Doctoral Thesis

Gathering Self-Interested People Together: a Strategic Perspective

PhD Program in Computer Science: XXXIII cycle

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April 2021

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List of Publications

This PhD thesis is based on the following works.

Conference proceedings:

- M. Flammini, B. Kodric, G.Varricchio. *Strategyproof Mechanisms for Friends and Enemies Games.* In proceedings of the 34th Conference on Artificial Intelligence, AAAI'20, pages 1950-1957;
- M. Flammini, B. Kodric, M. Olsen, G.Varricchio. *Distance Hedonic Games.* (extended abstract) In proceedings of the 19th International Conference on Autonomous Agents and Multiagent Systems, AAMAS '20, pages 1846-1848;
- M. Flammini, B. Kodric, M. Olsen, G.Varricchio. *Distance Hedonic Games.* In proceedings of the 47th International Conference on Current Trends in Theory and Practice of Computer Science, SOFSEM'21.

Accepted with minor revision:

- M. Flammini, B. Kodric, G.Varricchio. *Strategyproof Mechanisms for Friends and Enemies Games.* Artificial Intelligence, AIJ.

In preparation:

- M. Flammini, G.Varricchio. *Strategyproof Mechanisms for the Additively Separable Group Activity Selection Problem.* Manuscript.

Abstract

Over the past decades, the understanding of how individuals spontaneously gather together received particular attention leading to the definition of several variants of *Coalition Formation Games*, including the so-called *Hedonic Games*. In these latter, the individuals (or agents) have to be split into disjoint coalitions and express preferences only on the coalition they belong to, and not on how the others aggregate. Subsequently, the more general class of the *Group Activity Selection Problem*, where agents' preferences depend also on the activity they are performing, has been introduced. In both these classes of games, the study of the existence, the computability, and the efficiency of suitable stability solution concepts as well as the elicitation of agents' preferences through strategyproof mechanisms have been addressed.

In this work, we consider both the aforementioned research directions. In particular, we put our attention on classes of games in which agents' preferences are expressed by a *utility function* and we evaluate the global agents' satisfaction in a given outcome by means of the *utilitarian social welfare*. Moreover, we often compare the social welfare of the considered solutions with the social optimum, that is the maximum achievable value of the social welfare.

We first introduce and study a new model in the Hedonic Games setting, called *Distance Hedonic Games*, and we focus on the computation and the efficiency of Nash stable outcomes, i.e. coalition structures in which no agent can unilaterally improve her gain by deviating to another coalition. We then turn our attention to the design of strategyproof mechanisms for two specific classes of games: namely, *Friends and Enemies Games* and the *Additively Separable Group Activity Selection Problem*. In both cases, we measure the performances of the proposed mechanisms by considering their approximation ratio with respect to the social optimum.

To my first teacher,
my father,
my Number Devil.

Acknowledgements

First of all, I would like to express my sincere gratitude to Michele Flammini, not only my PhD advisor but also a guide and a friend, one of the best persons one can ever meet.

My deepest thank goes to Martin Olsen and, in particular, to Bojana Kodric for the valuable collaboration.

I can only thank everyone who has made these past three years special. Starting from the department members to all the people I have met in L'Aquila and at GSSI. Among the others Sara and Carlo, without them this experience would have not been the same.

Special thanks to my mother and my brother, as well as to all my big family and to my friends who have been there for me, each one in her/his own way: naming all of them would be impossible, I just hope they know how much I care about them.

Finally, many things I earned from the past three years, the most important is Diego's love; simply thanking him might not be enough.

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Chapter 1

Introduction

In many real-world scenarios, individuals prefer to perform tasks or activities by gathering together rather than being on their own. Simply think about researchers or employees working on different projects, politicians forming parties, but also people attending social events, and so forth. The increasing interest in economical, political, and social contexts in which individuals attempt common goals by splitting into groups led to the definition of *Coalition Formation Games*.

Coalition Formation Games model multi-agent systems where selfish agents form coalitions and have preferences over the possible outcomes of the game. More specifically, an outcome of a Coalition Formation Game is a partition of the agents into disjoint coalitions, referred to as a coalition structure, and agents express their satisfaction with the outcomes through preference relations. When agents' preferences only depend on the coalition they belong to, and not on how the other agents aggregate, we talk about *Hedonic Games*, introduced by Dreze and Greenberg in [36]. Hedonic Games have been widely studied [10] and, according to the properties of agents' preferences or other possible constraints, numerous subclasses have been defined.

While on the one hand, Hedonic Games offer a model sophisticated enough to describe a large variety of settings, on the other hand, they are not able to capture the satisfaction that agents have for the task they are eventually supposed to perform together. To this aim, the *Group Activity Selection Problem*, a proper generalization of the Hedonic Games, was introduced in [31]. Here, a set of possible tasks (or activities) is available, and the agents in a coalition have to perform a common task, one for each coalition. Thus, agents' preferences are based on both the coalition they belong to and the activity they are supposed to perform.

In the Hedonic Games and Group Activity Selection Problem literature different approaches have been considered, that can be broadly attributed to one of the two following research scenarios:

- the understanding of which kind of coalition structures (outcomes) the selfish behavior of the agents leads to;
- the elicitation of the agents' real preferences while maintaining good properties for the designed outcome of the game.

Regarding the first research direction, agents' preferences are known in advance and one of the main goals is to understand the outcomes that may be reached by the agents, both from an existential and an algorithmic perspective. To this aim, several stability concepts have been defined based either on individual or group deviations. As an example, *Nash stability* is one of the well-known solution concepts in Coalition Formation Games; namely, an outcome is Nash stable if no agent wants to leave her current coalition to join an already formed one or to form a new coalition alone. In addition, when agents' preferences are expressed by utilities, also the quality of the outcomes is considered by means of a *social welfare* function. Hence, it is possible to measure the quality of stable outcomes by comparing them to the *optimum* (the maximum achievable social welfare): classical examples are the ratios between the optimum and the worst/best Nash stable outcome, known as the *Price of Anarchy* and the *Price of Stability*, respectively.

Regarding the second research direction, agents' preferences are private information and must be communicated by the agents themselves to a designer that splits them accordingly. Since agents are selfish, they may misreport their preferences so as to maximize their satisfaction with the final outcome of the game. Thus the designer, solely based on the reported preferences, must compute an outcome of the game in order to induce a truthful behavior of the agents while satisfying good properties in the computed outcome, as for instance a good approximation of the maximum social welfare. This is known as the *Mechanism*

Design framework. In particular, in this setting, the outcome of the game is computed by an algorithm commonly called *mechanism*; if it has the property that no agent has incentive to misreport her preferences, then it is said to be a *strategyproof* mechanism.

In this thesis, we consider both the two above scenarios. The original contributions for the two different lines are described in the next section.

1.1 Contribution

We focus on games with *cardinal preferences*, that is, agents express their satisfaction with a given outcome by assigning it a real value, also called *utility*. Under this assumption, for a given outcome it is possible to determine a global value measuring the overall agents' satisfaction. In our specific case, we measure it through the *utilitarian social welfare* function, i.e., the sum of all the agents' utilities.

As our first contribution, we define a new large class of games, called *Distance Hedonic Games*, where preferences are based on a social graph. This model properly generalizes previously existing Hedonic Games like *Fractional Hedonic Games* and *Social Distance Games*. Here the agents evaluate the coalition they belong to according to their centrality in the subgraph induced by the coalition itself. Such a centrality is defined as a suitable function of the distances from the other agents in the coalition. Different centrality measures can be defined, leading to different games. In particular, we assume the existence of a scoring vector whose components represent the contribution of agents at a fixed distance. In this setting, we focus on Nash stable outcomes in the arising games. In particular, we give NP-hardness and inapproximability results on the problems of finding a social optimum and a best Nash stable outcome for the two natural scenarios in which the scoring vector has non-decreasing or non-increasing entries. Moreover, we evaluate the performance of the Nash stable outcomes in terms of the achieved social welfare, relying on the notions of the Price of Anarchy and of the Price of Stability.

We then turn our attention to the Mechanism Design framework in two relevant classes of games, called *Friends and Enemies Games* and *Additively Separable Group Activity Selection Problem*, respectively. In both cases we focus on mechanisms that achieve a good welfare approximation.

More precisely, we first investigate strategyproof mechanisms for *Friends and Enemies Games*, a subclass of Hedonic Games in which every agent classifies any other one as a *friend* or as an *enemy*. In this setting, we consider the two classical scenarios proposed in the literature, called *Friends Appreciation* and *Enemies Aversion* preferences profiles. Roughly speaking, in the former setting each agent gives priority to the number of friends in her coalition, while in the latter to the number of enemies. We provide strategyproof mechanisms achieving bounded approximation ratio. In particular, for both the aforementioned preference profiles we provide deterministic mechanisms that linearly (in the number of agents) approximate the optimum; in the specific case of enemies aversion preferences, we also show that no approximation better than linear can be achieved in polynomial time. Moreover, we show that at some cost (namely relying on either randomized or exponential time algorithms), it is possible to accomplish strategyproofness and constant approximation ratio.

Finally, we investigate strategyproof mechanisms for the Group Activity Selection Problem with the additively separable property: agents have values, simply called preferences, one for each activity and individual weights for the other agents. Thus, the utility of an agent in an outcome is given by the sum of the individual weights for the agents in her coalition and of the preference for the activity she is performing. In this setting, we exploit different scenarios for the values attributed to the activities and between agents. As a first result, we show that in general, when preferences are non-negative reals, no deterministic strategyproof can have bounded approximation ratio. To circumvent this impossibility result, we at first rely on randomization. In this case, we are able to provide a randomized mechanism that linearly, in the number of activities, approximates the optimum; moreover, we show that, when activities are copyable, that means different agents may perform the same activity simultaneously but non necessarily together, a constant approximation is possible. We then turn our attention to deterministic mechanisms when agents' preferences among activities are either boolean or publicly known, meaning that agents cannot manipulate the value given to the activities: in both these cases, we show that it is possible to provide strategyproof mechanisms with bounded approximation ratio.

1.2 Outline of the Thesis

The rest of the thesis is organized as follows. In Chapter 2 we provide an overview of the notation and the basic definitions that will be useful in this work. In Chapter 3 we introduce and investigate Distance Hedonic Games. We then turn our attention to strategyproof mechanisms for both Friends and Enemies games and the Additively Separable Group Activity Selection Problem, in Chapter 4 and 5, respectively. Finally, in Chapter 6 we give some conclusive remarks and some future research directions.

Chapter 2

Background

In this chapter, we present the basic notions that will be useful in the rest of the thesis. In particular, in Section 2.1 we introduce the classes of games that are the object of our study: namely, the Hedonic Games and the Group Activity Selection Problem. We then turn our attention to two different research directions, in Section 2.2 and Section 2.3, respectively. More precisely, in the former section, we describe classical (stable) solution concepts that have been studied in the literature, in the latter, we focus on the Mechanism Design framework, having both these aspects a central role in this work.

2.1 Games Definition

Let us now formally define two different, but nonetheless closely related, classes of games that have the common goal of splitting agents into disjoint coalitions. We start by describing the Hedonic Games, and we then introduce the more general class of the Group Activity Selection Problem.

2.1.1 Hedonic Games

Hedonic Games (HGs) represent a subclass of Coalition Formation Games where agents have *hedonic preferences*, that is, their preferences depend only on the coalition they are involved in and not on how the other agents aggregate in the other coalitions.

Formally, we are given a set $N = \{1, \dots, n\}$ of selfish agents, and the goal of the game is to partition them into a collection of disjoint coalitions $\mathcal{C} =$

$\{C_1, \ldots, C_m\}$ such that $\cup_{i=1}^{m} C_i = N$. Such a partition is also called an *outcome* or a *coalition structure*. The *grand coalition* GC is a coalition structure that consists of only one coalition containing all the agents and a *singleton coalition* is any coalition of size 1. We denote by $\mathscr{C}$ the set of all possible outcomes, and by $\mathcal{C}(i)$ the coalition that agent i belongs to in outcome $\mathcal{C} \in \mathscr{C}$. Each agent i has a *preference relation*, or simply called *preference*, $\succeq_i$ over $N_i = \{S \subseteq N \mid i \in S\}$ the family of subsets of N containing i. According to $\succeq_i$, for every $X, Y \in N_i$, we say that agent i prefers or equally prefers X to Y whenever $X \succeq_i Y$, and we say that agent i equally prefers X to Y whenever $X \sim Y$. We are now ready to define an HG instance.

Definition 2.1. A *Hedonic Game instance* is given by a pair (N, P), where N is the set of agents and $P = \{\succeq_i\}_{i \in N}$ is the *preference profile*, that is, the collection of agents' preferences. Moreover, for every agent $i \in N$ the preference relation $\succeq_i$ is a reflexive, complete and a binary relation on N_i.

According to the properties satisfied by P different types of HGs can be defined, some examples will be described in Subsection 2.1.3.

2.1.2 Group Activity Selection Problem

In the previous subsection, we assumed that agents gather together into coalitions and express preferences only on their own group members, without providing any preference on what they are supposed to do. In fact, in the HG model, the agents that split into coalitions either do not care about what they are doing or must perform the same task regardless of the coalition they belong to. However, there are contexts in which, once grouped together, they will perform some task among a set of available ones or they will participate to some activity.

A model able to capture such an aspect, i.e. how the performed task impacts on agents' preferences, has been introduced in the literature as the Group Activity Selection Problem (GASP), where it is assumed that agents in the coalitions can perform different activities (one for each coalition). More formally, in the GASP we are given a set of agents N and a set $A = \{a_1, \ldots, a_k\}$ of k *activities*, and agents' preferences also depend on the activity they perform. Agents have to be split into disjoint coalitions and each coalition perform a common activity among the ones in A; moreover, if an agent does not perform any real activity in A, then she is assumed to perform the *void activity* $a_\emptyset$ alone. We denote by $A^* = A \cup \{a_\emptyset\}$ the set of all the activities and the void activity. An agent i expresses her preferences, through $\succeq_i$, over the set of alternatives $X_i = \{(a, S) \,|\, a \in A,\, S \in N_i\} \cup \{a_\emptyset\}$, where an alternative for an agent is either a pair of the activity she is performing and the coalition she is involved in or the void activity. An *outcome*, or *assignment*, $\mathbf{z}$ is a map from the set of agents to the set of all the activities, $a_\emptyset$ included, that is $\mathbf{z} : N \to A^*$. Thus, given an agent $i \in N$ $\mathbf{z}(i)$, or simply z_i, stands for agent i is performing activity $z_i \in A$; moreover, if $z_i = z_j$ for some $i, j \in N$, $i \neq j$, then i and j are performing the same activity, and thus i, j are in the same coalition, unless z_i and z_j are the void activity. We denote by z_i^{-1} the set of the agents that are participating to the same activity of i in the assignment $\mathbf{z}$, i included; if $z_i = a_\emptyset$ then we set $z_i^{-1} = \{i\}$.

Definition 2.2. A *Group Activity Selection Problem instance* is given by a triple (N, A, P), where N is the set of agents, A is the set of activities and $P = \{\succeq_i\}_{i \in N}$ is the *preference profile*, that is the collection of agents' preferences over alternatives. Moreover, for every agent $i \in N$ the preference relation $\succeq_i$ is a reflexive, complete and binary relation on X_i.

As already mentioned, HGs can be seen as a special subclass of the GASP where agents' preferences are *activity independent*, that is, for any $i \in N$, for any $a, a' \in A$ and for any $S \in N_i$ we have $(a, S) \sim (a', S)$, and the number of the activities is at least the number of the agents, meaning that the number of the formed coalitions in an outcome may be any.

2.1.3 Succinct Representation and Cardinal Preferences

In the HGs and in the GASP agents' preferences are a binary relation over N_i and X_i respectively; however, these sets have exponentially large size in the number of agents, implying that, in general, the game instances may be exponentially large as well. In this regard, a good portion of the literature focused on those game instances that admit a *succinct representation*, i.e. the game instance is polynomial in the number of agents. Such an assumption is crucial in a computer science perspective, where the computation of an outcome of the game, with respect to some criterion, should be computationally tractable. To this aim several specific classes have been introduced and studied. Among the others, HGs [20] and the GASP [31] with anonymous preferences; in this setting, agents do not care about which but only about how many agents are involved in their coalition (or are participating to their same activity). Thus, the representation is rather simple since it is sufficient for an agent to express preferences over $\{1, \dots, n\}$ in HGs and over $\{(a, i) \,|\, a \in A\,, i = 1, \dots, n\} \cup \{a_\emptyset\}$ in the GASP. The anonymity assumption has also been considered together with further constraints like graph constraints on coalition structures [47, 49] or additional restriction on preferences [24, 28, 32].

Beyond the aforementioned classes with anonymous preferences, other relevant succinct representable classes of games are known in the literature [9]. In this thesis, we are particularly interested in those classes for which not only preferences are succinct representable but also can be expressed in a cardinal form. Such a requirement gives the possibility to understand how much an agent is satisfied with a given outcome and also to measure the overall agents' satisfaction, which will be discussed in the next section. When preferences are in a cardinal form they are described through a *utility function*, instead of a preference relation, that is, in general, a map $u_i : N_i \to \mathbb{R}$ in HGs and $u_i : X_i \to \mathbb{R}$ in the GASP, for each $i \in N$. Without loss of generality, we assume the preference profile P to be either the collection of agents preferences $\{\succeq_i\}_{i \in N}$ or the collection of agents utilities $\{u_i\}_{i \in N}$, depending on the game instance.

In HGs there are two well known examples of this kind of preferences: HGs with the *additively separable property*, also known as *Additively Separable Hedonic Games* (ASHGs) [13, 20], and *Fractional Hedonic Games* (FHGs) [4]. In both these models, each agent $i \in N$ has a *valuation function* $w_i : N \setminus \{i\} \to \mathbb{R}$, where $w_i(j)$, or simply $w_{i,j}$, is the value that agent i gives to agent j. Given a coalition structure $\mathcal{C}$ the utility agent i achieves in the coalition $\mathcal{C}(i)$ is given by $u_i(\mathcal{C}(i)) = \sum_{j \in \mathcal{C}(i) \setminus \{i\}} w_i(j)$, in ASHGs, and by $u_i(\mathcal{C}(i)) = \frac{1}{|\mathcal{C}(i)|} \sum_{j \in \mathcal{C}(i) \setminus \{i\}} w_i(j)$, in FHGs. Other than ASHGs and FHGs also the class of *modified Fractional Hedonic Games* (MFHGs) [39, 55, 58] have been considered, where, differently from FHGs, the utility of an agent is normalized by the coalition size minus one.

In the special case where $w_{i,j} = w_{j,i}$, $\forall\, i, j \in N$ with $i \neq j$, the aforementioned classes of games are said to be *symmetric*.

Similarly to the ASHGs, the GASP with the additively separable property, ASGASP in short, has been considered in [17]. In this case, not only the values $w_{i,j}$ are provided but also a *preference function* which expresses the value that agent i gives to each activity in A: for each $i \in V$ we have the corresponding preference function $p_i : A \to \mathbb{R}$. Moreover, $p_i(a_\emptyset) = 0$ for each $i \in N$. Given an assignment $\mathbf{z}$, $\delta_i(\mathbf{z}) = \sum_{j \in z_i^{-1} : j \neq i} w_{i,j}$ is the overall evaluation of i for all the agents participating to z_i. Thus, the utility achieved by i in the assignment $\mathbf{z}$ is defined as $u_i(\mathbf{z}) = \delta_i(\mathbf{z}) + p_i(z_i)$. Notice that, if i is associated to the void activity $a_\emptyset$, by definition the utility of i is 0, independently of the other agents assigned to $a_\emptyset$.

Graph Representation If agents' preferences among the other agents are expressed by single values, then we can represent such preferences by means of a directed and weighted graph where the agents are the vertices and the weight of a directed edge (i, j) is exactly the value $w_{i,j}$. More formally, an ASHG or FHG instance may be described by a directed graph $G = (V, E, w)$ where $V = N$ and $w_{i,j} = w_i(j)$ for each $(i, j) \in E$; moreover, $(i, j) \notin E$ if and only if $w_i(j) = 0$. This graph can also be used to partially describe an ASGASP instance; indeed, such a graph represents only the valuation functions of the agents and not their preferences over the activities. To completely describe an ASGASP instance we relay on an extension of G that is obtained by adding the activities as vertices and a weighted edge form each agent $i \in N$ to any activity $a \in A$ having weight $p_i(a)$.

2.2 Solution Concepts

In HGs and in the GASP, part of the literature focused on which outcomes the selfish behavior of the agents leads to. To this aim, several solution concepts and stability notions have been defined. In what follows, we provide an overview of some of these solution concepts. We often refer to a game instance by simply writing instance $\mathcal{I}$, if specifying whether it is an HG or a GASP instance is not necessary; similarly, we will refer to a coalition structure or an assignment writing an outcome $O \in \mathcal{O}$, where $\mathcal{O}$ is the collection of all the possible outcomes for the instance $\mathcal{I}$.

2.2.1 Single and Group Deviations

Traditionally, in CFGs the focus has been put onto the existence and efficiency of several solution concepts, based either on individual [19, 41, 44] or group deviations [13, 20, 40, 44, 48], Nash stability and core stability being the main non-cooperative and cooperative notions, respectively.

Below we informally describe some of the stability concepts that have been introduced. We distinguish between solutions based on single-agent deviations and solutions based on group deviations.

Stability Concepts based on Single-Agent Deviations An outcome $O \in \mathcal{O}$ for an instance $\mathcal{I}$ is said to be:

- *Individually rational*, if no agent benefits by moving from her current coalition in O and forming a new coalition by herself.
- *Nash stable*, if no agent benefits by either moving from her current coalition in O to an already formed one or forming a new coalition by herself.
- *Individually stable*, if no agent benefits by moving from her current coalition in O to another, possibly empty, while not making the members of the new coalition worse off.
- *Contractually individually stable*, if no agent benefits by moving from her current coalition in O, possibly empty, to another while not making the members of the new and old coalition worse off.

In particular, changing an outcome O into O' makes an agent i worse off if O' is a strictly less preferred outcome than O, that is, $O' \prec_i O$.

$$
\begin{tikzcd}
 & \text{Nash stable} \arrow[d, Rightarrow] & \\
\text{strictly core stable} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{individually stable} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \begin{array}{c}\text{contractually}\\ \text{individually stable}\end{array} \\
\text{core stable} & \text{individually rational} &
\end{tikzcd}
$$

FIGURE 2.1: Relations between stability concepts.

Stability Concepts based on Group Deviations An outcome $O \in \mathcal{O}$ for an instance $\mathcal{I}$ is said to be:

- *Core stable*, if no subset of agents can improve the gain of each member by forming together a new coalition.
- *Strictly core stable*, if no subset of agents can improve the gain of at least one member and not decrease for the others by forming together a new coalition.

The relations between these solution concepts are summarized in Figure 2.1, where for any couple of solution concept A, B, $A \Rightarrow B$ stands for A implies B.

Finally, we formally define the Pareto optimality solution concept. An outcome O is said to be *Pareto optimal* (PO) if there is no other outcome O' such that for each $i \in N$ $O \preceq_i O'$ and for at least one agent the inequality is strict. In other words, an outcome is PO if it is not possible to gather the agents in such a way the new outcome is more or equally preferred by all the agents and for at least one agent it is strictly better.

All the aforementioned solution concepts can also be defined when preference relations are expressed by utilities; in this case, it is sufficient to compare the utility achieved by the agents in the outcomes. What we are going to describe next are solution concepts and quality measures that can be studied when preferences are expressed in a utilitarian form.

2.2.2 Social Optimum and Quality of Equilibria

In Section 2.1 we introduced cardinal preferences, that is, agents express preferences among the possible outcomes by utility functions. In this specific setting, it is possible to measure the overall agents' satisfaction by using the *social welfare* (SW) function. In this work, we always assume the SW to be utilitarian, also called *utilitarian social welfare*. The utilitarian SW for an outcome O is computed by adding up the utilities of all the agents for that specific outcome, that is, $\mathrm{SW}(O) = \sum_{i \in N} u_i(O)$. Moreover, let $\mathcal{O}$ be the set of all the possible outcomes for the game instance $\mathcal{I}$, we define the *social optimum*, or simply the *optimum*, as the maximum achievable SW among all the possible outcomes, that is, $\mathbf{opt}(\mathcal{I}) = \max_{O \in \mathcal{O}} SW(O)$. When the instance is clear from the context we simply write **opt**. The computation of the optimum and the outcome(s) achieving the optimum received particular attention in the literature [6, 16–18].

The SW may also be used to measure the quality of the equilibria. In particular, as regards to Nash stable outcomes, two measures to understand how good they are with respect to the social optimum have been provided: the *Price of Anarchy* [51, 59] and the *Price of Stability* [2], that are the wort/best case ratio between the optimum and the social welfare achievable by a Nash stable outcome. More formally, denoted by $\mathsf{NS}(\mathcal{I}) \subseteq \mathcal{O}$ the set containing all the Nash stable outcomes for the instance $\mathcal{I}$

$$\mathrm{PoA}(\mathcal{I}) = \max_{O \in \mathsf{NS}(\mathcal{I})} \frac{\mathbf{opt}(\mathcal{I})}{\mathrm{SW}(O)} \qquad \text{and} \qquad \mathrm{PoS}(\mathcal{I}) = \min_{O \in \mathsf{NS}(\mathcal{I})} \frac{\mathbf{opt}(\mathcal{I})}{\mathrm{SW}(O)} \ .$$

Thus, for a class of games $\mathcal{G}$ the Price of Anarchy and the Price of Stability are defined as the worst value $\mathrm{PoA}(\mathcal{I})$ and $\mathrm{PoS}(\mathcal{I})$ can achieve among all the instances $\mathcal{I}$ of the class $\mathcal{G}$, that is,

$$\mathrm{PoA} = \sup_{\mathcal{I} \in \mathcal{G}} \mathrm{PoA}(\mathcal{I}) \qquad \text{and} \qquad \mathrm{PoS} = \sup_{\mathcal{I} \in \mathcal{G}} \mathrm{PoS}(\mathcal{I}) \ .$$

From now on we will always assume that agents express preferences using utilities.

2.3 Mechanism Design Framework

While most of the literature in GASP and HGs concentrated on investigating suitable stability criteria, like Nash and core stability, assuming that agents' preferences are known, in this section, we consider the setting in which preferences are private information of the agents. In particular, we focus on Mechanism Design without Money. We give some motivations in Section 2.3.1, then we formally define the setting in Section 2.3.2.

2.3.1 The Need for Truthfulness

In problems like HGs and the GASP we can assume agents to report their preferences, or the utility functions, and a *central authority* upon the declared preferences has to compute a stable solution with respect to some stability notion or an optimal solution according to the defined SW. However, agents are selfish and they may strategically misreport preferences in order to improve their gain. Thus, the central authority wants to avoid these situations in which agents try to maximize their utility instead of attempting global satisfaction. For this scope, a careful designed mechanism incentives a truthful behavior among players. A *mechanism* is an algorithm that, taken in input the declared preferences, computes an outcome of the game. Thus, the central authority proposes a mechanism to the agents and it must not be possible for the agents to strategically misreport their preferences so as to increase their satisfaction with the outcome. In other words, truthfully reporting their real preferences is a dominant strategy for all the agents.

Although several truthful mechanisms use payments as incentive to truthfully report private information, it is not always the case that money exchanges are permitted both for legal and ethical issues [56], or simply because allowing payments is not feasible [60]. Thus, the mechanism can only use the allocation rule as incentive to truthfulness; in this case we talk about *Mechanism Design without Money*.

2.3.2 Mechanisms Design without Money

In the classical *Mechanism Design without Money* setting there are n selfish agents, stored in N, and each agent $i \in N$ has a private information v_i; moreover, each agent may declare an information d_i that is possibly different from v_i. A mechanism $\mathcal{M}$ computes an outcome $\mathcal{M}(\mathbf{d})$ taking into account agents' declarations stored in $\mathbf{d} = (d_1, \dots, d_n)$. Denoted by $\mathbf{d}_{-i}$ the declarations of all the agents except of i, we may write $\mathbf{d}$ as the pair $(\mathbf{d}_{-i}, d_i)$. Thus, a *deterministic mechanism* $\mathcal{M}$ is an algorithm that, for any declaration $\mathbf{d}$, outputs an outcome of the game $\mathcal{M}(\mathbf{d})$, while a *randomized mechanism* $\mathcal{M}$ maps every declaration $\mathbf{d}$ to a distribution Δ over all the possible outcomes. In this case, the expected utility of agent i is given by $\mathbb{E}\left[u_i(\mathcal{M}(\mathbf{d}))\right] = \mathbb{E}_{O\sim\Delta}[u_i(O)]$.

Definition 2.3. A deterministic (resp. randomized) mechanism is said to be *strategyproof* if for any $i \in V$, $\mathbf{d}_{-i}$, d_i, and true v_i, it satisfies $u_i(\mathcal{M}(\mathbf{d}_{-i}, v_i)) \geq u_i(\mathcal{M}(\mathbf{d}_{-i}, d_i))$ (resp. $\mathbb{E}\left[u_i(\mathcal{M}(\mathbf{d}_{-i}, v_i))\right] \geq \mathbb{E}\left[u_i(\mathcal{M}(\mathbf{d}_{-i}, d_i))\right]$).

A deterministic (resp. randomized) mechanism is said to be *manipulable* if it is not strategyproof.

Since payments are not permitted, new problems come out: properties such as monotonicity and weak monotonicity [3, 52] no longer suffice for truthfulness. Moreover, mechanisms like the VCG require payments for truthfulness [37]; thus, it is not possible to apply such mechanisms in this setting.

Mechanisms Design without Money has been widely studied in the literature; a classical example is the *House Allocation problem*. In the House Allocation problem there are n selfish agents interested in n different houses; their evaluations of the houses are expressed by their preference relations among them. The central authority has to select an outcome, i.e. a matching between agents and houses, that guarantees the truthfulness. Gale's Top Trading Cycle [62] is a well known truthful implementation for the House Allocation problem that returns the unique core stable matching of the game.

Another challenge for the central authority is to provide not only a mechanism inducing a truthful behavior but also achieving a good solution with respect to the SW function. However, it may occur that the mechanism has to return an approximate solution for two different reasons: on the one hand, the underlaying optimization problem may not be computationally tractable, on the other hand, returning the optimal solution does not guarantee strategyproofness [60]. Hence, it is possible to evaluate the performance of a mechanism $\mathcal{M}$ through the corresponding approximation ratio with respect to the optimum, that is, $r^{\mathcal{M}} = \sup_{\mathbf{d}} \frac{\mathbf{opt}(\mathbf{d})}{\mathrm{SW}(\mathcal{M}(\mathbf{d}))}$ if $\mathcal{M}$ is deterministic, and $r^{\mathcal{M}} = \sup_{\mathbf{d}} \frac{\mathbf{opt}(\mathbf{d})}{\mathbb{E}[\mathrm{SW}(\mathcal{M}(\mathbf{d}))]}$ if $\mathcal{M}$ is randomized.

An example related to this perspective, and also very close to the GASP, is Mechanism Design problem for the strategic variant of *Generalized Assignment Problem* without using payments, developed in [37]. In this work, the authors design, for several variants of the Assignment Problem, mechanisms that incentive agents to truthfully report their preferences among the available items while approximating the utilitarian SW function.

Chapter 3

Distance Hedonic Games

In this chapter, we introduce and study a new model of HGs where agents are nodes in a given graph and evaluate their coalition according to their (induced) distance from the other participants. In this new setting, we show hardness and inapproximability results for the problem of finding a best Nash stable outcome; moreover, we provide bounds on both Price of Anarchy and Price of Stability for different classes of instances.

3.1 Introduction

In many natural scenarios, the relations between agents can be modeled as a graph, and the utilities that agents receive from their coalition strongly depend on the graph structure. Classical examples are ASHGs [20], FHGs [4] and Social Distance Games (SDGs) [23]. We introduce a broader class of games called *Distance Hedonic Games* (DHGs), that properly generalizes both symmetric unweighted FHGs and SDGs. In particular, while in SDGs each agent x contributes to the utility of another agent y in her coalition in an inversely proportional fashion with respect to their distance, and in symmetric unweighted FHGs only if they are neighbors, in DHGs we assume the existence of a scoring vector, in which the i-th coefficient expresses the extent to which x contributes to the utility of y if they are at distance i. The scoring vector is assumed to be the same for all agents. We focus on the most natural types of scoring vectors whose coefficients have a monotone growth and distinguish between decreasing and increasing vectors. In the case of decreasing vectors, if the distance between two agents increases, the interest of being together decreases. Thus, a decreasing vector models situations in which agents want to aggregate themselves into coalitions in which they are close together. In the case of increasing vectors, the relationships in the graph may represent competition between agents or anti-sympathy. Note that this does not violate the hedonic nature of the game. In fact, the graph structure does not necessarily represent the will of being together (or friendship) but it can represent, for example, the physical distance between agents. Thus, the idea of increasing vectors is to reach far away nodes while relying only on internal communication.

In this chapter we may refer to symmetric FHGs simply writing FHGs.

Unweighted FHGs and SDGs are in a way related to each other. In fact, while in unweighted FHGs the utility of an agent is proportional to her degree centrality, in SDGs the degree centrality measure is substituted by the harmonic centrality. Other than degree and harmonic centrality, DHGs encompass also any other node centrality measure that is a function of the node distances. For instance, the Dangalchev centrality measure [25] and its generalized version [26] also fit into our model. On the other hand, if the goal is to create working teams where agents want to participate but do not want a central role in the group (as an agent's centrality increases, so do the responsibilities), then an increasing scoring vector seems suitable. A further example for increasing vectors is the setting in which agents are nodes (that send and receive information) in a wireless network. The underlying graph then represents the multi-hop communication network in which faraway nodes must rely on intermediate ones to exchange messages. In particular, an edge between two nodes means that they can communicate, because they are in their mutual ranges. However if they transmit at the same time, they also create interference. Thus, in this sense it is better to have few neighbors in a coalition. At the same time, nodes in the same coalition have no incentive of being disconnected, since then there is no way to communicate with each other, even in a multi-hop fashion. A related example is influence spreading and collecting information, where an agent wants to reach far away nodes in the network, but needs to have a path to each of them in order to increase her utility. Here, it is also natural that closer agents are less valuable but necessary for reaching further away, thus more valuable, agents.

In conclusion, by using our model with varying scoring vectors, one can investigate how agents aggregate in different scenarios, including both interesting new and already studied ones.

3.1.1 Previous Work

As already mentioned, FHGs are a subclass of HGs in which the utility of an agent is given by the sum of her preferences for each single member of her coalition, divided by the size of the coalition. The special case of agents' preferences being in $\{0,1\}$ is termed unweighted FHGs. FHGs have also been investigated both with respect to group deviations [4, 8, 22] and with respect to individual deviations [15, 16, 58]. In particular, in [16] different aspects of FHGs are exhibited, showing that the existence of a stable outcome is not guaranteed if the graph has negative weights, while the grand coalition is always stable when weights are non-negative. Moreover, the paper studies the efficiency of equilibria, providing bounds on the Price of Anarchy and the Price of Stability while considering different topologies and different assumptions on the weights. Finally, the authors focus on the complexity of computing a best Nash stable outcome and show that this problem is NP-hard for weighted as well as unweighted graphs.

SDGs have been introduced in [23], where they are studied with respect to group deviations. In particular, the work focuses on core stability. Furthermore, the paper shows the NP-hardness of finding a coalition structure with maximum social welfare and provides a 2-approximation algorithm for the problem. In the work [11], SDGs are investigated with respect to individual deviations, and the NP-hardness of finding a best Nash stable outcome is proven. The focus is then set on determining the performance of stable outcomes. Bounds on both the Price of Anarchy and the Price of Stability are provided, with a particular focus on graphs with girth of at least 4. Some of the bounds are improved by [50]. Finally, Pareto optimality and stability has also been considered in the paper [12].

Scoring Vector	PoA	PoS	
General	∞ (Prop. 3.3)	∞	(Prop. 3.3)
Non-negative	∞ (Prop. 3.4)	$\Theta(n)$	
Non-negative and Normalized	$\leq M_\alpha(n-1)$ (Thm. 3.9)	$\leq \min\{\frac{M_\alpha}{m_\alpha}, \frac{n}{2} \cdot M_\alpha\}$	(Thm. 3.9)
Constant	1	1	
Decreasing normalized	$n-1$ (Thm. 3.9)	$\leq n \cdot \frac{\alpha_2+\sqrt{\alpha_2^2+(1-\alpha_2)^2}}{1+2\alpha_2\cdot(n-1)}$	for $\alpha_2 \leq \frac{1}{2}$, girth ≥ 5 (Thm. 3.17)
		$\frac{(2-\alpha_2)^2}{(1-\alpha_2)(1+2\alpha_2\cdot(n-1))}$	on trees (Thm. 3.22)
Increasing normalized	$\Theta\left(\frac{\mathrm{SW}(\mathsf{P}_n)}{n}\right)$ (Thm. 3.24)	$\Theta\left(\frac{\mathrm{SW}(\mathsf{P}_n)}{n}\right)$	(Thm. 3.24)
Increasing with $\alpha_1 = 0$	∞	$\Theta(n)$	

TABLE 3.1: Upper Bounds on the PoA and the PoS, where m_α and M_α denote the min and the max component of the scoring vector α, respectively. By $\mathrm{SW}(\mathsf{P}_n)$ we denote the social welfare achieved by a path of n nodes.

3.1.2 Our Contribution

We introduce and study DHGs, a natural generalization of both unweighted FHGs and SDGs. We give a broad picture of both the PoA and the PoS, distinguishing the following scoring vectors: general (with possibly negative coefficients), non-negative, non-negative and normalized (i.e., the first component is equal to 1), and constant. The focus of the paper, however, is set on non-negative scoring vectors with monotonically decreasing and monotonically increasing coefficients. We give improved PoS bounds for decreasing normalized scoring vectors whose second component is $\leq \frac{1}{2}$ and the underlying graph has girth ≥ 5 or when it is a tree, and for increasing normalized scoring vectors. Our bound on the PoS for decreasing normalized scoring vectors, with second component is $\leq \frac{1}{2}$ and girth ≥ 5, is in fact a generalization of the one obtained in [11]. Indeed, when the second component is exactly $\frac{1}{2}$, the bound equals $\frac{1}{2} + \frac{1}{\sqrt{2}}$. Finally, we give NP-hardness and inapproximability results for the problems of finding a social optimum and a best Nash stable outcome (the one with the highest social

Scoring Vector	MSW	MSW-S
Decreasing Normalized	NP-hard [8],[23]	NP-hard for $\alpha_2 \leq \frac{1}{2}$ (Thm. 3.6)
Increasing normalized	no poly-time with approx.$< \frac{2^{n+1}}{n^2+1}$ (Thm.3.8)	
Increasing with $\alpha_1 = 0$	no poly-time with approx. > 0 (Thm. 3.7)	no poly-time with approx. > 0 (Thm. 3.7)

TABLE 3.2: Hardness and inapproximability results for finding the optimal (MSW) and best Nash stable (MSW-S) outcomes.

welfare).

Our results are summarized in Table 3.1 and 3.2.

3.2 Model and Preliminaries

In our model, we consider HGs instances where the set of agents N coincides with the vertices V of a given undirected and unweighted graph $G = (V, E)$.

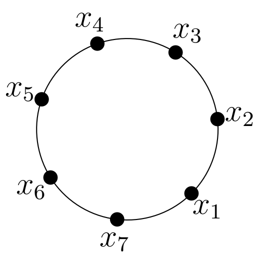
Definition 3.1. A *Distance Hedonic Game* (DHG) is defined by an undirected and unweighted graph $G = (V, E)$ and a scoring vector $\alpha \in \mathbb{R}^{n-1}$ where (i) V is a set of n agents and (ii) the *utility* of an agent $x \in V$ in a given coalition structure $\mathcal{C}$ is the evaluation of x for being in $\mathcal{C}(x)$, and it is given by

$$u_x^\alpha(\mathcal{C}(x)) = \frac{1}{|\mathcal{C}(x)|} \sum_{y \in \mathcal{C}(x) \setminus \{x\}} \alpha_{d_{\mathcal{C}(x)}(x,y)},$$

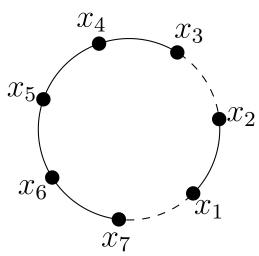
where $d_{\mathcal{C}(x)}(x, y)$ is the distance induced by the subgraph $G_{\mathcal{C}(x)} = (\mathcal{C}(x), E_{\mathcal{C}(x)})$, $E_{\mathcal{C}(x)} = \{(y, z) \in E\ : y, z \in \mathcal{C}(x)\}$. If x and y are disconnected in a coalition C, then $d_C(x, y) = \infty$ and we define $\alpha_\infty = 0$.

Let us provide a simple example showing a game instance and how to compute agents' utilities.

Example 3.1. *Let G be a cycle of 7 nodes, namely $x_1, \ldots, x_7$, and let $\mathcal{C} = \{\{x_1, x_2\}, \{x_3, \ldots, x_7\}\}$ be a coalition structure, as depicted in Figure 3.1(b). If the scoring vector is $\alpha = (1, 2, 3, 4, \ldots)$, then the utility of agent x_3 in the*



(a) The grand coalition



(b) An non-trivial outcome

FIGURE 3.1: Two coalition structures on a cycle of 7 nodes. Dashed edges are in the cut between two different coalitions.

coalition $\mathcal{C}(x_3)$ is $u^{\alpha}_{x_3}(\mathcal{C}(x_3)) = \frac{1}{5}(1+2+3+4) = 2$. On the other hand, in the grand coalition, as seen in Figure 3.1(a), the utility of each agent, so then also of agent x_3, is equal to $\frac{1}{7}(2\cdot 1 + 2\cdot 2 + 2\cdot 3) = \frac{12}{7}$.

We define a scoring vector α to be *decreasing* if $\alpha_i \geq \alpha_{i+1}$ for each $i \in [n-2]$. Similarly, α is *increasing* if $\alpha_i \leq \alpha_{i+1}$ for each $i \in [n-2]$. We say that an increasing or decreasing scoring vector α is *normalized* if $\alpha_1 = 1$.

When α is clear from context, we will simply write $u_x(\mathcal{C}(x))$. Furthermore, the social welfare of a coalition structure $\mathcal{C}$ is then given by $\mathrm{SW}(\mathcal{C}) = \sum_{x\in V} u_x(\mathcal{C}(x))$. Given a coalition $C \in \mathcal{C}$, we denote by $up(i, C)$ the number of unordered pairs that are at distance i in C. More precisely,

$$up(i, C) = \left|\{\{x, y\} \mid x, y \in C \subseteq V \wedge d_C(x, y) = i\}\right| .$$

Thus, the social welfare of a coalition C of size k is equal to

$$\mathrm{SW}(C) = \frac{2}{k}\sum_{i=1}^{k-1} \alpha_i \cdot up(i, C). \tag{3.1}$$

It is easy to check that $\sum\limits_{i=1}^{k-1} up(i, C) = \frac{k(k-1)}{2}$, and in particular for the GC it holds $\sum\limits_{i=1}^{n-1} up(i, V) = \frac{n(n-1)}{2}$.

A game instance $\langle G, \alpha\rangle$ is a pair of an undirected graph G and a scoring vector $\alpha \in \mathbb{R}^{n-1}$. Given a game instance $\langle G, \alpha\rangle$, we denote by $\mathsf{NS}_{\langle G,\alpha\rangle}$ the set of its Nash stable outcomes. A Nash stable outcome with maximum social welfare among the ones in $\mathsf{NS}_{\langle G,\alpha\rangle}$ is said to be a *best Nash stable* outcome.

Example 3.2. *Let us revisit the scenario described in Example 3.1. In the coalition structure $\mathcal{C}$, the utility of agent x_1 is given by $u^{\alpha}_{x_1}(\mathcal{C}(x_1)) = \frac{1}{2}$, while, if she deviates to the coalition of x_7 she gets $u^{\alpha}_{x_1}(\mathcal{C}(x_7) \cup \{x_1\}) = \frac{5}{2}$. In conclusion, $\mathcal{C}$ is not Nash stable. Since the scoring vector α is increasing, agent x_1 prefers to join agent x_7. Similarly, if we consider the same coalition structure $\mathcal{C}$ on G, but with $\alpha' = (1, \frac{1}{2}, \frac{1}{3}, \dots)$, that is decreasing, then agent x_7 will prefer to join agent x_1. Again, $\mathcal{C}$ is not Nash stable.*

The GC on the other hand is Nash stable for both α and α', as deviating from the GC provides a utility of 0, while for any scoring vector whose components *are non-negative, the grand coalition provides a non-negative utility. This fact, which is independent of* G*, will be stressed later on in Observation 1.*

For α *there exists no further Nash stable outcome, as agents always want to join a coalition that is of the same size or bigger than the coalition they currently belong to and this stops being possible only in the grand coalition. The grand coalition achieves a social welfare of* 12 *which is also the optimum, as the only outcome that could possibly have a higher social welfare keeps one agent in a singleton and places all other agents together, achieving a utility of* $\frac{1}{6}(2\cdot(1+2+3+4+5)+2\cdot(2\cdot 1+2+3+4)+2\cdot(2\cdot 1+2\cdot 2+3)) = \frac{35}{3} < 12$.

For α' *there exist further Nash stable outcomes other than the grand coalition, which has a social welfare of* $2\cdot 1+2\cdot\frac{1}{2}+2\cdot\frac{1}{3}=\frac{11}{3}$*. In particular, as the utility of an agent in a coalition consisting of a path of* 2 *nodes is* $\frac{1}{2}$*, as is the utility of an outer agent in a coalition consisting of a path of* 3 *nodes, and the utility of an outer agent in a coalition consisting of a path of* 4 *nodes is* $\frac{11}{24} < \frac{1}{2}$*, all coalition structures consisting of paths of* 2 *and* 3 *nodes are Nash stable. Such coalition structures again have a social welfare of* $\frac{11}{3}$*. Another group of Nash stable outcomes are the ones that consist of one path of* 3 *and one path of* 4 *nodes, with a social welfare of* $\frac{23}{6}$*, which is also the social optimum.*

Our aim is to bound the performance of Nash stable outcomes for a fixed scoring vector α by considering the *Price of Anarchy* (PoA) and the *Price of Stability* (PoS). Formally, for a fixed α

$$\mathrm{PoA}(\alpha) = \sup_{G} \max_{\mathcal{C}\in\mathsf{NS}_{\langle G,\alpha\rangle}} \frac{\mathbf{opt}}{\mathrm{SW}(\mathcal{C})} \qquad \mathrm{PoS}(\alpha) = \sup_{G} \min_{\mathcal{C}\in\mathsf{NS}_{\langle G,\alpha\rangle}} \frac{\mathbf{opt}}{\mathrm{SW}(\mathcal{C})}.$$

When for some α there exists a graph G and $\mathcal{C}\in\mathsf{NS}_{\langle G,\alpha\rangle}$ such that $\mathrm{SW}(\mathcal{C})=0$, we say that $\mathrm{PoA}(\alpha)$ is unbounded. Similarly, if for some α there exists a graph G s.t. $\forall\mathcal{C}\in\mathsf{NS}_{\langle G,\alpha\rangle}, \mathrm{SW}(\mathcal{C})=0$, we say that $\mathrm{PoS}(\alpha)$ is unbounded.

We assume that the underlying graph G is connected, as the connected components of a graph can otherwise be considered separately for bounding the PoA and PoS.

In some cases, our analysis will rely on the specific graph structure induced by the coalitions. For this reason, the introduction of the following topologies will be useful.

3.2.1 Stars

A k-star is a coalition of k nodes that form a tree with a diameter of at most 2. The *root* of a star is the node which is at distance 1 from all the others; the remaining nodes are called *leaves*. When the star is made of just two nodes, we will interchangeably refer to them as root or leaf. We denote a k-star by S_k and a coalition structure that is a star partition, i.e., in which each coalition is a star, by $\mathcal{S}$.

Given a graph G, we denote by $\Sigma(G)$ the family of all possible star partitions, where each star in the partition needs to have size of at least 2. For a fixed G, we can define a total order on $\Sigma(G)$ in the following way: given a star partition $\mathcal{S} \in \Sigma(G)$, let $n_i(\mathcal{S})$ be the number of stars in $\mathcal{S}$ of size i. Then, for any $\mathcal{S}, \mathcal{S}' \in \Sigma(G)$ we say that $\mathcal{S}$ is *co-lexicographically smaller* than $\mathcal{S}'$ if $n_j(\mathcal{S}) < n_j(\mathcal{S}')$, where j is the highest index for which $n_i(\mathcal{S})$ and $n_i(\mathcal{S}')$ differ. In that case, we write $\mathcal{S} <_{colex} \mathcal{S}'$.

While in general it is possible that $\Sigma(G) = \emptyset$, when G has girth[1]> 3, the existence of at least one star partition is always guaranteed. Moreover, if G has girth ≥ 5, any node of a star can be connected to at most one node of another star. In particular, if two roots are connected, then there are no further edges between the two star coalitions in the graph.

3.2.2 Paths

Since coalitions whose underlying subgraph is a path will be relevant to our investigation later on, here we give some formulas for computing their social welfare. We denote by P_ℓ a path of ℓ nodes.

Lemma 3.2. *Given a coalition $C \subseteq V$, if $C = \mathsf{P}_\ell$ with $\ell \geq 1$ nodes, then $SW(\mathsf{P}_\ell) = \frac{2}{\ell}\sum_{i=1}^{\ell-1} \alpha_i(\ell - i)$. Moreover, $SW(\mathsf{P}_{\ell+1}) - SW(\mathsf{P}_\ell) = \frac{2}{\ell(\ell+1)}\sum_{i=1}^{\ell} \alpha_i \cdot i$.*

[1]The girth of a graph is the length of the shortest cycle contained in it.

Proof. We start by observing that the utility of the node in position j is equal to $\frac{1}{\ell}\left(\sum_{i=1}^{j-1}\alpha_i + \sum_{i=1}^{\ell-j}\alpha_i\right)$.
Thus, the social welfare achieved by a path of ℓ nodes is given by

$$\begin{aligned}\mathsf{SW}(\mathsf{P}_\ell) &= \frac{1}{\ell}\sum_{j=1}^{\ell}\left(\sum_{i=1}^{j-1}\alpha_i + \sum_{i=1}^{\ell-j}\alpha_i\right) = \frac{1}{\ell}\left(\sum_{j=1}^{\ell}\sum_{i=1}^{j-1}\alpha_i + \sum_{j=1}^{\ell}\sum_{i=1}^{\ell-j}\alpha_i\right)\\ &= \frac{2}{\ell}\sum_{j=1}^{\ell}\sum_{i=1}^{j-1}\alpha_i = \frac{2}{\ell}\sum_{i=1}^{\ell-1}\alpha_i\,(\ell-i)\,.\end{aligned}$$

The second claim is now an immediate consequence. $\square$

3.2.3 Model Discussion

Let us briefly discuss the placement of our model among previous work and our modeling choices.

In symmetric unweighted FHGs, each agent $i \in N$ has a valuation function $w_i : N \setminus \{i\} \to \{0,1\}$ where $w_i(j) = w_j(i)$ for each $i, j \in N$ with $i \neq j$. Given a coalition structure $\mathcal{C}$ the utility agent i achieves in the coalition $\mathcal{C}(i)$ is given by

$$u_i(\mathcal{C}(i)) = \frac{1}{|\mathcal{C}(i)|}\sum_{j\in\mathcal{C}(i)} w_i(j) \ \ .$$

Furthermore, symmetric unweighted FHGs are usually represented by means of undirected and unweighted graphs where the agents are nodes and there exists an edge between two nodes if and only if the corresponding agents evaluate each other with 1.

In SDG instances, we are given an undirected and unweighted graph, where the nodes represent the agents, as input. In this case, given a coalition structure $\mathcal{C}$, the utility that agent i achieves in the coalition $\mathcal{C}(i)$ is given by

$$u_i(\mathcal{C}(i)) = \frac{1}{|\mathcal{C}(i)|}\sum_{j\in\mathcal{C}(i)} \frac{1}{d_{\mathcal{C}(i)}(i,j)} \ \ ,$$

Both symmetric unweighted FHGs and SDGs are a subclass of DHGs, as it can be verified by setting $\alpha = (1, 0, \ldots, 0)$ and $\alpha = (1, \frac{1}{2}, \ldots, \frac{1}{n-1})$, respectively. We also observe that these vectors, thus, induce utilities that are proportional to degree and harmonic centrality measures of nodes in coalitions. Moreover, other centrality measures fit into the DHG model, such as the Dangalchev centrality measure, where the i-th component of the scoring vector equals $1/2^i$, and its generalized version, where the i-th component equals β^i for any fixed β in $(0, 1)$. When $\beta = 1$, each node's centrality measure is simply the number of nodes in its coalition that it is connected to.

We observe that, both in our model and the preexisting ones, the utility of the agents depends on the distance induced by their coalitions and not on the original distance in the graph. This assumption relies not only on the hedonic character of the game, meaning that agents only care about the coalition they are involved with, but also, as it is traditional in clustering settings, on the fact that centrality measures are defined on the sole basis of the induced subgraph. On the other hand, considering original distances can be seen as a particular case of (weighted) FHGs. In fact, one can build another graph in which, if two nodes are at a fixed distance d, there is an edge between them with weight α_d.

We also want to point out that we set $\alpha_\infty = 0$, both for increasing and decreasing vectors, as we wish the agents to achieve positive utility only from the nodes they are connected to. In a way, this reflects the fact that a coalition should be independent from the others, with internal communication being private and not having to rely on external agents.

3.2.4 Preliminary Results: Quality of Equilibria

Before proceeding with our analysis we discard some classes of scoring vectors, not only because they do not represent natural scenarios, but also because of their low efficiency of Nash equilibria.

Proposition 3.3. *Given a scoring vector α that has negative components, both PoA(α) and PoS(α) can be unbounded.*

Proof. Let G be a star of $k > 4$ nodes and $\alpha = (-1, 1, 0, \dots, 0)$. Then, since the root does not want to be in any coalition, every Nash stable outcome has a social welfare of 0. On the other hand, the grand coalition achieves the social optimum, $\text{SW(GC)} = \frac{(k-1)(k-4)}{k}$. We conclude that both PoA and PoS are unbounded. □

Proposition 3.4. *Given a scoring vector α such that $\alpha_j \geq 0$ for each $j \in [n-1]$, but α is not monotone, $PoA(\alpha)$ can be unbounded.*

Proof. Consider again a star of size $k > 2$ and a scoring vector α such that $\alpha_1 = 0$, $\alpha_2 \neq 0$ and all the other components are 0. Every node being in its own coalition is a Nash stable outcome with a social welfare of 0. Instead, the social optimum is $\alpha_2 \cdot \frac{(k-1)(k-2)}{k}$. □

Notice that the proof of Proposition 3.4 also shows that the PoA can be unbounded when $\alpha_1 = 0$.

Motivated by the above arguments, and since we are here interested in classes of vectors that admit bounded PoA and PoS, in the sequel we only focus on non-negative scoring vectors, for which the following observation holds.

Observation 1. Given a scoring vector α such that $\alpha_j \geq 0$ for each $j \in [n-1]$, the grand coalition is always Nash stable.

Furthermore, we will assume that the scoring vector components are monotone, non-increasing and non-decreasing. While other scoring vector classes can still be of theoretical interest, the ones we focus on, seem also to be the most natural choices when edges in G represent friendship, or competition and antisympathy. Without loss of generality we will assume that α is normalized in such a way that $\alpha_1 = 1$. In fact, such a property can be obtained by a simple scaling argument.

We first provide hardness results concerning the determination of the best Nash stable outcomes in Section 3.3, and then we move to the analysis of the PoA and the PoS in Section 3.4.

3.3 Hardness Results

We now present computational complexity results concerning the problem of finding good solutions.

Definition 3.5. MSW (resp. MSW-S) is the problem of computing, given a DHG instance $\langle G, \alpha \rangle$, the coalition structure $\mathcal{C}^*$ with a maximum social welfare (resp. the Nash stable coalition structure with a maximum social welfare).

We show intractability or inapproximability results for the two problems, considering separately the case of decreasing and increasing scoring vectors.

3.3.1 Decreasing Vectors

Notice that, while for decreasing vectors the NP-hardness is already implied by the previous results on FHGs and SDGs, we show here a stronger result: the intractability is maintained for a large class of *fixed* scoring vectors, that is, for any fixed α with $\alpha_2 < \frac{1}{2}$.

Theorem 3.6. *MSW-S problem is NP-hard when restricted to a fixed normalized decreasing vector* α *with* $\alpha_2 < \frac{1}{2}$.

Proof. We will use a reduction from a variant of the Max-Clique Problem in which, given a graph $G = (V, E)$, a node $u \in V$, and an integer $k > \frac{2}{3} \cdot (|N(u)| + 1)$, we ask whether there exists a clique in G of size $\geq k$ containing u, where $N(u)$ is the set of nodes that are neighbors of u in G. It is easy to see that this restriction on k does not influence the NP-hardness, as for smaller k we can easily transform the instance into the desired form by adding dummy nodes. Without loss of generality, as for any fixed $\alpha_2 < 1/2$ one can choose k s.t. $\alpha_2 < (k-3)/2(k-1) < 1/2$, we prove the claim for $\alpha_2 < \frac{k-3}{2(k-1)}$.

Let n' denote the size of $N(u)$. We build the reduced MSW-S instance $G' = (V', E')$ in the following way:

- u and its neighborhood $v_1, \ldots, v_{n'}$ are vertices in the graph G' and $(u, v_i) \in E', \forall i \in [n']$;
- $(v_i, v_j) \in E \implies (v_i, v_j) \in E'$, $\forall v_i, v_j \in N(u)$;
- n' cliques $K_1, \ldots, K_{n'}$ of size $k-1$ are added to G';
- v_i is connected to all nodes in K_i, for each $i \in [n']$.

The obtained graph G' is depicted in Figure 3.2.

Our aim is to show that the following two statements are equivalent: 1) there exists a clique of size k containing u in G, and 2) in the best Nash stable outcome $\mathcal{C}$ for G', $\mathcal{C}(u)$ is a clique of size at least k.

Let us consider the following Nash stable outcomes:
a) n' coalitions where v_i is in the coalition with K_i for each $i \in [n']$, and u is in one of these coalitions and,
b) $\mathcal{C}(u)$ is a clique of size $\geq k$ and for each $v_i \notin \mathcal{C}(u)$, $\mathcal{C}(v_i) = \{v_i\} \cup K_i$.

We notice that if there exists a clique of size at least k in G then outcome b) exists and achieves a better social welfare than outcome a). Next, we show that no Nash stable outcome can achieve a social welfare better than a) and b).

We start by observing that $\forall x \in K_i$, if $\mathcal{C}$ is a Nash stable outcome, then $K_i \subseteq \mathcal{C}(x)$. Indeed, if there exist $x, y \in K_i$ such that $\mathcal{C}(x) \neq \mathcal{C}(y)$ and $u_x^\alpha(\mathcal{C}(x)) \geq u_y^\alpha(\mathcal{C}(y))$, then $u_y^\alpha(\mathcal{C}(x) \cup \{y\}) > u_y^\alpha(\mathcal{C}(y))$. Moreover, if $\mathcal{C}$ is an optimal Nash stable outcome, then K_i and K_j cannot be in the same coalition, $\forall i, j \in [n'], i \neq j$. In fact, since the vector is decreasing with $\alpha_2 < \frac{1}{2}$, and the utility is normalized by the size of the coalition, then outcome a) would achieve a better social welfare.

Let us now show that if $\mathcal{C}$ is a Nash stable outcome and $K_i \not\subseteq \mathcal{C}(u), \forall i \in [n']$, then $\mathcal{C}(u)$ is a clique of size $\geq k$. Assume otherwise, meaning that there exists at least one agent in $\mathcal{C}(u)$ which is not connected to some $v_i \in \mathcal{C}(u)$. Let m be the size of $\mathcal{C}(u)$ and j the number of agents in $\mathcal{C}(u) \setminus \{v_i\}$ which are not connected to v_i. Since $m \leq n' + 1$, the utility of v_i is $\frac{m-1-j+j\cdot\alpha_2}{m} \leq \frac{m-2+\alpha_2}{m} \leq \frac{n'-1+\alpha_2}{n'+1}$. On the other hand, v_i could deviate and become a member of a clique of size k, achieving a utility of $\frac{k-1}{k}$. Since $\alpha_2 < \frac{1}{2}$ and $k > \frac{2}{3} \cdot (n' + 1)$, this implies $\frac{n'-1+\alpha_2}{n'+1} < \frac{k-1}{k}$, which contradicts the stability of $\mathcal{C}$. Moreover, $\mathcal{C}(u)$ indeed has to be a clique of size at least k. Otherwise, every $v_i \in \mathcal{C}(u)$ prefers to deviate to the coalition of K_i. Finally, the best Nash stable outcome $\mathcal{C}$ is either a) or b) (using similar arguments as above, we exclude the possibility that two or more neighbors of

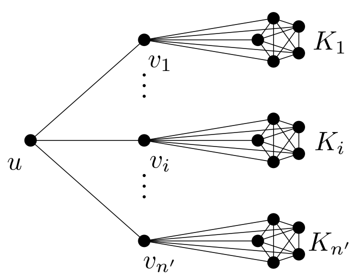


FIGURE 3.2: Reduction from MAX-CLIQUE to MSW-S problem for decreasing vectors.

u form a coalition not containing u). Moreover, since $\alpha_2 < \frac{k-3}{2(k-1)}$, outcome b) achieves a better social welfare than a).

In conclusion, if there exists a clique of size at least k in G containing u, then the best Nash stable outcome is outcome b), and otherwise it is outcome a). On the other hand, if b) is the best Nash stable outcome for the instance, then there exists a clique in G of size at least k containing u. $\square$

3.3.2 Increasing Vectors

Even though we focus mostly on normalized vectors in this work, Theorem 3.7 is presented here as it gives a nice inapproximability result for general, meaning not normalized, vectors. For $\alpha_1 = 1$, we prove a slightly weaker result of the same flavor in Theorem 3.8.

Theorem 3.7. *If $NP \neq P$, there is no poly-time algorithm with approximation ratio $f(n)$ for the MSW and MSW-S problems restricted to scoring vectors with increasing $0/1$ coefficients, where f is any strictly positive function.*

Proof. We prove the theorem for MSW-S. The same reduction can be used for MSW. Assume that there is a polynomial time algorithm A for the MSW-S problem with an approximation ratio $f(n)$, where f is a strictly positive function. We provide a reduction from the LONGEST INDUCED PATH problem, which is NP-complete [46]. An instance (G, L) of the LONGEST INDUCED PATH problem consists of a graph G and a number L and the question is whether G contains an induced subgraph that is a path of length at least L.

We transform a LONGEST INDUCED PATH instance (G, L) into the MSW-S instance $\langle G', \alpha\rangle$ where $\alpha = (\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_{L-1} = 0, \alpha_L = 1, \alpha_{L+1} = 1, \ldots, \alpha_{n-1} = 1)$ and G' is the graph obtained by adding a node x to G that connects to all original nodes from G. Note that $\mathrm{SW}(\mathcal{C}(x)) = 0$.

We now show that the following two statements are equivalent: 1) There is an induced path in G of length L, and 2) The algorithm A returns a positive value for the MSW-S instance. The direction 2) $\Rightarrow$ 1) is obvious.

To prove the other direction, assume that there is an induced path in G of length L. Consider the coalition structure $\{C_1, C_2\}$ where C_1 is formed by the nodes on the path of length L and $C_2 = V \setminus C_1$. If this coalition structure is not stable, then there must be a member z of C_2 that will obtain a positive utility by moving to C_1. We let z move to C_1 and repeat the argument recursively. After finitely many steps, we will reach a stable coalition structure with positive social welfare (C_2 shrinks at every step). □

Let us now focus on normalized scoring vectors α, i.e. $\alpha_1 = 1$.

Theorem 3.8. *If $NP \neq P$, then there is no poly-time algorithm for the MSW problem with an approximation ratio less than $\frac{2^{n+1}}{n^2+1}$.*

Proof. Let us assume that there is a polynomial time algorithm A for the MSW problem with an approximation ratio $\frac{2^{n+1}}{n^2+1}$. We transform a Longest Induced Path instance (G, L) into the MSW instance $\langle G, (\alpha_1 = 1, \alpha_2 = 1, \ldots, \alpha_{L-1} = 1, \alpha_L = 2^n, \alpha_{L+1} = 2^n, \ldots, \alpha_{n-1} = 2^n)\rangle$.

We now show that the following two statements are equivalent: 1) There is an induced path in G with length L, and 2) The algorithm A returns a value strictly greater than n for the MSW instance. The direction 2) $\Rightarrow$ 1) is obvious.

To prove the other direction, assume that there is an induced path in G of length L. Consider the coalition structure $\mathcal{C} = \{C_1, C_2\}$ where C_1 is made by the nodes on the path of length L and $C_2 = V \setminus C_1$. The welfare of $\mathcal{C}$ is at least $2 \cdot \frac{2^n}{n}$, already by considering the two nodes in C_1 that are the endpoints of the path of length L. This implies that A will return a value greater than $2 \cdot \frac{2^n}{n} \cdot \frac{n^2+1}{2^{n+1}} > n$. □

3.4 Price of Anarchy and Price of Stability

In the previous section we showed that the best (stable) outcomes cannot be easily computed. As the next step, we want to estimate the quality of stable outcomes by providing bounds on both the price of anarchy and the price of stability. We will focus on non-negative scoring vectors where $\alpha_1 > 0$. W.l.o.g. we then assume that α is normalized.

For instances where α is any normalized vector, we show that PoS and PoA are indeed bounded.

Theorem 3.9. *Given a scoring vector α, $PoA(\alpha) \leq M_\alpha \cdot (n-1)$ and $PoS(\alpha) \leq \min\left\{\frac{M_\alpha}{m_\alpha},\ \frac{n}{2} \cdot M_\alpha\right\}$, where m_α(resp. M_α) is the minimum (resp. max) component of α.*

Proof. For the PoS, note that the social welfare of the best Nash stable outcome can be lower bounded by SW(GC). Moreover, another lower bound for the social welfare achieved by the grand coalition is $\mathrm{SW(GC)} \geq \frac{2}{n} \cdot \sum_{i=1}^{n-1} m_\alpha \cdot up(i, V) = m_\alpha(n-1)$. However, this bound is not useful when m_α is close to 0 and in that case we can use that $\mathrm{SW(GC)} \geq \frac{1}{n} \cdot \sum_{x \in V} \delta_x = \frac{2}{n} \cdot |E| \geq 2 \cdot \frac{n-1}{n}$, where δ_x denotes the number of neighbors of node $x \in V$. On the other hand, if the optimum is made by m coalitions $\{C_1, \ldots, C_m\}$ of sizes $k_1, \ldots, k_m$, respectively, then,

$$\mathbf{opt} \leq \sum_{j=1}^{m} \frac{2}{k_j} \cdot \sum_{i=1}^{k_j - 1} M_\alpha \cdot up(i, C_j) \leq M_\alpha \cdot (n-1) \ .$$

By the definition of PoS the claimed bound follows.

Regarding PoA, in any Nash stable outcome any agent achieves a utility of at least $\frac{\alpha_1}{n} = \frac{1}{n}$. Thus, the social welfare of any Nash stable outcome is at least 1. By the definition of PoA, also in this case the bound follows. ☐

In particular, if α is the constant and normalized vector, then $\frac{M_\alpha}{m_\alpha} = 1$ and $\mathrm{PoS}(\alpha) = 1$. Moreover, in this case also $\mathrm{PoA}(\alpha)$ is equal to 1. This holds because the grand coalition is not only the optimum but also the only Nash stable outcome.

Lemma 3.10. *Given an instance $\langle G, \alpha \rangle$ where α is a normalized and constant vector, the grand coalition is the only Nash stable outcome.*

Proof. Assume there is a Nash stable outcome different from the grand coalition. Then, let us consider an agent i such that $i \in C_1 = C(i)$ and i also has at least one edge towards an agent j in a different coalition which we denote by $C_2 = C(j)$. Note that we can find such agent i since the graph is connected. If the coalition structure is stable, then both i and j do not want to deviate. This means that both $\frac{|C_1|-1}{|C_1|} > \frac{|C_2|}{|C_2|+1}$ and $\frac{|C_2|-1}{|C_2|} > \frac{|C_1|}{|C_1|+1}$ must hold, which is a contradiction. ☐

3.4.1 Decreasing Vectors

We now consider instances $\langle G, \alpha \rangle$ with α decreasing and normalized. By Theorem 3.9, the following bounds hold.

Corollary 3.11. *Given a decreasing scoring vector α, $PoA(\alpha) \leq n-1$ and $PoS(\alpha) \leq \frac{n}{2}$.*

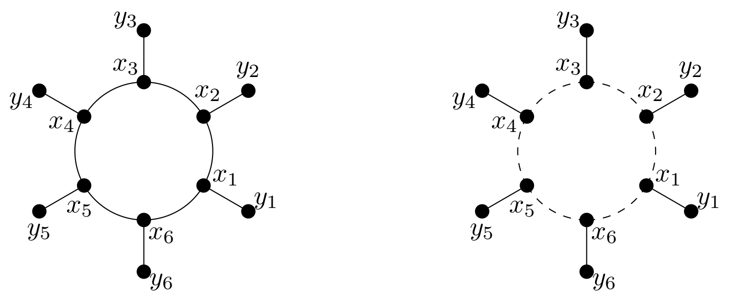
In what follows, we explore the connection between the upper bound on the price of stability and the existence of stable star partitions in the underlying graph.

3.4.1.1 Star Partitions on Graphs with Girth grater than 5

A key ingredient for arriving at a lower price of stability is studying instances which allow for stable star partitions. In fact, we turn our attention to star partitions, since star subgraphs provide high social welfare. A possible interpretation of good performance and stability of star partitions is that this is a natural way for agents to aggregate when they value closer agents significantly more than further ones (we will see that this intuition corresponds to $\alpha_2 \leq \frac{1}{2}$). In such a setting, a star can be understood as a group of agents gathered around a leader, the root of the star. In case where the agents are more curious about more distant agents ($\alpha_2 > \frac{1}{2}$), indeed we will see that the existence of a stable star partition cannot be guaranteed.

Proposition 3.12. *Given an instance $\langle G, \alpha \rangle$ where G admits a star partition $\mathcal{S}$ and α is decreasing, $SW(\mathcal{S}) \geq \frac{1}{2} + \alpha_2 \cdot (n-1)$. In particular, if $\alpha_2 \geq \frac{1}{2}$, then $SW(\mathcal{S}) \geq \frac{n}{2}$.*

Proof. Let $\mathcal{S} = \{\mathsf{S}_{k_1}, \ldots, \mathsf{S}_{k_m}\}$ be a star partition made of m stars respectively of size $k_1, \ldots, k_m$. Then, $\mathrm{SW}(\mathcal{S}) = \sum_{i=1}^{m} \mathrm{SW}(\mathsf{S}_{k_i})$ and this is equal to $\sum_{i=1}^{m} \frac{k_i-1}{k_i} + \sum_{i=1}^{m} \frac{(k_i-1)(1+\alpha_2 \cdot (k_i-2))}{k_i}$. First, note that $\sum_{i=1}^{m} \frac{k_i-1}{k_i} \geq \frac{m}{2}$. The second summation is $\geq \alpha_2 \cdot (n-m)$. By minimizing the whole expression w.r.t. m, the claim follows. $\square$



(a) Example of G_k, $k = 6$. (b) The star partition of G_6.

FIGURE 3.3: An example of a graph that admits no stable star partition for $\alpha_2 > \frac{1}{2}$. Dashed edges are in the cut between two different coalitions.

Not every star partition, however, is necessarily a Nash stable outcome. In what follows, we provide an analysis of stability of star partitions. We know that their existence is guaranteed when the girth of G is ≥ 4. Unfortunately, when the girth of G is exactly 4, no star partition need be stable. This can be easily checked by considering the complete bipartite graph with 2 left and 3 right nodes. Even if the girth of the graph is ≥ 5, we cannot ensure stability of star partitions for every α_2.

Proposition 3.13. *For any decreasing vector α with $\alpha_2 > \frac{1}{2}$ and every $k \geq 5$, there exists a graph G_k of girth k such that no star partition is Nash stable.*

Proof. For a fixed $k \geq 5$ we build the graph $G_k = (V_k, E_k)$ where

- $V_k = \{x_1, \ldots, x_k\} \cup \{y_1, \ldots y_k\}$,
- $x_1, \ldots, x_k$ are in a ring,
- and $(x_i, y_i) \in E_k$ for each $i \in [k]$.

The only possible star partition in G_k is given by $\mathcal{S} = \{\{x_i, y_i\} : i = 1 \ldots k\}$, but $\mathcal{S}$ is not stable. An example of G_k and its unique star partition are depicted in Figure 3.3. Indeed, any x_i is interested in deviating to an adjacent star, since her current utility is $\frac{1}{2}$ and she can achieve $\frac{1+\alpha_2}{3} > \frac{1}{2}$ by deviating. $\square$

Next, we show that when the girth is ≥ 5 and $\alpha_2 \leq \frac{1}{2}$, there always exists a Nash stable star partition (Theorem 3.15). Then, we provide an upper bound on

opt for girth ≥ 5 (Proposition 3.16). Finally, Theorem 3.17 establishes a better bound on the price of stability. We first state a simple lemma that will be used in the proof of Theorem 3.15.

Lemma 3.14. *If $\alpha_2 \leq \frac{1}{2}$, in any star partition $\mathcal{S} \in \Sigma(G)$*

1. *no root wants to deviate;*
2. *given $S_1, S_2 \in \mathcal{S}$ such that a leaf x of S_1 is connected to the root $y \in S_2$, x does not want to deviate iff $|S_1| > |S_2| + 1$.*

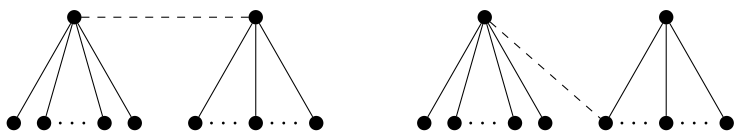
Proof. In this proof we consider two possible connections between stars in a star partition: root to root connection (see Figure 3.4(a)) and root to leaf connection (see Figure 3.4(b))

Case 1: For each $k, h \geq 2$ the root of a k-star connected to the root of a h-star does not want to leave the coalition when $\frac{k-1}{k} \geq \frac{1+(h-1)\alpha_2}{h+1}$. This is satisfied for all $k, h \geq 2$ if and only if $\alpha_2 \leq \frac{1}{2}$. Moreover, if the root of a k-star is connected to the leaf of a h-star, then by deviating it would achieve a utility of $\frac{1+\alpha_2+\alpha_3(h-2)}{h+1}$, which is smaller than $\frac{1+(h-1)\alpha_2}{h+1}$. Thus, the claim holds.

Case 2: For each $k, h \geq 2$, a leaf of the k-star does not want to leave its coalition and join the h-star when $\frac{1+(k-2)\alpha_2}{k} \geq \frac{1+(h-1)\alpha_2}{h+1}$. If $\alpha_2 \leq \frac{1}{2}$, this implies that $k \leq h - 1$. □

Theorem 3.15. *Given an instance $\langle G, \alpha \rangle$ where G has girth ≥ 5 and α is decreasing with $\alpha_2 \leq \frac{1}{2}$, there always exists a Nash stable star partition.*

Proof. We show that the minimum star partition according to the co-lexicographic order, which we denote by $\mathcal{S}$, is always a Nash stable outcome. First, observe



(a) Case 1: root to root　(b) Case 2: root to leaf

FIGURE 3.4: Connections between stars in a star partition.

that a star of size ≤ 3 is always stable. Now, assume that $\mathcal{S}$ is not Nash stable. By Lemma 3.14, this means that there exists a leaf x of a star $\mathsf{S}_k \in \mathcal{S}$, $k > 3$, that wants to deviate to another star $\mathsf{S}_h \in \mathcal{S}$. We distinguish two cases, depending on whether the node y to which x is connected to is the root or a leaf in S_h: 1) If y is the root of S_h, x would become a leaf of S'. Furthermore, Lemma 3.14 implies that $k \geq h + 1 > h$. Thus, the resulting partition $\mathcal{S}'$ is also a star partition s.t. $n_i(\mathcal{S}) = n_i(\mathcal{S}')$ for $i = k+1, \ldots, n$ and $n_k(\mathcal{S}) < n_k(\mathcal{S}')$. This implies $\mathcal{S}' <_{colex} \mathcal{S}$, contradicting the minimality of $\mathcal{S}$. 2) If y is a leaf in S_h, we can build a new star partition $\mathcal{S}'$ where both x, y deviate from their current coalitions and form a star of size 2. Also in this case, $n_i(\mathcal{S}) = n_i(\mathcal{S}')$ for $i = k+1, \ldots, n$ and $n_k(\mathcal{S}) < n_k(\mathcal{S}')$, again a contradiction. □

Proposition 3.16. *Given an instance $\langle G, \alpha \rangle$, where G has girth ≥ 5 and α is decreasing with $\alpha_2 \leq \frac{1}{2}$,* $\mathbf{opt} \leq \frac{n}{2} \cdot \left(\alpha_2 + \sqrt{\alpha_2^2 + (1-\alpha_2)^2}\right)$.

Proof. Let $\mathcal{C}^* = \{C_1^*, \ldots, C_m^*\}$ be a coalition structure with optimal social welfare and let k_i be the size of C_i^* $\forall i \in [m]$. We bound the utility of an agent counting the contribution of another member of her coalition 1 if they are neighbors and α_2 otherwise. Thus,

$$\mathbf{opt} \leq \sum_{i=1}^{m} \sum_{x \in C_i^*} \frac{\delta_{C_i^*}(x) + \alpha_2 \cdot (k_i - 1 - \delta_{C_i^*}(x))}{k_i} \ ,$$

which is equal to $2(1-\alpha_2) \cdot \sum_{i=1}^{m} \frac{|E_i|}{k_i} + \alpha_2 \cdot (n - m)$, where E_i denotes the set of edges in C_i^*, $i \in [m]$. When the graph has girth ≥ 5 the number of edges is upper bounded by $\frac{n\sqrt{n-1}}{2}$ [38]. Applying this bound to all the coalitions in $\mathcal{C}$,

$$\mathbf{opt} \leq 2(1-\alpha_2) \cdot \sqrt{nm - m^2} + \alpha_2 \cdot (n - m) \ .$$

Moreover, since $m \leq \frac{n}{2}$, the previous expression is maximized when m is equal to $\frac{n}{2} \cdot \left(1 - \frac{\alpha_2}{\alpha_2^2 + (1-\alpha_2)^2}\right)$. □

Theorem 3.17. *Given a decreasing scoring vector α with $\alpha_2 \leq \frac{1}{2}$, if we restrict our attention to graphs with girth ≥ 5, then we know that*

$$PoS(\alpha) \leq n \cdot \frac{\alpha_2 + \sqrt{\alpha_2^2 + (1-\alpha_2)^2}}{1 + 2\alpha_2 \cdot (n-1)} \ .$$

Proposition 3.12 and 3.16 together with Theorem 3.15 imply Theorem 3.17. We note that this result generalizes the result for SDGs (a decreasing vector with $\alpha_2 = \frac{1}{2}$) obtained in [11]. That is, PoS $\leq \frac{1}{2} + \frac{1}{\sqrt{2}}$.

The bound provided in Theorem 3.17 is better than the bound of $\frac{n}{2}$ from Corollary 3.11 if and only if $\alpha_2 \geq \frac{\sqrt{4n^2-12n+6}-n}{2((n-2)^2-2)} \approx \frac{1}{2n}$ and this quantity tends to 0 as n increases.

3.4.1.2 Star Partitions on Trees

We start by observing that the result shown in Proposition 3.13, for graphs with girth ≥ 5, holds even if the underlying graph is a tree.

Proposition 3.18. *For any decreasing vector α with $\alpha_2 > \frac{1}{2}$ there exist instances $\langle G, \alpha \rangle$ where G is a tree and no star partition is Nash stable.*

Proof. Let us consider a game instance $\langle G, \alpha \rangle$ where α is decreasing and normalized with $\alpha_2 > \frac{1}{2}$, and $G = \mathsf{P}_4$. It is easy to see that the only possible star partition, where no star is made by just one node, is made by two stars of size 2. Namely, if the agents are numbered from 1 up to 4, starting from the left side of the path to the right, then the only non trivial star partition is $\{\{1,2\},\{3,4\}\}$. Here, both 2 and 3 want to deviate, reaching the other stare, since the would achieve a utility of $\frac{1+\alpha_2}{3} > \frac{1}{2}$. $\square$

Hence, in what follows, we focus on decreasing scoring vectors whose second component is $\leq \frac{1}{2}$.

In [16], for unweighted FHGs where $\alpha = (1, 0, \ldots, 0)$, authors provide a polynomial time algorithm which computes a Nash stable star partition on trees. In particular, they show that an optimal coalition structure, according to the utilitarian social welfare, is made of coalitions with diameter at most 2. Moreover, the authors show that in this setting if a coalition structure is optimal, then it also Nash stable. Our aim, is to generalize this result for trees when the decreasing normalized vector α is such that $\alpha_2 \leq \frac{1}{2}$. While for $\alpha_2 = \frac{1}{2}$ it is easy to check that any star partition is Nash stable (it directly follows from the result in [11]), for $\alpha_2 < \frac{1}{2}$ only the existence of at least one stable star partition is guaranteed by Theorem 3.15. We next show that if for a fixed tree T and the scoring vector $\alpha = (1, 0, \ldots, 0)$ a star partition $\mathcal{S}$ is stable, then $\mathcal{S}$ is also stable for T and any other normalized decreasing vector α' such that $\alpha'_2 \leq \frac{1}{2}$. However, the same does not hold for optimality. More precisely, if $\mathcal{S}$ is an optimal coalition structure for T with $\alpha = (1, 0, \ldots, 0)$, $\mathcal{S}$ is not necessarily optimal for T with another scoring vector α' for which $\alpha'_2 \leq \frac{1}{2}$.

Lemma 3.19. *If $\mathcal{S}$ is a Nash stable star partition for $\langle G, \alpha \rangle$ where G is a tree and $\alpha = (1, 0, \ldots, 0)$, then $\mathcal{S}$ is stable for any instance $\langle G, \alpha' \rangle$ such that $\alpha'_2 \leq \frac{1}{2}$.*

Proof. Let be $\mathcal{S}$ be a Nash stable star partition for $\langle G, \alpha \rangle$. Let us assume that $\mathcal{S}$ is not Nash stable for $\langle G, \alpha' \rangle$. This means that there exists an agent $x \in V$ such that x wants to deviate from $\mathcal{S}(x)$. In that case, by Lemma 3.14, x must be a leaf in the star $\mathcal{S}(x)$. Furthermore, there exists an agent $y \in V$ such that $(x, y) \in E$ and, if $k = |\mathcal{S}(x)|$ and $h = |\mathcal{S}(y)|$, it holds that $k > h+1$. On the other hand, since $\mathcal{S}$ is stable for $\langle G, \alpha \rangle$, it has to hold that $\frac{1}{k} = u_\alpha(x, \mathcal{S}(x)) \geq u_\alpha(x, \mathcal{S}(y) \cup \{x\}) = \frac{1}{h+1}$. Since this is equivalent to $k \leq h + 1$, we reached a contradiction. □

Theorem 3.20. *Given an instance $\langle G, \alpha \rangle$ where G is a tree α is such that $\alpha_2 \leq \frac{1}{2}$, it is always possible to compute a Nash stable star partition in polynomial time.*

Proof. By Lemma 3.19 it is enough to compute a Nash stable star partition for the instance $\langle G, (1, 0, \ldots, 0) \rangle$, which can be done with the algorithm provided in [16]. □

Proposition 3.21. *Given an instance $\langle G, \alpha \rangle$ where G is a tree,* $\mathbf{opt} \leq \frac{n}{8} \cdot \frac{(2-\alpha_2)^2}{1-\alpha_2}$.

Proof. Let $\mathcal{C}^* = \{C_1^*, \ldots, C_m^*\}$, $k_i = |C_i^*|$, and we denote by E_i the set of edges in C_i^*, $i \in [m]$. As in the proof of Proposition 3.16, we arrive at

$$\mathbf{opt} \ \leq \ 2(1 - \alpha_2) \cdot \left(\sum_{i=1}^{m} \frac{|E_i|}{k_i} \right) + \alpha_2 \cdot (n - m).$$

Now, because C_i is a tree $\forall i \in [m]$, it follows that

$$\begin{aligned} \mathbf{opt} &\leq 2(1 - \alpha_2) \cdot \left(\sum_{i=1}^{m} \frac{k_i - 1}{k_i} \right) + \alpha_2 \cdot (n - m) \\ &\leq 2(1 - \alpha_2) \cdot \frac{m}{n}(n - m) + \alpha_2 \cdot (n - m) \end{aligned}$$

where the last inequality holds by Jensen's Inequality[2] applied on the concave function $f(x) = \frac{x-1}{x}$. Moreover, the last expression is maximized for $m = \frac{n}{4} \cdot \frac{2-3\alpha_2}{1-\alpha_2}$ and the claim follows. $\square$

Theorem 3.22. *Given a decreasing scoring vector α with $\alpha_2 \leq \frac{1}{2}$, if we restrict our attention to trees,*

$$PoS \leq \frac{n}{4} \cdot \frac{(2-\alpha_2)^2}{(1-\alpha_2)(1+2\alpha_2 \cdot (n-1))}$$

Proof. The claim follows from Proposition 3.12 and Proposition 3.21. $\square$

We next show that on trees and for a restricted range of decreasing scoring vectors, we can say something about the structure of the optimal outcome.

Theorem 3.23. *For any decreasing vectors α s.t. $\alpha_2 \leq 1/3$ and if G is a tree, the optimal outcome $\mathcal{C}^*$ is a star partition.*

Proof. Let us denote by C_k a coalition of k nodes that is not necessarily a star. Since $\mathrm{SW}(\mathsf{S}_k) = \frac{(k-1)(2+\alpha_2(k-2))}{k}$ and $\mathrm{SW}(C_k) = \frac{2}{k}\sum_{i=1}^{k-1} \alpha_i \cdot up(i, C)$, it is easy to see that on trees the coalition of k nodes that maximizes the social welfare is exactly S_k. In other words, $\mathrm{SW}(\mathsf{S}_k) > \mathrm{SW}(C_k)$ for any C_k that is not a star. Now, let $C_k \in \mathcal{C}^*$ and assume that $C_k \neq \mathsf{S}_k$. We show that in this case, we can split C_k into smaller coalitions whose sum of the social welfares is greater than the social welfare of C_k.

We prove the claim by induction over the diameter of C_k. Let us first consider the case in which the diameter of C_k equals 3, we observe that if the diameter is lower than 3 then C_k is necessarily a star. Then, we can split C_k along the diameter into two stars S_h and S_{k-h} for some $h \in \{2, \ldots, k-2\}$. Furthermore, for $\alpha_2 \leq 1/3$ it holds that $\mathrm{SW}(\mathsf{S}_h) + \mathrm{SW}(\mathsf{S}_{k-h}) \geq \mathrm{SW}(\mathsf{S}_k) > \mathrm{SW}(C_k)$. Now, let us assume that the claim holds for diameters up to and including m. If we consider a coalition C_k with diameter $m+1$, we can start splitting it into smaller coalitions along the diameter. In each step we remove a star and after a

[2] The Jansen's inequality states that for any concave function ψ, and for any sequence of reals $x_1, \ldots, x_m$, in the domain of ψ, it holds $\psi\left(\frac{1}{m} \cdot \sum_{i=1}^{m} x_i\right) \geq \frac{1}{m} \cdot \sum_{i=1}^{m} \psi(x_i)$.

finite number of steps, which we denote by i, we know that the diameter of the remaining coalition has to drop below $m+1$. Then,

$$\begin{aligned}
&\mathrm{SW}(\mathsf{S}_{h_1}) + \mathrm{SW}(\mathsf{S}_{h_2}) + \cdots + \mathrm{SW}(\mathsf{S}_{h_i}) + \mathrm{SW}(C_{k-h_1-h_2-\cdots-h_i}) \\
&\geq \mathrm{SW}(\mathsf{S}_{h_1+h_2+\cdots+h_i}) + \mathrm{SW}\left(C_{k-(h_1+h_2+\cdots+h_i)}\right) \\
&> \mathrm{SW}(\mathsf{S}_{h_1+h_2+\cdots+h_i}) + \mathrm{SW}(C_k) - \mathrm{SW}(\mathsf{S}_{h_1+h_2+\cdots+h_i}) \\
&= \mathrm{SW}(C_k),
\end{aligned}$$

where the first inequality follows from the fact that $\alpha_2 \leq 1/3$ and the second from the assumption of the induction. □

3.4.2 Increasing Vectors

We focus on instances $\langle G, \alpha \rangle$ where α is increasing and normalized. The main contribution of this section is summarized by the following theorem.

Theorem 3.24. *Given an increasing α, $PoA(\alpha) \leq \frac{2}{n} SW(\mathsf{P}_n)$ and $PoS(\alpha) \leq \frac{1}{n-1} SW(\mathsf{P}_n)$. Moreover, these bounds are tight.*

Our first objective is to prove that if there are n agents, the highest possible social welfare is achieved by the grand coalition on a graph that is a path on n nodes. To this aim, in Lemma 3.25 we show that for a fixed coalition size the underlying graph that provides the highest social welfare is a path. In Proposition 3.26 we conclude that any coalition structure on any underlying graph has a social welfare lower than the grand coalition on a graph that is a path on n nodes.

Lemma 3.25. *Given an instance $\langle G, \alpha \rangle$, where α is increasing, $SW(C) \leq SW(\mathsf{P}_k)$, for any $C \subseteq V$ of size k.*

Proof. Since the utility of any coalition C of size k is given by Equation 3.1, we want to prove that, when k is fixed, $\mathrm{SW}(C)$ is maximized for $C = \mathsf{P}_k$. Since we know that $\sum_{i=1}^{k-1} up(i, C) = \binom{k}{2}$, meaning that the sum of all unordered pairs at all possible distances is fixed, and α is increasing, we conclude that $\mathrm{SW}(C) \leq \mathrm{SW}(\mathsf{P}_k)$. □

Proposition 3.26. *Given an instance $\langle G, \alpha \rangle$, where α is increasing, $SW(\mathcal{C}) \leq SW(\mathsf{P}_n)$ for any coalition structure $\mathcal{C}$.*

Proof. Given any outcome $\mathcal{C} = \{C_1, \ldots, C_m\}$, if k_i is the size of coalition C_i, then by Lemma 3.25 $\mathrm{SW}(\mathcal{C}) \leq \sum_{i=1}^{m} \mathrm{SW}(\mathsf{P}_{k_i})$. Lastly, the summation is a convex function of $k_1, \ldots, k_m$. $\square$

Next, in Lemma 3.27 and 3.28 we give a lower bound on the social welfare of any Nash stable outcome and a tighter bound for the social welfare of the grand coalition, respectively. These two lemmas, together with Proposition 3.26, allow us to arrive at the PoA and PoS upper bounds of Theorem 3.24.

Lemma 3.27. *Given an instance $\langle G, \alpha \rangle$, where α is increasing, if $\mathcal{C} \in \mathsf{NS}_{\langle G, \alpha \rangle}$, then $SW(\mathcal{C}) \geq \frac{n}{2}$.*

Proof. Let us assume that $\mathcal{C}$ consists of k coalitions $C_1, \ldots, C_k$ that are of sizes $i_1, \ldots, i_k$, respectively. Since α is normalized w.r.t. the first component and increasing, if x is in the coalition C_j, then $u_x(C_j) \geq \frac{i_j - 1}{i_j}$. Thus, $\mathrm{SW}(\mathcal{C}) \geq \sum_{j=1}^{k} i_j - 1 = n - k$. Moreover, if the outcome is stable, then the coalitions are made of at least 2 nodes. This implies that $\mathrm{SW}(\mathcal{C}) \geq n - \frac{n}{2} = \frac{n}{2}$. $\square$

Lemma 3.28. *Given an instance $\langle G, \alpha \rangle$ where α is an increasing scoring vector, $SW(GC) \geq n - 1$.*

Proof. Since $\alpha_i \geq 1$ for every $i = 1, \ldots, n-1$, $u_x(\mathrm{GC}) \geq \frac{n-1}{n}$ for any $x \in V$. Thus, $\mathrm{SW}(\mathrm{GC}) \geq \sum_{x \in V} \frac{n-1}{n} = n - 1$ $\square$

Finally, we show an example that guarantees the tightness (up to a constant) of the provided upper bounds. To this aim, we consider the graph $G = (V, E)$ where $V = \{x_1, \ldots, x_n\}$, $(x_i, x_{i+1}) \in E, \forall i \in [n-2]$ and $(x_n, x_i) \in E, \forall i \in [n-1]$.

Since $\mathcal{C}^* = \{\{x_1, \ldots, x_{n-1}\}, \{x_n\}\}$, $\mathbf{opt} = \mathrm{SW}(\mathsf{P}_{n-1})$.

Lemma 3.29. *In the just described graph, if $\alpha_2 = 1 + \varepsilon$ and $\alpha_{i+1} - \alpha_i > \varepsilon, \forall i = 2, \ldots, n-2$, then there exists a constant c_ε such that $\sup\limits_{\mathcal{C} \in \mathsf{NS}_{\langle G, \alpha \rangle}} SW(\mathcal{C}) \leq c_\varepsilon (n-1)$.*

Proof. We set $\varepsilon = 2\varepsilon'$ and denote by $\mathcal{C}(x_n) = \{x_{i_1}, \dots, x_{i_k}\}$ the coalition of x_n. Given any x_{i_j} and $x_{i_{j+1}}$ in $\mathcal{C}(x_n)$, since $\mathcal{C}$ is a Nash stable outcome, then either $i_{j+1} = i_j + 1$ or $i_{j+1} - i_j - 1 > 2$ (in a stable outcome a node cannot be isolated). Now, we want to prove that if $\mathcal{C}$ is a Nash stable outcome, then $i_{j+1} - i_j - 1 \leq l_{\varepsilon'}$ for some $l_{\varepsilon'}$. If so, $SW(\mathcal{C}) \leq \alpha_{l_{\varepsilon'}}(n-1)$, and the claim follows. Let us assume $i_{j+1} - i_j - 1 = l$, meaning that there is a coalition between x_{i_j} and $x_{i_{j+1}}$ which is a path P_l on l nodes by the stability of $\mathcal{C}$. Since neither x_{i_j} nor $x_{i_{j+1}}$ wants to deviate if

$$\frac{1 + (k-2)\alpha_2}{k} \geq \frac{\sum_{i=1}^{l} \alpha_i}{l+1} \ ,$$

as $\frac{1+(k-2)\alpha_2}{k} < \alpha_2 = 1 + 2\varepsilon'$ and

$$\frac{\sum_{i=1}^{l} \alpha_i}{l+1} \geq \frac{\sum_{i=1}^{l} 1 + 2\varepsilon'(i-1)}{l+1} = \frac{l - 1 + \varepsilon'(l-1)l}{l+1} \ ,$$

then $1 + 2\varepsilon' \geq \frac{l+\varepsilon'(l-1)l}{l+1}$. Thus, $\frac{1}{\varepsilon'} \geq l^2 - 3l - 2$ must hold. In conclusion, if the outcome is stable, then $1 < l < l_{\varepsilon'}$, where $l_{\varepsilon'} = \frac{3+\sqrt{9+8(1+\frac{1}{2\varepsilon'})}}{2}$. $\square$

The tightness can be shown to hold even for bipartite graphs, as it can be checked in the above proof by connecting x_n only once every two nodes in the path.

3.5 Open Problems

A question worth investigating is giving a complete characterization of the classes of networks admitting low PoS. Another natural open question is the existence of other kinds of partitions that provide a good social welfare when star partitions are not Nash stable. A further interesting question is giving a lower bound on the PoS for graphs with girth 4.

Moreover, one might further investigate some of the examples we have provided in the introduction, concerning various scenarios falling within our DHGs model and identifying other relevant specific scoring vectors that are worth to be investigated. We also see considering weighted graphs as another promising avenue. The aim here would be to find a way to express the agents' utilities according to the weights of the graph, for instance by substituting the scoring vector with a distance scoring function. In fact, the DHGs model we introduced generalizes the centrality measures used on unweighted graphs (degree centrality, harmonic centrality, Dangalchev centrality) to any centrality measure that is a function of the node distances. Going from unweighted to weighted graphs would allow representing centrality measures to their full extent.

Finally, we focused on Nash stability, but other stability notions traditionally considered in these games remain to be investigated, both those considering individual and group deviations.

Chapter 4

Strategyproof Mechanisms for Friends and Enemies Games

In this chapter, we investigate strategyproof mechanism for Friends and Enemies Games a special subclass of ASHGs where agents split the others into friends and enemies. Two different types of preference profiles have been introduced in the literature, namely, Friends Appreciation and Enemies Aversion. We provide for both of them deterministic mechanisms that guarantee linear approximation ratio in the number of agents. We then show that, in order to achieve a constant approximation ratio, we should rely on either a randomized mechanism, under Friends Appreciation, or an exponential time mechanism, under Enemies Aversion. Finally, we show how to extend our results when agents may consider the others also neutral, meaning that these latter have no impact on their preferences.

4.1 Introduction

As it often happens, individuals are inclined to have either a positive or a negative opinion about others, or simply, they consider their acquaintances to be either friends or enemies. A natural question is, once individuals are gathered together, how the opinion on the others affects the one on the formed group. This scenario has been modeled as an HG and called *Friends and Enemies Games* [34]. In this work two different types of preference profiles have been considered *Friends Appreciation* (FA) and *Enemies Aversion* (EA). Under FA, agents prefer coalitions with a higher number of friends. When the number of friends is the same,

FA profiles	Deterministic	Randomized
U.B.	n (Mech. $\mathcal{M}_2$)	$4 \cdot (1+\varepsilon)$ w.h.p. (Mech. $\mathcal{M}_3$)
L.B.	2 (Thm. 4.9)	> 1 (Prop. 4.4)

EA profiles	Poly-time	Exp-time
U.B.	$(1+\sqrt{2})n$ (Mech. $\mathcal{M}_4$)	$1+\sqrt{2}$ (Mech. $\mathcal{M}_5$)
L.B.	$\Omega(n)$ (Thm. 4.16)	> 1 (Prop. 4.18)

TABLE 4.1: Lower and upper approximation bounds for strategyproof mechanisms.

they prefer a coalition with a smaller number of enemies. Conversely, under EA, agents always prefer coalitions with a smaller number of enemies and, in case of a tie, the ones with a bigger number of friends.

In this chapter, our aim is to design strategyproof mechanisms for both FA and EA preference profiles, improving upon the previously proposed solutions of [35] in terms of either the social welfare of the computed outcomes or the time complexity.

4.1.1 Our Contribution

For FA profiles we provide both a deterministic and a randomized truthful mechanism. While the deterministic mechanism has an approximation ratio of n, where n is the number of agents, the randomized one has an expected approximation ratio of 4 and of $4(1+\varepsilon)$ with high probability, for any fixed $\varepsilon > 0$.

For EA profiles, we first show that no polynomial time algorithm can have an approximation ratio that is in $O(n^{1-\varepsilon})$, since this problem is as hard to approximate as the MAXCLIQUE problem. Then, we give a simple polynomial time mechanism reaching an approximation ratio of $(1+\sqrt{2})n$, which is in turn asymptotically tight. Then, if time complexity is not a concern, we show that a deterministic strategyproof mechanism achieving constant approximation ratio exists. More precisely, we prove that the mechanism proposed in [35] achieves approximation $1+\sqrt{2}$.

Finally, we show how to extend our results in the presence of neutrals, and we discuss anonymity.

Our main results are listed in Table 4.1.

4.1.2 Related Work

Friends and Enemies Games with FA and EA preference profiles are an example of ASHGs. This model has been introduced in [34], where the authors focus on weak and strong core stability notions. While for FA it is always possible to compute a strict core stable coalition structure in polynomial time, for EA, even if it always exists, it is NP-hard to find a core stable outcome. Moreover, in [63] it is shown that for EA determining whether a coalition structure is core stable is co-NP complete. In [35] individual deviations are considered: the authors study Nash stable, individually and contractually individually stable outcomes. While Nash existence is not always guaranteed, individually stable and thus also contractually individually stable outcomes always exist.

Subsequent works on Friends and Enemies Games still focus on the core and strict core stability notions, but allow the presence, beside friends and enemies, also of neutrals that have no impact on agents' preferences [57]. In [14], on the contrary, neutrals can have a lower order positive or negative impact on the preferences. More specifically, the authors focus on FA preference profiles and distinguish Friends Appreciation with extroverted and with introverted agents, where for the same number of friends and enemies it is more preferable having a higher or lower number of neutrals, respectively. They consider core stable and individually stable outcomes, also investigating the hardness of deciding their existence.

In [65] the authors focus on strategyproof mechanisms without money for ASHGs with only positive preferences. They propose a mechanism returning the grand coalition, and investigate the same problem with constraints on the coalition size and with respect to (approximate) envy-freeness. A study of the properties of strategyproof core stable solutions for hedonic games is also provided in [61]. Deterministic and randomized strategyproof mechanisms for ASHGs and FHGs are provided in [43], where different types of valuations are considered.

Of particular interest for our study are ASHGs with individual *duplex* valuations, that is, with values in $\{-1, 0, 1\}$, for which the study of strategyproof mechanism has been considered in [43]. They can be seen as an HG with friends and enemies, where friends have the same impact as enemies. Since in our EA setting enemies have a negative effect that is higher than the positive effect of friends, as we will see in the sequel strategyproof mechanisms that work in our model with the inclusion of neutrals are also strategyproof for duplex valuations.

Finally, in [35] the authors also propose strategyproof mechanisms for both FA and EA preference profiles. However, they do not consider their efficiency (both in terms of quality of the returned solutions and of time complexity). We significantly improve over their results. For FA, the algorithm from [35] has an unbounded approximation ratio, while we provide a deterministic n-approximation and a randomized $4(1+\varepsilon)$-approximation (both in expectation and with high probability). For EA, [35] only gives a non-polynomial algorithm, while we provide a polynomial one that has bounded asymptotically tight approximation. More precisely, we show that it has linear approximation and that a sublinear approximation cannot be achieved in polynomial time. Moreover, we determine the approximation bound of their exponential deterministic mechanism.

4.2 Model and Preliminaries

In *Friends and Enemies games* every agent $i \in N$ partitions the other agents into a *set of friends* F_i and a *set of enemies* E_i, with $F_i \cup E_i = N \setminus \{i\}$ and $F_i \cap E_i = \emptyset$.

Example 4.1. *Let $N = \{1, 2, 3\}$ be the set of agents, and let $F_1 = \{2\}$, $F_2 = \{3\}$, $F_3 = \{2\}$ and $E_1 = \{3\}$, $E_2 = \{1\}$, $E_3 = \{1\}$ be the agents' sets of friends and enemies, respectively. The sets N, F_i and E_i, $i = 1, 2, 3$, define an instance of friends and enemies game. The just described instance is depicted in Fig. 4.1(a), where a directed edge from agent i to agent j represents i's opinion of j. Moreover, solid edges and dashed edges represent friend and enemy relations, respectively.*

Given such a friend and enemy partition, different settings can be defined. In particular, for each $i \in N$ and for every $X, Y \in N_i$, if agent i more or equally prefers coalition X then Y, we write $X \succeq_i Y$. A preference profile P is based on *Friends Appreciation* (FA) when $X \succeq_i Y$ iff

$$|X \cap F_i| > |Y \cap F_i| \quad \text{or} \quad |X \cap F_i| = |Y \cap F_i| \text{ and } |X \cap E_i| \leq |Y \cap E_i|,$$

and it is based on *Enemies Aversion* (EA) when $X \succeq_i Y$ iff

$$|X \cap E_i| < |Y \cap E_i| \quad \text{or} \quad |X \cap E_i| = |Y \cap E_i| \text{ and } |X \cap F_i| \geq |Y \cap F_i|.$$

In other words, under FA, a coalition is preferred over another one if it contains a higher number of friends; if the number of friends is the same, the coalition with less enemies is preferred. On the other hand, under EA, a coalition is preferred if it contains a smaller number of enemies; if the number of enemies is the same, the coalition with more friends is preferred.

Friends and Enemies Games are a proper subclass of ASHGs, because each agent i can be seen as having a suitable valuation $w_i(j)$ for every other agent j (according to the fact that she is "friend" or an "enemy"), and her utility for being in a given coalition C is $u_i(C) = \sum_{j \in C \setminus \{i\}} w_i(j)$. In the FA case, such valuation functions for each agent $i \in N$ can be defined as

$$w_i(j) = \begin{cases} 1 & \text{, if } j \in F_i, \\ -\frac{1}{n} & \text{, if } j \in E_i, \end{cases}$$

which, as it can be easily checked, correctly encodes the setting. In other words, the positive effect of one friend is greater than the overall possible negative effect due to enemies. Similarly, in the EA case, the valuations can be set in such a way that, for every agent $i \in N$,

$$w_i(j) = \begin{cases} \frac{1}{n} & \text{, if } j \in F_i, \\ -1 & \text{, if } j \in E_i. \end{cases}$$

Example 4.2. *Let us consider again the instance described in Example 4.1 and the grand coalition $GC = \{\{1, 2, 3\}\}$ as a possible outcome. Then, under* FA *preferences profiles, $u_1(GC) = w_1(2) + w_1(3) = 1 - \frac{1}{3} = \frac{2}{3}$. Similarly, $u_2(GC) = u_3(GC) = \frac{2}{3}$. Under* EA *preferences profiles, instead, $u_1(GC) = w_1(2) + w_1(3) = \frac{1}{3} - 1 = -\frac{2}{3}$, and $u_2(GC) = u_3(GC) = -\frac{2}{3}$.*

The just presented valuation functions were already assumed (albeit differently normalized) in the work that introduced Friends and Enemies Games [34]. Since these games are a proper subclass of ASHGs, we can resort to the notions defined for general ASHGs, that have been widely investigated in the literature, as they are both able to capture several realistic scenarios in coalition formation games, and they allow for a succinct graph representation in which the node set is N, i.e., nodes represent agents, and every arc (i, j) has weight $w_i(j)$. In this work, we will use the valuation functions defined above and the aforementioned graph representation.

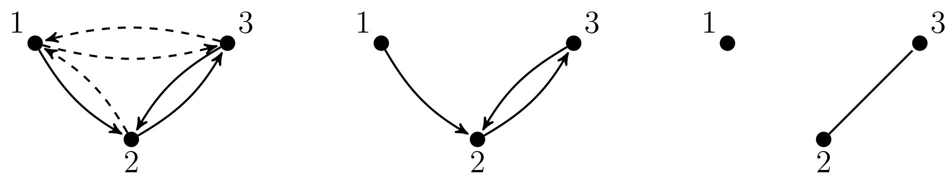
In Friends and Enemies Games, each agent i has to somehow communicate her preferences. Without loss of generality we assume that, both in the FA and the EA case, this is accomplished by declaring $d_i = (F_{d_i}, E_{d_i})$, where F_{d_i} is the declared set of friends and $E_{d_i} = N \setminus F_{d_i}$ the declared set of enemies of agent i. Notice that, as agents are self-interested entities, each declaration may differ from the actual sets of friends and enemies of agent i, $v_i = (F_i, E_i)$, i.e., it is possible that $F_i \neq F_{d_i}$ and $E_i \neq E_{d_i}$. The collection of the reported informations is stored in $\mathbf{d}$.

We are interested in strategyproof mechanisms that perform well with respect to the goal of maximizing the classical utilitarian social welfare, that is, the sum of the utilities achieved by all the agents. To this aim, we measure the performance of a mechanism $\mathcal{M}$ by considering its approximation ratio $r^{\mathcal{M}}$ introduced in Section 2.3. In what follows, given an FA or an EA profile $\mathbf{d}$, we will use $\mathcal{C}^*(\mathbf{d})$ to denote an optimal outcome for the game instance expressed by $\mathbf{d}$, and $\mathbf{opt}(\mathbf{d})$, or simply $\mathbf{opt}$, to denote its social welfare.

We say that a deterministic mechanism $\mathcal{M}$ is *admissible* if it always guarantees a non negative social welfare, i.e., $\mathrm{SW}(\mathcal{M}(\mathbf{d})) \geq 0$ for any possible list of preferences $\mathbf{d}$. Similarly, a randomized mechanism $\mathcal{M}$ is *admissible* if $\mathbb{E}\left[\mathrm{SW}(\mathcal{M}(\mathbf{d}))\right] \geq 0$ holds for every $\mathbf{d}$. In the following, we will always implicitly restrict our attention to admissible mechanisms. In fact, a simple admissible strategyproof mechanism can be trivially obtained by putting every agent into a separate singleton coalition, regardless of the declared valuations.

4.2.1 Graph Representation

As already mentioned, Friends and Enemies Games, being a proper subclass of ASGHs, can be suitably represented by means of graphs. More specifically, the



(a) Instance/Profile **d** (b) $G^{\rightarrow}_{\mathbf{d}}$ (c) $G^{\leftrightarrows}_{\mathbf{d}}$

FIGURE 4.1: Example of a Friends and Enemies Game instance and the corresponding graphs $G^{\rightarrow}_{\mathbf{d}}$ and $G^{\leftrightarrows}_{\mathbf{d}}$. Solid (resp. dashed) edges represent friend (resp. enemy) relations.

following representations will be useful for our purposes. For a given profile **d**, $G^{\rightarrow}_{\mathbf{d}} = (N, F_{\mathbf{d}})$ is a directed graph with the edge set

$$F_{\mathbf{d}} = \{(i,j) \mid i,j \in N,\ j \in F_{d_i}\} \quad ,$$

i.e., $G^{\rightarrow}_{\mathbf{d}}$ contains only directed edges corresponding to friendship relations; $G^{\leftrightarrows}_{\mathbf{d}} = (N, F^{\leftrightarrows}_{\mathbf{d}})$ is an undirected graph with the edge set

$$F^{\leftrightarrows}_{\mathbf{d}} = \Big\{\{i,j\} \,|\, i,j \in N,\ j \in F_{d_i} \ \wedge\ i \in F_{d_j}\Big\} \quad ,$$

i.e., each edge corresponds to a mutual friendship relation. Graphs $G^{\rightarrow}_{\mathbf{d}}$ and $G^{\leftrightarrows}_{\mathbf{d}}$ for the instance given in Example 4.1 can be seen in Fig. 4.1(b) and Figure 4.1(c), respectively.

Given a directed graph G and a pair of nodes x, y in G, x and y are *weakly connected* if they are connected in the undirected version of G. Moreover, a *weakly connected component* in G is a maximal subset of pairwise weakly connected nodes in G. Similarly, given a directed graph G and a pair of nodes x, y in G, x and y are *strongly connected* if there exists a directed path from x to y and a directed path from y to x in G. A *strongly connected component* in G is a maximal subset of strongly connected nodes in G.

In what follows, for the sake of simplicity, we will often identify a coalition C with the subgraph it induces in $G^{\rightarrow}_{\mathbf{d}}$ or $G^{\leftrightarrows}_{\mathbf{d}}$. Furthermore, we will denote by $f := |F_{\mathbf{d}}|$ the overall number of friendship relations in the profile **d**.

4.3 Friends Appreciation

FA preference profiles can be suitably analyzed exploiting the representation graph $G_{\vec{\mathbf{d}}}$. The following simple lemma will be useful in this section.

Lemma 4.1. *Let C be any coalition inducing a subgraph of g edges in $G_{\vec{\mathbf{d}}}$, or analogously containing g positive relations, and let $k = |C|$. Then,*

$$SW(C) = g \cdot \left(1 + \frac{1}{n}\right) - \frac{k(k-1)}{n} \quad .$$

Proof. The social welfare $\mathrm{SW}(C)$ of C is, by definition, equal to the sum of the utilities of all the agents in C. Each friend relation, that is each arc (i, j) in the subgraph induced by C in $G_{\vec{\mathbf{d}}}$, contributes exactly one to $\mathrm{SW}(C)$, while each negative relation contributes $-1/n$. Therefore, since there are g friend relations in C and all the remaining $k(k-1)-g$ ones are negative, $\mathrm{SW}(C) = g - \frac{k(k-1)-g}{n} = g \cdot \left(1 + \frac{1}{n}\right) - \frac{k(k-1)}{n}$. ☐

A useful topology for proving some of our results is given by a *directed star* containing n nodes, consisting of one central node and $n-1$ remaining leaf ones having an edge towards it (see Figure 4.2(a)). The following lemma concerns optimal solutions in directed stars.

Lemma 4.2. *Given an FA profile $\mathbf{d}$, if $G_{\vec{\mathbf{d}}}$ is a directed star of n agents, then $\frac{n^2-1}{4n} \leq \mathbf{opt} \leq \frac{n}{4}$.*

Proof. We show that an optimal solution $\mathcal{C}^*(\mathbf{d})$ can be obtained by an outcome with one coalition containing the central node and $\frac{n}{2}$ or $\frac{n \pm 1}{2}$ leaf agents (depending on the parity of n), while all other agents are in singleton coalitions. To this aim, notice first that any coalition without the central agent cannot have a strictly positive social welfare.

Given a coalition C_k of size k that forms a star (or any tree, because the graph structure and the direction of the edges does not play a role here) in $G_{\vec{\mathbf{d}}}$, by applying Lemma 4.1 with $g = k - 1$ we obtain

$$u(C_k) = (k-1)\left(1 - \frac{k-1}{n}\right) \tag{4.1}$$

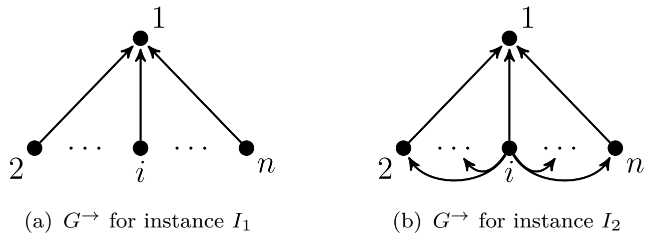


FIGURE 4.2: The lower bound instances for deterministic strategyproof mechanisms in the FA case.

A standard mathematical argument shows that $u(C_k)$ is maximized for $k = \frac{n}{2}+1$ if n is even and $k = \frac{n\pm1}{2}+1$ if n is odd, thus yielding the lemma. □

Finally, before proceeding to the design of strategyproof mechanisms, we provide the following useful lemma, which for n agents and a fixed number $g \leq n(n-1)$ characterizes the worst social welfare achievable by a coalition structure on an input profile $\mathbf{d}$ for which the number of friendship relations within the coalitions equals g.

Lemma 4.3. *Given a coalition structure $\mathcal{C}$ whose coalitions are weakly connected components and such that g is the number of friendship relations within the coalitions, then the worst possible scenario for social welfare occurs when:*

- *$G^{\rightarrow}$ is weakly connected and $\mathcal{C}$ is the grand coalition, if $g \geq n-1$;*
- *there is only one coalition C in $\mathcal{C}$ that is not a singleton coalition and the subgraph of C forms a tree, if $g < n-2$.*

Proof. Let us assume that $\mathcal{C}$ consists of h coalitions of size $k_1, \ldots, k_h$ and $\ell_1, \ldots, \ell_h$ positive relations contained in each, where $\sum_{i=1}^{h} k_i = n$ and $\sum_{i=1}^{h} \ell_i = g$. Using Lemma 4.1 and assuming $g \geq n-1$, we know that $\mathrm{SW}(\mathcal{C}) = g - \frac{1}{n}\sum_{i=1}^{h} k_i(k_i-1)$. Applying simple minimization arguments, we conclude that the last expression is minimized when $h = 1$.

To show the second statement we apply the same arguments simply not considering the singleton coalitions in $\mathcal{C}$. □

4.3.1 Strategyproof Mechanisms

In [35] the authors show that for FA returning the strongly connected components of $G_{\vec{\mathbf{d}}}$ is a strategyproof mechanism. However, this mechanism has a very bad performance. In fact, if we consider the example of a directed star depicted in Fig. 4.2(a), its achieved social welfare is 0, since it will split the agents into singleton coalitions. On the other hand, as shown in Lemma 4.2, the social optimum is linear in the number of agents, and thus the mechanism that returns the strongly connected components has an unbounded approximation ratio. In addiction, the directed star depicted in Fig. 4.2(a) also shows that it is not possible to have a mechanism which returns an individually rational outcome. Indeed, split the agents into singleton coalitions is also the only possible individually rational outcome.

One might wonder whether returning the social optimum is strategyproof. Unfortunately, this is not the case.

Proposition 4.4. *A mechanism that for every* FA *profile* $\mathbf{d}$ *returns an optimal outcome* $\mathcal{C}^*(\mathbf{d})$ *is not strategyproof.*

Proof. Let us consider again the directed star in Fig. 4.2(a). As shown in the proof of Lemma 4.2, the optimal solution consists of placing half of the leaf agents in a coalition with the central node and all others into singleton coalitions. If we alter the instance in such a way that a leaf node i that was placed alone now declares a positive valuation towards all the nodes in the graph, as depicted in Fig. 4.2(b), the grand coalition becomes the optimal solution. To verify this, first notice that any coalition that does not contain agent 1 nor agent i cannot have positive social welfare. Furthermore, the social welfare of a coalition that contains both agents 1 and i and $m-2$ other agents achieves a social welfare of $2m-3-\frac{m^2-3m+3}{n}$, and this expression is maximized for $m=n$, amounting to $n-\frac{3}{n}$. Finally, any outcome where agents 1 and i are not together, and the sizes of the respective coalitions are k_1 and k_2, achieves a social welfare of $(k_1-1)-\frac{(k-1)^2}{n}+(k_2-1)-\frac{(k_2-1)^2}{n} \leq n-\frac{2}{n}-\frac{k_1^2+k_2^2}{n} \leq n-\frac{3}{n}$. In the grand coalition the utility of agent i is $1-\frac{n-2}{n}>0$, and therefore we can conclude that the mechanism that returns the social optimum is not strategyproof. $\square$

Let us start with one of the simplest examples of a deterministic strategyproof mechanism.

Mechanism $\mathcal{M}_1$. Given an FA preference profile $\mathbf{d}$, return the grand coalition GC.

Mechanism $\mathcal{M}_1$ is strategyproof, as the outcome does not depend on the agents' declared profile $\mathbf{d}$. However, it does not always return an admissible outcome, e.g., for any profile with a total number of friendship relations $f \leq n-2$, the returned outcome is not admissible. Indeed, by applying Lemma 4.1 with $g = f$ and $k = n$, we have that $u(\mathrm{GC}) = f \cdot \left(1 + \frac{1}{n}\right) - (n-1) < 0$.

Our goal is not only to find admissible, but also, and more importantly, to find good approximation mechanisms. To this aim, we will first identify a broad class of strategyproof mechanisms to which our aforementioned deterministic and randomized mechanism for FA profiles will belong to.

Definition 4.5. We define $\mathscr{M}$ as the class of mechanisms $\mathcal{M}$ which, given an FA preference profile $\mathbf{d}$, work as follows:

1. select independently from $\mathbf{d}$, deterministically or at random, a partition $\mathcal{P}$ of N;
2. compute a coalition structure $\mathcal{C}$ s.t., for each agent $i \in N$, $\mathcal{C}(i)$ is the (maximal) weakly connected component containing i in the subgraph of $G_{\mathbf{d}}^{\rightarrow}$ induced by $\mathcal{P}(i)$.

In other words, once $\mathcal{P}$ is computed, agent i will be assigned to a subset $\mathcal{C}(i)$ of $\mathcal{P}(i)$ containing all friends she has in $\mathcal{P}(i)$, $\mathcal{C}(i)$ being the minimal subset of $\mathcal{P}(i)$ guaranteeing the same property for all the other agents that it contains.

Theorem 4.6. *$\mathscr{M}$ is an admissible and strategyproof class for* FA *profiles.*

Proof. Since each coalition C in the outcome is weakly connected, the number of friendship relations f_C in C is at least $k-1$, where k is the coalition size. Thus, applying Lemma 4.1, $\mathrm{SW}(C) = f_C \cdot \left(1 + \frac{1}{n}\right) - \frac{k(k-1)}{n} \geq (k-1)\left(1 - \frac{k-1}{n}\right) > 0$. Therefore, $\mathcal{M}$ is admissible

Next, note that by the definition of the class $\mathscr{M}$, the declaration d_i of i cannot influence the partition $\mathcal{P}$ selected by $\mathcal{M}$. Furthermore, given any subprofile $\mathbf{d}_{-i}$ declared by the remaining agents, consider the weakly connected components in the subgraph of $G_{\mathbf{d}}^{\rightarrow}$ induced by the agents in $\mathcal{P}(i)$ and $\mathbf{d}_{-i}$, that is, the weakly connected components that remain in $\mathcal{P}(i)$ after deleting the outgoing edges of i in $G_{\mathbf{d}}^{\rightarrow}$. By the definition of $\mathscr{M}$, all the agents belonging to each of these components will never be split into different coalitions. Since one friend contributes more to the utility of i than all her enemies subtract, the best utility that agent i can hope to achieve is obtained when she is put together with all the agents in the weakly connected component determined by $\mathcal{P}(i), \mathbf{d}_{-i}$ and F_i, which contains all her friends in $\mathcal{P}(i)$. But this exactly is the outcome selected by $\mathcal{M}$ if i simply declares her true valuation v_i, and therefore reporting any different declaration d_i cannot yield a better utility. □

While every mechanism in the class $\mathscr{M}$ is always strategyproof, it is not true that every strategyproof mechanism is in $\mathscr{M}$. A simple counter-example is given by the mechanism of [35], which computes the strongly connected components.

4.3.1.1 Deterministic Mechanisms

We now present a mechanism from the just described class $\mathscr{M}$ and give a bound on its approximation ratio.

Mechanism $\mathcal{M}_2$. Given an FA preference profile $\mathbf{d}$, output as coalitions the weakly connected components of $G_{\mathbf{d}}^{\rightarrow}$.

Theorem 4.7. *Mechanism $\mathcal{M}_2$ is admissible, strategyproof and has approximation ratio $r^{\mathcal{M}_2} \leq n$.*

Proof. Mechanism $\mathcal{M}_2$ belongs to the class $\mathscr{M}$ with partition $\mathcal{P}$ simply being the grand coalition GC, and thus, by Theorem 4.6, it is admissible and strategyproof.

Regarding its approximation ratio, let us first assume that $G_{\mathbf{d}}^{\rightarrow}$ is weakly connected, so that $f \geq n-1$. Since, by Lemma 4.1, $\mathrm{SW}(\mathcal{M}_2(\mathbf{d})) = f \cdot \left(1 + \frac{1}{n}\right) - (n-1)$ and $\mathbf{opt} \leq f$, then $r^{\mathcal{M}_2} \leq \frac{f}{f \cdot \left(1+\frac{1}{n}\right)-(n-1)}$. Such a ratio is decreasing in f, and at most equal to n for $f = n-1$, thus, implying the claim.

If $G_{\mathbf{d}}^{\rightarrow}$ is not weakly connected, the same argument can be separately applied to each weakly connected component, with the worst case being reached exactly when $G_{\mathbf{d}}^{\rightarrow}$ is weakly connected. □

Observe that, although $\mathcal{M}_2$ is admissible, the returned outcome is not individually rational; however, it fail to be individually rational only if there exists at least one agent which has no friend and it is considered as a friend by at least another agent. Moreover, the computed outcome is Pareto optimal; this immediately follows considering that every agent is in a coalition together with all her friends. Thus, the only way to improve the utility of an agent is to split a weakly connected component removing some of her enemies; however, by the definition of weakly connected component, this implies that at least one agent will loose at least one friend. In conclusion, in the returned outcome there is no way to improve the utility of one agent without decreasing the one of another.

Notice that the poor performance of mechanism $\mathcal{M}_2$ is due to instances for which the number of friendship relations f is very close to n. However, as soon as $G_{\mathbf{d}}^{\rightarrow}$ becomes denser, and in particular when f is at least a $c \cdot n$ for any fixed constant $c > 1$, the ratio becomes constant. In particular, for $f \geq 2(n-2)$, the ratio is $\frac{f}{f \cdot \left(1+\frac{1}{n}\right)-(n-1)} < 2$. More generally, the following theorem holds.

Theorem 4.8. *Mechanism $\mathcal{M}_2$ has a constant approximation ratio on any* FA *preference profile* $\mathbf{d}$ *for which $f = (1 \pm \varepsilon)n$ with $\varepsilon > 1/n$. More precisely, the approximation ratio is bounded by $\frac{1+\varepsilon}{\varepsilon}$ and $\frac{1}{\varepsilon}$ for $f = (1+\varepsilon)n$ and $f = (1-\varepsilon)n$, respectively.*

Proof. Let us first assume that $f = (1-\varepsilon)n$. Since $f = (1-\varepsilon)n < \left(1 - \frac{1}{n}\right) n \leq n-1$, then $G_{\mathbf{d}}^{\rightarrow}$ is not weakly connected and we know by Lemma 4.1 that $\mathrm{SW}(\mathcal{M}_2) \geq f(1+\frac{1}{n}) - \frac{(f+1)f}{n} = f\left(1 - \frac{f}{n}\right)$. Here we used that in the worst case (by Lemma 4.3) the only non-singleton coalition returned by $\mathcal{M}_2$ has size of at most $f+1$. But then

$$r^{\mathcal{M}_2} \leq \frac{f}{f\left(1-\frac{f}{n}\right)} = \frac{n}{n-f} = \frac{n}{n-(1-\varepsilon)n} = \frac{1}{\varepsilon},$$

thus proving the claim.

Now, we turn to the case where $f = (1+\varepsilon)n > n$. This in particular means that in the worst case (again by Lemma 4.3) $G^{\rightarrow}_{\mathbf{d}}$ is weakly connected, and proceeding as in the proof of Theorem 4.7 we get

$$r^{\mathcal{M}_2} \leq \frac{(1+\varepsilon)n}{(1+\varepsilon)n \cdot \left(1+\frac{1}{n}\right) - (n-1)} = \frac{(1+\varepsilon)n}{\varepsilon(n-1) + 2(1+\varepsilon)} \leq \frac{(1+\varepsilon)n}{\varepsilon(n-1)} \leq \frac{1+\varepsilon}{\varepsilon}.$$

Notice that we did not need to use the condition $\varepsilon > 1/n$, and indeed, for $f = (1+\varepsilon)n$, $r^{\mathcal{M}_2} \leq \frac{1+\varepsilon}{\varepsilon}$ holds for any $\varepsilon > 0$. However, when $\varepsilon \leq \frac{1}{n}$, $\frac{1+\varepsilon}{\varepsilon} \geq n+1 > n$, so the bound on the approximation ratio of $\mathcal{M}_2$ that we obtain in this case is weaker than the one we already know holds by Theorem 4.7. Therefore, in the theorem statement also for $f = (1+\varepsilon)n$ we restrict our attention to $\varepsilon > 1/n$. □

From Theorem 4.8 we see that mechanism $\mathcal{M}_2$ guarantees an approximation ratio of at most 2 for all instances for which $f \notin (n/2, 2n)$. Furthermore, we next show a lower bound of 2 on the approximation ratio achievable by any deterministic mechanism. The lower bound example uses the two instances shown in Figs. 4.2(a) and 4.2(b), which differ only by the reported preferences of agent i.

Theorem 4.9. *No deterministic strategyproof mechanism for* FA *profiles can have an approximation ratio less than* 2*.*

Proof. Let $\mathcal{M}$ be any deterministic strategyproof mechanism, and let $\mathbf{d}$ be the profile corresponding to the directed star instance I_1 in Figure 4.2(a). If $\mathcal{M}$ returns the grand coalition, from Eq. (4.1) $\mathrm{SW}(\mathcal{M}(\mathbf{d}))) = \frac{n-1}{n}$, and since by Lemma 4.2 it is $\mathbf{opt} \geq \frac{n^2-1}{4n}$, the theorem follows.

Next, assume that $\mathcal{M}$ does not return the grand coalition. In order for $\mathrm{SW}(\mathcal{M}(\mathbf{d}))$ to be strictly positive, there has to exist $C \in \mathcal{M}(\mathbf{d})$ that contains the central node and some further leaf agents. Let $k > 1$ be the size of C. Since by assumption C is not the grand coalition, $k < n$ and there exists at least one leaf agent $i \notin C$. Therefore, $u_i(\mathcal{M}(\mathbf{d})) = 0$.

Let us now consider instance I_2 in Figure 4.2(b), in which the only difference with respect to I_1 is the declared valuation d_i of agent i, according to which i considers all the other agents to be friends, i.e., $F_{d_i} = N \backslash \{i\}$. In this instance, the optimum is in any case lower bounded by the social welfare of the grand coalition GC (in the proof of Proposition 4.4 we actually saw that the grand coalition GC is the optimal solution), which by applying Lemma 4.1 with $g = 2n-3$ is equal to

$$\mathrm{SW(GC)} = (2n-3)\left(1+\frac{1}{n}\right) - (n-1) = n - \frac{3}{n}.$$

Mechanism $\mathcal{M}$, in order to be strategyproof, needs to return a coalition C' of size $n' < n$ that contains the central star node from I_1, agent 1, and such that $i \notin C'$. Since C' induces a star of n' nodes, and the remaining agents in $N \setminus C'$ induce a star of $n - n'$ nodes centered at i, by Equation (4.1) the overall social welfare achieved by $\mathcal{M}$ is at most $(n'-1)\left(1 - \frac{n'-1}{n}\right) + (n-n'-1)\left(1 - \frac{n-n'-1}{n}\right) \leq \frac{n}{2} - \frac{2}{n}$, the first term being maximized for $n' = n/2$. Hence, the approximation ratio is

$$\frac{\mathbf{opt}}{\mathrm{SW}(\mathcal{M}(\mathbf{d}))} \geq \frac{\mathrm{SW}\,(\mathrm{GC})}{\mathrm{SW}(\mathcal{M}(\mathbf{d}))} \geq \frac{n-3/n}{n/2-2/n} = \frac{2n^2-3}{n^2-4} > 2\ .$$

$\square$

4.3.1.2 Randomized Mechanisms

So far, we have provided a deterministic strategyproof mechanism for FA profiles with a linear approximation ratio. We now present an improved randomized mechanism, which is in fact able to reach a constant approximation factor both in expectation and with high probability.

Mechanism $\mathcal{M}_3$. Given an FA preference profile $\mathbf{d}$, if $n < 4$ run $\mathcal{M}_2$. Otherwise, at first randomly generate a partition $\mathcal{P} = \{P_1, P_2\}$ by placing every agent $i \in N$ in P_1 or P_2 uniformly and independently at random (i.e., i is put in P_1 with probability $\frac{1}{2}$, and in P_2 otherwise), and then output the weakly connected components of P_1 and P_2 in $G_{\vec{\mathbf{d}}}$.

The idea underlying the definition of $\mathcal{M}_3$ is first of all that of maintaining the good performance of the deterministic mechanism $\mathcal{M}_2$ when $f \geq 2(n-2)$. This is accomplished by retaining, with respect to the coalitions formed by $\mathcal{M}_2$, each friendship relation with probability $1/2$ and each enemy relation with probability at most $1/2$, thus yielding an expected social welfare which is at least half of the one of $\mathcal{M}_2$. At the same time, $\mathcal{M}_3$ avoids the pathological instances causing $\mathcal{M}_2$

to have an approximation ratio linear in n. Such instances occur when $f \approx n$ and the graph contains a large weakly connected component of size close to n. In these cases, it is possible to achieve a better social welfare by splitting the large weakly connected component into smaller coalitions. This can be accomplished by creating a partition $\mathcal{P}$ whose expected number of agents on one side is $n/2$. A paradigmatic example of the difference between $\mathcal{M}_2$ and $\mathcal{M}_3$ is considering directed stars, for which $\mathcal{M}_2$ achieves an approximation ratio of n, while $\mathcal{M}_3$ returns an almost optimal solution in expectation and also with high probability. In fact, as shown in Lemma 4.2, the optimal solution for a star is obtained by putting roughly half of the leaves in the same coalition with the central node and the other leaves in singletons. Mechanism $\mathcal{M}_3$ does the same by exploiting the partition $\mathcal{P}$.

Theorem 4.10. *$\mathcal{M}_3$ is admissible, strategyproof and $r^{\mathcal{M}_3} \leq 4$ in expectation. Moreover, for any fixed $\varepsilon > 0$, $r^{\mathcal{M}_3} \leq 4(1+\varepsilon)$ with high probability.*

Proof. The mechanism belongs to the class $\mathscr{M}$, and therefore it is strategyproof. For $n < 4$, $\mathcal{M}_3$ executes $\mathcal{M}_2$ and the approximation ratio of $\mathcal{M}_2$ is in this case bounded by 3 by Theorem 4.7. We now prove the upper bound on the approximation ratio for $n \geq 4$.

Within this proof, we will always consider the worst case scenario for the social welfare while assuming a fixed number of friendship relations within the coalitions of a coalition structure. Thus, in all the case studies we will apply Lemma 4.3.

We start by showing the claimed expected approximation ratio. Given an FA profile $\mathbf{d}$ as input, let $\overline{f}$ be the expected number of positive relations with both endpoints either in P_1 or in P_2. Then, because each positive relation has a probability of $1/2$ of having both of its endpoints ending up in the same side of the partition, by linearity of expectation $\overline{f} = f/2$. Moreover, the expected number of agents in P_1 is $n/2$ and the same holds for P_2. We distinguish 3 cases based on the range which $\overline{f}$ is in, keeping in mind that $\mathbf{opt} \leq f$.

Case 1: $\overline{f} \leq \frac{n}{2} - 1$ (and thus, $f \leq n-2$). In this scenario the worst case outcome $\mathcal{I}$ occurs when all of the $\overline{f}$ positive relations contribute to just one weakly connected component T that forms a tree on one side of the partition (either in P_1 or P_2), while the other side is completely disconnected. Then, $\mathrm{SW}(\mathcal{I}) = \mathrm{SW}(T)$ and, according to Eq. (4.1), $\mathrm{SW}(\mathcal{T}) = \overline{f}\left(1 - \frac{\overline{f}}{n}\right)$.

Thus, $r^{\mathcal{M}_3} \leq \frac{2}{1-\frac{f}{2n}} \leq \frac{2}{1-\frac{n}{2}\cdot\frac{1}{n}} = 4$.

Case 2: $\frac{n}{2} \leq \overline{f} \leq n-1$ (and thus, $n \leq f \leq 2n-2$). Here, the worst case outcome $\mathcal{I}$ occurs when $\frac{n}{2}-1$ of the $\overline{f}$ positive relations form a tree T_1 connecting all the $\frac{n}{2}$ nodes on one side of the partition and on the other side the remaining $\overline{f}-\frac{n}{2}+1$ positive relations form a tree T_2 of $\overline{f}-\frac{n}{2}+2$ nodes. Thus, applying Eq. (4.1), the social welfare of the outcome is $\frac{n}{4}-\frac{1}{n}+\left(\overline{f}-\frac{n}{2}+1\right)\left(1-\frac{\overline{f}-\frac{n}{2}+1}{n}\right) \geq 2\overline{f}\left(1+\frac{1}{n}\right)-\frac{n}{2}-\frac{\overline{f}^2}{n}$, so that $r^{\mathcal{M}_3} \leq \frac{f}{f\left(1+\frac{1}{n}\right)-\frac{n}{2}-\frac{f^2}{4n}}$. This ratio is a convex function for $n \leq f \leq 2n-2$, and takes values $4\frac{n}{n+4}$ and $\frac{4(n-1)}{n+4-\frac{6}{n}}$ for $f=n$ and $f=2n-2$, respectively. Thus, $r^{\mathcal{M}_3} \leq 4$.

Case 3: $\overline{f} \geq n$ (and thus, $f \geq 2n$). The worst case outcome $\mathcal{I}$ occurs when both P_1 and P_2 are weakly connected by the $\overline{f}$ retained positive relations. In that case, $\mathrm{SW}(\mathcal{I}) \geq \overline{f}\left(1+\frac{1}{n}\right)-\frac{n}{2}+1 \geq \frac{1}{2}(f-n)$ and, thus, $r^{\mathcal{M}_3} \leq \frac{2f}{f-n} \leq 4\frac{n-1}{n-2} \leq 4$.

We will now prove that the $4(1+\varepsilon)$ approximation bound holds with probability of at least $1-\frac{1+\varepsilon}{(\sqrt{1+\varepsilon}-1)^2}\cdot\left(\frac{2}{f}+\frac{1}{n}\right)$. To this aim, it is sufficient to show that $r^{\mathcal{M}_3} \leq \frac{4}{(1-c)^2}$ with probability of at least $1-\frac{1}{c^2}\cdot\left(\frac{2}{f}+\frac{1}{n}\right)$, for a small constant $c \leq 1-\frac{1}{\sqrt{1+\varepsilon}}$.

Let us consider the random variables N_1 and N_2, which are defined as the number of nodes in P_1 and P_2, respectively. Both of these random variables have expected value of $\frac{n}{2}$ and variance of $\frac{n}{4}$. We define

$$A = \{(i,j) \,|\, i,j \in N,\ j \in F_i \wedge i \in E_j\}$$

as the set of the pairs in which only one agent considers the other as a friend, and

$$B = \{(i,j) \,|\, i,j \in N, j > i,\ j \in F_i \wedge i \in F_j\}$$

as the set of the pairs in which both agents consider each other as friends.

Let us consider the Boolean random variables $\{X_{ij}\}_{(i,j)\in A}$ and $\{Y_{ij}\}_{(i,j)\in B}$ that take value 1 if the pair (i,j) is in the same side of the partition P and 0 otherwise. We observe that the only difference between X_{ij} and Y_{ij} is that the former is defined on pairs in A and the latter on pairs in B. This distinction, however, is important for our analysis, as it enables us to get rid of unwanted dependencies. Finally, we define F as the random variable counting the number of positive links within P_1 and P_2, and we note that it can be expressed as the sum of the already defined Boolean random variables, i.e., $F = \sum_{(i,j)\in A} X_{ij} + 2\sum_{(i,j)\in B} Y_{ij}$. It is easy to see that $\mathbb{E}[F] = f/2$, because any two agents have probability of $1/2$ to be placed in the same side of the partition. Since $\{X_{ij}\}_{(i,j)\in A}$ and $\{Y_{ij}\}_{(i,j)\in B}$ are both pairwise independent, if we define $x = |A|$ and $y = |B|$, we can easily compute the variance of F as

$$\mathrm{Var}(F) = \sum_{(i,j)\in A} \mathrm{Var}(X_{ij}) + 4\sum_{(i,j)\in B} \mathrm{Var}(Y_{ij}) = \frac{x}{4} + y \leq \frac{f}{2} \tag{4.2}$$

where the last inequality holds since the number of positive links can be expressed as $f = x + 2y$.

Applying the Chebyshev's inequality[1] to both N_h, $h = 1, 2$, and F we get

$$\Pr\left[\left|N_h - \frac{n}{2}\right| \geq \frac{1}{2}n \cdot c\right] \leq \frac{1}{c^2 \cdot n} \tag{4.3}$$

and

$$\Pr\left[\left|F - \frac{f}{2}\right| \geq \frac{1}{2}f \cdot c\right] \leq \frac{2}{c^2 \cdot f}. \tag{4.4}$$

Without loss of generality assume that $N_1 \leq N_2$, meaning that P_2 is the larger partition. Then, by Eq. (4.3) and Eq. (4.4), $\frac{1-c}{2}n \leq N_1 \leq \frac{n}{2}$, $\frac{n}{2} \leq N_2 \leq \frac{1+c}{2}n$ and $\frac{1-c}{2}f \leq F \leq \frac{1+c}{2}f$ with high probability.

In order to bound the social welfare of the outcomes returned by $\mathcal{M}_2$ we shall distinguish three different cases:

1. $F \leq N_2 - 1$,

[1]**Chebyshev's inequality.** Given a random variable Z with expected value μ and variance σ^2, for any $\gamma > 0$, $\Pr[|Z - \mu| \geq \gamma] \leq \frac{\sigma^2}{\gamma^2}$.

2. $N_2 - 1 < F \leq n - 2$,
3. $F > n - 2$.

We inspect the worst case scenario for the social welfare of the returned outcome in each of these cases separately. Generally speaking, since the mechanism always outputs coalitions that are weakly connected components in $G^{\rightarrow}$, the worst case occurs when within these coalitions both the number of participants and the number of negative relations are maximized. The exact formula, however, is different for each of the cases.

In case 1, the worst case occurs when there is only one non trivial connected component (a tree) in the P_2 side of the partition and all the other nodes, if they exist, are isolated. In case 2, the worst case arises when $N_2 - 1$ positive relations form a tree connecting all the nodes in P_2 and on the other side the remaining positive relations also form a tree in P_1. In case 3, the worst case occurs when both P_1 and P_2 are weakly connected by the retained positive relations.

Therefore, we can use the following formulas, which represent the worst social welfare achievable in cases 1, 2 and 3, respectively, as a lower bound on the social welfare achieved by the outcome returned by $\mathcal{M}_3$:

$$F\left(1 - \frac{F}{n}\right) \tag{Eq. 1}$$

$$\begin{aligned}(N_2 - 1)\left(1 - \frac{N_2 - 1}{n}\right) + (F - N_2 + 1)\left(1 - \frac{F - N_2 + 1}{n}\right)\\ = F - \frac{(N_2 - 1)^2 + (F - N_2 + 1)^2}{n}\end{aligned} \tag{Eq. 2}$$

$$F(1 + \frac{1}{n}) - \frac{N_1(N_1 - 1) + N_2(N_2 - 1)}{n} > F\left(1 + \frac{1}{n}\right) - \frac{N_1^2 + N_2^2}{n} \tag{Eq. 3}$$

We develop our analysis considering the three disjoint cases one by one, always exploiting the fact that **opt** $\leq f \leq \frac{2}{1-c}F$ with high probability.

Case 1: In this case, applying (Eq. 1) and the fact that $F \leq N_2 - 1 < N_2 \leq \frac{1+c}{2}n$, we have that

$$r^{\mathcal{M}_3} \leq \frac{\frac{2}{1-c}F}{F\left(1-\frac{F}{n}\right)} = \frac{2n}{(1-c)(n-F)} < \frac{4}{(1-c)^2} \quad .$$

Case 2: It is easy to see that, for $F > N_2 - 1$, it is

$$F - \frac{(N_2-1)^2 + (F - N_2 + 1)^2}{n} > F\left(1 - \frac{F}{n}\right) \quad ,$$

i.e., the lower bound on the social welfare given by (Eq. 2) is strictly higher than the one given by (Eq. 1). Therefore, we can also use (Eq. 1) as a lower bound on the social welfare and thus

$$r^{\mathcal{M}_3} < \frac{\frac{2}{1-c}F}{F\left(1-\frac{F}{N}\right)} = \frac{2n}{(1-c)(n-F)} \leq \frac{4}{1-c},$$

where for the last inequality we used the fact that in this case $F \geq N_2 \geq n/2$.

Case 3: As in the previous cases, we use the corresponding lower bound on the social welfare given by (Eq. 3), and observe that

$$F\left(1+\frac{1}{n}\right) - \frac{N_1^2 + N_2^2}{n} \geq F\left(1+\frac{1}{n}\right) - \frac{n}{2}(1+c^2),$$

as the first term is minimized for $N_1 = \frac{1-c}{2}n$ and $N_2 = \frac{1+c}{2}n$. Therefore,

$$r^{\mathcal{M}_3} < \frac{\frac{2}{1-c}F}{F\left(1+\frac{1}{n}\right) - \frac{n}{2}(1+c^2)} = \frac{1}{1-c} \cdot \frac{4F}{2F - n(1+c^2)}.$$

Since the last expression is decreasing in F, by recalling that in this case $F \geq n-1$, we get

$$r^{\mathcal{M}_3} < \frac{1}{1-c} \cdot \frac{4n^2+4}{n^2(1-c^2)+4n+4} \leq \frac{1}{1-c} \cdot \frac{4n^2}{n^2(1-c^2)} =$$
$$= \frac{4}{(1-c)(1-c^2)} = \frac{4}{(1-c)^2(1+c)}.$$

In conclusion, denoted by A and B the events $N_2 \leq \frac{1+c}{2}n$ and $\left|F - \frac{f}{2}\right| \leq \frac{1}{2}f \cdot c$ respectively, if both A and B are satisfied then we have

$$r^{\mathcal{M}_3} < \frac{4}{(1-c)^2} \tag{4.5}$$

which holds with probability

$$\begin{aligned}
\Pr\left[A \ \wedge \ B\right] &\geq 1 - \Pr\left[N_2 \geq \frac{1+c}{2}n \ \vee \ \left|F - \frac{f}{2}\right| \geq \frac{1}{2}f \cdot c\right] \\
&\geq 1 - \Pr\left[\left|N_2 - \frac{n}{2}\right| \geq \frac{1}{2}n \cdot c\right] - \Pr\left[\left|F - \frac{f}{2}\right| \geq \frac{1}{2}f \cdot c\right] \\
&\geq 1 - \frac{1}{c^2} \cdot \left(\frac{1}{n} + \frac{2}{f}\right).
\end{aligned}$$

In order to prove the initial claim of the theorem, we suffices to take $c \leq 1 - \frac{1}{\sqrt{1+\varepsilon}}$. □

Before concluding the section, let us remark that, even if we did not state it explicitly, all of our mechanisms for FA profiles are efficient, that is, they compute their outcomes in polynomial time.

4.4 Enemies Aversion

In this section we consider EA preference profiles. Unlike in the FA case, here we will mostly use the undirected graph representation $G_{\mathbf{d}}^{\leftrightarrows}$. Again, we will commonly identify a coalition C with the subgraph it induces in $G_{\mathbf{d}}^{\leftrightarrows}$.

Differently from the FA case, we show that for EA profiles good approximation mechanisms with polynomial time complexity cannot be found. This is not due to strategyproofness, but because of the inherent hardness of approximating the optimal solution. More precisely, we show that the optimum cannot be approximated below a ratio linear in n. We provide an approximation preserving reduction from the MaxClique problem, in order to transfer the well known $O(n^{1-\varepsilon})$ inapproximability result to our problem. To this aim, the following definitions and facts will be useful in the sequel.

Definition 4.11. Given an undirected graph $G = (V, E)$, a *(disjoint) clique partition* of G is a collection of cliques $\mathcal{K} = \{K_1, \dots, K_m\}$ such that $\forall i, j \in [m]$ s.t. $i \neq j$, $K_i \cap K_j = \emptyset$ and $\bigcup_{i \in [m]} K_i = V$.

We are interested in clique partitions inducing good outcomes for EA profiles.

Definition 4.12. A best clique partition for an EA profile $\mathbf{d}$ is any clique partition $\mathcal{K}^*$ of $G_{\mathbf{d}}^{\leftrightarrows}$ such that $\mathcal{K}^*$ achieves the highest possible social welfare in the instance induced by $\mathbf{d}$, among all possible clique partitions.

The following two lemmas concern the structure of optimal outcomes.

Lemma 4.13. *Given any* EA *preference profile* $\mathbf{d}$*, no agent in an optimal outcome* $\mathcal{C}^*(\mathbf{d})$ *can be involved in more than one negative relation in her coalition.*

Proof. Given any EA preference profile $\mathbf{d}$, let $\mathcal{C}^*(\mathbf{d}) = \{C_1^*, \dots, C_m^*\}$ be an optimal outcome for $\mathbf{d}$. Assume, for the purpose of reaching a contradiction, that there exists an agent $i \in N$ involved in two negative relations in her coalition $C_\ell^* \in \mathcal{C}^*(\mathbf{d})$. Then, if we set $k = |C_\ell^*|$, putting i in a separate singleton coalition will 1) delete from C_ℓ^* at least two negative relations, increasing the social welfare by at least 2, 2) delete at most $2(k-1)-2$ positive relations, decreasing the social welfare by at most $(2(k-1)-2) \cdot \frac{1}{n} < 2$. Therefore, $\mathrm{SW}(C_\ell^* \setminus \{i\}) + \mathrm{SW}(\{i\}) = \mathrm{SW}(C_\ell^* \setminus \{i\}) > \mathrm{SW}(C_\ell^*)$, thus contradicting the optimality of $\mathcal{C}^*(\mathbf{d})$. □

Lemma 4.14. *Given an* EA *preference profile* $\mathbf{d}$*, there exists at most one coalition in* $\mathcal{C}^*(\mathbf{d})$ *that is not a clique in* $G_{\mathbf{d}}^{\leftrightarrows}$*.*

Proof. Let us assume that $C^* \in \mathcal{C}^*(\mathbf{d})$ is not a clique in $G_{\mathbf{d}}^{\leftrightarrows}$, and let $k = |C^*|$ and k_1 be the number of negative relations in C^*. Since by Lemma 4.13 such relations do not share any endpoint, picking one endpoint per relation and putting it in a new coalition K_1, we can split C^* into two cliques K_1 and K_2 of sizes k_1 and $k_2 = k - k_1$, respectively, so that all the negative relations now lie in the cut between K_1 and K_2. Thus, as this deletes exactly k_1 negative relations and exactly $2k_1(k-k_1) - k_1$ positive relations from C^*, $\mathrm{SW}(C^*)$ is equal to $\mathrm{SW}(K_1) + \mathrm{SW}(K_2) + \frac{1}{n} \cdot (2k_1(k-k_1) - k_1) - k_1$.

Since C^* belongs to $\mathcal{C}^*(\mathbf{d})$, $\frac{1}{n}(2k_1(k-k_1)-k_1)-k_1 \geq 0$ must hold, which implies $k \geq \frac{n+1}{2}+k_1$. Therefore, since a coalition in $\mathcal{C}^*(\mathbf{d})$ that is not a clique must contain more than half of the agents from N, it must also be unique. $\square$

By Lemma 4.14, even if in general the optimal solution for an EA preference profile $\mathbf{d}$ is not a partition into cliques of $G^{\leftrightarrows}_{\mathbf{d}}$, it is "almost a clique partition". So, a natural next step is to try to quantify how far **opt** is from the social welfare of a best clique partition for an EA profiles.

Proposition 4.15. *Given an* EA *preference profile* $\mathbf{d}$ *and a best clique partition* $\mathcal{K}^*$ *for* $G^{\leftrightarrows}_{\mathbf{d}}$, $\mathbf{opt} \leq \left(\frac{1+\sqrt{2}}{2}\right) \cdot SW(\mathcal{K}^*)$. *Furthermore, this bound is tight.*

Proof. Let us assume that $\mathcal{C}^*(\mathbf{d}) = \{C_1^*, \ldots, C_m^*\}$. By Lemma 4.14, there exists at most one coalition in $\mathcal{C}^*(\mathbf{d})$ that is not a clique. W.l.o.g. let us assume that C_1^* is such a coalition, and let $k = |C_1^*|$. Recalling the proof of Lemma 4.14, it is possible to split C_1^* into two cliques K_1 and K_2 of respective sizes k_1 and k_2, where k_1 is the number of negative relations in C_1^*, and

$$\mathrm{SW}(C_1^*) = \mathrm{SW}(K_1) + \mathrm{SW}(K_2) + \frac{1}{n}(2k_1k_2 - k_1) - k_1.$$

Therefore, taking into account that $\{K_1, K_2, C_2^*, \ldots, C_m^*\}$ is a clique partition of $G^{\leftrightarrows}_{\mathbf{d}}$, we know that $\mathrm{SW}(\mathcal{K}^*) \geq \mathrm{SW}(K_1) + \mathrm{SW}(K_2) + \sum_{\ell=2}^m \mathrm{SW}(C_\ell^*)$, since $\mathcal{K}^*$ is a best clique partition. Thus,

$$\frac{\mathbf{opt}}{\mathrm{SW}(\mathcal{K}^*)} \leq \frac{\mathrm{SW}(C_1^*) + \sum_{\ell=2}^m \mathrm{SW}(C_\ell^*)}{\mathrm{SW}(K_1) + \mathrm{SW}(K_2) + \sum_{\ell=2}^m \mathrm{SW}(C_\ell^*)} \leq \frac{\mathrm{SW}(C_1^*)}{\mathrm{SW}(K_1) + \mathrm{SW}(K_2)} = 1 + s,$$

where

$$s = \frac{\frac{1}{n} \cdot (2k_1k_2 - k_1 - k_1 n)}{\frac{1}{n} \cdot (k_1(k_1-1) + k_2(k_2-1))} = \frac{2k_1k_2 - k_1 - k_1 n}{k_1(k_1-1) + k_2(k_2-1)}.$$

Let $\alpha < 1$ be such that $k_1 = \alpha k$ and $k_2 = (1-\alpha)k$. Then,

$$\begin{aligned} s &= \frac{2\alpha(1-\alpha)k - \alpha - \alpha n}{\alpha(\alpha k - 1) + (1-\alpha)((1-\alpha)k - 1)} \\ &\leq \frac{2\alpha(1-\alpha)n - \alpha - \alpha n}{(\alpha^2 + (1-\alpha)^2)n - 1} \\ &\leq \frac{\alpha(1-2\alpha)}{2\alpha(\alpha-1)+1} \leq \frac{1}{\sqrt{2}} - \frac{1}{2}. \end{aligned}$$

As regards to the tightness of the bound, we now prove that, for every fixed $\varepsilon > 0$, there exists an EA preference profile $\mathbf{d}$ such that

$$\mathbf{opt} \geq \left(\frac{1}{2} + \frac{1}{\sqrt{2}} - \varepsilon\right) \cdot \mathrm{SW}(\mathcal{K}^*),$$

where $\mathcal{K}^*$ is a best clique partition for $G_{\mathbf{d}}^{\leftrightarrows}$.

We now show that the just provided upper bound is also tight. Consider the EA preference profile $\mathbf{d}$ inducing an instance in which all the possible ordered pairs (i, j) of agents are friends, except for the k pairs $(1, 2), (3, 4), \ldots, (2k - 1, 2k)$, where we set $k = \frac{2n^2-2n-\sqrt{2n^4-6n^3+6n^2-2n}}{2(n+1)} < \frac{n-1}{2}$. Since the negative relations do not share any endpoint and thus form a matching, as in the proof of Lemma 4.14, we can construct a clique partition that splits all the agents into two cliques of respective sizes k and $n - k$. We denote this clique partition of $G_{\mathbf{d}}^{\leftrightarrows}$ by $\mathcal{K}$. Note that $\mathrm{SW}(\mathrm{GC}) = n-1-k(1+\frac{1}{n}) > \frac{2k^2}{n}-2k+n-1 = \mathrm{SW}(\mathcal{K})$ if and only if $k < \frac{n-1}{2}$. Moreover, $\mathcal{K}$ is the best clique partition in $G_{\mathbf{d}}^{\leftrightarrows}$, which implies $\mathrm{SW}(\mathcal{K}^*) = \mathrm{SW}(\mathcal{K})$. Therefore, since $\frac{\mathbf{opt}}{\mathrm{SW}(\mathcal{K}^*)} \geq \frac{\mathrm{SW}(\mathrm{GC})}{\mathrm{SW}(\mathcal{K})}$ and

$$\lim_{n\to\infty} \frac{\mathrm{SW}(\mathrm{GC})}{\mathrm{SW}(\mathcal{K})} = \lim_{n\to\infty} \frac{n - 1 - k(1 + \frac{1}{n})}{\frac{2k^2}{n} - 2k + n - 1} = \frac{1}{2} + \frac{1}{\sqrt{2}} \ ,$$

where we used the definition of k in terms of n to arrive at the final result. Thus, the claim follows. □

Even if the best clique partition gives a $\frac{1}{\sqrt{2}} + \frac{1}{2}$ approximation of the optimum, no polynomial time algorithm can compute it. In fact, we now prove that the social welfare maximization problem for EA profiles is as hard to approximate as the MaxClique problem.

Theorem 4.16. *For any $\varepsilon > 0$ no polynomial time algorithm can approximate the optimal social welfare for* EA *preference profiles with an approximation ratio better than $O(n^{1-\varepsilon})$, unless $P = NP$.*

Proof. For a fixed $\varepsilon > 0$, let us assume that there exists a polynomial time algorithm A that, given as input an EA preference profile $\mathbf{d}$, always returns a partition $\mathcal{C}^A(\mathbf{d})$ such that $\mathrm{SW}(\mathcal{C}^A(\mathbf{d}))$ approximates the optimal social welfare

opt within an approximation ratio $o(n^{1-\varepsilon})$. We now show that algorithm A can be exploited to find a good approximation for MaxClique.

To this aim, consider any instance G of MaxClique, and let $\mathbf{d}$ be an EA profile such that $G_{\mathbf{d}}^{\leftrightarrows} \equiv G$. Let k^* be the size of a maximum clique in $G_{\mathbf{d}}^{\leftrightarrows}$. Since a maximum clique of $G_{\mathbf{d}}^{\leftrightarrows}$ completed with singleton coalitions containing all the other agents is a possible outcome, we know that $k^*(k^*-1)/n \leq \mathbf{opt}$.

Consider now the outcome $\mathcal{C}^A(\mathbf{d}) = \{C_1^A, \ldots, C_m^A\}$ returned by A when run on profile $\mathbf{d}$. W.l.o.g. assume that no agent in $\mathcal{C}^A(\mathbf{d})$ participates in more than one negative relation in her coalition, otherwise we can put such an agent alone in a singleton coalition, increasing the overall social welfare (as shown in the proof of Lemma 4.13). Then, similarly as in the proof of Lemma 4.14, we can split every coalition $C_i^A \in \mathcal{C}^A(\mathbf{d})$ that contains negative relations into two cliques K_i^1 and K_i^2. If C_i^A is already a clique, we define $K_i^1 = C_i^A$ and $K_i^2 = \emptyset$. Let $\mathcal{K}$ be the coalition structure consisting of all the cliques K_i^1 and $K_i^2 \neq \emptyset$.

By exploiting the same arguments as in the proof of Proposition 4.15, it follows that $\frac{\mathrm{SW}(C_i^A)}{\mathrm{SW}(K_i^1)+\mathrm{SW}(K_i^2)} \leq \frac{1}{2} + \frac{1}{\sqrt{2}}$ and then consequently $\mathrm{SW}(\mathcal{C}^A(\mathbf{d})) \leq \left(\frac{1+\sqrt{2}}{2}\right) \cdot \mathrm{SW}(\mathcal{K})$. Now, if

$$k_{max} = \max_{\substack{i\in[m],\\ j\in\{1,2\}}} \left\{|K_i^j|\right\} \text{ and } q = |\{K_i^j \mid K_i^j \neq \emptyset, i \in [m], j \in \{1,2\}\}|,$$

we have that

$$\mathrm{SW}(\mathcal{K}) \leq q \cdot \frac{k_{max}(k_{max}-1)}{n} \leq k_{\max}(k_{\max}-1).$$

By assumption, for every $c > 0$, there exists $n_c \in \mathbb{N}$ such that $\frac{\mathbf{opt}}{\mathrm{SW}(\mathcal{C}^A(\mathbf{d}))} < cn^{1-\varepsilon}$, $\forall\ n > n_c$ and $\mathbf{d} = (d_1, \ldots, d_n)$. Therefore, as $k_{\max} \leq k^*$, for any fixed

$c > 0$ and $n > n_{c'}$ with $c' = c^2 \cdot \frac{2}{1+\sqrt{2}}$

$$\left(\frac{k^*}{k_{\max}}\right)^2 \cdot \frac{1}{n} \leq \frac{k^*(k^*-1)}{k_{max}(k_{max}-1)} \cdot \frac{1}{n} \leq \frac{\mathbf{opt}}{\mathrm{SW}(\mathcal{K})}$$
$$\leq \left(\frac{1+\sqrt{2}}{2}\right) \frac{\mathbf{opt}}{\mathrm{SW}(\mathcal{C}^A(\mathbf{d}))}$$
$$< \left(\frac{1+\sqrt{2}}{2}\right) \cdot c' n^{1-\varepsilon},$$

meaning that $k^*/k_{max} < c \cdot n^{1-\frac{\varepsilon}{2}}$.

In other words, by using algorithm A, we can extract in polynomial time from $\mathcal{C}^A(\mathbf{d})$ a clique of size k_{max} in G that approximates the optimal solution of MaxClique with an approximation ratio $o(n^{1-\frac{\varepsilon}{2}})$, which is not possible unless P = NP. □

4.4.1 Strategyproof Mechanisms

We start this subsection by first focusing on efficient mechanisms, that is, running in polynomial time.

Mechanism $\mathcal{M}_4$. Given an EA preference profile $\mathbf{d}$,

1. enumerates the agents in N from 1 to n;
2. set $\mathcal{C} = \emptyset$;
3. for $i = 1$ to n
   - if there exists $j > i$ in the neighborhood of i in $G_{\mathbf{d}}^{\leftrightarrows}$ not matched yet, then $\mathcal{C} = \mathcal{C} \cup \{i, j\}$,
   - otherwise, $\mathcal{C} = \mathcal{C} \cup \{i\}$;
4. return $\mathcal{C}$.

According to Theorem 4.16, the approximation factor of mechanism $\mathcal{M}_4$ proven in the following theorem is asymptotically optimal.

Theorem 4.17. *$\mathcal{M}_4$ is strategyproof and $r^{\mathcal{M}_4} \leq (1+\sqrt{2}) \cdot n$.*

Proof. To show the strategyproofness, we observe that only unassigned agents are interested in manipulating the mechanism. Let i be an unassigned agent. By the definition of $\mathcal{M}_4$, this means that all her neighbors in $G^{\leftrightarrows}_{\mathbf{d}}$ have been assigned in the previous rounds. Agent i can manipulate in the following ways: 1) declare an enemy j as a friend, or, 2) declare a friend j as an enemy. In case 1) either i will still not be assigned by the mechanism or, even worse, she will be assigned to her enemy j. In case 2) the outcome does not change. Thus, $\mathcal{M}_4$ is strategyproof.

In order to establish the approximation ratio of $\mathcal{M}_4$, we first observe that, given any EA profile $\mathbf{d}$, the returned coalition structure $\mathcal{C} = \{C_1, \ldots, C_\ell\}$ forms a maximal matching in $G^{\leftrightarrows}_{\mathbf{d}}$, and that, as it is well-known, such a matching has at least half of the edges of a maximum matching of $G^{\leftrightarrows}_{\mathbf{d}}$. Moreover, given a best clique partition $\mathcal{K}^* = \{K^*_1, \ldots, K^*_m\}$ for $G^{\leftrightarrows}_{\mathbf{d}}$, in a maximum matching there are at least $\sum_{i=1}^m \lfloor \frac{k^*_i}{2} \rfloor$ edges, where $k^*_i = |K^*_i|, i \in [m]$. Therefore, since each coalition $C_j \in \mathcal{C}, j \in [\ell]$, contains a pair of mutual positive relations and thus contributes $2/n$ to the overall social welfare, $\mathrm{SW}(\mathcal{C}) \geq \frac{2}{n} \cdot \frac{1}{2} \sum_{i=1}^m \lfloor \frac{k^*_i}{2} \rfloor$, so that

$$
\begin{aligned}
\left(\frac{2}{1+\sqrt{2}}\right) \cdot \frac{\mathbf{opt}}{\mathrm{SW}(\mathcal{M}_4(\mathbf{d}))} &\leq \frac{\mathrm{SW}(\mathcal{K}^*)}{\mathrm{SW}(\mathcal{C})} \\
&\leq \frac{\frac{1}{n}\sum_{i=1}^m k^*_i(k^*_i - 1)}{\frac{1}{n}\sum_{i=1}^m \lfloor \frac{k^*_i}{2} \rfloor} \\
&\leq \frac{k^*_{max}\sum_{i=1}^m (k^*_i - 1)}{\sum_{i=1}^m \lfloor \frac{k^*_i}{2} \rfloor} \leq 2 \cdot k^*_{max} \leq 2n,
\end{aligned}
$$

where k^*_{max} is the size of the maximum clique in $\mathcal{K}^*$, and the first inequality holds by Proposition 4.15 while the second to last inequality holds since $\lfloor \frac{k^*_i}{2} \rfloor \geq \frac{1}{2}(k^*_i - 1)$. Rearranging the terms, we get the claimed approximation ratio. $\square$

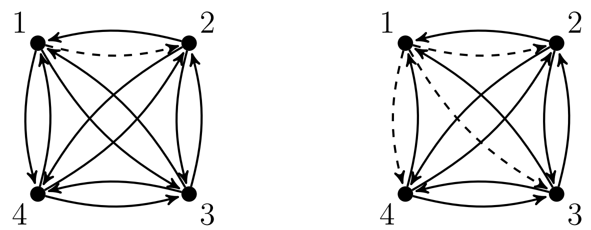
In contrast to mechanism $\mathcal{M}_2$, mechanism $\mathcal{M}_4$ always outputs an individually rational, but not necessarily Pareto optimal, outcome. Individual rationality simply follows by considering that matched agents achieve a utility grater than 0. Regarding Pareto optimality it is sufficient to consider as counter example an instance with 4 agents where the corresponding $G^{\leftrightarrows}$ is a clique; in this case, independently of the agents' ordering, the mechanism creates two coalitions of size 2 where the utility of each agent is equal to $\frac{1}{4}$, on the other hand, in the grand coalition every agent improves her gain by achieving a utility of $\frac{3}{4}$.

As already observed, the approximation ratio of the above mechanism is high due to the inherent difficulty of computing a sublinear approximation in polynomial time. However, if efficiency is not a concern, one might wonder whether a mechanism that always returns an optimal outcome is strategyproof. The following proposition shows that this is not the case.

Proposition 4.18. *A mechanism that for every* EA *profile* $\mathbf{d}$ *returns an optimal outcome* $\mathcal{C}^*(\mathbf{d})$ *is not strategyproof.*

Proof. Let $N = \{1, 2, 3, 4\}$ and consider the instance I_1 depicted in Fig. 4.3(a), having only one negative relation $(1, 2)$. The grand coalition GC is the optimal outcome for I_1, even though agent 1 has a negative utility. On the other hand, if we consider the instance I_2 depicted in Fig. 4.3(b), that differs from I_1 only by the fact that agent 1 now declares all the other agents as enemies, the optimal solution becomes $\{\{1\}, \{2, 3, 4\}\}$. Here, agent 1 is in a singleton coalition, and thus has utility 0. In conclusion, an algorithm that always returns an optimal outcome is manipulable, since in I_1 agent 1 can improve her utility by misreporting her preferences. ☐

Although returning the optimal outcome is not strategyproof, we next show that a constant non-polynomial approximation mechanism exists. In particular, the following mechanism has been described and shown to be strategyproof in [35] (without the analysis of its approximation ratio).



(a) Instance I_1 (b) Instance I_2

FIGURE 4.3: Returning the social optimum in the EA case is not strategyproof. Dashed edges represent negative relations.

(a) Outcome of the mechanism (b) Social optimum

FIGURE 4.4: Lower bound instance for Mechanism $\mathcal{M}_5$.

Mechanism $\mathcal{M}_5$. Given an EA preference profile $\mathbf{d}$,

1. set $G = G_{\mathbf{d}}^{\leftrightarrows}$ and $\mathcal{C} = \emptyset$;
2. while $G \neq \emptyset$
   - compute the first maximum clique K of G in lexicographical order,
   - $\mathcal{C} = \mathcal{C} \cup \{K\}$;
   - $G = G \setminus K$;
3. return $\mathcal{C}$.

We will now analyze the approximation ratio of mechanism $\mathcal{M}_5$. To this end we first prove the following lemma.

Lemma 4.19. *$\mathcal{M}_5$ returns an outcome that is a 2-approximation of the social welfare achieved by a best clique partition $\mathcal{K}^*$. Moreover, this bound is tight.*

Proof. Let $\mathcal{K}^*$ be a best clique partition of graph $G_{\mathbf{d}}^{\leftrightarrows}$ and let K be a clique in $\mathcal{K}^*$. We denote by $\overline{\mathcal{K}}$ the clique partition returned by mechanism $\mathcal{M}_5$ and by $K_i \in \overline{\mathcal{K}}$ the clique selected in the i-th iteration of $\mathcal{M}_5$.

Let us enumerate the elements of $K = \{v_1, \dots, v_k\}$ where $k = |K|$ in such a way that for each $i < j$, v_i has been selected either in a previous or in the same iteration as v_j by mechanism $\mathcal{M}_5$.

Due to the greedy approach of $\mathcal{M}_5$, we know that it has to assign agent v_i to a clique of size at least $k - i + 1$. Indeed, $v_i, \dots, v_k$ are not yet selected and they form a clique of size $k - i + 1$, while mechanism $\mathcal{M}_5$ selects a maximum clique in the still available subgraph.

Thus, the contribution of v_i to the social welfare will be at least $\frac{k-i}{n}$. Summing up over all the agents of K, their global contribution to $\mathrm{SW}(\overline{\mathcal{K}})$ is at least $\frac{k(k-1)}{2 \cdot n}$, while $\mathrm{SW}(K) = \frac{k(k-1)}{n}$. Repeating the same argument for every clique in the best clique partition, we finally get $\mathrm{SW}(\mathcal{K}^*) \leq 2 \cdot \mathrm{SW}(\overline{\mathcal{K}})$ and consequently $r^{\mathcal{M}_5} \leq 2$.

For the tightness of the bound, it is sufficient to consider the example shown in Figure 4.4. □

Lemma 4.19 and Proposition 4.15, together with the strategyproofness of $\mathcal{M}_5$ shown in [35], prove the following theorem.

Theorem 4.20. *$\mathcal{M}_5$ is strategyproof and $r^{\mathcal{M}_5} \leq (1 + \sqrt{2})$.*

4.5 Neutrals

So far we allowed each agent to split the other ones into either friends or enemies, that is, $F_i \cup E_i = N \setminus \{i\}$ for each $i \in N$. In this section we consider a more general setting where $F_i \cup E_i \subseteq N \setminus \{i\}$ for each $i \in N$. In this case, any agent that is neither a friend nor an enemy of $i \in N$ is said to be a *neutral*, meaning that she will have no impact on i's utility. More precisely, for each $i \in N$, if $j \in N \setminus (F_i \cup E_i \cup \{i\})$, then $v_i(j) = 0$.

It is easy to see that all the mechanisms provided in the previous sections, both for FA and EA, are strategyproof also in this extended setting. However, in general their approximation ratios do not necessarily transfer. In particular, while the mechanisms for FA preferences profiles, namely $\mathcal{M}_2$ and $\mathcal{M}_3$, maintain the same approximation ratio, the mechanisms for EA profiles do not. Indeed, $\mathcal{M}_4$ and $\mathcal{M}_5$ have an unbounded approximation ratio when neutrals are allowed. In what follows, we elaborate on these claims.

4.5.1 Friends Appreciation in the Presence of Neutrals

We start by analyzing the presence of neutrals in the FA case. Given the agents' preferences $\mathbf{d}$, $G_{\mathbf{d}}^{\rightarrow}$ is still computed by just considering the positive relations between agents.

We first observe that $\mathscr{M}$ is a class of strategyproof mechanism also in this setting. Indeed, the proof of Theorem 4.6 can be easily generalised to allow the presence of neutrals.

Theorem 4.21. *$\mathscr{M}$ is an admissible and strategyproof class of mechanisms for* FA *profiles also in presence of neutrals.*

Proof. The claim follows by exactly the same arguments in the proof of Theorem 4.6. □

Furthermore, we can generalize the main results of Section 4.3 to the case of neutrals. More precisely, both our deterministic mechanism $\mathcal{M}_2$ and randomized mechanism $\mathcal{M}_3$ can be applied to this more general setting.

Theorem 4.22. *Mechanism $\mathcal{M}_2$ is admissible, strategyproof and has approximation ratio $r^{\mathcal{M}_2} \leq n$ also in the presence of neutrals.*

Proof. From Theorem 4.21, we know that the mechanism is admissible and strategyproof.

Regarding the approximation ratio, given a preference profile $\mathbf{d}$, let us denote by f and e the number of positive and negative relations in $\mathbf{d}$, respectively.

If $f = 0$, the outcome returned by $\mathcal{M}_2$ is the coalition structure containing only singleton coalitions. Since in this case such an outcome is also the optimal one, the claim follows.

If $f > 0$, the worst case scenario occurs when the graph $G^{\rightarrow}$ is weakly connected. The social welfare achieved by the mechanism is at least $f - \frac{e}{n} \geq f - \frac{1}{n}(n(n-1) - f) = f\left(1 + \frac{1}{n}\right) - (n-1)$, where e is the number of negative relationships, meaning that the social welfare can only be higher than the one without neutrals. Thus, the same bound on the approximation ratio as in Theorem 4.7 applies here as well. □

Theorem 4.23. *In the presence of neutrals $\mathcal{M}_3$ is admissible, strategyproof and $r^{\mathcal{M}_3} \leq 4$ in expectation. Moreover, for any fixed $\varepsilon > 0$, $r^{\mathcal{M}_3} \leq 4(1+\varepsilon)$ with high probability.*

By Theorem 4.21 $\mathcal{M}_3$ is admissable and strategyproof. For $n < 4$ the approximation ratio is bounded by 3 by Theorem 4.22.

For $n \geq 4$, it is enough to notice that the social welfare of the outcome returned by the mechanism in this setting is always lower bounded by the social welfare of the outcome returned on the modified instance in which every neutral relation in our original input is replaced by an enemy relation. Now, the claim follows from Theorem 4.10.

4.5.2 Enemies Aversion in the Presence of Neutrals

We now turn our attention to EA profiles, showing that unfortunately in this case neutrals have a bad impact on the performance of our strategyproof mechanisms.

Example 4.3. *Consider the following instance: for each $i = 1, \ldots, n$, $j \in F_i$ for every $j > i$, while $E_i = \emptyset$. In this case, both $\mathcal{M}_4$ and $\mathcal{M}_5$ return the all singletons coalition structure achieving social welfare 0, while the social optimum is given by the grand coalition and has social welfare $\frac{n+1}{2}$.*

As it can be seen from Example 4.3, different mechanisms are needed to handle this new extended setting. The main problem of our mechanisms is that they rely on the graph $G^{\leftrightarrows}_{\mathbf{d}}$, where only mutual friendship relations are represented. However, when neutrals appear, also pairs i, j with $j \in F_i$ and $i \notin E_j$ should be considered.

In this extended setting, providing a strategyproof mechanism with a bounded approximation ratio is indeed possible. More precisely, we can resort on the mechanism of [43] for ASHGs with $-1, 0, 1$ valuations. We provide it below as mechanism $\mathcal{M}_6$, reformulated for our setting. It always returns a coalition structure in which all the coalitions are singletons, except possibly one – containing a pair of nodes having at least one positive relation between them, and no negative one.

In more detail, the mechanism examines the agents one by one in an arbitrary but fixed order and in each step tries to create a coalition consisting of two agents, after which it places all the other agents into singletons and terminates. To create a coalition with two agents, it focuses on the preferences of the currently examined agent i. First, it tries to find another agent j such that i and j mutually consider each other to be friends (step $1a$). If such an agent does not exist, the mechanism next tries to find an agent j such that i considers j to be a friend and such that every other agent k that j considers to be a friend, in return considers j to be an enemy (step $1b$). This condition implies that the mechanism tries to pair i with an agent j for whom it holds that being paired with an agent that she sees as neutral is the best she can achieve. Finally, if both of the previous steps do not apply, the mechanism tries to pair the currently examined agent i with an already examined agent j that she considers to be a friend, and j sees i as neutral (step $1c$). This condition ensures that the mechanism pairs i with an agent j for whom it holds that she cannot be paired with an agent she considers to be a friend anymore. If none of the matching steps apply for any of the agents, the mechanism returns all agents in singleton coalitions. Possible ties are resolved by considering the ordering in which the mechanism is processing the agents. More specifically, if the i-th agent could be placed in a coalition both with the j-th and the k-th agent, $\mathcal{M}_6$ pairs it with the j-th agent if and only if $j < k$.

Unfortunately, the approximation ratio of this mechanism is $O(n^2)$, both in the setting considered in [43] and in our setting.

Mechanism $\mathcal{M}_6$. Given an EA preference profile $\mathbf{d}$,

1. For $i = 1$ to n

   a If there exists $j \in N$ s.t. $j \in F_i \wedge i \in F_j$: put agents i and j together into a coalition and all the others into a singletons, and terminates,

   b If there exists $j \in N$ s.t. $j \in F_i \wedge i \in N \setminus \{F_j \cup E_j \cup \{j\}\} \wedge \forall k \in F_j, j \in E_k$: put agents i and j together into a coalition and any other agent into a singleton, and terminates,

   c If there exists $j \in N$ s.t. $j \in F_i \wedge i \in N \setminus \{F_j \cup E_j\}$ and $j < i$: put agents i and j together into a coalition and any other agent into a singleton, and terminates;

2. If no coalition of two agents has been created during the previous steps, put all the agents into singletons.

Theorem 4.24. *Mechanism $\mathcal{M}_6$ is admissible, strategyproof and $r^{\mathcal{M}_6} \in \Theta(n^2)$.*

Proof. Since mechanism $\mathcal{M}_6$ never pairs agents who have a negative relation between them, it is admissible. Regarding strategyproofness, we follow the same arguments of [43]. First of all, we notice that no agent can achieve utility higher than $1/n$ in any outcome of the mechanism, and that the utility of each agent is guaranteed to be ≥ 0 when she reports her preferences truthfully. Therefore, the only agents who might be interested in manipulating are the ones that have utility exactly 0 when reporting truthfully. Let us first assume that such an agent i is placed into a singleton coalition. Under this assumption, i cannot modify her declared preferences so as to increase her utility, as in the best case she will be placed into a coalition with an enemy or a neutral. This is so because i was placed in a singleton either because all agents that i considers as friends consider her in return to be an enemy, or the problem lies in the ordering in which the mechanism examines the agents, which she cannot influence. Similarly, if i is placed into a coalition with a neutral, by lying she cannot manage to be placed with a friend. In fact, an agent gets placed with a neutral only when she either does not have any friend for which she is also a friend, or when she already had a chance to be paired but failed. Both of these reasons are beyond what can be manipulated by changing one's own preference profile.

For the approximation ratio, we start by observing that the mechanism always returns a coalition structure where at most one coalition has size 2 and all others are singletons.

If the mechanism returns only singleton coalitions, then the achieved social welfare is 0. However, in this case also the optimum social welfare equals 0, and thus the mechanism is optimal. Indeed, if $\mathcal{M}_6$ puts every agent into a singleton, then for each pair of agents there exists a negative relation between them. Therefore, any coalition structure containing a coalition of size larger than 1 has a strictly negative social welfare.

If the mechanism matches one pair of agents, the achieved social welfare is $1/n$. On the other hand, the social welfare of an optimal outcome is at most $n-1$. Thus, $r^{\mathcal{M}_6} \leq n \cdot (n-1)$. This bound is also tight, as it can be checked considering the instance in which $F_i = N \setminus \{i\}, \forall i \in N$. $\square$

Even if mechanism $\mathcal{M}_6$ seems rather trivial, improving its approximation ratio of $\Theta(n^2)$ appears to be a challenging task. In fact, a matching mechanism with a better approximation ratio would immediately yield also a mechanism with a better approximation ratio than the one in [43] for ASHGs with $-1, 0, 1$ valuations. This follows by observing that, while in our setting positive and negative relations contribute $1/n$ and -1 to the social welfare, respectively, in ASHGs with $-1, 0, 1$ valuations these values are $1 > 1/n$ and -1. At the same time, the optimum is bounded by $n - 1$ in both settings. Furthermore, strategyproofness transfers from our setting to ASHGs with $-1, 0, 1$ valuations, as the incentives are exactly the same and the only difference being that the only possible positive utility for an agent becomes 1 (instead of $1/n$).

4.6 Open Problems

The most natural open problem is reducing our approximation gaps, especially in the FA deterministic case. This appears to be a challenging task, and bears similarities with analogous gaps in [43]. A further interesting extension of our work might be allowing neutrals to have a negligible yet non-null positive or negative effect on the utilities.

Beyond the efficiency in terms of the approximation ratio, anonymity is another interesting feature of strategyproof mechanisms. Roughly speaking, a mechanism is anonymous if it does not rely on agents' identities. Anonymity is a desirable property, as it in some sense guarantees equal treatment of the agents, so it makes the mechanism fairer. Unfortunately, not in all settings a mechanism that is at the same time anonymous, strategyproof and admissible exists. While for FA our mechanisms are anonymous, for EA they are not. In fact, for EA it is not possible to have a deterministic mechanism that is both admissible and anonymous. This can be verified by considering an instance with 3 agents where $(1, 3), (3, 1)$ are the negative relations, and all others are positive. In this example, only the outcomes $\{\{1, 2\}, \{3\}\}$ and $\{\{1\}, \{2, 3\}\}$ achieve a positive social welfare. However, both of these outcomes cannot be consistently reached by an anonymous mechanism. Thus, under EA, a deterministic anonymous mechanism cannot be admissible. Anonymity in randomized mechanisms remains an interesting open question.

Chapter 5

Strategyproof Mechanisms for the Additively Separable Group Activity Selection Problem

In this chapter we investigate strategyproof mechanisms for the Group Activity Selection Problem with the additively separable property. Here the agents have a distinct value for each activity and for any other agent. Thus, we develop our analysis taking into account the possible values given to the activities and to the agents.

5.1 Introduction

In the *Group Activity Selection Problem* (GASP) [32] we are given a set of agents and a set of activities, and each agent must be assigned to one of the available activities according to her own preferences. This problem is able to represent several realistic scenarios, such as workers that must be split into teams for performing specific tasks, or employers that must be located in different sites, or students that must be assigned to classrooms, and so forth.

An interesting subclass of the GASP is the one with the additively-separable property (ASGASP) [17], in which each agent has a specified preference value for each activity and an individual weight representing her appreciation for each other individual agent. The utility of an agent in a given outcome is the sum of the preference for her assigned activity and of the weights she has for all the other agents participating in such an activity. Although it is a rather simple representation, agents may not be inclined to complex computations, and thus, summation represents the simplest type of computation one can think about.

In this work, we also consider two types of activities. Indeed, according to the problem's nature, activities can be assumed to be either *copyable* or *non-copyable*. If we consider the problem of splitting tasks among workers, the same task may be performed by different team groups independently; being tasks "abstract" activities might be assumed to be copyable. On the other hand, for the problem of splitting students among classrooms, we cannot assume a given classroom to be copyable, since it is not duplicable. Thus, students assigned to the same classroom will necessarily stay together, meaning that activities are non-copyable.

5.1.1 Our contribution

In this chapter we investigate strategyproof mechanisms with good welfare guarantees for the ASGASP. In particular, we evaluate the performance of our mechanisms through their approximation ratio with respect to the maximum utilitarian social welfare. We develop our analysis by taking into account the possible values of the preferences among the activities and of the individual weights between the agents. Namely, they can be either non-negative reals or unitary, that is 0 or 1. Unless not specified otherwise, our results listed below hold for the more standard non-copyable case.

We first consider non-negative preferences and show that a quite negative result holds: regardless of the restrictions on the possible values of the individual weights, no deterministic mechanism can achieve a bounded approximation ratio. Because of this impossibility result, we then turn our attention to randomized mechanisms, and design a simple k-approximate mechanism, where k is the number of activities. We also provide a constant lower bound holding for any randomized mechanism. Moreover, we provide a $\left(2 - \frac{1}{k}\right)$-approximate randomized mechanism for the copyable case.

We then focus on the case where preferences can only be unitary, and provide a k-approximate deterministic mechanism, together with an $\Omega\left(\sqrt{k}\right)$ lower bound. Moreover, when also the weights are unitary, we improve this ratio by providing a 2-approximate deterministic mechanism.

Table 5.1: Lower and upper bounds for strategyproof mechanisms. (D=Deterministic, R=Randomized)

		Non-neg. preferences		Unitary preferences		Public preferences	
		LB	UB	LB	UB	LB	UB
Non-neg. weights	D	∞ (Th. 5.3)		$\Omega\left(\sqrt{k}\right)$ (Th. 5.12)	k (Mech. $\mathcal{M}_{10}$)	$\Omega\left(\sqrt{k}\right)$ (Th. 5.12)	k (Mech. $\mathcal{M}_{10}$)
	R	$2-\frac{2}{k+1}$ (Th. 5.7)	k (Mech. $\mathcal{M}_8$)	$2-\frac{2}{k+1}$ (Th. 5.7)	k (Mech. $\mathcal{M}_8$)	$2-\frac{2}{k+1}$(Th. 5.7)	$2-\frac{1}{k}$ (Mech. $\mathcal{M}_{12}$)
Unitary weights	D	∞ (Th. 5.9)		>1 (Prop. 5.1)	2 (Mech. $\mathcal{M}_{11}$)		
	R	4/3 (Th. 5.10)	k (Mech. $\mathcal{M}_8$)	>1 (Prop. 5.1)	2 (Mech. $\mathcal{M}_{11}$)		

Finally, we show how to extend our results to public preferences, that is when agents have limited power to manipulate, since they may only misreport their individual weights. Interestingly enough, under such an assumption we can guarantee the same results holding for unitary preferences. Furthermore, we are able to give a constant-approximate randomized mechanism.

We want to highlight that while, on the one hand, all our mechanisms work in polynomial time, on the other hand, the provided lower bounds do not depend on time complexity. Indeed, independently of computation strategyproof mechanisms cannot guarantee optimal outcomes in general, and in some specific cases not even constant.

A summary our results is given in Table 5.1.

5.1.2 Related work

In the past decade, considerable attention has been devoted to the GASP [32], as an interesting generalization of the well-known HGs [36]. Most of the literature in this setting has focused on the subclasses of GASP with anonymous preferences that allow a succinct representation. In particular, [32] introduced the GASP and the approval-based version aGASP, while [27] proposed the ordinal preferences version called oGASP. Other investigated classes of GASP have been sGASP [30], where there is a lower and an upper bound on the number of participants per activity, and gGASP [49], where feasible coalitions must be connected components of a given interaction graph. In the above papers, individual and group deviations have been considered, providing several hardness results concerning the existence of respective stable solutions, plus positive ones for particular cases. Some parameterized complexity studies in these subclasses have been given in [45, 47, 53]. In the same spirit of ASHGs [7], in [17] the ASGASP has been investigated, and results on the complexity of determining Nash stable outcomes and on the performance of equilibria in terms of price of anarchy and stability have been given. Moreover, the NP-hardness of computing the optimal solution has been proven, together with a $(2 - 1/k)$-approximate algorithm. A nice introduction to GASP and to the various subclasses can be found in [33].

Some research in HGs and GASP concentrated on the realistic setting in which agents' preferences are not known, and the assignment of agents should be done so as to encourage their truthful reporting, as typically computed in mechanism design. In Chapter 4, we already mentioned the effort made in this direction for the ASHGs. In [29] the manipulability (non-strategyproofness) of several reasonable mechanisms (rules) with the requirement of individual rationality has been shown for three possible extensions of oGASP. Finally, in [54] strategyproof mechanisms for GASP with single-picked preferences are provided.

Another related stream of research is the study *one-sided Markets* (also known as the *House Allocation Problem*), where n self-interested agents have preferences among n available items and each agent must be assigned to exactly one item. In this setting, the *Random Serial Dictatorship* (RSD) [21] guarantees strategyproofness and also good properties such as Pareto efficiency and fairness. Moreover, in some cases, RSD provides a bounded approximation with respect to the optimal social welfare. In particular, it offers (in expectation) a 3-approximation if preferences are dichotomous, and a $\sqrt{e \cdot n}$-approximation if preferences are unit-range [1]. When preferences are unit-sum, it has been shown that RSD guarantees a $\sqrt{n}$-approximation [42].

Finally, for the *indivisible goods* problem, the available items are m and agents may receive more than one item; in this case, it has been shown that RSD is the only mechanism that is strategyproof, neutral, and nonbossy [64].

More related to our problem, in [5] it has been shown that the serial dictatorship is strategyproof in general HGs where agents have no unacceptable coalitions. As we will see, RSD can also be applied to our setting while still guaranteeing strategyproofness. However, other mechanisms are able to achieve a better approximation ratio under the reasonable assumption that the number of activities is smaller than the number of agents.

5.2 Model and Preliminaries

We recall that an ASGASP instance $\mathcal{I} = (G, A, p)$ is given by a directed weighted graph $G = (V, E, w)$, where V is a set of n *agents*, A is a set of k *activities*, and p is a collection of agents' *preferences*. Namely, for each $i \in V$, the *preference function* $p_i : A \to \mathbb{R}$ expresses the value that agent i gives to each activity in A. Moreover, for each arc $(i, j) \in E$, weight $w_{i,j}$ represents the appreciation that agent i has for agent j. We implicitly assume $w_{i,j} = 0$ if $(i, j) \notin E$. For each $i \in V$, we denote by $W_i(G) = \sum_{(i,j)\in E} w_{i,j}$ the overall evaluation that i has for all the other agents and by $W(G) = \sum_{i\in V} W_i(G)$ the sum of all the arc weights of G.

Each agent is assigned to at most one of the k available activities; if she is not assigned to any activity $a \in A$, she is assumed to be assigned alone to a distinguished *void activity* $a_\emptyset$, that has value 0 for all the agents.

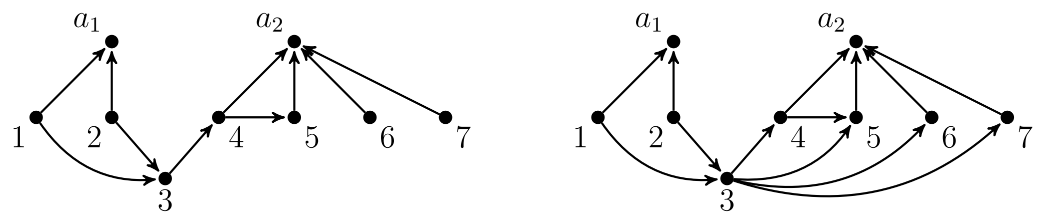
In this work, we always assume both the preferences and weights to be non-negative. In particular, we consider preferences (resp. weights) which either take values in $\mathbb{R}_{\geq 0}$ or are unitary (i.e. in $\{0, 1\}$). In order to evaluate the performance of an assignment, we use the classical definition of *utilitarian social welfare*, given by $\text{SW}(\mathbf{z}) = \sum_{i\in V} u_i(\mathbf{z})$. Furthermore, we often express $\text{SW}(\mathbf{z})$ as the sum $x_\mathbf{z} + y_\mathbf{z}$, where $x_\mathbf{z} = \sum_{i\in V} \delta_i(\mathbf{z})$ and $y_\mathbf{z} = \sum_{i\in V} p_i(z_i)$. Given an instance $\mathcal{I}$, the *social optimum*, denoted by $\mathbf{opt}(\mathcal{I})$, is the maximum social welfare achievable by an assignment, and we denote by $\mathbf{o}(\mathcal{I})$ such an assignment. When the instance is clear from the context, we refer to the social optimum and an optimal outcome simply as $\mathbf{opt}$ and $\mathbf{o}$, respectively.

Also in this chapter we focus on strategyproof mechanisms with good welfare guarantees with respect to the utilitarian social welfare and we measure the performance of a mechanism $\mathcal{M}$ through the approximation ratio $r^\mathcal{M}$. Moreover, a mechanism is said to be *bounded* if there exists a bounded function f such that $r^\mathcal{M} \leq f(n, k)$. In what follows, we often identify agents' declaration $\mathbf{d}$ with the instance $\mathcal{I}(\mathbf{d})$ built according to $\mathbf{d}$. Moreover, for the sake of simplicity, we usually refer to $\mathcal{I}(\mathbf{d})$ as $\mathcal{I}$.

So far, we assumed the activities to be *non-copyable*, i.e., for any assignment $\mathbf{z}$ and for each couple $i, j \in V$, $i \neq j$, participating to the same activity, contribute to each other's utility since they necessarily participate together to the activity. If activities are *copyable*, instead, they may be performed by different subgroups of agents simultaneously; thus, each agent utility will count only for her own group members. More formally, given an assignment $\mathbf{z}$, each agent $i \in N$ is assigned to $z_i \in \overline{A} \cup \{a_\emptyset\}$, where $\overline{A}$ is the set containing all the activities' copies, that is, for each $a \in A$ and for each $j \in [n]$, $a(j) \in \overline{A}$ and it is the j-th copy of a. Thus, an agent i evaluates a copy $a(j)$ as much as she evaluates a, i.e. $p_i(a(j)) = p_i(a)$; moreover, the existence of n copies of an activity guarantees the possibility of assigning all the agents to that activity alone. We observe that, in terms of social welfare, the copyability assumption does not lead to better solutions, being preferences and weights non-negative. However, it will be a useful feature to design efficient strategyproof mechanisms.

5.2.1 Social Optimum and Strategyproofness

We first observe that, since preferences and weights are non-negative, for $k = 1$ a trivial optimal strategyproof mechanism consists in assigning all agents to the single available activity. One might wonder whether a mechanism returning the optimum is also strategyproof when $k \geq 2$. In [17], it is shown that for $k \geq 3$ it is NP-hard to find the optimal solution even if preferences and weights are unitary. We next show that, besides the fact that finding the optimum is not computationally tractable, a mechanism which returns an optimal solution is anyway not strategyproof for every $k \geq 2$.



(a) Instance $\mathcal{I}_1$ (b) Instance $\mathcal{I}_2$

FIGURE 5.1: An instance showing that returning the optimum is not strategyproof. We represent unitary preferences and weights by means of a directed unweighted graph: a directed edge from an agent i to an agent j (resp. activity a) stands for $w_{ij} = 1$ (resp. $p_i(a) = 1$). All the other values in the instance are set to 0.

Proposition 5.1. *A mechanism which returns the optimal solution is not strategyproof even if preferences and weights are unitary and* $k = 2$.

Proof. Let us consider the instance $\mathcal{I}_1$ shown in Figure 5.1(a), with unitary preferences and weights, having set of agents $V = \{1, 2, 3, 4, 5, 6, 7\}$, set of activities $A = \{a_1, a_2\}$, preferences $p_1(a_1) = p_2(a_1) = p_4(a_2) = p_5(a_2) = p_6(a_2) = p_7(a_2) = 1$ and weights $w_{1,3} = w_{2,3} = w_{3,4} = w_{4,5} = 1$; all the other values are set to 0. We observe that the optimum for $\mathcal{I}_1$ is achieved only by the assignment where $1, 2, 3$ are assigned to a_1 and $4, 5, 6, 7$ to a_2.

Let us now consider the $\mathcal{I}_2$ shown in Figure 5.1(b), differing from $\mathcal{I}_1$ only for the declaration of agent 3, that is $w_{3,5} = w_{3,6} = w_{3,7} = 1$. In this case, in an optimal solution agent 3 must always be assigned to activity a_2 together with agent 4. Since for agent 3 in $\mathcal{I}_1$ it is preferable to get the optimal assignment for $\mathcal{I}_2$, rather than the one for $\mathcal{I}_1$, a mechanism that returns the optimal assignment is manipulable. Thus, the thesis follows. $\square$

We want to highlight that this proof does not rely on preferences manipulation, and thus Proposition 5.1 holds even if agents can misreport only weights.

By the above proposition, in this paper we will focus on mechanisms that return good approximate solutions.

5.3 Non-Negative Preferences

In this section, we consider instances in which agents' preferences can be positive reals. We distinguish the two subcases in which weights can be arbitrary positive numbers or only unitary values (0 or 1) in two separate subsections.

5.3.1 Non-Negative Weights

In what follows, we show that every deterministic strategyproof mechanism has an unbounded approximation ratio for any number of activities $k \geq 2$. Thus, we turn our attention to randomized mechanisms whose approximation ratio is instead bounded. To this aim, we first provide a necessary condition for achieving a bounded approximation ratio.

Lemma 5.2. *Given a deterministic strategyproof mechanism $\mathcal{M}$, for any instance $\mathcal{I} = (G, A, p)$ with non-negative preferences and non-negative weights, if $r^{\mathcal{M}}$ is bounded then $u_i(\mathbf{z}^{\mathcal{M}}(\mathcal{I})) \geq \frac{1}{n} \cdot (W_i(G) + \max_{a \in A} p_i(a))$ holds $\forall i \in V$.*

Proof. By contradiction, let us assume that there exists an instance $\mathcal{I}$ and an agent $i \in V$ such that, in the returned assignment $\mathbf{z}^{\mathcal{M}}$ by the mechanism $\mathcal{M}$, $u_i(\mathbf{z}^{\mathcal{M}}(\mathcal{I})) < \frac{1}{n} \cdot (W_i(G) + \max_{a \in A} p_i(a))$. Then, there must exist an activity $a \in A$ such that $a \neq z_i^{\mathcal{M}}$ and $u_i(\mathbf{z}^{\mathcal{M}}) < p_i(a)$ (case 1), or an agent $j \in V \setminus \{i\}$ such that $z_i^{\mathcal{M}} \neq z_j^{\mathcal{M}}$ and $u_i(\mathbf{z})^{\mathcal{M}} < w_{i,j}$ (case 2). Let us now consider a new instance $\mathcal{I}(M)$ where in case 1 (resp. case 2) agent i changes only the value she gives to the activity a (resp. to the agent j) and sets it to a suitably large positive number M. Since $\mathcal{M}$ is strategyproof, agent i will never be assigned to activity a (resp. to the same activity of j), otherwise agent i will achieve a better outcome in $\mathcal{I}$ by modifying it into $\mathcal{I}(M)$. Thus, as M increases, the ratio $r^{\mathcal{M}}(\mathcal{I}(M)) = \frac{\mathbf{opt}(\mathcal{I}(M))}{\mathrm{SW}(\mathcal{M}(\mathcal{I}(M)))} \geq \frac{M}{\mathrm{SW}(\mathcal{M}(\mathcal{I}(M)))}$ increases as well, contradicting the assumption that $r^{\mathcal{M}}$ is bounded. Indeed, given any function $f(n, k)$ for upper bounding $r^{\mathcal{M}}$, for suitably large M it is $r^{\mathcal{M}}(\mathcal{I}(M)) > f(n, k)$. □

Exploiting the above property, we now show that is impossible to provide a deterministic mechanism with bounded approximation ratio even if $k = 2$.

Theorem 5.3. *No deterministic strategyproof mechanism can be bounded when preferences and weights are non-negative, even if $k = 2$.*

Proof. Given any mechanism $\mathcal{M}$ with bounded approximation ratio and a suitably large real number $M \gg 1$, let us consider the instance $\mathcal{I}_1$ depicted in Fig. 5.2(a). Since $\mathcal{M}$ is bounded, according to Lemma 5.2, it assigns both the agents to the same activity; without loss of generality, we assume it to be activity a_2.

Let us now consider instance $\mathcal{I}_2$ depicted in Fig. 5.2(b), where $\alpha > 2$. Due to Lemma 5.2, agent 2 must be assigned by $\mathcal{M}$ to the same activity of 1. However, 1 and 2 cannot be assigned together to activity a_1. In fact, if not, agent 1 in instance $\mathcal{I}_1$ can achieve a better outcome changing her preference for activity a_1 and thus contradicting the strategyproofness of $\mathcal{M}$. Therefore, in order to be strategyproof, $\mathcal{M}$ in $\mathcal{I}_2$ must assign both the agents to activity a_2, violating Lemma 5.2 for agent 1.

In conclusion, a deterministic strategyproof mechanism achieving a bounded approximation ratio cannot exist. $\square$

We want to highlight that Theorem 5.3 holds even if activities are copyable. Since no deterministic and strategyproof mechanism can be bounded, we now turn our attention to randomized mechanisms. We first observe that in this case a bounded approximation ratio in expectation can be achieved by means of the following suitable version of the RSD mechanism discussed in the introduction.

Mechanism $\mathcal{M}_7$. [RSD] It scans the agents according to a predefined random order. At each step, the current agent, if not yet selected by another agent in a previous step, can select an activity as well as all the unassigned agents she likes and wants to be with her.

Theorem 5.4. *The RSD mechanism ($\mathcal{M}_7$) is strategyproof. Moreover, $r^{\mathcal{M}_7} = n$.*

Proof. (Strategyproofness) The strategyproofness simply follows by observing that the selection of the dictator agent does not depend on the declarations, and once an agent is selected, the mechanism maximizes her personal utility according to her declaration. Hence, no agent has any convenience to lie.

(Approximation) Given an instance $\mathcal{I}$, let $\mathbf{z}$ and $\mathbf{o}$ be the assignments provided by RSD and an optimal solution for the instance $\mathcal{I}$ respectively.
We now provide an upper bound on the approximation ratio based on this simple argument: the probability for an agent of being the first dictator is $\frac{1}{n}$. Thus, for each agent $i \in V$, with probability at least $\frac{1}{n}$ she is assigned to her preferred activity and all the agents she likes. This implies,

$$\mathbb{E}\left[u_i(\mathbf{z})\right] \geq \frac{1}{n} \cdot \left(\max_{a \in A} p_i(a) + W_i(G)\right) \geq \frac{1}{n} \cdot u_i(\mathbf{o}) \ .$$

Summing up among all the agents we get $\mathbb{E}\left[\mathrm{SW}(\mathbf{z})\right] \geq \frac{1}{n} \cdot \mathrm{SW}(\mathbf{o})$, and then the thesis follows.

We now show that this bound is also tight. Let us consider the following instance where $V = \{1, \ldots, n\}$ is the set of agents and $A = \{a_1, a_2\}$ is the set of activities; moreover, $p_1(a_1) = M$, for a suitable large M, and $p_i(a_2) = w_{i,1} = 1$ for each $i = 2, \dots, n$; all the other values are set to 0. In such an instance $\mathbf{opt} = M + n - 1$, while the expected social welfare is at most $\frac{1}{n} \cdot M + 2(n-1)$. As $M \to \infty$ we get that the approximation ratio is at least n. $\square$

We now present a simple randomized mechanism with a better ratio, and then provide a lower bound holding for any randomized strategyproof mechanism.

Mechanism $\mathcal{M}_8$. Select uniformly at random one activity and then assign all the agents to it.

Theorem 5.5. *$\mathcal{M}_8$ is strategyproof and $r^{\mathcal{M}_8} = k$.*

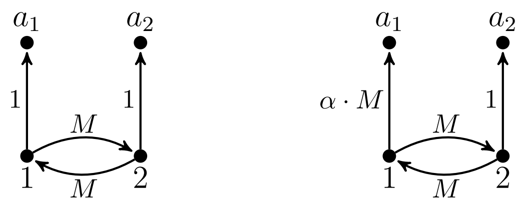
Proof. (Strategyproofness) Since the assignment does not depend on the agents' declarations, the strategyproofness of $\mathcal{M}_8$ trivially follows.

(Approximation) Given an instance $\mathcal{I}$, let $\mathbf{z}$ and $\mathbf{o}$ be the assignments provided by $\mathcal{M}_8$ and an optimal solution for the instance $\mathcal{I}$ respectively.
According to how the mechanism works, $\mathbb{E}\left[x_{\mathbf{z}}\right] = W(G) \geq x_{\mathbf{o}}$ and

$$\mathbb{E}\left[y_{\mathbf{z}}\right] = \frac{1}{k} \sum_{i \in V} \sum_{a \in A} p_i(a) \geq \frac{1}{k} \sum_{i \in V} p_i(o_i) = \frac{1}{k} \cdot y_{\mathbf{o}} \quad .$$

Since $x_{\mathbf{o}} \leq \mathbb{E}\left[x_{\mathbf{z}}\right] \leq k \cdot \mathbb{E}\left[x_{\mathbf{z}}\right]$ and $y_{\mathbf{o}} \leq k \cdot \mathbb{E}\left[y_{\mathbf{z}}\right]$, $r^{\mathcal{M}} = \frac{x_{\mathbf{o}} + y_{\mathbf{o}}}{\mathbb{E}[x_{\mathbf{z}} + y_{\mathbf{z}}]} \leq k$.

It remains to show that this upper bound is tight. Let us consider an instance $\hat{\mathcal{I}} = (G, A, p)$ with $A = \{a_1, \dots, a_k\}$, $V = \{1, \dots, k\}$, $W(G) = 0$, $p_i(a_j) = 1$ if and only if $i = j$, and $p_i(a_j) = 0$ otherwise. In this instance $\mathbf{opt} = k$, while $\mathbb{E}\left[SW(\mathcal{M}_8(\hat{\mathcal{I}}))\right] = 1$. Thus the claim follows. $\square$



(a) Instance $\mathcal{I}_1$ (b) Instance $\mathcal{I}_2$

FIGURE 5.2: Lower bound for both deterministic and randomized mechanisms.

The result of Theorem 5.5 can be improved if activities are copyable by means of the following mechanism.

Mechanism $\mathcal{M}_9$. Select $\mathbf{z}_1$ with probability $\frac{k}{2k-1}$ and $\mathbf{z}_2$ otherwise, where $\mathbf{z}_1$ is the outcome where all the agents are assigned to the same activity selected uniformly at random, and $\mathbf{z}_2$ is the outcome where each agent is assigned alone to one of the activities she prefers most.

Theorem 5.6. *If activities are copyable $\mathcal{M}_9$ is strategyproof and $r^{\mathcal{M}_9} = 2 - \frac{1}{k}$.*

Proof. (Strategyproofness) We start by observing that $\mathcal{M}_9$ outputs either $\mathbf{z}_1$ or $\mathbf{z}_2$ independently from the agents' declarations. Moreover, for both these assignments, no input manipulation by an agent can improve her gain. Thus, $\mathcal{M}_9$ is strategyproof.
(Approximation) Given an instance $\mathcal{I}$, let $\mathbf{z}$ be the outcome returned by $\mathcal{M}_9$ and $\mathbf{o}$ be the optimal solution. Then, $\mathbb{E}\left[\mathrm{SW}(\mathbf{z})\right] = \frac{k}{2k-1} \cdot \mathbb{E}\left[\mathrm{SW}(\mathbf{z}_1)\right] + \frac{k-1}{2k-1} \cdot \mathrm{SW}(\mathbf{z}_2)$. Moreover, $x_{\mathbf{o}} + \frac{1}{k} \cdot y_{\mathbf{o}} \leq \mathbb{E}\left[\mathrm{SW}(\mathbf{z}_1)\right]$ and $y_{\mathbf{o}} \leq \mathrm{SW}(\mathbf{z}_2)$. In conclusion,

$$\mathbf{opt} \leq \mathbb{E}\left[\mathrm{SW}(\mathbf{z}_1)\right] + \frac{k-1}{k} \cdot \mathrm{SW}(\mathbf{z}_2) = \left(2 - \frac{1}{k}\right) \cdot \mathbb{E}\left[\mathrm{SW}(\mathbf{z})\right].$$

Such a bound is also tight, it can be easily checked considering the instance $\hat{\mathcal{I}}$ in the proof of Theorem 5.5. □

The randomized mechanisms just described provide an upper bound of k and $2 - \frac{1}{k}$ in the non-copyabe and copyable case, respectively. We now show a lower bound that holds for any randomized strategyproof mechanism, both in the copyable and the non-copyable case respectively.

Theorem 5.7. *For any given randomized strategyproof mechanism $\mathcal{M}$, $r^{\mathcal{M}} > 2 - \frac{2}{k+1}$ both in the copyable and the non-copyable case.*

Proof. Given $V = \{1, \dots, k, x\}$ and $A = \{a_1, \dots, a_k\}$, let us consider the family of instances $\{\mathcal{I}_\alpha\}_{\alpha \in \mathbb{R}^+}$, where for any $i, j \in \{1, \dots, k\}$, $p_i(a_j) = 1$ if and only if $i = j$ and $w_{x,i} = \alpha$ for every $i \in \{1, \dots, k\}$. All the other values are set to 0.

Let ε be any strictly positive real number such that $\varepsilon \ll \frac{1}{k}$, and let us consider the corresponding instance $\mathcal{I}_\varepsilon$. We observe that the optimal solution for $\mathcal{I}_\varepsilon$

achieves a social welfare of $\mathbf{opt}(\mathcal{I}_\varepsilon) = k+\varepsilon$, and it is obtained by assigning agent i to activity a_i and agent x to any activity.

Given any randomized strategyproof mechanism $\mathcal{M}$, we denote by $c(k) = r^{\mathcal{M}}$ its approximation ratio. Let q_m be the probability that agent x is assigned to an activity together with $m \in \{0, 1, \dots, k\}$ other agents. Then, the expected utility of x in instance $\mathcal{I}_\varepsilon$ is given by the expected number of agents in her activity times ε, that is $\mathbb{E}\left[u_x(\mathcal{M}(\mathcal{I}_\varepsilon))\right] = \varepsilon \cdot \sum_{m=0}^{k} m \cdot p_m$. Moreover, it is possible to give a trivial upper bound to the expected social welfare assuming that: (a) if an agent i is not with x, then it is assigned to the activity she evaluates 1, and (b) one agent assigned together with x evaluates 1 the activity they belong to. Thus, taking into account assumptions (a) and (b), we get the following expression:

$$\mathbb{E}\left[\mathrm{SW}(\mathcal{M}(\mathcal{I}_\varepsilon))\right] \leq \sum_{m=0}^{k} (\varepsilon \cdot m + 1 + (k-m)) \cdot p_m = k + 1 - (1-\varepsilon) \cdot \sum_{m=0}^{k} m \cdot p_m \, .$$

We now use the strategyproofness assumption to give a lower bound on the expected number of agents assigned to the same activity of x. To this aim, we consider instance $\mathcal{I}_M$ and define p'_m as the probability that x is assigned to an activity together with other $m \in \{0, 1, \dots, k\}$ agents. In this case we can express the expected utility of agent x as $\mathbb{E}\left[u_x(\mathcal{M}(\mathcal{I}_M))\right] = M \cdot \sum_{m=0}^{k} m \cdot p'_m$.

Since $\mathcal{M}$ is strategyproof, $\sum_{m=0}^{k} m \cdot p_m = \sum_{m=0}^{k} m \cdot p'_m$ must hold. Indeed, if $\sum_{m=0}^{k} m \cdot p_m < \sum_{m=0}^{k} m \cdot p'_m$, for agent x in instance $\mathcal{I}_\varepsilon$ it would be possible to increase her expected utility by modifying $\mathcal{I}_\varepsilon$ into $\mathcal{I}_M$, thus contradicting the strategyproofness of $\mathcal{M}$. Similar arguments can be applied if we assume $\sum_{m=0}^{k} m \cdot p_m > \sum_{m=0}^{k} m \cdot p'_m$. Thus, bounding $\sum_{m=0}^{k} m \cdot p_m$ is equivalent to bounding $\sum_{m=0}^{k} m \cdot p'_m$, as they are the same quantity.

Let us assume $M \gg 1$. In this case $\mathbf{opt}(\mathcal{I}_M) = M \cdot k + 1$, as it can be checked by assigning all the agents to the same activity. Moreover, $\mathbb{E}\left[\mathrm{SW}(\mathcal{M}(\mathcal{I}_M))\right] \leq M \cdot \sum_{m=0}^{k} m \cdot p'_m + k$. Since for any large enough $M \gg 1$ it is

$$c(k) = r^{\mathcal{M}} \geq \frac{kM+1}{M \cdot \sum_{m=0}^{k} m \cdot p'_m + k} \overset{M \to \infty}{\longrightarrow} \frac{k}{\sum_{m=0}^{k} m \cdot p'_m} \, ,$$

we have that $\sum_{m=0}^{k} m \cdot p'_m \geq \frac{k}{c(k)}$. Applying the just provided lower bound to the expected social welfare for instance $\mathcal{I}_\varepsilon$, we get

$$\mathbb{E}\left[\mathrm{SW}(\mathcal{M}(\mathcal{I}_\varepsilon))\right] \leq k+1-(1-\varepsilon)\cdot\sum_{m=0}^{k} m \cdot p_m \leq k+1-(1-\varepsilon)\cdot\frac{k}{c(k)} \ .$$

Thus, according to the definition of approximation ratio, $\frac{k+\varepsilon}{k+1-(1-\varepsilon)\cdot\frac{k}{c(k)}} \leq c(k)$. Solving the inequality for $\varepsilon \to 0$, we finally obtain $c(k) \geq 2 \cdot \frac{k}{k+1} = 2 - \frac{2}{k+1}$. □

While this result shows that Mechanism $\mathcal{M}_9$ is tight for the copyable case, it remains an interesting open question to close the gap between Mechanism $\mathcal{M}_8$ and the just provided lower bound. We want to highlight that the proof of Theorem 5.7 does not rely on the non-copyability assumption; thus, in order to close such a gap, it would be necessary to either design a different mechanism achieving a much better approximation ratio, or to provide a new lower bound proof that strongly relies on non-copyability. Finally, let us observe that Theorem 5.7 holds also for unitary preferences and public preferences, being all the agents preferences set to 1 and only weights manipulation considered.

5.3.2 Unitary Weights

Let us now consider the hypothesis of unitary weights, i.e., when $w_{i,j} \in \{0,1\}$ for every $i,j \in V$. Unfortunately, the same negative result of Theorem 5.3 holds also in this case. In fact, Theorem 5.9 shows that no bounded deterministic mechanism can be found. To this aim, let us first give the following necessary condition whose proof is similar to the one of Lemma 5.2.

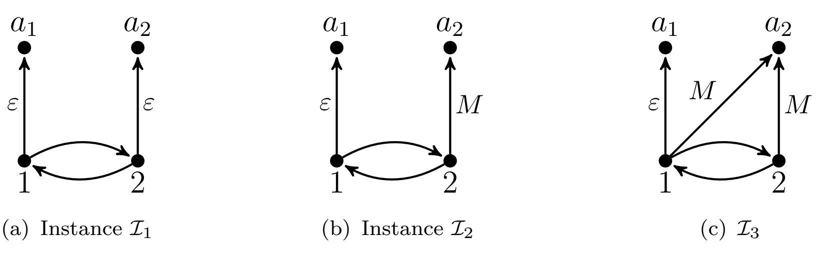
Lemma 5.8. *Given a deterministic strategyproof mechanism $\mathcal{M}$, for any instance $\mathcal{I} = (G, A, p)$ with non-negative preferences and unitary weights, if $r^{\mathcal{M}}$ is bounded, then $u_i(\mathbf{z}^{\mathcal{M}}(\mathcal{I})) \geq \max_{a\in A} p_i(a)$ holds $\forall i \in V$.*

Proof. By contradiction, let us assume that there exists an instance $\mathcal{I}$ and an agent $i \in V$ such that, in the assignment $\mathbf{z}^{\mathcal{M}}$ returned by mechanism $\mathcal{M}$, $u_i(\mathbf{z}^{\mathcal{M}}(\mathcal{I})) < \max_{a\in A} p_i(a)$. Then, there must exist an activity $a \in A$ such that $a \neq z_i^{\mathcal{M}}$ and $u_i(\mathbf{z}^{\mathcal{M}}) < p_i(a)$. Let us now consider a new instance $\mathcal{I}(M)$ where agent i changes only the value she gives to activity a and sets it to a suitably large positive number M. Since $\mathcal{M}$ is strategyproof, agent i will never be assigned to activity a, otherwise she would achieve a better outcome in $\mathcal{I}$ by modifying it into $\mathcal{I}(M)$. Thus, as M increases, the ratio $r^{\mathcal{M}}(\mathcal{I}(M)) = \frac{\mathbf{opt}(\mathcal{I}(M))}{\mathrm{SW}(\mathcal{M}(\mathcal{I}(M)))} \geq \frac{M}{\mathrm{SW}(\mathcal{M}(\mathcal{I}(M)))}$ increases as well, contradicting the assumption that $r^{\mathcal{M}}$ is bounded. □

Theorem 5.9. *No deterministic mechanism can be bounded for non-negative preferences and unitary weights, even if $k = 2$.*

Proof. Let $\mathcal{M}$ be a bounded and deterministic strategyproof mechanism, and $\mathcal{I}_1$ be the instance depicted in Fig. 5.3(a). According to Lemma 5.8, there are only two possible outcomes: (a) both agents are assigned to the same activity (either a_1 or a_2); (b) 1 is assigned to a_1 and 2 is assigned to a_2. We first observe that, if $\mathcal{M}$ returns outcome (b), then it cannot have a bounded approximation ratio, since such an outcome has social welfare 2ε and $\mathbf{opt} = 1 + \varepsilon$. Thus, by the assumption that $\mathcal{M}$ is bounded, it has to return outcome (a).

Let us now focus on outcome (a). In this case, we can assume that both agent 1 and 2 are assigned to activity a_1 (a symmetric argument applies to a_2). In this outcome, it would be preferable for agent 2 to be assigned to activity a_2 together with 1. Consider then the instance $\mathcal{I}_2$ depicted in Fig. 5.3(b). Here, in order for the mechanism to be bounded, agent 2 must be assigned to a_2. Moreover, agent 1 must be assigned to a_1, otherwise agent 2 could manipulate $\mathcal{M}$ by transforming $\mathcal{I}_1$ into $\mathcal{I}_2$. However, if we consider the instance $\mathcal{I}_3$ in Fig. 5.3(c), since $\mathcal{M}$ is bounded and strategyproof, by Lemma 5.8 the only possible outcome is the one in which both the agents are assigned to a_2. This implies that $\mathcal{M}$ is manipulable, since agent 1 can achieve a better outcome by modifying $\mathcal{I}_2$ into $\mathcal{I}_3$. Thus, $\mathcal{M}$ cannot be both strategyproof and bounded. □



(a) Instance $\mathcal{I}_1$ (b) Instance $\mathcal{I}_2$ (c) $\mathcal{I}_3$

FIGURE 5.3: Lower bound under non-negative preferences and unitary weights.

Notice that, while on the one hand the lower bounds showed in this subsection for unitary weights seem to generalize the ones for non-negative weights, on the other hand they do not hold for the copyable case. Hence, finding a bounded deterministic mechanism for unitary weights and copyable activities still remains an interesting open question. Furthermore, the lower bound provided in Theorem 5.7 do not apply in this case, thus, let us provide a specific lower bound for a randomized mechanism under non-negative preferences and unitary weights.

Theorem 5.10. *For any randomized and strategyproof mechanism $\mathcal{M}$, $r^{\mathcal{M}} \geq \frac{4}{3}$.*

Proof. Let us consider the instance $\mathcal{I}_1$ where $V = \{1, 2\}$, $A = \{a_1, a_2\}$, $p_1(a_1) = p_2(a_2) = \varepsilon$, for some real number $\varepsilon \ll 1$, and $w_{1,2} = w_{2,1} = 1$; all the other values are set to 0. Let q_1 and q_2 be the probabilities that the agents are assigned together to activity a_1 and to activity a_2, respectively, and let $q = q_1 + q_2 \leq 1$ be the overall probability they are assigned together to the same activity by $\mathcal{M}$.

Let us assume without loss of generality that $q_1 \leq q_2$, so that $q_1 \leq \frac{1}{2}$, and consider instance $\mathcal{I}_2$ differing from $\mathcal{I}_1$ only for $p_1(a_1) = 2$. Let s_1 and s_2 be defined for $\mathcal{I}_2$ similarly to q_1 and q_2 in $\mathcal{I}_1$. By the strategyproofness of $\mathcal{M}$, it must be $q_1 \geq s_1$, otherwise agent 1 would gain by lying in $\mathcal{I}_1$. Thus, $s_1 \leq \frac{1}{2}$.

The social optimum in $\mathcal{I}_2$ is $\mathbf{opt}(\mathcal{I}_2) = 4$, while the expected social welfare achieved by the mechanism is upper bounded as

$$\mathbb{E}(SW(\mathcal{M}(\mathcal{I}_2))) \leq 4 \cdot s_1 + (1 - s_1)(2 + \varepsilon) \leq 3 + \frac{\varepsilon}{2} \ .$$

Thus, $r^{\mathcal{M}} \geq \frac{4}{3+\frac{\varepsilon}{2}}$, and for $\varepsilon \to 0$ the claim holds. □

5.4 Unitary Preferences

Given the previous negative results for arbitrary non-negative preferences, we now focus on unitary ones. Again, we distinguish between the cases of non-negative and unitary weights, and we show that in both cases it is possible to provide strategyproof mechanisms with bounded approximation ratio.

5.4.1 Non-Negative Weights

Let us define by $a^* \in A = \{a_1, \ldots, a_k\}$ as the activity achieving the highest total preference of the agents, i.e., $a^* = \arg\max_{a \in A} \sum_{i \in V} p_i(a)$, and in case of ties the one coming first in the input order.

Mechanism $\mathcal{M}_{10}$. Assign all the agents to activity a^*.

While it is possible to check that $\mathcal{M}_{10}$ is not strategyproof under non-negative preferences, fortunately this is not the case for unitary ones.

Theorem 5.11. *$\mathcal{M}_{10}$ is strategyproof under unitary preferences and non-negative weights. Moreover, $r^{\mathcal{M}_{10}} = k$.*

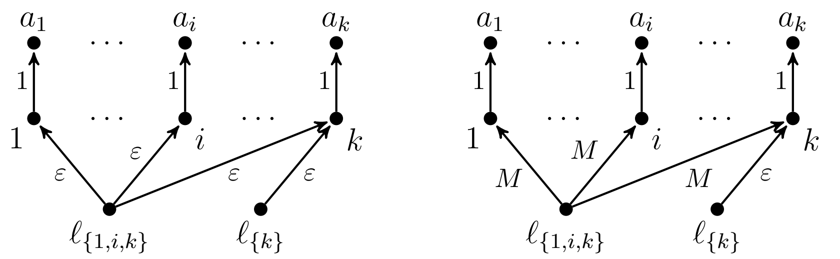
Proof. (Strategyproofness) An agent can be interested in manipulating the outcome only if she evaluates 0 the activity she is assigned to. However, this cannot be done in such a way she will be assigned to an activity she evaluates 1. In fact, not declaring her true preferences, she cannot improve the number of agents evaluating 1 one of the activities she likes.
(Approximation) Let y be $\max_{a \in A} \sum_{i \in V} p_i(a)$. Since $x_{\mathbf{o}} \leq W(G)$ and $y_{\mathbf{o}} \leq k \cdot y$, $r^{\mathcal{M}_{10}} \leq k$ directly follows. Moreover, the bound is tight, as it can be checked by considering the instance $\hat{\mathcal{I}}$ described in the proof of Theorem 5.5. □

We now provide a lower bound on the approximation ratio of any deterministic strategyproof mechanism.

Theorem 5.12. *Any bounded deterministic strategyproof mechanism $\mathcal{M}$ for unitary preferences and non-negative weights has approximation ratio $r^{\mathcal{M}} = \Omega\left(\sqrt{k}\right)$.*

Proof. Given any $\varepsilon \leq \frac{1}{k \cdot 2^k}$, let us consider consider the instance $\mathcal{I}_\varepsilon = (G, A, p)$ constructed as follows. The set of the activities is $A = \{a_1, \ldots, a_k\}$, and $\forall\, S \subseteq \{1, \ldots, k\} = [k]$, $S \neq \emptyset$, there is a corresponding agent ℓ_S. The set of all the agents is $V = [k] \cup \{\ell_S | S \subseteq [k], S \neq \emptyset\}$. Moreover, $p_i(a_i) = 1\ \forall\ i \in [k]$, and $w_{\ell_S, i} = \varepsilon$ for each $S \subseteq [k]$ and $i \in S$; all the other preferences and weights are set to 0. The just described instance is depicted in Figure 5.4(a) .

Let $\mathcal{M}(\mathcal{I}_\varepsilon)$ be the assignment returned by mechanism $\mathcal{M}$ for instance $\mathcal{I}_\varepsilon$, and let $\{i_1, \ldots, i_m\}$ with $m \in [k]$ be the set of the agents assigned to the activity



(a) Instance $\mathcal{I}_\varepsilon$ (b) Instance $\mathcal{I}_S(M)$ with $S = \{1, i, k\}$

FIGURE 5.4: Deterministic lower bound for unitary preferences and non-negative weights.

they evaluate 1 in $\mathcal{M}(\mathcal{I}_\varepsilon)$. Then, $\mathrm{SW}(\mathcal{M}(\mathcal{I}_\varepsilon)) \leq m + c \cdot \varepsilon$ for some $c \geq 0$, while $\mathbf{opt}(\mathcal{I}_\varepsilon) \geq k$, thus implying $r^{\mathcal{M}} \geq \frac{k}{m+c\cdot\varepsilon} \xrightarrow{\varepsilon \to 0} \frac{k}{m}$.

Let us now consider $S = \{i_1, \ldots, i_m\}$ and the corresponding agent ℓ_S, that can be necessarily assigned to only one activity and thus with only one of the agents in S. Such an agent may try to manipulate the outcome by changing the instance $\mathcal{I}_\varepsilon$ into $\mathcal{I}_S(M)$, depicted in Figure 5.4(b), where $w_{\ell_S,i} = M$ for each $i \in S$ and all other preferences and weights are the same of $\mathcal{I}_\varepsilon$, with M suitably large. However, since $\mathcal{M}$ is strategyproof, this cannot lead to a better outcome for ℓ_S. Therefore, also in this instance ℓ_S cannot be assigned to an activity together with more than one agent in S. This means that $\mathrm{SW}(\mathcal{M}(\mathcal{I}_S(M))) \leq M + k + 1$, while $\mathbf{opt}(\mathcal{I}_S(M)) \geq m \cdot M$. Therefore, $r \geq \frac{m\cdot M}{M+k+1} \xrightarrow{M\to\infty} m$.

In conclusion, both $r \geq \frac{k}{m}$ and $r \geq m$ must hold. Minimizing the maximum of the two terms with respect to m, we finally get $r \geq \sqrt{k}$. □

5.4.2 Unitary Weights

We now consider instances with unitary preferences and weights. In this setting, $\mathcal{M}_{10}$ is still strategyproof and achieves an approximation ratio of k. In what follows we show that under the unitary assumption a better mechanism exists.

We start by observing that when weights are unitary the graph G of an instance $\mathcal{I} = (G, A, p)$ can be seen as a directed unweighted graph, and thus we can identify the set $\mathcal{C}_G$ of its maximal weakly connected components.

Mechanism $\mathcal{M}_{11}$. Fix a distinguished activity $a \in A$; for each $C \in \mathcal{C}_G$, if there exists $\hat{a} \in A$ such that $p_i(\hat{a}) = 1$ $\forall i \in C$, it assigns all the agents in C to $\hat{a}$, while it assigns them to a otherwise.

Theorem 5.13. *$\mathcal{M}_{11}$ is strategyproof under unitary preferences and weights, and $r^{\mathcal{M}_{11}} = 2$.*

Proof. (Strategyproofness) An agent i that truthfully reports her values is assigned with all the agents for which she has weight 1, thus, is not convenient for her misreporting her preferences among the other agents. If she lies to get an activity she evaluates 1, she necessarily loses at least one of her neighbors, thus falling into an assignment not providing her a better utility.

(Approximation) We evaluate the overall utility reached by all the agents in a same component $C \in \mathcal{C}_G$ separately. If there exists $\hat{a} \in A$ such that $p_i(\hat{a}) = 1$ $\forall i \in C$, then the overall utility of all the agents agents in C is the maximum possible one. If agents in C do not have a common preferred activity, their overall utility is at least the total weight $W(C) \geq |C| - 1$ of the edges induced by C in G, while the overall utility they can achieve in any solution is at most $W(C) + |C| - 1 \leq 2W(C)$. In fact, either they are assigned to the same activity, for which at least one of them has preference 0, or they are split among different activities, loosing at least 1 in terms of global weight. $\square$

5.5 Public Preferences

In this section, we show how to extend some of the previous results when preferences and weights are non-negative, but agents have limited power in manipulating. In particular, they cannot strategically misreport all the values, as we assume that preferences are public. Namely, the specific rewards that agents get for the single activities are a priori known. We start by observing that the lower bound provided in Theorem 5.12 still holds in this setting. Indeed, the used arguments in the proof rely only on the manipulations of the weights and not of the preferences, showing the following.

Proposition 5.14. *Any bounded and deterministic strategyproof mechanism $\mathcal{M}$ for non-negative public preferences and non-negative weights has $r^{\mathcal{M}} = \Omega\left(\sqrt{k}\right)$.*

Similarly, Theorem 5.11 holds also in this case.

Proposition 5.15. *$\mathcal{M}_{10}$ is strategyproof when both preferences and weights are non-negative and preferences are public. Moreover, $r^{\mathcal{M}_{10}} = k$.*

The strategyproofness follows by observing that the only way to modify the outcome is to change the preferences among the activities, that is not possible. The approximation ratio can be proven as in the proof of Theorem 5.11.

We now show a randomized mechanism similar to $\mathcal{M}_9$ for the copyable case.

Mechanism $\mathcal{M}_{12}$. Select $\mathbf{z}_1$ with probability $\frac{k}{2k-1}$, $\mathbf{z}_2$ otherwise, where in $\mathbf{z}_1$ all the agents are assigned to the same activity selected uniformly at random, and in $\mathbf{z}_2$ each agent is assigned to one of the activities she prefers most.

We observe that $\mathcal{M}_{12}$ differs from $\mathcal{M}_9$ only for the assignment $\mathbf{z}_2$. In fact, for such an outcome, $\mathcal{M}_9$ assigns agents to their preferred activities alone using the copying feature, while $\mathcal{M}_{12}$ together with the other agents. Fortunately, this difference does not affect the strategyproofness of $\mathcal{M}_{12}$, since preferences are public, and thus not manipulable. Moreover, its strategyproofness and approximation ratio can be proven as in Theorem 5.6

Proposition 5.16. *$\mathcal{M}_{12}$ is strategyproof under public preferences, and $r^{\mathcal{M}_{12}} = 2 - \frac{1}{k}$.*

5.6 Open Problems

Besides reducing the gaps between the lower and upper bounds, an interesting open question remains the existence of a bounded deterministic mechanism for copyable activities under non-negative preferences and unitary weights. It would be also worth considering other suitable subclasses of ASGASP and GASP with cardinal utilities. For instance, what about the case in which agents are friends or enemies (as a natural extension of the problem considered in Chapter 4) or utilities are defined similarly as in FHGs?

Finally, Pareto optimality is an interesting feature for strategyproof mechanisms that has not been considered in this work and its investigation is a promising future research direction.

Chapter 6

Conclusions

In this thesis, we considered different aspects of HGs and GASP. We mainly focused on the following two research directions: 1) the study of the existence and efficiency of equilibria and 2) the elicitation of the agents' preferences.

For what concerns the first research direction, in Chapter 3 we introduced and studied a new hedonic model, the so-called DHGs, that is able to capture a large class of settings in which agents' preferences depend on their distances from the other group members. Indeed, it is assumed the existence of an underlying graph structure representing the connections (friendship or anti-sympathy) between agents; thus, agents evaluate their own group members according to the induced distances in their coalitions. To this aim, the model introduces a scoring vector which stores in its components the agents' contribution at every fixed distance. In this setting, we focused on the existence and, in particular, on the efficiency of Nash stable outcomes. Not surprisingly, we discovered that, while for a large number of scoring vectors Nash stable outcomes always exist, it is NP-hard to find a best Nash stable outcome. Due to these hardness results, we then focused on the efficiency of Nash stable outcomes. We analyzed both the PoA and PoS for different types of game instances, trying to figure out which scoring vectors and/or graph structures induce good Nash stable solutions. We want to highlight that DHGs establish a connection between the well-known classes of unweighted FHGs and SDGs. In particular, this connection is maintained also for the computational hardness of best Nash stable solutions. Indeed, the result of Theorem 3.6 shows that is NP-hard to find such a solution for a large number of scoring vectors lying between the ones of unweighted FHGs and SDGs, i.e., normalized and decreasing scoring vectors whose second component is smaller than $\frac{1}{2}$.

Some questions remain open: first of all determining the existence of Nash stable outcomes for general vectors, whose components may be any real numbers; moreover, understanding which kind of coalition structures produce good Nash stable outcomes when star partitions are non guaranteed to be stable. Finally, to our opinion it would be interesting to find new worth investigating classes of games that may be represented by increasing vectors. While we mostly focused on this case because of its theoretical interest, we would like to determine specific realistic games and scenarios fulfilling this assumption.

In Chapters 4 and 5 we focused on the second research direction we mentioned, in particular, we studied strategyproof mechanisms for Friends and Enemies games and ASGASP. These two classes of games are somehow related to each other, being a sub- and a super-class of ASHGs, respectively. In this sense, we enriched the results obtained in [43], this work representing a starting point for both the aforementioned chapters of this thesis.

Friends and Enemies Games are related to the *duplex valuations* case considered in [43] by Flammini et al., where the values given to the other agents can only belong to the set $\{-1, 0, 1\}$. Indeed, in Friends and Enemies Games, enemies may have a lower/higher impact than friends in a coalition, that is, the values given to the others can be either $\left\{1, -\frac{1}{n}\right\}$ or $\left\{\frac{1}{n}, -1\right\}$ under FA and EA, respectively. For duplex valuations Flammini et al. provided a strategyproof mechanism achieving an n^2-approximation of the optimal social welfare and proved that an approximation factor better than n is not possible under the strategyproofness constraint. In contrast, we have shown that an n-approximation can be attained for FA preferences profiles even in presence of neutrals, that is, when it is also possible to evaluate 0 the other agents. For EA preferences profiles, we provided a $\Theta(n)$-approximation solution in a strategyproof manner. However, such a mechanism no longer guarantees bounded approximation when neutrals appear. In this case, the mechanism provided for duplex valuations still works also for EA in presence of neutrals, maintaining the same welfare approximation ratio of n^2.

Regarding the ASGASP, it is a natural generalization of the ASHGs. For this latter, Flammini et al. showed that, while on the one hand for general valuations, possibly negative, no deterministic strategyproof mechanism can achieve a bounded approximation ratio, on the other hand, when these values are non-negative, returning the optimum is always strategyproof. Because of this, we focused on ASGASP under the assumption of non-negative values for both the activities and the other agents, hoping for good results also in this setting. Unfortunately, this was not the case, as we discovered that, even if the available activities are only two, no deterministic mechanism can achieve a bounded approximation ratio. Thus, we exploited different scenarios, other than non-negative values on both agents and activities, with the aim of understanding how to circumvent this impossibility result. We first relied on randomization and showed that bounded mechanisms in this case exist. For deterministic mechanisms, instead, we identified the presence of multiple activities as the main cause of an unbounded approximation ratio. This motivated us to investigate deterministic mechanisms in the following two scenarios: the value given to the activities are boolean, or they are publicly known. In such cases, we are able to provide strategyproof mechanisms with bounded approximation ratio.

What was not considered in our work, and might be a future research direction, is to introduce payments to achieve strategyproofness. Since an agent may manipulate a good approximating mechanism by modifying the values she gives to the activities, we could consider the reasonable assumption that agents should pay for participating to an activity. However, this idea needs further development.

We believe that the strategyproof aspect in CFGs, and in particular in HGs and GASP, merits to be further investigated. In our work we also provided lower bounds for the approximation ratio of strategyproof mechanisms for both Friends and Enemies Games and ASGASP. However, many gaps between the lower and upper bounds are left open in our work as well as in [43]. It is still not clear whether better mechanisms can be provided or if more involved techniques have to be considered to increase the theoretical lower bounds. In any case, these issues should be further investigated.

Bibliography

[1] Adamczyk, M., Sankowski, P., and Zhang, Q. Efficiency of truthful and symmetric mechanisms in one-sided matching. In *Proceedings of the 7th International Symposium on Algorithmic Game Theory (SAGT)*.

[2] Anshelevich, E., Dasgupta, A., Kleinberg, J. M., Tardos, É., Wexler, T., and Roughgarden, T. (2004). The price of stability for network design with fair cost allocation. In *Proceedings of the 45th Symposium on Foundations of Computer Science (FOCS)*, pages 295–304.

[3] Archer, A. and Tardos, É. (2001). Truthful mechanisms for one-parameter agents. In *Proceedings of the 42th Symposium on Foundations of Computer Science (FOCS)*, pages 482–491.

[4] Aziz, H., Brandl, F., Brandt, F., Harrenstein, P., Olsen, M., and Peters, D. (2019). Fractional hedonic games. *ACM Transactions on Economics and Computation (TEAC)*, 7(2):1–29.

[5] Aziz, H., Brandt, F., and Harrenstein, P. (2013). Pareto optimality in coalition formation. *Games and Economic Behavior*, 82:562–581.

[6] Aziz, H., Brandt, F., and Seedig, H. G. (2011a). Optimal partitions in additively separable hedonic games. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence(IJCAI)*, pages 43–48.

[7] Aziz, H., Brandt, F., and Seedig, H. G. (2011b). Stable partitions in additively separable hedonic games. In *Proceedings of the 10th International Conference of Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 183–190.

[8] Aziz, H., Gaspers, S., Gudmundsson, J., Mestre, J., and Täubig, H. (2015). Welfare maximization in fractional hedonic games. In *Proceedings of the 24th*

International Joint Conference on Artificial Intelligence(IJCAI), pages 461–467.

[9] Aziz, H., Harrenstein, P., and Pyrga, E. Individual-based stability in hedonic games depending on the best or worst players. In *Proceedings of the 11th International Conference of Autonomous Agents and Multi-Agent Systems (AAMAS).*

[10] Aziz, H. and Savani, R. Hedonic games. In *Handbook of Computational Social Choice.*

[11] Balliu, A., Flammini, M., Melideo, G., and Olivetti, D. (2017a). Nash stability in social distance games. In *Proceedings of the 31st Conference on Artificial Intelligence (AAAI)*, pages 342–348.

[12] Balliu, A., Flammini, M., and Olivetti, D. (2017b). On pareto optimality in social distance games. In *Proceedings of the 31st Conference on Artificial Intelligence (AAAI)*, pages 349–355.

[13] Banerjee, S., Konishi, H., and Sönmez, T. (2001). Core in a simple coalition formation game. *Social Choice & Welfare*, 18(1):135–153.

[14] Barrot, N., Ota, K., Sakurai, Y., and Yokoo, M. (2019). Unknown agents in friends oriented hedonic games: Stability and complexity. In *Proceedings of the 33rd Conference on Artificial Intelligence (AAAI)*, pages 1756–1763.

[15] Bilò, V., Fanelli, A., Flammini, M., Monaco, G., and Moscardelli, L. (2015). On the price of stability of fractional hedonic games. In *Proceedings of the 14th International Conference of Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 1239–1247. ACM.

[16] Bilò, V., Fanelli, A., Flammini, M., Monaco, G., and Moscardelli, L. (2018). Nash stable outcomes in fractional hedonic games: Existence, efficiency and computation. *J. Artif. Intell. Res.*, 62:315–371.

[17] Bilò, V., Fanelli, A., Flammini, M., Monaco, G., and Moscardelli, L. (2019). Optimality and nash stability in additively separable generalized group activity selection problems. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence(IJCAI)*, pages 102–108.

[18] Bilò, V., Gourvès, L., and Monnot, J. (2019). On a simple hedonic game with graph-restricted communication. In *Proceedings of the 12th International Symposium on Algorithmic Game Theory (SAGT)*, volume 11801, pages 252–265. Springer.

[19] Bloch, F. and Diamantoudi, E. (2011). Noncooperative formation of coalitions in hedonic games. *Int. J. Game Theory*, 40(2):263–280.

[20] Bogomolnaia, A., Jackson, M. O., et al. (2002). The stability of hedonic coalition structures. *Games and Economic Behavior*, 38(2):201–230.

[21] Bogomolnaia, A. and Moulin, H. (2001). A new solution to the random assignment problem. *Journal of Economic theory*, 100(2):295–328.

[22] Brandl, F., Brandt, F., and Strobel, M. (2015). Fractional hedonic games: Individual and group stability. In *Proceedings of the 14th International Conference of Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 1219–1227. ACM.

[23] Brânzei, S. and Larson, K. (2011). Social distance games. In *Proceedings of the 22nd International Joint Conference on Artificial Intelligence(IJCAI)*, pages 91–96.

[24] Bredereck, R., Elkind, E., and Igarashi, A. (2019). Hedonic diversity games. pages 565–573.

[25] Dangalchev, C. (2006). Residual closeness in networks. *Physica A: Statistical Mechanics and its Applications*, 365(2):556–564.

[26] Dangalchev, C. (2011). Residual closeness and generalized closeness. *Int. J. Found. Comput. Sci.*, 22(8):1939–1948.

[27] Darmann, A. (2015). Group activity selection from ordinal preferences. In *International Conference on Algorithmic Decision Theory*, pages 35–51. Springer.

[28] Darmann, A. (2018). Stable and pareto optimal group activity selection from ordinal preferences. volume 47, pages 1183–1209.

[29] Darmann, A. (2019). Manipulability in a group activity selection problem. *Social Choice & Welfare*, 52(3):527–557.

[30] Darmann, A., Döcker, J., Dorn, B., Lang, J., and Schneckenburger, S. (2017). On simplified group activity selection. In *International Conference on Algorithmic Decision Theory*, pages 255–269. Springer.

[31] Darmann, A., Elkind, E., Kurz, S., Lang, J., Schauer, J., and Woeginger, G. (2012). Group activity selection problem. In *Proceedings of the 8th International Conference on Web and Internet Economics (WINE)*, pages 156–169. Springer.

[32] Darmann, A., Elkind, E., Kurz, S., Lang, J., Schauer, J., and Woeginger, G. J. (2018). Group activity selection problem with approval preferences. *Int. J. Game Theory*, 47(3):767–796.

[33] Darmann, A. and Lang, J. (2016). Group activity selection problems. In *Handbook of Computational Social Choice*.

[34] Dimitrov, D., Borm, P., Hendrickx, R., and Sung, S. C. (2006). Simple priorities and core stability in hedonic games. *Social Choice & Welfare*, 26(2):421–433.

[35] Dimitrov, D. and Sung, S. C. (2004). Enemies and friends in hedonic games: individual deviations, stability and manipulation. *CentER Discussion Paper Series*.

[36] Dreze, J. H. and Greenberg, J. (1980). Hedonic coalitions: Optimality and stability. *Econometrica: Journal of the Econometric Society*, pages 987–1003.

[37] Dughmi, S. and Ghosh, A. (2010). Truthful assignment without money. In *Proceedings of the 11th Conference on Economics and Computation (EC)*, pages 325–334. ACM.

[38] Dutton, R. D. and Brigham, R. C. (1991). Edges in graphs with large girth. *Graphs and Combinatorics*, 7(4):315–321.

[39] Elkind, E., Fanelli, A., and Flammini, M. (2016). Price of pareto optimality in hedonic games. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 30.

[40] Elkind, E. and Wooldridge, M. J. (2009). Hedonic coalition nets. In *Proceedings of the 8th International Conference of Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 417–424.

[41] Feldman, M., Lewin-Eytan, L., and Naor, J. (2015). Hedonic clustering games. *TOPC*, 2(1):4:1–4:48.

[42] Filos-Ratsikas, A., Frederiksen, S. K. S., and Zhang, J. (2014). Social welfare in one-sided matchings: Random priority and beyond. In *Proceedings of the 7th International Symposium on Algorithmic Game Theory (SAGT)*, pages 1–12. Springer.

[43] Flammini, M., Monaco, G., and Zhang, Q. (2017). Strategyproof mechanisms for additively separable hedonic games and fractional hedonic games. In *Proceedings of the 15th International Workshop on Approximation and Online Algorithms (WAOA)*, pages 301–316.

[44] Gairing, M. and Savani, R. (2019). Computing stable outcomes in symmetric additively separable hedonic games. *Math. Oper. Res.*, 44(3):1101–1121.

[45] Ganian, R., Ordyniak, S., and Rahul, C. (2019). Group activity selection with few agent types. In *Proceedings of the 27th European Symposium on Algorithms (ESA)*.

[46] Garey, M. R. and Johnson, D. S. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman.

[47] Igarashi, A., Bredereck, R., and Elkind, E. (2017a). On parameterized complexity of group activity selection problems on social networks. In *Proceedings of the 16th International Conference of Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 1575–1577. ACM.

[48] Igarashi, A. and Elkind, E. (2016). Hedonic games with graph-restricted communication. In *Proceedings of the 15th International Conference of Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 242–250. ACM.

[49] Igarashi, A., Peters, D., and Elkind, E. (2017b). Group activity selection on social networks. In *Proceedings of the 31st Conference on Artificial Intelligence (AAAI)*, pages 565–571. AAAI Press.

[50] Kaklamanis, C., Kanellopoulos, P., and Patouchas, D. (2018). On the price of stability of social distance games. In *Proceedings of the 11th International Symposium on Algorithmic Game Theory (SAGT)*, volume 11059, pages 125–136.

[51] Koutsoupias, E. and Papadimitriou, C. H. (2009). Worst-case equilibria. *Comput. Sci. Rev.*, 3(2):65–69.

[52] Lavi, R., Mu'alem, A., and Nisan, N. (2003). Towards a characterization of truthful combinatorial auctions. In *Proceedings of the 44th Symposium on Foundations of Computer Science (FOCS)*, pages 574–583.

[53] Lee, H. and Williams, V. V. (2017). Parameterized complexity of group activity selection. In *Proceedings of the 16th International Conference of Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 353–361.

[54] Long, Y. (2019). Strategy-proof group selection under single-peaked preferences over group size. *Economic Theory*, 68(3):579–608.

[55] Monaco, G., Moscardelli, L., and Velaj, Y. (2020). Stable outcomes in modified fractional hedonic games. *Auton. Agents Multi Agent Syst.*, 34(1):4.

[56] Nisan, N., Roughgarden, T., Tardos, É., and Vazirani, V. V., editors (2007). *Algorithmic Game Theory*. Cambridge University Press.

[57] Ohta, K., Barrot, N., Ismaili, A., Sakurai, Y., and Yokoo, M. (2017). Core stability in hedonic games among friends and enemies: Impact of neutrals. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence(IJCAI)*, pages 359–365.

[58] Olsen, M. (2012). On defining and computing communities. In *Eighteenth Computing: The Australasian Theory Symposium, CATS 2012, Melbourne, Australia, January 2012*, volume 128 of *CRPIT*, pages 97–102. Australian Computer Society.

[59] Papadimitriou, C. H. (2011). Games, algorithms, and the internet. In *Proceedings of the 20th International Conference on World Wide Web, (WWW)*, pages 5–6. ACM.

[60] Procaccia, A. D. and Tennenholtz, M. (2013). Approximate mechanism design without money. *ACM Trans. Economics and Comput.*, 1(4):18:1–18:26.

[61] Rodríguez-Álvarez, C. (2009). Strategy-proof coalition formation. *Int. J. Game Theory*, 38(3):431–452.

[62] Shapley, L. and Scarf, H. (1974). On cores and indivisibility. *Journal of mathematical economics*, 1(1):23–37.

[63] Sung, S. C. and Dimitrov, D. (2007). On core membership testing for hedonic coalition formation games. *Oper. Res. Lett.*, 35(2):155–158.

[64] Svensson, L.-G. (1999). Strategy-proof allocation of indivisible goods. *Social Choice & Welfare*, 16(4):557–567.

[65] Wright, M. and Vorobeychik, Y. (2015). Mechanism design for team formation. In *Proceedings of the 29th Conference on Artificial Intelligence (AAAI)*, pages 1050–1056.