m}} \right)^2 + \varsigma_m (\lambda_m^+)^{-1} \left(\vartheta_\gamma(t)^{-1} \mu^{\overline{m}} \right)^3 \Delta_\gamma^2 \right] \\ &\leq C_\sharp [\varsigma_m^2 \mu^{3\overline{m}} \cdot \{ \Delta_\gamma, \varsigma_m (\lambda_m^+ \vartheta_\gamma)^{-1} \Delta_\gamma^2 \}^+] \vartheta_\gamma^{-2}. \end{aligned} \tag{5.5.24}$$

Finally, again by (5.4.46), the last line is estimated by

$$\begin{aligned} &C_\sharp (\lambda_m^+)^{-1} \mu^{\overline{m}} \vartheta_\gamma^{-1} \overline{M}_0(m,t) + C_\sharp \left(\mathbb{c} + \left[\varsigma_m + \chi_u \mu^{\overline{m}} \Delta_\gamma \vartheta_\gamma^{-1} \right] \vartheta_\gamma^{-1} \right)^2 \\ &\leq C_\sharp \{ \overline{M}_0(m,t), \vartheta_\gamma^{-2} \}^+ \cdot \left\{ (\lambda_m^+ \vartheta_\gamma)^{-1} \mu^{\overline{m}}, \left[\mathbb{c} + \varsigma_m + \chi_u \mu^{\overline{m}} \Delta_\gamma \vartheta_\gamma^{-1} \right]^2 \right\}^+. \end{aligned} \tag{5.5.25}$$

Collecting the above estimates we obtain

$$\begin{aligned} \left| \frac{\tilde{\Psi}''_{\nu_{\overline{m}}}}{\tilde{\Psi}_{\nu_{\overline{m}}}} \right| &\lesssim \left\{ (\lambda_m^+ \vartheta_\gamma)^{-1} \mu^{\overline{m}}, \left[\mathbb{c} + \varsigma_m + \chi_u \mu^{\overline{m}} \Delta_\gamma \vartheta_\gamma^{-1} \right]^2, \varsigma_m \mu^{3\overline{m}} \Delta_\gamma^2 \{ \varsigma_m, (\lambda_m^+ \vartheta_\gamma)^{-1} \}^+ \right\}^+ \\ &\quad \cdot \{ \vartheta_\gamma^{-2}, \| \overline{M}_0(m, \cdot) \|_\infty, (\lambda_m^+)^2 \}^+. \end{aligned}$$

We finally have, setting $\overline{M}_0(m) := \| \overline{M}_0(m, \cdot) \|_\infty$,

$$\sum_{\nu_m \in F^m \gamma} \| \tilde{\Psi}_{\nu_{\overline{m}}} \|_{\mathcal{C}^2} \leq C_\sharp \left\| \frac{\tilde{\Psi}''_{\nu_{\overline{m}}}}{\tilde{\Psi}_{\nu_{\overline{m}}}} \right\|_{\mathcal{C}^0} \sum_{\nu_{\overline{m}} \in F^{\overline{m}} \gamma} \| \tilde{\Psi}_{\nu_{\overline{m}}} \|_{\mathcal{C}^0} \leq C_\sharp O_\star \cdot \{ \vartheta_\gamma^{-2}, \overline{M}_0(m), (\lambda_m^+)^2 \}.$$

By the above equation and (5.5.16) we then obtain the statement. □

Corollary 5.5.4. *For each $n \in \mathbb{N}$*

$$\| \mathcal{L}^n 1 \|_{L^\infty(\mathbb{T}^2)} \leq C_{\mu,n} \mu^{2n}. \tag{5.5.26}$$

Proof. For any $x \in \mathbb{T}^2$ we want to estimate the quantity

$$\mathcal{L}^n 1(x) = \sum_{y \in F^{-n} x} \frac{1}{|\det D_y F^n|}. \tag{5.5.27}$$

Recall the notation in Section 5.2 and take $y \in \gamma$, where $\gamma \in \Gamma$ is an admissible central curve. Then, for every $x \in F^{-n}(y)$, there exist $t \in \mathbb{T}$ and $\nu \in F^{-n}\gamma$ such that $x = \nu(h_n(t)) = \hat{\nu}(t)$. Hence

$$\sup_{y\in\gamma} \sum_{x\in F^{-n}(y)} \left| \frac{1}{\det D_x F^n} \right| \leq \sum_{\nu \in F^{-n}\gamma} \left\| \frac{h'_{\nu,n}}{\det D_{\hat{\nu}} F^n} \right\|_{\mathcal{C}^0} \|(h'_n)^{-1}\|_{\mathcal{C}^0}.$$

By equations (5.4.9) and (4.1.5) we know that $\|(h'_n)^{-1}\|_{\mathcal{C}^0} \leq C_\sharp \mu^n$, for every ν and n. Moreover, Lemma 5.5.2 gives the bound

$$\sum_{\nu \in F^{-n}\gamma} \left\| \frac{h'_\nu}{\det D_{\hat{\nu}} F^n} \right\|_{\mathcal{C}^0} \leq C_{\mu,n} \mu^{2n}.$$

□

Remark 5.5.5. *With some extra work the estimate (5.5.26) can be made sharper, however the above bound is good enough for our current purposes. We will need an improvement, provided in Lemma 10.2.5, in Section 10.*

Chapter 6

A first Lasota-Yorke inequality

We define a class of geometric norms inspired by [43] and [4]. Given $u \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})$ and an integer $\rho < r$, we denote by $\mathcal{B}_\rho$ the completion of $\mathcal{C}^r(\mathbb{T}^2, \mathbb{R})$ with respect to the norm:

$$\|u\|_\rho := \max_{|\alpha| \leq \rho} \sup_{\gamma \in \Gamma} \sup_{\substack{\phi \in \mathcal{C}^{|\alpha|}(\mathbb{T}) \\ \|\phi\|_{\mathcal{C}^{|\alpha|}} = 1}} \int_{\mathbb{T}} \phi(t)(\partial^\alpha u)(\gamma(t))dt. \tag{6.0.1}$$

This defines a decreasing sequence of Banach spaces continuously embedded in L^1, namely

$$\|u\|_{L^1} \leq C\|u\|_{\rho_1} \leq C\|u\|_{\rho_2}, \qquad \text{for every} \quad 0 \leq \rho_1 \leq \rho_2 \leq r-1. \tag{6.0.2}$$

To see this we observe that, since $\sigma_x(t) = (x, t) \in \Gamma$,

$$\begin{aligned} \|u\|_{L^1} = \sup_{\|\phi\|_{\mathcal{C}^0(\mathbb{T}^2)} \leq 1} \int_{\mathbb{T}} dx \int_{\mathbb{T}} dy \phi(x,y)u(x,y) &\leq \int_{\mathbb{T}} dx \sup_{\|\phi\|_{\mathcal{C}^0(\mathbb{T}^2)}} \int_{\mathbb{T}} dy \phi(x,y)u(x,y) \\ &\leq \int_{\mathbb{T}} dx \sup_{\|\phi\|_{\mathcal{C}^0(\mathbb{T})}} \int_{\mathbb{T}} dt\, \phi(t)u(\sigma_x(t)) \leq \int_{\mathbb{T}} dx \|u\|_0 = \|u\|_0. \end{aligned}$$

The above proves the first inequality of (6.0.2), the others being trivial. We start with a Lasota-Yorke type inequality between the spaces $\mathcal{B}_\rho$ and $\mathcal{B}_{\rho-1}$.

Theorem 6.0.1. *Let $F \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$ be a SVPH. Let $\mathcal{L} := \mathcal{L}_F$ be the transfer operator defined in (4.3.1), and $\bar{n}$ be the integer given in Lemma 5.4.1. For each $\rho \in [1, r-1]$ and $n > \bar{n}$, there exists $C_{n,\rho}$ such that*

$$\|\mathcal{L}^n u\|_0 \leq C_{\mu,n} \mu^n \|u\|_0 \tag{6.0.3}$$

$$\|\mathcal{L}^n u\|_\rho \leq \frac{C_{\mu,n}^{\bar{a}_\rho} \mu^{\bar{b}_\rho n}}{\lambda_-^{\rho n}} \|u\|_\rho + C_{n,\rho}\|u\|_{\rho-1} \tag{6.0.4}$$

where $\bar{a}_\rho = 1 + a_\rho(\rho^2 + \rho(\rho+1)/2 + 1)$ and $\bar{b}_\rho = 1 + \rho!(2\rho^2 + \rho/2 + 1)$.

We postpone the proof of Theorem 6.0.1 to section 6.2. First we need to develop several results on the commutators between differential operators and transfer operators which will be needed throughout the paper.

6.1 Differential Operators

For $s, \rho \in \mathbb{N}$ we denote by $P_{s,\rho}$ a differential operator of order at most ρ defined as a finite linear combination of compositions of at most ρ vector fields, and we write

$$P_{s,\rho} u = \sum_{j=0}^{s} \sum_{\alpha \in A \subset \mathbb{N}^j} v_{j,\alpha_1} \cdots v_{j,\alpha_j} u, \tag{6.1.1}$$

where A is a finite set and for every $i \leq j$, v_{j,α_i} are vector fields in $\mathcal{C}^{\rho+j-s}$, with the convention that $v_{j,\alpha_1} \cdots v_{j,\alpha_j} u = u$ if $j = 0$. We denote by $\Psi^{s,\rho}$ the set of differential operators $P_{s,\rho}$. For a function $u \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})$ and a smooth vector field v, we denote $\partial_v u(x) = \langle \nabla_x u, v(x) \rangle$.

We start by studying the structure of the commutator between $\mathcal{L}$ and the differential operators. Next, we will estimate the coefficients of the commutator.

Proposition 6.1.1. *Given smooth vector fields $v_1, \cdots, v_s \in \mathcal{C}^\rho$, we have*

$$\partial_{v_s} \cdots \partial_{v_1} \mathcal{L}^n = \mathcal{L}^n \partial_{F^{n*} v_s} \cdots \partial_{F^{n*} v_1} + \mathcal{L}^n P_{s-1,\rho},$$

where $F^* v(x) := (D_x F)^{-1} v(F(x))$ is the pullback of v by the map F and $P_{s-1,\rho} \in \Psi^{s-1,\rho}$ whose coefficients may depend on n.

Proof. Let us start with $s = 1$. Let $v_1 \in \mathcal{C}^\rho(\mathbb{T}^2, \mathbb{T}^2)$ and define

$$J_n(p) = (\det D_p F^n)^{-1}; \quad \phi_n(p) = \log |\det D_p F^n|. \tag{6.1.2}$$

For each $\mathfrak{h} \in \mathfrak{H}^n$ we have

$$\begin{aligned} \langle \nabla \left[J_n \circ \mathfrak{h} \cdot u \circ \mathfrak{h} \right], v_1 \rangle &= \langle J_n \circ \mathfrak{h} (D\mathfrak{h})^* \nabla u \circ \mathfrak{h}, v_1 \rangle - \langle (D\mathfrak{h})^* \nabla (\det DF^n) \circ \mathfrak{h} J_n^2 \circ \mathfrak{h} u \circ \mathfrak{h}, v_1 \rangle \\ &= J_n \circ \mathfrak{h} \langle (D\mathfrak{h})^* \nabla u \circ \mathfrak{h}, v_1 \rangle - J_n \circ \mathfrak{h} \langle (D\mathfrak{h})^* \nabla \phi_n \circ \mathfrak{h} u \circ \mathfrak{h}, v_1 \rangle. \end{aligned}$$

Then, since $DF^n \circ \mathfrak{h} D\mathfrak{h} = \mathrm{Id}_{\mathcal{R}_\mathfrak{h}}$, for each $\mathfrak{h} \in \mathfrak{H}^n$ and $x \in \mathcal{D}_\mathfrak{h}$ [1]

$$\langle \nabla \left[J_n \circ \mathfrak{h} \cdot u \circ \mathfrak{h} \right](x), v_1(x) \rangle = J_n \circ \mathfrak{h}(x) \left[\partial_{F^{n*} v_1} u - \partial_{F^{n*} v_1} \phi_n u \right] \circ \mathfrak{h}(x). \tag{6.1.3}$$

Observing that

$$\mathcal{L}^n u = \sum_{\mathfrak{h} \in \mathfrak{H}^n} u \circ \mathfrak{h} J_n \circ \mathfrak{h} \mathbb{1}_{\mathcal{R}_\mathfrak{h}} \circ \mathfrak{h}, \tag{6.1.4}$$

it follows

$$\langle \nabla_x \mathcal{L}^n u, v_1(x) \rangle = \mathcal{L}^n \left(\partial_{F^{n*} v_1} u \right)(x) - \mathcal{L}^n (\partial_{F^{n*} v_1} \phi_n \cdot u)(x), \tag{6.1.5}$$

which prove the result since the multiplication operator $P_{0,\rho} := -\partial_{F^{n*} v_1} \phi_n \in \Psi^{0,\rho}$. Next, we argue by induction on s:

$$\begin{aligned} \partial_{v_{s+1}} \cdots \partial_{v_1} \mathcal{L}^n u &= \partial_{v_{s+1}} \left[\mathcal{L}^n \partial_{F^{n*} v_s} \cdots \partial_{F^{n*} v_1} u + \mathcal{L}^n P_{s-1,\rho} u \right] \\ &= \mathcal{L}^n \partial_{F^{n*} v_{s+1}} \cdots \partial_{F^{n*} v_1} u + \mathcal{L}^n (\partial_{F^{n*} v_{s+1}} \phi_n \cdot \partial_{F^{n*} v_s} \cdots \partial_{F^{n*} v_1} u) \\ &\quad + \mathcal{L}^n \partial_{F^{n*} v_{s+1}} P_{s-1,\rho} u + \mathcal{L}^n (\partial_{F^{n*} v_{s+1}} \phi_n \cdot P_{s-1,\rho} u), \end{aligned} \tag{6.1.6}$$

[1] Recall that $\mathcal{D}_\mathfrak{h}, \mathcal{R}_\mathfrak{h}$ indicate respectively the domain and the range of $\mathfrak{h}$.

which yields the Lemma with

$$P_{s,\rho} = \partial_{F^{n*}v_{s+1}} P_{s-1,\rho} + \partial_{F^{n*}v_{s+1}} \phi_n \cdot \left[\partial_{F^{n*}v_s} \cdots \partial_{F^{n*}v_1} + P_{s-1,\rho}\right] + \partial_{F^{n*}v_{s+1}} P_{s-1,\rho}. \tag{6.1.7}$$

□

In the case $v_j \in \{e_1, e_2\}$ for each j, we have the following Corollary as an immediate iterative application of formulae (6.1.3) and (6.1.5).

Corollary 6.1.2. *For each $t \geq 1$, $n \in \mathbb{N}$ $\alpha = (\alpha_1, .., \alpha_t) \in \{1,2\}^t$ and $\mathfrak{h} \in \mathfrak{H}^n$,*

$$\partial^\alpha [J_n \circ \mathfrak{h} \cdot u \circ \mathfrak{h}] = J_n \circ \mathfrak{h} \cdot [P^\alpha_{n,t} u] \circ \mathfrak{h}, \tag{6.1.8}$$

in particular

$$\partial^\alpha \mathcal{L}^n u = \mathcal{L}^n P^\alpha_{n,t} u, \tag{6.1.9}$$

the operators $P^\alpha_{n,t}$ being defined by the following relations, for each $u \in \mathcal{C}^t$,

$$\begin{cases} P^\alpha_{n,0} u = u, \\ P^\alpha_{n,1} u = A^{\alpha_1}_n u - A^{\alpha_1}_n \phi_n \cdot u, \\ P^\alpha_{n,t} u = A^\alpha_{n,1} u - \sum_{k=1}^t A^\alpha_{n,k+1}((A^{\alpha_k}_n \phi_n) \cdot P^\alpha_{n,k-1} u) \quad \text{for } t \geq 2, \end{cases} \tag{6.1.10}$$

where $A^{\alpha_i}_n = \partial_{F^{n*}e_{\alpha_i}}$, $A^\alpha_{n,k} := A^{\alpha_t}_n \cdots A^{\alpha_k}_n$, $A^\alpha_{n,t+1} = Id$ and ϕ_n is defined in (6.1.2).

Proposition 6.1.3. *For each $n \in \mathbb{N}$ let $P^\alpha_{n,t} \in \Psi^{t,t}$ given by* (6.1.10). *For any $1 \leq t < r$, $\psi \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{C})$ with supp$\psi \subset U = \mathring{U} \subset \mathbb{T}^2$, $\nu \in \Gamma(\mathbb{c})$ such that $DF^{n-m}\nu' \in \mathbf{C}_c$, $\varphi \in \mathcal{C}^t(\mathbb{T}, \mathbb{C})$ with $\|\varphi\|_{\mathcal{C}^t} \leq 1$, multi-index α, $|\alpha| = t$ and $u \in \mathcal{C}^r(\mathbb{T}^2)$ we have*

$$\int_{\mathbb{T}} \varphi(\tau) P^\alpha_{n,t}(\psi u)(\nu(\tau)) d\tau \leq \tilde{C}(t,n,m) \|\psi\|_{\mathcal{C}^t(U)} \|u\|_t, \tag{6.1.11}$$

where [2]

$$\tilde{C}(t,n,m) \leq \begin{cases} C_\sharp \mu^{2n} \sup_{t \in \operatorname{supp}\varphi} \{\varsigma^2_n \circ \nu(t) \bar{\varsigma}_{n,m} \circ \nu(t) + \mu^n \varsigma_n \circ \nu(t) \mathbb{c}\} & t = 2, \\ C_\sharp \Lambda^{c_\sharp n} & t > 2. \end{cases} \tag{6.1.12}$$

Proof. For simplicity we set $\partial_k = \partial_{x_k}$ for $k \in \{1,2\}$. First of all notice that, if we set $d_{k,i} = \langle (DF^n)^{-1} e_k, e_i \rangle$, then $A^{\alpha_j}_n = \sum_{i=1}^2 d_{\alpha_j,i} \partial_{x_i}$. Furthermore, by formula (5.1.4), $\|d_{j,i}\|_{\mathcal{C}^t} \leq \|(DF^n)^{-1}\|_{\mathcal{C}^t} \leq \Lambda^n$, for each $2 \leq t \leq r$. We are going to prove (6.1.11) by induction on t. For $t = 0$ it is obvious, let us assume it for any $k \leq t-1$. By (6.1.10) the integral in (6.1.11) splits into[3]

$$\begin{aligned} &\int \varphi(\tau) P^\alpha_{n,t}(\psi u)(\nu(\tau)) d\tau \\ &= \int \varphi \left[A^{\alpha_t}_n \cdots A^{\alpha_1}_n (\psi u)\right] \circ \nu - \int \varphi \sum_{k=1}^t \left[A^\alpha_{n,k+1}((A^{\alpha_k}_n \phi_n) \cdot P^\alpha_{n,k-1}(\psi u))\right] \circ \nu. \end{aligned} \tag{6.1.13}$$

[2] Recall equations (5.3.3) and (5.3.4) for the notations.

[3] Unless differently specified, in the following all the integrals are on $\mathbb{T}$.

The first integral is equal to

$$\sum_{\substack{i_1,\cdots,i_t\\ i_l\in\{1,2\}}}\sum_{J,J_0,J_1,..,J_t}\int\varphi\cdot(\prod_{j\in J}\partial_j u)(\prod_{j\in J_0}\partial_j\psi)(\prod_{j\in J_1}\partial_j d_{\alpha_1,i_1})\cdots(\prod_{j\in J_t}\partial_j d_{\alpha_t,i_t}),\tag{6.1.14}$$

where the second sum is made over all the partitions $J, J_0, J_1, .., J_t$ of $\{1, .., t\}$ such that $J_j \subset \{j+1, .., t\}, j \geq 1$.[4] Note that $\left\|(\prod_{k=1}^t \Pi_{j\in J_k}\partial_j)d_{\alpha_k,i_k}\right\|_{\mathcal{C}^t_\nu} \leq \Lambda^{n\sum_{k=1}^t \sharp J_k}$ and $\|\Pi_{j\in J_0}\partial_j\psi\|_{\mathcal{C}^{\sharp J}_\nu} \lesssim \|\psi\|_{\mathcal{C}^{\sharp J+\sharp J_0}} \leq \|\psi\|_{\mathcal{C}^t}$. Consequently, from (6.1.14) and the definition (6.0.1), we have

$$\left|\int\varphi(\tau)A^\alpha_{n,1}(\psi u)(\nu(\tau))d\tau\right| \leq C_\sharp\Lambda^{c_\sharp n}\|\psi\|_{\mathcal{C}^t}\|u\|_t.\tag{6.1.15}$$

To bound the second integral in (6.1.13) we first note that

$$\begin{aligned}A^{\alpha_k}_n\phi_n(x) &= \sum_{j=0}^{n-1}\langle(D_xF^j)^*\nabla\phi_1\circ F^j(x),(D_xF^n)^{-1}e_{\alpha_k}\rangle\\ &= \sum_{j=0}^{n-1}\langle\nabla\phi_1,(DF^{n-j})^{-1}e_{\alpha_k}\rangle\circ F^j(x),\end{aligned}\tag{6.1.16}$$

thus (5.1.3) implies

$$\|A^{\alpha_k}_n\phi_n\|_{\mathcal{C}^l} \leq C_\sharp\sum_{j=0}^{n-1}\|(DF^{n-j})^{-1}\|_{\mathcal{C}^l}\Lambda^{nl} \leq C_\sharp\sum_{j=0}^{n-1}\Lambda^{c_\sharp(n-j+l)} \leq C_\sharp\Lambda^{c_\sharp n}.\tag{6.1.17}$$

We can then use (6.1.15) to estimate

$$\begin{aligned}\left|\int\varphi A^\alpha_{n,k+1}((A^{\alpha_k}_n\phi_n)\cdot P^\alpha_{n,k-1}(\psi u))\right| &\leq C_\sharp\Lambda^{c_\sharp n}\|A^{\alpha_k}_n\phi_n\|_{\mathcal{C}^{t-k-1}_\nu}\|P^\alpha_{n,k-1}(\psi u))\|_{t-k-1}\\ &\leq C_\sharp\Lambda^{c_\sharp n}\|P^\alpha_{n,k-1}(\psi u))\|_{t-k-1}.\end{aligned}\tag{6.1.18}$$

To bound the last term we take $\phi \in \mathcal{C}^{t-k-1}$, $\|\phi\|_{\mathcal{C}^{t-k-1}} = 1, \gamma \in \Gamma$, and we consider

$$\int\phi\partial^{t-k-1}[P^\alpha_{n,k-1}(\psi u)]\circ\gamma.$$

We can then split the integral as in (6.1.13), although this time $\alpha = (\alpha_1, \cdots, \alpha_{k-1})$. For the first term we take $t-k-1$ derivatives in (6.1.14) and, arguing as we did to prove (6.1.15), we have

$$\left|\int\phi(\tau)\partial^{t-k-1}A^\alpha_{n,1}(\psi u)(\gamma(\tau))d\tau\right| \leq C_\sharp\Lambda^{c_\sharp n}\|\psi\|_{\mathcal{C}^t}\|u\|_t.$$

[4]We use the conventions $\prod_{j\in\emptyset}\partial_j A = A$ and $\sharp B$ denote the cardinality of the set B.

The second term is estimated in the same way, using the inductive assumption. The first statement of the Lemma then follows using this in (6.1.18).
In the special case $t = 2$, for $\alpha = (\alpha_1, \alpha_2)$,[5]

$$\begin{aligned}
P^{\alpha}_{n,2}(\psi u) &= A^{\alpha}_{n,1}(\psi u) - A^{\alpha_2}_{n}(\phi_n)A^{\alpha_1}_{n}(\psi u) - A^{\alpha}_{n,1}(\phi_n)\psi u - A^{\alpha_1}_{n}\phi_n A^{\alpha_2}_{n}(\psi u)\\
&\quad - A^{\alpha_1}_{n}\phi_n A^{\alpha_2}_{n}\phi_n \cdot \psi u\\
&= \left\{A^{\alpha}_{n,1}\psi - A^{\alpha_2}_{n}\phi_n A^{\alpha_1}_{n}\psi - (A^{\alpha_1}_{n}\phi_n A^{\alpha_2}_{n}\phi_n + A^{\alpha}_{n,1}\phi_n)\psi\right\} u\\
&\quad - \left\{A^{\alpha_2}_{n}\psi + \psi A^{\alpha_2}_{n}\phi_n\right\} A^{\alpha_1}_{n} u - \left\{A^{\alpha_1}_{n}\psi + \psi A^{\alpha_1}_{n}\phi_n\right\} A^{\alpha_2}_{n} u + \psi A^{\alpha}_{n,1} u\\
&=: \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4.
\end{aligned}$$

We then want to integrate the above terms along the curve ν against a test function $\varphi \in \mathcal{C}^2$. Recalling that the coefficients of the differential operators $A^{\alpha_j}_n$ have $\mathcal{C}^r$ norm bounded by $\|(DF^n)^{-1}\|_{\mathcal{C}^r}$, we thus have

$$\begin{aligned}
\int \varphi \Phi_1 \circ \nu \le C_\sharp \max_{i,j}\{&\|A^{\alpha}_{n,1}\psi\|_{\mathcal{C}^0_\nu}, \|\psi A^{\alpha}_{n,1}\phi_n\|_{\mathcal{C}^0_\nu},\\
&(1 + \|A^{\alpha_i}_{n}\phi_n\|_{\mathcal{C}^0_\nu})^2\|\psi\|_{\mathcal{C}^0}, \|A^{\alpha_i}_{n}\phi_n A^{\alpha_j}_{n}\psi\|_{\mathcal{C}^0_\nu}\}\|u\|_0.
\end{aligned}$$

The bounds for Φ_2 and Φ_3 are similar:

$$\int \varphi \Phi_2 \circ \nu \le C_\sharp \|(DF^n)^{-1}\|_{\mathcal{C}^1_\nu} \max_i\{\|A^{\alpha_i}_{n}\psi\|_{\mathcal{C}^1_\nu}, \|A^{\alpha_i}_{n}\phi_n\|_{\mathcal{C}^1_\nu}\|\psi\|_{\mathcal{C}^1}\}\|u\|_1.$$

Next, for any two vector $v, w \in \mathbb{R}^2$, $i, j \in \{1, 2\}$ and $x = (x_1, x_2) \in \mathbb{T}^2$,[6]

$$\begin{aligned}
&\partial_{F^*v}(\partial_{F^*w}u) = \partial_{F^*v}(\langle \nabla u, (DF)^{-1}w\rangle) = \langle \nabla(\langle \nabla u, (DF)^{-1}w\rangle), (DF)^{-1}v\rangle\\
&= \sum_{j,k} \partial^2_{x_j x_k} u \cdot \left[(DF)^{-1}v\right]_k [(DF)^{-1}w]_j + \sum_{j,k} \partial_{x_k} u\, \partial_{x_j}[(DF)^{-1}w]_k \cdot [(DF)^{-1}v]_j.
\end{aligned}$$

Recalling the properties of the $\|\cdot\|_\rho$ norm and (5.3.6) we have

$$\int \varphi \Phi_4 \circ \nu \le C_\sharp \{\mu^{2n}\|\psi\|_{\mathcal{C}^2}, \mu^n \|(DF^n)^{-1}\|_{\mathcal{C}^2_\nu}\|\psi\|_{\mathcal{C}^1}, \|(DF^n)^{-1}\|^2_{\mathcal{C}^1_\nu}\|\psi\|_{\mathcal{C}^1}\}\|u\|_2.$$

It follows by the property of the $\mathcal{C}^r$ norm and (6.0.2) that

$$\begin{aligned}
&\int \varphi P^{\alpha}_{2,n}(\psi u) \circ \nu \le C_\sharp \{\|A^{\alpha}_{n,1}\phi_n\|_{\mathcal{C}^0_\nu}\|\psi\|_{\mathcal{C}^0}, \|A^{\alpha_i}_{n}\phi_n\|^2_{\mathcal{C}^0_\nu}\|\psi\|_{\mathcal{C}^0},\\
&\|(DF^n)^{-1}\|_{\mathcal{C}^1_\nu}\|A^{\alpha_1}_{n}\phi_n\|_{\mathcal{C}^1_\nu}\|\psi\|_{\mathcal{C}^1}, \mu^n\|(DF^n)^{-1}\|_{\mathcal{C}^2_\nu}\|\psi\|_{\mathcal{C}^2}, \|(DF^n)^{-1}\|^2_{\mathcal{C}^1_\nu}\|\psi\|_{\mathcal{C}^2}\}\|u\|_2.
\end{aligned}$$

We have thus proved that

$$\begin{aligned}
\tilde{C}(2, n) = C_\sharp \big\{&\|A^{\alpha}_{n,1}\phi_n\|_{\mathcal{C}^0_\nu}, \max_{i\in\{1,2\}} \|A^{\alpha_i}_{n}\phi_n\|^2_{\mathcal{C}^0_\nu}, \|(DF^n)^{-1}\|_{\mathcal{C}^1_\nu}\|A^{\alpha_1}_{n}\phi_n\|_{\mathcal{C}^1_\nu},\\
&\mu^n\|(DF^n)^{-1}\|_{\mathcal{C}^2_\nu}, \|(DF^n)^{-1}\|^2_{\mathcal{C}^1_\nu}\big\}^+.
\end{aligned}$$

[5] We use the following notation: Φ_1 equals the third line from the bottom, the other Φ_i are, ordered, the terms in the second line from the bottom.

[6] Here we denote $\left[(DF)^{-1}w\right]_k := \langle (DF)^{-1}w, e_k\rangle$.

To conclude we give a bound of the above quantity. By Proposition 5.3.3 it is enough to find estimates for $\|(A^{\alpha}_{n,1}\phi_n)\|_{\mathcal{C}^0_\nu}$ and $\|A^{\alpha_1}_n\phi_n\|_{\mathcal{C}^1_\nu}\cdot\|(DF^n)^{-1}\|_{\mathcal{C}^1_\nu}$. First we can use formulae (6.1.16) and (5.3.6),

$$|\partial_{F^{n^*}e_\ell}\phi_n(x)| \leq C_\sharp \sum_{j=0}^{n-1}\mu^{n-j} \leq C_{\mu,n}\mu^n. \tag{6.1.19}$$

In particular $\|A^{\alpha_1}_n\phi_n\|_{\mathcal{C}^0_\nu} \leq C_{\mu,n}\mu^n$. Next we take another derivative of (6.1.19) in the direction of $F^{n^*}e_q$ and, setting $g_{\ell,n,j}(x) = \langle\nabla\phi_1, (DF^{n-j})^{-1}e_\ell\rangle(x)$, we have

$$\begin{aligned}\partial_{F^{n^*}e_q}(\partial_{F^{n^*}e_\ell}\phi_n(x)) &= \sum_{j=0}^{n-1}\langle\nabla(g_{\ell,n,j}\circ F^j(x)), (D_xF^n)^{-1}e_q\rangle\\ &= \sum_{j=0}^{n-1}\langle(D_xF^j)^*\nabla g_{\ell,n,j}\circ F^j(x), (D_xF^n)^{-1}e_q\rangle\\ &= \sum_{j=0}^{n-1}\langle\nabla g_{\ell,n,j}\circ F^j(x), (D_xF^{n-j})^{-1}e_q\rangle.\end{aligned} \tag{6.1.20}$$

By a direct computation we see that, recalling (5.3.10),

$$\|\nabla g_{\ell,n,j}\| \leq C_\sharp \max_i\{\|\partial_{x_i}(DF^{n-j})^{-1}\|\} \leq C_\sharp\varsigma_{n-j}(x)\mu^{n-j}.$$

We use this in (6.1.20) obtaining

$$|\partial_{F^{n^*}e_q}(\partial_{F^{n^*}e_\ell}\phi_n(x))| \leq C_\sharp\mu^{2n}\varsigma_n(x).$$

Hence, $\|A^{\alpha}_{n,1}\phi_n\|_{\mathcal{C}^0_\nu} \leq C_\sharp\mu^{2n}\varsigma_n$. Finally, we compute

$$\begin{aligned}\left|\frac{d}{dt}(A^{\alpha_1}_n\phi_n\circ\nu)\right| &\leq \sum_{j=0}^{n-1}|\langle(D_\nu F^j)^*D(\nabla\phi_1)\circ F^j\nu', (D_\nu F^n)^{-1}e_{\alpha_1}\rangle|\\ &\quad + |\langle\nabla\phi_1\circ F^j(\nu), [(D_\nu F^n)^{-1}]'e_{\alpha_1}\rangle|\\ &\leq C_{\mu,n}\mu^n + C_\sharp\varsigma_{n,m}\circ\nu,\end{aligned}$$

so that, using (5.3.6) and the definition of $\varsigma_{n,m}$ in (5.3.2), we obtain

$$\|A^{\alpha_1}_n\phi_n\|_{\mathcal{C}^1_\nu}\cdot\|(DF^n)^{-1}\|_{\mathcal{C}^1_\nu} \leq C_\sharp\mu^{2n-m}\varsigma^2_{n,m}.$$

The Lemma follows collecting all the above estimates and recalling again (5.3.6) for the estimate of $\mu^n\|(DF^n)^{-1}\|_{\mathcal{C}^2_\nu}$. □

6.2 Proof of Theorem 6.0.1

Proof. Given Lemma 5.5.2 the proof of Theorem 6.0.1 is almost exactly the same as in [43], hence we provide the full proof for $\rho = 0, 1$ and give a sketched proof for the case $1 < \rho \leq r - 1$.

Let us prove (6.0.3) first, since it is an immediate consequence of Lemma 5.5.2 and Definition 6.0.1 in the case $\rho = 0$. Indeed, by changing the variables and recalling the notation of Section 5.2 and Lemma 5.5.2, we have

$$\int_{\mathbb{T}} \phi(t)\mathcal{L}^n u(\gamma(t))dt = \sum_{\nu \in F^{-n}\gamma} \int_{\mathbb{T}} |\det D_{\nu(t)} F^n|^{-1} \cdot (u \circ \nu)(t) \cdot \phi(t)dt$$
$$= \sum_{\nu \in F^{-n}\gamma} \int_{\mathbb{T}} |\det D_{\hat{\nu}} F^n|^{-1} \cdot (u \circ \hat{\nu})(t) \cdot (\phi \circ h_n)(t) h_n'(t)dt$$
$$\leq \sum_{\nu \in F^{-n}\gamma} \left\| h_n' \left|\det D_{\hat{\nu}} F^n\right|^{-1} \right\|_{\mathcal{C}^0} \|u\|_0 \lesssim C_{\mu,n}\mu^n \|u\|_0.$$

Let us now proceed with the case $\rho = 1$, from which we deduce the general case by similar computations. We must bound the quantity

$$\int_{\mathbb{T}} \phi(t)(\partial_v \mathcal{L}^n u)(\gamma(t))dt = \int_{\mathbb{T}} \phi(t)\langle \nabla(\mathcal{L}^n u)(\gamma(t)), v\rangle dt,$$

where now $\phi \in \mathcal{C}^1(\mathbb{T})$ with norm one and v is a unitary $\mathcal{C}^r$ vector field. From Proposition 6.1.1 the above quantity is equal to the sum over $\nu \in F^{-n}\gamma$ of

$$\int \frac{1}{|\det D_\nu F^n|} \phi \cdot \partial_{F^{n*}v} u(\nu) + \int \mathcal{L}^n(P_0 u)\phi, \tag{6.2.1}$$

where P_0 is an operator of multiplication by a $\mathcal{C}^\rho$ function.
By Proposition 6.1.3 applied with $\psi = 1$, plus the result for $\rho = 0$, the last term is then bounded by $C_n\|u\|_1$. In order to bound the first term of (6.2.1) we need an analogous of Lemma 6.5 in [43]. The idea is to decompose the vector field v into a vector tangent to the central curve γ and a vector field approximately in the unstable direction so that the first one can be integrated by parts, while for the other we can exploit the expansion. The proof of the following Lemma follows that of the aforementioned paper, since the key point is the splitting of the tangent space in two directions, one of which is expanding. Once more, however, the presence of the central direction creates difficulties. For completeness we give the proof adapted to our case in Appendix B.

Lemma 6.2.1. *Let $\bar{n}$ be the integer provided by Lemma 5.4.1. For every $n > \bar{n}$, $\gamma \in \Gamma(\mathbb{c})$, $\nu \in F^{-n}\gamma$, and any vector field $v \in \mathcal{C}^r$, with $\|v\|_{\mathcal{C}^r} \leq 1$, defined in some neighborhood $M(\gamma)$ of γ, there exist a neighborhood $M'(\gamma)$ of γ and a decomposition*

$$v = \hat{v}^c + \hat{v}^u, \tag{6.2.2}$$

where $\hat{v}^c$ and $\hat{v}^u$ are $\mathcal{C}^r(M'(\gamma))$ vector fields such that, setting $F^n(N(\nu)) = M'(\gamma)$,

- $\hat{v}^c(\gamma(t)) = g(t)\gamma'(t)$, *where $g \in \mathcal{C}^r$ and $\|g\|_{\mathcal{C}^\rho} \leq C_\sharp \mathbb{c}^{\rho!} C_{\mu,n}^{a_\rho} \mu^{\rho! n}$,*
- $\|(F^n)^*\hat{v}^u\|_{\mathcal{C}^\rho(N(\nu))} \leq \lambda_-^{-n} C_{\mu,n}^{\rho a_\rho} \mu^{\rho\rho! n}$,
- $\|(F^n)^*\hat{v}^c\|_{\mathcal{C}^\rho(N(\nu))} \leq C_{\mu,n}^{2\rho a_\rho + 1} \mu^{(\rho+1)(2\rho!+1)n}$,
- $\|\hat{v}^u\|_{\mathcal{C}^\rho(M'(\gamma))} + \|\hat{v}^c\|_{\mathcal{C}^\rho(M'(\gamma))} \leq C_n$.

By the above decomposition, the first term in (6.2.1) becomes

$$\int \frac{1}{|\det D_\nu F^n|}\phi \cdot \partial_{F^{n*}\hat{v}^c} u(\nu) + \int \frac{1}{|\det D_\nu F^n|}\phi \cdot \partial_{F^{n*}\hat{v}_u} u(\nu). \tag{6.2.3}$$

Since $\gamma(t) = F^n\nu(t)$ we have $g(t)D_{\nu(t)}F^n \cdot \nu'(t) = \hat{v}^c(F^n\nu(t))$, hence:

$$g(t)\nu'(t) = (D_{\nu(t)}F^n)^{-1} \cdot \hat{v}^c(F^n\nu(t)) = F^{n^*}\hat{v}^c(\nu(t)).$$

Accordingly,

$$\begin{aligned}
&\int \frac{\phi(t)}{|\det D_{\nu(t)}F^n|}\partial_{F^{n*}\hat{v}^c} u(\nu(t))dt = \int \frac{g(t)\phi(t)}{|\det D_{\nu(t)}F^n|}\frac{d}{dt}(u(\nu(t)))dt \\
&= \int \frac{g(t)\phi(t)}{|\det D_{\hat{\nu}\circ h_n^{-1}(t)}F^n|}\left[\frac{d}{dt}(u\circ\hat{\nu})\right]\circ h_n^{-1}(t)\left[h_n^{-1}(t)\right]' dt \\
&= \int \frac{[g\phi]\circ h_n(t)}{|\det D_{\hat{\nu}(t)}F^n|}(u\circ\hat{\nu})'(t)dt = -\int \frac{d}{dt}\left(\frac{[g\phi]\circ h_n(t)}{|\det D_{\hat{\nu}(t)}F^n|}\right)u(\hat{\nu}(t))dt \\
&\leq \left\|\frac{[g\phi]\circ h_n}{\det D_{\hat{\nu}}F^n}\right\|_{\mathcal{C}^1}\|u\|_0.
\end{aligned}$$

Summing over $\nu \in F^{-n}\gamma$ and using Lemma 5.5.2 we obtain

$$\sum_{\nu\in F^{-n}\gamma}\int \phi(t)\partial_{F^{n*}\hat{v}^c} u(\nu(t))dt \lesssim \mathbb{c}^2 C^2_{\mu,n}\mu^{3n}\|u\|_0. \tag{6.2.4}$$

The second term of (6.2.3) is

$$\begin{aligned}
\int \frac{\phi}{|\det D_\nu F^n|}\partial_{F^{n*}\hat{v}^u} u(\nu) &= \int \frac{\phi}{|\det D_\nu F^n|}\langle\nabla u, F^{n^*}\hat{v}^u\rangle\circ\nu \\
&\leq C_\sharp\left\|\frac{\phi\circ h_n h_n'}{\det D_{\hat{\nu}}F^n}\right\|_{\mathcal{C}^1}\|F^{n^*}\hat{v}^u\circ\hat{\nu}\|_{\mathcal{C}^1}\|u\|_1 \\
&\leq C_\sharp\|h_n\|_{\mathcal{C}^1}\left\|\frac{h_n'}{\det D_{\hat{\nu}}F^n}\right\|_{\mathcal{C}^1}\lambda_-^{-n}C_{\mu,n}\mu^n\|u\|_1,
\end{aligned} \tag{6.2.5}$$

where we made the usual change of variables $t = h_n(s)$ and used Lemma 6.2.1. Finally, using (6.2.4) and (6.2.5) in (6.2.3), and recalling (5.4.1), we have by Lemma 5.5.2, with $\rho = 1$,

$$\|\mathcal{L}^n u\|_1 \leq \lambda_-^{-n}C_{\mu,n}\mu^{2n}\|u\|_1 + C_n\|u\|_0. \tag{6.2.6}$$

For the general case $1 \leq \rho \leq r-1$ one has to control the term

$$\int_{\mathbb{T}}\phi(t)\partial_{v_s}\cdots\partial_{v_1}\mathcal{L}^n u(\nu(t))dt,$$

for vector fields $v_j \in \mathcal{C}^\rho, j = 1, ..., s$ and $s \leq \rho$. Using again Propositions 6.1.1 and 6.1.3, the latter is bounded by

$$\sum_{\nu\in F^{-n}\gamma}\int \frac{1}{|\det D_\nu F^n|}\phi\cdot\partial_{F^{n^*}v_s\cdots F^{n^*}v_1}u(\nu) + C_{n,\rho}\|u\|_{\rho-1}. \tag{6.2.7}$$

Now the strategy is exactly the same as before. We use Lemma 6.2.1 to decompose each $v_j = \hat{v}_j^u + \hat{v}_j^c$. We take $\sigma \in \{u, c\}^s$, $k = \#\{i|\sigma_i = c\}$ and let π be a permutation of $\{1, \dots, s\}$ such that $\pi\{1, \dots, k\} = \{i|\sigma_i = c\}$. Using integration by parts, we can write the integral in (6.2.7) as

$$\int \frac{\phi}{\det D_\nu F^n} \partial_{F^{n*} v_s} \dots \partial_{F^{n*} v_1} u(\nu) = \sum_{\sigma \in \{u,c\}^s} \int \frac{\phi}{\det D_\nu F^n} \left(\prod_{s=1}^{1} \partial_{F^{n*} \hat{v}_i^{\sigma_i}} \right) u(\nu)$$

$$= \sum_{\sigma \in \{u,c\}^s} \int \frac{\phi}{\det D_\nu F^s} \prod_{i=s}^{k} \partial_{F^{n*} \hat{v}^c_{\pi(i)}} \prod_{i=k+1}^{1} \partial_{F^{n*} \hat{v}^u_{\pi(i)}} u(\nu) + C_{n,\rho} \|u\|_{\rho-1}$$

$$= \sum_{\sigma \in \{u,c\}^s} (-1)^k \int \prod_{i=k+1}^{s} \partial_{F^{n*} \hat{v}^u_{\pi(i)}} u(\nu) \prod_{i=k}^{1} \partial_{F^{n*} \hat{v}^c_{\pi(i)}} \left(\frac{\phi}{\det D_{\nu(t)} F^n} \right) + C_{n,\rho} \|u\|_{\rho-1}.$$

By Lemma 6.2.1,

$$\|F^{n^*} \hat{v}^c_{\pi(i)}\|_{\mathcal{C}^\rho(\nu)} \le C_{\mu,n}^{2\rho a_\rho + 1} \mu^{(\rho+1)(2\rho!+1)n}$$

while

$$\| \prod_{i=k+1}^{s} F^{n^*} \hat{v}^u_{\pi(i)} \|_{\mathcal{C}^\rho(\nu)} \le C \lambda_-^{-(s-k)n} (C_{\mu,n}^{\rho a_\rho} \mu^{\rho\rho! n})^{s-k}.$$

It follows by Lemma 5.5.2, equation (5.4.1) and the fact that $\|\phi\|_{\mathcal{C}^r} \le 1$, that[7]

$$\sum_{\nu \in F^{-n}\gamma} \int \frac{\phi}{\det DF^n} \partial_{F^{n*} v_1} \dots \partial_{F^{n*} v_\rho} u \circ \nu$$

$$\le \lambda_-^{-\rho n} C_{\mu,n}^{\rho^2 a_\rho} \mu^{\rho^2 \rho! n} \|h_n\|_{\mathcal{C}^\rho} \sum_{\nu \in F^{-n}\gamma} \left\| \frac{h'_n}{\det D_{\hat{\nu}} F^n} \right\|_{\mathcal{C}^\rho} \|u\|_\rho + C_{n,\rho} \|u\|_{\rho-1}$$

$$\lesssim \lambda_-^{-\rho n} C_{\mu,n}^{\rho^2 a_\rho + a_\rho + \tilde{a}_\rho} \mu^{n(\rho^2 \rho! + \rho! + \tilde{b}_\rho)} \|u\|_\rho + C_{n,\rho} \|u\|_{\rho-1},$$

hence (6.0.4) with $\bar{a}_\rho = 1 + a_\rho(\rho^2 + \rho(\rho+1)/2 + 1)$ and $\bar{b}_\rho = 1 + \rho!(2\rho^2 + \rho/2 + 1)$. □

The last result of this section is a Corollary of Theorem 6.0.1 which provides the inequality we are truly interested in.

Corollary 6.2.2. *Let us assume that, for every integer* $1 \le \rho \le r - 1$,

$$\mu^{\bar{b}_\rho} \lambda_-^{-\frac{\rho}{2}} < 1, \tag{6.2.8}$$

where $\bar{b}_\rho$ *given in Theorem 6.0.1. Let* $\delta_* \in (\lambda_-^{-\frac{1}{2}}, 1)$. *Then, for each* $n \in \mathbb{N}$,

$$\|\mathcal{L}^n u\|_\rho \le C_\sharp \delta_*^n \|u\|_\rho + C_{\mu.n} \mu^n \|u\|_0. \tag{6.2.9}$$

[7]Notice that the coefficient in front of the strong norm is obtained in the case $s = \rho$ and $k = 0$, while all the other terms are bounded again by $C_{n,\rho} \|u\|_{\rho-1}$.

Proof. Let us set $\delta := \lambda_-^{-\frac{1}{2}}$ and take $\bar{n} \in \mathbb{N}$ large enough to guarantee that $C_{\mu,n}^{\bar{a}_\rho} \mu^{\bar{b}_\rho \bar{n}} \lambda_-^{-\rho\bar{n}} < \delta^{\rho\bar{n}}$ for every $\rho \in [1, r-1]$. Notice that this is possible by the definition of $C_{\mu,n}$ and (6.2.8). Let us proceed by induction on ρ. For $\rho = 1$ the statement is simply (6.0.3). Let us assume it true for each integer smaller then or equal to $\rho - 1$. By Theorem 6.0.1 and (6.2.8), we have

$$\|\mathcal{L}^{\bar{n}} u\|_\rho \leq C_\sharp \delta^{\rho\bar{n}} \|u\|_\rho + C_{\bar{n}} \|u\|_{\rho-1}. \tag{6.2.10}$$

For every $m \in \mathbb{N}$ we write $m = \bar{n}q + r$, $0 \leq r < \bar{n}$, and iterate (6.2.10) to have

$$\|\mathcal{L}^m u\|_\rho = \|\mathcal{L}^{\bar{n}}(\mathcal{L}^{m-\bar{n}} u)\|_\rho \leq C_\sharp \delta^{\rho\bar{n}} \|\mathcal{L}^{m-\bar{n}} u\|_\rho + C_{\bar{n}} \|\mathcal{L}^{m-\bar{n}} u\|_{\rho-1} \leq \cdots$$
$$\cdots \leq C_\sharp \delta^{q\rho\bar{n}} \|\mathcal{L}^r u\|_\rho + C_\sharp \sum_{k=0}^{q-1} \delta^{k\rho\bar{n}} \|\mathcal{L}^{m-(k+1)\bar{n}} u\|_{\rho-1} \leq C_\sharp \delta^{\rho m} \|u\|_\rho + C_{\mu,m} \mu^m \|u\|_{\rho-1},$$

where we used $\|\mathcal{L}^{m-(k+1)\bar{n}} u\|_{\rho-1} \leq C_{\mu,m} \mu^{m-(k+1)} \|u\|_{\rho-1}$ by the inductive assumption. We iterate the last inequality ρ times and obtain

$$\begin{aligned} \|\mathcal{L}^{\rho m} u\|_\rho &\leq C_{\mu,m}^{\rho-1} (\mu^{\rho-1}\delta^\rho)^m \|u\|_\rho + C_{\mu,m}^\rho \mu^{\rho m} \|u\|_0 \\ &\leq C_{\mu,m}^\rho (\mu^\rho \delta^\rho)^m \|u\|_\rho + C_{\mu,m}^\rho \mu^{\rho m} \|u\|_0. \end{aligned}$$

We then consider the above inequality for m such that $\rho m = \bar{n}$, so that

$$C_{\mu,m}^\rho (\mu^\rho \delta^\rho)^m < \delta^{\bar{n}},$$

as $\mu^\rho \delta^\rho \leq \mu^{\bar{b}_\rho} \lambda_-^{-\frac{\rho}{2}} < 1$ by assumption. Hence,

$$\|\mathcal{L}^{\bar{n}} u\|_\rho \leq \delta^{\bar{n}} \|u\|_\rho + C_{\mu,\bar{n}} \mu^{\bar{n}} \|u\|_0. \tag{6.2.11}$$

Finally, we iterate once again (6.2.11) and we obtain the result for some $\delta_* \in (\delta, 1) = (\lambda_-^{-\frac{1}{2}}, 1)$. $\square$

Remark 6.2.3. *Although Corollary 6.2.2 provides a Lasota-Yorke inequality, a fundamental ingredient is missing. Indeed the embedding of $\mathcal{B}_\rho$ in $\mathcal{B}_0$ is not compact.*

Chapter 7

A second Lasota-Yorke type inequality: preliminaries

The main result of the following two sections is the second step towards the proof of Theorem 4.3.2, namely a Lasota-Yorke type inequality between the Hilbert space $\mathcal{H}^s$ and $\mathcal{B}_\rho$.[1] We will see in Corollary 9.0.2 that this solves the compactness problem mentioned in Remark 6.2.3. First we state some result on the $\mathcal{H}^s$-norm of the transfer operator.

7.1 $\mathcal{H}^s$-norm of $\mathcal{L}$

Lemma 7.1.1. *Let $F \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$ satisfying* **(H1)**. *For each $n \in \mathbb{N}$ and $1 \leq s \leq r$, there exist $A_s, Q(n,s) > 0$ such that, for every $u \in \mathcal{H}^s(\mathbb{T}^2, \mathbb{R})$,*

$$\|\mathcal{L}^n u\|_{L^2} \leq \|\mathcal{L}^n 1\|_\infty^{\frac{1}{2}} \|u\|_{L^2} \tag{7.1.1}$$

$$\|\mathcal{L}^n u\|_{\mathcal{H}^s}^2 \leq A_s \mu^{2sn} \|\mathcal{L}^n 1\|_\infty \|u\|_{\mathcal{H}^s}^2 + Q(n,s)\|u\|_{\mathcal{H}^{s-1}}^2, \tag{7.1.2}$$

where $Q(n,1) \leq C_{\mu,n}^{\frac{3}{2}} \mu^{2n}$.

Proof. First of all notice that

$$\begin{aligned}\|\mathcal{L}^n u\|_{L^2}^2 &\leq \|u\|_{L^2} \left(\int (\mathcal{L}^n u \circ F^n)^2\right)^{\frac{1}{2}} \leq \|u\|_{L^2} \left(\int (\mathcal{L}^n u)^2 \mathcal{L}^n 1\right)^{\frac{1}{2}} \\ &\leq \|u\|_{L^2} \|\mathcal{L}^n 1\|_\infty^{\frac{1}{2}} \|\mathcal{L}^n u\|_{L^2},\end{aligned} \tag{7.1.3}$$

hence (7.1.1). Next, by (6.1.9) and (6.1.10) we have, for each $v_i \in \{e_1, e_2\}$,

$$\begin{aligned}\|\partial_{v_s} \cdots \partial_{v_1} \mathcal{L}^n u\|_{L^2}^2 \leq{}& \|\mathcal{L}^n(\partial_{F^{n*}v_s} \cdots \partial_{F^{n*}v_1}) u\|_{L^2}^2 \\ &+ \sum_{k=1}^{s} \|\mathcal{L}^n(A_{n,k}^\alpha((A_n^{\alpha_k}\phi_n) \cdot P_{k-1}^\alpha u))\|_{L^2}^2.\end{aligned} \tag{7.1.4}$$

[1]See Appendix C for definitions and properties of $\mathcal{H}^s(\mathbb{T}^2)$.

Let us analyse the first term above when $s = 2$. Notice that

$$\begin{aligned}\partial_{F^{n*}v_2}(\partial_{F^{n*}v_1}u) &= \langle \nabla \left(\langle \nabla u, (DF^n)^{-1}v_1 \rangle \right), (DF^n)^{-1}v_2 \rangle \\ &= \langle (DF^n)^{-1}v_1 D^2 u, (DF^n)^{-1}v_2 \rangle + \langle D((DF^n)^{-1}v_1)\nabla u, (DF^n)^{-1}v_2 \rangle.\end{aligned}$$

where $D^2 f$ indicates the Hessian of a function f and $D(V)$ is the Jacobian of the vector field V. The term with higher derivatives of u has coefficients bounded by $\|(DF^n)^{-1}\|^2$, while the other term is a differential operator of order one applied to u. In the general case we can find some $P_{s-1,\rho}$ such that

$$|\mathcal{L}^n(\partial_{F^{n*}v_s} \cdots \partial_{F^{n*}v_1})u| \le \|(DF^n)^{-1}\|^s \mathcal{L}^n(|\partial_{v_s} \cdots \partial_{v_1} u|) + |\mathcal{L}^n P_{s-1,\rho} u|. \quad (7.1.5)$$

Hence, by (7.1.1), (C.0.4) and (5.3.5), there exists a constant $C_1(n, s)$ such that

$$\|\partial_{v_s} \cdots \partial_{v_1} \mathcal{L}^n u\|_{L^2}^2 \le C_\sharp \|\mathcal{L}^n 1\|_\infty \mu^{2sn} \|u\|_{\mathcal{H}^s}^2 + C_1(n, s)\|u\|_{\mathcal{H}^{s-1}}^2. \quad (7.1.6)$$

Similarly there exists $C_2(s, n)$ such that

$$\sum_{k=1}^{t} \|\mathcal{L}^n(A_{n,k}^\alpha((A_n^{\alpha_k}\phi_n) \cdot P_{n,k-1}^\alpha u))\|_{L^2}^2 \le C_2(n, s)\|u\|_{\mathcal{H}^{s-1}}^2. \quad (7.1.7)$$

By (7.1.4), (7.1.6) and (7.1.7) we obtain

$$\|\mathcal{L}^n(\partial_{F^{n*}v_1} \cdots \partial_{F^{n*}v_s})u\|_{L^2}^2 \le C_\sharp \|\mathcal{L}^n 1\|_\infty \mu^{2sn} \|u\|_{\mathcal{H}^s}^2 + Q(n, s)\|u\|_{\mathcal{H}^{s-1}}^2.$$

It remains to prove that in the case $s = 1$ we have an explicit bound on $Q(n, 1)$. Recall that by (6.1.5) and (7.1.1) we have, for any $v \in \{e_1, e_2\}$,

$$\begin{aligned}\|\langle \nabla \mathcal{L}^n u, v \rangle\|_{L^2} &\le \|\mathcal{L}^n \langle \nabla u, (DF^n)^{-1} v \rangle\|_{L^2} + \|\mathcal{L}^n(\langle \nabla \phi_n, (DF^n)^{-1} v \rangle u)\|_{L^2}, \\ &\le \|\mathcal{L}^n 1\|_\infty^{\frac{1}{2}} \left(\|\langle \nabla u, (DF^n)^{-1} v \rangle)\|_{L^2} + \|(\langle \nabla \phi_n, (DF^n)^{-1} v \rangle u)\|_{L^2} \right).\end{aligned} \quad (7.1.8)$$

A bound for the first term is straightforward, since by (5.3.5)

$$\|\langle \nabla u, (DF^n)^{-1} v \rangle)\|_{L^2} \le C_\sharp \mu^n \|\nabla u\|_{L^2}. \quad (7.1.9)$$

For the second term we use formula (6.1.16) and we have

$$\begin{aligned}\|(\langle \nabla \phi_n, (DF^n)^{-1} v \rangle u)\|_{L^2} &\le \sum_{j=0}^{n-1} \|\langle \nabla \phi_1 \circ F^j(x), (D_{F^j x} F^{n-j})^{-1} v \rangle\|_\infty \|u\|_{L^2} \\ &\le C_\sharp \sum_{j=0}^{n-1} \mu^{n-j} \|u\|_{L^2} \le C_{\mu,n} \mu^n \|u\|_{L^2},\end{aligned} \quad (7.1.10)$$

By (7.1.8), (7.1.9), (7.1.10) and (5.5.26) we obtain (7.1.2) for $s = 1$. □

7.2 Transversality

In this Section we give some useful definitions and results related to the quantities $\mathcal{N}_F, \widetilde{\mathcal{N}}_F$ defined in section 4.2. Recall that

$$\begin{aligned} \mathcal{N}_F(n) &= \sup_{y\in\mathbb{T}^2} \sup_{z_1\in F^{-n}(y)} \mathcal{N}_F(n,y,z_1) \\ \widetilde{\mathcal{N}}_F(n) &= \sup_{y\in\mathbb{T}^2} \sup_{L} \widetilde{\mathcal{N}}_F(n,y,L). \end{aligned} \tag{7.2.1}$$

Both $\mathcal{N}_F$ and $\widetilde{\mathcal{N}}_F$ depend on the map F, however in the following we will drop the F dependence to ease notation. An important advantage of $\widetilde{\mathcal{N}}$ over $\mathcal{N}$ is the following

Proposition 7.2.1. *$\widetilde{\mathcal{N}}(n)$ is sub-multiplicative, i.e $\widetilde{\mathcal{N}}(n+m) \le \widetilde{\mathcal{N}}(n)\widetilde{\mathcal{N}}(m)$, for every $n, m \in \mathbb{N}$.*

Proof. For any $z \in \mathbb{T}^2$, let us call L' the line obtained applying $(DF^n(z))^{-1}$ to L. Then

$$\begin{aligned} \widetilde{\mathcal{N}}(y,L,n+m) &= \sum_{\substack{z\in F^{-n-m}(y)\\ DF^{n+m}(z)\mathbf{C}_u\supset L}} |\det DF^{n+m}(z)|^{-1} \\ &= \sum_{\substack{\hat z\in F^{-n}(y)\\ DF^{n}(\hat z)\mathbf{C}_u\supset L}} \sum_{\substack{z\in F^{-m}(\hat z)\\ DF^{m}(z)\mathbf{C}_u\supset (DF^n(\hat z))^{-1}L}} \frac{1}{|\det DF^m(z)\det DF^n(\hat z)|} \\ &\le \sum_{\substack{\hat z\in F^{-n}(y)\\ DF^{n}(\hat z)\mathbf{C}_u\supset L}} \frac{1}{|\det DF^n(\hat z)|} \sup_{\tilde z}\sup_{L'} \sum_{\substack{\tilde z\in F^{-m}(\hat z)\\ DF^{m}(z)\mathbf{C}_u\supset L'}} \frac{1}{|\det DF^m(z)|}, \end{aligned}$$

taking the sup over $y \in \mathbb{T}^2$ and L we get the claim. □

Remark 7.2.2. *The above Proposition, in spite of its simplicity, turns out to be pivotal. The sub-multiplicativity of the sequence $\widetilde{\mathcal{N}}(n)$ implies the existence of $\lim_{n\to\infty}\widetilde{\mathcal{N}}(n)^{\frac{1}{n}}$. Also, an estimate of $\widetilde{\mathcal{N}}(n_0)$ for some $n_0 \in \mathbb{N}$ yields an estimate for all $n \in \mathbb{N}$.*

The result below, inspired by [16], provides the relation between $\mathcal{N}$ and $\widetilde{\mathcal{N}}$.

Lemma 7.2.3. *Let $\alpha = \frac{\log(\lambda_-\mu^{-2})}{\log(\lambda_+)} \in (0,1)$ and $m_0 = m_0(n) = \lceil \alpha n\rceil$ we have, for all $n \in \mathbb{N}$*

$$\mathcal{N}(n)^{\frac{1}{n}} \le \|\mathcal{L}^{n-m_0}1\|_\infty^{\frac{1}{n}} \left(\widetilde{\mathcal{N}}(m_0)^{\frac{1}{m_0}}\right)^{\alpha}.$$

Proof. Given $y \in \mathbb{T}^2$, we consider $z_1, z_2 \in F^{-n}(y)$ such that $D_{z_1}F^n\mathbf{C}_u \cap D_{z_2}F^n\mathbf{C}_u \neq \{0\}$ and the line $L := L(z_1) := D_{z_1}F^n(\mathbb{R}\times\{0\})$. Let $v_\pm = (1, \pm\chi_u) \in \mathbf{C}_u$ and $\theta_n := \measuredangle(D_{z_1}F^n e_1, D_{z_1}F^n v_\pm)$. Clearly, for each $n \in \mathbb{N}$, $|\cos\theta_n|^{-1} \le a_0$, for some uniform $a_0 \geq 1$.[2] On the other hand, by formula (5.3.1), Proposition 5.3.1 and condition (4.1.5) we have

$$\begin{aligned}|\tan\theta_n| \leq a_0|\sin\theta_n| &= a_0|\sin\measuredangle(e_1, v_\pm)|\frac{|\det D_{z_1}F^n|\|v_\pm\|}{\|D_{z_1}F^n e_1\|\|D_{z_1}F^n v_\pm\|} \\ &\leq |\sin(\arctan\chi_u)|a_0 C_* C_\star^2 \mu_+^n \lambda_-^{-n} \\ &= a_0 C_* C_\star^2 \chi_u \mu_+^n \lambda_-^{-n},\end{aligned} \tag{7.2.2}$$

where we have used that $\sin(\arctan x) = x(\sqrt{1+x^2})^{-1}$. Next, note that in the projective space $\mathbb{RP}^2$ the cones are canonically identified with two intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$, while the line is a point that we also denote by L. From the assumption on the cones, and (7.2.2), we have that the projective distance between L and each one of the extremal points of I_2 is bounded by

$$\min\{\operatorname{dist}(L, a_2), \operatorname{dist}(L, b_2)\} \leq a_0 C_* C_\star^2 \chi_u \lambda_-^{-n}\mu_+^n. \tag{7.2.3}$$

Let us now take $m < n$ to be chosen later and, for $\tilde{z} = F^{n-m}(z_2)$, consider the cone $D_{\tilde{z}}F^m\mathbf{C}_u$ corresponding to the interval I_3 in the projective space. By the forward invariance of the unstable cone it is clear that $D_{\tilde{z}}F^m\mathbf{C}_u \supset D_{z_2}F^n\mathbf{C}_u$, meaning that $I_3 \supset I_2$. We are going to prove that $L \in I_3$. Let $w_{n,m} := D_{z_2}F^{n-m}v_\pm$. Arguing as before, but remembering also condition (4.1.4), we have

$$\begin{aligned}|\sin\measuredangle(D_{\tilde{z}}F^m w_{n,m}, D_{\tilde{z}}F^m v_\pm)| &= |\sin\measuredangle(w_{n,m}, v_\pm)|\frac{|\det D_{\tilde{z}}F^m|\|v_\pm\|\|w_{n,m}\|}{\|D_{\tilde{z}}F^m w_{n,m}\|\|D_{\tilde{z}}F^m v_\pm\|} \\ &\geq C_*^{-1}C_\star^{-2}|\sin(\arctan(\iota_\star\chi_u))\lambda_+^{-m}\mu_-^n \\ &= C_*^{-1}C_\star^{-2}\frac{\iota_\star\chi_u}{\sqrt{1+(\iota_\star\chi_u)^2}}\lambda_+^{-m}\mu_-^n.\end{aligned} \tag{7.2.4}$$

It follows that, setting $B_{\iota_\star} := a_0 C_*^2 C_\star^4 \iota_\star^{-1} \geq 1$, if $\lambda_+^{-m}\mu_-^n \geq B_{\iota_\star}\lambda_-^{-n}\mu_+^n$, then $L \in I_3$. By a direct computation, and recalling that $\mu := \{\mu_+, \mu_-^{-1}\}^+$, we see that the choice $m = \lceil \alpha n - \beta_{\iota_\star}\rceil$, with $\alpha := \frac{\log(\lambda_-\mu^{-2})}{\log(\lambda_+)} > 0$ and $\beta_{\iota_\star} := \frac{\log B_{\iota_\star}}{\log \lambda_+} \geq 0$ yields the wanted inequality. Also, note that $\alpha < 1$ since $\lambda_- < \lambda_+\mu^2$.

The above computation shows that, given $z_1 \in F^{-n}(y)$, for every $z_2 \in F^{-n}(y)$ which is non-transversal to z_1, the line L is contained in the cone $D_{\tilde{z}}F^m\mathbf{C}_u$, for $\tilde{z} = F^{n-m}(z_2)$. In particular, for every $y \in \mathbb{T}^2$, one has

$$\begin{aligned}\sup_{z_1\in F^{-n}(y)} \sum_{\substack{z_2\in F^{-n}(y)\\ z_2 \not\pitchfork z_1}} |\det D_{z_2}F^n|^{-1} &\leq \sup_{L\subset\mathbb{RP}^2} \sum_{\substack{z_2\in F^{-n}(y)\\ D_{\tilde{z}}F^m\mathbf{C}_u\supset L}} |\det D_{z_2}F^n|^{-1} \\ &\leq \sup_{L\subset\mathbb{RP}^2} \sum_{\substack{z\in F^{-m}(y)\\ D_zF^m\mathbf{C}_u\supset L}} |\det D_zF^m|^{-1} \sum_{z_2\in F^{-n+m}(z)} |\det D_{z_2}F^{n-m}|^{-1} \\ &\leq \mathcal{L}^{n-m}1(z) \sup_{L\subset\mathbb{RP}^2} \sum_{\substack{z\in F^{-m}(y)\\ D_zF^m\mathbf{C}_u\supset L}} |\det D_zF^m|^{-1},\end{aligned}$$

[2] Notice that, for $n = 0$, $|\cos\theta_0|^{-1} = |\cos(\arctan(\chi_u))|^{-1} = \sqrt{1+\chi_u^2} \leq 2$.

where we have used (4.3.2). The above inequality then implies

$$\mathcal{N}(n) \leq \|\mathcal{L}^{n-m}1\|_\infty \widetilde{\mathcal{N}}(m) \leq \|\mathcal{L}^{\lceil n(1-\alpha)\rceil}1\|_\infty \widetilde{\mathcal{N}}(\lceil \alpha n \rceil). \quad \square$$

Chapter 8

A second Lasota-Yorke inequality: Results

To state the main result we need a few definitions. From Appendix C we recall that, for positive integers $N \in \mathbb{N}$ and $s \geq 1$, and for $u \in \mathcal{C}^r(\mathbb{T}^2)$,

$$\|\mathcal{L}^N u\|_{\mathcal{H}^s}^2 = \sum_{\xi \in \mathbb{Z}^2} |\langle \xi \rangle^s \mathcal{F}\mathcal{L}^N u(\xi)|^2, \tag{8.0.1}$$

where $\langle \xi \rangle = \sqrt{1 + \|\xi\|^2}$. Since we will work in Fourier space, it is convenient to introduce the notion of the dual of a cone in $\mathbb{R}^2$ by:

$$\mathbf{C}^{\perp} = \{v \in \mathbb{R}^2 \quad \text{s.t} \quad \exists u \in \mathbf{C} : \langle v, u \rangle = 0\}, \tag{8.0.2}$$

and if $\xi \in \mathbb{Z}^2$ we define $\xi^* := (\xi_1^*, \xi_2^*)$ to be the unit vector normal to ξ with the usual orientation. In addition, we define $\rho(\xi^*) = |\xi_2^*|/|\xi_1^*|$, for $\xi_1^* \neq 0$, and $\rho(\pm e_2) = \infty$, and

$$\vartheta(\xi^*) := \{\rho(\xi^*), \chi_u\}^+. \tag{8.0.3}$$

Let us also define the sequence

$$\mathbb{L}_n := \|\mathcal{L}^n 1\|_\infty. \tag{8.0.4}$$

Finally, to state the main result one last key assumption is needed. Let us define

$$n_0(F) := \min\{n \in \mathbb{N} : \forall p \in \mathbb{T}^2 \quad \exists z_1, z_2 \in F^{-n}p : z_1 \pitchfork z_2\}. \tag{8.0.5}$$

We will always assume that the map F satisfies

$$n_0(F) < \infty. \tag{8.0.6}$$

For simplicity, in the following we will just use the notation n_0 instead of $n_0(F)$.

Remark 8.0.1. *In [53] it is proven that assumption (8.0.6) is generic. More precisely, the author proves that for surface partially hyperbolic systems F, the quantity $\mathcal{N}_F(n)$ is generically strictly smaller than 1, for n large. Nevertheless, in Section 10 we will introduce an open set of systems for which (8.0.5) is always satisfied.*

The scope of this Section is to prove the following Theorem.

Theorem 8.0.2. *Let m_{χ_u} and n_0 be the integers given in (5.2.4) and* (8.0.5) *respectively. There exist uniform constants $C_1, c_\sharp > 0$, $\Lambda > 2$ and $\sigma > 1$ such that, for each $q_0 \geq n_0$ and any $1 \leq s < r$, if $M = \sigma m_{\chi_u}$ and $N = M + q_0$,*

$$\|\mathcal{L}^N u\|_{\mathcal{H}^s} \leq C_1 \left(\sqrt{[\mathbb{L}_M \mathcal{N}(q_0)]^{\frac{1}{N}} \mu^{2s}} \right)^N \|u\|_{\mathcal{H}^s} + \Theta_{\chi_u}(M, s) \|u\|_{s+2}. \tag{8.0.7}$$

where $\Theta_{\chi_u}(N, s) \lesssim C_{q_0} Q(M, s) C_{\mu,M} \Lambda^{c_\sharp M}$ and $Q(M, s)$ is the constant given in Lemma 7.1.1. In addition, if the map F satisfies the following condition

$$\chi_u^{-1} \|\omega\|_{\mathcal{C}^r} \leq C_5, \tag{8.0.8}$$

for some uniform $C_5 > 0$, then there exist $\beta_3, \beta_4 \in \mathbb{R}^+$ which depend only on C_5 such that

$$\Theta_{\chi_u}(M, 1) \leq C_\sharp C_{q_0} \chi_u^{-\frac{11}{2} c_\sharp \ln \mu} C_{\mu,M}^{\beta_3} \mu^{\beta_4 M} M^{\frac{1}{2}}. \tag{8.0.9}$$

We will prove Theorem 8.0.2 in Section 8.4, after several steps.

8.1 Partitions of unity

We will use notations and definitions given in Section 5.2. First of all we want to decompose the transfer operator using suitable partitions of unity. For each point $z \in \mathbb{T}^2$, and $q_0 \geq n_0$, let us set $\delta_{q_0}(z) := \mu_{q_0}^-(z) \lambda_{q_0}^+(z)^{-1}$,[1] and define

$$\mathcal{U}_{z,q_0} = \{y \in \mathbb{T}^2 \ : \ \|y - z\| \leq \mathbb{d} \epsilon \delta_{q_0}(z)\}, \tag{8.1.1}$$

where[2]

$$\mathbb{d} = \mathbb{d}(\chi_u) = L_\star(q_0, \chi_u)^{-1} C_0 \chi_u, \tag{8.1.2}$$

for some uniform constant C_0 to be chosen later. By Besicovitch covering theorem there exists a finite subset $\mathcal{A}$ and points $\{z_\alpha\}_{\alpha \in \mathcal{A}}$ such that $\mathbb{T}^2 \subset \bigcup_\alpha \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha = 5\mathcal{U}_{z_\alpha, q_0}$, and such that the number of intersections is bounded by some fixed constant $C_\sharp$. We then define a family of smooth function $\{\psi_\alpha\}_\alpha$ supported on $\mathcal{U}_\alpha$ such that $\sum_\alpha \psi_\alpha = 1$. Next we construct a refinement of the above partition using the inverse branches introduced in Section 5.2. For $\alpha \in \mathcal{A}$ we pick two curves $\gamma_\alpha, \tilde{\gamma}_\alpha \in \Upsilon$ such that $\mathcal{U}_\alpha \cap \gamma_\alpha = \{\emptyset\}$ and, recalling $\mathfrak{H}_{\gamma_\alpha,1} = \{\mathfrak{h} \in \mathfrak{H} : \mathcal{D}_\mathfrak{h} = \mathbb{T}^2 \setminus \gamma_\alpha\}$, for each $\mathfrak{h} \in \mathfrak{H}_{\gamma_\alpha,1} \cup \mathfrak{H}_{\tilde{\gamma}_\alpha,1}$ either $\mathfrak{h}(\mathbb{T}^2) \cap \gamma_\alpha = \emptyset$ or $\mathfrak{h}(\mathbb{T}^2) \cap \tilde{\gamma}_\alpha = \emptyset$. Note that the cardinality of $\mathfrak{H}_{\alpha,0} := \mathfrak{H}_{\gamma_\alpha,1}$ and $\mathfrak{H}_{\tilde{\gamma}_\alpha,1}$ is exactly d.

We can then consider the set $\mathfrak{H}_\alpha^n = \{(\mathfrak{h}_1, \cdots, \mathfrak{h}_n) \in \mathfrak{H}^n : \mathfrak{h}_j \in \mathfrak{H}_{\gamma_{j-1},\alpha}, j \in \{1, \cdots, n\}\}$ where $\gamma_j = \gamma_\alpha$ if $\mathfrak{h}_{j-1}(\mathbb{T}^2) \cap \gamma_\alpha = \emptyset$ and $\gamma_j = \tilde{\gamma}_\alpha$ if $\mathfrak{h}_{j-1}(\mathbb{T}^2) \cap \gamma_\alpha \neq \emptyset$. Note that the $\mathfrak{H}_\alpha^n$ has an element for each equivalence class of $\mathfrak{H}_{*,\gamma_\alpha}^n$, defined in equation (5.2.2), hence it is isomorphic to $\mathfrak{H}_{\gamma_\alpha,n}$ and has exactly d^n elements. To simplify notation, given $\alpha \in \mathcal{A}$ and $q_0 \in \mathbb{N}$, in the following we will denote $\mathfrak{H}^{q_0} := \mathfrak{H}_{*,\gamma_\alpha}^{q_0}$ which is then a set with finite cardinality.

[1]The functions μ_n^- and λ_n^+ are defined in (4.1.3).

[2]Recall that $L_\star$ is the Lipschitz constant of the unstable cone field given in (E.0.2).

Next, let

$$\psi_{\alpha,\mathfrak{h}}(z) = \psi_\alpha \circ F^{q_0}(z)\mathbb{1}_{\mathfrak{h},\alpha}(z),\ \forall \mathfrak{h} \in \mathfrak{H}^{q_0}, z \in \mathbb{T}^2, \tag{8.1.3}$$

where $\mathbb{1}_{\mathfrak{h},\alpha} := \mathbb{1}_{\mathcal{U}_{\alpha,\mathfrak{h}}}$, and $\mathcal{U}_{\alpha,\mathfrak{h}} := \mathfrak{h}(\mathcal{U}_\alpha)$. Notice that (8.1.3) defines again a $\mathcal{C}^\infty$ partition of unity, supported on $\{\mathcal{U}_{\alpha,\mathfrak{h}}\}_{\mathfrak{h}\in\mathfrak{H}^{q_0}}$, which have intersection multiplicity bounded by $C_\sharp$. We have the following result from [4, Lemma 9], whose proof is adapted to our case.

Lemma 8.1.1. *For each $u \in \mathcal{C}^r(\mathbb{T}^2)$*

$$\|u\|^2_{\mathcal{H}^s} \le C_\sharp \sum_{\alpha\in\mathcal{A}} \|u\psi_\alpha\|^2_{\mathcal{H}^s} \tag{8.1.4}$$

$$\sum_{\alpha\in\mathcal{A}} \sum_{\mathfrak{h}\in\mathfrak{H}^{q_0}} \|u\psi_{\alpha,\mathfrak{h}}\|^2_{\mathcal{H}^s} \le C_\sharp \|u\|^2_{\mathcal{H}^s} + C_\psi(s)\|u\|^2_{L^1}, \tag{8.1.5}$$

where $C_\psi(s)$ depends on ψ_α. However, $C_\psi(1) \lesssim C_{q_0}(\mathrm{d}\epsilon)^{-2}$.

Proof. For the first inequality note that

$$\|u\|^2_{\mathcal{H}^s} = \Big\| \sum_{\alpha\in\mathcal{A}} u\psi_\alpha \Big\|^2_{\mathcal{H}^s} = \sum_{(\alpha,\alpha')\in\mathcal{A}\times\mathcal{A}} \langle \psi_\alpha u, \psi_{\alpha'} u\rangle_s.$$

By the definition of the $\langle\cdot,\cdot\rangle_s$ the above sum is zero if the supports of ψ_α and $\psi_{\alpha'}$ do not intersect. For the other terms, denoting with $\mathcal{A}^*$ the set of elements in $\mathcal{A}\times\mathcal{A}$ for which the above supports intersect, we have:

$$\sum_{\mathcal{A}^*} \langle \psi_\alpha u, \psi_{\alpha'} u\rangle_s \le \sum_{\mathcal{A}^*} \frac{\|\psi_\alpha u\|^2_{\mathcal{H}^s} + \|\psi_{\alpha'} u\|^2_{\mathcal{H}^s}}{2} \le C_\sharp \sum_{\alpha\in\mathcal{A}} \|\psi_\alpha u\|^2_{\mathcal{H}^s}.$$

We now prove (8.1.5). By formula (C.0.4) we have

$$\begin{aligned}
\sum_{\alpha,\mathfrak{h}} \|u\psi_{\alpha,\mathfrak{h}}\|^2_{\mathcal{H}^s} &\lesssim \sum_{\alpha,\mathfrak{h}} \sum_{|\beta|\le s} \|\partial^\beta (u\psi_{\alpha,\mathfrak{h}})\|^2_{L^2} \\
&\lesssim \sum_{\alpha,\mathfrak{h}} \sum_{|\beta|\le s} \int_{\mathbb{T}^2} |\partial^\beta u|^2 |\psi_{\alpha,\mathfrak{h}}|^2 + \sum_{\alpha,\mathfrak{h}} \sum_{|\beta|\le s} \sum_{|\gamma|<|\beta|} C_{\beta,\gamma} \int_{\mathbb{T}^2} |\partial^\gamma u|^2 |\partial^{|\beta|-|\gamma|}\psi_{\alpha,\mathfrak{h}}|^2 \\
&\le C_\sharp \|u\|^2_{\mathcal{H}^s} + C_\psi(s)^{\frac{1}{2}} \|u\|^2_{\mathcal{H}^{s-1}} \le C_\sharp \|u\|^2_{\mathcal{H}^s} + C_\psi(s)\|u\|^2_{L^1},
\end{aligned}$$

where in the last line we used the fact that the $\psi_{\alpha,\mathfrak{h}}$ are partitions of unity and Lemma C.0.1. This proves (8.1.5) in the general case $s \ge 1$. Next we compute explicitly the second summation in the second line above for $s = 1$, which is bounded by:

$$\begin{aligned}
&\sum_{\alpha,\mathfrak{h}} \int_{\mathbb{T}^2} |u|^2 |\nabla(\psi_{\alpha,\mathfrak{h}})|^2 \le \sum_{\alpha,\mathfrak{h}} \int_{\mathbb{T}^2} |u|^2 |(DF^{q_0})^t \nabla\psi_\alpha \circ F^{q_0} \mathbb{1}_{\mathfrak{h},\alpha}|^2 \\
&\le \|(DF^{q_0})^t\|^2_\infty \sum_\alpha \int_{\mathbb{T}^2} \sum_{\mathfrak{h}\in\mathfrak{H}^{q_0}} |\nabla\psi_\alpha|^2 \circ F^{q_0} \mathbb{1}_{\mathfrak{h},\alpha} |u|^2 \le C_{q_0} \sup_\alpha \|\psi_\alpha\|^2_{\mathcal{C}^1} \|u\|^2_{L^2}.
\end{aligned}$$

Finally, since ψ_α is supported on $\mathcal{U}_\alpha$ which has diameter bounded by $\mathbb{d}\epsilon\mu^{-q_0}\lambda_+^{-q_0}$, it is easy to see that there exists C_{q_0} such that

$$\|\psi_\alpha\|_{\mathcal{C}^1} \lesssim C_{q_0}\,(\mathbb{d}\epsilon)^{-1}\,,$$

hence $C_\psi(1)^{\frac{1}{2}} \lesssim C_{q_0}\,(\mathbb{d}\epsilon)^{-1}$, from which we conclude. □

Remark 8.1.2. *Note that, under condition* (8.0.8), *recalling* (E.0.2), *we have*

$$L_\star(q_0,\chi_u) = C_\sharp C_{q_0}\chi_u^{1-c_\sharp \ln\mu},$$

which implies that, by (8.1.2), $C_\psi(1) \le \epsilon^{-2}C_{q_0}\chi_u^{-c_\sharp\ln\mu}$.

The next Proposition is the main ingredient for the proof of Theorem 8.0.2.

Proposition 8.1.3. *Let σ as in* (5.4.43) *and n_0 as in* (8.0.5). *For each $\xi \in \mathbb{Z}^2$, $m \in \mathbb{N}$, $q_0 \ge n_0$, $\tilde{\mathfrak{h}} \in \mathfrak{H}^\infty$, $\alpha \in \mathcal{A}$ and $\mathfrak{h} \in \mathfrak{H}^{q_0}$ such that $\bar{\mathfrak{h}} = \tilde{\mathfrak{h}} \circ \mathfrak{h}$ is well-defined, $D_p\bar{\mathfrak{h}}_m\xi^* \in \mathbf{C}_{\epsilon,c}$ for each $p \in supp\psi_{\alpha,\mathfrak{h}}$ and $D_p\bar{\mathfrak{h}}_{n_0}\xi^* \notin \mathbf{C}_u$, there exists M_ξ, depending only on ξ and m, such that $\sigma m \le M_\xi < \infty$, $\sup_{\xi:\xi^*\notin\mathbf{C}_u} M_\xi < \infty$ and, for each $t \ge 2$,*

$$\langle\xi\rangle^t|\mathcal{F}\mathcal{L}^{q_0}(\psi_{\alpha,\mathfrak{h}}\mathcal{L}^{M_\xi}u)(\xi)| \le K_1(t,M_\xi,m)\|u\|_t, \tag{8.1.6}$$

where $K_1(t,M_\xi,m) \le C_\sharp C_{\psi,q_0}\Lambda^{c_\sharp M_\xi}$, with C_{ψ,q_0} a constant which depends on $\psi_{\alpha,\mathfrak{h}}$. In addition, if the map satisfies condition (8.0.8), *then there exist $C_{\epsilon,q_0}, \beta_1, \beta_2 > 0$ such that*[3]

$$K_1(2,M_\xi,m) \le C_\sharp C_{\epsilon,q_0}\chi_u^{-c_\sharp\ln\mu}C_{\mu,M_\xi}^{\beta_1}\mu^{\beta_2 M_\xi}\vartheta(\xi^*)^{-6}. \tag{8.1.7}$$

Proof. Let $\xi = (\xi_1,\xi_2)$, let $j \in \{1,2\}$ such that $\|\xi\| \le 2|\xi_j|$, and $M_\xi > 0$ to be chosen later. Since $\xi_j\mathcal{F}u = -i\mathcal{F}\partial_{x_j}u$, $\|\mathcal{F}u\|_\infty \lesssim \|u\|_{L^1}$ and using (6.0.1) we have, for each $t \ge 1$ and setting $u_{\alpha,\mathfrak{h}}^{M_\xi} = \psi_{\alpha,\mathfrak{h}}\mathcal{L}^{M_\xi}u$,

$$\begin{aligned}\langle\xi\rangle^t|\mathcal{F}\mathcal{L}^{q_0}(\psi_{\alpha,\mathfrak{h}}\mathcal{L}^{M_\xi}u)(\xi)| &\lesssim \|\mathcal{L}^{q_0}(u_{\alpha,\mathfrak{h}}^{M_\xi})\|_{L^1} + |\xi_j|^t|\mathcal{F}\mathcal{L}^{q_0}(u_{\alpha,\mathfrak{h}}^{M_\xi})|\\ &\lesssim \|u\|_t + |\mathcal{F}\partial_{x_j}^t\mathcal{L}^{q_0}(u_{\alpha,\mathfrak{h}}^{M_\xi})|.\end{aligned} \tag{8.1.8}$$

Let us estimate the last term. Letting $J_k(p) = (\det D_pF^k)^{-1}$ we have

$$\begin{aligned}&\left[\mathcal{F}\partial_{x_j}^t\mathcal{L}^{q_0}(u_{\alpha,\mathfrak{h}}^{M_\xi})\right](\xi) = \int_{\mathcal{U}_\alpha} dz\, e^{-2\pi i\langle z,\xi\rangle}\partial_{z_j}^t\left[J_{q_0}\psi_{\alpha,\mathfrak{h}}\mathcal{L}^{M_\xi}u\right]\circ\mathfrak{h}(z)\\ &= \sum_{|\eta_1|+|\eta_2|=t} C_{\eta_1,\eta_2}\int_{\mathcal{U}_\alpha} dz\, e^{-2\pi i\langle z,\xi\rangle}\partial^{\eta_1}\left[\psi_{\alpha,\mathfrak{h}}\right]\circ\mathfrak{h}(z)\cdot\partial^{\eta_2}\left[J_{q_0}\mathcal{L}^{M_\xi}u\right]\circ\mathfrak{h}(z).\end{aligned} \tag{8.1.9}$$

Operating the change of variables $\gamma_\ell(\tau) = z + \ell\xi + \tau\xi^\perp$, where $\xi^\perp$ is the unit vector perpendicular to ξ and $\ell,\tau \in I_{q_0} = [-\mathbb{d}\epsilon\delta_{q_0}(z_\alpha), \mathbb{d}\epsilon\delta_{q_0}(z_\alpha)]$,[4] we have

$$\left|\mathcal{F}\partial_{x_j}^t\mathcal{L}^{q_0}(u_{\alpha,\mathfrak{h}}^{M_\xi})\right| \le C_\sharp \sup_{|\eta_1|+|\eta_2|=t}\int_{I_{q_0}} d\ell\left|\int_{I_{q_0}} d\tau\left\{\partial^{\eta_1}\psi_\alpha\cdot\partial^{\eta_2}\left[J_{q_0}\mathcal{L}^{M_\xi}u\right]\circ\mathfrak{h}\right\}(\gamma_\ell(\tau))\right|.$$

[3] Recall the definition of $\vartheta(\xi^*)$ in (8.0.3).

[4] This is because γ_ℓ is supported in some $5\mathcal{U}_{z_\alpha,q_0}$ given in (8.1.1), i.e. the integrand is supported on an interval depending on q_0.

Let $\overline{m}(z, \tilde{\mathfrak{h}} \circ \mathfrak{h})$ satisfies (5.4.44) with $n_\star = \min\{m, \bar{c}_2 \log \chi_u^{-1}\}$, and set $\overline{m}(\tilde{\mathfrak{h}}) = \sup_\ell \sup_{z\in\gamma_\ell} \overline{m}(z, \tilde{\mathfrak{h}} \circ \mathfrak{h})$.[5] We then define

$$M_\xi = \sup_{\tilde{\mathfrak{h}}\in\mathfrak{H}^\infty} \overline{m}(\tilde{\mathfrak{h}}), \tag{8.1.10}$$

and we observe that, by Lemma (5.4.4), $M_\xi \geq \sigma m$, where σ is defined in (5.4.43). Moreover, the assumption $D_p\bar{\mathfrak{h}}_{n_0}\xi^* \notin \mathbf{C}_u$ and condition (4.1.4) imply that $M_\xi < \infty$. At the end of the proof we will show that $\sup_{\xi^*\notin fC_u} M_\xi < \infty$.
Next, we define $\hat{\mathfrak{H}}^*_\alpha = \{\mathfrak{h}_{\overline{m}_\alpha(\bar{\mathfrak{h}})}\}_{\bar{\mathfrak{h}}\in\mathfrak{H}^\infty}$, $\hat{\mathfrak{H}}_\alpha = \{\hat{\mathfrak{h}} \;:\; \hat{\mathfrak{h}} \circ \mathfrak{h} \in \hat{\mathfrak{H}}^*_\alpha\}$, $v_{\alpha,\hat{\mathfrak{h}}} = \mathcal{L}^{M_\xi - \overline{m}(\hat{\mathfrak{h}})+q_0} u$ and write

$$\mathfrak{H}^{M_\xi} = \bigcup_{\hat{\mathfrak{h}}\in\hat{\mathfrak{H}}_\alpha} \left\{\mathfrak{h}' \circ \hat{\mathfrak{h}} \;:\; \mathfrak{h}' \in \mathfrak{H}^{M_\xi - \overline{m}(\hat{\mathfrak{h}})+q_0}\right\} \tag{8.1.11}$$

which allows to define the decomposition

$$\mathcal{L}^{M_\xi} u = \sum_{\hat{\mathfrak{h}}\in\hat{\mathfrak{H}}_\alpha} J_{\overline{m}(\hat{\mathfrak{h}})-q_0} \circ \hat{\mathfrak{h}} \cdot \left[\mathcal{L}^{M_\xi - \overline{m}(\hat{\mathfrak{h}})+q_0} u\right] \circ \hat{\mathfrak{h}} = \sum_{\hat{\mathfrak{h}}\in\hat{\mathfrak{H}}_\alpha} J_{\overline{m}(\hat{\mathfrak{h}})-q_0} \circ \hat{\mathfrak{h}} \cdot v_{\alpha,\hat{\mathfrak{h}}} \circ \hat{\mathfrak{h}}.$$

Thus, recalling (6.1.8),

$$\begin{aligned}&\left|\mathcal{F}\partial^t_{x_j}\mathcal{L}^{q_0}(u^{M_\xi}_{\alpha,\mathfrak{h}})\right| \\ &\leq C_\sharp \sup_{|\eta_1|+|\eta_2|=t} \sum_{\hat{\mathfrak{h}}\in\hat{\mathfrak{H}}_\alpha} \int_{I_{q_0}} d\ell \left| \int_{I_{q_0}} d\tau \left\{\partial^{\eta_1}\psi_\alpha \cdot J_{\overline{m}(\hat{\mathfrak{h}})} \circ \hat{\mathfrak{h}} \circ \mathfrak{h} \left[P^{\eta_2}_{\overline{m}(\hat{\mathfrak{h}}),|\eta_2|} v_{\alpha,\hat{\mathfrak{h}}}\right] \circ \hat{\mathfrak{h}} \circ \mathfrak{h}\right\} (\gamma_\ell(\tau))\right|\end{aligned}$$

Next, we apply Lemma E.0.1 to γ_ℓ with $\delta = \mathbb{d}\epsilon\delta_{q_0}(z_\alpha)$, note that the hypotheses of the Lemma are satisfied thanks to the assumptions on ξ. We thus obtain closed curves $\tilde{\gamma}_\ell$ with $j+1$ derivative bounded by $C_{q_0,j}\Delta^j_{\tilde{\gamma}}$. It follows

$$\begin{aligned}&\left|\mathcal{F}\partial^t_{x_j}\mathcal{L}^{q_0}(u^{M_\xi}_{\alpha,\mathfrak{h}})\right| \\ &\leq C_\sharp \sup_{|\eta_1|+|\eta_2|=t} \sum_{\hat{\mathfrak{h}}\in\hat{\mathfrak{H}}_\alpha} \int_{I_{q_0}} d\ell \left| \int_{\mathbb{T}} d\tau \left\{\partial^{\eta_1}\psi_\alpha \cdot J_{\overline{m}(\hat{\mathfrak{h}})} \circ \hat{\mathfrak{h}} \circ \mathfrak{h} \left[P^{\eta_2}_{\overline{m}(\hat{\mathfrak{h}}),|\eta_2|} v_{\alpha,\hat{\mathfrak{h}}}\right] \circ \hat{\mathfrak{h}} \circ \mathfrak{h}\right\} (\tilde{\gamma}_\ell(\tau))\right|\end{aligned}$$

Next, we apply, for each inverse branch $\hat{\mathfrak{h}} \circ \mathfrak{h}$, Lemma 5.4.4 to the curves $\tilde{\gamma}_\ell$ and obtain admissible central curves $\hat{\nu}_\ell = \nu_\ell \circ h_{\ell,\overline{m}}$.[6] Thus, we can rewrite the integrals in the right hand side of the above equation as follows

$$\begin{aligned}&\int_{\mathbb{T}} d\tau \left\{\partial^{\eta_1}\psi_\alpha \cdot J_{\overline{m}(\hat{\mathfrak{h}})} \circ \hat{\mathfrak{h}} \circ \mathfrak{h} \left[P^{\eta_2}_{\overline{m}(\hat{\mathfrak{h}}),|\eta_2|} v_{\alpha,\hat{\mathfrak{h}}}\right] \circ \hat{\mathfrak{h}} \circ \mathfrak{h}\right\} (\tilde{\gamma}_\ell(\tau)) \\ &= \int_{\mathbb{T}} d\tau \Psi_{\hat{\nu}_\ell}(\tau) \left\{(\partial^{\eta_1}\psi_\alpha) \circ F^{\overline{m}(\hat{\mathfrak{h}})} \cdot \left[P^{\eta_2}_{\overline{m}(\hat{\mathfrak{h}}),|\eta_2|} v_{\alpha,\hat{\mathfrak{h}}}\right]\right\} (\hat{\nu}_\ell(\tau)),\end{aligned}$$

[5] Notice that $\overline{m}$ depends on ξ through γ_ℓ. Also, it would be more precise to call it $\overline{m}(\tilde{\mathfrak{h}} \circ \mathfrak{h})$, but we keep the notation as simple as possible.

[6] Notice that ν_ℓ depends on $\hat{\mathfrak{h}}$, but we drop this dependence for simplicity.

where $\Psi_{\hat\nu_\ell}(\tau) = h'_{\ell,\overline{m}}[\det D_{\hat\nu_\ell(\tau)}F^{\overline{m}(\hat{\mathfrak{h}})}]^{-1}$. By Proposition 6.1.3 applied with $n = \overline{m}(\hat{\mathfrak{h}})$, $\varphi = \Psi_{\hat\nu_\ell}(\partial^{\eta_1}\psi_\alpha)\circ F^{\overline{m}(\hat{\mathfrak{h}})}\circ\hat\nu_\ell\|\Psi_{\hat\nu_\ell}(\partial^{\eta_1}\psi_\alpha)\circ F^{\overline{m}(\hat{\mathfrak{h}})}\|^{-1}_{\mathcal{C}^t_{\hat\nu_\ell}}$, $\psi = 1$, $u = v_{\alpha,\hat{\mathfrak{h}}}$ and $U = \mathbb{T}^2$, the above integral is bounded by

$$\tilde C(t,\overline{m}(\hat{\mathfrak{h}}),m)\|\Psi_{\hat\nu_\ell}(\partial^{\eta_1}\psi_\alpha)\circ F^{\overline{m}(\hat{\mathfrak{h}})}\|_{\mathcal{C}^t_{\hat\nu_\ell}}\|v_{\alpha,\hat{\mathfrak{h}}}\|_t|I_{q_0}|, \tag{8.1.12}$$

where $|I_{q_0}| \le 2\mathbb{d}\epsilon\delta_{q_0}(z_\alpha) \le 2\mathbb{d}\epsilon\lambda_+^{-q_0}\mu^{q_0}$. Accordingly,

$$\begin{aligned}&\left|\mathcal{F}\partial^t_{x_j}\mathcal{L}^{q_0}(u^{M_\xi}_{\alpha,\mathfrak{h}})\right|\\ &\le C_\sharp \sup_{|\eta_1|+|\eta_2|=t}\sum_{\hat{\mathfrak{h}}\in\hat{\mathfrak{H}}_\alpha}\tilde C(t,\overline{m}(\hat{\mathfrak{h}}),m)\|\Psi_{\hat\nu_\ell}(\partial^{\eta_1}\psi_\alpha)\circ F^{\overline{m}(\hat{\mathfrak{h}})}\|_{\mathcal{C}^t_{\hat\nu_\ell}}\|v_{\alpha,\hat{\mathfrak{h}}}\|_t|I_{q_0}|.\end{aligned} \tag{8.1.13}$$

By (6.1.12), $\tilde C(t,\overline{m}(\hat{\mathfrak{h}}),m) \le C_\sharp\Lambda^{c_\sharp M_\xi}$ and

$$\tilde C(2,\overline{m}(\hat{\mathfrak{h}}),m) \le C_\sharp\mu^{2M_\xi}\sup_{s\in\operatorname{supp}\varphi}[\lambda_m^+\circ\hat\nu_\ell(s)\varsigma^2_{\overline{m}(\hat{\mathfrak{h}})}\circ\hat\nu_\ell(s)+\varsigma_{\overline{m}(\mathfrak{h})}\circ\hat\nu_\ell C_{q_0}\Delta_{\tilde\gamma}]. \tag{8.1.14}$$

Note that, by Corollary 6.2.2

$$\|v_{\alpha,\mathfrak{h}}\|_t = \|\mathcal{L}^{M_\xi-\overline{m}_\alpha(\mathfrak{h})+q_0}u\|_t \le C_{\mu,M_\xi}\mu^{M_\xi}\|u\|_t \tag{8.1.15}$$

and, by Lemma 5.5.3, for each $\alpha\in\mathcal{A}$

$$\sum_{\hat{\mathfrak{h}}\in\hat{\mathfrak{H}}_\alpha}\|\Psi_{\hat\nu(\hat{\mathfrak{h}})}\|_{\mathcal{C}^t} \le \sum_{\hat{\mathfrak{h}}\in\mathfrak{H}^{M_\xi}}\|\Psi_{\hat\nu_\ell(\hat{\mathfrak{h}})}\|_{\mathcal{C}^t} \le A_u(t,M_\xi,m), \tag{8.1.16}$$

where

$$A_u(\tau,M_\xi,m) := \begin{cases} C_\sharp\left(\Delta_{\tilde\gamma}+\mathbb{I}_{\gamma,\overline{m}(\hat{\mathfrak{h}}),m}\vartheta^{-1}_{\tilde\gamma}\right)\mu^{\overline{m}}\mathbb{J}_{\gamma,m} & \tau=0\\ (C_{\mu,\overline{m}(\hat{\mathfrak{h}})}\mu)^{2\overline{m}(\hat{\mathfrak{h}})}\left(\Delta_{\tilde\gamma}+\mathbb{I}_{\gamma,\overline{m}(\hat{\mathfrak{h}}),m}\vartheta^{-1}_{\gamma}\right)^2\mu^{\overline{m}}\mathbb{J}_{\gamma,m} & \tau=1\\ O_\star(M_\xi,m)\cdot\{\vartheta_\gamma^{-2},\|\overline{M}_0(m,\cdot)\|_\infty,(\lambda_m^+)^2\}^+ & \tau=2\\ C_\sharp\Lambda^{c_\sharp M_\xi} & \tau>2.\end{cases} \tag{8.1.17}$$

Since $\|(\partial^{\eta_1}\psi_\alpha)\circ F^{\overline{m}(\hat{\mathfrak{h}})}\|_{\mathcal{C}^t_{\hat\nu_\ell}} \le C_\sharp C_{\psi,q_0}\Lambda^{c_\sharp M_\xi}$, this concludes the case $t>2$.

It remains to prove (8.1.7). In this case we assume (8.0.8) and we estimate the terms in (8.1.12) for $t=2$. Arguing as in Remark 8.1.2 we first have $\|\psi_\alpha\|_{\mathcal{C}^r} \le C_\sharp C_{\epsilon,q_0}\chi_u^{-c_\sharp\ln\mu}$.
Next, setting temporarily $\overline{m}=\overline{m}(\hat{\mathfrak{h}})$, $g_\alpha=\partial^{\eta_1}\psi_\alpha$, and $G_\alpha(s)=g_\alpha\circ\tilde\gamma_\ell\circ h_{\ell,\overline{m}}(s)$, and recalling that $F^{\overline{m}}\hat\nu_\ell=\tilde\gamma_\ell\circ h_{\ell,\overline{m}}$,

$$\begin{aligned}G'_\alpha &= \langle\nabla g_\alpha\circ\tilde\gamma_\ell\circ h_{\ell,\overline{m}},\tilde\gamma'_\ell\circ h_{\ell,\overline{m}}h'_{\ell,\overline{m}}\rangle\\ G''_\alpha &= \langle(D\nabla g_\alpha)\tilde\gamma'_\ell\circ h_{\ell,\overline{m}},\tilde\gamma'_\ell\circ h_{\ell,\overline{m}}\rangle(h'_{\ell,\overline{m}})^2\\ &\quad+\langle\nabla g_\alpha\circ\tilde\gamma_\ell\circ h_{\ell,\overline{m}},\tilde\gamma''_\ell\circ h_{\ell,\overline{m}}(h'_{\ell,\overline{m}})^2+\tilde\gamma'_\ell\circ h_{\ell,\overline{m}}h''_{\ell,\overline{m}}\rangle.\end{aligned}$$

Then, by (5.4.46) (with $h_{\overline{m}} = h_{\overline{m},\ell}$) and since (E.0.1), (E.0.2) imply $\Delta_{\tilde{\gamma}} \leq C_{q_0,\epsilon}\chi_u^{-c_\sharp \ln \mu}\mu^m$,

$$\begin{aligned} &\|(\partial^{\eta_1}\psi_\alpha)\circ F^{\overline{m}(\hat{\mathfrak{h}})}\|_{\mathcal{C}^1_{\hat{\nu}_\ell}} \leq C_{q_0,\epsilon}\vartheta_{\tilde{\gamma}}^{-1}\mu^{M_\xi} \\ &\|(\partial^{\eta_1}\psi_\alpha)\circ F^{\overline{m}(\hat{\mathfrak{h}})}\|_{\mathcal{C}^2_{\hat{\nu}_\ell}} \leq C_{\epsilon,q_0}\|\overline{M}_{q_0}(m,\cdot)\|_\infty + C_{\epsilon,q_0}\mu^{2M_\xi}\chi_u^{-c_\sharp \ln \mu}\vartheta_{\tilde{\gamma}}^{-2}. \end{aligned} \tag{8.1.18}$$

Since $(\Psi_{\hat{\nu}_\ell}G_\alpha)'' = \Psi''_{\hat{\nu}_\ell}G_\alpha + 2\Psi'_{\hat{\nu}_\ell}G'_\alpha + \Psi_{\hat{\nu}_\ell}G''_\alpha$, by Lemma 5.5.3, (8.1.16) and (8.1.18)

$$\begin{aligned} &\sum_{\hat{\mathfrak{h}}\in\hat{\mathfrak{H}}_\alpha} \|\Psi_{\hat{\nu}_\ell}(\partial^{\eta_1}\psi_\alpha)\circ F^{\overline{m}(\hat{\mathfrak{h}})}\|_{\mathcal{C}^2_{\hat{\nu}_\ell}} \\ &\leq C_\sharp C_{\epsilon,q_0}\chi_u^{-c_\sharp \ln\mu} A_u(2,M_\xi,m) + A_u(1,M_\xi,m)C_{q_0,\epsilon}\vartheta_{\tilde{\gamma}}^{-1}\mu^{M_\xi} \\ &+ A_u(0,M_\xi,m)C_{\epsilon,q_0}\left[\|\overline{M}_{q_0}(m,\cdot)\|_\infty + \mu^{2M_\xi}\chi_u^{-c_\sharp\ln\mu}\vartheta_{\tilde{\gamma}}^{-2}\right]. \end{aligned} \tag{8.1.19}$$

To conclude, we need to relate all the quantities to $\vartheta(\xi^*)$. First we notice that, by (5.4.41) and Lemma E.0.1, $\vartheta_{\tilde{\gamma}}^{-1} \leq \vartheta_\gamma^{-1} = [\{\rho(\xi^*),\chi_u\}^+]^{-1} =: \vartheta(\xi^*)^{-1}$. Therefore, recalling (5.4.57), it follows that $\lambda_m^+\circ\hat{\nu}_\ell(s) \leq C_\sharp\vartheta(\xi^*)^{-1}\mu^m$, for each $s\in\mathbb{T}^2$. Next, choosing $n_\star = \min\{\bar{c}_2\log\chi_u^{-1}, m\}$ in Lemma 5.4.2, we can check that

$$a^{\overline{m}}_{n_\star} + c^{\overline{m}}_{n_\star} \leq C_\sharp\ ; \quad b^{\overline{m}}_{n_\star} \leq C_{\mu,n_\star} = C_{\mu,M_\xi}\ ; \quad \mathbb{s}_{n_\star} = C_{\mu,n_\star}\mu^{6n_\star}$$

since, by (5.3.2), $\varsigma_{n_\star} \leq [C_{\mu,n_\star} + C_\sharp\chi_u\vartheta(\xi^*)^{-1}] \leq C_{\mu,n_\star}$. Similarly $\varsigma_m \leq C_{\mu,m}$. We can use this to compute, in (5.4.45),[7]

$$\begin{aligned} \|M_{n_0}(m,\cdot)\|_\infty &\leq \{C_{q_0,\epsilon}\chi_u^{-c_\sharp\ln\mu}\mu^{2m}, (1+C_\sharp\mu^{2m})\vartheta(\xi^*)^{-1}\mu^m\}^+ \\ &\leq \{C_{q_0,\epsilon}\vartheta(\xi^*)^{-c_\sharp\ln\mu}\mu^{2m}, (1+C_\sharp\mu^{2m})\vartheta(\xi^*)^{-1}\mu^m\}^+ \\ &\leq C_{q_0,\epsilon}\mu^{3m}\vartheta(\xi^*)^{-\{1,c_\sharp\ln\mu\}^+}. \end{aligned}$$

Consequently, we can also compute

$$\begin{aligned} \|\overline{M}_{n_0}(m,\cdot)\|_\infty &\leq \{\mu^{6m}C_{q_0,\epsilon}\chi_u^{-c_\sharp\ln\mu}, C_{q_0,\epsilon}\mu^{3m}\vartheta(\xi^*)^{-\{1,c_\sharp\ln\mu\}^+}, \vartheta(\xi^*)^{-2}, C_\sharp\vartheta(\xi^*)^{-2}\mu^m\}^+ \\ &\leq C_{q_0,\epsilon}\mu^{3m}\vartheta(\xi^*)^{-1-\{1,c_\sharp\ln\mu\}^+}. \end{aligned}$$

Finally, by the above estimates and condition (5.4.44),[8]

$$\begin{aligned} \lambda^+_{\overline{m}}\circ\hat{\nu}_\ell(s) \leq C_\sharp\frac{c_\flat a^{\overline{m}}_{n_\star}\mu^{3\overline{m}}\overline{M}_{n_0}(m,t)}{\mathbb{s}_{n_\star}} &\leq C_\sharp\frac{\mu^{3(M_\xi+m)}}{C^2_{\mu,n_\star}\mu^{6n_\star}}\vartheta(\xi^*)^{-1-\{1,c_\sharp\ln\mu\}^+} \\ &\leq C_\sharp\mu^{3M_\xi}\vartheta(\xi^*)^{-1-\{1,c_\sharp\ln\mu\}^+}, \end{aligned} \tag{8.1.20}$$

so that $\varsigma_{M_\xi} \leq C_\sharp\mu^{3M_\xi}\vartheta(\xi)^{-1-\{1,c_\sharp\ln\mu\}^+}$, and we immediately have by (6.1.12)

$$\tilde{C}(2,\overline{m}(\mathfrak{h}),m) \leq C_\sharp\mu^{9M_\xi}\vartheta(\xi^*)^{-3-\{1,c_\sharp\ln\mu\}}. \tag{8.1.21}$$

[7]Recall that $q_0 \geq n_0$.

[8]Note that, since $\overline{m}$ is the smaller integer such that the second of (5.4.44) holds, there exists $C_\sharp$ such that $\bar{\eta}_{n_\star}(\overline{m}+1,m;t)\overline{M}_{n_0}(m,t) \geq C_\sharp s_{n_\star}$

We can now conclude. Using the above estimates it follows that there are $a, b > 0$ such that

$$\begin{aligned}
&\mathbb{J}_{\gamma,m} \le C_{\sharp},\\
&\mathbb{I}_{\tilde{\gamma},\overline{m},m} \le C_{q_0,\epsilon}\{C_{\mu,m}, \mu^{M_\xi}\chi_u^{-c_\sharp \ln\mu}\}^+\\
&O_\star(M_\xi, m) \le C_\sharp C_{q_0,\epsilon} C^a_{\mu,M_\xi}\mu^{bM_\xi},
\end{aligned}$$

which imply

$$\begin{aligned}
&A_u(0, M_\xi, m) \le C_{q_0,\epsilon}\mu^{2M_\xi}\chi_u^{-c_\sharp \ln\mu}\vartheta(\xi^*)^{-1},\\
&A_u(1, M_\xi, m) \le C_{q_0,\epsilon}C_{\mu,M_\xi}\mu^{7M_\xi}\chi_u^{-c_\sharp \ln\mu}\vartheta(\xi^*)^{-2}\\
&A_u(2, M_\xi, m) \le C_{q_0,\epsilon}C_\sharp C^a_{\mu,M_\xi}\mu^{bM_\xi}\vartheta(\xi^*)^{-1-\{1,c_\sharp \ln\mu\}^+}.
\end{aligned} \tag{8.1.22}$$

Using this in (8.1.19) we find $\beta_1, \beta_2 > 0$ such that

$$\sum_{\hat{\mathfrak{h}}\in\hat{\mathfrak{H}}_\alpha} \|\Psi_{\hat{\nu}_\ell}(\partial^{\beta_1}\psi_\alpha)\circ F^{\overline{m}(\hat{\mathfrak{h}})}\|_{\mathcal{C}^2_{\hat{\nu}_\ell}} \le C_{q_0,\epsilon}C^{\beta_1}_{\mu,M_\xi}\mu^{\beta_2 M_\xi}\vartheta(\xi^*)^{-3}\chi_u^{-c_\sharp \ln\mu}. \tag{8.1.23}$$

Hence, by (8.1.13), (8.1.21), and (8.1.23), we have

$$\left|\mathcal{F}\partial^2_{x_j}\mathcal{L}^{q_0}(u^{M_\xi}_{\alpha,\mathfrak{h}})\right| \le C_{q_0,\epsilon}C^{\beta_1}_{\mu,M_\xi}\mu^{\beta_2 M_\xi}\vartheta(\xi^*)^{-6}\chi_u^{-c_\sharp \ln\mu},$$

which concludes the proof of (8.1.7), recalling equation (8.1.8).
We still ow the reader the proof that $\sup_{\xi^*\notin\mathbf{C}_u} M_\xi < \infty$. We notice that, by equations (8.1.20), (8.1.10) and (4.1.5), for each $\xi^* \notin \mathbf{C}_u$,

$$\lambda_-^{M_\xi} \le C_\star C_\sharp \mu^{3M_\xi}\vartheta(\xi^*)^{-1-\{1,c_\sharp \ln\mu\}^+},$$

which yields $M_\xi \le \log(\lambda_-\mu^{-3})^{-1}\log(C_\star C_\sharp\vartheta(\xi^*)^{-1-\{1,c_\sharp \ln\mu\}^+})$, which is finite for each $\xi^* \notin \mathbf{C}_u$, by the definition of $\vartheta(\xi^*)$ in (8.0.3).

□

We henceforth consider $\sigma > 1$ as in (5.4.43) and m_{χ_u} as in (5.2.4), and we define

$$\overline{m}_{\chi_u} = \sigma m_{\chi_u}. \tag{8.1.24}$$

8.2 Decomposition in Fourier space

Let $\mathcal{Z}_u = \{\xi : \xi^* \in \mathbf{C}_u\}$ and $\mathcal{Z}_u^c = \mathbb{Z}^2\backslash\mathcal{Z}_u$. Recalling that $\rho(\xi^*) = |\xi_2^*||\xi_1^*|^{-1}$, $\rho(e_2) = \infty$,

$$\mathcal{Z}_u = \{\xi : \rho(\xi^*) \le \chi_u\} \quad ; \quad \mathcal{Z}_u^c = \{\xi : \rho(\xi^*) > \chi_u\}.$$

Next, take $N = q_0 + M$, for some $M \in \mathbb{N}$ to be chosen shortly. For simplicity, it is convenient to introduce the following notation for $A \subset \mathbb{Z}^2, \mathfrak{h}, \mathfrak{h}' \in \mathfrak{H}^{q_0}$:

$$S^\alpha_{q_0,M}(A, \mathfrak{h}, \mathfrak{h}') = \sum_{\xi\in\mathbb{Z}^2} \mathbb{1}_A(\xi)\langle\xi\rangle^{2s}[\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}})](\xi)\overline{[\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})]}(\xi), \tag{8.2.1}$$

where $u^M_{\alpha,\mathfrak{h}} = \psi_{\alpha,\mathfrak{h}}\mathcal{L}^M u$. Then, by equation (8.1.4) we have

$$
\begin{aligned}
\|\mathcal{L}^N u\|^2_{\mathcal{H}^s} &\le C_\sharp \sum_\alpha \Big\| \sum_{\mathfrak{h}\in\mathfrak{H}^{q_0}} \mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}})\Big\|^2_{\mathcal{H}^s} \\
&= C_\sharp \sum_\alpha \sum_{(\mathfrak{h},\mathfrak{h}')\in\mathfrak{H}^{q_0}\times\mathfrak{H}^{q_0}} \langle \mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}}), \mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})\rangle_s \\
&= C_\sharp \sum_\alpha \sum_{(\mathfrak{h},\mathfrak{h}')\in\mathfrak{H}^{q_0}\times\mathfrak{H}^{q_0}} \sum_{\xi\in\mathbb{Z}^2} \langle\xi\rangle^{2s}[\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}})](\xi)\overline{[\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})]}(\xi) \\
&= C_\sharp \sum_\alpha \sum_{(\mathfrak{h},\mathfrak{h}')\in\mathfrak{H}^{q_0}\times\mathfrak{H}^{q_0}} S^\alpha_{q_0,M}(\mathcal{Z}_u,\mathfrak{h},\mathfrak{h}') + C_\sharp \sum_\alpha \sum_{(\mathfrak{h},\mathfrak{h}')\in\mathfrak{H}^{q_0}\times\mathfrak{H}^{q_0}} S^\alpha_{q_0,M}(\mathcal{Z}^c_u,\mathfrak{h},\mathfrak{h}').
\end{aligned}
\tag{8.2.2}
$$

We start estimating the second term in the above equation, in the next section we will treat the term with $\xi \in \mathcal{Z}_u$.

Lemma 8.2.1 (Bound on $\mathcal{Z}^c_u$). *Let $\overline{M} := \sup_{\xi\in\mathcal{Z}^c_u} M_\xi$.*[9] *For each $M \ge \overline{M}$, $1 \le s \le r-1$, $\mathfrak{h} \in \mathfrak{H}^{q_0}$ and $N = q_0 + M$,*

$$
\sum_{\xi\in\mathbb{Z}^2} |\langle\xi\rangle^s \mathbb{1}_{\mathcal{Z}^c_u}(\xi)[\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}})](\xi)|^2 \lesssim \Theta_s\|u\|^2_{s+2},
\tag{8.2.3}
$$

where $\Theta_s = (C_{\psi,q_0}C_{\mu,M}\mu^M\Lambda^{c_\sharp M})^2$ and, under condition (8.0.8),

$$
\Theta_1 := (C_{\epsilon,q_0}C^{\beta_1}_{\mu,M}\mu^{\beta_2 M})^2\chi_u^{-11-c_\sharp\log\mu}M.
\tag{8.2.4}
$$

Proof. Since $\xi^* \notin \mathbf{C}_u$, by the definition of m_{χ_u} in (5.2.4), for each $p \in \mathbb{T}^2, \mathfrak{h} \in \mathfrak{H}^\infty$

$$
\begin{aligned}
&D_p\mathfrak{h}_{m_{\chi_u}}\xi^* \in \mathbf{C}_{c,\epsilon} \\
&D_p\mathfrak{h}_{n_0}\xi^* \notin \mathbf{C}_u,
\end{aligned}
\tag{8.2.5}
$$

so the hypothesis of Proposition 8.1.3 are satisfied with $m = m_{\chi_u}$. We will treat the case $s > 1$ first. For each $M \ge \overline{M}$, we have

$$
\begin{aligned}
&\sum_{\xi\in\mathbb{Z}^2} |\langle\xi\rangle^s \mathbb{1}_{\mathcal{Z}^c_u}\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})|^2 = \sum_{\xi\in\mathbb{Z}^2}\langle\xi\rangle^{-4}|\langle\xi\rangle^{s+2}\mathbb{1}_{\mathcal{Z}^c_u}\mathcal{F}\mathcal{L}^{q_0}(\psi_{\alpha,\mathfrak{h}'}\mathcal{L}^{M_\xi}(\mathcal{L}^{M-M_\xi}u))|^2 \\
&\lesssim (C_{\psi,q_0}\Lambda^{c_\sharp M})^2\|\mathcal{L}^{M-\overline{M}}u\|^2_{s+2}.
\end{aligned}
\tag{8.2.6}
$$

where we used the fact that $\Lambda > 2$ and the convergence of the series. The statement (8.2.3) for $s > 1$ then follows since, by Corollary 6.2.2,

$$
\|\mathcal{L}^{M-\overline{M}}u\|^2_t \le C^2_{\mu,M}\mu^{2M}\|u\|^2_t, \quad \forall t \ge 1.
$$

[9] Recall that thisi is finite by Proposition 8.1.3.

Let us move on the $s=1$ case. For any $R>0$ let $B_R=\{\xi\in\mathbb{Z}^2:\|\xi\|\le R\}$ and $B_R^c=\mathbb{Z}^2\setminus B_R$. Then

$$\begin{aligned}&\sum_{\xi\in\mathbb{Z}^2}|\langle\xi\rangle^s\mathbb{1}_{\mathcal{Z}_u^c}\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})|^2\\&=\sum_{\xi\in\mathcal{Z}_u^c\cap B_R}\langle\xi\rangle^{-2}|\langle\xi\rangle^{s+1}\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})|^2+\sum_{\xi\in\mathcal{Z}_u^c\cap B_R^c}\langle\xi\rangle^{-3}|\langle\xi\rangle^{s+1}\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})\langle\xi\rangle^{s+2}\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})|.\end{aligned}$$

For each $\xi\in\mathcal{Z}_u^c$ we can apply Proposition 8.1.3 to have

$$\sum_{\xi\in\mathcal{Z}_u^c\cap B_R}|\langle\xi\rangle\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})|^2\le C^2_{\mu,M}\mu^{2M}\|u\|_2^2\sum_{\xi\in\mathcal{Z}_u^c\cap B_R}\langle\xi\rangle^{-2}K_1(2,M_\xi,m_{\chi_u})^2 \quad (8.2.7)$$

and

$$\begin{aligned}&\sum_{\xi\in\mathcal{Z}_u^c\cap B_R^c}|\langle\xi\rangle\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})|^2\\&\le C^2_{\mu,M}\mu^{2M}\|u\|_2\|u\|_3\sum_{\xi\in\mathcal{Z}_u^c\cap B_R^c}\langle\xi\rangle^{-3}K_1(2,M_\xi,m_{\chi_u})K_1(3,M_\xi,m_{\chi_u}).\end{aligned} \quad (8.2.8)$$

We use the estimate of $K_1(2,M_\xi,m_{\chi_u})$ in (8.1.7) for the sum in (8.2.7), with $\vartheta(\xi^*)=\rho(\xi^*)$, since $\xi\in\mathcal{Z}_u^c$, and we have

$$\begin{aligned}&\sum_{\xi\in\mathcal{Z}_u^c\cap B_R}\langle\xi\rangle^{-2}K_1(M_\xi,2)^2\lesssim(C_{\epsilon,q_0}\chi_u^{-c_\sharp\ln\mu}C^{\beta_1}_{\mu,M}\mu^{\beta_2M})^2\sum_{\xi\in\mathcal{Z}_u^c\cap B_R}\langle\xi\rangle^{-2}\rho(\xi^*)^{-12}\\&\lesssim(C_{\epsilon,q_0}C^{\beta_1}_{\mu,M}\mu^{\beta_2M})^2\chi_u^{-11-c_\sharp\ln\mu}\log R,\end{aligned} \quad (8.2.9)$$

since

$$\sum_{\xi\in\mathcal{Z}_u^c\cap B_R}\langle\xi\rangle^{-2}\rho(\xi^*)^{-12}\lesssim\int_0^R\int_{\{\tan\theta>\chi_u\}}\frac{1}{1+\rho^2}\frac{1}{(\tan\theta)^{12}}\rho d\rho d\theta\lesssim\chi_u^{-11}\log R. \quad (8.2.10)$$

Similarly, for the sum in (8.2.8), we have

$$\begin{aligned}&\sum_{\xi\in\mathcal{Z}_u^c\cap B_R^c}\langle\xi\rangle^{-3}K_1(2,M_\xi,m_{\chi_u})K_1(3,M_\xi,m_{\chi_u})\\&\lesssim C_{\epsilon,q_0}\chi_u^{-c_\sharp\ln\mu}C^{\beta_1}_{\mu,M}\mu^{\beta_2M}\sum_{\xi\in\mathcal{Z}_u^c\cap B_R^c}\langle\xi\rangle^{-3}\rho(\xi^*)^{-6}\Lambda^{c_\sharp M_\xi}\\&\lesssim C_{\epsilon,q_0}C^{\beta_1}_{\mu,M}\mu^{\beta_2M}\chi_u^{-5-c_\sharp\ln\mu}R^{-1}\Lambda^{c_\sharp M}.\end{aligned} \quad (8.2.11)$$

Choosing $R=\Lambda^{c_\sharp M}$ by (8.2.7) and (8.2.8) we have the following estimate:

$$\sum_{\xi\in\mathbb{Z}^2}|\langle\xi\rangle\mathbb{1}_{\mathcal{Z}_u^c}\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})|^2\lesssim(C_{\epsilon,q_0}C^{\beta_1}_{\mu,M}\mu^{\beta_2M})^2\chi_u^{-11-c_\sharp\ln\mu}M\|u\|_3^2, \quad (8.2.12)$$

from which we conclude the proof of (8.2.3) also for the case $s=1$. □

8.3 The case $\xi^* \in \mathbf{C}_u$

In this case we cannot apply Proposition 8.1.3 directly as we did in the previous section. The main reason is that there could be "bad" vectors ξ^* which are in an unstable direction, so (8.2.5) may fail. Here transversality plays a major role.

Lemma 8.3.1 (Bound on $\mathcal{Z}_u$)**.** *If there exists $q_0 \in \mathbb{N}$ such that for each $\xi \in \mathbb{Z}$ the hypothesis of Proposition 8.1.3 are satisfieded, then there exist C_{q_0} such that, for each $M \geq \overline{m}_{\chi_u}$,*

$$\sum_{\alpha} \sum_{(\mathfrak{h},\mathfrak{h}')\in\mathfrak{H}^{q_0}\times\mathfrak{H}^{q_0}} \sum_{\xi\in\mathbb{Z}^2} \mathbb{1}_{\mathcal{Z}_u}(\xi)\langle\xi\rangle^{2s}[\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}})](\xi)\overline{[\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})]}(\xi)$$
$$\leq \mathcal{N}(q_0)\mu^{2sq_0}\sum_{\mathfrak{h}\in\mathfrak{H}^{q_0}}\|u^M_{\alpha,\mathfrak{h}}\|^2_{\mathcal{H}^s} + C_{q_0}\sum_{\mathfrak{h}\in\mathfrak{H}^{q_0}}\|u^M_{\alpha,\mathfrak{h}}\|^2_{\mathcal{H}^{s-1}} + C_{q_0}Q(M,s)\sqrt{\Theta_s}\|u\|_{s+2}\|u\|_{\mathcal{H}^s},$$

where $Q(M,s)$ is given in (7.1.2) and Θ_s in Lemma 8.2.1.

The rest of this Section is devoted to the proof of the above Lemma. We divide the argument in three Steps.

Step I (Local transversality)

We need a definition of transversality uniform on the elements the partition of unity (8.1.3):

Definition 8.3.2. *Given $n \in \mathbb{N}$ and $\mathfrak{h},\mathfrak{h}' \in \mathfrak{H}^n$ we say that $\mathfrak{h} \pitchfork^n_\alpha \mathfrak{h}'$ ($\mathfrak{h}$ is transversal to $\mathfrak{h}'$ on α at time n) if for every $z \in \mathfrak{h}(\mathcal{U}_\alpha)$ and $w \in \mathfrak{h}'(\mathcal{U}_\alpha)$ such that $F^n(z) = F^n(w) \in \mathcal{U}_\alpha$:*

$$D_zF^n\mathbf{C}_{\epsilon,u} \cap D_wF^n\mathbf{C}_{\epsilon,u} = \{0\}. \tag{8.3.1}$$

Next, we relate the (pointwise) Definition 4.2.1 to the (local) Definition 8.3.2.

Lemma 8.3.3. *The constant C_0 in* (8.1.2) *can be chosen such that: for all $\alpha \in \mathcal{A}$, $p \in \mathcal{U}_\alpha \subset \mathbb{T}^2$ and $\mathfrak{h},\mathfrak{h}' \in \mathfrak{H}^{q_0}$, if $z_1 = \mathfrak{h}(p)$ and $z_2 = \mathfrak{h}'(p)$, then $z_1 \pitchfork z_2$ implies $\mathfrak{h} \pitchfork^{q_0}_\alpha \mathfrak{h}'$.*

Proof. Recall that $z_1 \pitchfork z_2$ means

$$D_{z_1}F^{q_0}\mathbf{C}_u \cap D_{z_2}F^{q_0}\mathbf{C}_u = \{0\}. \tag{8.3.2}$$

As $\mathbf{C}_{u,\epsilon} \Subset \mathbf{C}_u$, clearly $D_{z_1}F^{q_0}\mathbf{C}_{u,\epsilon} \Subset D_{z_1}F^{q_0}\mathbf{C}_u$. So the above implies also

$$D_{z_1}F^{q_0}\mathbf{C}_{u,\epsilon} \cap D_{z_2}F^{q_0}\mathbf{C}_{u,\epsilon} = \{0\}.$$

Let $\tilde{p} \in \mathcal{U}_\alpha$, $\tilde{p} \neq p$, and define $\tilde{z}_1 = \mathfrak{h}(\tilde{p})$ and $\tilde{z}_1 = \mathfrak{h}'(\tilde{p})$. We claim that, for each $v \in \mathbf{C}_{u,\epsilon}$, the difference between $D_{z_1}F^{q_0}v$ and $D_{\tilde{z}_1}F^{q_0}v$ is smaller than the opening of $D_{z_1}F^{q_0}\mathbf{C}_u$, provided we choose $\mathcal{U}_\alpha$ small enough. This suffices to conclude the argument.

We compute a lower bound for the opening of the connected components of $D_{z_1}F^{q_0}\mathbf{C}_u \setminus D_{z_1}F^{q_0}\mathbf{C}_{u,\epsilon}$. By Proposition 5.3.1, and by formula (5.3.1), we deduce that, for each unitary vectors $v \in \mathbf{C}_{u,\epsilon}$ and $w \notin \mathbf{C}_u \cup \mathbf{C}_c$,

$$\measuredangle(D_{z_1}F^{q_0}v, D_{z_1}F^{q_0}w) = \frac{|\det D_{z_1}F^{q_0}|\measuredangle(v,w)}{\|D_{z_1}F^{q_0}v\|\|D_{z_1}F^{q_0}w\|} \geq \frac{C_*\chi_u\epsilon}{\mu_{q_0}^-(z)\lambda_{q_0}^+(z)} = C_*\chi_u\epsilon\delta_{q_0}(z_1).$$

On the other hand let us recall that $u_{\mathfrak{h},q_0}(p)$ defined in (5.2.8) gives the slope of the boundary of the cone $D_{\mathfrak{h}_\alpha(p)}F^{q_0}\mathbf{C}_u$, and it is a Lipschitz function of p. In particular Lemma E.0.1 provides an estimate for the Lipschitz constant $L_\star(q_0)$ given in (E.0.2). Then, by the definition of $\mathcal{U}_{z,q_0}$ in (8.1.1) and (8.1.2), we have the claim, since

$$\begin{aligned}\|D_{z_1}F^{q_0}v - D_{\tilde{z}_1}F^{q_0}v\| &\leq L_\star(q_0)\|z_1 - \tilde{z}_1\| \leq L_\star(q_0)L_\star(\chi_u, q_0)^{-1}C_0\chi_u\epsilon\delta_{q_0}(z_1)\\ &\leq C_\sharp C_0\chi_u\epsilon\delta_{q_0}(z_1).\end{aligned}$$

Clearly the same is true replacing $z_1, \tilde{z}_1, \mathfrak{h}$ with $z_2, \tilde{z}_2, \mathfrak{h}'$, and the result follows. □

Lemma 8.3.4. *Let m_{χ_u} given in (5.2.4). For every $p \in \mathcal{U}_\alpha$, $M > m_{\chi_u}$ and $\tilde{z}, \tilde{w} \in F^{-q_0}(p)$ such that $\tilde{z} \pitchfork \tilde{w}$ we have*

$$\mathbb{R}^2 = \left((D_zF^{M+q_0})^*\right)^{-1}\mathbf{C}_{\epsilon,c}^\perp \cup \left((D_wF^{M+q_0})^\perp\right)^{-1}\mathbf{C}_{\epsilon,c}^\perp, \tag{8.3.3}$$

for every $z \in \mathfrak{h}(\tilde{z})$ and $w \in \mathfrak{h}'(\tilde{w})$, $\mathfrak{h}, \mathfrak{h}' \in \mathfrak{H}^M$.

Proof. By assumption $D_{\tilde{z}}F^{q_0}\mathbf{C}_u \cap D_{\tilde{w}}F^{q_0}\mathbf{C}_u = \{0\}$ which, together with condition (5.2.4) implies that for every $z_1 \in F^{-M}(\tilde{z})$ and $z_2 \in F^{-M}(\tilde{w})$

$$D_{\tilde{z}}F^{q_0}(D_{z_1}F^M(\mathbb{R}^2 \setminus \mathbf{C}_c)) \cap D_{\tilde{w}}F^{q_0}(D_{z_2}F^M(\mathbb{R}^2 \setminus \mathbf{C}_c)) = \{0\}. \tag{8.3.4}$$

Therefore, setting $N = q_0 + M$, there are $z, w \in F^{-N}(p)$ such that

$$D_zF^N(\mathbb{R}^2 \setminus \mathbf{C}_c) \cap D_wF^N(\mathbb{R}^2 \setminus \mathbf{C}_c) = \{0\}. \tag{8.3.5}$$

Now we can conclude the argument showing that the above implies the statement. Indeed, equation (8.3.5) obviously implies

$$\left(D_zF^N(\mathbb{R}^2 \setminus \mathbf{C}_c)\right)^\perp \cap \left(D_wF^N(\mathbb{R}^2 \setminus \mathbf{C}_c)\right)^\perp = \{0\}.$$

For any cone $\mathcal{K} \subset \mathbb{R}^2$ and any $z \in \mathbb{T}^2$, one has $(D_zF^N\mathcal{K})^* = \left((D_zF^N)^*\right)^{-1}\mathcal{K}^\perp$ and $(\mathbb{R}^2 \setminus \mathcal{K})^\perp = \mathbb{R}^2 \setminus \mathcal{K}^\perp$. We then have

$$\left((D_zF^N)^*\right)^{-1}(\mathbb{R}^2 \setminus \mathbf{C}_c^\perp) \cap \left((D_wF^N)^*\right)^{-1}(\mathbb{R}^2 \setminus \mathbf{C}_c^\perp) = \{0\},$$

which in turn implies that $\mathbb{R}^2 = \left((D_zF^N)^*\right)^{-1}\mathbf{C}_c^\perp \cup \left((D_wF^N)^*\right)^{-1}\mathbf{C}_c^\perp$. The conclusion then follows using Lemma 8.3.3 and obtaining the statement for the smaller cones $\mathbf{C}_{\epsilon,c}$. □

Using Definition 8.3.2, and recalling notation (8.2.1), we have the following decomposition into transversal and non transversal terms:

$$\sum_{(\mathfrak{h},\mathfrak{h}')\in\mathfrak{H}^{q_0}\times\mathfrak{H}^{q_0}} S^{\alpha}_{q_0,M}(\mathcal{Z}_u,\mathfrak{h},\mathfrak{h}') = \sum_{\mathfrak{h}\pitchfork^{q_0}_{\alpha}\mathfrak{h}'} S^{\alpha}_{q_0,M}(\mathcal{Z}_u,\mathfrak{h},\mathfrak{h}') + \sum_{\mathfrak{h}\not\pitchfork^{q_0}_{\alpha}\mathfrak{h}'} S^{\alpha}_{q_0,M}(\mathcal{Z}_u,\mathfrak{h},\mathfrak{h}'). \tag{8.3.6}$$

Step II (Estimate of transversal terms)

In this step we will prove that

$$\sum_{\mathfrak{h}\pitchfork^{q_0}_{\alpha}\mathfrak{h}'} S^{\alpha}_{q_0,M}(\mathcal{Z}_u,\mathfrak{h},\mathfrak{h}') \leq C_\sharp C_{q_0} Q(M,s)\sqrt{\Theta_s}\|u\|_{s+2}\|u\|_{\mathcal{H}^s}, \tag{8.3.7}$$

where Θ_s is given in Lemma 8.2.1.
If $M > m_{\chi_u}$, for $\mathfrak{h} \pitchfork^{q_0}_{\alpha} \mathfrak{h}'$, Lemma 8.3.3 and Lemma 8.3.4 imply that, for any $\xi \in \mathbb{Z}^2$, either $(D_zF^N)^*\xi \in (\mathbf{C}_{\epsilon,c})^{\perp}$ for every $z \in \operatorname{supp}(\psi_{\alpha,\mathfrak{h}})$, or $(D_zF^N)^*\xi \in (\mathbf{C}_{\epsilon,c})^{\perp}$ for every $z \in \operatorname{supp}(\psi_{\alpha,\mathfrak{h}'})$. We then decompose $\mathcal{Z}_u = Z_1 \cup Z_2$, where

$$Z_1 = \{\xi \in \mathcal{Z}_u : (D_zF^N)^*\xi \in \mathbf{C}^*_{\epsilon,c} \quad \forall z \in \operatorname{supp}\psi_{\alpha,\mathfrak{h}}\}, \qquad Z_2 = \mathcal{Z}_u \setminus Z_1,$$

and we write

$$S^{\alpha}_{q_0,M}(\mathcal{Z}_u,\mathfrak{h},\mathfrak{h}') = S^{\alpha}_{q_0,M}(\mathcal{Z}_u\cap Z_1,\mathfrak{h},\mathfrak{h}') + S^{\alpha}_{q_0,M}(\mathcal{Z}_u\cap Z_2,\mathfrak{h},\mathfrak{h}'). \tag{8.3.8}$$

It is enough to estimate the first addend, the second being analogous. Notice that for each $\xi \in Z_i, i \in \{1,2\}$ we can apply Proposition (8.1.3) with $m = m_{\chi_u}$. By the Cauchy-Schwartz inequality we have

$$|S^{\alpha}_{q_0,M}(\mathcal{Z}_u\cap Z_1,\mathfrak{h},\mathfrak{h}')| \lesssim \left(\sum_{\xi\in\mathbb{Z}^2} |\langle\xi\rangle^s \mathbb{1}_{\mathcal{Z}_u\cap Z_1}\mathcal{F}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}})|^2\right)^{\frac{1}{2}} \|\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})\|_{\mathcal{H}^s}. \tag{8.3.9}$$

Moreover, by (7.1.2), $\|\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'})\|_{\mathcal{H}^s} \leq CQ(M,s)\|u\|_{\mathcal{H}^s}$. On the other hand, we can bound the sum inside the square root using the same argument of the proof of Lemma 8.2.1, since the key condition (8.2.5) is now replaced by $\xi \in Z_1$, with the difference that this time $\vartheta(\xi^*) = \chi_u$, since $\xi \in \mathcal{Z}_u$, so we use the estimate

$$\sum_{\xi\in\mathcal{Z}_u\cap Z_1\cap B_R} \langle\xi\rangle^{-2} \leq C_\sharp \chi_u \log R$$

instead of (8.2.10). We thus have

$$|S^{\alpha}_{q_0,M}(\mathcal{Z}_u\cap Z_1,\mathfrak{h},\mathfrak{h}')| \lesssim Q(M,s)\sqrt{\Theta_s}\|u\|_{s+2}\|u\|_{\mathcal{H}^s}.$$

Of course the same computation is valid for the second term of (8.3.8) from which, summing over $\mathfrak{h}\not\pitchfork^{q_0}_{\alpha}\mathfrak{h}'$, we conclude the proof of (8.3.7).

Step III (Estimate of non-transversal terms).

We now want to estimate the sum in (8.3.6) for $\mathfrak{h} \not\pitchfork_{\alpha}^{q_0} \mathfrak{h}'$. We are going to prove that, for $N = q_0 + M$,

$$\sum_{\mathfrak{h} \not\pitchfork_{\alpha}^{q_0} \mathfrak{h}'} \langle \mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}}), \mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'}) \rangle_s \lesssim \mathcal{N}(q_0)\mu^{2sq_0} \sum_{\mathfrak{h}\in\mathfrak{H}^{q_0}} \|u^M_{\alpha,\mathfrak{h}}\|^2_{\mathcal{H}^s} + C_{q_0} \sum_{\mathfrak{h}\in\mathfrak{H}^{q_0}} \|u^M_{\alpha,\mathfrak{h}}\|^2_{\mathcal{H}^{s-1}}. \tag{8.3.10}$$

Keeping the same notation used previously, we write

$$\sum_{\mathfrak{h} \not\pitchfork_{\alpha}^{q_0} \mathfrak{h}'} \langle \mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}}), \mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'}) \rangle_s = \sum_{\mathfrak{h}\in\mathfrak{H}^{q_0}} \sum_{\mathfrak{h}' : \mathfrak{h}' \not\pitchfork_{\alpha}^{q_0} \mathfrak{h}} \langle \mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}}), \mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'}) \rangle_s. \tag{8.3.11}$$

By equation (C.0.4) and the definition of the inner product (C.0.3), there are $C_{\gamma,\beta}$ such that

$$\langle \mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}}), \mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'}) \rangle_s = \sum_{\gamma+\beta=s} C_{\gamma,\beta} \langle \partial^\gamma_{x_1}\partial^\beta_{x_2}(\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}}), \partial^\gamma_{x_1}\partial^\beta_{x_2}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'}) \rangle_{L^2}. \tag{8.3.12}$$

We then use equation (7.1.5) and we have, for every γ, β such that $\gamma + \beta = s$

$$|\partial^\gamma_{x_1}\partial^\beta_{x_2}(\mathcal{L}^{q_0}u^M_{\alpha,\mathfrak{h}})| \le \|(DF^{q_0})^{-1}\|^s_\infty \mathcal{L}^{q_0}(|\partial^\gamma_{x_1}\partial^\beta_{x_2}u^M_{\alpha,\mathfrak{h}}|) + \mathcal{L}^{q_0}(P^{q_0}_{s-1}u^M_{\alpha,\mathfrak{h}})$$

where $P^{q_0}_{s-1}$ is a differential operator of order $s-1$. By (5.3.5) $\|(DF^{q_0})^{-1}\|^s_\infty \le C\mu^{sq_0}$. Clearly the same inequality holds for $\mathfrak{h}'$ and we use this in (8.3.12) to obtain

$$\begin{aligned} &\sum_{\gamma+\beta=s} C_{\gamma,\beta} \langle \partial^\gamma_{x_1}\partial^\beta_{x_2}(\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}}), \partial^\gamma_{x_1}\partial^\beta_{x_2}\mathcal{L}^{q_0}(u^M_{\alpha,\mathfrak{h}'}) \rangle_{L^2} \\ &\lesssim \mu^{2sq_0} \sum_{\gamma+\beta=s} C_{\gamma,\beta} \langle \mathcal{L}^{q_0}(|\partial^\gamma_{x_1}\partial^\beta_{x_2}u^M_{\alpha,\mathfrak{h}}|), \mathcal{L}^{q_0}(|\partial^\gamma_{x_1}\partial^\beta_{x_2}u_{\alpha,\mathfrak{h}'}|) \rangle_{L^2} \\ &\quad + C_{q_0}\|u^M_{\alpha,\mathfrak{h}}\|_{\mathcal{H}^{s-1}}\|u^M_{\alpha,\mathfrak{h}'}\|_{\mathcal{H}^{s-1}}. \end{aligned} \tag{8.3.13}$$

Since $u^M_{\alpha,\mathfrak{h}}$ and $u^M_{\alpha,\mathfrak{h}'}$ are supported on invertibility domains of F^{q_0},

$$\mathcal{L}^{q_0}|(\partial^\gamma_{x_1}\partial^\beta_{x_2}u^M_{\alpha,\tau}|) = \frac{|\partial^\gamma_{x_1}\partial^\beta_{x_2}u^M_{\alpha,\tau}| \circ \tau}{|\det DF^{q_0}| \circ \tau}, \quad \tau \in \{\mathfrak{h}, \mathfrak{h}'\}. \tag{8.3.14}$$

We define $\chi_\tau := |\partial^\gamma_{x_1}\partial^\beta_{x_2}u^M_{\alpha,\tau}| \circ \tau$ and $g_\tau := |\det DF^N| \circ \tau$ and we have

$$\begin{aligned} \langle \mathcal{L}^{q_0}(|\partial^\gamma_{x_1}\partial^\beta_{x_2}u^M_{\alpha,\mathfrak{h}}|), \mathcal{L}^{q_0}(|\partial^\gamma_{x_1}\partial^\beta_{x_2}u^M_{\alpha,\mathfrak{h}'}|) \rangle_{L^2} &= \int_{\mathbb{T}^2} \frac{\chi_{\mathfrak{h}}\chi_{\mathfrak{h}'}}{\sqrt{g_{\mathfrak{h}}g_{\mathfrak{h}'}}\sqrt{g_{\mathfrak{h}}g_{\mathfrak{h}'}}} \\ &\le \frac{1}{2}\int_{\mathbb{T}^2} \frac{\chi^2_{\mathfrak{h}}}{g_{\mathfrak{h}'}g_{\mathfrak{h}}} + \frac{1}{2}\int_{\mathbb{T}^2} \frac{\chi^2_{\mathfrak{h}'}}{g_{\mathfrak{h}'}g_{\mathfrak{h}}}, \end{aligned} \tag{8.3.15}$$

where we used the elementary inequality $ab \le \frac{1}{2}(a^2+b^2)$ with $a = \frac{\chi_{\mathfrak{h}}}{\sqrt{g_{\mathfrak{h}}g_{\mathfrak{h}'}}}, b = \frac{\chi_{\mathfrak{h}'}}{\sqrt{g_{\mathfrak{h}}g_{\mathfrak{h}'}}}$. In order to obtain (8.3.10), we need to sum equation (8.3.13) over $\mathfrak{h} \in \mathfrak{H}^{q_0}$

and $\mathfrak{h}' \not\sim_\alpha^{q_0} \mathfrak{h}$. Let us begin with the first term. Consider one of the integrals in (8.3.15), for example the first one. By Definition 7.2.1 of $\mathcal{N}(q_0)$ and Lemma 8.3.3 it follows that

$$\begin{aligned}\sum_{\mathfrak{h}} \sum_{\mathfrak{h}':\mathfrak{h}' \not\sim_\alpha^{q_0} \mathfrak{h}} \int_{\mathbb{T}^2} \frac{\chi_{\mathfrak{h}}^2}{g_{\mathfrak{h}'} g_{\mathfrak{h}}} &\le \sum_{\mathfrak{h}} \int_{\mathbb{T}^2} \frac{\chi_{\mathfrak{h}}^2}{g_{\mathfrak{h}}} \sum_{\mathfrak{h}':\mathfrak{h}' \not\sim_\alpha^{N} \mathfrak{h}} \frac{1}{g_{\mathfrak{h}'}} \le \mathcal{N}(q_0) \sum_{\mathfrak{h}} \int_{\mathbb{T}^2} \frac{|\partial_{x_1}^\gamma \partial_{x_2}^\beta u_{\alpha,\mathfrak{h}}^M|^2 \circ \mathfrak{h}}{|\det DF^N| \circ \mathfrak{h}} \\ &= \mathcal{N}(q_0) \sum_{\mathfrak{h}} \|\mathcal{L}^{q_0} |\partial_{x_1}^\gamma \partial_{x_2}^\beta u_{\alpha,\mathfrak{h}}^M|^2\|_{L^1} \le \mathcal{N}(q_0) \sum_{\mathfrak{h}} \|\partial_{x_1}^\gamma \partial_{x_2}^\beta u_{\alpha,\mathfrak{h}}^M\|_{L^2}^2. \end{aligned} \tag{8.3.16}$$

By symmetry we have

$$\begin{aligned}&\mu^{2sq_0} \sum_{\mathfrak{h} \not\sim_\alpha^{q_0} \mathfrak{h}'} \sum_{\gamma+\beta=s} C_{\gamma,\beta} \langle \mathcal{L}^{q_0}(|\partial_{x_1}^\gamma \partial_{x_2}^\beta u_{\alpha,\mathfrak{h}}^M|), \mathcal{L}^{q_0}(|\partial_{x_1}^\gamma \partial_{x_2}^\beta u_{\alpha,\mathfrak{h}'}|)\rangle_{L^2} \\ &\lesssim \mu^{2sq_0} \mathcal{N}(q_0) \sum_{\mathfrak{h} \in \mathfrak{H}^{q_0}} \sum_{\gamma+\beta=s} \|\partial_{x_1}^\gamma \partial_{x_2}^\beta u_{\alpha,\mathfrak{h}}^M\|_{L^2}^2 \le C_\sharp \mu^{2sq_0} \mathcal{N}(q_0) \sum_{\mathfrak{h} \in \mathfrak{H}^{q_0}} \|u_{\alpha,\mathfrak{h}}^M\|_{\mathcal{H}^s}^2,\end{aligned} \tag{8.3.17}$$

which corresponds to the first addend of the r.h.s. of (8.3.10).
Finally we sum the second term of (8.3.13) over $\mathfrak{h} \in \mathfrak{H}^{q_0}$, and we write

$$\begin{aligned}C_{q_0} \sum_{\mathfrak{h}' \not\sim_\alpha^{q_0} \mathfrak{h}} \|u_{\alpha,\mathfrak{h}}^M\|_{\mathcal{H}^{s-1}} \|u_{\alpha,\mathfrak{h}'}^M\|_{\mathcal{H}^{s-1}} &\le C_{q_0} \sum_{\mathfrak{h}' \not\sim_\alpha^{q_0} \mathfrak{h}} \frac{\|u_{\alpha,\mathfrak{h}}^M\|_{\mathcal{H}^{s-1}}^2 + \|u_{\alpha,\mathfrak{h}'}^M\|_{\mathcal{H}^{s-1}}^2}{2} \\ &\lesssim C_{q_0} \sum_{\mathfrak{h} \in \mathfrak{H}^{q_0}} \|u_{\alpha,\mathfrak{h}}^M\|_{\mathcal{H}^{s-1}}^2,\end{aligned} \tag{8.3.18}$$

which yields the second addend of (8.3.10) and, together with (8.3.7), conclude the proof of Lemma 8.3.1.

We are finally ready to prove Theorem 8.0.2.

8.4 Proof of Theorem 8.0.2

By (8.2.2) and Lemmata 8.2.1 and 8.3.1,[10] we have

$$\begin{aligned}\|\mathcal{L}^N u\|_{\mathcal{H}^s}^2 &\lesssim \Theta_s \|u\|_{s+2}^2 + C_{q_0} Q(M,s) \sqrt{\Theta_s} \|u\|_{s+2} \|u\|_{\mathcal{H}^s} \\ &\quad + \mathcal{N}(q_0) \mu^{2sq_0} \sum_\alpha \sum_{\mathfrak{h} \in \mathfrak{H}^{q_0}} \|u_{\alpha,\mathfrak{h}}^M\|_{\mathcal{H}^s}^2 + C_{q_0} \sum_\alpha \sum_{\mathfrak{h} \in \mathfrak{H}^{q_0}} \|u_{\alpha,\mathfrak{h}}^M\|_{\mathcal{H}^{s-1}}^2.\end{aligned} \tag{8.4.1}$$

Recalling that $u_{\alpha,\mathfrak{h}}^M = \psi_{\alpha,\mathfrak{h}} \mathcal{L}^M u$, we can use equations (8.1.5) and (7.1.2) to write,[11]

$$\begin{aligned}\sum_\alpha \sum_{\mathfrak{h} \in \mathfrak{H}^{q_0}} \|u_{\alpha,\mathfrak{h}}^M\|_{\mathcal{H}^s}^2 &= \sum_\alpha \sum_{\mathfrak{h} \in \mathfrak{H}^{q_0}} \|\psi_{\alpha,\mathfrak{h}} \mathcal{L}^M u\|_{\mathcal{H}^s}^2 \le C \|\mathcal{L}^M u\|_{\mathcal{H}^s}^2 + C_\psi \|\mathcal{L}^M u\|_{L^1}^2 \\ &\le C A_s \|\mathcal{L}^M 1\|_\infty \mu^{2sM} \|u\|_{\mathcal{H}^s}^2 + Q(M,s) \|u\|_{\mathcal{H}^{s-1}}^2 + C_\psi \|u\|_{L^1}^2\end{aligned} \tag{8.4.2}$$

[10] Note that, to use (8.2.3) in (8.2.2), we just use an inequality analogous to (8.3.18).
[11] We also use repeatedly $\|\mathcal{L}^n u\|_{L^1} \le \|u\|_{L^1}$.

and

$$\begin{aligned}&\sum_{\alpha}\sum_{\mathfrak{h}\in\mathfrak{H}^{q_0}}\|u^M_{\alpha,\mathfrak{h}}\|^2_{\mathcal{H}^{s-1}}\\&\leq C\|\mathcal{L}^M u\|^2_{\mathcal{H}^{s-1}}+C_\psi\|\mathcal{L}^M u\|^2_{L^1}\leq CQ(M,s)\|u\|^2_{\mathcal{H}^{s-1}}+C_\psi\|u\|^2_{L^1}.\end{aligned} \tag{8.4.3}$$

Next, by Lemma C.0.1

$$\|u\|^2_{\mathcal{H}^{s-1}}\leq\varsigma\|u\|^2_{\mathcal{H}^s}+\varsigma^{-1}C\|u\|^2_{L^1},\quad\forall\varsigma>0. \tag{8.4.4}$$

If we chose $\varsigma=\mathcal{N}(q_0)\mu^{2sq_0}Q(M,s)^{-1}C_{q_0}^{-1}$, using (8.4.2) and (8.4.3) in (8.4.1), setting $\overline{Q}(M,s)=\{Q(M,s),C_\psi\}^+$, and recalling (8.0.4) for the definition of $\mathbb{L}_M$, we obtain

$$\begin{aligned}\|\mathcal{L}^N u\|^2_{\mathcal{H}^s}&\leq C_\sharp\mathbb{L}_M\mathcal{N}(q_0)\mu^{2sN}\|u\|^2_{\mathcal{H}^s}\\&\quad+\Theta_s\|u\|^2_{s+2}+C_{q_0}Q(M,s)\sqrt{\Theta_s}\|u\|_{s+2}\|u\|_{\mathcal{H}^s}+C_{q_0}\overline{Q}(M,s)\|u\|^2_{L^1}.\end{aligned} \tag{8.4.5}$$

Finally we note that $\|u\|^2_{L^1}\lesssim\|u\|_{\mathcal{H}^s}\|u\|_{s+2}$ and, as $\sqrt{ab}\leq\sqrt{\frac{\bar\epsilon}{2}}a+\sqrt{\frac{1}{2\bar\epsilon}}b$ for each $a,b,\bar\epsilon>0$, we have $\sqrt{\|u\|_{\mathcal{H}^s}\|u\|_{s+2}}\leq\sqrt{\frac{\bar\epsilon}{2}}\|u\|_{\mathcal{H}^s}+\sqrt{\frac{1}{2\bar\epsilon}}\|u\|_{s+2}$. We apply this with $\bar\epsilon:=\Theta_s^{-\frac12}Q(M,s)^{-1}\varsigma$, for ς arbitrarily small so that, taking the square root of (8.4.5), there exist a uniform constant $C_1>0$ and $C_{q_0}>0$ such that[12]

$$\|\mathcal{L}^N u\|_{\mathcal{H}^s}\leq C_1\left(\sqrt{[\mathbb{L}_M\mathcal{N}(q_0)]^{\frac1N}\mu^{2s}}\right)^N\|u\|_{\mathcal{H}^s}+C_{q_0}\overline{Q}(M,s)\sqrt{\Theta_s}\|u\|_{s+2},$$

from which we obtain (8.0.7) in the case $s>1$.

The case $s=1$

It remains to prove (8.0.9) for $s=1$. First, by Lemma 7.1.1, $Q(M,1)\lesssim C_{\mu,M}^{\frac32}\mu^{2M}$. Recalling Remark 8.1.2, $C_\psi(1)\leq C_\sharp C_{q_0}\chi_u^{-c_\sharp\ln\mu}$. Finally, using also (8.2.4), we can find $\beta_3,\beta_4>0$ such that

$$\overline{Q}(M,1)\sqrt{\Theta_1}\leq C_{\mu,M}^{\beta_3}\mu^{\beta_4 M}\chi_u^{\frac{-11}{2}-c_\sharp\ln\mu}M^{\frac12},$$

which concludes the proof of Theorem 8.0.2. □

[12]Here we use $\|u\|_{L^1}\lesssim\|u\|_{s+2}$.

Chapter 9

The final Lasota-Yorke Inequality

In this section we state and prove our main technical Theorem which implies the Theorems stated in section 4. For each integer $1 \le s \le r-1$ we define the following norm

$$\|\cdot\|_{s,*} := \|\cdot\|_{\mathcal{H}^s} + \|\cdot\|_{s+2}.$$

Theorem 9.0.1. *Let $F \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$ be an SVPH an and $\alpha = \frac{\log(\lambda_- \mu^{-2})}{\log(\lambda_+)}$. Let $\overline{m}_{\chi_u}$ be as in* (8.1.24)*, n_0 as in* (8.0.5) *and $C_1 > 0$ provided in Theorem* (8.0.2)*. We assume that there exist: a constant $K > 0$, an integer $n_1 \ge n_0$, and uniform constants $\tau_0 \ge 1, c \ge 0, \kappa_1, \kappa_0 \in \mathbb{N}$ such that, for some $1 \le s+2 \le r-1$,*[1]

$$\sup_{m \le n} \|\mathcal{L}^m 1\|_\infty \le K\mu^{cn^{\tau_0}}, \qquad \forall n < \kappa_1 n_1 + \overline{m}_{\chi_u}, \tag{9.0.1}$$

$$\left\{\mu^{\zeta_s}\lambda_-^{-\frac{1}{2}}, \sqrt{\widetilde{\mathcal{N}}_F(\lceil \alpha n_1 \rceil)\mu^{\alpha_s n_1^{\tau_0} + \beta_s \overline{m}_{\chi_u}^{\tau_0}}}\right\}^+ \le \nu_0 < 1, \tag{9.0.2}$$

$$(KC_1)^{\frac{1}{\kappa_0 n_1 + \overline{m}_{\chi u}}} \nu_0^{\frac{\kappa_0^{\tau_0}}{\kappa_0 n_1 + \overline{m}_{\chi u}}} < 1, \tag{9.0.3}$$

where $\widetilde{\mathcal{N}}_F$ is given in (4.2.2), $\alpha_s = c[(1-\alpha)^{\tau_0} + 1] + 2s$, $\beta_s = 2(s+c)$ and ζ_s given in (4.3.3)*. Moreover, for $\kappa \in (\kappa_0, \kappa_1)$, choose*

$$\sigma_\kappa \in (\{\lambda_-^{-\frac{1}{4}}, (C_1 K)^{\frac{1}{\kappa_0 n_1 + \overline{m}_{\chi u}}} \nu_0^{\frac{\kappa^{\tau_0}}{\kappa n_1 + \overline{m}_{\chi u}}}\}, 1). \tag{9.0.4}$$

Then, for each $n \in \mathbb{N}$ and $\bar{\sigma}_\kappa \in (\sigma_\kappa, 1)$ we have

$$\|\mathcal{L}^n u\|_{s,*} \le C_\sharp A(\kappa, n_1, \overline{m}_{\chi_u}, s)\sigma_\kappa^n \|u\|_{s,*} + C_\sharp A(\kappa, n_1, \overline{m}_{\chi_u}, s)\mu^n \|u\|_0 \tag{9.0.5}$$

$$\|\mathcal{L}^n u\|_{s,*} \le C_\sharp A(\kappa, n_1, \overline{m}_{\chi_u}, s)\overline{\sigma}_\kappa^n \|u\|_{s,*} + C_{\bar{\sigma}_\kappa} A(\kappa, n_1, \overline{m}_{\chi_u}, s)^3 \mu^{3n} \|u\|_{L^1}, \tag{9.0.6}$$

where $A(\kappa, n_1, \overline{m}_{\chi_u}, s) = \Theta_{\chi_u}(\kappa n_1 + \overline{m}_{\chi_u}, s)$, Θ_{χ_u} as in (8.0.9) with C_{κ,n_1} instead of C_{q_0} .

Proof. We will use Theorem 8.0.2 with $N = q_0 + \overline{m}_{\chi_u}$, where $q_0 = \kappa n_1 \ge n_0$ and $\kappa \in (\kappa_0, \kappa_1)$. First, by conditions (9.0.1) and (9.0.2) and Lemma 7.2.3 , we

[1] Notice that (9.0.1) defines κ_1, (9.0.2) defines n_1 and (9.0.3) defines κ_0.

observe that

$$
\begin{aligned}
[\mathbb{L}_{\overline{m}_{\chi_u}}\mathcal{N}(q_0)]^{\frac{1}{N}}\mu^{2s} &\leq [K\mu^{c\overline{m}_{\chi_u}^{\tau_0}}\mathcal{N}(q_0)]^{\frac{1}{N}}\mu^{2s} \leq \\
&\leq \left(K\mu^{c\overline{m}_{\chi_u}^{\tau_0}}\mathbb{L}_{q_0-\lceil\alpha q_0\rceil}\widetilde{\mathcal{N}}(\lceil\alpha q_0\rceil)\right)^{\frac{1}{N}}\mu^{2s} \\
&\leq \left(K^2\mu^{q_0^{\tau_0}(1-\alpha)^{\tau_0}+c\overline{m}_{\chi_u}^{\tau_0}}\widetilde{\mathcal{N}}(\lceil\alpha q_0\rceil)\right)^{\frac{1}{N}}\mu^{2s} \\
&\leq (K^2\widetilde{\mathcal{N}}(\lceil\alpha q_0\rceil)\mu^{q_0^{\tau_0}\alpha_s+\beta_s\overline{m}_{\chi_u}^{\tau_0}}))^{\frac{1}{N}}.
\end{aligned}
\tag{9.0.7}
$$

Therefore, by equation (8.0.7),

$$
\|\mathcal{L}^N u\|_{\mathcal{H}^s} \leq C_1 K\left((\widetilde{\mathcal{N}}(\lceil\alpha q_0\rceil)\mu^{q_0^{\tau_0}\alpha_s+\beta_s\overline{m}_{\chi_u}^{\tau_0}}))^{\frac{1}{N}}\right)^{\frac{N}{2}}\|u\|_{\mathcal{H}^s} + C_{q_0}\Theta_{\chi_u}(N,s)\|u\|_{s+2}. \tag{9.0.8}
$$

Moreover by the sub-multiplicativity of $\widetilde{\mathcal{N}}$

$$
\widetilde{\mathcal{N}}(\lceil\alpha q_0\rceil) = \widetilde{\mathcal{N}}(\lceil\alpha\kappa n_1\rceil) \leq \widetilde{\mathcal{N}}(\lceil\alpha n_1\rceil)^{\kappa}.
$$

It follows by the definition of ν_0 that

$$
\sqrt{(\widetilde{\mathcal{N}}(\lceil\alpha q_0\rceil)\mu^{q_0^{\tau_0}\alpha_s+\beta_s\overline{m}_{\chi_u}^{\tau_0}})^{\frac{1}{N}}} \leq \sqrt{[(\widetilde{\mathcal{N}}(\lceil\alpha n_1\rceil)\mu^{\frac{\alpha_s q_0^{\tau_0}+\beta_s\overline{m}_{\chi_u}^{\tau_0}}{\kappa}}]^{\kappa\tau_0})^{\frac{1}{N}}} \leq \nu_0^{\frac{\kappa\tau_0}{\kappa n_1+\overline{m}_{\chi_u}}}.
$$

Accordingly

$$
\|\mathcal{L}^N u\|_{\mathcal{H}^s} \leq \sigma_\kappa^N\|u\|_{\mathcal{H}^s} + C_{q_0}\Theta_{\chi_u}(N,s)\|u\|_{s+2}. \tag{9.0.9}
$$

On the other hand, the assumption $\mu^{\zeta_s}\lambda_-^{-\frac{1}{2}} \leq \nu_0$ implies (6.2.8), so that we can choose δ_* in (6.2.9) such that, for all $n \in \mathbb{N}$,

$$
\|\mathcal{L}^n u\|_{s+2} \leq C\sigma_\kappa^{2n}\|u\|_{s+2} + CC_{\mu,n}\mu^n\|u\|_0, \tag{9.0.10}
$$

where $C_{\mu,n}$ is defined in (5.3.2). Iterating (9.0.9) by multiple of N and using (9.0.10) yields

$$
\|\mathcal{L}^n u\|_{s,*} \leq C_\sharp\sigma_\kappa^n(\|u\|_{\mathcal{H}^s} + A(\kappa,n_1,\overline{m}_{\chi_u},s)\|u\|_{s+2}) + C_\sharp A(\kappa,n_1,\overline{m}_{\chi_u},s)\mu^n\|u\|_0, \tag{9.0.11}
$$

from which we deduce (9.0.5).

Next, we want to compare the norm $\|\cdot\|_0$ with the L^1-norm. Let us fix $\ell > 0$. Take an admissible central curve γ and notice that, for any $\phi \in \mathcal{C}^0(\mathbb{T})$ with $\|\phi\|_\infty = 1$, we have

$$
\left|\int_{\mathbb{T}}\phi(t)(u)(\gamma(t)+\ell e_1)dt - \int_{\mathbb{T}}\phi(t)(u)(\gamma(t))dt\right| = \left|\int_0^\ell ds\int_{\mathbb{T}}\phi(t)\partial_z u(\gamma(t)+se_1)dt\right|.
$$

Writing $\gamma(t) = (\sigma(t),t)$ we can make the change of variables $\psi(s,t) = \gamma(t)+se_1 = (\sigma(t)+s,t)$. Since $\det(D\psi) = -1$ and setting $D_\ell = \{\psi(s,t)\ ;\ t \in \mathbb{T}, s \in [0,\ell]\}$, we have

$$
\begin{aligned}
\left|\int_{\mathbb{T}}\phi(t)(u)(\gamma(t)+\ell e_1)dt - \int_{\mathbb{T}}\phi(t)(u)(\gamma(t))dt\right| &= \left|\int_{D_\ell}\phi(z)\partial_z u(x,z)dxdz\right| \\
&\leq \|\phi\|_{L^\infty}\sqrt{\ell}\|u\|_{\mathcal{H}^1}.
\end{aligned}
$$

Hence

$$\int_{\mathbb{T}} \phi(t)(u)(\gamma(t) + se_1)dt \geq \int_{\mathbb{T}} \phi(t)(u)(\gamma(t))dt - \sqrt{s}\|u\|_{\mathcal{H}^1}.$$

Integrating in $s \in [0, \ell]$ and taking the sup on γ and ϕ yields

$$\|u\|_0 \leq \ell^{-1}\|u\|_{L^1} + \frac{2\ell^{\frac{1}{2}}}{3}\|u\|_{\mathcal{H}^1}. \tag{9.0.12}$$

Applying the above formula to (9.0.5) with $\ell = C_\sharp \sigma_\kappa^2 \mu^{-2n}$ yields

$$\|\mathcal{L}^n u\|_{s,*} \leq C_\sharp A(\kappa, n_1, \overline{m}_{\chi_u}, s)\sigma_\kappa^n \|u\|_{s,*} + C_\sharp A(\kappa, n_1, \overline{m}_{\chi_u}, s)\sigma_\kappa^{-2n}\mu^{3n}\|u\|_{L^1}$$

Next, for each $\bar{\sigma}_\kappa \in (\sigma_\kappa, 1)$, let n_κ be the smallest integer such that

$$C_\sharp A(\kappa, n_1, \overline{m}_{\chi_u}, s)\sigma_\kappa^{n_\kappa} \leq \bar{\sigma}_\kappa^{n_\kappa}.$$

For each $n \in \mathbb{N}$, write $n = kn_\kappa + m$ with $m < \kappa$, then iterating the above equation yields

$$\begin{aligned}\|\mathcal{L}^n u\|_{s,*} &\leq \bar{\sigma}_\kappa^{kn_\kappa}\|\mathcal{L}^m u\|_{s,*} + C_\sharp A(\kappa, n_1, \overline{m}_{\chi_u}, s)\mu^{3kn_\kappa}\sigma_\kappa^{-2n_\kappa}\sum_{j=0}^{k-1}\bar{\sigma}_\kappa^{n_\kappa}\|u\|_{L^1} \\ &\leq C_\sharp A(\kappa, n_1, \overline{m}_{\chi_u}, s)\bar{\sigma}_\kappa^n\|u\|_{s,*} + C_\sharp \frac{A(\kappa, n_1, \overline{m}_{\chi_u}, s)^3\mu^{3n}}{\bar{\sigma}_\kappa^{2n_\kappa}(1 - \bar{\sigma}_\kappa^{n_\kappa})}\|u\|_{L^1}\end{aligned}$$

which implies (9.0.6). □

Corollary 9.0.2. *Under the assumptions of Theorem 9.0.1 there exists a Banach space $\mathcal{B}_{s,*}$ such that $\mathcal{C}^{r-1}(\mathbb{T}^2) \subset \mathcal{B}_{s,*} \subset \mathcal{H}^s(\mathbb{T}^2)$ on which the operator $\mathcal{L} : \mathcal{B}_{s,*} \to \mathcal{B}_{s,*}$ has spectral radius one and is quasi compact with essential spectral radius bounded by σ_κ.*

Proof. We call $\mathcal{B}_{s,*}$ the completion of $\mathcal{C}^{r-1}(\mathbb{T}^2)$ with respect to the norm $\|\cdot\|_{s,*}$, then $\mathcal{C}^{r-1}(\mathbb{T}^2) \subset \mathcal{B}_{s,*} \subset \mathcal{H}^s(\mathbb{T}^2)$. Iterating (9.0.6), and since $\mathcal{L}$ is a L^1 contraction, implies that the spectral radius is bounded by one, but since the adjoint of $\mathcal{L}$ has eigenvalue one, so does $\mathcal{L}$, hence the spectral radius is one.

To bound the essential spectral radius note that the immersion $\mathcal{B}_{s,*} \hookrightarrow \mathcal{H}^s$ is continuous by definition of the norm. Moreover the immersion $\mathcal{H}^s \hookrightarrow L^1$ is compact for every s by Sobolev embeddings theorems, hence $\mathcal{B}_{s,*} \hookrightarrow L^1$ is compact. Hence by (9.0.6) and Hennion theorem [37] follows that the essential spectral radius is bounded by $\bar{\sigma}_\kappa$ and hence the claim by the arbitrariness of $\bar{\sigma}_\kappa$. □

Proof of Theorem 4.3.2. According to Corollary 9.0.2, it is enough to check the conditions of Theorem 9.0.1. Since $\mu > 1$, Corollary 5.5.4 implies $\sup_{k\leq n}\|\mathcal{L}^k 1\|_\infty \leq K\mu^{2n}$ for each $n \in \mathbb{N}$, with $K = (\mu - 1)^{-1}$,[2] hence (9.0.1) is satisfied with $c = 2$ and $\tau_0 = 1$ and arbitrary $\kappa_1 \in \mathbb{N}$. Next, $\mu^{\zeta_s}\lambda_-^{\frac{1}{2}} < 1$ is

[2]Recall that $C_{\mu,n} \leq (\mu - 1)^{-1}$ (see also Remark 5.3.2).

implied by hypothesis **(H3)**. Therefore, condition (4.3.4) coincides with (9.0.2) with $\alpha_s, \beta_s, \zeta_s$ given in (4.3.3). Finally, choosing any κ_0 such that

$$\kappa_0 > -\frac{\ln(C_1 K^{\frac{1}{2}})}{\ln \nu_0}, \tag{9.0.13}$$

we have also (9.0.3), whereby we conclude. $\square$

Part III

Application to Fast Slow Systems

Chapter 10

The map F_ε

In this section we check that we can apply Theorem 9.0.1 to the family of maps F_ε given in (4.5.1) and we prove Theorems 4.5.2 and 4.5.4.

10.1 The F_ε are SVPH

Let $(1, \varepsilon u) \in \mathbf{C}^u_\varepsilon$, for $p = (x, \theta) \in \mathbb{T}^2$. In this case equation (4.4.2) yields

$$D_pF_\varepsilon(1, \varepsilon u) = (\partial_x f + \varepsilon u \partial_\theta f)(1, \varepsilon \Xi_\varepsilon(p, u)), \tag{10.1.1}$$

where

$$\Xi_\varepsilon(p, u) = \frac{\partial_x \omega + \varepsilon u \partial_\theta \omega + u}{\partial_x f + \varepsilon u \partial_\theta f}. \tag{10.1.2}$$

We have also a more explicit formula for iteration of the map Ξ_ε. For any $k \geq 0$ and $p \in \mathbb{T}^2$, let us denote $p_k = F^k_\varepsilon(p)$. Then we have the recursive formula:

$$\Xi^{(n)}_\varepsilon(p, u) = \Xi_\varepsilon(p_{n-1}, \Xi^{(n-1)}(p, u)). \tag{10.1.3}$$

On the other hand, recalling (4.4.3):

$$\partial_u \Xi_\varepsilon(p, u) = \frac{\partial_x f + \varepsilon(\partial_\theta \omega \partial_x f - \partial_\theta f \partial_x \omega)}{(\partial_x f + \varepsilon u \partial_\theta f)^2}. \tag{10.1.4}$$

Now we use Lemma 4.4.1, applied with ω replaced by $\omega\varepsilon$, to check that the maps given in (4.5.1) are SVPHS for ε small enough. Conditions (2) and (3) are immediate. In particular, $\mathbf{C}^u_\varepsilon = \{(\xi, \eta) \in \mathbb{R}^2 : |\eta| \leq \varepsilon u_\star |\xi|\}$[1] and $\mathbf{C}^c = \{(\xi, \eta) \in \mathbb{R}^2 : |\xi| \leq \chi_c |\eta|\}$ satisfy $D_pF_\varepsilon(\mathbf{C}^u_\varepsilon) \Subset \mathbf{C}^u_\varepsilon$ and $D_pF^{-1}_\varepsilon(\mathbf{C}^c) \Subset \mathbf{C}^c$ if

$$u_\star = 2\|\partial_x \omega\|_\infty =: \varepsilon^{-1}\chi_u \qquad \text{and} \qquad \chi_c := \frac{1}{2}. \tag{10.1.5}$$

In fact, with the above choice of $u_\star$ and using (10.1.2), for ε small enough,

$$D_pF_\varepsilon(\mathbf{C}^u_\varepsilon) \subset \{(\eta, \xi) : |\xi| \leq \frac{3}{2}\lambda^{-1}\chi_u|\eta|\},$$

[1] Observe that in this special case $\chi_u(\varepsilon) = \varepsilon u_\star$, thus we have an unstable cone of size ε.

hence condition (4.1.4) with $\iota_\star = \frac{3}{2}\lambda^{-1} < 1$.[2] Next, we note that condition (5) of Lemma 4.4.1 implies conditions (1) and (4) for each ε small enough. Moreover, in (4.4.8) it is shown that for some $\bar{c} > 0$, $\mu_\pm = e^{\pm\bar{c}\varepsilon}$, which implies (6) for sufficiently small ε. In particular, by (4.4.10), it follows that condition (4.1.5) holds with $C_\star = 1$, $\lambda_+ = 2\sup_{\mathbb{T}^2}\partial_x f$ and $\lambda_- = 2\lambda/3$.
The above discussion shows that all the quantities χ_c, $\iota_\star$ and $C_\star$ are independent of ε, and μ_+ and $(\lambda_- - \mu_+)^{-1}$ are bounded uniformly in ε. Therefore, a constant which depends on these quantities, will be uniform according to the notation given at the beginning. Finally, it is useful to note that, if we set $\psi(p) = \langle \nabla\omega, (-\frac{\partial_\theta f}{\partial_x f}, 1)\rangle(p)$, for every $p \in \mathbb{T}^2$ and $n \in \mathbb{N}$ we have

$$\det D_p F_\varepsilon^n = \prod_{k=0}^{n-1} \det D_{F_\varepsilon^k p} F_\varepsilon = \prod_{k=0}^{n-1} \left[\partial_x f(F_\varepsilon^k p)(1 + \varepsilon\psi(F_\varepsilon^k p))\right],$$

hence

$$e^{-\bar{c}\varepsilon n}\lambda^n \le \det D_p F_\varepsilon^n \le e^{\bar{c}\varepsilon n}\Lambda^n, \quad \forall p \in \mathbb{T}^2, \forall n \in \mathbb{N}. \tag{10.1.6}$$

10.2 A non-transversality argument

The aim is to prove the following theorem which guarantees that, after some fix time which does not depend on ε, for each point we have at least one couple of pre-images with transversal unstable cones, provided ω satisfies some checkable conditions. We will see that this corresponds to proving the existence of the integer n_1 required by Theorem 9.0.1.

In the following we denote as $\mathfrak{H}_\varepsilon$ the set of the inverse branches of F_ε.[3] Moreover, $\mathfrak{H}_\varepsilon^n$ will be the set of elements of the form $\mathfrak{h}_1 \circ \cdots \mathfrak{h}_n$, for $\mathfrak{h}_j \in \mathfrak{H}_\varepsilon$ and $\mathfrak{H}_\varepsilon^\infty := \mathfrak{H}_\varepsilon^{\mathbb{N}}$, in particular, for $\mathfrak{h} \in \mathfrak{H}_\varepsilon^\infty$ the symbol h_n will denote the restriction of $\mathfrak{h}$ on $\mathfrak{H}_\varepsilon^n$.

Remark 10.2.1. *Since F_0 and F_ε are homotopic coverings they are isomorphic, that is there exist $I_\varepsilon : \mathbb{T}^2 \to \mathbb{T}^2$ such that $F_\varepsilon = F_0 \circ I_\varepsilon$. This induces an isomorphism $\mathcal{I}_\varepsilon : \mathfrak{H}_0 \to \mathfrak{H}_\varepsilon$ defined by $\mathcal{I}_\varepsilon\mathfrak{h} = I_\varepsilon^{-1} \circ \mathfrak{h}$. Hence the same is true for the sets $\mathfrak{H}_\varepsilon^n = \mathfrak{H}^n$ and $\mathfrak{H}_\varepsilon^\infty$. In the following we will then identify inverse branches of F_ε^n and F_0^n by these isomorphisms, and drop the script ε from the notation when it is not necessary.*

The main result of this section is the following.

Proposition 10.2.2. *If ω is not $x-$constant with respect to F_0 (see Definition 4.5.1), then there exist $\varepsilon_0 > 0$ and $n_1 \in \mathbb{N}$ such that, for every $\varepsilon \le \varepsilon_0$, $p \in \mathbb{T}^2$ and vector $v \in \mathbb{R}^2$, there exists $q \in F_\varepsilon^{-n_1}(p)$ such that $v \notin D_q F_\varepsilon^{n_1}\mathbf{C}_\varepsilon^u$.*

[2] Recall that $\lambda = \inf \partial_x f > 2$.
[3] Accordingly $\mathfrak{H}_0$ is the set of inverse branches of F_0.

Proof. We argue by contradiction and suppose that for every $\varepsilon_0 > 0$ and $\ell \in \mathbb{N}$ there exist $\varepsilon_\ell \in [0, \varepsilon_0]$, $p_\ell \in \mathbb{T}^2$ and $v_\ell = (1, \varepsilon_\ell u_\ell)$ with $|u_\ell| \le u_\star$ such that[4]

$$D_q F^\ell_{\varepsilon_\ell} \mathbf{C}^u_{\varepsilon_\ell} \supset v_\ell, \quad \forall q \in F^{-\ell}_{\varepsilon_\ell}(p_\ell), \tag{10.2.1}$$

namely, all the above cones have a common direction. Since the sequence $\{p_\ell, u_\ell\} \subset \mathbb{T}^2 \times [-u_\star, u_\star]$, it has an accumulation point (p_*, u_*). In analogy with (5.2.7), for $p \in \mathbb{T}^2$ and $u \in [-u_\star, u_\star]$ we define

$$\Phi^n_\varepsilon(p, u) = \left(F^n_\varepsilon(p), \Xi^{(n)}_\varepsilon(p, u)\right), \tag{10.2.2}$$

where $\Xi^{(n)}_\varepsilon$ is given by formula (10.1.3). Condition (10.2.1) in terms of this dynamics says that the slope u_ℓ is contained in the interval $\Xi^{(\ell)}_{\varepsilon_\ell}(q, [-u_\star, u_\star])$ for every $\ell \in \mathbb{N}$ and $q \in F_{\varepsilon_\ell}(p_\ell)$. Hence, it can be written as:

$$\forall \ell \in \mathbb{N}, \quad \exists (p_\ell, u_\ell): \quad \pi_2 \circ \Phi^\ell_{\varepsilon_\ell}(q, [-u_\star, u_\star]) \supset \{u_\ell\}, \quad \forall q \in F^{-\ell}_{\varepsilon_\ell}(p_\ell), \tag{10.2.3}$$

where $\pi_2 : \mathbb{T}^2 \times [-u_\star, u_\star] \to [-u_\star, u_\star]$ is the projection on the second coordinate. Now, for $m \in \mathbb{N}$, $\varepsilon \in [0, \varepsilon_0]$, $u_0 \in [-u_\star, u_\star]$ and $\mathfrak{h} \in \mathfrak{H}^\infty$, let us define

$$u^\varepsilon_{\mathfrak{h},m}(p) = \pi_2 \circ \Phi^m_\varepsilon(\mathfrak{h}_m(p), u_0) : \mathbb{T}^2 \to [-u_\star, u_\star]. \tag{10.2.4}$$

Next, we prove the following result, which will allow us to conclude the proof.

Sublemma 10.2.3. *The sequence of functions defined in* (10.2.4) *satisfies:*

(i) For every $\varepsilon \in [0, \varepsilon_0]$ and $\mathfrak{h} \in \mathfrak{H}^\infty$, there exists $u^\varepsilon_{\mathfrak{h},\infty}(q) := \lim_{m\to\infty} u^\varepsilon_{\mathfrak{h},m}(q)$, and the limit is uniform in $q \in \mathbb{T}^2$.

(ii) For every $\mathfrak{h} \in \mathcal{H}^\infty$, the sequence $\{u^\varepsilon_{\mathfrak{h},\infty}\}_\varepsilon$ converges to $\overline{u}_{\mathfrak{h},\infty}$ uniformly.

(iii) The functions $\overline{u}_{\mathfrak{h},\infty}$ are independent of $\mathfrak{h}$, we call them $\tilde{u}$. In addition, $\tilde{u}$ satisfies

$$\tilde{u}(F_0(q)) = \Xi_0(q, \tilde{u}(q)), \quad \forall q \in \mathbb{T} \times \{\theta_*\}. \tag{10.2.5}$$

Proof. Applying Lemma D.0.1 with $u = u' \equiv u_0 \in [-u_\star, u_\star]$, $\varepsilon_0 = 1$, $A = 2\chi_c u_\star$ and $B = 0$ we have that there exists $\nu \in (0, 1)$ such that, for each $\mathfrak{h} \in \mathfrak{H}^\infty$, $q \in \mathbb{T}^2$, $\varepsilon, \varepsilon' \in [0, 1)$, $m \in \mathbb{N}$ and $n > m$,[5]

$$\begin{aligned} |u^\varepsilon_{\mathfrak{h},m}(q) - u^{\varepsilon'}_{\mathfrak{h},m}(q)| &\le C_\sharp \mu^{3m} |\varepsilon - \varepsilon'| \\ |u^\varepsilon_{\mathfrak{h},n}(q) - u^\varepsilon_{\mathfrak{h},m}(q)| &\le C_\sharp \nu^m. \end{aligned} \tag{10.2.6}$$

It follows that there exists $u^\varepsilon_{\mathfrak{h},\infty}(q) := \lim_{m\to\infty} u^\varepsilon_{\mathfrak{h},m}(q)$, and the limit is uniform in q. Next, for each $\delta > 0$, we choose ε_* and m such that $C_\sharp \mu^{3m} \varepsilon_* \le \frac{\delta}{4}$ and $\nu^m \le \frac{\delta}{4}$, then, for each $\varepsilon, \varepsilon' \le \varepsilon_*$ and $q \in \mathbb{T}^2$

$$\begin{aligned} |u^\varepsilon_{\mathfrak{h},\infty}(q) - u^{\varepsilon'}_{\mathfrak{h},\infty}(q)| &\le |u^\varepsilon_{\mathfrak{h},\infty}(q) - u^{\varepsilon'}_{\mathfrak{h},m}(q)| + |u^\varepsilon_{\mathfrak{h},m}(q) - u^{\varepsilon'}_{\mathfrak{h},m}(q)| + |u^{\varepsilon'}_{\mathfrak{h},m}(q) - u^{\varepsilon'}_{\mathfrak{h},\infty}(q)| \\ &\le 2\nu^m + C_\sharp \mu^{3n} |\varepsilon - \varepsilon'| \le \delta. \end{aligned}$$

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[4] We use the notation with subscript ℓ for a generic object that depends on ℓ through ε_ℓ, but we keep the notation as simple as possible when there is no need to specify.

[5] The second equation of (10.2.6) is a direct consequence of (D.0.5) which implies that $\Xi_\varepsilon(p, \cdot)$ is a contraction.

The above proves the first two items. Let us proceed with the third one. First we claim that, for $q \in \mathbb{T}^2$, if $\mathfrak{h}_q$ is such that $q = \mathfrak{h}_q(F_\varepsilon(q))$, then

$$u^\varepsilon_{\mathfrak{h}\circ\mathfrak{h}_q,\infty}(F_\varepsilon(q)) = \Xi_\varepsilon(q, u^\varepsilon_{\mathfrak{h},\infty}(q)), \quad \forall q \in \mathbb{T}^2. \tag{10.2.7}$$

Indeed, since $u^\varepsilon_{\mathfrak{h},\infty}$ belongs to the unstable cone, by (10.2.4), for every $\mathfrak{h} \in \mathfrak{H}^\infty$ and $q \in \mathbb{T}^2$,

$$\left(F_\varepsilon(q), \Xi_\varepsilon(q, u^\varepsilon_{\mathfrak{h},\infty}(q))\right) = \Phi_\varepsilon(q, u^\varepsilon_{\mathfrak{h},\infty}(q)) = \left(F_\varepsilon(q), u^\varepsilon_{\mathfrak{h}\circ\mathfrak{h}_q,\infty}(F_\varepsilon(q))\right),$$

which implies the claim taking the projection on the second coordinate.

For every $\ell \in \mathbb{N}$, let us now consider ε_ℓ, p_ℓ and u_ℓ as given in (10.2.3) and let ℓ_j so that (p_{ℓ_j}, u_{ℓ_j}) is a convergent sequence. Equation (10.2.1) implies

$$|u_{\ell_j} - u^{\varepsilon_{\ell_j}}_{\mathfrak{h},n_{\ell_j}}(p_{\ell_j})| \le C_\sharp \nu^{n_{\ell_j}}. \tag{10.2.8}$$

Taking the limit for $j \to \infty$ in the above inequality yields[6]

$$u_* = \lim_{j\to\infty} u_{\ell_j} = \lim_{j\to\infty} u^{\varepsilon_{\ell_j}}_{\mathfrak{h},n_{\ell_j}}(p_{\ell_j}) = \overline{u}_{\mathfrak{h},\infty}(p_*), \tag{10.2.9}$$

regardless of the choice of the inverse branch $\mathfrak{h} \in \mathfrak{H}^\infty$. Let $\mathfrak{h}_q$ be the inverse branch such that $q = \mathfrak{h}_q(F_\varepsilon(q))$, and set $q_\ell = \mathfrak{h}_q(p_\ell)$ in equation (10.2.7) to obtain:

$$u^{\varepsilon_\ell}_{\mathfrak{h}\circ\mathfrak{h}_q,\infty}(p_\ell) = \Xi_{\varepsilon_\ell}(q_\ell, u^{\varepsilon_\ell}_{\mathfrak{h},\infty}(q_\ell)). \tag{10.2.10}$$

By item (ii) above, and by the continuity of the map F_ε, we can take the limit as $\ell_j \to \infty$ in the last equation and obtain

$$\overline{u}_{\mathfrak{h}\circ\mathfrak{h}_q,\infty}(p_*) = \Xi_0(q_*, \overline{u}_{h,\infty}(q_*)),$$

where q_* is such that $F_0(q_*) = p_*$. By (10.2.9), the above equation becomes $u_* = \Xi_0(q_*, \overline{u}_{\mathfrak{h},\infty}(q_*))$, and, since $\Xi_0(q_*, \cdot)$ is invertible, this implies that there exists $u_*(q_*)$ independent of $\mathfrak{h} \in \mathcal{H}^\infty$ such that

$$u_*(q_*) = \bar{u}_\mathfrak{h}(q_*) = \lim_{j\to\infty} u^{\varepsilon_{\ell_j}}_{\mathfrak{h},\infty}(q_{\ell_j}).$$

Hence, by induction, $\overline{u}_{\mathfrak{h},\infty}(q)$ is independent on $\mathfrak{h}$ for each $q \in \bigcup_{k\in\mathbb{N}} F_0^{-k}(p_*) =: \Lambda_{\theta_*}$, let us call it $u_*(q)$. Taking the limit in equation (10.2.7) we have, for each $q \in \Lambda_{\theta_*}$,

$$u_*(F_0(q)) = \Xi_0(q, u_*(q)). \tag{10.2.11}$$

Note that the $\overline{u}_{\mathfrak{h},\infty}$ are uniform limits of continuous functions and hence are continuous functions such that $\overline{u}_{\mathfrak{h},\infty}|_{\Lambda_{\theta_*}} = u_*$. Since Λ_{θ_*} is dense in $\mathbb{T} \times \{\theta_*\}$.[7] It follows that the $\overline{u}_{\mathfrak{h},\infty}$ equal some continuous function $\tilde{u}$ defined on $\mathbb{T} \times \{\theta_*\}$ and independently of $\mathfrak{h}$. In addition, $\tilde{u}$ satisfies (10.2.5).[8] □

[6] Recall that (p_*, u_*) is an accumulation point of the sequence (p_ℓ, u_ℓ) given in (10.2.3)

[7] It follows from the expansivity of $f(\cdot, \theta_*)$ that the preimmages of any point form a dense set.

[8] Just approximate any point with a sequence $\{q_j\} \subset \Lambda_{\theta_*}$ and take the limit in (10.2.11).

We can now conclude the proof of Proposition 10.2.2. By Sub-Lemma 10.2.3 we can find a function $\tilde{u} : \mathbb{T}^2 \to \mathbb{R}$ and $\theta_* \in \mathbb{T}^1$ such that (10.2.5) holds, namely:

$$\tilde{u}(F_0(q)) = \frac{\partial_x \omega(q) + \tilde{u}(q)}{\partial_x f(q)}, \quad q \in \mathbb{T}^1 \times \{\theta_*\} \tag{10.2.12}$$

Let us use the notation $g_\theta(x)$ for a function $g(x,\theta)$ and observe that, integrating (10.2.12) and recalling that ω is periodic by hypothesis, we have

$$\begin{aligned}\int_0^1 \tilde{u}_{\theta_*}(x)dx &= \int_0^1 f'_{\theta_*}(x)\tilde{u}_{\theta_*}(f_{\theta_*}(x))dx - \int_0^1 \partial_x\omega(x,\theta_*)dx \\ &= \sum_{i=0}^{d-1}\int_{U_i} f'_{\theta_*}(x)\tilde{u}_{\theta_*}(f_{\theta_*}(x))dx = d\int_0^1 \tilde{u}_{\theta_*}(t)dt,\end{aligned}$$

where U_i are the invertibility domains of f_{θ_*}, and $d > 1$ its topological degree. Hence $\int_{\mathbb{T}} \tilde{u}_{\theta_*}(x)dx = 0$. So there is a potential given by $\Psi_{\theta_*}(x) = \int_0^x \tilde{u}_{\theta_*}(z)dz$. Finally, integrating equation (10.2.12) from 0 to x, there exists $c > 0$ such that

$$\omega_{\theta_*}(x) = \Psi_{\theta_*}(f_{\theta_*}(x)) - \Psi_{\theta_*}(x) + c,$$

which contradicts the assumption on ω whereby proving the Proposition. □

For reasons which will be clear in a moment, we introduce a further quantity related to $\mathcal{N}_{F_\varepsilon}$ and $\widetilde{\mathcal{N}}_{F_\varepsilon}$ which can be interpreted as a kind of normalization of the latter one. The following definition is inspired by [16].

Definition 10.2.4. *For each $p = (x,\theta) \in \mathbb{T}^2$, $v \in \mathbb{R}^2$, $n \in \mathbb{N}$ and $\varepsilon > 0$ we define*

$$\widetilde{\mathfrak{N}}(x,\theta,v,n) := \frac{1}{h_*(x,\theta)} \sum_{\substack{(y,\eta)\in F_\varepsilon^{-n}(x,\theta) \\ DF_\varepsilon^n(y,\eta)\mathbf{C}_\varepsilon^u \ni v}} \frac{h_*(y,\theta)}{|\det DF_\varepsilon^n(y,\eta)|}, \tag{10.2.13}$$

where, for every $\theta \in \mathbb{T}$, $h_*(\cdot,\theta) =: h_{*\theta}(\cdot)$ is the density of the unique invariant measure of $f(\cdot,\theta)$. As before we will denote $\widetilde{\mathfrak{N}}(n) := \sup_p \sup_v \widetilde{\mathfrak{N}}(p,v,n)$.

The motivation to introduce this quantity is twofold. One reason lies in Lemma 10.2.5 below in which, using a shadowing argument similar to [21, Appendix B], we exploit the following fact: for each $\theta \in \mathbb{T}$, setting $f_\theta(\cdot) = f(\cdot,\theta)$, we have

$$\frac{1}{h_{*\theta}(x)} \sum_{y\in f_\theta(x)} \frac{h_{*\theta}(y)}{(f_\theta^n)'(y)} = 1, \quad \forall x \in \mathbb{T}. \tag{10.2.14}$$

On the other hand it is easy to see that $\widetilde{\mathfrak{N}}$ has the same properties of $\widetilde{\mathcal{N}}_{F_\varepsilon}$. In particular, arguing exactly in the same way as in Proposition 7.2.1 and Lemma 7.2.3, one can show that

$$\widetilde{\mathfrak{N}}(n) \text{ is submultiplicative,} \tag{10.2.15}$$

$$\mathcal{N}_{F_\varepsilon}(n)^{\frac{1}{n}} \le C_\sharp \|\mathcal{L}_{F_\varepsilon}^{n\lfloor 1-\alpha\rfloor} 1\|_\infty^{\frac{1}{n}} \left(\widetilde{\mathfrak{N}}(\lfloor \alpha n\rfloor)^{\frac{1}{\lfloor \alpha n\rfloor}}\right)^\alpha, \quad \text{for some } \alpha \in (0,1) \tag{10.2.16}$$

$$\widetilde{\mathfrak{N}}(n) \le \sup_{(x,\theta)\in\mathbb{T}^2} \frac{1}{h_*(x,\theta)} (\mathcal{L}_{F_\varepsilon}^n h_*)(x,\theta). \tag{10.2.17}$$

This implies that we can check condition (9.0.2) of Theorem (9.0.1) with $\widetilde{\mathcal{N}}$ replaced by $\widetilde{\mathfrak{N}}$.
To ease notation in the following we set $\mathcal{L}_{F_\varepsilon} =: \mathcal{L}_\varepsilon$.

Lemma 10.2.5. *There are constants $C, c_* > 0$ such that, for each $n < C\varepsilon^{-\frac{1}{2}}$,*

$$\sup_{(x,\theta)\in\mathbb{T}^2} \frac{1}{h_*(x,\theta)}(\mathcal{L}_\varepsilon^n h_*)(x,\theta) \le e^{c_* n^2 \varepsilon}. \tag{10.2.18}$$

Proof. Let $F_\varepsilon^n(q) = (x,\theta)$ and define $q_k = (x_k,\theta_k) = F_\varepsilon^k(q)$, for every $0 \le k \le n$. Then,

$$|\theta - \theta_k| \le \sum_{j=k}^{n-1} \varepsilon\|\omega\|_\infty \le C_\sharp(n-k)\varepsilon. \tag{10.2.19}$$

Let us set $f_\theta(y) = f(y,\theta)$. Since f_θ is homotopic to f_{θ_k}, for each k, there is a correspondence between inverse branches, hence there exists x_* such that $|f_\theta^k(x_*) - x_k| \le \lambda^{-1}$. Moreover, let $\xi_k = f_\theta^k(x_*) - x_k$. Since f is expanding, by the mean value theorem and (10.2.19), there is $(\bar{x},\bar{\theta})$ such that

$$|\xi_{k+1}| = |\langle \nabla f(\bar{x},\bar{\theta}), (\xi_k, \theta_k - \theta)\rangle| \ge \lambda|\xi_k| - C_\sharp n\varepsilon.$$

Since $\xi_n = 0$, we find by induction $|\xi_k| \le \sum_{j=k}^{n-1} \lambda^{-j+k} C_\sharp \varepsilon n \le C_\sharp \varepsilon n$. Moreover, since h_* is differentiable[9] we also have

$$|h_*(x_k,\theta_k) - h_{*\theta}(f_\theta^k(x_*))| \le C_\sharp \varepsilon n.$$

Next, since $|\det D_q F_\varepsilon - \partial_x f(q)| \le C_\sharp\varepsilon$,

$$\frac{(f_\theta^n)'(x_*)}{\det DF_\varepsilon^n(x_0,\theta_0)} = \prod_{k=0}^{n-1} \frac{f_\theta'(f_\theta^k(x_*))}{\det DF_\varepsilon(x_k,\theta_k)} \le \prod_{k=0}^{n-1} \frac{f_\theta'(f_\theta^k(x_*))}{\det DF_\varepsilon(f_\theta^k(x_*),\theta)}[1 + C_\sharp n\varepsilon] \le e^{c_\sharp n^2\varepsilon}.$$

It follows that,

$$\frac{1}{h_*(x,\theta)} \sum_{(y,\vartheta)\in F_\varepsilon^{-n}(x,\theta)} \left(\frac{h_*(y,\vartheta)}{|\det DF_\varepsilon^n(y,\vartheta)|}\right) \le \frac{e^{c_\sharp n^2\varepsilon}}{h_{*\theta}(x)} \sum_{x_*\in f_\theta^{-n}(x)} \frac{h_{*\theta}(x_*)}{(f_\theta^n)'(x_*)} = e^{c_\sharp n^2\varepsilon},$$

where we have used (10.2.14). □

10.3 Proof of Theorem 4.5.2.

By the results of section 10.1 F_ε is SVPH for ε small enough. We now prove conditions of Theorem 9.0.1 for F_ε, under the assumption that ω is not x-constant. In this case the existence of n_1 independent of ε is guaranteed by Proposition 10.2.2. Notice that $\chi_u = u_\star\varepsilon$, i.e the unstable cone $\mathbf{C}_\varepsilon^u$ is of order ε while the center cone $\mathbf{C}^c$ is of order one. Hence, by (5.2.5), there exist $c_0 > 0$ such that

[9]See [20] for the details.

$\overline{m}_{\chi_u} \leq \lfloor c_0 \log \varepsilon^{-1} \rfloor$. [10] We then take any $\kappa_1 \leq c_1 \log \varepsilon^{-1}$, for some $c_1 > 0$ and, by Lemma 10.2.5, we have

$$\sup_{m \leq n} \|\mathcal{L}_\varepsilon^m 1\|_\infty \leq \frac{1}{|\inf h_*|} \sup_{m \leq n} \|\mathcal{L}_\varepsilon^m h_*\|_\infty \leq \frac{1}{|\inf h_*|} e^{c_* \varepsilon n^2}, \quad \forall n \leq \{c_0, c_1\}^+ \log \varepsilon^{-1},$$

hence condition (9.0.1) with $K = \frac{1}{|\inf h_*|}$, $\tau_0 = 2$ and $c = c_*/\bar{c}$. Next, let n_1 be the integer as in Proposition 10.2.2. We prove that there exists a uniform constant ν_0 such that

$$\left\{ e^{\bar{c}\varepsilon\zeta_s} \lambda^{-\frac{1}{2}}, \widetilde{\mathfrak{N}}(\lceil \alpha n_1 \rceil) e^{c\varepsilon\left(\alpha_s n_1^2 + \beta_s m_{\chi_u}^2\right)} \right\}^+ \leq \nu_0 < 1, \tag{10.3.1}$$

i.e condition (9.0.1) with $\widetilde{\mathcal{N}}_F$ replaced by $\widetilde{\mathfrak{N}}$ which, as we already observed, implies (9.0.2) for F_ε. Obviously there exists $\varepsilon_1 > 0$ such that, for each $1 \leq s \leq r-1$

$$\mu^{\zeta_s} \lambda^{-\frac{1}{2}} = e^{\bar{c}\varepsilon\zeta_s} \lambda^{-\frac{1}{2}} < 1, \quad \forall \varepsilon \in (0, \varepsilon_1). \tag{10.3.2}$$

Let n_1 and ε_0 be as in Proposition 10.2.2. Accordingly, for every $p = (x, \theta) \in \mathbb{T}^2$ and $v \in \mathbb{R}^2$, there exists $q_* \in F^{-n_1}(p)$ such that

$$\frac{1}{h_*(x,\theta)} \sum_{\substack{(y,\theta) \in F_\varepsilon^{-n_1}(p) \\ DF_\varepsilon^{n_1}(y,\eta)\mathbf{C}_\varepsilon^u \supset v}} \frac{h_*(y,\theta)}{|\det D_q F_\varepsilon^{n_1}|} \leq \frac{1}{h_*(x,\theta)} (\mathcal{L}_\varepsilon^{n_1} h_*)(x,\theta) - \frac{\Bbbk}{|\det D_{q_*} F_\varepsilon^{n_1}|},$$

where $\Bbbk = \frac{\inf h_*}{\sup h_*}$. By Lemma 10.2.5 and equation (10.1.6), the last expression is bounded by $e^{cn_1^2\varepsilon} - \frac{C}{\Lambda^{n_1}}$. Choosing $\varepsilon_2 < \min\left(\varepsilon_0, \frac{1}{cn_1^2} \log(1 + C\Lambda^{-n_1})\right)$, we have that $\widetilde{\mathfrak{N}}(n_1) \leq \bar{\sigma} < 1$ for every $\varepsilon \in [0, \varepsilon_2]$. Consequently there exists ε_3 such that

$$\widetilde{\mathfrak{N}}(\lceil \alpha n_1 \rceil) e^{\bar{c}\varepsilon\left(\alpha_s n_1^2 + \beta_s m_{\chi_u}\right)^2} \leq \bar{\sigma} e^{\bar{c}\varepsilon\left(\alpha_s n_1^2 + \beta_s m_{\chi_u}^2\right)} < 1, \quad \forall \varepsilon \in (0, \varepsilon_3). \tag{10.3.3}$$

By (10.3.2) and (10.3.3) we deduce (10.3.1) taking $\varepsilon_* = \min\{\varepsilon_1, \varepsilon_3\}$. Finally, condition (9.0.3) is satisfied choosing κ_0 as in (9.0.13), since C_1, ν_0 and K are all uniform. Thus Theorem 9.0.1 applies and Theorem 4.5.2 follows by Corollary 9.0.2. □

10.4 Eigenfunctions regularity (quantitative)

As we have already seen in 9.0.2, the main consequence of Theorem 8.0.2 is that there exists a Banach space $\mathcal{B}_{s,*} \subset \mathcal{H}^s$ on which the transfer operator $\mathcal{L}_\varepsilon$ is quasi compact for each $\varepsilon < \varepsilon_*$. In addition, using inequality (9.0.5), we can say much more about the constants, paying the price of having a bigger essential spectral radius. Indeed for each $n, \kappa \in \mathbb{N}$

$$\|\mathcal{L}_\varepsilon^n u\|_{s,*} \leq C_\sharp A(\kappa, n_1, m_{\chi_u}, s) {\sigma_\kappa}^n \|u\|_{s,*} + B_\sharp A(\kappa, n_1, m_{\chi_u}, s) \mu^n \|u\|_0,$$

[10] For simplicity in the following we drop the $\lfloor \cdot \rfloor$ notation.

where $m_{\chi_u} = c_0 \log \varepsilon^{-1}$ and σ_κ given in (9.0.4). The choice $\kappa = C_\sharp \log \varepsilon^{-1}$ yields a spectral radius uniform in ε, but we have no control on the constant $A(\kappa, n_1, m_{\chi_u}, s)$. On the contrary, the choice $\kappa = 2\kappa_0 \in \mathbb{N}$ (independent of ε) implies, for some $c_\star > 0$,

$$\sigma_{\kappa_0} \in (1 - \left(c_\star \log \varepsilon^{-1}\right)^{-1}, 1),$$

hence a lesser information on the size of the essential spectrum but allows a control of the constants, especially in the case $s = 1$. Indeed, observe that by (5.3.2)

$$C_{\mu, n_1 + m_{\chi_u}} \leq C_\sharp \min\{\log \varepsilon^{-1}, \varepsilon^{-1}\} = C_\sharp \log \varepsilon^{-1}.$$

In addition, since $\kappa = 2\kappa_0$ and n_1 do not depend on ε, in this case the constant C_{κ, n_1} in Theorem 9.0.1 is independent on ε. Hence, it follows by (8.0.9) that we can find $\beta_3, \beta_3, C_\sharp > 0$ and $\epsilon > 0$ such that

$$\Theta_{u\varepsilon}(n_1 + u\varepsilon, 1) \leq C_\sharp \varepsilon^{-\frac{11}{2}} (\log \varepsilon^{-1})^{\beta_1} e^{\beta_2 \varepsilon \log \varepsilon^{-1}}. \tag{10.4.1}$$

Thus, for $s = 1$ and for each $\alpha > \frac{11}{2}$ and provided ϵ is chosen small enough, we have, for all $n \in \mathbb{N}$,

$$\begin{aligned} &\|\mathcal{L}_\varepsilon u\|_0 \leq C e^{\bar{c} n \varepsilon} \|u\|_0 \\ &\|\mathcal{L}_\varepsilon^n u\|_{1,*} \leq C_\alpha \varepsilon^{-\alpha} e^{-\frac{\bar{c} n}{\ln \varepsilon^{-1}}} \|u\|_{1,*} + B_\alpha \varepsilon^{-\alpha} \|u\|_0. \end{aligned} \tag{10.4.2}$$

Proof of Theorem 4.5.4. Let $c_\star = \bar{c}$ and $\mathcal{L}_\varepsilon u = \nu u$ with $\nu^n > e^{-\mathfrak{r} \frac{\bar{c} n}{\ln \varepsilon^{-1}}}$, $\mathfrak{r} < 1$, then

$$\|u\|_{1,*} = \nu^{-n} \|\mathcal{L}_\varepsilon^n u\|_{1,*} \leq C_\alpha \varepsilon^{-\alpha} \nu^{-n} e^{-\frac{\bar{c} n}{\ln \varepsilon^{-1}}} \|u\|_{1,*} + B_\alpha \nu^{-n} \varepsilon^{-\alpha} \|u\|_0.$$

We choose n to be the smallest integer such that $C_\alpha \varepsilon^{-\alpha} e^{-\frac{(1-\mathfrak{r})\bar{c} n}{\ln \varepsilon^{-1}}} \leq \frac{1}{2}$, which yields

$$\|u\|_{\mathcal{H}^1} \leq \|u\|_{1,*} \leq C_\alpha \varepsilon^{-(1+\mathfrak{r})\alpha} \|u\|_0$$

which concludes the proof. □

10.5 Proof of Theorem 4.5.3

Let $\sigma_{ph}(\mathcal{L}_{F_\varepsilon}) = \{z \in \mathbb{C} : |z| = 1\}$ be the peripheral spectrum. If $e^{i\vartheta} \in \sigma_{ph}(\mathcal{L}_{F_\varepsilon})$, then by Theorem 4.5.2 it is point spectrum of finite multiplicity. In addition, since the operator is power bounded, there cannot exists Jordan blocks, thus the algebraic and geometric multiplicity are equal.

In fact, see [9, Section 5] for a proof which applies verbatim to the present context, the eigenvectors associated to the eigenvalue one are the physical measure.

Hence there is $N \in \mathbb{N}$ and $\{\vartheta_j, h_j, \ell_j\}_{j=1}^N$ such that $\vartheta_0 = 1$, $\ell_0(\varphi) = \int_{\mathbb{T}^2} \varphi$, $\vartheta_j \in [0, 2\pi)$, $h_j \in \mathcal{B}_{*,s}$, $\ell_j \in \mathcal{B}'_{*,s}$ and $\mathcal{L}_{F_\varepsilon} h_j = e^{i\vartheta_j} h_j$, $\ell_j(\mathcal{L}_{F_\varepsilon} \varphi) = e^{i\vartheta_j} \ell_j(\varphi)$ for all

$\varphi \in \mathcal{B}_{*,s}$. On the other hand, for each j let $\varphi_j \in \mathcal{C}^\infty$ be such that $\int_{\mathbb{T}^2} h_k\varphi_j = \delta_{kj}$, then

$$|\ell_j(h)| = \left|\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} e^{-i\vartheta_j k}\int_{\mathbb{T}^2}\varphi_j\mathcal{L}_{F_\varepsilon}^k h\right| \leq \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_{\mathbb{T}^2}|\varphi_j\circ F_\varepsilon^k|\,|h| \leq \|\varphi_j\|_\infty\|h\|_{L^1}.$$

Which implies that there exists $\tilde{\ell}_j \in L^\infty$ such that

$$\ell_j(h) = \int_{\mathbb{T}^2}\tilde{\ell}_j h.$$

Note that the above also implies $\tilde{\ell}_j \circ F_\varepsilon = e^{i\vartheta_j}\tilde{\ell}_j$. The above means that, for all $l \in \mathbb{N}$,

$$\int_{\mathbb{T}^2}\tilde{\ell}_j^l\mathcal{L}_{F_\varepsilon}h = \int_{\mathbb{T}^2}\tilde{\ell}_j^l\circ F_\varepsilon h = e^{i\vartheta_j l}\int_{\mathbb{T}^2}\tilde{\ell}_j^l h.$$

This implies that $e^{i\vartheta_j l}$ belongs to the spectrum of $(\mathcal{L}_{F_\varepsilon})'$, hence of $\mathcal{L}_{F_\varepsilon}$. Since there can be only finitely many elements of $\sigma_{ph}(\mathcal{L}_{F_\varepsilon})$, it must be $\vartheta_j = \frac{2\pi p}{q}$ for some $p, q \in \mathbb{N}$, that is the $\{\vartheta_j\}$ form a finite group.

It follows that we have the following spectral decomposition

$$\mathcal{L}_{F_\varepsilon} = \sum_j e^{i\vartheta_j}\Pi_j + Q \tag{10.5.1}$$

where $\Pi_j h = h_j\ell_j(h)$, $\Pi_j\Pi_k = \delta_{jk}\Pi_j$ and Q has spectral radius strictly smaller than one.

In addition,

$$|h_j| \leq \mathcal{L}_{F_\varepsilon}|h_j|.$$

Since $h_j \in \mathcal{H}^3$ it follows that $h_j \in \mathcal{C}^1$, so $|h_j|$ is Lipschitz, hence $h_j^* = |h_j| \in \mathcal{H}^1 \cap \mathcal{C}^0$. Hence,

$$0 = \int_{\mathbb{T}^2}\mathcal{L}_{F_\varepsilon}h_j^* - h_j^*$$

which implies $h_j^* = \mathcal{L}_{F_\varepsilon}h_j^*$. It follows that h_j^* is an eigenvector of $\mathcal{L}_{F_\varepsilon}$ associated to the eigenvalue one. Next, we prove that 1 is the only eigenvalue on the unit circle or, in other words, that in the decomposition (10.5.1) the ϑ_j are all null. Setting $V_j := \{z \in \mathbb{T}^2 : h_j^*(z) = 0\}$ and $A_j := \mathbb{T}^2 \setminus V_j$, we observe that A_j is an F_ε-invariant set Lebesgue almost surely. Indeed, if $z \in \mathbb{T}^2 \setminus A_j$, then

$$0 = h_j^*(z) = \mathcal{L}_{F_\varepsilon}h_j^*(z) = \sum_{y\in F_\varepsilon^{-1}(z)}(\det DF_\varepsilon)^{-1}\cdot h_j^*)(y),$$

which implies $h_j^*(y) = 0$, for each $y \in F_\varepsilon^{-1}(z)$, by the positivity of $\mathcal{L}_{F_\varepsilon}$. Hence, $F_\varepsilon^{-1}(\mathbb{T}^2\setminus A_j) \subset \mathbb{T}^2\setminus A_j$ which implies in turn $F_\varepsilon(A_j) \subset A_j$ Lebesgue almost surely. By the previous discussion we can have $h_j = h_j^* e^{i\beta_j}$, where $\beta_j \in \mathcal{C}^0(A_j)$, as both h_j, and h_j^* are continuous. Next, since $\mathcal{L}_{F_\varepsilon}h_j = e^{i\vartheta_j}h_j$, we have

$$\mathcal{L}_{F_\varepsilon}e^{i(\beta_j - \vartheta_j - \beta_j\circ F_\varepsilon)}h_j^* - h_j^* = 0.$$

Taking the real part and integrating we get

$$\int_{\mathbb{T}^2} (1 - \cos(\vartheta_j + \beta_j - \beta_j \circ F_\varepsilon))\, h_j^* = 0. \tag{10.5.2}$$

Equation (10.5.2) implies that there exists a function $k : A_j \to \mathbb{N}$ such that

$$2\pi k(z) = \vartheta_j + \beta_j(z) - \beta_j \circ F_\varepsilon(z), \tag{10.5.3}$$

Lebesgue almost surely in A_j. But $\beta_j \in \mathcal{C}^0$, hence k must be a constant function.

Therefore, multiplying (10.5.3) by h_j^* and integrating over A_j, we obtain $\vartheta_j = 2\pi k$, which implies $\vartheta_j = 0$, since $\theta_j \in [0, 2\pi)$. We conclude that $\mathcal{L}_{F_\varepsilon} h_j = h_j$ and Π_0 is the projector associated to the eigenvalue 1. In particular

$$\Pi_0 g = \sum_j h_j \int \tilde{\ell}_j g, \quad \tilde{\ell}_j \in L^\infty, \quad h_j \in \mathcal{B}_{1,*}, \tag{10.5.4}$$

and $\Pi_0 = \mathcal{L}_{F_\varepsilon} \Pi_0$. We claim that $\tilde{\ell}_j = \mathbb{1}_{\tilde{B}_j}$ for some set $\tilde{B}_j$. Let us assume, without loss of generality, that $\int \tilde{\ell}_j = 1$. Equation (10.5.4) implies that, for each $g \in \mathcal{B}_{1,*}$,

$$\sum_j h_j \int \tilde{\ell}_j g = \sum_j h_j \int (\tilde{\ell}_j \circ F_\varepsilon) g.$$

Therefore $\tilde{\ell}_j = \tilde{\ell}_j \circ F_\varepsilon$, Leb-a.s. This implies that, for each set B_i,

$$\int \mathbb{1}_{B_i} \circ F_\varepsilon \tilde{\ell}_j h_i = \int \mathbb{1}_{B_i} \circ F_\varepsilon \tilde{\ell}_j \circ F_\varepsilon h_i = \int \mathbb{1}_{B_i} \mathcal{L}_{F_\varepsilon} h_i.$$

Hence, if B_i is the basin of the measure with density h_i, the above equation implies $\int \tilde{\ell}_i \tilde{\ell}_j = \int \mathbb{1}_{B_i} h_i \ell_j = \delta_{i,j}$. For $i = j$ it follows that $\int \tilde{\ell}_j^2 = \int \ell_j$, therefore ℓ_j can have only two values $\{0, 1\}$, from which the claim follows.

Next we would like to better understand the structure of the peripheral spectrum, and prove equations (4.5.2) and (4.5.3).

Let $(x_k, \theta_k) = F_\varepsilon^k(x, \theta)$ and $f_\theta(x') = f(x', \theta)$. By [21, Lemma 4.2] there exists Y_n such that $\pi_2(F_\varepsilon^n(x, \theta)) = f_\theta^n(Y_n(x))$ and, for all $k \le n$,

$$\begin{aligned}
&\|x_k - f_\theta^k \circ Y_n\|_\infty \le C_\sharp \varepsilon k \\
&|\theta_k - \theta| \le C_\sharp k \varepsilon \\
&\|1 - \partial_x Y_n\|_\infty \le C_\sharp \varepsilon n^2.
\end{aligned}$$

Let $\mathcal{L}_\theta$ be the transfer operator associated to f_θ and $h_*(\cdot, \theta)$ the associated unique invariant probability density. The operator $\mathcal{L}_\theta$ has a uniform spectral gap $1 - \sigma$ in $\mathcal{H}^1(\mathbb{R})$, hence we have, for each $n \in \{C_\sharp \ln \varepsilon^{-1}, \ldots, C_\sharp \varepsilon^{-\frac{1}{2}}\}$,

$$\begin{aligned}
\int_{\mathbb{T}^2} \varphi \mathcal{L}_{F_\varepsilon}^n h &= \int_{\mathbb{T}^2} \varphi \circ F_\varepsilon^n h = \int_{\mathbb{T}^2} \varphi(f_\theta^n(Y_n(x)), \theta) h(x, \theta) + \mathcal{O}(\varepsilon n \|\varphi\|_{\mathcal{C}^1} \|h\|_{L^1}) \\
&= \int_{\mathbb{T}^2} \varphi(f_\theta^n(x, \theta) h \circ Y_n^{-1}(x, \theta) + \mathcal{O}(\varepsilon n^2 \|\varphi\|_{\mathcal{C}^1} \|h\|_{L^1}) \\
&= \int_{\mathbb{T}^2} \varphi(x, \theta) [\mathcal{L}_\theta^n (h \circ Y_n^{-1})](x, \theta) + \mathcal{O}(\varepsilon n^2 \|\varphi\|_{\mathcal{C}^1} \|h\|_{L^1}).
\end{aligned}$$

Let $\mathcal{L}_\theta$ be the transfer operator associated to f_θ and $h_*(\cdot,\theta)$ the associated unique invariant density. Then, for each $\theta \in \mathbb{T}$, $\mathcal{L}_\theta$ has a uniform spectral gap $\sigma \in (0,1)$ on the Sobolev space $W^{1,1}(\mathbb{T})$ with norm $\|g\|_{W^{1,1}} = \|g\|_{L^1} + \|g'\|_{L^1}$. Thus

$$\int_{\mathbb{T}} \left| [\mathcal{L}_\theta^n (h \circ Y_n^{-1})](x,\theta) - h_*(x,\theta) \int_{\mathbb{T}} (h \circ Y_n^{-1})(y,\theta) dy \right| dx \le C_\sharp \sigma^n \|h \circ Y_n^{-1}\|_{W^{1,1}}.$$

Since $\|h \circ Y_n^{-1}(\cdot,\theta)\|_{W^{1,1}} \le C_\sharp \|h(\cdot,\theta)\|_{W^{1,1}}$, we have, setting $\|h\|_\dagger = \int_{\mathbb{T}} d\theta \|h(\cdot,\theta)\|_{W^{1,1}}$,

$$\begin{aligned}\int_{\mathbb{T}^2} \varphi \mathcal{L}_{F_\varepsilon}^n h = & \int_{\mathbb{T}^2} dx \varphi(x,\theta) h_*(x,\theta) \int_{\mathbb{T}} dy h(y,\theta) + \mathcal{O}(\varepsilon n^2 \|\varphi\|_{\mathcal{C}^1} \|h\|_{L^1}) \\ + & \quad \mathcal{O}\left(\sigma^n \|\varphi\|_{\mathcal{C}^0} \|h\|_\dagger\right) \\ = & \int_{\mathbb{T}^2} \bar{\varphi} h + \mathcal{O}(\varepsilon n^2 \|\varphi\|_{\mathcal{C}^1} \|h\|_{L^1} + \sigma^n \|\varphi\|_{\mathcal{C}^0} \|h\|_\dagger)\end{aligned} \tag{10.5.5}$$

where $\bar{\varphi}(\theta) := \int_{\mathbb{T}} \varphi(x,\theta) h_*(x,\theta) dx$. Let

$$Ph(x,\theta) = h_*(x,\theta) \int_{\mathbb{T}} h(y,\theta) dy. \tag{10.5.6}$$

We can then choose $n = c \ln \varepsilon^{-1}$, for c large enough, and obtain

$$\|\mathcal{L}_{F_\varepsilon}^{c_\sharp \ln \varepsilon^{-1}} h - Ph\|_{(\mathcal{C}^1)'} \le C_\sharp \varepsilon [\ln \varepsilon^{-1}]^2 \|h\|_{L^1} + C_\sharp \varepsilon^{1000} \|h\|_{\mathcal{H}^1} \tag{10.5.7}$$

Equation (9.0.6) yields

$$\|\mathcal{L}_{F_\varepsilon}^k h\|_{\mathcal{H}^1} \le C_\alpha \varepsilon^{-\alpha} e^{-\frac{c_\sharp k}{\ln \varepsilon^{-1}}} \|h\|_{\mathcal{B}_{1,*}} + C_\alpha \varepsilon^{-3\alpha} \ln \varepsilon^{-1} \|h\|_{L^1}. \tag{10.5.8}$$

Hence, by equation (10.5.7) we have that, for each $\varphi \in \mathcal{C}^1$ and $h \in \mathcal{B}_{1,*}$,

$$\begin{aligned}&\int_{\mathbb{T}^2} \varphi \Pi_0 h = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{\mathbb{T}^2} \varphi \mathcal{L}_{F_\varepsilon}^k h \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n - c_\sharp \ln \varepsilon^{-1}} \int_{\mathbb{T}^2} \varphi \mathcal{L}_{F_\varepsilon}^{c_\sharp \ln \varepsilon^{-1}} \mathcal{L}_{F_\varepsilon}^k h \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n - c_\sharp \ln \varepsilon^{-1}} \int_{\mathbb{T}^2} \varphi P \mathcal{L}_{F_\varepsilon}^k h + \mathcal{O}(\varepsilon [\ln \varepsilon^{-1}]^2 \|\varphi\|_{\mathcal{C}^1} \|h\|_{L^1} + \varepsilon^{50} \|h\|_{L^1}) \\ &= \int_{\mathbb{T}^2} \varphi P \Pi_0 h + \mathcal{O}(\varepsilon [\ln \varepsilon^{-1}]^2 \|\varphi\|_{\mathcal{C}^1} + \varepsilon^{50}) \|h\|_{L^1}.\end{aligned} \tag{10.5.9}$$

Hence, by the density of $\mathcal{B}_{1,*}$ in L^1 and since Π_0 extends naturally to a bounded operator on L^1, we have

$$\|\Pi_0 - P\Pi_0\|_{L^1 \to (\mathcal{C}^1)'} \le C_\sharp \varepsilon [\ln \varepsilon^{-1}]^2. \tag{10.5.10}$$

It remains to prove equation (4.5.3). For each $\tau > 0$ consider $h \in \mathcal{B}_{*,1}$ such that $\mathcal{L}_{F_\varepsilon} h = \nu h$ with $|\nu| \geq e^{-\varepsilon^\tau}$. Then, for all $\varphi \in \mathcal{C}^1$ and $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\int_{\mathbb{T}^2} \varphi h &= \nu^{-n} \int_{\mathbb{T}^2} \varphi \mathcal{L}_{F_\varepsilon}^n h = \nu^{-n} \int_{\mathbb{T}^2} \varphi \circ F_\varepsilon^n h \\
&= \nu^{-n} \int_{\mathbb{T}^2} \varphi(f_\theta^n \circ Y_n(x), \theta) h(x, \theta) + \mathcal{O}\left(\nu^{-n} n\varepsilon \|\varphi\|_{\mathcal{C}^1} \|h\|_{L^1}\right) \\
&= \nu^{-n} \int_{\mathbb{T}^2} \varphi(f_\theta^n(x), \theta) h(Y_n^{-1}(x), \theta) + \mathcal{O}\left(\nu^{-n} n^2 \varepsilon \|\varphi\|_{\mathcal{C}^1} \|h\|_{L^1}\right) \\
&= \nu^{-n} \int_{\mathbb{T}^2} \varphi(x, \theta) (\mathcal{L}_\theta^n [h_\theta \circ Y_n^{-1}])(x) + \mathcal{O}\left(\nu^{-n} n^2 \varepsilon \|\varphi\|_{\mathcal{C}^1} \|h\|_{L^1}\right)
\end{aligned}
$$

where $h_\theta(x) = h(x, \theta)$. Note that $\|h_\theta \circ Y_n\|_{W^{1,1}} = \|h_\theta\|_{W^{1,1}}$ and $\int_{\mathbb{T}^1} d\theta \|h_\theta\|_{W^{1,1}} \leq \|h\|_{\mathcal{H}^1}$ and, by inequality (10.5.8), we have $\|h\|_{\mathcal{H}^1} \leq C_\sharp \varepsilon^{-3\alpha} \ln \varepsilon^{-1} \|h\|_{L^1}$. Thus

$$
\begin{aligned}
\int_{\mathbb{T}^2} &\varphi(x, \theta) (\mathcal{L}_\theta^n [h_\theta \circ Y_n^{-1}])(x) = \\
&= \int_{\mathbb{T}^2} dx d\theta \varphi(x, \theta) h_*(x, \theta) \int_{\mathbb{T}} dy h(Y_n^{-1}(y), \theta) + \mathcal{O}(\sigma^n \|\varphi\|_{\mathcal{C}^0} \|h\|_{\mathcal{H}^1}) \\
&= \int_{\mathbb{T}^2} \varphi P h + \mathcal{O}(\left[\sigma^n \varepsilon^{-3\alpha} \ln \varepsilon^{-1} + n^2 \varepsilon\right] \|\varphi\|_{\mathcal{C}^0} \|h\|_{L^1}).
\end{aligned}
$$

To conclude we choose $n = c \ln \varepsilon^{-1}$, with c large enough, and obtain

$$
\int_{\mathbb{T}^2} \varphi h = \int_{\mathbb{T}^2} \varphi P h + \mathcal{O}\left(\varepsilon (\ln \varepsilon^{-1})^2 \|\varphi\|_{\mathcal{C}^1} \|h\|_{L^1}\right).
$$

It follows that there exists a $\beta_h \in \mathcal{H}^1(\mathbb{T})$, $\beta_h(\theta) = \int_{\mathbb{T}^1} dy h(y, \theta)$, such that

$$
\|h - h_* \beta_h\|_{(\mathcal{C}^1)'} \leq C_\sharp \varepsilon (\ln \varepsilon^{-1})^2 \|h\|_{L^1}. \qquad \square
$$

Appendix A

Proof of Lemma 5.1.1

We start considering $\varphi, \psi \in \mathcal{C}^\rho(\mathbb{T}^2, \mathbb{R})$. First we prove, by induction on ρ, that

$$\sup_{|\alpha|=\rho} \|\partial^\alpha(\varphi\psi)\|_{\mathcal{C}^0} \le \sum_{k=0}^{\rho} \binom{\rho}{k} 2^{\rho-k} \sup_{|\beta|=\rho-k} \|\partial^\beta \varphi\|_{\mathcal{C}^0} \sup_{|\gamma|=k} \|\partial^\gamma \psi\|_{\mathcal{C}^0}. \tag{A.0.1}$$

Indeed, it is trivial for $\rho = 0$ and

$$\begin{aligned}
&\|\partial_{x_i}\partial^\alpha(\varphi\psi)\|_{\mathcal{C}^0} = \|\partial^\alpha(\psi\partial_{x_i}\varphi + \varphi\partial_{x_i}\psi)\| \\
&\le \sum_{k=0}^{\rho} \binom{\rho}{k} \sup_{|\beta|=\rho-k} \|\partial^\beta \partial_{x_i}\varphi\|_{\mathcal{C}^0} \sup_{|\gamma|=k} \|\partial^\gamma \psi\|_{\mathcal{C}^0} \\
&+ \sum_{k=0}^{\rho} \binom{\rho}{k} \sup_{|\beta|=\rho-k} \|\partial^\beta \partial_{x_i}\psi\|_{\mathcal{C}^0} \sup_{|\gamma|=k} \|\partial^\gamma \varphi\|_{\mathcal{C}^0} \\
&\le \sum_{k=0}^{\rho} \binom{\rho}{k} \sup_{|\beta|=\rho-k} \|\partial^\beta \varphi\|_{\mathcal{C}^0} \sup_{|\gamma|=k} \|\partial^\gamma \psi\|_{\mathcal{C}^0} \\
&+ \sum_{k=0}^{\rho+1} \binom{\rho}{\rho+1-k} \sup_{|\beta|=\rho-k} \|\partial^\beta \varphi\|_{\mathcal{C}^0} \sup_{|\gamma|=k} \|\partial^\gamma \psi\|_{\mathcal{C}^0},
\end{aligned}$$

from which (A.0.1) follows taking the sup on α, i and since $\binom{\rho}{k} + \binom{\rho}{\rho+1-k} = \binom{\rho+1}{k}$. We then have the first statement of the Lemma, indeed

$$\begin{aligned}
\|\varphi\psi\|_{\mathcal{C}^r} &\le \sum_{k=0}^{\rho} 2^{\rho-k} \sum_{j=0}^{k} \binom{k}{j} \sup_{|\beta|=k-j} \|\partial^\beta \varphi\|_{\mathcal{C}^0} \sup_{|\gamma|=j} \|\partial^\gamma \psi\|_{\mathcal{C}^0} \\
&\le \sum_{j=0}^{\rho} \sum_{l=0}^{\rho-j} \binom{\rho}{j} 2^{\rho-j-l} \sup_{|\beta|=l} \|\partial^\beta \varphi\|_{\mathcal{C}^0} \sup_{|\gamma|=j} \|\partial^\gamma \psi\|_{\mathcal{C}^0} \le \|\varphi\|_{\mathcal{C}^r} \|\psi\|_{\mathcal{C}^r}
\end{aligned}$$

since $\binom{\rho}{j} \le 2^\rho$. The extension to function with values in the matrices follows trivially since we have chosen a norm in which the matrices form a norm algebra. To prove the second statement we proceed again by induction on ρ. The case $\rho = 0$ is immediate since $\mathcal{K}_{0,0}$ contains only the zero string. Let us assume that the statement is true for every $k \le \rho$ and prove it for $\rho+1$. By equation (A.0.1) and the inductive hypothesis (5.1.3), we have, for each $|\alpha| = \rho+1$,

$$\begin{aligned}
|\partial^\alpha(\varphi\circ\psi)| &\le C_\sharp \sup_{|\beta|=\rho} \sup_{|\tau_1|,|\tau_2|=1} \left|\partial^\beta \left[(\partial^{\tau_1}\varphi)\circ\psi\cdot\partial^{\tau_2}\psi\right]\right| \\
&\le C_\rho \sup_{|\tau_1|,|\tau_2|=1} \sup_{|\alpha_0|+|\alpha_1|=\rho} \|\partial^{\alpha_0}\left[(\partial^{\tau_1}\varphi)\circ\psi\right]\|_{\mathcal{C}^0}\|\partial^{\alpha_1}\partial^{\tau_2}\psi\|_{\mathcal{C}^0} \\
&\le C_\rho \sup_{|\tau_1|=1} \sup_{\alpha_0\le\rho} \|(\partial^{\tau_1}\varphi)\circ\psi\|_{\mathcal{C}^{\alpha_0}}\|D\psi\|_{\mathcal{C}^{\rho-\alpha_0}} \\
&\le C_\rho C_\rho^\star \sup_{\alpha_0\le\rho}\sum_{s=0}^{\alpha_0}\|\varphi\|_{\mathcal{C}^{s+1}}\sum_{k\in\mathcal{K}_{\alpha_0,s}}\prod_{l\in\mathbb{N}}\|D\psi\|_{\mathcal{C}^{l-1}}^{k_l}\cdot\|D\psi\|_{\mathcal{C}^{\rho-\alpha_0}} \\
&\le C_\rho C_\rho^\star \sup_{\alpha_0\le\rho}\sum_{s=0}^{\alpha_0}\|\varphi\|_{\mathcal{C}^{s+1}}\sum_{k\in\mathcal{K}_{\rho+1,s+1}}\prod_{l\in\mathbb{N}}\|D\psi\|_{\mathcal{C}^{l-1}}^{k_l} \\
&\le C_\rho C_\rho^\star \sum_{s=0}^{\rho+1}\|\varphi\|_{\mathcal{C}^{s}}\sum_{k\in\mathcal{K}_{\rho+1,s}}\prod_{l\in\mathbb{N}}\|D\psi\|_{\mathcal{C}^{l-1}}^{k_l}.
\end{aligned}$$

The result follows by choosing $C_{\rho+1}^\star$ large enough. □

Appendix B

Proof of Lemma 6.2.1

This appendix is devoted to the proof of Lemma 6.2.1.

As usual we use the notation $F^k\hat{\nu}_k = \gamma \circ h_k$, $F^k\nu_k = \gamma$. As the computation is local it suffices to consider $p_n \in \hat{\nu}_n$ and $p_0 \in \gamma$ such that $F^n(p_n) = p_0$. Let $p_k = F^{n-k}p_n$. To ease notation we use a translation to reparmetrize the curves so that $\nu_k(0) = \hat{\nu}_k(0) = p_k$, note that $h_k(0) = 0$. Before discussing the splitting of the vector field we need some notations and few estimates.

It is convenient to perform the changes of variables $\phi_k^{-1}(x, y) = (x, 0) + \hat{\nu}_k(y)$ and set

$$\widetilde{F}^k = \phi_0 \circ F^k \circ \phi_k^{-1} \ ; \quad \widetilde{F}_k = \phi_{k-1} \circ F \circ \phi_k^{-1}$$

Note that $\widetilde{F}^k = \widetilde{F}_k \circ \cdots \circ \widetilde{F}_1$ and $\widetilde{F}^n(0, y) = \phi_0 \circ F^n(\hat{\nu}_n(y)) = \phi_0(\gamma \circ h_n(y)) = (0, h_n(y))$, this implies that

$$D_{(0,y)}\widetilde{F}^n = \begin{pmatrix} a^n(y) & 0 \\ c^n(y) & d^n(y) \end{pmatrix} \ ; \quad D_{(0,y)}\widetilde{F}_k = \begin{pmatrix} a_k(y) & 0 \\ c_k(y) & d_k(y) \end{pmatrix} \ ; \quad D\phi_k^{-1} = \begin{pmatrix} 1 & (\hat{\nu}_k')_1 \\ 0 & 1 \end{pmatrix},$$

with $d^n(y) = h_n'(y)$ and $d_k(y) = h_k^*(y)$. Thus, we have the estimates on the $\mathcal{C}^\rho$ norms of d^k by Lemma 5.4.1, also the changes of coordinates ϕ_k have uniformly bounded $\mathcal{C}^\rho$ norms. From the above we easily get the formulae:

$$a^{k+1}(y) = a^k(y)a_{k+1}(h_k(y)) \tag{B.0.1}$$

$$d^{k+1}(y) = d_{k+1}(h_k(y))d^k(y) \tag{B.0.2}$$

$$c^k(y) = \sum_{j=1}^{k} d_k(h_{k-1}(y)) \cdots d_{j+1}(h_j(y))c_j(h_{j-1}(y))a_{j-1}(h_{j-2}(y)) \cdots a_1(y). \tag{B.0.3}$$

Moreover,

$$DF^k = \begin{pmatrix} a^k + (\hat{\nu}_0')_1 c^k & (\hat{\nu}_0')_1 d^k - (\hat{\nu}_k')_1 \left[a^k + (\hat{\nu}_0')_1 c^k\right] \\ c^k & d^k - (\hat{\nu}_k')_1 c^k \end{pmatrix}$$

which, setting $y_k = h_k(y)$, yields the alternative representations and estimates

$$\begin{aligned} &c_k(y_{k-1}) = \langle e_2, D_{(0,y_{k-1})}Fe_1\rangle \\ &a_k(y_{k-1}) = \langle e_1, D_{(0,y_{k-1})}Fe_1\rangle - \nu'_{k-1}(y_{k-1})_1\langle e_2, D_{(0,y_{k-1})}Fe_1\rangle \\ &|c^k(y)| = |\langle e_2, D_{(0,y)}F^k e_1\rangle| \le \lambda_k^+\chi_u \\ &\frac{\lambda_k^-}{\sqrt{1+\chi_u^2}} - \chi_c\chi_u\lambda_k^+ \le |a^k(y)| \le \lambda_k^+ + \chi_c\chi_u\lambda_k^+ \end{aligned} \tag{B.0.4}$$

Also, for further use,

$$\left(D\widetilde{F}^k\right)^{-1} = \begin{pmatrix} a^k(y)^{-1} & 0 \\ -d^k(y)^{-1}a^k(y)^{-1}c^k(y) & d^k(y)^{-1}\end{pmatrix}. \tag{B.0.5}$$

We are now ready to describe the splitting of the vector field. We do it in the new coordinates. Consider the subspace $E_n(y) = \{(\eta, u^n(y)\eta)\}_{\eta\in\mathbb{R}}$, where $u_n(y) = a^n(y)^{-1}c^n(y)$, which is a $\mathcal{C}^r$ approximation of the unstable direction. Given a vector $v \in \mathbb{R}^2$ let us call $\tilde{v} = D\phi_0 v$ the vector in the new coordinates. Next, we decompose a vector $\tilde{v}$ as

$$\tilde{v} = (1, u_n \circ \mathfrak{h}_n)\tilde{v}_1 + (\tilde{v}_2 - \tilde{v}_1 u_n \circ \mathfrak{h}_n)e_2$$

where $\mathfrak{h}_n \circ \widetilde{F}^n(0,y) = (0,y)$. Thus, setting $V(t) = v_1(\gamma(t)) - \gamma'(t)_1 v_2(\gamma(t))$, we have the decomposition (6.2.2) with

$$\begin{aligned} v^u(\gamma(t)) &= V(t)(1 + \gamma'(t)_1 u_n \circ \mathfrak{h}_n(0,t), u_n \circ \mathfrak{h}_n(0,t)) \\ v^c(\gamma(t)) &= [v_2(\gamma(t)) - u_n \circ \mathfrak{h}_n(0,t)V(t)]\gamma'(t), \end{aligned} \tag{B.0.6}$$

which gives, in particular, $v^c(\gamma(t)) = g(t)\gamma'(t)$ with $g(t) = v_2(\gamma(t)) - u_n \circ \mathfrak{h}_n(0,t)V(t)$.

To extend the above decomposition in a neighborhood of γ we will proceed as in [43, Lemma 6.5].[1] First, we compute the derivatives along the curve, to this end note that in the new coordinates $t = y_n$. Differentiating (B.0.1) we have

$$\partial_y a^k(y)^{-1} = \left[\partial_y a^{k-1}(y)^{-1}\right] a_k(y_{k-1})^{-1} + a^{k-1}(y)^{-1}\partial_{y_{k-1}} a_k(y_{k-1})^{-1}\tilde{d}^{k-1}, \tag{B.0.7}$$

and, by (B.0.4) and and Lemma 5.4.1,

$$\begin{aligned} &|\partial_{y_{k-1}} a_k(y_{k-1})| \le C_\sharp(1 + \|\nu''_{k-1}\|) \le C_\sharp(1 + \mathbb{c}) \\ &\|\partial_y a_k\|_{\mathcal{C}^\rho} \le C_\sharp\|\nu_{k-1}\|_{\mathcal{C}^{\rho+1}} \le C_\sharp \mathbb{c}^{\rho!}. \end{aligned}$$

Next, using (B.0.7), we prove by induction that $\|(a^n)^{-1}\|_{\mathcal{C}^\rho} \le C_\sharp\lambda_-^{-n}\mathbb{c}^{\rho\rho!}C_{\mu,n}^{\rho a_\rho}\mu^{\rho\rho!n}$:[2]

$$\begin{aligned} \|[a^n]^{-1}\|_{\mathcal{C}^\rho} &\le C_\sharp\lambda_-^{-n} + \lambda^{-1}\|[a^{n-1}]^{-1}\|_{\mathcal{C}^\rho} + C_\sharp\|[a^{n-1}]^{-1}\|_{\mathcal{C}^{\rho-1}}\mathbb{c}^{\rho!}C_{\mu,n-1}^{a_\rho}\mu^{\rho!(n-1)} \\ &\le C_\sharp\lambda_-^{-n} + \mathbb{c}^{\rho!}C_\sharp\sum_{j=0}^{n-1}\lambda_-^{j-n}\|[a^j]^{-1}\|_{\mathcal{C}^{\rho-1}}C_{\mu,j}^{a_\rho}\mu^{\rho!j} \\ &\le C_\sharp\lambda_-^{-n}\mathbb{c}^{\rho\rho!}C_{\mu,n}^{\rho a_\rho}\mu^{\rho\rho!n}. \end{aligned} \tag{B.0.8}$$

[1]In the mentioned paper the authors need more regularity for the extended vector field. Here it is enough to obtain a vector field which is $\mathcal{C}^\rho$.

[2]Here a_ρ is the one given by Lemma 5.4.1.

To compute $\|(d^n)^{-1}\|_{\mathcal{C}^\rho}$ we can use formula (5.1.3) and recall (5.4.1) and (5.4.9):

$$\|(d^n)^{-1}\|_{\mathcal{C}^\rho} = \|(h_n')^{-1}\|_{\mathcal{C}^\rho} \leq C_\sharp \mu^{(\rho+1)n} C_{\mu,n}^{a_{\rho+1}} \mu^{(\rho+1)!n} = C_\sharp C_{\mu,n}^{a_{\rho+1}} \mu^{(\rho+1)(\rho!+1)n}. \tag{B.0.9}$$

Next, by (B.0.1), (B.0.2) and (B.0.3) we have

$$[a^n(y)]^{-1}c^n(y) = \sum_{j=1}^n d_n(h_{n-1}(y)) \cdots d_{j+1}(h_j(y)) c_j(h_{j-1}(y)) \cdot \\ [a_n(h_{n-1}(y)) \cdots a_j(h_{j-1}(y))]^{-1},$$

$$[d^n(y)a^n(y)]^{-1}c^n(y) = \sum_{j=1}^n [d_{j-1}(h_{j-2}(y)) \cdots d_1(y))]^{-1} c_j(h_{j-1}(y)) \cdot \\ [a_n(h_{n-1}(y)) \cdots a_j(h_{j-1}(y))]^{-1}.$$

Hence, by (B.0.8), (B.0.9) and the first of (B.0.4), we obtain, using (A.0.1),

$$\begin{aligned} \|[d^n a^n]^{-1} c^n\|_{\mathcal{C}^\rho} &\leq C_\sharp \mathbb{c}^{\rho\rho!} C_{\mu,n}^{2\rho a_\rho+1} \mu^{(\rho+1)(2\rho!+1)n} \\ \|[a^n]^{-1} c^n\|_{\mathcal{C}^\rho} &\leq C_\sharp \mathbb{c}^{\rho!} C_{\mu,n}^{a_\rho} \mu^{\rho!n}. \end{aligned} \tag{B.0.10}$$

We are ready to conclude. Since

$$(D_{\hat\nu_n(y)} F^n)^{-1} = D_{(0,y)} \phi_n^{-1} (D_{(0,y)} \widetilde{F}^n)^{-1} D_{\gamma \circ h_n(y)} \phi_0,$$

by (B.0.6) and (B.0.5) it follows

$$\begin{aligned} &(D_{\hat\nu_n(y)} F^n)^{-1} v^u(\gamma \circ h_n(y)) = V(h_n(y)) \left(a^n(y)^{-1}, 0\right), \\ &(D_{\hat\nu_n(y)} F^n)^{-1} v^c(\gamma \circ h_n(y)) = d^n(y)^{-1} \cdot [v_2 - u_n v_1] \circ \gamma(h_n(y)) \big((\hat\nu_n')_1(y), 1\big). \end{aligned}$$

Recalling that $u_n(y) = a^n(y)^{-1}c^n(y)$, by (B.0.8), (B.0.9), (B.0.10), and since $\gamma \in \Gamma(\mathbb{c})$ and $\|v\|_{\mathcal{C}^r} \leq 1$, we have the result for the vector field along the curve. Finally, we extend v^u to a neighborhood of γ. It turns out the be more convenient to define first the extension

$$w(x, y) = F^{n^*} v^u(\hat\nu_n(y))$$

then $\hat v^u = \mathfrak{h}_n^* w$ and $F^{n^*} \hat v^u = w$. By these definitions it follows

$$\begin{aligned} &\|F^{n^*} \hat v^u\|_{\mathcal{C}^\rho(N(\nu))} = \|F^{n^*} v^u\|_{\mathcal{C}^\rho_\nu} \leq \lambda_-^{-n} C_{\mu,n}^{\rho a_\rho} \mu^{\rho\rho!n} \\ &\|\hat v^u\|_{\mathcal{C}^\rho(M'(\gamma))} \leq C_n. \end{aligned}$$

The definition of $\hat v^c$ and relative estimates are analogous. □

Appendix C

The space $\mathcal{H}^s$

Let $u \in \mathcal{C}^\infty(\mathbb{T}^2)$. The *Fourier Transform* of u and its inverse are

$$\mathcal{F}u(\xi) = \int_{\mathbb{T}^2} e^{-i2\pi x\xi} u(x)dx, \quad \xi \in \mathbb{Z}^2, \tag{C.0.1}$$

$$u(x) = \sum_{\xi \in \mathbb{Z}^2} \mathcal{F}u(\xi) e^{i2\pi x\xi}, \quad x \in \mathbb{T}^2. \tag{C.0.2}$$

Then $\mathcal{H}^s$ is the completion of $\mathcal{C}^\infty(\mathbb{T}^2)$ with respect to the inner product

$$\langle u, v\rangle_s = \sum_{\xi \in \mathbb{Z}^2} \langle \xi \rangle^{2s} \mathcal{F}u(\xi)\overline{\mathcal{F}v(\xi)}, \quad \langle \xi \rangle := \sqrt{1 + \|\xi\|^2}. \tag{C.0.3}$$

Notice that, by formula (7.9.2) of [38], there is $C > 0$ such that

$$C^{-1} \sum_{\gamma+\beta=s} C_{\gamma,\beta} \|\partial_{x_1}^\gamma \partial_{x_2}^\beta u\|_{L^2}^2 \leq \|u\|_{\mathcal{H}^s}^2 \leq C \sum_{\gamma+\beta=s} C_{\gamma,\beta} \|\partial_{x_1}^\gamma \partial_{x_2}^\beta u\|_{L^2}^2. \tag{C.0.4}$$

Below we recall some standard important results about the space $\mathcal{H}^s$:[1]

- $\mathcal{C}^\infty(\mathbb{T}^2) \subset H^s(\mathbb{T}^2) \subset \mathcal{C}^\infty(\mathbb{T}^2)'$
- $H^0(\mathbb{T}^2) = L^2(\mathbb{T}^2)$;
- $H^s(\mathbb{T}^2) = \{u \in L^2(\mathbb{T}^2) : \partial^\beta \varphi \in L^2(\mathbb{T}^2), \text{ for any } |\beta| \leqslant s\}$;
- $H^s(\mathbb{T}^2) \subset H^{s'}(\mathbb{T}^2)$ if $s > s'$;
- $C^s(\mathbb{T}^2) \subset H^s(\mathbb{T}^2)$, and if $s > 1$, then $H^s(\mathbb{T}^2) \subset C^{s-1-\gamma}(\mathbb{T}^2)$, for each $\gamma > 0$.

[1] We denote with $\mathcal{C}^\infty(\mathbb{T}^2)'$ the dual space of $\mathcal{C}^\infty(\mathbb{T}^2)$.

Lemma C.0.1. *For every* $\varsigma \in (0,1)$ *and* $1 \le s < r$ *there exists constants* C_s *such that*

$$\|u\|^2_{\mathcal{H}^{s-1}} \le \varsigma \|u\|^2_{\mathcal{H}^s} + \frac{C_s}{\varsigma}\|u\|^2_{L^1}, \quad u \in \mathcal{C}^r(\mathbb{T}^2).$$

Proof. By definition of the norm we have, for all $\tau \in (1,2)$,

$$\|u\|^2_{\mathcal{H}^{s-1}} = \sum_{\xi\in\mathbb{Z}^2} |\mathcal{F}u(\xi)|^2 \langle\xi\rangle^{2(s-1)} = \sum_{\xi\in\mathbb{Z}^2} |\mathcal{F}u(\xi)|^2 \langle\xi\rangle^{2s-2+\tau}\langle\xi\rangle^{-\tau} \tag{C.0.5}$$

By Young inequality $ab \le \frac{\varsigma a^p}{p} + \frac{\varsigma^{-\frac{p}{q}} b^q}{q}$, for every $\varsigma > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. We apply this with $a = \langle\xi\rangle^{2s-2+\tau}$, $b = \langle\xi\rangle^{-\tau}$ and $p = \frac{2s}{(2s-2+\tau)}, q = \frac{2s}{2-\tau}$ to obtain:

$$\langle\xi\rangle^{2s-2+\tau}\langle\xi\rangle^{-\tau} \le \left(1 - \frac{2-\tau}{2s}\right)\varsigma\langle\xi\rangle^{2s} + \varsigma^{-1}\frac{(2-\tau)\langle\xi\rangle^{-\frac{2s\tau}{2-\tau}}}{2s}.$$

Using this fact in (C.0.5) and recalling that $\|\mathcal{F}u\|_\infty \le C\|u\|_{L^1}$, we get

$$\|u\|^2_{\mathcal{H}^{s-1}} \le \varsigma \sum_{\xi\in\mathbb{Z}^2} |\mathcal{F}u(\xi)|^2\langle\xi\rangle^{2s} + \frac{C_s}{\varsigma}\|\mathcal{F}u\|^2_\infty \le \varsigma\|u\|^2_{\mathcal{H}^s} + \frac{C_s}{\varsigma}\|u\|^2_{L^1}.$$

□

Appendix D

Vector Field regularity

This appendix is devoted to proving the following regularity results on the iteration of a vector field.

Lemma D.0.1. *Let* $\varepsilon_0 \in (0,1]$*,* $A \in [0,1/2]$*,* $B > 0$ *and* $u, u' \in \mathcal{C}^1(\mathbb{T}^2, \mathbb{R})$ *such that* $\|u\|_\infty, \|u'\|_\infty \le A\varepsilon_0^{-1}$ *and* $\|\nabla u\|_\infty, \|\nabla u'\|_\infty \le B\varepsilon_0^{-1}$*. Consider a family of vertically partially hyperbolic map* F_ε*,* $\varepsilon \le \varepsilon_0$ *such that*

$$\begin{aligned} &\left\| \frac{\partial_\theta f(p)}{\partial_x f(p)} \right\|_\infty \le 1 \\ &\partial_x f(p) \left[1 - A \left\| \frac{\partial_\theta f(p)}{\partial_x f(p)} \right\|_\infty \right] \ge 2(1 + \varepsilon_0 \|\partial_x \omega\|_\infty). \end{aligned} \tag{D.0.1}$$

For each $\mathfrak{h} \in \mathfrak{H}^\infty$ *and* $k \le n \in \mathbb{N}$*, we define the sequence of functions*[1]

$$\begin{aligned} \bar{u}_0(p, \varepsilon) &= u(\mathfrak{h}_n(p)) \\ \bar{u}_k(p, \varepsilon) &= \pi_2 \circ \Phi_\varepsilon^k(\mathfrak{h}_n(p), u(\mathfrak{h}_n(p))), \end{aligned}$$

and similarly for $\bar{u}'_k$*. Then, for each* $p, p' \in \mathbb{T}^2$ *and* $\varepsilon, \varepsilon' < \varepsilon_0$*,*

$$\begin{aligned} |\bar{u}_n(p, \varepsilon) - \bar{u}_n(p', \varepsilon')| \le C_\sharp e^{4A} \mu^{3n} \Big\{ &\lambda_n^+(\mathfrak{h}_n(p))^{-1} \|u - u'\|_\infty \\ &+ (\|\omega\|_{\mathcal{C}^2} + \mu^{2n} \lambda_n^+(\mathfrak{h}_n(p))^{-1} C_{\mu,n} |u'|) \|p - p'\| \\ &+ \left[1 + \lambda_n^+(\mathfrak{h}_n(p))^{-1} |u'|^2 \right] |\varepsilon - \varepsilon'| \Big\}. \end{aligned} \tag{D.0.2}$$

Proof. Let $p_k(p, \varepsilon) = \mathfrak{h}_k(p)$, for $\mathfrak{h} \in \mathcal{H}^\infty$, $p \in \mathbb{T}^2$. By (5.3.6) (or see [20] for details) we have

$$\|\partial_p p_k\| \le \|(D_{\mathfrak{h}_k(p)} F_\varepsilon^k)^{-1}\| \le C_\sharp \mu^k \le C_\sharp e^{c_\sharp \varepsilon k}. \tag{D.0.3}$$

For each $u > 0$ and for $k \le n$ let

$$\begin{aligned} \lambda(p, \varepsilon) &= \frac{|\partial_x f(p)|}{1 + \varepsilon \left(\|\partial_\theta \omega\|_\infty + \|\partial_x \omega\|_\infty \right)} \ge |\partial_x f(p)| \mu^{-1} \\ u_k(p, \varepsilon, u) &= \Xi_\varepsilon(p_{n-k+1}(p, \varepsilon), u_{k-1}(p, \varepsilon, u)), \end{aligned} \tag{D.0.4}$$

[1] See (10.2.2) for the definition of Φ_ε^n.

where in the first line we have used Remark 4.4.1. Note that $\bar{u}_n(p,\varepsilon) = u_n(p,\varepsilon,u_0(p,\varepsilon))$.

Using (10.1.2) and (10.1.4) a direct computation yields, for $|u| \leq A\varepsilon_0^{-1}$,

$$
\begin{aligned}
|\Xi_\varepsilon(p,u)| &\leq \frac{|u|\,(1+\varepsilon\|\partial_\theta\omega\|_\infty)}{|\partial_x f(p)|\left[1-\varepsilon|u|\|\frac{\partial_\theta f(p)}{\partial_x f(p)}\|_\infty\right]} + \frac{\|\partial_x\omega\|_\infty}{\partial_x f(p)\left[1-A\|\frac{\partial_\theta f(p)}{\partial_x f(p)}\|_\infty\right]}\\
&\leq \frac{1}{|\partial_x f(p)|}\mu e^{2\varepsilon_0|u|}|u| + \frac{1}{2}\|\partial_x\omega\|_\infty\\
|\partial_u\Xi_\varepsilon(p,u)| &\leq \frac{1}{\lambda(p,\varepsilon)}\left[1-\varepsilon\left|u\frac{\partial_\theta f(p)}{\partial_x f(p)}\right|\right]^{-2}\\
\|\partial_p\Xi_\varepsilon(p,u)\| &\leq C_\sharp(\|\omega\|_{\mathcal{C}^2}+|u|)\\
|\partial_\varepsilon\Xi_\varepsilon(p,u)| &\leq C_\sharp(1+|u|)|u|.
\end{aligned}
\tag{D.0.5}
$$

The first line of the (D.0.5) and the second of (D.0.1) imply

$$
|u_k(p,\varepsilon,u)| \leq 2^{-k}|u| + \|\partial_x\omega\|_\infty.
$$

We can get a sharper bound defining

$$
\begin{aligned}
&\Lambda_{k,j}(p) := \prod_{i=k+1}^{j}\lambda(p_i,\varepsilon)\ ; \quad \overline{\Lambda}_{k,j}(p) := \prod_{i=k+1}^{j}|\partial_x f(p_i)|\\
&\Delta := \|\partial_x\omega\|_\infty\left\|\frac{\partial_\theta f(p)}{\partial_x f(p)}\right\|_\infty,
\end{aligned}
$$

then

$$
|u_k(p,\varepsilon,u)| \leq \Lambda_{n-k,n}(p)^{-1}|u| + \|\partial_x\omega\|_\infty. \tag{D.0.6}
$$

Moreover, setting $u_j = u_j(p,u,\varepsilon)$, $u_j' = u_j(p',u',\varepsilon')$, with $|u|,|u'| \leq \frac{A}{\varepsilon_0}$, and recalling (D.0.3), (D.0.4), (D.0.6), we have

$$
\begin{aligned}
&|u_{k+1}(p,\varepsilon,u) - u_{k+1}(p',\varepsilon',u')| = |\Xi_\varepsilon(p_{n-k},u_k) - \Xi_{\varepsilon'}(p'_{n-k},u'_k)|\\
&\leq C_\sharp(\|\omega\|_{\mathcal{C}^2}+|u'_k|)\|p_{n-k}-p'_{n-k}\| + C_\sharp(1+|u'_k|)|u'_k||\varepsilon-\varepsilon'|\\
&\quad + \Lambda_{n-k-1,n-k}(p)^{-1}e^{2^{-k+1}A+2\varepsilon_0\Delta}|u_k-u'_k|\\
&\leq C_\sharp(\|\omega\|_{\mathcal{C}^2}+\overline{\Lambda}_{n-k,n}(p')^{-1}\mu^k|u'|)\big[\mu^{n-k}\|p-p'\|\\
&\quad + (1+\overline{\Lambda}_{n-k,n}(p')^{-1}\mu^k|u'|)|\varepsilon-\varepsilon'|\big]\\
&\quad + \Lambda_{n-k-1,n-k}(p)e^{2^{-k+1}A+2\varepsilon_0\Delta}|u_k-u'_k|.
\end{aligned}
\tag{D.0.7}
$$

We can then iterate the above equation and obtain

$$
\begin{aligned}
&|u_n(p,\varepsilon,u) - u_n(p',\varepsilon',u')| = \overline{\Lambda}_{0,n}(p)^{-1}\mu^n e^{4A+2n\varepsilon_0\Delta}|u-u'|\\
&+ C_\sharp\sum_{k=0}^{n-1}\overline{\Lambda}_{0,n-k}(p)^{-1}\mu^{n-k}e^{4A+2\varepsilon_0(n-k)\Delta}(\|\omega\|_{\mathcal{C}^2}+\overline{\Lambda}_{n-k,n}(p')^{-1}\mu^k|u'|)\mu^{n-k}\|p-p'\|\\
&+ C_\sharp\sum_{k=0}^{n-1}\overline{\Lambda}_{0,n-k}(p)^{-1}\mu^{n-k}e^{4A+2\varepsilon_0(n-k)\Delta}(1+\overline{\Lambda}_{n-k,n}(p')^{-2}\mu^k|u'|^2)|\varepsilon-\varepsilon'|.
\end{aligned}
$$

In addition equations (10.1.1) and (5.3.5) imply

$$\overline{\Lambda}_{j,n}(p) \geq C_\sharp \lambda^+_{n-j}(p_n)$$

$$|\partial_p \overline{\Lambda}_{j,n}(p)| \leq \sum_{j=l+1}^{n} \left| \overline{\Lambda}_{l,n}(p) \frac{\partial_x^2 f(p_l)}{[\partial_x f(p_l)]^2} \overline{\Lambda}_{j,l-1}(p) \right| \|\partial_p p_l\| \leq C_\sharp C_{\mu,n} \mu^n \overline{\Lambda}_{j,n}(p).$$

Thus,

$$|u_n(p,\varepsilon,u) - u_n(p',\varepsilon',u')| \leq C_\sharp e^{4A+2n\varepsilon_0\Delta} \mu^n \Big\{ \lambda^+_n(p_n)^{-1}|u-u'|$$

$$+ (\|\omega\|_{\mathcal{C}^2} + \mu^{2n}\lambda^+_n(p_n)^{-1} C_{\mu,n}|u'|)\|p-p'\| + \left[1 + \lambda^+_n(p_n)^{-1}|u'|^2\right] |\varepsilon - \varepsilon'| \Big\}.$$

The Lemma follows since, by Remark 4.4.1 and our hypotheses, $e^{\varepsilon_0 \Delta} \leq \mu$. □

Appendix E

Extension of curves

In this section we explain how to extend a segment to a close curve of homotopy class $(0, 1)$ with precise dynamical properties and explicit bounds on the derivatives.

Lemma E.0.1. *There exist constants δ_0, $C_{n_0,j} > 0$ and $L_\star \geq 1$ such that for each line segment $\gamma(t) = \gamma(0) + (1, v)t$ of length $\delta \leq \delta_0$ and $n_0 \in \mathbb{N}$ such that $\gamma'(t) \notin \cup_{z \in F^{n_0}(\gamma(t))} D_z F^{n_0} \mathbf{C}_u$ we can extend γ to a closed curve $\tilde{\gamma}$, parametrized by arclength, of homotopy class $(0, 1)$ with the following properties:*

- *let $\gamma_-(t) = \gamma(0) + \frac{1}{2}e_1 + e_2 t$, then for each $\mathfrak{h} \in \mathfrak{H}^\infty_{\gamma_-}$ and $k \in \mathbb{N}$ we have $\tilde{\gamma} \in \operatorname{Dom}(\mathfrak{h}_k)$ and $\mathfrak{h}_k \circ \tilde{\gamma}$ is a closed curve in the homotopy class $(0, 1)$.*
- $\vartheta_\gamma \leq \vartheta_{\tilde{\gamma}}$.
- *For all $p \in \mathbb{T}^2$ and $m \geq n_0 \in \mathbb{N} \cup \{0\}$, if $D_p \mathfrak{h}_{n_0} \gamma' \notin \mathbf{C}_u$ and $D_p \mathfrak{h}_m \gamma' \in \mathbf{C}_{\epsilon,c}$, then $D_p \mathfrak{h}_{n_0} \tilde{\gamma}' \notin \mathbf{C}_{C_\sharp \epsilon, u}$ and $D_p \mathfrak{h}_m \tilde{\gamma}' \in \mathbf{C}_c$.*
- *For each $j \in \{1, \dots, r-1\}$ and $t \in \mathbb{R}$,*

$$\|\tilde{\gamma}^{(j+1)}(t)\| \leq C_{n_0,j} \frac{(L_\star \{L_\star, 1\}^+ \epsilon^{-1} \mu^m)^j}{(\chi_u + |\pi_2(\tilde{\gamma}'(t))|)^j} := C_{n_0,j} \Delta^j_{\tilde{\gamma}}. \tag{E.0.1}$$

Moreover, if the conditions of Lemma D.0.1 are satisfied, then (E.0.1) *holds true with*

$$\begin{aligned} L_\star(n) &= \sup_{|v| \leq 1} L_\star(n, v), \\ L_\star(n, v) &= C_\sharp \bar{\kappa}^{-c_\sharp \ln \mu} C_{\mu, n_0} (\|\omega\|_{\mathcal{C}^2} + \bar{\kappa}^{1 - c_\sharp \ln \mu}) \; ; \quad \bar{\kappa} = |v| + \chi_u. \end{aligned} \tag{E.0.2}$$

Proof. By an isometric change of variables we can assume, without loss of generality, that $\gamma(0) = 0$. Hence $\gamma(t) = (1, v)t$ for $t \in [-\delta, \delta]$ and $\gamma'(t) = (1, v) =: \bar{v}$. Note that we can assume $|v| \leq 1$ since otherwise the Lemma is trivial.

Before getting to the extension per se, we need some results on the dynamics of the tangent vectors seen as elements of a projective space. We write a vector outside the central cone as $(1, \zeta)$, so ζ can be interpreted as a projective coordinate. Then, in analogy with (4.4.2), we have, for each $p \in \mathbb{T}^2$ and $\zeta \in \mathbb{R}$,

$$
\begin{aligned}
D_pF(1,\zeta) &= (\partial_x F_1 + \partial_\theta F_1 \zeta)(1, \Xi(p,\zeta)) \\
\Xi(p,\zeta) &= \frac{\partial_x F_2 + \partial_\theta F_2 \zeta}{\partial_x F_1 + \partial_\theta F_1 \zeta}.
\end{aligned}
$$

Also, computing as in (4.4.3),

$$
\partial_\zeta \Xi(p,\zeta) = \frac{\det(D_pF)}{(\partial_x F_1 + \partial_\theta F_1 \zeta)^2}.
$$

Next, for each $q \in \mathbb{T}^2$, let $q_n = F^{n_0}(q)$, $z_0(\zeta) = \zeta$, $z_1(q,\zeta) = \Xi(q, z_0(\zeta))$ and, for $j \geq 1$, $z_{j+1}(q_j, \zeta) = \Xi(q_j, z_j(q_{j-1}, \zeta))$. In particular, if $\mathfrak{h} \in \mathfrak{H}^\infty$, $p \in \mathbb{T}^2$ and $\Gamma_j(p) = D_{\mathfrak{h}_j(p)} F^j \mathbf{C}_c$, then $\Gamma_j(p) = \{(1, \bar{z}_j(p,\zeta)) \ : \ |\zeta| \leq \chi_c\}$ where $\bar{z}_j(p,\zeta) := z_j(\mathfrak{h}_j(p), \zeta)$. Note that for all j such that $\bar{z}_j \notin \mathbf{C}_u$ we have

$$
|\bar{z}_j(p,\chi_c)| \leq C_\sharp \lambda_j^-(\mathfrak{h}_j(p))^{-1} \chi_c.
$$

In the following we need an estimate of $|\bar{z}_j(p, \pm\chi_c) - \bar{z}_j(p, \pm\chi_c(1-\epsilon))|$. Since

$$
\partial_\zeta \bar{z}_j(p,\zeta) = \partial_{\bar{z}} \Xi(\mathfrak{h}_j(p), \bar{z}_{j-1}(p,\zeta)) \partial_\zeta \bar{z}_{j-1}(p,\zeta)
$$

iterating the above identities and recalling Propositions 5.3.1, 5.3.3 we have

$$
C_\sharp \frac{\mu^{-j}}{\lambda_j^-(\mathfrak{h}_j(p))} \leq |\partial_\zeta \bar{z}_j(p,\zeta)| \leq C_\sharp \frac{\lambda_j^+(\mathfrak{h}_j(p))\mu^j}{\lambda_j^-(\mathfrak{h}_j(p))^2} \leq C_\sharp \mu^{n_0} \lambda_-^{-n_0}.
$$

It follows that $|\bar{z}_j(p, \pm\chi_c) - \bar{z}_j(p, \pm\chi_c(1-\epsilon))| \geq C_\sharp \epsilon \mu^{-j} |\bar{z}_j(p, \pm\chi_c)|$.

If, for some $v_0 > 0$, $v_0 \geq \bar{z}_j(p, \chi_c(1-\epsilon)) \geq \bar{z}_j(p,\chi_c) \geq 0$, then either $\bar{z}_j(p,\chi_c) \leq \frac{1}{2} v_0$, then $|\bar{z}_j(p,\chi_c) - v_0| \geq \frac{1}{2} v_0$; otherwise

$$
|\bar{z}_j(p,\chi_c) - v_0| \geq |\bar{z}_j(p,\chi_c) - \bar{z}_j(p,\chi_c(1-\epsilon))| \geq C_\sharp \epsilon \mu^{-j} |\bar{z}_j(p,\chi_c)| \geq C_\sharp \epsilon \mu^{-j} v_0.
$$

Accordingly, $|\bar{z}_j(p, \pm\chi_c) - v_0| \geq C_\sharp \epsilon \mu^{-j} v_0$. Let $L_\star$ be the maximal Lipschitz constant of the $\bar{z}_j(p, \pm\chi_c)$ for $m - n_0 \leq C_\sharp$. If the hypotheses of Lemma D.0.1 are satisfied, then we can provide and explicit estimate for $L_\star$: in a finite number of steps n_1 (depending only on the derivatives of F) we can have $\bar{z}_{n_1} \leq 1/2$, we can thus apply Lemma D.0.1 $\varepsilon_0 = \varepsilon = \varepsilon' = 1$, $A = 1/2$, $B \leq C_\sharp$ and $u = u' = \bar{z}_{n_1}(p)$, we have

$$
\begin{aligned}
&|\bar{z}_{m-n_0}(p,\chi_c) - \bar{z}_{m-n_0}(p',\chi_c)| \leq L_{m-n_0} \|p - p'\| \\
&L_j = C_\sharp \mu^{3j} (\|\omega\|_{\mathcal{C}^2} + \mu^{2j} \lambda_j^+(p)^{-1} C_{\mu,j}/2).
\end{aligned}
$$

Since $D\mathfrak{h}_{n_0} v \notin \mathbf{C}_u$ we have, for $n_0 = 0$,

$$
|\,|\bar{z}_m(p, \pm\chi_c)| - v| \geq C_\sharp \epsilon \mu^{-m} v,
$$

while, for $n_0 > 0$, applying the above considerations to $v_0 = D\mathfrak{h}_{n_0} \bar{v}$ yields

$$
\left| |\bar{z}_{m-n_0}(\mathfrak{h}_{n_0}(p), \pm\chi_c)| - \frac{\pi_2(D\mathfrak{h}_{n_0}\bar{v})}{\pi_1(D\mathfrak{h}_{n_0}\bar{v})} \right| \geq C_\sharp \epsilon \mu^{m-n_0} \chi_u.
$$

Hence,

$$||\bar{z}_m(\gamma(t), \pm\chi_c)| - v| \geq C_n^{-1}\epsilon\mu^{-m}\bar{\kappa}. \tag{E.0.3}$$

Next, note that, by usual distortion arguments, it must be $\lambda^+_{m-n_0} \geq C_\sharp \mu^{n_0}(\chi_c\bar{\kappa})^{-1}$ and $m - n \leq C_\sharp \ln \bar{\kappa}^{-1}$, thus

$$L_{m-n_0} \leq C_\sharp \mu^{c_\sharp \ln \bar{\kappa}^{-1}}(\|\omega\|_{\mathcal{C}^2} + C_{\mu,n_0}\mu^{c_\sharp \ln \bar{\kappa}^{-1}}\bar{\kappa}) = L_\star(v).$$

We are finally ready to extend our segment. We discuss only the case $v \geq \bar{z}_m(\gamma(\delta), \chi_c)$ and $t \geq 0$ since the other cases can be treated similarly.

For $\varphi \in \mathbb{R}$, let $w(\varphi) = (\cos\varphi, \sin\varphi)$, $\theta = \tan v$ and $a = \sqrt{1+v^2}$. Then $\bar{v} = aw(\theta)$. We start by extending the curve to the interval $(\delta, \delta + A)$, with $A = \frac{1}{2a}C_n^{-1}\epsilon\mu^{-m}\bar{\kappa}L_\star^{-1} < 1$.

Next, let $b \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ be a bump function with $b(t) = 0$ for $t \leq 0$ and $b(t) = 1$ for $t \geq 1$. Also, let $B = \{aL_\star, 16\theta_c, \Bbbk\}$, for some $\Bbbk \geq 1$ to be chosen later, and where $\theta_c := \arctan(2\chi_c^{-1})$, and define

$$\hat{\gamma}'(t) = aw\big(\theta + b((t-\delta)A^{-1})B(t-\delta)\big) =: aw(\tilde{\theta}(t)). \tag{E.0.4}$$

Note that, by construction, $\tilde{\theta}(t) \geq \theta$. Moreover, for $t \in [\delta, \delta + A]$, we have

$$\|\hat{\gamma}(t) - \hat{\gamma}(\delta)\| \leq \int_\delta^{\delta+A} \|aw(\tilde{\theta}(ts))\| ds \leq Aa.$$

Thus, recalling (E.0.3),

$$\arctan \bar{z}_m(\hat{\gamma}(t), \chi_c) \leq \arctan \bar{z}_m(\hat{\gamma}(\delta), \chi_c) + L_\star aA \leq \theta - \frac{\epsilon\bar{\kappa}}{2C_n}\mu^{-m} + L_\star Aa < \theta \leq \tilde{\theta}(t),$$

which implies that $D_{\hat{\gamma}(t)}\mathfrak{h}_m\hat{\gamma}'(t) \in \mathbf{C}_c$. In addition, for $t \geq \delta + A$, we have

$$|\frac{d}{dt}\tan\tilde{\theta}(t)| \geq B \geq aL_\star \geq |\frac{d}{dt}\bar{z}_m(\tilde{\gamma}(t)|.$$

Next, let $T > 0$ be such that $\tilde{\theta}(T) = \theta_c$ so that $\hat{\gamma}'(T)$ is well inside the central cone. This implies $T \leq \delta + \theta_c B^{-1}$ and

$$|\pi_1(\hat{\gamma})| \leq C_\sharp T \leq C_\sharp \delta + B^{-1} \leq C_\sharp(\delta_0 + \Bbbk^{-1}) < 1/2,$$

provided δ_0 and $\Bbbk^{-1}$ are small enough. It is then a simple exercise to construct an extension $\hat{\gamma} : [0, S] \to \mathbb{T}^2$ such that $\hat{\gamma}'(t) \in \mathbf{C}_c$, $\|\hat{\gamma}'\| = a$, for all $t \in [T, S]$ and $\hat{\gamma}(S) = (0, 1/2)$, $|\pi_1(\hat{\gamma})| \leq C_\sharp(\delta_0 + \Bbbk^{-1})$, $\hat{\gamma}'(S) = (-\chi_c/2, 1)$, $\hat{\gamma}^{(j)}(S) = 0$ for all $j > 1$ and $\sup_{t\in[T,S]} \|\hat{\gamma}^{(j)}(t)\| \leq C_\sharp$. By symmetry we have a closed curve $\hat{\gamma}$ of homotopy class $(0, 1)$. It suffices to ask $C_\sharp(\delta_0 + \Bbbk^{-1}) \leq \frac{1}{4}$, to insure that $\hat{\gamma} \in \mathrm{Dom}(\mathfrak{h}_k)$ for each $\mathfrak{h} \in \mathfrak{H}^\infty_{\gamma_-}$ and $k \in \mathbb{N}$. Then Lemma 5.2.1 implies that there exists inverse branches $\{\mathfrak{h}_{k,i}\}_{i=1}^{d^k}$, where d is the degree of F, such that $F^{-k}\hat{\gamma} = \bigcup_{i=1}^{d^k} \mathfrak{h}_{k,i} \circ \hat{\gamma}$. Since $\mathfrak{h}_{k,i}$ is a diffeomorphism, $\mathfrak{h}_{k,i} \circ \hat{\gamma}$ is a closed curve. In addition it must be of homotopy type $(0, 1)$, otherwise it would intersect an horizontal segment in more than one point and the image, under F^k, of the interval between two intersection points would be an unstable curve going from

$\hat{\gamma}$ to itself. Since such a curve would be transversal to $\hat{\gamma}$ by hypothesis, it follows that it would have to wrap around the torus horizontally an hence intersect γ_- contradicting the fact that it is in the domain of $\mathfrak{h}_{k,i}$.

Recalling (E.0.4), formula (5.1.3) gives, for all $j \geq 2$, [1]

$$\|\hat{\gamma}\|_{\mathcal{C}^{j+1}} \leq C_\sharp \|w \circ \tilde{\theta}\|_{\mathcal{C}^j} \leq C_\sharp \sum_{s=0}^{j} \|w\|_{\mathcal{C}^s} \sum_{k \in \mathcal{K}_{j,s}} \prod_{l \in \mathbb{N}}^{j} \|\tilde{\theta}\|_{\mathcal{C}^l}^{k_l}$$
$$\leq C_\sharp \sum_{s=0}^{j} \sum_{k \in \mathcal{K}_{j,s}} \prod_{l \in \mathbb{N}} \left(A^{-l}B\right)^{k_l} \leq A^{-j} \sum_{s=0}^{j} B^s.$$

Thus, since $\|\hat{\gamma}'\| = a$, we can reparametrize the curve by arc-length. Calling $\tilde{\gamma}$ the reparametrized curve we obtain

$$\|\tilde{\gamma}^{(j)}(t)\| \leq \begin{cases} 0 & \text{if } |t| \leq \delta \\ C_\sharp A^{-j+1} B^{j-1} & \text{if } \delta \leq |t| \leq \delta + A \\ C_\sharp B^{j-1} & \text{if } |t| \geq \delta + A, \end{cases}$$

which yields (E.0.1) since

$$|\pi_2(\tilde{\gamma}'(t))| \geq \begin{cases} |v| & \text{if } |t| \leq \delta + A \\ C_\sharp(|v| + B(t - \delta)) & \text{if } |t| \geq \delta + A. \end{cases}$$

□

[1] Notice that, as $\|\tilde{\theta}\|_{\mathcal{C}^l} \leq C_\sharp A^{-l} B$, recalling the definition of $\mathcal{K}_{j,s}$ we have

$$\sum_{k \in \mathcal{K}_{j,s}} \prod_{l \in \mathbb{N}} (A^{-l}B)^{k_l} \lesssim \sum_{k \in \mathcal{K}_{j,s}} A^{-\sum_{l=1}^{\infty} l k_l} B^{\sum_{l=1}^{\infty} k_l} \lesssim A^{-j} B^s.$$

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