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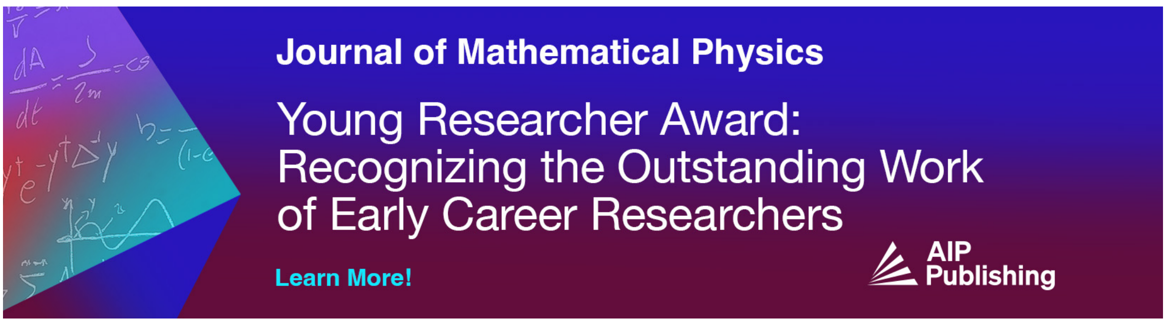

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Serena Cenatiempo[a)]

AFFILIATIONS

Gran Sasso Science Institute, Viale Francesco Crispi 7, 67100 L'Aquila, Italy

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[a)]**Electronic mail:** serena.cenatiempo@gssi.it

ABSTRACT

In 1947, Bogoliubov suggested a heuristic theory to compute the excitation spectrum of weakly interacting Bose gases. Such a theory predicts a linear excitation spectrum and provides expressions for the thermodynamic functions which are believed to be correct in the dilute limit. Thus far, there are only a few cases where the predictions of Bogoliubov can be obtained by means of rigorous mathematical analysis. A major challenge is to control the corrections beyond Bogoliubov theory, namely, to test the validity of Bogoliubov's predictions in regimes where the approximations made by Bogoliubov are not valid. In these notes, we discuss how this challenge can be addressed in the case of a system of N interacting bosons trapped in a box with volume one in the Gross-Pitaevskii limit, where the scattering length of the potential is of the order $1/N$ and N tends to infinity. This is a recent result obtained in Boccato *et al.* [Commun. Math. Phys. (to be published); preprint arXiv:1812.03086 and Acta Math. **222**, 219–335 (2019); e-print arXiv:1801.01389], which extends a previous result obtained in Boccato *et al.* [Commun. Math. Phys. **359**, 975 (2018)], removing the assumption of a small interaction potential.

I. INTRODUCTION

Since the early experiments on superfluidity in liquid helium,[1,2] and even more after the first experimental realizations of Bose-Einstein condensation in cold atomic gases,[3–5] the understanding of the low temperature properties of systems of interacting bosons has stimulated several theoretical and mathematical investigations. The aim of these notes is to report on a recent result establishing the equilibrium properties of the interacting Bose gas in one of the regimes which are relevant for the description of condensation in low-interacting and dilute atomic gases, the so called *Gross-Pitaevskii regime*. As a prelude, we will start by reviewing the progress made so far in the comprehension of the equilibrium properties of the interacting Bose gas in the thermodynamic limit. This preliminary discussion will set the stage to clarify the mathematical difficulties posed by the Gross-Pitaevskii regime and to compare the main result obtained in Refs. 6 and 7 with related results achieved in different parameters regimes.

In the course of these notes, we are going to consider systems of N bosons in a three dimensional box Λ of side length L with periodic boundary conditions. The Hamilton operator describing the system has the form

$$H_{N,\Lambda} = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} V(x_i - x_j) \tag{1}$$

and acts on the Hilbert space $L^2_s(\Lambda^N)$, with the subspace of $L^2(\Lambda^N)$ consisting of functions that are symmetric with respect to permutations of the N particles. We require V to be non-negative, radial, and to have finite zero energy scattering length $\mathfrak{a}_0$. The latter is defined as

$$\mathfrak{a}_0 = (8\pi)^{-1} \int V(x) f(x), \tag{2}$$

with $f(x)$ solution of the zero energy scattering equation $(-\Delta + \frac{1}{2} V(x))f(x) = 0$, with boundary condition $f(x) \to 1$ as $|x| \to \infty$. We will first discuss the equilibrium properties of the system in the *thermodynamic limit*, where the density of the system $\rho = N/|\Lambda|$ is kept constant and the side of box Λ is sent to infinity. It is well known that in absence of interaction, the systems exhibit Bose-Einstein condensation; in particular, at zero temperature, all particles are in the ground state of the kinetic energy operator, which is given by the zero momentum mode. A long-standing goal is to understand what happens to the system when we take into account the interaction among particles. Does the system still exhibit condensation? Can we provide expressions for the ground state energy and excitation spectrum, at least in some weakly interacting regime? Can we explain the emergence of superfluidity, as observed in experiments?

The usual picture of Bose-Einstein condensation in the homogeneous interacting case is based on an approximate exactly solvable model due to Bogoliubov[8] [see also Ref. 9 (Appendix A) for a review]. Bogoliubov rewrote the Hamilton operator (1) in momentum space, using the formalism of second quantization. Since he expected low-energy states to exhibit Bose-Einstein condensation (at least for sufficiently weak interaction), he replaced all creation and annihilation operators associated with the zero-momentum mode by factors $N^{1/2}$. The resulting Hamiltonian contains constant terms (describing the interaction among particles in the condensate), terms that are quadratic in creation and annihilation operators associated with modes with momentum $p \neq 0$ (describing the kinetic energy of the excitations as well as the interaction between excitations and the condensate), and terms that are cubic and quartic (describing interactions among excitations). Neglecting all cubic and quartic contributions, Bogoliubov obtained a quadratic Hamiltonian that he could diagonalize explicitly, obtaining the following expression for the ground state energy:

$$E_{N,\Lambda} = \frac{N}{2}\rho\widehat{V}(0) - \frac{1}{4}\sum_{\substack{p\in\frac{2\pi}{L}\mathbb{Z}^3\\ p\neq 0}} \frac{(\rho\widehat{V}(p))^2}{p^2} - \frac{1}{2}\sum_{\substack{p\in\frac{2\pi}{L}\mathbb{Z}^3\\ p\neq 0}}\left[p^2 + \rho\widehat{V}(p) - \sqrt{p^4+2\rho\widehat{V}(p)p^2} - \frac{1}{2}\frac{(\rho\widehat{V}(p))^2}{p^2}\right] \tag{3}$$

and an excitation spectrum of the form[10]

$$\sum_{p\in\frac{2\pi}{L}\mathbb{Z}^3} \sqrt{p^4 + 2\rho\widehat{V}(p)p^2}\, n_p \tag{4}$$

for finitely many $n_p \in \mathbb{N}$. Remarkably, Bogoliubov recognized that, after having taken the thermodynamic limit, the expressions

$$\mathfrak{a}_0^{(0)} = (8\pi)^{-1}\widehat{V}(0), \quad \mathfrak{a}_0^{(1)} = -\int \frac{d^3p}{(2\pi)^3}\frac{\widehat{V}(p)^2}{2p^2}$$

appearing on the rhs of (3) were just the first and second Born approximations of the infinite volume scattering length $\mathfrak{a}_0$. By replacing the sum $\mathfrak{a}_0^{(0)} + \mathfrak{a}_0^{(1)}$ by $\mathfrak{a}_0$ on the r.h.s. of Eq. (3), and $\widehat{V}(0)$ by $8\pi\mathfrak{a}_0$ in the integral obtained from the sum on the r.h.s. of the same equation,[11] Bogoliubov obtained the following formula for the ground state energy for particle of a dilute Bose gas in the thermodynamic limit:

$$\lim_{\substack{N,|\Lambda|\to\infty\\ \rho=N/|\Lambda|}} \frac{E_{N,\Lambda}}{N} = 4\pi\rho\mathfrak{a}_0\left[1 + \frac{128}{15\sqrt{\pi}}(\rho\mathfrak{a}_0^3)^{1/2} + o((\rho\mathfrak{a}_0^3)^{1/2})\right], \tag{5}$$

which is known as the Lee-Huang-Yang formula.[12,13] A similar substitution is expected to give the correct expression for the velocity of sound

$$v_s = \lim_{p\to 0}\frac{\sqrt{p^4 + 16\pi\rho\mathfrak{a}_0 p^2}}{p} = \sqrt{16\pi\rho\mathfrak{a}_0},$$

obtained by substituting $\widehat{V}(p)$ by $8\pi\mathfrak{a}_0$ in the dispersion relation provided by (4). As an additional example, one can compute within Bogoliubov's approximation the expected density of particles outside the condensate in the ground state (the so called *condensate depletion*), obtaining

$$\rho_+ = \sum_{\substack{p\in\frac{2\pi}{L}\mathbb{Z}^3\\ p\neq 0}}\left[\frac{p^2 + \rho\widehat{V}(p) - \sqrt{p^4 + 2\rho\widehat{V}(p)p^2}}{2\sqrt{p^4 + 2\rho\widehat{V}(p)p^2}}\right]. \tag{6}$$

Taking the thermodynamic limit of (6) and substituting $\widehat{V}(p)$ with $8\pi\mathfrak{a}_0$, one obtains the prediction

$$\frac{\rho_+}{\rho} = \frac{8}{3\sqrt{\pi}}\sqrt{\rho\mathfrak{a}_0^3}. \tag{7}$$

It is worth stressing that the Bogoliubov model is based on the very strong assumption (not *a priori* justified) that the interacting system exhibits condensation, plus a quite rough truncation of the Hamiltonian. Nevertheless, Bogoliubov's predictions are believed to be correct in the dilute limit $\rho \mathfrak{a}_0^3 \ll 1$. Indeed, in his final replacement [the one discussed after (4)], Bogoliubov might compensate exactly for all terms (cubic and quartic in creation and annihilation operators) that he neglected in his analysis.[14] Then, it is not surprising that Bogoliubov's '47 paper was followed by several attempts to study in a systematic way the corrections to Bogoliubov theory; that is, to understand the role of the cubic and quartic contributions neglected in Bogoliubov's approximation. Unfortunately, perturbation theory around the Bogoliubov model is plagued by ultraviolet and infrared divergences, whose meaning could be in principle that the interacting system has completely different physical properties with respect to the ones predicted by Bogoliubov. A few partial results confirming Bogoliubov's picture have been obtained in the late '60s on the basis on diagrammatic techniques borrowed from quantum field theory,[12,13,15–22] but they were all based on the summations of special classes of diagrams selected from the divergent perturbation theory.

More recently, a study of the whole perturbation theory around Bogoliubov's model for weak repulsive interactions (and/or at low densities) in three dimensions, and the proof of its order by order convergence after proper resummations, has been obtained by Benfatto.[8] This work provides a strong indication of the stability of the three dimensional Bose-Einstein condensate at zero temperature and a confirmation of the expression (4) with a renormalized speed of sound.[23] It is worth stressing that even though the method used by Benfatto is taken from the constructive theory, the resulting bounds are not enough for constructing the theory: they are enough for deriving finite bounds at all orders in renormalized perturbation theory, growing like $n!$ at the nth order (*n! bounds*), but the possible Borel summability of the series remains a great challenge for the current century. A long term program addressing this issue has been started by Balaban-Feldman-Knörrer-Trubowitz; see Ref. 24 for the state of the art of this project.

Even though a full control of the corrections to Bogoliubov's approximation is to date beyond reach of rigorous analysis, several results are available if we focus on Bogoliubov's predictions for the ground state energy. Indeed, mathematically, the validity of Bogoliubov's approach in three-dimensional Bose gases has been first established by Lieb and Solovej for the computation of the ground state energy of bosonic jellium in Ref. 25 and of the two-component charged Bose gas in Ref. 26 (upper bounds were later given by Solovej in Ref. 27). Extending the ideas of Refs. 25 and 26, Giuliani and Seiringer established in Ref. 28 the validity of the Lee-Huang-Yang formula (5) for Bose gases interacting through potentials scaling with the density to approach a simultaneous weak coupling and high density limit. This result was later improved by Brietzke and Solovej in Ref. 29 to include a certain class of weak coupling and low density limits. It is worth stressing that in the regimes considered in Refs. 25, 26, 28, and 29, the difference between the first and second Born approximations and the full scattering length is small, and it only gives negligible contributions to the energy. In other words, in the above mentioned regimes, cubic and quartic contributions neglected in Bogoliubov's analysis can be proved to be small; this is crucial to make Bogoliubov's approach rigorous.

An upper bound for the ground state energy in the thermodynamic limit coinciding with (5) up to second order was established in Ref. 30 (improving a previous result by Ref. 31). Very recently, a lower bound for the ground state energy of a dilute Bose gas which confirms the Lee-Huang-Yang formula for a broad class of repulsive pair-interactions in three dimensions has been obtained in Ref. 32 (improving a previous result by Ref. 33). Those results represent a confirmation of the validity of the predictions of Bogoliubov theory for the ground state energy of dilute Bose gases in the thermodynamic limit.

It remains an ambitious open problem to verify Bogoliubov's prediction for the excitation spectrum in the thermodynamic limit.

A. The Gross-Pitaevskii regime

A natural question arising from the discussion in Sec. I is whether some of the results predicted by Bogoliubov theory can be validated in regimes different from the thermodynamic limit, but still physically relevant for the description of Bose-Einstein condensates. This is the case of the so called *scaling regimes*, where the bosons are confined in a box of side length one (in the more general case the bosons are trapped by an external confining potential) and the interaction is allowed to depend on the number of particles. In the three-dimensional case, it turns out interesting to consider systems of N bosons in the box $\Lambda = [-\frac{1}{2}; \frac{1}{2}]^{\times 3}$, described by the Hamilton operator

$$H_N^\beta = \sum_{j=1}^N -\Delta_{x_j} + \frac{\kappa}{N} \sum_{i<j}^N N^{3\beta} V(N^\beta(x_i - x_j)) \tag{8}$$

for a parameter $\beta \in [0; 1]$, a coupling constant $\kappa > 0$, and a short range potential $V \geq 0$. Hamilton operators of the form (8) interpolate between the *mean-field regime* associated with $\beta = 0$ (effectively describing bosons interacting through weak and long range interactions) and the *Gross-Pitaevskii regime* corresponding to $\beta = 1$ (depicting a situation where interactions among the particles are strong and very short range). Note that, denoting with $\mathfrak{a}_N$ the scattering length of the potential $N^{3\beta-1}V(N^\beta x)$, for any $\beta \in [0, 1]$, we have $\rho \mathfrak{a}_N^3 = N^{-2}$, which corresponds to a diluteness condition. Hence, we may reasonably expect the predictions of Bogoliubov theory to hold for systems of bosons described by (8).

Since the Born series for the scattering length is an expansion in the ratio between the parameter $\int V$ and the range of the potential, a simple computation shows that, in the regimes described by the Hamilton operator (8), replacing first and second Born approximations with the scattering length produces an error in the ground state energy of the order $N^{2\beta-1}$. Hence, one may guess that Bogoliubov's truncation of the Hamiltonian can be rigorously justified for any $\beta < 1/2$. Indeed, starting from the pioneering work,[34] several results have confirmed Bogoliubov's picture in the mean-field limit $\beta = 0$, both in the homogeneous and nonhomogeneous setting.[35–42] On the other side, for $\beta \geq 1/2$, Bogoliubov's approximation fails. Nevertheless, in Ref. 43, the predictions of Bogoliubov theory where rigorously justified for any $0 < \beta < 1$

(the proof in Ref. 43 holds for κ sufficiently small, but can be extended to any κ using the strategy recently developed for the Gross-Pitaevskii regime in Ref. 6). The key idea to achieve this result was to understand the emergence of the scattering length as a consequence of correlations among the particles.

The Gross-Pitaevskii regime, where $\beta = 1$, is even more challenging from a mathematical point of view. Indeed, in this regime, the ratio between $\mathfrak{a}_N$ and the range of the potential is of order one, and all terms in the Born series of the scattering length contribute to the same order in N. From a physical point of view, the Gross-Pitaevskii regime owes its success to the fact that it represents a good description for the strong and short range interactions among atoms in dilute, cold atomic gases. Moreover, the Gross-Pitaevskii equation, widely used in the physics literature to effectively describe the dynamics of Bose-Einstein condensates, arises from a microscopic description where the interaction among particles scales as in the Gross-Pitaevskii regime.[44] Finally, it is easy to see that H_N^β for $\beta = 1$ is equivalent to the Hamiltonian for N bosons in a box with $L = N$ interacting through a fixed potential V; hence, the Gross-Pitaevskii regime corresponds to a regime where the size of box is sent to infinity, but the system has in this limit a very low density $\rho = N/L^3 = N^{-2}$.

It follows from the results of Refs. 45–48 that the ground state energy E_N of the Gross-Pitaevskii Hamiltonian

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)) \tag{9}$$

defined on $L^2_s(\Lambda^N)$ is such that

$$\lim_{N\to\infty} \frac{E_N}{N} = 4\pi\mathfrak{a}_0 .$$

Furthermore, for any sequence of approximate ground states, i.e., for any sequence $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and

$$\lim_{N\to\infty} \frac{1}{N}\langle \psi_N, H_N \psi_N\rangle = 4\pi\mathfrak{a}_0 ,$$

the reduced density matrices $\gamma_N = \text{tr}_{2,\ldots,N}|\psi_N\rangle\langle\psi_N|$ are such that

$$\lim_{N\to\infty} \text{tr}\,|\gamma_N - |\varphi_0\rangle\langle\varphi_0|| = 0, \tag{10}$$

where $\varphi_0 \in L^2(\Lambda)$ is the zero momentum mode defined by $\varphi_0(x) = 1$ for all $x \in \Lambda$. The aim of these notes is to discuss how to go beyond those results and compute the ground state energy and the low-lying excitation spectrum of (9), up to errors vanishing in the limit $N \to \infty$. This is the content of our main theorem.

Theorem I.1. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, spherically symmetric, and compactly supported. Then, in the limit $N \to \infty$, the ground state energy E_N of the Hamilton operator (9) is given by*

$$E_N = 4\pi(N-1)\mathfrak{a}_0 + e_\Lambda \mathfrak{a}_0^2 - \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right] + \mathcal{O}(N^{-1/4}), \tag{11}$$

with $\mathfrak{a}_0$ as the scattering length of V. Here, we introduced the notation $\Lambda_+^* = 2\pi\mathbb{Z}^3\backslash\{0\}$ and we defined

$$e_\Lambda = 2 - \lim_{M\to\infty} \sum_{\substack{p\in\mathbb{Z}^3\backslash\{0\}:\\ |p_1|,|p_2|,|p_3|\le M}} \frac{\cos(|p|)}{p^2},$$

where, in particular, the limit exists. Moreover, the spectrum of $H_N - E_N$ below a threshold ζ consists of eigenvalues given, in the limit $N \to \infty$, by

$$\sum_{p\in\Lambda_+^*} n_p\sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} + \mathcal{O}(N^{-1/4}(1+\zeta^3)) . \tag{12}$$

Here, $n_p \in \mathbb{N}$ for all $p \in \Lambda_+^*$ and $n_p \neq 0$ for finitely many $p \in \Lambda_+^*$ only.

Remarks.

- By comparing (11) with the prediction (3) from Bogoliubov theory, we see that in the Gross-Pitaevskii regime the leading order term contains the scattering length $\mathfrak{a}_0$, rather than the sum of the first two terms on the rhs of (3). Moreover, the first term in the Born approximation of the scattering length $(8\pi)^{-1}\widehat{V}(0)$ has been replaced by $\mathfrak{a}_0$ both in the sum on the r.h.s. of (11) and in the expression for the spectrum (12) [to be compared with (4)].
- The term $e_\Lambda \mathfrak{a}_0^2$ in (11) arises as a correction to the scattering length $\mathfrak{a}_0$, due to the finiteness of box Λ. For small interaction potentials, we can define a finite volume scattering length $\mathfrak{a}_\Lambda$ through the convergent Born series

$$8\pi\mathfrak{a}_\Lambda = \widehat{V}(0) + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2N)^k} \sum_{p_1,\dots,p_k\in\Lambda_+^*} \frac{\widehat{V}(p_1/N)}{p_1^2} \left(\prod_{i=1}^{k-1} \frac{\widehat{V}((p_i-p_{i+1})/N)}{p_{i+1}^2} \right) \widehat{V}(p_k/N) .$$

In this case, one can check that

$$\lim_{N\to\infty} 4\pi(N-1)[\mathfrak{a}_0 - \mathfrak{a}_\Lambda] = e_\Lambda \mathfrak{a}_0^2 .$$

Observe that, if we replace the potential V by a rescaled interaction $V_R(x) = R^{-2}V(x/R)$ with scattering length $\mathfrak{a}_R = \mathfrak{a}_0 R$ then, for large R (increasing R makes the effective density larger), the order one contributions to the ground state energy scale as $e_\Lambda \mathfrak{a}_0^2 R^2$ and, respectively, as

$$\begin{aligned} -\frac{1}{2} \sum_{p\in 2\pi\mathbb{Z}^3\setminus\{0\}} & \left[p^2 + 8\pi\mathfrak{a}_0 R - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 R p^2} - \frac{(8\pi\mathfrak{a}_0 R)^2}{p^2} \right] \\ &= \frac{R}{2} \sum_{p\in \frac{2\pi}{\sqrt{R}}\mathbb{Z}^3\setminus\{0\}} \left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{p^2} \right] \\ &\simeq \frac{R^{5/2}}{2(2\pi)^3} \int_{\mathbb{R}^3} \left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{p^2} \right] dp \\ &= \frac{4\pi R^{5/2}(16\pi\mathfrak{a}_0)^{5/2}}{15(2\pi)^3} = 4\pi\mathfrak{a}_0 \cdot \frac{128}{15\sqrt{\pi}} \mathfrak{a}_0^{3/2} R^{5/2}. \end{aligned} \tag{13}$$

In particular, letting $R \to \infty$ (independently of N), consequently the finite volume correction becomes subleading, compared with (13). From this point of view, Theorem I.1 establishes the analog of the Lee-Huang-Yang formula for the ground state energy in the Gross-Pitaevskii regime.
- Theorem I.1 gives precise information on the low-lying eigenvalues of (9). The approach in Ref. 7 combined with standard arguments, also gives information on the corresponding eigenvectors. In Ref. 7, we provide a norm approximation of eigenvectors associated with the low-energy spectrum of (9). As an application, we can compute the condensate depletion in the ground state ψ_N of (1), confirming Bogoliubov's prediction.

In the rest of these notes, we are going to describe the strategy leading to Theorem I.1, as obtained in Refs. 6 and 7. In particular, we will focus on the new ideas which are needed to remove the assumption of the small interaction potential that was previously used in Ref. 49.

II. A FOCK SPACE REPRESENTATION FOR EXCITED PARTICLES

The first step in the proof of Theorem I.1 consists in a rigorous version of the substitution of the creation and annihilation operators in the condensate by scalar numbers, which is the first step in Bogoliubov theory. Following an idea from Ref. 36, we use the Fock space to describe orthogonal excitations with respect to the condensate wave function.[50] More precisely, we write any arbitrary N-particle wave function $\psi \in L_s^2(\Lambda^N)$ as

$$\psi_N = \alpha_0 \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes(N-1)} + \cdots + \alpha_N \tag{14}$$

with $\alpha_j \in L^2_{s,\perp}(\Lambda^j)$ for all $j = 0, 1, \dots, N$. Here, $\varphi_0(x)$ is the condensate wave function in (10). Moreover $L^2_\perp(\Lambda)$ denotes the orthogonal complement of the one-dimensional subspace spanned by φ_0 in $L^2(\Lambda)$, and $L^2_{s,\perp}(\Lambda^j)$ is the symmetric tensor product of j copies of $L^2_\perp(\Lambda)$. It is easy to check that the decomposition (14) defines a unitary map U_N from the space $\in L_s^2(\Lambda^N)$ to the truncated Fock space constructed over $L^2_\perp(\Lambda)$,

$$\mathscr{F}_+^{\leq N} = \bigotimes_{j=0}^{N} L^2_{s,\perp}(\Lambda^j).$$

For a $\psi_N \in L^2_s(\Lambda^N)$, we denote with ξ_N the corresponding excitation vector

$$\xi_N := U_N \psi_N = \{\alpha_0, \alpha_1, \ldots, \alpha_N\} \in \mathscr{F}_+^{\leq N}.$$

The action of the unitary operator U_N on products of a creation and an annihilation operator [products of the form $a_p^* a_q$ can be thought of as operators mapping $L^2_s(\Lambda^N)$ to itself] is reminiscent of Bogoliubov's substitution. Indeed, for any $p, q \in \Lambda_+^* = 2\pi\mathbb{Z}^3\backslash\{0\}$, we find (see Ref. 36)

$$\begin{aligned} U_N\, a_0^* a_0\, U_N^* &= N - \mathcal{N}_+, \\ U_N\, a_p^* a_0\, U_N^* &= a_p^* \sqrt{N - \mathcal{N}_+}, \\ U_N\, a_0^* a_p\, U_N^* &= \sqrt{N - \mathcal{N}_+}\, a_p, \\ U_N\, a_p^* a_q\, U_N^* &= a_p^* a_q, \end{aligned} \tag{15}$$

with $\mathcal{N}_+ = \sum_{p \in \Lambda_+^*} a_p^* a_p$ as the operator counting the number of excited particles (here, a_p^* and a_p are the usual creation and annihilation operators defined on the bosonic Fock space $\mathscr{F} = \otimes_{j\geq 0} L^2_s(\Lambda^j)$ and satisfying canonical commutation relations $[a_p, a_q^*] = \delta_{pq}$ and $[a_p, a_q] = [a_p^*, a_q^*] = 0$).

Using U_N, we can define an excitation Hamiltonian $\mathscr{L}_N := U_N H_N U_N^*$, acting on a dense subspace of $\mathscr{F}_+^{\leq N}$. To compute the operator $\mathscr{L}_N$, we first write the Hamiltonian (9) in momentum space, in terms of creation and annihilation operators. We find

$$H_N = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{1}{2N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}, \tag{16}$$

where

$$\widehat{V}(k) = \int_{\mathbb{R}^3} V(x) e^{-ik\cdot x} dx$$

is the Fourier transform of V, defined for all $k \in \mathbb{R}^3$. Using (15), we conclude that

$$\mathscr{L}_N = \mathscr{L}_N^{(0)} + \mathscr{L}_N^{(2)} + \mathscr{L}_N^{(3)} + \mathscr{L}_N^{(4)} \tag{17}$$

with

$$\begin{aligned} \mathscr{L}_N^{(0)} &= \frac{N-1}{2N} \widehat{V}(0)(N - \mathcal{N}_+) + \frac{\widehat{V}(0)}{2N} \mathcal{N}_+ (N - \mathcal{N}_+), \\ \mathscr{L}_N^{(2)} &= \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p + \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) a_p^* a_p \left(\frac{N - \mathcal{N}_+}{N} \right) \\ &\quad + \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left[a_p^* a_{-p}^* \sqrt{\frac{N - 1 - \mathcal{N}_+}{N} \frac{N - \mathcal{N}_+}{N}} + \text{h.c.} \right], \\ \mathscr{L}_N^{(3)} &= \frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N) \left[a_{p+q}^* a_{-p}^* a_q \sqrt{\frac{N - \mathcal{N}_+}{N}} + \text{h.c.} \right], \\ \mathcal{L}_N^{(4)} &= \frac{1}{2N} \sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq -p,-q} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}. \end{aligned}$$

As in Bogoliubov theory, conjugation with U_N extracts, from the original quartic interaction, constant, quadratic, cubic, and quartic terms in creation and annihilation operators a_p^* and a_p associated with momenta $p \in \Lambda_+^* = 2\pi\mathbb{Z}^3\backslash\{0\}$, collected in $\mathscr{L}_N^{(0)}$, $\mathscr{L}_N^{(2)}$, $\mathscr{L}_N^{(3)}$, and $\mathscr{L}_N^{(4)}$ respectively. The challenge of the Gross-Pitaevskii regime is that, due to the slow decay of $\widehat{V}(p/N)$ for large momenta, we cannot neglect the cubic and quartic contributions in (17) for $N \to \infty$. This fact can be understood from different points of view.

- It is well known that the ground state energy of bosons in the Gross-Pitaevskii regime is heavily affected by correlations. Indeed, the ground state vector is characterized by a correlation structure which varies on the length scale of the scattering length of the interaction

$\mathfrak{a}_N \sim N^{-1}$, and which can be modeled by the solution of the zero energy scattering equation. This is the key ingredient to show upper and lower bounds consistent with (11) at leading order,[45] and to establish the results in Refs. 30–33. The same correlation structure has to be included in any approach aimed to show the emergence of the Gross-Pitaevskii equation as an effective description for the evolution of initially trapped Bose-Einstein condensates which evolves under the dynamics generated by (9) [see Ref. 51 (Chap. 5) and references therein]. On the contrary, the application of the unitary map U_N only factors out the condensate, but does not remove the short scale correlation structure that, as we will see below, still carries an energy of order N.

- From a renormalization group perspective, studying $\mathcal{L}_N$ corresponds to carry out perturbation theory around Bogoliubov's approximation for momenta larger than 2π. However, as already commented in the Introduction, such a theory is divergent in the ultraviolet, and to get a well defined theory, we need to renormalize both the quadratic and the cubic vertices of the theory.

Before describing how to include the correlation structure into our analysis, let us explain the guiding idea behind our overall strategy, which can be easily illustrated in the simpler case where we substitute $\widehat{V}(p/N)$ by a mean-field potential $\kappa\widehat{V}(p)$ with intensity $\kappa > 0$ sufficiently small.

A. A sketch of the strategy in the mean field case

In the approach we are going to follow, the key ingredient used to investigate the validity of Bogoliubov theory is the proof of optimal bounds on the number and energy of excitations in low energy states. With this goal in mind, let us denote with $\mathcal{L}_N^{\text{mf}}$ an excitation Hamiltonian identical to (17) except that $\widehat{V}(p/N)$ is substituted by $\kappa\widehat{V}(p)$. It is easy to check that there exists a constant $C > 0$ such that

$$\mathcal{L}_N^{\text{mf}} \geq \frac{N}{2}\kappa\widehat{V}(0) + \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p + \frac{\kappa}{2N}\sum_{\substack{p,q\in\Lambda_+^*,\\ r\in\Lambda^*:\\ r\neq -p,-q}} \widehat{V}(r) a_{p+r}^* a_q^* a_p a_{q+r} - C\kappa(\mathcal{N}_+ + 1), \tag{18}$$

where we used that $\mathcal{N}_+ \leq N$. Using positivity of the interaction and the gap in the kinetic energy $\sum_{p\in\Lambda_+^*} p^2 a_p^* a_p \geq (2\pi)^2\mathcal{N}_+$, we obtain that for sufficiently small κ, there exists $C > 0$ such that

$$\mathcal{L}_N^{\text{mf}} \geq \frac{N}{2}\kappa\widehat{V}(0) + c\mathcal{N}_+ - C. \tag{19}$$

This also implies the lower bound $\mathcal{L}_N^{\text{mf}} \geq \frac{N}{2}\kappa\widehat{V}(0) + C$. On the other side by using the vacuum state in $\mathcal{F}_+^{\leq N}$ as a trial state, we obtain the upper bound

$$\mathcal{L}_N^{\text{mf}} \leq \frac{N}{2}\kappa\widehat{V}(0).$$

This allows us to conclude that the ground state energy of our mean-field Hamiltonian

$$H_N^{\text{mf}} = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{2N}\sum_{p,q,r\in\Lambda^*} \widehat{V}(r) a_{p+r}^* a_q^* a_p a_{q+r}$$

satisfies the bound $|E_N^{\text{mf}} - N\kappa\widehat{V}(0)/2| \leq C$. Moreover for any N-particle wave function $\psi_N \in L_s^2(\Lambda^N)$ such that

$$\langle \psi_N, H_N^{\text{mf}}\psi_N\rangle \leq \frac{N}{2}\kappa\widehat{V}(0) + \zeta, \tag{20}$$

we have that the corresponding excitation vector $\xi_N = U^*\psi_N$ satisfies

$$\langle \xi_N, \mathcal{L}_N^{\text{mf}}\xi_N\rangle \leq \frac{N}{2}\kappa\widehat{V}(0) + \zeta,$$

and hence, through (19),

$$\langle \xi_N, \mathcal{N}_+\xi_N\rangle \leq C(1+\zeta).$$

Hence, low energy states have a bounded number of excitations. Additionally, denoting

$$\mathcal{K} = \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p$$

as the kinetic energy of excitations and

$$\mathcal{V}_N^{\text{mf}} = \frac{\kappa}{2N} \sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*: \\ r\neq -p,-q}} \widehat{V}(r) a_{p+r}^* a_q^* a_p a_{q+r}$$

as the potential energy in the mean-field scaling, from (18), we also obtain

$$\mathcal{L}_N^{\text{mf}} \geq \frac{N}{2}\kappa\widehat{V}(0) + \frac{1}{2}(\mathcal{V}_N^{\text{mf}} + \mathcal{K}) - C. \tag{21}$$

This implies that any excitation vector ξ associated with low energy states in the sense of (20) has a bounded excitation energy, namely, satisfies

$$\langle \xi_N, \mathcal{H}_N^{\text{mf}} \xi_N \rangle \leq C(1+\zeta),$$

with $\mathcal{H}_N^{\text{mf}} := \mathcal{K} + \mathcal{V}_N^{\text{mf}}$.

We can derive even stronger bounds on the excitation vector ξ_N associated with normalized N-particle wave function ψ_N, if instead of imposing the condition (20), we require ψ_N to belong to the spectral subspace of H_N associated with energies below $\frac{N}{2}\kappa\widehat{V}(0) + \zeta$. To this aim, we define

$$\tilde{\mathcal{L}}_N = \mathcal{L}_N^{\text{mf}} - \frac{N}{2}\kappa\widehat{V}(0).$$

Then,

$$\langle \xi_N, \mathcal{N}_+ \mathcal{H}_N^{\text{mf}} \xi_N \rangle = \langle \xi_N, \mathcal{N}_+^{1/2} \mathcal{H}_N^{\text{mf}} \mathcal{N}_+^{1/2} \xi_N \rangle \leq \langle \xi_N, \mathcal{N}_+^{1/2} (\tilde{\mathcal{L}}_N + C) \mathcal{N}_+^{1/2} \xi_N \rangle \leq \langle \xi_N, \mathcal{N}_+^{1/2} [\tilde{\mathcal{L}}_N, \mathcal{N}_+^{1/2}] \xi_N \rangle + \langle \xi_N, \mathcal{N}_+ (\tilde{\mathcal{L}}_N + C) \xi_N \rangle,$$

where in the second line we used (21). Using the assumption of ξ_N being in the spectral subspace of $\tilde{\mathcal{L}}_N$ associated with energies below ζ, we get

$$\langle \xi_N, \mathcal{N}_+ \tilde{\mathcal{L}}_N \xi_N \rangle \leq \langle \xi_N, \mathcal{N}_+ \xi_N \rangle^{1/2} \langle \tilde{\mathcal{L}}_N \xi_N, \mathcal{N}_+ \tilde{\mathcal{L}}_N \xi_N \rangle^{1/2} \leq C(1+\zeta)^{1/2} \langle \tilde{\mathcal{L}}_N \xi_N, \mathcal{H}_N^{\text{mf}} \tilde{\mathcal{L}}_N \xi_N \rangle^{1/2} \leq C(1+\zeta^2).$$

On the other side, to bound the term $\langle \xi_N, \mathcal{N}_+^{1/2} [\tilde{\mathcal{L}}_N, \mathcal{N}_+^{1/2}] \xi_N \rangle$, we use that the commutator $[\tilde{\mathcal{L}}_N, \mathcal{N}_+^{1/2}]$ can be computed explicitly. More precisely, one can show that the operator $A = (\mathcal{H}_N^{\text{mf}} + 1)^{-1/2} [\tilde{\mathcal{L}}_N, \mathcal{N}_+^{1/2}] (\mathcal{H}_N^{\text{mf}} + 1)^{-1/2}$ is a self-adjoint operator on $\mathcal{F}_+^{\leq N}$ whose norm is bounded uniformly in N, leading to the bound

$$\langle \xi_N, \mathcal{N}_+^{1/2} [\tilde{\mathcal{L}}_N, \mathcal{N}_+^{1/2}] \xi_N \rangle \leq \langle \xi_N, \mathcal{N}_+ (\mathcal{H}_N^{\text{mf}} + 1) \xi_N \rangle^{1/2} \langle \xi_N, (\mathcal{H}_N^{\text{mf}} + 1) \xi_N \rangle^{1/2} \leq \delta \langle \xi_N, \mathcal{N}_+ \mathcal{H}_N^{\text{mf}} \xi_N \rangle + C(1+\zeta).$$

We conclude that

$$\langle \xi_N, \mathcal{N}_+ \mathcal{H}_N^{\text{mf}} \xi_N \rangle \leq C(1+\zeta^2).$$

By induction, similar bounds can be proved for expectations of products of the form $(\mathcal{H}_N^{\text{mf}} + 1)(\mathcal{N}_+ + 1)^k$ onto excitation vectors which are in the spectral subspace of $\tilde{\mathcal{L}}_N$ associated with energy below ζ, for any $k \in \mathbb{N}$.

Armed with these stronger bounds, one can analyze the excitation Hamiltonian $\mathcal{L}_N^{\text{mf}}$ from a different perspective and show that the cubic and quartic terms in $\mathcal{L}_N^{\text{mf}}$ are bounded by $CN^{-1/2}(\mathcal{N}+1)^2$ and are therefore negligible on low energy states, according to Bogoliubov's picture.

If now we want to apply the strategy sketched above to the Gross-Pitaevskii regime, already in the simpler case of sufficiently small unscaled potential, we find two main difficulties. First of all, the ground state energy in the Gross-Pitaevskii regime is given at leading order in N by $4\pi\mathfrak{a}_0 N$, which is strictly smaller than $\frac{N}{2}\kappa\widehat{V}(0)$. Moreover the quadratic non diagonal term are large (of order N), due to the slow decay of the interaction. Both problems are related to the fact that we need to extract from $\mathcal{L}_N^{(3)}$ and $\mathcal{L}_N^{(4)}$ important contributions to the energy of low energy states. This is what we are going to describe in Sec. III.

III. CORRELATIONS BETWEEN CONDENSATE AND EXCITATION PAIRS

In Sec. II, we emphasized many times that in order to deal with the Gross-Pitaevskii regime, we should take into account correlations among particles. We include correlations in $\mathcal{F}_+^{\leq N}$ by means of a suitable unitary operator which models the creation (annihilation) of excitation pairs out of the condensate. The idea of factoring out correlations using unitary operators in the Fock space (and in particular Bogoliubov transformations) dates back to Ref. 52. In our setting, to make sure that the truncated Fock space $\mathcal{F}_+^{\leq N}$ remains invariant, we will have to use generalized Bogoliubov transformations. For $\mu > 0$, we define the operator

$$T(\eta_H) = \exp\left[\frac{1}{2}\sum_{|p|\geq\mu}\eta_p\left(b_p^* b_{-p}^* - b_p b_{-p}\right)\right], \tag{22}$$

where we introduced generalized creation and annihilation operators

$$b_p^* = a_p^*\sqrt{\frac{N-\mathcal{N}_+}{N}}, \quad \text{and} \quad b_p = \sqrt{\frac{N-\mathcal{N}_+}{N}}\,a_p$$

for all $p \in \Lambda_+^*$. To understand the role of the b_p and b_p^* operators, it is sufficient to observe that, by (15),

$$U_N^* b_p^* U_N = a_p^*\frac{a_0}{\sqrt{N}}, \quad U_N^* b_p U_N = \frac{a_0^*}{\sqrt{N}}a_p.$$

In other words, b_p^* creates a particle with momentum $p \in \Lambda_+^*$ but, at the same time, it annihilates a particle from the condensate; it creates an excitation, preserving the total number of particles in the system. This guarantees that $T(\eta_H)$ is an operator from $\mathcal{F}_+^{\leq N}$ into itself. The action of $T(\eta_H)$ on b_p and b_p^* is reminiscent of the action of a usual Bogoliubov transformation [which would be defined as (22), but with usual creation and annihilation operators]. Indeed, for any p such that $|p| \geq \mu$, we have

$$\begin{aligned} T^*(\eta_H)b_pT(\eta_H) &= \cosh(\eta_p)b_p + \sinh(\eta_p)b_{-p}^* + d_p,\\ T^*(\eta_H)b_p^*T(\eta_H) &= \cosh(\eta_p)b_p^* + \sinh(\eta_p)b_{-p} + d_p^*, \end{aligned} \tag{23}$$

where it is possible to prove that the operators d_p and d_{-p} produce small contributions on states with a bounded number of excitations, as those we are going to consider. We refer the reader to Ref. 6 (Lemma 3.4) for the precise estimates satisfied by the d_p operators and a discussion of this point. For the sake of these notes it will be sufficient to think that the operators b_p and b_p^* act as usual creation/annihilation operators.

It remains to discuss the role of the function η_p appearing in (22). The correct choice for this function results to be

$$\eta_p = \frac{1}{N^2}\widehat{(1-f_N)}(p/N), \tag{24}$$

with $f_N(x)$ the solution of the Neumann problem

$$\left(-\Delta + \frac{1}{2}N^2V(Nx)\right)f_N(x) = \lambda_N f_N(x)$$

on the ball $|x| \leq 1/2$, with $f_N(x) = 1$ and $\partial_{|x|}f_N(x) = 0$ for $|x| = 1/2$. The function $f_N(x)$ is a slight modification of the zero-energy infinite volume scattering length and, in particular, we have

$$\left|\int N^3\, V(Nx)f_N(x) - 8\pi\mathfrak{a}_0\right| \leq \frac{C\mathfrak{a}_0^2}{N}, \tag{25}$$

to be compared with (2). The properties of $f_N(x)$ can be found in Ref. 6 (Lemma 3.1). What is relevant for the next analysis is that as a consequence of the definition (24), we have

$$|\eta_p| \leq \frac{C}{|p|^2}e^{-|p|/N}$$

for some constant $C > 0$. Hence, defining

$$\eta_H(p) = \eta_p \chi(|p| \geq \mu),$$

we have $\|\eta_H\|_2 \leq C\mu^{-1/2}$ (in the following, $\mu^{-1/2}$ will play the role of the small parameter in the general situation where the unscaled potential is not small). Note that conjugation by $T(\eta_H)$ does not change substantially the number of excitations. Indeed, by using (23), it is easy to check that the number of excitations on a excitation state $\xi_N = T(\eta)\Omega_N$, with $\Omega_N = \{1, 0, 0, \ldots, 0\} \in \mathscr{F}_+^{\leq N}$ as the vacuum state, is given by

$$\left\langle T(\eta_H)\Omega_N, \mathcal{N}_+ T(\eta_H)\Omega_N \right\rangle \leq C\|\eta_H\|^2.$$

More in general, one can show the following lemma [see Ref. 53 (Lemma 3.1)]:

Lemma III.1. For every $n \in \mathbb{N}$, there exists a constant $C > 0$ such that, on $\mathscr{F}_+^{\leq N}$,

$$T^*(\eta_H)(\mathcal{N}_+ + 1)^n T(\eta_H) \leq Ce^{C\|\eta_H\|}(\mathcal{N}_+ + 1)^n.$$

On the other side, since $\|\eta\|_{H^1} \leq C\sqrt{N}$, we expect the few excitations that we are introducing through $T(\eta_H)$ to carry a large (order N) contribution to the energy. Therefore, conjugation with $T(\eta_H)$ has a chance to decrease the vacuum expectation of the excitation Hamiltonian $\mathscr{L}_N$ to $4\pi\mathfrak{a}_0 N$ (to leading order). With this motivation in mind, we define a new excitation Hamiltonian $\mathscr{G}_N : \mathscr{F}_+^{\leq N} \to \mathscr{F}_+^{\leq N}$ by setting

$$\mathscr{G}_N = T^*(\eta_H)\mathscr{L}_N T(\eta_H) = T^*(\eta_H)U_N H_N U_N^* T(\eta_H).$$

The outcome of the action of $T(\eta_H)$ on $\mathscr{L}_N$ is summarized by the next proposition, which was proved in Ref. 6 (Proposition 4.2) [note that in Ref. 6, the high-momentum cutoff was chosen to be $\mu = \ell^{-\alpha}$, for some $\alpha > 3$ and $\ell \in (0; 1/2)$ a sufficiently small parameter].

Proposition III.2. Let $V \in L^3(\mathbb{R}^3)$ be compactly supported, pointwise non-negative, and spherically symmetric. Then,

$$\begin{aligned}
\mathscr{G}_N &= 4\pi\mathfrak{a}_0 N + \mathscr{H}_N \\
&\quad + [2\widehat{V}(0) - 8\pi\mathfrak{a}_0] \sum_{p \in \Lambda_+^* : |p| \leq \mu} a_p^* a_p (1 - \mathcal{N}_+/(2N)) \\
&\quad + 4\pi\mathfrak{a}_0 \sum_{p \in \Lambda_+^* : |p| \leq \mu} \left[b_p^* b_{-p}^* + b_p b_{-p}\right] \\
&\quad + \frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N)\left[b_{p+q}^* a_{-p}^* a_q + h.c.\right] \\
&\quad + \mathscr{E}_{\mathscr{G}_N},
\end{aligned} \tag{26}$$

where there exist constants $C, \alpha, \beta > 0$ such that

$$\pm\mathscr{E}_{\mathscr{G}_N} \leq \frac{C}{\mu^\alpha}\mathscr{H}_N + C\mu^\beta.$$

We see that indeed the action of $T(\eta_H)$ renormalizes the constant part of the energy (at leading order) and the nondiagonal quadratic contributions. Let us quickly explain the mechanism behind the outcome of Proposition III.2. Writing $T = e^{B(\eta_H)}$, with $B(\eta_H) = (1/2)\sum_{p \in \Lambda_+^*} \eta_H(p)\left(b_p^* b_{-p}^* - b_p b_{-p}\right)$, we observe that

$$\begin{aligned}
\mathscr{G}_N &= T^*(\eta_H)\mathscr{L}_N T(\eta_H) = e^{-B(\eta_H)}\mathscr{L}_N e^{B(\eta_H)} \\
&\simeq \mathscr{L}_N + [\mathscr{L}_N, B(\eta_H)] + \frac{1}{2}[[\mathscr{L}_N, B(\eta_H)], B(\eta_H)] + \ldots.
\end{aligned} \tag{27}$$

The commutator $[\mathscr{L}_N, B(\eta_H)]$ contains the contributions $[\mathscr{K}, B(\eta_H)]$ and $[\mathscr{V}_N, B(\eta_H)]$. Up to small errors, we find

$$[\mathscr{K}, B(\eta_H)] \simeq \sum_{\substack{p \in \Lambda_+^* \\ |p| \geq \mu}} p^2 \eta_p \left[b_p^* b_{-p}^* + b_p b_{-p}\right] \tag{28}$$

and

$$[\mathcal{V}_N, B(\eta_H)] \simeq \frac{1}{2N} \sum_{\substack{p,q\in\Lambda_+^* \\ |p+q|\geq\mu}} \widehat{V}(q/N)\eta_{q+p}\big[b_p^* b_{-p}^* + b_p b_{-p}\big]. \tag{29}$$

In fact, the commutator $[\mathcal{V}_N, B(\eta_H)]$ is approximately quartic in creation and annihilation operators. Rearranging it in normal order, however, we obtain the quadratic contribution (29) (the remaining, normally ordered, quartic term is negligible). With the appropriate choice of the coefficients η_p [given by (24)], we can combine the large term

$$\frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\big[b_p^* b_{-p}^* + \text{h.c.}\big]$$

with (28) and (29) so that their sum can be estimated by $C\|\eta_H\|(\mathcal{N}_+ + 1)$. At the same time, the second commutator $[[\mathcal{L}_N, B(\eta_H)], B(\eta_H)]$ produces new constant terms that, again with the choice (24) of η_p, change the vacuum expectation to its correct value $4\pi\mathfrak{a}_0 N$.

An important remark is that conjugation by $T(\eta_H)$ leaves the cubic term and the quadratic diagonal terms in $\mathcal{L}_N$ unchanged. For interactions κV with sufficiently small intensity $\kappa > 0$, we can bound all the diagonal quadratic terms by $C\kappa(\mathcal{H}_N + 1)$. Moreover, after writing the cubic term in position space, we get

$$\begin{aligned}
&\Big|\Big\langle \xi, \frac{1}{\sqrt{N}}\int dxdy N^3 \kappa V(N(x-y)) b_x^* a_y^* a_x \xi\Big\rangle\Big| \\
&\leq \Big[\int dxdy N^2 \kappa V(N(x-y)) \|a_x a_y \xi\|^2\Big]^{1/2} \\
&\qquad \times \Big[\int dxdy N^3 \kappa V(N(x-y)) \|a_x \xi\|^2\Big]^{1/2} \\
&\leq C\kappa^{1/2}\langle \xi, \mathcal{V}_N \xi\rangle^{1/2}\langle \xi, \mathcal{N}_+ \xi\rangle^{1/2} \\
&\leq C\kappa^{1/2}\langle \xi, \mathcal{H}_N \xi\rangle.
\end{aligned}$$

Hence, for weak interaction potentials, Proposition III.2 immediately implies the lower bound

$$\mathcal{G}_N \geq 4\pi\mathfrak{a}_0 N + \frac{1}{2}\mathcal{H}_N - C_{\mu,\kappa}\mathcal{N}_+ - C,$$

where the constant $C_{\mu,\kappa} > 0$ can be chosen to be sufficiently small by choosing μ^{-1} and κ sufficiently small. Hence, the error term proportional to the number of particles operator can be controlled by the gap in the kinetic energy, and one can repeat for the Gross-Pitaevskii interaction, the same strategy sketched in Sec. II A. This is the approach used in Ref. 49 to show condensation with an optimal rate for bosons in the Gross-Pitaevskii regime, under the assumption of a small unscaled potential.

For large potentials, it is clear that conjugation by $T(\eta_H)$ is not enough to take advantage of the kinetic energy gap. In fact, we can only show the following proposition [see Ref. 6 (Proposition 4.2) for a proof].

Proposition III.3. Let $V \in L^3(\mathbb{R}^3)$ be compactly supported, pointwise non-negative, and spherically symmetric. Then,

$$\mathcal{G}_N = 4\pi\mathfrak{a}_0 N + \mathcal{H}_N + \theta_{\mathcal{G}_N},$$

where for every $\delta > 0$, there exists constants C, α, $\beta > 0$ such that

$$\pm\theta_{\mathcal{G}_N} \leq \delta\mathcal{H}_N + C\mu^{\alpha}(\mathcal{N}_+ + 1)$$

and the improved lower bound

$$\theta_{\mathcal{G}_N} \geq -\delta\mathcal{H}_N - C\mathcal{N}_+ - C\mu^{\beta} \tag{30}$$

holds true for μ (of order one) sufficiently large and $N \in \mathbb{N}$ large enough.

The remaining part of these notes are devoted to explain how to extend our analysis to large potentials. Looking at (26), it might appear evident that one possible route to this extension is to take into account for additional correlations, to renormalize the cubic term on the rhs of (26). Mathematically, this is achieved by conjugating $\mathcal{G}_N$ with an additional unitary operator, given by the exponential of an operator cubic in creation and annihilation operators, as described in Sec. IV.

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IV. CORRELATIONS DUE TO TRIPLETS

To renormalize the cubic term on the rhs of (26), we include correlations due to triplets. For a parameter $0 < \nu < \mu$, we define the low-momentum set

$$P_L = \{p \in \Lambda_+^* : |p| \le \nu\}.$$

Notice that the high-momentum set entering in the quadratic operator $T(\eta_H)$

$$P_H = \{p \in \Lambda_+^* : |p| \ge \mu\}$$

and P_L are separated by a set of intermediate momenta $\nu < |p| < \mu$. We introduce the operator $A : \mathcal{F}_+^{\le N} \to \mathcal{F}_+^{\le N}$, by

$$A = \frac{1}{\sqrt{N}} \sum_{\substack{r \in P_H \\ v \in P_L}} \eta_r \big[b^*_{r+v} a^*_{-r} a_v - \text{h.c.} \big]. \tag{31}$$

While the generalized Bogoliubov transformation $T(\eta_H)$ used in the definition of $\mathcal{G}_N$ describes scattering processes involving two excitations with momenta p and $-p$ and two particles in the condensate (i.e., two particles with zero momentum), the cubic operator A corresponds to processes involving two excitations with large momenta p and $p + v$, an excitation with small momentum v, and a particle in the condensate. Here, large and small refer to the expected value of the sound velocity $\sqrt{16\pi\mathfrak{a}_0}$, which represents the separation between momenta for which we expect a linear spectrum of excitations and momenta for which the quasiparticles behave as free particles; see Fig. 1.

Similarly to what discussed for $T(\eta_H)$, conjugation with e^A does not substantially change the number of excitations. Indeed, the following lemma is proved in Ref. 6 (Sec. V).

Lemma IV.1. Suppose that A is defined as in (31). For any $k \in \mathbb{N}$, there exists a constant $C > 0$ such that the operator inequality

$$e^{-A}(\mathcal{N}_+ + 1)^k e^A \le C(\mathcal{N}_+ + 1)^k$$

holds true on $\mathcal{F}_+^{\le N}$, for all $\mu > \nu > 0$, and N large enough.

We use now the cubic phase e^A to introduce a new excitation Hamiltonian, defining

$$\mathcal{R}_N := e^{-A}\, \mathcal{G}_N\, e^A$$

on a dense subset of $\mathcal{F}_+^{\le N}$. The definition of the excitation Hamiltonian $\mathcal{R}_N$ corresponds to rewrite N-particle wave functions in the form

$$\psi_N = U^* e^A\, T(\eta_H)\xi_N, \tag{32}$$

with $\xi_N \in \mathcal{F}_+^{\le N}$. Conjugation with e^A renormalizes the diagonal quadratic term and the cubic term on the rhs of (26), effectively replacing the singular potential $\widehat{V}(p/N)$ by a potential decaying already on momenta of order one. The mechanism for this renormalization is similar to the one described around (27). Again, expanding to second order, we find

$$\mathcal{R}_N = e^{-A}\mathcal{G}_N e^A \simeq \mathcal{G}_N + [\mathcal{G}_N, A] + \frac{1}{2}[[\mathcal{G}_N, A], A] + \dots. \tag{33}$$

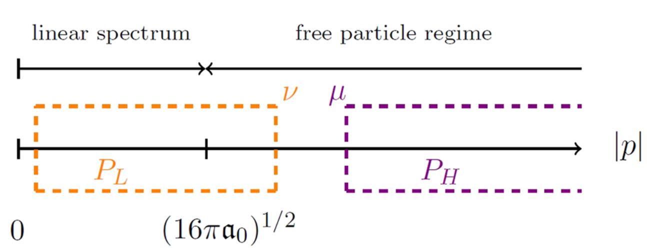

FIG. 1. Schematical picture of the high and low momenta sets entering in the definition of the cubic operator A defined in (31). The arrowed line represents the energy scale of the problem going from zero energy to high energy (ultraviolet). There are two energy scales in our problem: the first is the inverse of the range of the potential, which in our case is of order N; the second is provided by the expected value of the velocity of sound, equal to $\sqrt{16\pi\mathfrak{a}_0}$, which is in our setting of order one. The latter corresponds to the scale below which the low energy excitation spectrum behaves linearly.

From the canonical commutation relations (ignoring the fact that A is cubic in generalized, rather than standard, field operators), we conclude that $[\mathscr{K}, A]$ and $[\mathscr{V}_N, A]$ are cubic and quintic in creation and annihilation operators, respectively. Some of the terms contributing to $[\mathscr{V}_N, A]$ are not in normal order; i.e., they contain creation operators lying to the right of annihilation operators. When we rearrange creation and annihilation operators to restore normal order, we generate an additional cubic contribution. There are therefore two cubic contributions arising from the first commutator $[\mathscr{G}_N, A]$ on the rhs of (33). Moreover, the first commutator between the cubic term left in $\mathscr{G}_N$ and A, and the second commutator $[[\mathscr{H}_N, A], A]$ produces the quadratic contributions that renormalizes the diagonal quadratic term in $\mathscr{G}_N$. Indeed, one ends up with the following proposition, whose proof can be found in Ref. 6 (Sec. 8.6).

Proposition IV.2. Let $V \in L^3(\mathbb{R}^3)$ be compactly supported, pointwise non-negative, and spherically symmetric. Then, for all choices of $\mu^{1/2} < \nu < \mu^{2/3}$, there exist $\kappa, \alpha > 0$ and a constant $C > 0$ such that

$$\mathscr{R}_N = 4\pi\mathfrak{a}_0 N - 4\pi\mathfrak{a}_0\frac{\mathcal{N}_+^2}{N} + \mathcal{H}_N + 8\pi\mathfrak{a}_0 \sum_{p\in\Lambda_+^*,\,|p|\le\mu} a_p^* a_p\left(1-\frac{\mathcal{N}_+}{N}\right) + 4\pi\mathfrak{a}_0 \sum_{p\in\Lambda_+^*,\,|p|\le\mu}\left[b_p b_{-p} + b_p^* b_{-p}^*\right] + \frac{8\pi\mathfrak{a}_0}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*\\ |p|\le\mu,\, p\neq -q}}\left[b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\right] + \mathscr{E}_{\mathscr{R}_N} \quad (34)$$

with

$$\pm\mathscr{E}_{\mathscr{R}_N} \le C\mu^{-\kappa}(\mathscr{H}_N + 1) + C\mu^{\alpha}.$$

We notice that $\mathscr{R}_N$ is almost an excitation Hamiltonian for a mean field potential $8\pi\mathfrak{a}_0\chi(|p|\le\mu)$. More precisely, we define the function $v_\mu \in L^\infty(\Lambda)$ by setting

$$v_\mu(x) := 8\pi\mathfrak{a}_0 \sum_{p\in\Lambda^*:\,|p|\le\mu} e^{ip\cdot x}.$$

In other words, v_μ is defined so that $\widehat{v}_\mu(p) = 8\pi\mathfrak{a}_0$ for all $p\in\Lambda^*$ with $|p|\le\mu$ and $\widehat{v}_\mu(p) = 0$ otherwise. Observe, in particular, that $\widehat{v}_\mu(p)\ge 0$ for all $p\in\Lambda^*$. Using (15), it is easy to check that most of the terms on the rhs of (34) can be obtained by computing $UN^{-1}\sum_{i<j}^N v_\mu(x_i - x_j)U^*$, and in fact, we obtain the lower bound

$$\mathscr{R}_N \ge \frac{1}{N} U\sum_{i<j}^N v_\mu(x_i - x_j)U^* + (1 - C\mu^{-\alpha})(\mathscr{H}_N + 1) - \frac{4\pi\mathfrak{a}_0}{N}\sum_{\substack{lp,q,r\in\Lambda_+^*\\ |r|\le\mu,\, r\neq -p,-q}} a_{p+r}^* a_q^* a_p a_{q+r} - C\frac{\mathcal{N}_+^2}{N} - C\mu^{\beta}.$$

Following a standard argument for mean field potentials with non-negative Fourier transform [e.g., Ref. 34 (Lemma 1)], we find

$$\frac{1}{N}\sum_{i<j} v_\mu(x_i - x_j) \ge 4\pi\mathfrak{a}_0 N - C\mu^3.$$

Using then the bound

$$\frac{4\pi\mathfrak{a}_0}{N}\sum_{\substack{p,q\in\Lambda_+^*,|r|\le\mu:\\ r\neq -p,-q}}\langle\xi, a_{p+r}^* a_q^* a_p a_{q+r}\xi\rangle \le \frac{C}{N}\sum_{\substack{p,q\in\Lambda_+^*,|r|\le\mu:\\ r\neq -p,-q}}\|a_{p+r}a_q\xi\|\,\|a_p a_{q+r}\xi\| \le \frac{C\mu^3}{N}\|\mathcal{N}_+\xi\|^2,$$

we conclude that there exists $\beta > 0$ such that

$$\mathscr{R}_N \ge 4\pi\mathfrak{a}_0 N + \frac{1}{2}\mathscr{H}_N - \mu^3\mathcal{N}_+^2/N - C\mu^{\beta}. \quad (35)$$

If we were on a subspace of $\mathscr{F}_+^{\le N}$ with $\mathcal{N}_+ \le cN$, for sufficiently small $c > 0$, we could conclude that

$$\mathscr{R}_N \ge 4\pi\mathfrak{a}_0 N + c\mathcal{N}_+ - C$$

thus allowing us to show that $\mathcal{N}_+$ is bounded on low energy states. This observation suggests us to apply localization techniques developed by Lewin-Nam-Serfaty-Solovej in Ref. 36 (inspired by previous work of Lieb-Solovej in Ref. 46) based on localization of the number of excitations. On sectors with few excitations, we can control all the error terms in $\mathscr{R}_N$ by the gap in the kinetic energy operator. On the other hand, on sectors with many excitations, we are going to use that we do not have condensation, and therefore the energy per particle must be strictly larger than $4\pi\mathfrak{a}_0 N$ [due to the estimate (10)], as described in Sec. V.

V. LOCALIZATION TECHNIQUES AND BOSE-EINSTEIN CONDENSATION

The aim of this section is to explain how the application of localization techniques from Ref. 36 allows us to show the optimal rate of condensation and similar bounds for the energy of excitations.

Let $f, g : \mathbb{R} \to [0;1]$ be smooth, with $f^2(x) + g^2(x) = 1$ for all $x \in \mathbb{R}$. Moreover, assume that $f(x) = 0$ for $x > 1$ and $f(x) = 1$ for $x < 1/2$. We fix $M = cN$ and we set $f_M = f(\mathcal{N}_+/M)$ and $g_M = g(\mathcal{N}_+/M)$. It follows from Ref. 6 (Proposition 4.3) that

$$\mathcal{G}_N - 4\pi\mathfrak{a}_0 N \geq f_M(\mathcal{G}_N - 4\pi\mathfrak{a}_0 N)f_M + g_M(\mathcal{G}_N - 4\pi\mathfrak{a}_0 N)g_M - \frac{C\mu^{1/2}}{N^2}(\mathcal{H}_N + 1) \tag{36}$$

for $\mu > 0$, $N \in \mathbb{N}$, and $M \in \mathbb{N}$ large enough. To bound $f_M\mathcal{G}_N f_M$, we conjugate $\mathcal{G}_N$ by e^{-A} and use the lower bound (35)

$$\begin{aligned} f_M\mathcal{G}_N f_M &\geq f_M e^{A}\,\mathcal{R}_N\, e^{-A} f_M \\ &\geq 4\pi\mathfrak{a}_0\, N f_M^2 + f_M e^{A}\left[\frac{1}{2}\mathcal{H}_N - \mu^3\mathcal{N}_+^2/N - C\mu^{\alpha}\right]e^{-A} f_M \\ &\geq 4\pi\mathfrak{a}_0\, N f_M^2 + f_M e^{A}\left[\frac{1}{2}\mathcal{H}_N - \mu^{\kappa}\mathcal{N}_+\right]e^{-A} f_M - C\mu^{\alpha} f_M^2, \end{aligned} \tag{37}$$

where we used Lemma IV.1 and chose $M = \mu^{-3-\kappa}N$. Using the gap in the kinetic energy and once more Lemma IV.1, we conclude that for μ large enough

$$f_M\mathcal{G}_N f_M \geq 4\pi\mathfrak{a}_0\, N f_M^2 + C f_M^2\mathcal{N}_+ - C\mu^{\alpha} f_M^2. \tag{38}$$

On g_M using (10), one can claim that there exists a constant $C > 0$ such that

$$g_M\mathcal{G}_N g_M \geq 4\pi\mathfrak{a}_0\, N g_M^2 + C g_M^2 N \tag{39}$$

for all N sufficiently large. Indeed, if this was not the case, one could build, starting from an excitation vector $\xi \in \mathcal{F}^{\leq N}_{\geq M/2}$ with at least $M/2 = \mu^{-3-\kappa}N/2$ particles, an approximate ground state of H_N. But this would contradict (10) since the ratio between the expected number of excitations on ξ and the total number of particles would not go to zero as $N \to \infty$. We refer the reader to Ref. 6 (Sec. 6) for details. From (39), using $\mathcal{N}_+ \leq N$, we get

$$g_M\mathcal{G}_N g_M \geq 4\pi\mathfrak{a}_0\, N g_M^2 + C\mathcal{N}_+ g_M^2. \tag{40}$$

Inserting (38) and (40) on the rhs of (36), we obtain that

$$\mathcal{G}_N \geq 4\pi\mathfrak{a}_0 N + C\mathcal{N}_+ - CN^{-2}\mathcal{H}_N - C \tag{41}$$

for N large enough [the constants C are now allowed to depend on μ and ν, since the cutoff has been fixed once and for always after (38)]. Interpolating (41) with the lower bound

$$\mathcal{G}_N \geq 4\pi\mathfrak{a}_0 N + \frac{1}{2}\mathcal{H}_N - C\mathcal{N}_+ - C,$$

obtained using (30), we get

$$\mathcal{G}_N \geq 4\pi\mathfrak{a}_0 N + c\,\mathcal{N}_+ - C. \tag{42}$$

The condensation bound follows easily from (42). Now let $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and

$$\langle \psi_N, H_N\psi_N\rangle \leq 4\pi\mathfrak{a}_0 N + \zeta.$$

Recalling that $\mathcal{G}_N = e^{-B(\eta_H)}U_N H_N U_N^* e^{B(\eta_H)}$ and defining the excitation vector $\xi_N = e^{-B(\eta_H)}U_N\psi_N$, we have

$$\langle \xi_N, \mathcal{N}_+ \xi_N \rangle \le C \langle \xi_N, (\mathcal{G}_N - 4\pi \mathfrak{a}_0 N)\xi_N \rangle + C \le C(1+\zeta). \tag{43}$$

Following a strategy similar to one described for the mean-field case in Sec. II A, we can show the following stronger bounds on excitation vectors; see Ref. 7 (Sec. IV) for their proof.

Proposition V.1. Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported, and spherically symmetric. Let E_N be the ground state energy of the Hamiltonian H_N defined in (16) [or, equivalently, in (1)]. Let $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ belong to the spectral subspace of H_N with energies below $E_N + \zeta$, for some $\zeta > 0$, i.e.,

$$\psi_N = \mathbf{1}_{(-\infty; E_N+\zeta]}(H_N)\psi_N . \tag{44}$$

Let $\xi_N = e^{-B(\eta)} U_N \psi_N$ be the renormalized excitation vector associated with ψ_N. Then, for any $k \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$\langle \xi_N, (\mathcal{N}_+ + 1)^k (\mathcal{H}_N + 1)\xi_N \rangle \le C(1 + \zeta^{k+1}) .$$

Using these bounds, we are now in the position to establish the validity of Bogolibov theory in the Gross-Pitaevskii regime, as pictured in Sec. VI. Notice also that the bound (43) also implies an improved bound for the trace norm convergence in (10). In fact, if γ_N denotes the one-particle reduced density matrix associated with ψ_N, we obtain

$$\begin{aligned} &1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \\ &= 1 - \frac{1}{N}\langle \psi_N, a^*(\varphi_0) a(\varphi_0)\psi_N \rangle \\ &= 1 - \frac{1}{N}\langle U_N^* e^{B(\eta_H)}\xi_N, a^*(\varphi_0) a(\varphi_0) U_N^* e^{B(\eta_H)}\xi_N \rangle \\ &= \frac{1}{N}\langle e^{B(\eta_H)}\xi_N, \mathcal{N}_+ e^{B(\eta_H)}\xi_N \rangle \\ &\le \frac{C}{N}\langle \xi_N, \mathcal{N}_+ \xi_N \rangle \le \frac{C(K+1)}{N}, \end{aligned}$$

where in the last line, we used Lemma III.1.

VI. BOGOLIUBOV THEORY

In this section, we sketch the proof of Theorem I.1, as proved in Ref. 7; we refer to the review Ref. 54 for a more extended presentation of this part, which is only slightly modified whenever we remove the assumption of smallness of the potential. The key idea is that using the bounds in Proposition V.1, we can give a second look to the excitation Hamiltonian $\mathcal{G}_N = e^{-B(\eta_H)} U_N H_N U_N^* e^{B(\eta_H)}$, and identify terms which go to zero as $N \to \infty$ on low energy states. More precisely, one finds

$$\mathcal{G}_N = C_N + \mathcal{Q}_N + \mathscr{C}_N + \mathscr{V}_N + \delta_N, \tag{45}$$

where C_N is a constant, $\mathcal{Q}_N$ is quadratic,

$$\mathscr{C}_N = \frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^*, p \ne -q} \widehat{V}(p/N)\, b^*_{p+q} b^*_{-p} \big(\gamma_q b_q + \sigma_q b^*_{-q}\big) + \text{h.c.}$$

with $\gamma_p = \cosh(\eta_H((p))$ and $\sigma_p = \sinh(\eta_H(p))$, and the error term satisfies the bound

$$\pm \delta_N \le \frac{C}{\sqrt{N}}\Big[(\mathscr{H}_N + 1)(\mathscr{N}_+ + 1) + (\mathcal{N}_+ + 1)^3\Big].$$

If there was no cubic term on the rhs of (45), we could obtain the ground state energy and spectrum of $\mathcal{G}_N$ just by diagonalizing a quadratic Hamiltonian. In fact, the quartic interaction can be bounded from above by CN^{-1}on suitable states [see Ref. 7 (Lemma 6.1)] and can be neglected due to positivity of the interaction as far as lower bounds are concerned.

Once more, the strategy to renormalize the large (order one) cubic term is to conjugate $\mathcal{G}_N$ by a suitable unitary operator, given by the exponential of the operator

$$\tilde{A} = \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^* \\ |r|\geq\sqrt{N}, \\ |v|<\sqrt{N}}} \eta_r\Big[\sigma_v b^*_{r+v} b^*_{-r}\big(\gamma_v b_v + \sigma_v b^*_{-v}\big) - \text{h.c.}\Big].$$

Notice that, as in the definition of A in (31), the operator $\tilde{A}$ describes the scattering between two excitations with high momenta, one excitation with low momenta and one particle in the condensate, but with a different notion of "high" and "small" momenta with respect to (31). We introduce a new excitation Hamiltonian $\mathcal{J}_N : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ through

$$\mathcal{J}_N = e^{-\tilde{A}} \mathcal{G}_N e^{\tilde{A}} = e^{-\tilde{A}} T^*(\eta_H) U_N H_N U_N^* T(\eta_H) e^{\tilde{A}}. \tag{46}$$

The latter can be decomposed as

$$\mathcal{J}_N = \tilde{C}_N + \tilde{\mathcal{Q}}_N + \mathcal{V}_N + \tilde{\delta}_N,$$

where $\tilde{C}_N$ and $\tilde{\mathcal{Q}}_N$ are constant and quadratic in annihilation and creation operators and where

$$\pm\tilde{\delta}_N \leq CN^{-1/4}\Big[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3\Big].$$

In particular, the expression of the quadratic part makes evident the effect of the renormalization obtained by conjugating $\mathcal{L}_N$ with the unitary operators $T(\eta)$ and $e^{-\tilde{A}}$. We have in fact

$$\tilde{\mathcal{Q}}_N = \sum_{p\in\Lambda_+^*} \Big[F_p b_p^* b_p + G_p\big(b_p^* b_{-p}^* + b_p b_{-p}\big)\Big]$$

with

$$F_p = p^2\big(\gamma_p^2 + \sigma_p^2\big) + \big(\widehat{V}(\cdot/N) * \widehat{f}_N\big)_p\big(\gamma_p + \sigma_p\big)^2$$
$$G_p = 2p^2\gamma_p\sigma_p + \big(\widehat{V}(\cdot/N) * \widehat{f}_N\big)_p\big(\gamma_p + \sigma_p\big)^2.$$

We see that the Fourier transform of the interaction potential $\widehat{V}(p/N)$ has been replaced everywhere by $\big(\widehat{V}(\cdot/N) * \widehat{f}_N\big)_p$, whose value for $p = 0$ is related to the scattering length $\mathfrak{a}_0$ through the relation (25). One can check that for all $p \in \Lambda_+^*$,

$$p^2/2 \leq F_p \leq C(1 + p^2), \quad |G_p| \leq C/p^2, \quad |G_p| < F_p,$$

and therefore, we can introduce coefficients $\tau_p \in \mathbb{R}$ such that

$$\tanh(2\tau_p) = -\frac{G_p}{F_p}$$

for all $p \in \Lambda_+^*$. Using these coefficients, we define the generalized Bogoliubov transformation $e^{B(\tau)} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ with

$$T(\tau) := \exp\Bigg[\frac{1}{2}\sum_{p\in\Lambda_+^*} \tau_p\big(b_{-p}^* b_p^* - b_{-p} b_p\big)\Bigg].$$

Notice that, since $|\tau_p| \leq C|p|^{-4}$ for all $p \in \Lambda_+^*$, we can show that [see Ref. 7 (Lemma 5.2)]

$$T^*(\tau)(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1)T(\tau) \leq C(\mathcal{N}_+ + 1)(\mathcal{H}_N + 1).$$

That is, the generalized Bogoliubov transformation $T(\tau)$ does not change substantially neither the number nor the energy of the excitations. Conjugation of the excitation Hamiltonian $\mathcal{J}_N$ defined in (46) with $T(\tau)$ leads to the excitation Hamiltonian $\mathcal{M}_N : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ defined as

$$\mathscr{M}_N = T^*(\tau)\mathcal{J}_N T(\tau) = T^*(\tau)e^{-\tilde{A}}\,T^*(\eta_H)U_N H_N U_N^* T(\eta_H)e^{\tilde{A}}\,T(\tau).$$

One finally finds

$$\begin{aligned}\mathscr{M}_N &= 4\pi\mathfrak{a}_0(N-1) + e_\Lambda \mathfrak{a}_0^2\\ &\quad + \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[\sqrt{p^4+16\pi\mathfrak{a}_0 p^2} - p^2 - 8\pi\mathfrak{a}_0 + \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right]\\ &\quad + \sum_{p\in\Lambda_+^*}\sqrt{p^4+16\pi\mathfrak{a}_0 p^2}\,a_p^* a_p + \mathcal{V}_N + \delta_N'\end{aligned}$$

with

$$\pm\delta_N' \le CN^{-1/4}\Big[(\mathcal{H}_N+1)(\mathcal{N}_++1) + (\mathcal{N}_++1)^3\Big].$$

Theorem I.1 follows from the min-max principle, since on low energy states of the diagonal Hamiltonian, we find $\mathcal{V}_N \le CN^{-1}(\zeta+1)^{7/2}$ [with ζ entering in the spectral assumption (44)].

The results of Theorem I.1, together with standard arguments as in Ref. 35 (Sec. 7), also provide an approximation for the eigenvectors corresponding to low energy states. In particular, if ψ_N denotes a ground state vector of the Hamiltonian H_N, one can show that there exists a phase $\omega \in [0;2\pi)$ such that

$$\left\|\psi_N - e^{i\omega}U_N^*\,T(\eta_H)\,e^{\tilde{A}}\,T(\tau)\Omega\right\|^2 \le \frac{C}{\theta_1-\theta_0}N^{-1/4}, \tag{47}$$

where $\theta_0 \le \theta_1 \le \ldots$ denote the ordered eigenvalues of H_N.

It is interesting to compare Eq. (47) with the approximation for the ground state vector within Bogoliubov's approximation that would have been of the form $U_N^*\tilde{T}(\tilde{\tau})\Omega$, with

$$\tilde{T}(\tilde{\tau}) = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^*}\tilde{\tau}_p(a_p^* a_{-p}^* - a_p a_{-p})\right]$$

as the usual Bogoliubov transformation and coefficients $\tilde{\tau}_p \in \mathbb{R}$ that in the Gross-Pitaevskii regime would be defined by

$$\tanh(2\tilde{\tau}_p) = -\frac{\widehat{V}(p/N)}{p^2+\widehat{V}(p/N)}$$

[see Ref. 9 (Appendix A)]. The unitary transformation $\tilde{T}(\tilde{\tau})$ is the one that diagonalizes the quadratic terms in the Bogoliubov Hamiltonian, and so it has the same role of the transformation $T(\tau)$ in our approach. On the other side, while the kernel $\tilde{\tau}_p$ has a large H^1- norm [similarly to the kernel η_p defined in (24)], the kernel τ_p has both the L^2 and H^1 norms uniformly bounded in N. Hence, the diagonalizing unitary transformation $T(\tau)$ does not change substantially neither the number nor the energy of excitations. We see that the trick to take into account for the correlations among excitations neglected in Bogoliubov theory was to implement two additional unitary transformations $T(\eta_H)$ and $e^{\tilde{A}}$ which have extracted the large energy contained in the cubic and quartic terms. In particular, one of the consequences of the action of $T(\eta_H)$ and $e^{\tilde{A}}$ is the renormalization of the quadratic terms of the excitation Hamiltonian $\mathcal{R}_N$, leading to the appearance of the convolution $\big(\widehat{V}(\cdot/N) * \widehat{f}_N\big)_p$ in the definition of the coefficients τ_p.

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