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Ballistic and superdiffusive scales in the macroscopic evolution of a chain of oscillators

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Abstract

We consider a one dimensional infinite acoustic chain of harmonic oscillators whose dynamics are perturbed by a random exchange of velocities, such that the energy and momentum of the chain are conserved. Consequently, the evolution of the system has only three conserved quantities: volume, momentum and energy. We show the existence of two space–time scales on which the energy of the system evolves. On the hyperbolic scale $(t\epsilon^{-1}, x\epsilon^{-1})$ the limits of the conserved quantities satisfy a Euler system of equations, while the thermal part of the macroscopic energy profile remains stationary. Thermal energy starts evolving at a longer time scale, corresponding to superdiffusive scaling $(t\epsilon^{-3/2}, x\epsilon^{-1})$, and follows a fractional heat equation. We also prove the diffusive scaling limit of the Riemann invariants—the so-called normal modes, corresponding to linear hyperbolic propagation.

Keywords: hydrodynamic limits, Euler equations, superdiffusion, fractional heat equation
Mathematics Subject Classification numbers: 60K35, 74A25, 82C22, 82C70

1. Introduction

Consider a chain of coupled anharmonic oscillators in one dimension. Denote by $\mathfrak{q}_x$ and $\mathfrak{p}_x$ the position and momentum of the particle labeled by $x \in \mathbb{Z}$. The interaction between particles x and $x-1$ is described by the potential energy $V(\mathfrak{r}_x)$ of an anharmonic spring, where the quantity

$$\mathfrak{r}_x := \mathfrak{q}_x - \mathfrak{q}_{x-1} \tag{1.1}$$

is called the inter-particle distance or *volume strain*. The (formal) Hamiltonian of the chain is given by

$$\mathcal{H}(\mathfrak{q},\mathfrak{p}) = \sum_x \mathfrak{e}_x(\mathfrak{q},\mathfrak{p}), \tag{1.2}$$

where the energy of the oscillator x is defined by

$$\mathfrak{e}_x(\mathfrak{q},\mathfrak{p}) := \frac{\mathfrak{p}_x^2}{2} + V(\mathfrak{r}_x). \tag{1.3}$$

The respective Hamiltonian dynamics are given by the solution of the equations:

$$\dot{\mathfrak{q}}_x(t) = \frac{\partial \mathcal{H}}{\partial \mathfrak{p}_x}(\mathfrak{q},\mathfrak{p}) = \mathfrak{p}_x, \tag{1.4}$$

$$\dot{\mathfrak{p}}_x(t) = -\frac{\partial \mathcal{H}}{\partial \mathfrak{q}_x}(\mathfrak{q},\mathfrak{p}), \quad x \in \mathbb{Z}.$$

There are three formally conserved quantities (also called balanced) for these dynamics

$$\sum_x \mathfrak{r}_x, \qquad \sum_x \mathfrak{p}_x, \qquad \sum_x \mathfrak{e}_x \tag{1.5}$$

that correspond to the total volume, momentum and energy of the chain. The corresponding equilibrium Gibbs measures $\nu_{\boldsymbol{\lambda}}$ are parametrized by $\boldsymbol{\lambda} = [\beta, p, \tau]$, with the respective components equal to the inverse temperature, velocity and tension. They are product measures given explicitly by formulas

$$\mathrm{d}\nu_{\boldsymbol{\lambda}} = \prod_x \exp\{-\beta(\mathfrak{e}_x - p\mathfrak{p}_x - \tau\mathfrak{r}_x) - \mathcal{G}(\boldsymbol{\lambda})\}\mathrm{d}\mathfrak{r}_x\, \mathrm{d}\mathfrak{p}_x, \tag{1.6}$$

where

$$\mathcal{G}(\boldsymbol{\lambda}) := \beta\frac{p^2}{2} + \log\left[\sqrt{2\pi\beta^{-1}}Z(\tau,\beta)\right] \tag{1.7}$$

and

$$Z(\tau,\beta) := \int_{\mathbb{R}} \exp\{-\beta(V(r) - \tau r)\}\mathrm{d}r. \tag{1.8}$$

The length and internal energy in the equilibrium can be expressed as functions of $\boldsymbol{\lambda}$ by

$$r(\tau,\beta) = \langle \mathfrak{r}_x \rangle_{\nu_{\boldsymbol{\lambda}}} = \beta^{-1}\partial_\tau \mathcal{G}(\boldsymbol{\lambda}), \qquad p = \beta^{-1}\partial_p \mathcal{G}(\boldsymbol{\lambda}) \tag{1.9}$$

$$u(\tau,\beta) = \langle \mathfrak{e}_x \rangle_{\nu_{\boldsymbol{\lambda}}} - \frac{p^2}{2} = \frac{1}{2\beta} + \langle V(\mathfrak{r}_x) \rangle_{\nu_{\boldsymbol{\lambda}}} = -\partial_\beta \mathcal{G}(\boldsymbol{\lambda}) + \tau r.$$

Therefore the tension $\tau(r,u)$ and inverse temperature $\beta(r,u)$ can be determined from the relations:

$$\beta = \partial_u S(u,r), \qquad \tau\beta = -\partial_r S(u,r), \tag{1.10}$$

where $S(u, r)$ is the *thermodynamic entropy* defined by the Legendre transform

$$S(u,r) = \inf_{\tau,\beta}\left[-\beta\tau r + \beta u + \mathcal{G}(\beta,0,\tau)\right]. \tag{1.11}$$

It is expected that after the hyperbolic rescaling of space and time $(t\epsilon^{-1}, x\epsilon^{-1})$, the empirical distribution of the balanced quantities

$$\mathfrak{w}_x^T(t) := [\mathfrak{r}_x(t), \mathfrak{p}_x(t), \mathfrak{e}_x(t)], \tag{1.12}$$

defined for a smooth function J with compact support by the formula:

$$\epsilon \sum_x J(\epsilon x) \mathfrak{w}_x\left(\frac{t}{\epsilon}\right)$$

converges, as $\epsilon \to 0$, to the solution $w^T(t, y) = [r(t, y), p(t, y), e(t, y)]$ of the compressible Euler system of equations:

$$\begin{aligned} \partial_t r &= \partial_y p, \\ \partial_t p &= \partial_y \tau(r, u), \\ \partial_t e &= \partial_y \left[p\tau(r, u) \right], \end{aligned} \tag{1.13}$$

where $u\,(t, y) := e\,(t, y) - p^2(t, y)/2$ is the local internal energy. For a finite macroscopic volume, this limit has been proven using the relative entropy method, provided the microscopic dynamics are perturbed by a random exchange of velocities between particles, see [8] and [2], in the regime when the system of Euler equations admits a smooth solution. After a sufficiently long time, the solution of (1.13) should converge, in an appropriately weak sense, to some mechanical equilibrium described by:

$$p(x) = p_0, \qquad \tau(r(x), u(x)) = \tau_0, \tag{1.14}$$

where p_0 and τ_0 are some constants.

Characterizing all possible stationary solutions of (1.14) could be a daunting task. Most likely, they are generically very irregular, but it is quite obvious that if we start with smooth initial conditions that satisfy (1.14), the respective solutions of (1.13) remain stationary. Also, by the same relative entropy method as used in [8] and [2], it follows that starting with such initial profiles, the corresponding empirical distributions of the balanced quantities converge, in the hyperbolic time scale, to the same initial stationary solution, i.e. they do not evolve in time. On the other hand, we do know that the system will eventually converge to a global equilibrium, so this implies that there exists a longer time scale, on which these profiles (stationary at the hyperbolic scale) will evolve.

There is a strong argument, stemming from both numerical evidence and quite convincing heuristics, suggesting the divergence of the Green–Kubo formula, defining the thermal diffusivity for a generic unpinned one-dimensional system, see [6, 7]. Therefore, we expect the aforementioned larger time scale (at which the evolution of these profiles is observed) to be superdiffusive. Furthermore, an argument by Spohn [9], based on fluctuating hydrodynamics and mode coupling, also suggests a superdiffusive evolution of the *heat mode* fluctuation field, when the system is in equilibrium. Consequently, one can conjecture the following scenario: after the space-time rescaling $(\epsilon^{-\alpha}t, \epsilon^{-1}x)$, the temperature $T(t, y) = \beta^{-1}(t, y)$ evolves following some fractional (possibly non-linear) heat equation. The choice of the exponent α may depend on the particular values of the tension and of the interaction potential $V(r)$.

In the present work we prove rigorously that this picture of two-time-scale evolution holds for the harmonic chain with a momentum exchange noise that conserves the total momentum and energy. We consider the quadratic Hamiltonian:

$$\mathcal{H}(\mathfrak{q}, \mathfrak{p}) = \sum_x \frac{1}{2}\mathfrak{p}_x^2 - \frac{1}{4}\sum_{x,x'} \alpha_{x-x'}(\mathfrak{q}_x - \mathfrak{q}_{x'})^2 \tag{1.15}$$

where the harmonic coupling coefficients $\alpha_{x-x'}$, between atoms labeled by x and x', are assumed to decay sufficiently quickly. Detailed hypotheses made about the potential are contained in section 2.1. Furthermore, we perturb the Hamiltonian dynamics, given by (1.4), with a random exchange of momentum between the nearest neighbor atoms, in such a way that the total energy and momentum of the chain are conserved, see the formula (2.7) below. We assume that the initial configuration is distributed according to a probability measure μ_ϵ such that its total energy grows like ϵ^{-1} (i.e. macroscopically: it is of order 1). Additionally, we suppose that the family of measures $(\mu_\epsilon)_{\epsilon>0}$ possesses a *macroscopic profile* $w^T(y) = [r(y), p(y), e(y)]$, with r, p, e belonging to $C_0^\infty(\mathbb{R})$—the space of all smooth and compactly supported functions. The above means that

$$\lim_{\epsilon\to 0+} \epsilon \sum_x \langle \mathfrak{w}_x \rangle_{\mu_\epsilon} J(\epsilon x) = \int_{\mathbb{R}} w^T(y) J(y) \mathrm{d}y, \tag{1.16}$$

for any test function $J \in C_0^\infty(\mathbb{R})$.

We decompose the initial configuration into phononic (low frequency) and thermal (high frequency) terms, i.e. the initial configuration $(\mathfrak{r}, \mathfrak{p})$ is assumed to be of the form

$$\mathfrak{r}_x = \mathfrak{r}_x' + \mathfrak{r}_x'' \quad \text{and} \quad \mathfrak{p}_x = \mathfrak{p}_x' + \mathfrak{p}_x'', \quad x \in \mathbb{Z}, \tag{1.17}$$

where the variance of $(\mathfrak{r}_x'', \mathfrak{p}_x'')$ around the macroscopic profile $(r(\epsilon x), p(\epsilon x))_{x\in\mathbb{Z}}$ vanishes, with $\epsilon \to 0+$ (see (2.28) and (2.29) below). This obviously implies that the macroscopic profiles corresponding to $(\mathfrak{r}_x', \mathfrak{p}_x')$ equal 0. Concerning the distribution of this configuration we suppose that its energy spectrum, defined in (2.23) below, is square integrable in the sense of condition (2.26). We call the respective energy profiles corresponding to $(\mathfrak{r}_x', \mathfrak{p}_x')$ and $(\mathfrak{r}_x'', \mathfrak{p}_x'')$ the thermal and phononic (or mechanical) ones, see definitions 2.2 and 2.3 below. In section 9 we give several examples of families of initial configurations satisfying the above hypotheses. Among them are inhomogeneous Gibbs measures with the temperature, momentum and tension profiles changing on the macroscopic scale, see section 9.2.3. These examples also include the families of measures corresponding to the local equilibrium states for the dynamics described by (1.2), see section 9.2.5.

In our first result, see the theorem 3.1 and corollary 3.2 below, we show that the empirical distribution of $\mathfrak{w}_x(t)$ converges, at the hyperbolic time-space scale, to a solution satisfying the linear version of (1.13) with $\tau(r,u) := \tau_1 r$, where the parameter τ_1, called *the speed of sound*, is given by the formula (2.5).

Notice that since we are working in infinite volume, relative entropy methods cannot be applied to obtain such hydrodynamic limits, as done in [2] and [8]. Instead, we obtain the limit result, using only the techniques based on the L^2 bounds on the initial configurations. After a direct computation it becomes clear that the function

$$T(y) := e(t,y) - e_{\mathrm{ph}}(t,y), \tag{1.18}$$

is stationary in time. The macroscopic phononic energy is given by

$$e_{\mathrm{ph}}(t,y) := \frac{p^2(t,y)}{2} + \tau_1 r^2(t,y), \tag{1.19}$$

Also, the entropy in this case equals

$$S(u,r) = C \log(u - \tau_1 r^2)$$

for some constant $C > 0$ depending on the coefficients (α_x). In fact, as it turns out, see theorem 3.1 below, $T(y)$ and $e_{\mathrm{ph}}(t,y)$, given by (1.18) and (1.19) above, are the respective limits of the thermal and phononic energy profiles.

Concerning the behavior of the thermal energy profile at the longer, superdiffusive time scale it has been shown in [4] that after space-time rescaling $(\epsilon^{-3/2}t, \epsilon^{-1}x)$, the thermal energy profile converges to the solution $T(t, y)$ of a fractional heat equation

$$\partial_t T(t,y) = -\hat{c}|\Delta_y|^{3/4} T(t,y), \quad \text{with } T(0,y) = T(y) \tag{1.20}$$

and the coefficient $\hat{c}$ given by the formula (5.50) below. This result is generalized in the present paper to configurations of the form (1.17).

Finally, in the theorem 3.4 we consider the empirical distributions of the microscopic estimators of the Riemann invariants of the linear wave equation system that describe the evolution of the macroscopic profiles of $(\mathfrak{r}_x(t), \mathfrak{p}_x(t))$—the so-called *normal modes*. We show that they evolve at the diffusive space-time scale $(t\epsilon^{-2}, y\epsilon^{-1})$. Similar behavior is conjectured for some anharmonic chains, e.g. those corresponding to the β-FPU potential at zero tension (see [9]).

Concerning the organization of the paper, in section 2 we rigorously introduce the basic notions that appear throughout the article. The main results are formulated in section 3. In sections 4–8 we present their respective proofs. Finally, in section 9 we show examples of the distributions of the initial data that are both of phononic and thermal types.

2. The stochastic dynamics

2.1. Hamiltonian dynamics with a noise

Concerning the interaction appearing in (1.15) we consider only the unpinned case; therefore, we let

$$\alpha_0 := -\sum_{x \neq 0} \alpha_x. \tag{2.1}$$

Define the Fourier transform of $(\alpha_x)_{x\in\mathbb{Z}}$ by

$$\hat{\alpha}(k) := \sum_x \alpha_x \exp\{-2\pi \mathrm{i} kx\}, \quad k \in \mathbb{T}, \tag{2.2}$$

where $\mathbb{T}$ is the unit torus that is the interval $[-1/2, 1/2]$ with the identified endpoints. Then, from (2.1), we have $\hat{\alpha}(0) = 0$.

Furthermore, we assume that:

(a1) coefficients $(\alpha_x)_{x\in\mathbb{Z}}$ are real-valued and symmetric and there exists a constant $C > 0$ such that

$$|\alpha_x| \leqslant C\mathrm{e}^{-|x|/C} \quad \text{for all } x \in \mathbb{Z}.$$

(a2) stability: $\hat{\alpha}(k) > 0$, $k \neq 0$,

(a3) the chain is acoustic: $\hat{\alpha}''(0) > 0$,

The above conditions imply that $\hat{\alpha}(k) = \hat{\alpha}(-k)$, $k \in \mathbb{T}$. In addition, $\hat{\alpha} \in C^\infty(\mathbb{T})$ and it can be written as

$$\hat{\alpha}(k) = 4\tau_1 \mathfrak{s}^2(k)\varphi^2(\mathfrak{s}^2(k)). \tag{2.3}$$

The function $\varphi : [0, +\infty) \to (0, +\infty)$ is smooth and $\varphi(0) = 1$. For the sake of abbreviation, we shall use the notation

$$\mathfrak{s}(k) := \sin(\pi k) \quad \text{and} \quad \mathfrak{c}(k) := \cos(\pi k), \quad k \in \mathbb{T}. \tag{2.4}$$

The parameter τ_1 appearing in (2.3), called the *sound speed*, is defined by

$$\tau_1 := \frac{\hat{\alpha}''(0)}{8\pi^2}. \tag{2.5}$$

Let also

$$\varphi_- := \inf \varphi \quad \text{and} \quad \varphi_+ := \sup \varphi. \tag{2.6}$$

Since the Hamiltonian is invariant under the translations of the positions $\mathfrak{q}_x$ of the atoms, the latter are not well defined, and the configuration space of our system is given by $((\mathfrak{r}_x, \mathfrak{p}_x))_{x\in\mathbb{Z}}$, where $\mathfrak{r}_x$ should be thought of as the distance between particle x and $x-1$. The evolution is given by a system of stochastic differential equations

$$\mathrm{d}\mathfrak{r}_x(t) = \nabla^*\mathfrak{p}_x(t)\,\mathrm{d}t, \tag{2.7}$$

$$\begin{aligned}\mathrm{d}\mathfrak{p}_x(t) = &\left\{-\alpha * \mathfrak{q}_x(t) - \frac{\gamma}{2}(\beta * \mathfrak{p}(t))_x\right\}\mathrm{d}t\\ &+ \gamma^{1/2}\sum_{z=-1,0,1}(Y_{x+z}\mathfrak{p}_x(t))dw_{x+z}(t), \quad x\in\mathbb{Z}.\end{aligned}$$

with the parameter $\gamma > 0$ determining the strength of the noise in the system. The vector fields $(Y_x)_{x\in\mathbb{Z}}$ are given by

$$Y_x := (\mathfrak{p}_x - \mathfrak{p}_{x+1})\partial_{\mathfrak{p}_{x-1}} + (\mathfrak{p}_{x+1} - \mathfrak{p}_{x-1})\partial_{\mathfrak{p}_x} + (\mathfrak{p}_{x-1} - \mathfrak{p}_x)\partial_{\mathfrak{p}_{x+1}}. \tag{2.8}$$

Here, $(w_x(t),\ t \geqslant 0)_{x\in\mathbb{Z}}$ are i.i.d. one-dimensional, real-valued, standard Brownian motions, that are non-anticipative over some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The symbol $*$ denotes the convolution, but in particular, since $\hat{\alpha}(0) = 0$, we can make sense of (remember that $\mathfrak{q}_x$ are not really defined)

$$\alpha * \mathfrak{q}_x := \sum_{x'}\alpha_{x-x'}\mathfrak{q}_{x',x},$$

where

$$\mathfrak{q}_{x',x} := \begin{cases}\displaystyle\sum_{x'<x''\leqslant x}\mathfrak{r}_{x''}, & \text{if } x' < x,\\ 0, & \text{if } x' = x,\\ \displaystyle\sum_{x<x''\leqslant x'}\mathfrak{r}_{x''}, & \text{if } x < x'.\end{cases} \tag{2.9}$$

Furthermore, $\beta_x = \Delta\beta_x^{(0)}$ and $\beta_x^{(1)} := \nabla^*\beta_x^{(0)}$, where

$$\beta_x^{(0)} = \begin{cases}-4, & x = 0,\\ -1, & x = \pm 1,\\ \ \ 0, & \text{if otherwise.}\end{cases} \tag{2.10}$$

The lattice Laplacian of a given $(g_x)_{x\in\mathbb{Z}}$ is defined as $\Delta g_x := g_{x+1} + g_{x-1} - 2g_x$. Let also $\nabla g_x := g_{x+1} - g_x$ and $\nabla^* g_x := g_{x-1} - g_x$. A simple calculation shows that

$$\hat{\beta}(k) = 8\mathfrak{s}^2(k)(1 + 2\mathfrak{c}^2(k)) = 8\mathfrak{s}^2(k) + 4\mathfrak{s}^2(2k), \tag{2.11}$$

and

$$\hat{\beta}^{(1)}(k) = 4\mathrm{i}e^{-2\mathrm{i}\pi k}\mathfrak{s}(k)(1 + 2\mathfrak{c}^2(k)). \tag{2.12}$$

The evolution equations are formulated on the Hilbert space ℓ_2 made of all real-valued sequences $((\mathfrak{r}_x, \mathfrak{p}_x))_{x\in\mathbb{Z}}$ that satisfy

$$\sum_x(\mathfrak{r}_x^2+\mathfrak{p}_x^2)<+\infty. \tag{2.13}$$

Concerning the initial data, we assume that it is distributed according to a probability measure μ_ϵ that satisfies

$$\sup_{\epsilon\in(0,1]}\epsilon\sum_x\left\langle \mathfrak{r}_x^2+\mathfrak{p}_x^2\right\rangle_{\mu_\epsilon}<+\infty. \tag{2.14}$$

The above means that the total macroscopic energy of the system is finite. Here, $\langle\cdot\rangle_{\mu_\epsilon}$ denotes the average with respect to μ_ϵ.

Denote by $\mathbb{E}_\epsilon$ the expectation with respect to the product measure $\mathbb{P}_\epsilon=\mu_\epsilon\otimes\mathbb{P}$. The existence and uniqueness of a solution to (2.7) in ℓ_2, with the aforementioned initial condition, can be easily concluded from the standard Hilbert space theory of stochastic differential equations, see e.g. chapter 6 of [3].

2.2. Energy density functional

For a configuration that satisfies (2.13), the energy per atom functional can be defined by:

$$\mathfrak{e}_x(\mathfrak{r},\mathfrak{p}):=\frac{\mathfrak{p}_x^2}{2}-\frac{1}{4}\sum_{x'}\alpha_{x-x'}\mathfrak{q}_{x,x'}^2. \tag{2.15}$$

Notice that $\sum_x\mathfrak{e}_x=\mathcal{H}(\mathfrak{q},\mathfrak{p})$. Here, we highlight the fact that although the total energy is non-negative, in light of the assumptions (a1)–(a3) made about the interaction potential, the energy per atom $\mathfrak{e}_x$ does not have a definite sign. However, we can prove the following fact (see appendix below).

Proposition 2.1. *We have*

$$c_-\sum_x(\mathfrak{r}_x^2+\mathfrak{p}_x^2)\leqslant\sum_x\mathfrak{e}_x(\mathfrak{r},\mathfrak{p})\leqslant c_+\sum_x(\mathfrak{r}_x^2+\mathfrak{p}_x^2),\quad(\mathfrak{r},\mathfrak{p})\in\ell_2, \tag{2.16}$$

with $c_-:=\min\{1/2,\tau_1\varphi_-^2\}$ *and* $c_+:=\max\{1/2,\tau_1\varphi_+^2\}$*. In addition,*

$$c_*\sum_x|\mathfrak{e}_x(\mathfrak{r},\mathfrak{p})|\leqslant\sum_x\mathfrak{e}_x(\mathfrak{r},\mathfrak{p})\leqslant\sum_x|\mathfrak{e}_x(\mathfrak{r},\mathfrak{p})| \tag{2.17}$$

and $c_*:=c_-\Big(\max\Big\{1/2,\sum_{z>0}z^2|\alpha_z|\Big\}\Big)^{-1}$.

Thanks to (2.14) and (2.16), the distribution of the initial data is such that the macroscopic energy of the chain at time $t=0$ is finite, i.e. that

$$K_0:=\sup_{\epsilon\in(0,1]}\epsilon\sum_x\langle\mathfrak{e}_x\rangle_{\mu_\epsilon}<+\infty. \tag{2.18}$$

We shall also assume that there exists a macroscopic profile for the family of measures $(\mu_\epsilon)_{\epsilon>0}$, i.e. functions $r(y)$, $p(y)$ and $e(y)$ belonging to $C_0^\infty(\mathbb{R})$ such that the respective $w^T(y)=[r(y), p(y), e(y)]$ satisfies (1.16).

2.3. Macroscopic profiles of temperature and phononic energy

The macroscopic temperature profile corresponding to the family of laws $(\mu_\epsilon)_{\epsilon\in(0,1]}$ is defined as a function $T:\mathbb{R}\to[0,+\infty)$ such that

$$\int_{\mathbb{R}} T(y)J(y)\mathrm{d}y = \lim_{\epsilon\to 0+} \epsilon\sum_x J(\epsilon x)\left\{\left\langle(\delta_\epsilon \mathfrak{p}_x)^2\right\rangle_{\mu_\epsilon} - \frac{1}{4}\sum_{x'}\alpha_{x-x'}\left\langle(\delta_\epsilon \mathfrak{q}_{x,x'})^2\right\rangle_{\mu_\epsilon}\right\} \tag{2.19}$$

for any $J\in C_0^\infty(\mathbb{R})$, where $\delta_\epsilon \mathfrak{p}_x := \mathfrak{p}_x - p(\epsilon x)$ and

$$\delta_\epsilon \mathfrak{q}_{x',x} := \begin{cases} \sum_{x'<x''\leqslant x} \delta_\epsilon \mathfrak{r}_{x''}, & \text{if } x'<x,\\ 0, & \text{if } x'=x,\\ \sum_{x<x''\leqslant x'} \delta_\epsilon \mathfrak{r}_{x''}, & \text{if } x<x', \end{cases} \tag{2.20}$$

with $\delta_\epsilon \mathfrak{r}_x := \mathfrak{r}_x - r(\epsilon x)$. After a simple calculation one gets the following identity:

$$T(y) = e(y) - e_{\mathrm{ph}}(y), \tag{2.21}$$

with

$$e_{\mathrm{ph}}(y) := \frac{1}{2}(p^2(y) + \tau_1 r^2(y)), \quad y\in\mathbb{R}, \tag{2.22}$$

the *phononic* (or *mechanical*) macroscopic energy profile.

2.4. Energy spectrum

The energy spectrum of the configuration distributed according to μ_ϵ is defined as

$$\mathfrak{w}_\epsilon(k) := \left\langle|\hat{\mathfrak{p}}(k)|^2\right\rangle_{\mu_\epsilon} + \frac{\hat{\alpha}(k)}{4\mathfrak{s}^2(k)}\left\langle|\hat{\mathfrak{r}}(k)|^2\right\rangle_{\mu_\epsilon}, \tag{2.23}$$

where $\hat{\mathfrak{p}}(k)$ and $\hat{\mathfrak{r}}(k)$ are the Fourier transforms of $(\mathfrak{p}_x)$ and $(\mathfrak{r}_x)$, respectively. Using (2.3) we get

$$\mathfrak{w}_\epsilon(k) = \left\langle|\hat{\mathfrak{p}}(k)|^2\right\rangle_{\mu_\epsilon} + \tau_1\varphi^2(\mathfrak{s}^2(k))\left\langle|\hat{\mathfrak{r}}(k)|^2\right\rangle_{\mu_\epsilon}, \quad k\in\mathbb{T}. \tag{2.24}$$

Assumption (2.18) is equivalent with

$$K_0 = \sup_{\epsilon\in(0,1]} \epsilon\int_{\mathbb{T}} \mathfrak{w}_\epsilon(k)\mathrm{d}k < +\infty. \tag{2.25}$$

Definition 2.2. The family of distributions $(\mu_\epsilon)_{\epsilon>0}$ is said to be of a thermal type if its energy spectrum $\mathfrak{w}_\epsilon(k)$ satisfies

$$K_1 := \sup_{\epsilon\in(0,1]} \epsilon^2\int_{\mathbb{T}} \mathfrak{w}_\epsilon^2(k)\mathrm{d}k < +\infty. \tag{2.26}$$

Remark. In section 5.1 we show that (2.26) implies that the respective macroscopic profiles $(r(y), p(y))$ vanish. Therefore, we conclude that the macroscopic phononic energy $e_{\mathrm{ph}}(y)\equiv 0$ (see (2.22)) and, as a result,

$$\lim_{\epsilon\to 0+} \epsilon\sum_x J(\epsilon x)\langle \mathfrak{e}_x\rangle_{\mu_\epsilon} = \int_{\mathbb{R}} J(y)T(y)\mathrm{d}y, \quad J\in C_0^\infty(\mathbb{R}). \tag{2.27}$$

We stress that although $r(y) \equiv 0$ and $p(y) \equiv 0$, in case condition (2.26) holds, configuration $(\mathfrak{r}, \mathfrak{p})$ need not necessarily be centered (in the measure μ_ϵ). Condition (2.26) is related to the issue of variability of the initial data on the microscopic scale, as can be seen in the examples presented in section 9.

Definition 2.3. The family (μ_ϵ) is said to be of a phononic (or mechanical) type if

$$\lim_{\epsilon \to 0+} \epsilon \sum_x \langle [\mathfrak{r}_x - r(\epsilon x)]^2 \rangle_{\mu_\epsilon} = 0 \tag{2.28}$$

and

$$\lim_{\epsilon \to 0+} \epsilon \sum_x \left\langle [\mathfrak{p}_x - p(\epsilon x)]^2 \right\rangle_{\mu_\epsilon} = 0, \tag{2.29}$$

where $r(x)$ and $p(x)$ are the macroscopic profiles of the strain and momentum, see (1.16).

Remark. Using (2.19) and (2.28) together with (2.29) it is straightforward to see that the temperature profile corresponding to phononic types of initial data vanishes, i.e. $T(y) \equiv 0$.

3. Statement of the main results

3.1. Hyperbolic scaling

Assume that the configuration $(\mathfrak{r}, \mathfrak{p})$ can be decomposed into two parts whose laws are of thermal and phononic types respectively. More precisely, we assume that

$$\mathfrak{r}_x = \mathfrak{r}'_x + \mathfrak{r}''_x, \quad \mathfrak{p}_x = \mathfrak{p}'_x + \mathfrak{p}''_x, \quad x \in \mathbb{Z}, \tag{3.1}$$

where

$$K'_1 := \sup_{\epsilon \in (0,1]} \epsilon^2 \int_{\mathbb{T}} (\mathfrak{w}'_\epsilon)^2(k)\mathrm{d}k < +\infty, \tag{3.2}$$

and $\mathfrak{w}'_\epsilon(k)$ is the energy spectrum of $(\mathfrak{r}', \mathfrak{p}')$, i.e.

$$\mathfrak{w}'_\epsilon(k) = \left\langle |\hat{\mathfrak{p}}'(k)|^2 \right\rangle_{\mu_\epsilon} + \tau_1 \varphi^2(\mathfrak{s}^2(k)) \left\langle |\hat{\mathfrak{r}}'(k)|^2 \right\rangle_{\mu_\epsilon}, \quad k \in \mathbb{T}. \tag{3.3}$$

with $(\hat{\mathfrak{p}}'(k), \hat{\mathfrak{r}}'(k))$ the respective Fourier transforms of $(\mathfrak{p}', \mathfrak{r}')$. In addition, we assume that the configuration $(\mathfrak{r}'', \mathfrak{p}'')$ is of the phononic type in the sense of definition 2.3.

Suppose that $(\mathfrak{r}'(t), \mathfrak{p}'(t))$ and $(\mathfrak{r}''(t), \mathfrak{p}''(t))$ describe the evolution of the respective initial configurations $(\mathfrak{r}'_x, \mathfrak{p}'_x)$ and $(\mathfrak{r}''_x, \mathfrak{p}''_x)$ under the dynamics (2.7). Define the microscopic temperature and phononic energy density profiles by

$$\mathfrak{e}_{\mathrm{th},x}(t) := \mathfrak{e}_x(\mathfrak{r}'(t), \mathfrak{p}'(t)) \quad \text{and} \quad \mathfrak{e}_{\mathrm{ph},x}(t) := \mathfrak{e}_x(\mathfrak{r}''(t), \mathfrak{p}''(t)) \tag{3.4}$$

Let

$$v^T(t, y) := (r(t, y), p(t, y)) \tag{3.5}$$

be the solution of the linear wave equation

$$\begin{cases} \partial_t r(t, y) = \partial_y p(t, y), \\ \partial_t p(t, y) = \tau_1 \partial_y r(t, y), \\ r(0, y) = r(y), \quad p(0, y) = p(y). \end{cases} \tag{3.6}$$

The macroscopic phononic energy at time t, see (2.22), is given by

$$e_{\rm ph}(t,y)=\frac{1}{2}(p^2(t,y)+\tau_1 r^2(t,y)). \tag{3.7}$$

The components of

$$w^T(t,y):=[r(t,y),p(t,y),e_{\rm ph}(t,y)]$$

evolve according to the system of linear Euler equations:

$$\begin{cases}\partial_t r(t,y)=\partial_y p(t,y),\\ \partial_t p(t,y)=\tau_1\partial_y r(t,y),\\ \partial_t e_{\rm ph}(t,y)=\tau_1\partial_y(r(t,y)p(t,y)),\\ w(0,y)=w(y).\end{cases} \tag{3.8}$$

The following result is proven in section 6.

Theorem 3.1. *Suppose that the initial configuration* $(\mathfrak{r},\mathfrak{p})$ *satisfies the assumptions formulated in the foregoing. Then,* $(\mathfrak{r}'(t\epsilon^{-1}),\mathfrak{p}'(t\epsilon^{-1}))$ *is of the thermal type for all* $t\geqslant 0$ *and*

$$\lim_{\epsilon\to 0+}\epsilon\sum_x\int_0^{+\infty}J(t,\epsilon x)\mathbb{E}_\epsilon\mathfrak{e}_{{\rm th},x}\left(\frac{t}{\epsilon}\right){\rm d}t=\int_0^{+\infty}{\rm d}t\int_{\mathbb{R}}T(y)J(t,y){\rm d}y \tag{3.9}$$

for any $J\in C_0^\infty([0,+\infty)\times\mathbb{R})$*. Configurations* $(\mathfrak{r}''(t\epsilon^{-1}),\mathfrak{p}''(t\epsilon^{-1}))$ *are of the phononic type for all* $t\geqslant 0$ *and*

$$\lim_{\epsilon\to 0+}\epsilon\sum_x\left|\mathbb{E}_\epsilon\mathfrak{e}_{{\rm ph},x}\left(\frac{t}{\epsilon}\right)-e_{\rm ph}(t,\epsilon x)\right|=0. \tag{3.10}$$

In addition,

$$\lim_{\epsilon\to 0+}\epsilon\sum_x\int_0^{+\infty}J(t,\epsilon x)\mathbb{E}_\epsilon\mathfrak{e}_x\left(\frac{t}{\epsilon}\right){\rm d}t=\int_0^{+\infty}\int_{\mathbb{R}}[e_{\rm ph}(t,y)+T(y)]\,J(t,y){\rm d}t{\rm d}y, \tag{3.11}$$

for any $J\in C_0^\infty([0,+\infty)\times\mathbb{R})$*.*

Concerning the macroscopic evolution of the vector $\mathfrak{w}_x(t)$, see (1.12), the above result implies the following.

Corollary 3.2. *Assume that*

$$\lim_{\epsilon\to 0+}\epsilon\sum_x J(\epsilon x)\langle\mathfrak{w}_x(0)\rangle_{\mu_\epsilon}=\int_{\mathbb{R}}J(y)w(y)\,{\rm d}y,\quad J\in C_0^\infty(\mathbb{R}). \tag{3.12}$$

Then, for any $J\in C_0^\infty([0,+\infty)\times\mathbb{R})$ *we have*

$$\lim_{\epsilon\to 0+}\epsilon\sum_x\int_0^{+\infty}J(t,\epsilon x)\mathbb{E}_\epsilon\mathfrak{w}_x\left(\frac{t}{\epsilon}\right){\rm d}t=\int_0^{+\infty}\int_{\mathbb{R}}J(t,y)w(t,y){\rm d}t{\rm d}y, \tag{3.13}$$

with the components of $w(t,\ y)$ *being the solutions of the linear Euler system* (3.8)*.*

3.2. Superdiffusive scaling

It follows from theorem 3.1 that the phononic energy evolution takes place on the hyperbolic scale that is described by the linear Euler equation. In consequence, it gets dispersed to infinity at time scales longer than $t\epsilon^{-1}$. On the other hand, it has been shown (see theorem 3.1 of [4]) that the respective temperature profile $T(y)$, see (2.19), evolves on a superdiffusive scale $(t\epsilon^{-3/2}, x\epsilon^{-1})$. Therefore, we have the following result.

Theorem 3.3. *Suppose that the distribution of the initial configuration* $(\mathfrak{r}, \mathfrak{p})$ *satisfies the condition* (*3.1*)*. Then, for any test function* $J \in C_0^\infty([0, +\infty) \times \mathbb{R})$ *we have:*

$$\lim_{\epsilon\to 0+} \epsilon \sum_x \int_0^{+\infty} J(t, \epsilon x)\mathbb{E}_\epsilon \mathfrak{e}_x\left(\frac{t}{\epsilon^{3/2}}\right)\mathrm{d}t = \int_0^{+\infty}\int_{\mathbb{R}} T(t, y)J(t, y)\mathrm{d}t\mathrm{d}y. \tag{3.14}$$

Here $T(t, y)$ *satisfies the fractional heat equation* (*5.49*) *with the initial condition* $T(0, y) = T(y)$*, given by* (2.19).

The proof of the above result is presented in section 7.

3.3. The evolution of normal modes at the diffusive time scale

Finally, we consider the Riemann invariants of the linear wave equation system (3.6). They are given by

$$f^{(\pm)}(t, y) := \bar{p}(t, y) \pm \sqrt{\tau_1}\,\bar{r}(t, y). \tag{3.15}$$

The quantities defined in (3.15) are constant along the characteristics of (3.6), which are given by the straight lines $y \pm \sqrt{\tau_1}\,t = \text{const}$. Therefore,

$$f^{(\pm)}(t, y) = f^{(\pm)}(y \pm \sqrt{\tau_1}\,t), \tag{3.16}$$

with $f^{(\pm)}(y)$ determined by the initial datum. This motivates the introduction of the microscopic *normal modes* given by

$$\begin{aligned} \mathfrak{f}_y^{(+)}(t) &:= \mathfrak{p}_y(t) + \sqrt{\tau_1}\left[1 + \frac{1}{2}\left(\frac{3\gamma}{\sqrt{\tau_1}} - 1\right)\nabla^*\right]\mathfrak{r}_y(t),\\ \mathfrak{f}_y^{(-)}(t) &:= \mathfrak{p}_y(t) - \sqrt{\tau_1}\left[1 - \frac{1}{2}\left(\frac{3\gamma}{\sqrt{\tau_1}} + 1\right)\nabla^*\right]\mathfrak{r}_y(t), \end{aligned} \tag{3.17}$$

that are the second order approximations (up to a diffusive scale) of $f^{(\pm)}(t, y)$. The particular form of $\mathfrak{f}^{(\pm)}(t)$ is determined by the fact that we are looking for quantities of the form $\mathfrak{p}(t) \pm \sqrt{\tau_1}(1 \pm c_\pm\nabla^*)\mathfrak{r}(t)$ that, at the hyperbolic scale, approximate the Riemann invariants (3.15) and possess limits at the diffusive time scale. The latter requirement determines that $c_\pm = (1/2)(3\gamma/\sqrt{\tau_1} \pm 1)$.

Remark. The normal modes $\mathfrak{f}^{(\pm)}(t)$ capture the fluctuations around the Riemann invariants of the linear hyperbolic system (1.13). Their analogs, called normal modes $\phi_\pm(t)$, are considered in appendix 1 (b) of [9], in the context of anharmonic chains with FPU-potential, see (8.18) of [9].

To state our result rigorously, assume that $(\mu_\epsilon)_\epsilon$—the initial laws of configurations $(\mathfrak{r}, \mathfrak{p})$ satisfy (1.16). Denote also the heat kernel

$$P_t(y) := \frac{1}{\sqrt{4\pi Dt}} \exp\left\{-\frac{y^2}{4Dt}\right\}, \tag{3.18}$$

where $D := 3\gamma$. Let $f_\iota^{(d)}(t,y) := P_t * f^{(\iota)}(y)$, $\iota = \pm$ and

$$f^{(\pm)}(y) := p(y) \pm \sqrt{\tau_1}\, r(y). \tag{3.19}$$

With the above notation we can formulate the following result.

Theorem 3.4. *Suppose that condition (1.16) is in force. Then, the phonon modes $\mathfrak{f}_y^{(\pm)}(t)$ satisfy*

$$\lim_{\epsilon\to 0+} \epsilon \sum_x J(\epsilon x)\mathbb{E}_\epsilon \mathfrak{f}_x^{(\pm)}\left(\frac{t}{\epsilon}\right) = \int_{\mathbb{R}} J(y) f^{(\pm)}(t,y)\mathrm{d}y. \tag{3.20}$$

for any $J \in C_0^\infty(\mathbb{R})$. In addition, for any $\iota \in \{-,+\}$

$$\lim_{\epsilon\to 0+} \epsilon \sum_x J\left(\epsilon x - \iota\sqrt{\tau_1}\frac{t}{\epsilon}\right)\mathbb{E}_\epsilon \mathfrak{f}_x^{(\iota)}\left(\frac{t}{\epsilon^2}\right) = \int_{\mathbb{R}} f_\iota^{(d)}(t,y)J(y)\mathrm{d}y, \tag{3.21}$$

The proof of this theorem is contained in section 8.

Remark. The results of the present paper are also valid for other stochastic dynamics obtained by a stochastic perturbation of the harmonic chain that conserves volume, energy and momentum. For example we can take a harmonic chain perturbed by a noise of the 'jump' type. More precisely, let $(N_{x,x+1}(t))_{x\in\mathbb{Z}}$ be i.i.d. Poisson processes with an intensity of $3\gamma/2$. The dynamics of the strain component $(\mathfrak{r}_x(t))_{x\in\mathbb{Z}}$ are the same as in (2.7), while the momentum $(\mathfrak{p}_x(t))_{x\in\mathbb{Z}}$ is a càdlàg process given by

$$\begin{aligned} d\mathfrak{p}_x(t) &= -(\alpha * \mathfrak{q}(t))_x \mathrm{d}t \\ &\quad + [\nabla \mathfrak{p}_x(t-)dN_{x,x+1}(t) + \nabla^* \mathfrak{p}_x(t-)\mathrm{d}N_{x-1,x}(t)], \quad x \in \mathbb{Z} \end{aligned} \tag{3.22}$$

where $(\alpha * \mathfrak{q})_x(t)$ is defined as in (2.9). With essentially the same arguments as the ones used in the present paper, one can show that theorems 3.1–3.4 also hold for these dynamics.

4. Asymptotics of the phononic ensemble

In this section we assume that the laws $(\mu_\epsilon)_{\epsilon>0}$ of the initial configuration $(\mathfrak{r},\mathfrak{p})$ are of the phononic type, i.e. they satisfy (2.28) and (2.29).

4.1. Approximation of the macroscopic phononic energy

The solution of (3.6) is given by

$$\begin{aligned} r(t,y) &= r(y+\sqrt{\tau_1}t) + r(y-\sqrt{\tau_1}t) + \frac{1}{\sqrt{\tau_1}}[p(y+\sqrt{\tau_1}t) - p(y-\sqrt{\tau_1}t)] \\ p(t,y) &= p(y+\sqrt{\tau_1}t) + p(y-\sqrt{\tau_1}t) + \sqrt{\tau_1}\,[r(y+\sqrt{\tau_1}t) - r(y-\sqrt{\tau_1}t)], \end{aligned} \tag{4.1}$$

where $r(y)$, $p(y)$ are the initial data in (3.6). Using the above formulas and (3.7) we can write $e_{\mathrm{ph}}(t,y)$ in terms of the profiles $r(y)$ and $p(y)$. We can easily infer the following.

Lemma 4.1. *Suppose that $r, p \in C_0^\infty(\mathbb{R})$. Then,*

$$\lim_{\epsilon\to 0+}\sup_{t\geqslant 0}\left|\epsilon\sum_x e_{\rm ph}(t,\epsilon x)-\int_{\mathbb{R}} e_{\rm ph}(t,y){\rm d}y\right|=0. \tag{4.2}$$

4.2. Evolution of the mean ensemble

Define the mean configuration

$$(\bar{\mathfrak{r}}_\epsilon(t),\bar{\mathfrak{p}}_\epsilon(t))=(\bar{\mathfrak{r}}_{\epsilon,x}(t),\bar{\mathfrak{p}}_{\epsilon,x}(t))_{x\in\mathbb{Z}},$$

where

$$\bar{\mathfrak{r}}_{\epsilon,x}(t):=\mathbb{E}_\epsilon\mathfrak{r}_x(t),\quad \bar{\mathfrak{p}}_{\epsilon,x}(t):=\mathbb{E}_\epsilon\mathfrak{p}_x(t). \tag{4.3}$$

The respective energy density is then defined as

$$\bar{\mathfrak{e}}_x^{(\epsilon)}(t):=\mathfrak{e}_x(\bar{\mathfrak{r}}_\epsilon(t),\bar{\mathfrak{p}}_\epsilon(t))=\frac{\bar{\mathfrak{p}}^2_{\epsilon,x}(t)}{2}-\frac14\sum_{x'}\alpha_{x-x'}\bar{\mathfrak{q}}^2_{\epsilon,x,x'}(t), \tag{4.4}$$

where $\bar{\mathfrak{q}}_{\epsilon,x,x'}(t)$ is the respective mean of $\mathfrak{q}_{x,x'}(t)$ defined by (2.9).

Let $\hat{\mathfrak{v}}^T(t,k):=[\hat{\mathfrak{r}}(t,k),\hat{\mathfrak{p}}(t,k)]$, $k\in\mathbb{T}$, where $\hat{\mathfrak{r}}(t,k),\hat{\mathfrak{p}}(t,k)$ are the Fourier transforms of the respective components of the configuration $(\mathfrak{r}_x(t),\mathfrak{p}_x(t)))$. Suppose that $\delta\geqslant 1$. Define

$$\hat{\bar{\mathfrak{r}}}_\epsilon(t,k):=\epsilon\mathbb{E}_\epsilon\hat{\mathfrak{r}}\left(\frac{t}{\epsilon^\delta},\epsilon k\right)\quad\text{and}\quad \hat{\bar{\mathfrak{p}}}_\epsilon(t,k):=\epsilon\mathbb{E}_\epsilon\hat{\mathfrak{p}}\left(\frac{t}{\epsilon^\delta},\epsilon k\right). \tag{4.5}$$

Conservation of energy and condition (2.18) imply that

$$E_*:=\sup_{t\geqslant 0,\epsilon\in(0,1]}\int_{\epsilon^{-1}\mathbb{T}}(|\hat{\bar{\mathfrak{r}}}_\epsilon(t,k)|^2+|\hat{\bar{\mathfrak{p}}}_\epsilon(t,k)|^2){\rm d}k<+\infty. \tag{4.6}$$

From (2.7) we obtain

$$\frac{\rm d}{{\rm d}t}\hat{\bar{\mathfrak{v}}}_\epsilon(t,k)=\frac{1}{\epsilon^{\delta-1}}A_\epsilon(k)\hat{\bar{\mathfrak{v}}}_\epsilon(t,k), \tag{4.7}$$

where

$$\hat{\bar{\mathfrak{v}}}_\epsilon^T(t,k)=[\hat{\bar{\mathfrak{r}}}_\epsilon(t,k),\hat{\bar{\mathfrak{p}}}_\epsilon(t,k)] \tag{4.8}$$

and

$$A_\epsilon(k):=\begin{bmatrix}0 & a\\ -\tilde{\tau}(\epsilon k)a^* & -b\end{bmatrix}, \tag{4.9}$$

with a^*—the complex conjugate of a and

$$\begin{aligned}&\tilde{\tau}(k):=\tau_1\varphi^2(\sin^2(\pi k)),\quad a:=\frac{1}{\epsilon}(1-\exp\{-2\pi{\rm i}\epsilon k\}),\\ &b:=\frac{\gamma}{\epsilon}|1-\exp\{-2\pi{\rm i}\epsilon k\}|^2[2+\cos(2\pi\epsilon k)].\end{aligned} \tag{4.10}$$

Observe that for a given $M>0$ we have

$$\lim_{\epsilon\to 0+}\sup_{|k|\leqslant M}|A_\epsilon(k)-A_0(k)|=0 \tag{4.11}$$

and

$$A_0(k):=\begin{bmatrix} 0 & 2\pi \mathrm{i}k \\ 2\pi\mathrm{i}\bar{\tau}_1 k & 0\end{bmatrix}.$$

Since both eigenvalues of $A_0(k)$ are imaginary, we have

$$D_*:=\sup_{(t,k)\in\mathbb{R}^2}\|\exp\{A_0(k)t\}\|<+\infty. \tag{4.12}$$

Here, $\|\cdot\|$ is the matrix norm defined as the norm of the corresponding linear operator on a euclidean space. In fact, an analog of (4.12) holds in the case of the linear dynamics governed by (4.7).

Lemma 4.2. *We have*

$$C_*:=\sup_{(t,k)\in\mathbb{R}^2,\epsilon\in(0,1]}\|\exp\{A_\epsilon(k)t\}\|<+\infty. \tag{4.13}$$

Proof. The eigenvalues of $A_\epsilon(k)$ equal

$$\lambda_\pm=\frac{-1}{2}\left\{b\pm\sqrt{b^2-4\tilde{\tau}(\epsilon k)|a|^2}\right\}.$$

Since both b and $\tilde{\tau}(\epsilon k)$ (see (4.10)) are real and non-negative, it is clear that $\mathrm{Re}\lambda_\pm\leqslant 0$ and the conclusion of the lemma follows. □

4.3. Asymptotics of the mean ensemble

Let $\hat{\bar{\mathfrak{v}}}^{(0)}_\epsilon(t,k)$ be the solution of the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\bar{\mathfrak{v}}}^{(0)}_\epsilon(t,k)=\frac{1}{\epsilon^{\delta-1}}A_0(k)\hat{\bar{\mathfrak{v}}}^{(0)}_\epsilon(t,k),\quad \hat{\bar{\mathfrak{v}}}^{(0)}_\epsilon(0,k)=\hat{\bar{\mathfrak{v}}}_\epsilon(0,k). \tag{4.14}$$

Also, let $\hat{v}(t,\ell)$ be the Fourier transform of $v(t,y)$ the solution of the linear wave equation system given in (3.5). It satisfies

$$\hat{v}(t,\ell)=\exp\{A_0(\ell)t\}\hat{v}(0,\ell). \tag{4.15}$$

Conditions (2.28) and (2.29) imply

$$\lim_{\epsilon\to 0+}\int_{\epsilon^{-1}\mathbb{T}}\left|\hat{\bar{\mathfrak{v}}}_\epsilon(0,\ell)-\hat{v}(0,\ell)\right|^2\mathrm{d}\ell=0. \tag{4.16}$$

Lemma 4.3. *Suppose that $\delta\in[1,2)$ and $M>0$. Then,*

$$\lim_{\epsilon\to 0+}\int_{-M}^{M}|\hat{\bar{\mathfrak{v}}}_\epsilon(t,k)-\hat{\bar{\mathfrak{v}}}^{(0)}_\epsilon(t,k)|^2\mathrm{d}k=0,\quad t\geqslant 0. \tag{4.17}$$

Proof. Note that $A_\epsilon(k)=A_0(k)+\epsilon B_\epsilon(k)$ (see (4.9)), where

$$b_*(M):=\sup_{\epsilon\in(0,1]}\sup_{|k|\leqslant M}\|B_\epsilon(k)\|<+\infty. \tag{4.18}$$

From (4.7) we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\hat{\mathfrak{v}}}_\epsilon(t,k)=\frac{1}{\epsilon^{\delta-1}}A_0(k)\hat{\hat{\mathfrak{v}}}_\epsilon(t,k)+\epsilon^{2-\delta}B_\epsilon(k)\hat{\hat{\mathfrak{v}}}_\epsilon(t,k), \tag{4.19}$$

Therefore, by Duhamel's formula, we conclude

$$\hat{\hat{\mathfrak{v}}}_\epsilon(t,k)-\hat{\hat{\mathfrak{v}}}^{(0)}_\epsilon(t,k)=\epsilon^{2-\delta}\int_0^t\exp\left\{\frac{t-s}{\epsilon^{\delta-1}}A_0(k)\right\}B_\epsilon(k)\hat{\hat{\mathfrak{v}}}_\epsilon(s,k)\mathrm{d}k. \tag{4.20}$$

The lemma then follows from (4.6), (4.18) and lemma 4.2. □

Lemma 4.4. *For any $t\geqslant 0$ and $\delta\in[1,2)$ we have*

$$\lim_{\epsilon\to0+}\int_{\epsilon^{-1}\mathbb{T}}\left|\hat{\hat{\mathfrak{v}}}_\epsilon(t,\ell)-\hat{v}\left(\frac{t}{\epsilon^{\delta-1}},\ell\right)\right|^2\mathrm{d}\ell=0 \tag{4.21}$$

and, in consequence,

$$\lim_{\epsilon\to0+}\epsilon\sum_x\left|\bar{\mathfrak{v}}_x\left(\frac{t}{\epsilon^\delta}\right)-v\left(\frac{t}{\epsilon^{\delta-1}},\epsilon x\right)\right|^2=0. \tag{4.22}$$

Proof. Formula (4.22) follows from (4.21) so we only focus on the proof of the latter. We have

$$\hat{\hat{\mathfrak{v}}}_\epsilon(t,\ell)=\exp\left\{A_\epsilon(\ell)\frac{t}{\epsilon^{\delta-1}}\right\}\hat{\hat{\mathfrak{v}}}_\epsilon(0,\ell).$$

Therefore, from the above and (4.15) we can write

$$\begin{aligned}&\limsup_{\epsilon\to0+}\int_{\epsilon^{-1}\mathbb{T}}\left|\hat{\hat{\mathfrak{v}}}_\epsilon(t,\ell)-\hat{v}\left(\frac{t}{\epsilon^{\delta-1}},\ell\right)\right|^2\mathrm{d}\ell\\ &\leqslant 2\limsup_{\epsilon\to0+}\int_{\epsilon^{-1}\mathbb{T}}\left|\exp\left\{A_\epsilon(\ell)\frac{t}{\epsilon^{\delta-1}}\right\}[\hat{\hat{\mathfrak{v}}}_\epsilon(0,\ell)-\hat{v}(0,\ell)]\right|^2\mathrm{d}\ell\\ &+2\limsup_{\epsilon\to0+}\int_{\epsilon^{-1}\mathbb{T}}\left|\exp\left\{A_\epsilon(\ell)\frac{t}{\epsilon^{\delta-1}}\right\}\hat{v}(0,\ell)-\exp\left\{A_0(\ell)\frac{t}{\epsilon^{\delta-1}}\right\}\hat{v}(0,\ell)\right|^2\mathrm{d}\ell.\end{aligned} \tag{4.23}$$

Using lemma 4.2 and (4.16) we conclude that the first term on the right-hand side vanishes. To estimate the second term, divide the domain of integration into the regions $|\ell|\leqslant M$ and its complement. On the first region we use lemma 4.3 to conclude that the respective limit vanishes. Therefore, we can write that with modulo multiplication by a factor of 2, the second term on the right- hand side of (4.23) is equal to

$$\begin{aligned}&\limsup_{\epsilon\to0+}\int_{M\leqslant|\ell|\leqslant1/(2\epsilon)}\left|\exp\left\{A_\epsilon(\ell)\frac{t}{\epsilon^{\delta-1}}\right\}\hat{v}(0,\ell)-\exp\left\{A_0(\ell)\frac{t}{\epsilon^{\delta-1}}\right\}\hat{v}(0,\ell)\right|^2\mathrm{d}\ell\\ &\leqslant 2\limsup_{\epsilon\to0+}\int_{M\leqslant|\ell|\leqslant1/(2\epsilon)}\left\{\left|\exp\left\{A_\epsilon(\ell)\frac{t}{\epsilon^{\delta-1}}\right\}\hat{v}(0,\ell)\right|^2+\left|\exp\left\{A_0(\ell)\frac{t}{\epsilon^{\delta-1}}\right\}\hat{v}(0,\ell)\right|^2\right\}\mathrm{d}\ell\\ &\leqslant 4(C_*+D_*)\limsup_{\epsilon\to0+}\int_{M\leqslant|\ell|\leqslant1/(2\epsilon)}|\hat{v}(0,\ell)|^2\mathrm{d}\ell,\end{aligned} \tag{4.24}$$

the last estimate following from lemma 4.2 and estimate (4.12). We can adjust $M > 0$ to become sufficiently large so that the utmost right-hand side of (4.24) can be arbitrarily small. The conclusion of the lemma therefore follows. □

Also, suppose that $(r(t, y), p(t, y))$ is the solution of the linear wave equation (3.6) and $\bar{e}_{\rm ph}(t, y)$ is the corresponding macroscopic phononic energy density profile, defined by (3.7). As a direct corollary from lemma 4.4 we conclude the following.

Corollary 4.5. *For any $t \geqslant 0$ we have*

$$\lim_{\epsilon\to 0+} \epsilon \sum_x \left[\bar{\mathfrak{r}}_{\epsilon,x}\left(\frac{t}{\epsilon}\right) - r(t, \epsilon x)\right]^2 = 0,$$
$$\lim_{\epsilon\to 0+} \epsilon \sum_x \left[\bar{\mathfrak{p}}_{\epsilon,x}\left(\frac{t}{\epsilon}\right) - p(t, \epsilon x)\right]^2 = 0. \tag{4.25}$$

Concerning the behavior of the energy functional $\bar{\mathfrak{e}}_x^{(\epsilon)}(t)$ corresponding to the average phononic ensemble (see (4.4)) we get the following.

Corollary 4.6. *For any $t \geqslant 0$ and $\delta \in [1, 2)$ we have*

$$\lim_{\epsilon\to 0+} \epsilon \sum_x \left|\bar{\mathfrak{e}}_x^{(\epsilon)}\left(\frac{t}{\epsilon^\delta}\right) - e_{\rm ph}\left(\frac{t}{\epsilon^{\delta-1}}, \epsilon x\right)\right| = 0, \tag{4.26}$$

where $e_{\rm ph}(t, y)$ is given by (3.7). In addition, for any $\delta \in (1, 2)$ we have

$$\lim_{\epsilon\to 0+} \epsilon \sum_x J(\epsilon x)\bar{\mathfrak{e}}_x^{(\epsilon)}\left(\frac{t}{\epsilon^\delta}\right) = 0, \quad J \in C_0^\infty(\mathbb{R}). \tag{4.27}$$

Proof. In light of lemma 4.4 only equality (4.27) requires a proof. Assume that $\delta \in (1, 2)$. Note that the left-hand side of (4.27) equals $\lim_{\epsilon\to 0+}[\mathcal{J}_1^{(\epsilon)}(t) + \mathcal{J}_2^{(\epsilon)}(t)]$, where

$$\begin{aligned}\mathcal{J}_1^{(\epsilon)}(t) &:= \frac{\epsilon}{2}\sum_x \bar{\mathfrak{p}}_{\epsilon,x}^2\left(\frac{t}{\epsilon^\delta}\right)J(\epsilon x)\\ &= \frac{1}{2}\int_{-1/(2\epsilon)}^{1/(2\epsilon)} \tilde{J}_\epsilon(\ell)\mathrm{d}\ell \int_{-1/(2\epsilon)}^{1/(2\epsilon)} \hat{\bar{\mathfrak{p}}}_\epsilon\left(\frac{t}{\epsilon^{\delta-1}}, -\ell - k\right)\hat{\bar{\mathfrak{p}}}_\epsilon\left(\frac{t}{\epsilon^{\delta-1}}, k\right)\mathrm{d}k\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}\mathcal{J}_2^{(\epsilon)}(t) &:= -\frac{\epsilon}{4}\sum_{x,x'} J(\epsilon x)\alpha_{x-x'}\bar{\mathfrak{q}}_{\epsilon,x}^2\left(\frac{t}{\epsilon^\delta}\right) = -\sum_z \frac{\alpha_z}{4}\int_{-1/(2\epsilon)}^{1/(2\epsilon)} \tilde{J}_\epsilon(\ell)\mathrm{d}\ell\\ &\times \int_{-1/(2\epsilon)}^{1/(2\epsilon)} h_z(-\epsilon(k+\ell))h_z(\epsilon k)\hat{\bar{\mathfrak{r}}}_\epsilon\left(\frac{t}{\epsilon^{\delta-1}}, -\ell - k\right)\hat{\bar{\mathfrak{r}}}_\epsilon\left(\frac{t}{\epsilon^{\delta-1}}, k\right)\mathrm{d}k,\end{aligned} \tag{4.29}$$

where

$$h_z(k) := \frac{\exp\{-2\pi \mathrm{i} kz\} - 1}{\exp\{-2\pi \mathrm{i} k\} - 1}, \quad z \in \mathbb{Z},\ k \in \mathbb{T}$$

and

$$\tilde{J}_\epsilon(\ell) := \sum_m \hat{J}\left(\ell + \frac{m}{\epsilon}\right).$$

Recall that $\hat{\hat{\mathfrak{v}}}_\epsilon(t,k)$ and $\hat{\hat{\mathfrak{v}}}^{(0)}_\epsilon(t,k)$ are given by (4.7) and (4.14) respectively. Using lemma 4.4, we conclude that

$$\lim_{\epsilon\to 0+} [\mathcal{J}_1^{(\epsilon)}(t) - \bar{\mathcal{J}}_1^{(\epsilon)}(t)] = 0 \quad \text{and} \quad \lim_{\epsilon\to 0+} [\mathcal{J}_2^{(\epsilon)}(t) - \bar{\mathcal{J}}_2^{(\epsilon)}(t)] = 0,$$

where

$$\bar{\mathcal{J}}_1^{(\epsilon)}(t) := \int_{\mathbb{R}} \hat{J}(\ell)\mathrm{d}\ell \int_{\mathbb{R}} \hat{p}\left(\frac{t}{\epsilon^{\delta-1}}, -\ell - k\right)\hat{p}\left(\frac{t}{\epsilon^{\delta-1}}, k\right)\mathrm{d}k \tag{4.30}$$

and

$$\mathcal{J}_2^{(\epsilon)}(t) := -\sum_z \frac{z^2\alpha_z}{4} \int_{\mathbb{R}} \hat{J}(\ell) \int_{\mathbb{R}} \hat{r}\left(\frac{t}{\epsilon^{\delta-1}}, -\ell - k\right)\hat{r}\left(\frac{t}{\epsilon^{\delta-1}}, k\right)\mathrm{d}\ell\mathrm{d}k. \tag{4.31}$$

Recall that

$$\hat{p}\left(\frac{t}{\epsilon^{\delta-1}}, k\right) = \left\langle \exp\left\{A_0(k)\frac{t}{\epsilon^{\delta-1}}\right\}\hat{v}(k), \mathrm{e}_2\right\rangle_{\mathbb{C}^2}$$

where $\mathrm{e}_2^T = [0,1]$ and $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$ is the scalar product on $\mathbb{C}^2$. Therefore,

$$\bar{\mathcal{J}}_1^{(\epsilon)}(t) = \int_{\mathrm{R}} \hat{J}(\ell)\mathrm{d}\ell \int_{\mathrm{R}} \left\langle \hat{v}(-k-l), \exp\left\{A_0^T(-k-\ell)\frac{t}{\epsilon^{\delta-1}}\right\}\mathrm{e}_2\right\rangle_{\mathbb{C}^2}$$
$$\left\langle \hat{v}(k), \exp\left\{A_0^T(k)\frac{t}{\epsilon^{\delta-1}}\right\}\mathrm{e}_2\right\rangle_{\mathbb{C}^2} \mathrm{d}k.$$

By an elementary application of the Riemann–Lebesgue theorem we get

$$\lim_{\epsilon\to 0+} \bar{\mathcal{J}}_j^{(\epsilon)}(t) = 0, \quad j = 1,2.$$

This ends the proof of (4.27). □

4.4. The evolution of conserved quantities at a hyperbolic scale

Our goal in this section is to prove the following result.

Theorem 4.7. *Suppose that (2.28) and (2.29) are in force. Then,*

$$\begin{aligned}
&\lim_{\epsilon\to 0+} \epsilon\sum_x \mathbb{E}_\epsilon\left[\mathfrak{r}_x\left(\frac{t}{\epsilon}\right) - r(t,\epsilon x)\right]^2 = 0,\\
&\lim_{\epsilon\to 0+} \epsilon\sum_x \mathbb{E}_\epsilon\left[\mathfrak{p}_x\left(\frac{t}{\epsilon}\right) - p(t,\epsilon x)\right]^2 = 0
\end{aligned} \tag{4.32}$$

and

$$\lim_{\epsilon\to 0+}\epsilon\sum_x\left|\mathbb{E}_\epsilon\mathfrak{e}_x\left(\frac{t}{\epsilon}\right)-e_{\rm ph}(t,\epsilon x)\right|=0,\qquad t\geqslant 0. \tag{4.33}$$

Proof. Equality (4.33) is a consequence of (4.32), and we need only substantiate the latter. Let $(\delta\mathfrak{r}_\epsilon(t),\delta\mathfrak{p}_\epsilon(t)):=(\delta\mathfrak{r}_{\epsilon,x}(t),\delta\mathfrak{p}_{\epsilon,x}(t))_{x\in\mathbb{Z}}$, where

$$\delta\mathfrak{r}_{\epsilon,x}(t):=\mathfrak{r}_{\epsilon,x}(t)-\bar{\mathfrak{r}}_{\epsilon,x}(t),\qquad \delta\mathfrak{p}_{\epsilon,x}(t):=\mathfrak{p}_{\epsilon,x}(t)-\bar{\mathfrak{p}}_{\epsilon,x}(t), \tag{4.34}$$

with $\bar{\mathfrak{r}}_{\epsilon,x}(t)$ and $\bar{\mathfrak{p}}_{\epsilon,x}(t)$ given by (4.3). We have (see (4.4))

$$\mathbb{E}_\epsilon\mathfrak{e}_x(\mathfrak{r}_\epsilon(t),\mathfrak{p}_\epsilon(t))=\mathbb{E}_\epsilon\mathfrak{e}_x(\delta\mathfrak{r}_\epsilon(t),\delta\mathfrak{p}_\epsilon(t))+\bar{\mathfrak{e}}_x^{(\epsilon)}(t). \tag{4.35}$$

By conservation of the total energy we obtain

$$\begin{aligned}
&\epsilon\sum_x\langle\mathfrak{e}_x(\mathfrak{r},\mathfrak{p})\rangle_{\mu_\epsilon}=\epsilon\sum_x\mathbb{E}_\epsilon\mathfrak{e}_x\left(\mathfrak{r}\left(\frac{t}{\epsilon}\right),\mathfrak{p}\left(\frac{t}{\epsilon}\right)\right)\\
&=\epsilon\sum_x\mathbb{E}_\epsilon\mathfrak{e}_x\left(\delta\mathfrak{r}_\epsilon\left(\frac{t}{\epsilon}\right),\delta\mathfrak{p}_\epsilon\left(\frac{t}{\epsilon}\right)\right)+\epsilon\sum_x\bar{\mathfrak{e}}_x^{(\epsilon)}\left(\frac{t}{\epsilon}\right)\\
&\geqslant\epsilon\sum_x\mathbb{E}_\epsilon\mathfrak{e}_x\left(\delta\mathfrak{r}_\epsilon\left(\frac{t}{\epsilon}\right),\delta\mathfrak{p}_\epsilon\left(\frac{t}{\epsilon}\right)\right)+\epsilon\sum_x\bar{e}_{\rm ph}(t,\epsilon x)-\epsilon\sum_x\left|\bar{\mathfrak{e}}_x^{(\epsilon)}\left(\frac{t}{\epsilon}\right)-\bar{e}_{\rm ph}(t,\epsilon x)\right|
\end{aligned} \tag{4.36}$$

Letting $\epsilon\to 0+$ and using assumptions (2.28) and (2.29) we conclude that the limit of the energy functional appearing on the utmost left-hand side equals

$$\int_{\mathbb{R}}e_{\rm ph}(0,y){\rm d}y\equiv\int_{\mathbb{R}}e_{\rm ph}(t,y){\rm d}y,\qquad t\in\mathbb{R}, \tag{4.37}$$

with $e_{\rm ph}(t,y)$ given by (3.7). Using the above, together with (4.26), we conclude that

$$\int_{\mathbb{R}}e_{\rm ph}(0,y){\rm d}y\geqslant\limsup_{\epsilon\to 0+}\epsilon\sum_x\mathbb{E}_\epsilon\mathfrak{e}_x\left(\delta\mathfrak{r}_\epsilon\left(\frac{t}{\epsilon}\right),\delta\mathfrak{p}_\epsilon\left(\frac{t}{\epsilon}\right)\right)+\int_{\mathbb{R}}e_{\rm ph}(t,y){\rm d}y.$$

From (4.37) we infer, therefore, that

$$\limsup_{\epsilon\to 0+}\epsilon\sum_x\mathbb{E}_\epsilon\mathfrak{e}_x\left(\delta\mathfrak{r}_\epsilon\left(\frac{t}{\epsilon}\right),\delta\mathfrak{p}_\epsilon\left(\frac{t}{\epsilon}\right)\right)=0. \tag{4.38}$$

From (2.16) applied to the configuration $(\delta\mathfrak{r}_\epsilon(t\epsilon^{-1}),\delta\mathfrak{p}_\epsilon(t\epsilon^{-1}))$ we conclude that there exists $c_*>0$ such that

$$\begin{aligned}
&c_-\epsilon\sum_x\mathbb{E}_\epsilon\left[\left(\delta\mathfrak{r}_{\epsilon,x}\left(\frac{t}{\epsilon}\right)\right)^2+\left(\delta\mathfrak{p}_{\epsilon,x}\left(\frac{t}{\epsilon}\right)\right)^2\right]\\
&\leqslant\epsilon\sum_x\mathbb{E}_\epsilon\mathfrak{e}_x\left(\delta\mathfrak{r}_\epsilon\left(\frac{t}{\epsilon}\right),\delta\mathfrak{p}_\epsilon\left(\frac{t}{\epsilon}\right)\right).
\end{aligned} \tag{4.39}$$

Equalities (4.32) then follow in a straightforward way from (4.39) and corollary 4.5. □

4.5. The evolution of energy density at the superdiffusive scale

In this section, we show that the energy density disperses to infinity at the time scale t/ϵ^{δ}, where $\delta \in (1,2)$.

Theorem 4.8. *Suppose that (2.28) and (2.29) are in force, and $\delta \in (1,2)$. Then,*

$$\lim_{\epsilon\to 0+} \epsilon \sum_x J(\epsilon x)\mathbb{E}_\epsilon \mathfrak{e}_x\left(\frac{t}{\epsilon^{\delta}}\right) = 0, \tag{4.40}$$

for any $J \in C_0^\infty(\mathbb{R})$, $t \geqslant 0$.

Proof. In analogy to (4.36) we write

$$\begin{aligned}
&\epsilon\sum_x \langle \mathfrak{e}_x(\mathfrak{r},\mathfrak{p})\rangle_{\mu_\epsilon} = \epsilon\sum_x \mathbb{E}_\epsilon \mathfrak{e}_x\left(\mathfrak{r}\left(\frac{t}{\epsilon^{\delta}}\right),\mathfrak{p}\left(\frac{t}{\epsilon^{\delta}}\right)\right)\\
&\geqslant \epsilon\sum_x \mathbb{E}_\epsilon \mathfrak{e}_x\left(\delta\mathfrak{r}_\epsilon\left(\frac{t}{\epsilon^{\delta}}\right),\delta\mathfrak{p}_\epsilon\left(\frac{t}{\varepsilon^{\delta}}\right)\right)\\
&+\epsilon\sum_x \bar{e}_{\rm ph}\left(\frac{t}{\epsilon^{\delta-1}},\epsilon x\right) - \epsilon\sum_x\left|\bar{\mathfrak{e}}_x^{(\epsilon)}\left(\frac{t}{\epsilon^{\delta}}\right) - \bar{e}_{\rm ph}\left(\frac{t}{\epsilon^{\delta-1}},\epsilon x\right)\right|.
\end{aligned} \tag{4.41}$$

Therefore, taking the limit as $\epsilon \to 0+$, in (4.41) we obtain (using (4.2) and (4.26))

$$\begin{aligned}
&\int_{\mathbb{R}} e_{\rm ph}(0,y){\rm d}y \geqslant \limsup_{\epsilon\to 0+} \epsilon\sum_x \mathbb{E}_\epsilon \mathfrak{e}_x\left(\delta\mathfrak{r}_\epsilon\left(\frac{t}{\epsilon^{\delta}}\right),\delta\mathfrak{p}_\epsilon\left(\frac{t}{\varepsilon^{\delta}}\right)\right)\\
&+\limsup_{\epsilon\to 0+}\int_{\mathbb{R}} e_{\rm ph}\left(\frac{t}{\epsilon^{\delta-1}},y\right){\rm d}y.
\end{aligned} \tag{4.42}$$

By (4.37) we conclude

$$\limsup_{\epsilon\to 0+} \epsilon\sum_x \mathbb{E}_\epsilon \mathfrak{e}_x\left(\delta\mathfrak{r}_\epsilon\left(\frac{t}{\epsilon^{\delta}}\right),\delta\mathfrak{p}_\epsilon\left(\frac{t}{\varepsilon^{\delta}}\right)\right) = 0. \tag{4.43}$$

Since for any $J \in C_0^\infty(\mathbb{R})$ we have

$$\begin{aligned}
&\epsilon\sum_x J(\epsilon x)\mathbb{E}_\epsilon \mathfrak{e}_x\left(\mathfrak{r}\left(\frac{t}{\epsilon^{\delta}}\right),\mathfrak{p}\left(\frac{t}{\epsilon^{\delta}}\right)\right)\\
&= \epsilon\sum_x J(\epsilon x)\mathbb{E}_\epsilon \mathfrak{e}_x\left(\delta\mathfrak{r}_\epsilon\left(\frac{t}{\epsilon^{\delta}}\right),\delta\mathfrak{p}_\epsilon\left(\frac{t}{\epsilon^{\delta}}\right)\right) + \epsilon\sum_x J(\epsilon x)\bar{\mathfrak{e}}_x^{(\epsilon)}\left(\frac{t}{\epsilon^{\delta}}\right).
\end{aligned} \tag{4.44}$$

The conclusion of the theorem follows directly from (4.43) and (4.27). □

5. The evolution of a thermal ensemble

In this section we investigate the limit of the dynamics of ensemble families whose initial laws $(\mu_\epsilon)_{\epsilon>0}$ satisfy condition (2.26). Then, as it turns out, the respective macroscopic profile of $(\mathfrak{r},\mathfrak{p})$ is trivial, i.e. $r(y) \equiv 0$ and $p(y) \equiv 0$. In fact, we show that condition (2.26) persists in time and the temperature profile remains stationary at the hyperbolic time scale.

5.1. Wave function

Define the wave function of a given configuration as

$$\psi_x := (\tilde{\omega} * \mathfrak{q})_x + \mathrm{i}\mathfrak{p}_x = \sum_{x'} \tilde{\omega}_{x-x'}\mathfrak{q}_{0,x'} + \mathrm{i}\mathfrak{p}_x, \quad x \in \mathbb{Z}, \tag{5.1}$$

where $\mathfrak{q}_{0,x'}$ is given by (2.9) and $(\tilde{\omega}_x)$ are the Fourier coefficients of the dispersion relation defined as, see (2.2),

$$\omega(k) := \hat{\alpha}^{1/2}(k) = 2\sqrt{\pi_1}\,|\mathfrak{s}(k)|\varphi(\mathfrak{s}^2(k)). \tag{5.2}$$

Using (5.2) we can rewrite (5.1) in the form

$$\psi_x = (\tilde{\omega}^{(1)} * \mathfrak{r})_x + \mathrm{i}\mathfrak{p}_x, \quad x \in \mathbb{Z},$$

with $(\tilde{\omega}^{(1)}_x)$ being the Fourier coefficients of

$$\omega^{(1)}(k) = -\mathrm{i}e^{\pi \mathrm{i}k}\mathrm{sign}(k)\sqrt{\pi_1}\,\varphi(\mathfrak{s}^2(k)), \quad k \in \mathbb{T}. \tag{5.3}$$

The Fourier transform $\hat{\psi}(k)$ of the wave function (ψ_x) can be written as

$$\hat{\psi}(k) = \omega^{(1)}(k)\hat{\mathfrak{r}}(k) + \mathrm{i}\hat{\mathfrak{p}}(k). \tag{5.4}$$

Therefore,

$$\langle|\hat{\psi}(k)|^2\rangle_{\mu_\epsilon} = \langle|\omega^{(1)}(k)\hat{\mathfrak{r}}(k)|^2 + |\hat{\mathfrak{p}}(k)|^2\rangle_{\mu_\epsilon} + 2\langle \mathrm{Im}(\hat{\mathfrak{r}}(k)\omega^{(1)}(k)\hat{\mathfrak{p}}^*(k))\rangle_{\mu_\epsilon}. \tag{5.5}$$

On the other hand, since $\hat{\mathfrak{r}}(-k) = \hat{\mathfrak{r}}^*(k)$ and $\hat{\mathfrak{p}}(-k) = \hat{\mathfrak{p}}^*(k)$ we obtain

$$\langle|\hat{\psi}(-k)|^2\rangle_{\mu_\epsilon} = \langle|\omega^{(1)}(k)\hat{\mathfrak{r}}(k)|^2 + |\hat{\mathfrak{p}}(k)|^2\rangle_{\mu_\epsilon} - 2\langle \mathrm{Im}(\hat{\mathfrak{r}}(k)\omega^{(1)}(k)\hat{\mathfrak{p}}^*(k))\rangle_{\mu_\epsilon}. \tag{5.6}$$

From (5.5) and (5.6) we obtain

$$\int_{\mathbb{T}} \mathfrak{w}^2_\epsilon(k)\mathrm{d}k = \int_{\mathbb{T}} \langle|\hat{\psi}(k)|^2\rangle^2_{\mu_\epsilon}\mathrm{d}k. \tag{5.7}$$

Condition (2.26) is therefore equivalent to

$$\sup_{\epsilon\in(0,1]} \epsilon^2 \int_{\mathbb{T}} \langle|\hat{\psi}(k)|^2\rangle^2_{\mu_\epsilon}\mathrm{d}k < +\infty. \tag{5.8}$$

Remark. Note that the macroscopic profile $(r(y), p(y))$ corresponding to a given configuration $(\mathfrak{r}, \mathfrak{p})$ satisfying (2.26) necessarily vanishes, i.e.

$$\lim_{\epsilon\to 0+} \epsilon\sum_x \langle \mathfrak{r}_x\rangle_{\mu_\epsilon} J(\epsilon x) = 0 \quad \text{and} \quad \lim_{\epsilon\to 0+} \epsilon\sum_x \langle \mathfrak{p}_x\rangle_{\mu_\epsilon} J(\epsilon x) = 0 \tag{5.9}$$

for any $J \in C_0^\infty(\mathbb{R})$. To show (5.9) it suffices to prove that

$$\lim_{\epsilon\to 0+} \epsilon\sum_x \langle \psi_x\rangle_{\mu_\epsilon} J(\epsilon x) = 0. \tag{5.10}$$

Indeed, from conditions (2.28) and (2.29), we obtain

$$\lim_{\epsilon\to 0+} \epsilon\sum_x \langle \psi_x\rangle_{\mu_\epsilon} J(\epsilon x) = \int_{\mathbb{R}} \psi(y)J(y)\mathrm{d}y, \tag{5.11}$$

where

$$\psi(y)=\sqrt{\pi_1}\mathcal{H}(r)(y)+\mathrm{i}p(y)$$

and

$$\mathcal{H}(f)(y):=-\frac{1}{\pi}\mathrm{p.v.}\int_{\mathbb{R}}\frac{f(y')\mathrm{d}y'}{y-y'}$$

is the Hilbert transform of a given function $f\in C_0^\infty(\mathbb{R})$. The integral on the right-hand side is understood in the sense of the principal value. In particular, $\psi(y)\equiv 0$ implies that both $p(y)\equiv 0$ and $r(y)\equiv 0$.

To show (5.10), observe that by the Plancherel identity we can write that the absolute value of the expression under the limit in (5.10) equals

$$\epsilon\left|\int_{\mathbb{T}}\langle\hat{\psi}(k)\rangle_{\mu_\epsilon}\hat{J}_\epsilon(k)\mathrm{d}k\right|, \tag{5.12}$$

where

$$\hat{J}_\epsilon(k):=\sum_x J(\epsilon x)\exp\{-2\pi\mathrm{i}kx\}\approx\frac{1}{\epsilon}\hat{J}\left(\frac{k}{\epsilon}\right)$$

and $\hat{J}(k)$ is the Fourier transform of $J(x)$. The expression in (5.12) is therefore estimated as follows:

$$\left|\int_{\mathbb{T}}\langle\hat{\psi}(k)\rangle_{\mu_\epsilon}\hat{J}\left(\frac{k}{\epsilon}\right)\mathrm{d}k\right|\leqslant\left[\int_{\mathbb{T}}\langle|\hat{\psi}(k)|^2\rangle_{\mu_\epsilon}^2\mathrm{d}k\right]^{1/4}\left[\int_{\mathbb{T}}\left|\hat{J}\left(\frac{k}{\epsilon}\right)\right|^{4/3}\mathrm{d}k\right]^{3/4},$$

where the estimate follows by the Hölder inequality. Using the change of variables $k':=k/\epsilon$ in the second integral on the right-hand side we conclude that it is bounded by $\epsilon K_1^{1/4}\|\hat{J}\|_{L^{4/3}(\mathbb{R})}$ for $\epsilon\in(0,1]$, which proves (5.10).

5.2. The evolution of the energy functional at the hyperbolic time scale

Define by $\mathfrak{w}_\epsilon(t,k)$ the energy spectrum (2.23) corresponding to the configuration $(\mathfrak{r}(t),\mathfrak{p}(t))$. The following result asserts that there is no evolution of the macroscopic temperature profile at the hyperbolic scale. Moreover, if the initial energy distribution is of the thermal type, it remains so at this time scale.

Theorem 5.1. *For any $J\in C_0^\infty([0,+\infty)\times\mathbb{R})$ we have*

$$\lim_{\epsilon\to 0+}\epsilon\sum_x\int_0^{+\infty}J(t,\epsilon x)\mathbb{E}_\epsilon\mathfrak{e}_x\left(\frac{t}{\epsilon}\right)\mathrm{d}t=\int_0^{+\infty}\int_{\mathbb{R}}J(t,y)T(y)\mathrm{d}t\mathrm{d}y. \tag{5.13}$$

Moreover, at any time $t>0$ the energy spectrum satisfies

$$K_1(t):=\sup_{\epsilon\in(0,1]}\epsilon^2\int_{\mathbb{T}}\mathfrak{w}_\epsilon^2\left(\frac{t}{\epsilon},k\right)\mathrm{d}k<+\infty \tag{5.14}$$

and

$$\lim_{\epsilon\to 0+}\epsilon\sum_x \mathbb{E}_\epsilon \mathfrak{r}\left(\frac{t}{\epsilon}\right)J(\epsilon x)=0 \quad \text{and} \quad \lim_{\epsilon\to 0+}\epsilon\sum_x \mathbb{E}_\epsilon \mathfrak{p}\left(\frac{t}{\epsilon}\right)J(\epsilon x)=0 \tag{5.15}$$

for any $J\in C_0^\infty(\mathbb{R})$*.*

The proof of this result is given in section 5.5.

5.3. Wigner transform

Define $\psi_x^{(\epsilon)}(t):=\psi_x(t\epsilon^{-1})$ and $\hat{\psi}^{(\epsilon)}(t,k)$ its Fourier transform (belonging to $L^2(\mathbb{T})$). By $W_\epsilon(t)$ we denote the (averaged) Wigner transform of $\psi^{(\epsilon)}(t)$, see (5.9) of [4], given by

$$\langle W_\epsilon(t),J\rangle := \int_{\mathbb{R}\times\mathbb{T}} \widehat{W}_\epsilon(t,p,k)\hat{J}^*(p,k)\mathrm{d}p\mathrm{d}k, \tag{5.16}$$

where $\widehat{W}_\epsilon(t,p,k)$ the Fourier–Wigner transform of $\hat{\psi}^{(\epsilon)}(t)$ given by

$$\widehat{W}_\epsilon(t,p,k) := \frac{\epsilon}{2}\mathbb{E}_\epsilon\left[(\hat{\psi}^{(\epsilon)})^*\left(t,k-\frac{\epsilon p}{2}\right)\hat{\psi}^{(\epsilon)}\left(t,k+\frac{\epsilon p}{2}\right)\right] \tag{5.17}$$

and J belongs to $\mathcal{S}$—the set of functions on $\mathbb{R}\times\mathbb{T}$ that are of C^∞ class and such that for any integers l, m, n we have

$$\sup_{y\in\mathbb{R},\,k\in\mathbb{T}} (1+y^2)^n|\partial_y^l\partial_k^m J(y,k)|<+\infty.$$

Let $\mathcal{A}$ be the completion of $\mathcal{S}$ under the norm

$$\|J\|_{\mathcal{A}} := \int_{\mathbb{R}} \sup_k |\hat{J}(p,k)|\mathrm{d}p. \tag{5.18}$$

In what follows, we shall also consider the Fourier–Wigner anti-transform of $\hat{\psi}^{(\epsilon)}(t)$ given by

$$\widehat{Y}_\epsilon(t,p,k) := \frac{\epsilon}{2}\mathbb{E}_\epsilon\left[\hat{\psi}^{(\epsilon)}\left(t,-k+\frac{\epsilon p}{2}\right)\hat{\psi}^{(\epsilon)}\left(t,k+\frac{\epsilon p}{2}\right)\right] \tag{5.19}$$

From the Cauchy–Schwartz inequality we get

$$|\langle W_\epsilon(t),J\rangle| \leqslant \frac{\epsilon}{2}\|J\|_{\mathcal{A}}\,\mathbb{E}_\epsilon\|\hat{\psi}^{(\epsilon)}(t)\|^2_{L^2(\mathbb{T})}.$$

Thanks to the energy conservation property of the dynamics and the Cauchy–Schwartz inequality, we get

$$\sup_{\epsilon\in(0,1]}\sup_{t\geqslant 0}(\|Y_\epsilon(t)\|_{\mathcal{A}'}+\|W_\epsilon(t)\|_{\mathcal{A}'})\leqslant 2K_0, \tag{5.20}$$

where K_0 is the constant appearing in condition (2.25). As a direct consequence of the above estimate, we infer that the family $(W_\epsilon(\cdot))_{\epsilon\in(0,1]}$ is $*$-weakly sequentially compact as $\epsilon\to 0+$ in any dual to $L^1([0,T];\mathcal{A})$, where $T>0$, i.e. for any $\epsilon_n\to 0+$ one can choose a subsequence $\epsilon_{n'}$ such that $(W_{\epsilon_{n'}}(\cdot))_{n'}$ that is $*$-weakly converging. In fact, using hypothesis (2.26) one can prove that the following estimate holds (see proposition 9.1 of [4]).

Proposition 5.2. *For any $M > 0$ there exists $C_1 > 0$ such that*

$$
\begin{aligned}
&\|\widehat{W}_\epsilon(t,p,\cdot)\|^2_{L^2(\mathbb{T})} + \|\widehat{Y}_\epsilon(t,p,\cdot)\|^2_{L^2(\mathbb{T})} \\
&\leqslant (\|\widehat{W}_\epsilon(0,p,\cdot)\|^2_{L^2(\mathbb{T})} + \|\widehat{Y}_\epsilon(0,p,\cdot)\|^2_{L^2(\mathbb{T})})\mathrm{e}^{C_1\epsilon t}, \quad \forall\ \epsilon\in(0,1],\ |p|\leqslant M,\ t\geqslant 0.
\end{aligned}
\tag{5.21}
$$

The right-hand side of (5.21) remains bounded for $\epsilon\in(0,1]$ thanks to (5.8).

In fact, asymptotically as $\epsilon\to 0+$, the function $|\psi_x^{(\epsilon)}(t)|^2$ coincides with the energy density $\mathfrak{e}_x(t\epsilon^{-1})$, which can be concluded from the following result, see proposition 5.3 of [4].

Proposition 5.3. *Suppose that condition (5.8) holds. Then,*

$$
\lim_{\epsilon\to 0+}\epsilon\sum_x J(\epsilon x)\mathbb{E}_\epsilon\left[\mathfrak{e}_x\left(\frac{t}{\epsilon}\right)-\frac{1}{2}\left|\psi_x^{(\epsilon)}(t)\right|^2\right]=0,\quad t\geqslant 0,\ J\in C_0^\infty(\mathbb{R}). \tag{5.22}
$$

From the above result we conclude that for a real-valued function $J(y,k)\equiv J(y)$ we can write

$$
\begin{aligned}
\lim_{\epsilon\to 0+}\langle W_\epsilon(t),J\rangle &= \lim_{\epsilon\to 0+}\frac{\epsilon}{2}\sum_x J(\epsilon x)|\psi_x^{(\epsilon)}(t)|^2 \\
&= \lim_{\epsilon\to 0+}\epsilon\sum_x J(\epsilon x)\mathbb{E}_\epsilon\mathfrak{e}_x\left(\frac{t}{\epsilon}\right).
\end{aligned}
\tag{5.23}
$$

5.4. Homogenization of the Wigner transform in the mode frequency domain

It turns out that in the limit, as $\epsilon\to 0+$, the Wigner transform $W_\epsilon(t,x,k)$ becomes independent of the k variable for any $t>0$. To avoid boundary layer considerations at the initial time $t=0$ we formulate this property for the Laplace–Fourier transform of the respective Wigner function. More precisely, let

$$
\bar{w}_{\epsilon,\pm}(\lambda,p,k) := \int_0^{+\infty}\mathrm{e}^{-\lambda t}\widehat{W}_\epsilon(t,p,\pm k)\mathrm{d}t, \tag{5.24}
$$

$$
\bar{u}_{\epsilon,\pm}(\lambda,p,k) := \int_0^{+\infty}\mathrm{e}^{-\lambda t}\widehat{U}_{\epsilon,\pm}(t,p,k)\mathrm{d}t,
$$

where

$$
\begin{aligned}
\widehat{U}_{\epsilon,+}(t,p,k) &:= \frac{1}{2}\left[\widehat{Y}_\epsilon(t,p,k)+\widehat{Y}^*_\epsilon(t,-p,k)\right], \\
\widehat{U}_{\epsilon,-}(t,p,k) &:= \frac{1}{2\mathrm{i}}\left[\widehat{Y}_\epsilon(t,p,k)-\widehat{Y}^*_\epsilon(t,-p,k)\right].
\end{aligned}
$$

Thanks to proposition 5.2 the Laplace transform is defined for any $\lambda>0$ and for any $M>0$, compact interval $I\subset(0,+\infty)$. In addition, we have

$$
C_I := \sup_{\epsilon\in(0,1]}\sup_{\lambda\in I,|p|\leqslant M}\left(\|\bar{w}_{\epsilon,\iota}(\lambda,p)\|_{L^2(\mathbb{T})}+\sum_{\iota\in\{-,+\}}\|\bar{u}_{\epsilon,\iota}(\lambda,p)\|_{L^2(\mathbb{T})}\right)<+\infty. \tag{5.25}
$$

Let

$$w_\epsilon^{(\pm)}(\lambda, p) := \langle \bar{w}_{\epsilon,+}(\lambda), \mathfrak{e}_\pm \rangle_{L^2(\mathbb{T})} = \langle \bar{w}_{\epsilon,-}(\lambda), \mathfrak{e}_\pm \rangle_{L^2(\mathbb{T})},$$

where

$$\mathfrak{e}_+(k) := \frac{8}{3}\mathfrak{s}^4(k), \quad \mathfrak{e}_-(k) := 2\mathfrak{s}^2(2k). \tag{5.26}$$

The following result holds, see theorem 10.2 of [4].

Theorem 5.4. *Suppose that the initial laws satisfy* (2.26)*. Then, for any* $M > 0$ *and a compact interval* $I \subset (\lambda_0, +\infty)$ *we have*

$$\lim_{\epsilon \to 0+} \sup_{\lambda \in I, |p| \leqslant M} \int_{\mathbb{T}} \left| \bar{w}_{\epsilon,+}(\lambda, p, k) - w_\epsilon^{(\pm)}(\lambda, p) \right| \mathrm{d}k = 0, \tag{5.27}$$

and

$$\lim_{\epsilon \to 0+} \sup_{\lambda \in I, |p| \leqslant M} \int_{\mathbb{T}} |\bar{u}_{\epsilon,\pm}(\lambda, p, k)| \mathrm{d}k = 0. \tag{5.28}$$

5.5. Proof of theorem 5.1

Applying (5.21) at $p = 0$ we conclude that

$$\sup_{\epsilon \in (0,1]} \epsilon^2 \int_{\mathbb{T}} \left(\mathbb{E}_\epsilon |\hat{\psi}^{(\epsilon)}(t, k)|^2 \right)^2 \mathrm{d}k < +\infty. \tag{5.29}$$

Condition (5.14) then follows directly from (5.7). In addition, equalities (5.15) can be inferred from (5.29) and (5.9).

Concerning the proof of the convergence of the energy functional in (5.13), recall that we already know that $(W_\epsilon(\cdot))$ is $*$-weakly sequentially compact in $(L^1([0, +\infty), \mathcal{A}))^*$. Therefore, for any $\epsilon_n \to 0$, as $n \to +\infty$, we can choose a subsequence, denoted in the same way, such that $*$-weakly converging to some $W \in (L^1([0, +\infty), \mathcal{A}))^*$.

In light of (5.20) we have

$$\sup_{t \geqslant 0} \|W_\epsilon(t)\|_{\mathcal{A}'} \leqslant K_0, \tag{5.30}$$

with K_0 the same as in (2.25). Therefore, the respective Laplace–Fourier transform $\bar{w}_\epsilon(\lambda, p, k)$ can be defined for any $\lambda > 0$. To identify W, it suffices to identify the $*$-weak limit in $\mathcal{A}'$ of the Laplace transforms $\bar{w}_{\epsilon_n}(\lambda)$, as $n \to +\infty$. Thanks to theorem 5.4, any limit $w(\lambda, p, k) \equiv w(\lambda, p)$ obtained in this way will be constant in k. In fact

$$w(\lambda, p) = \lim_{n \to +\infty} w_{\epsilon_n}^{(\pm)}(\lambda, p).$$

In fact, we claim that for any $J \in C_0(\mathbb{R})$

$$\langle w(\lambda), J \rangle = \int_{\mathbb{R}} J(y) T(y) \mathrm{d}y, \tag{5.31}$$

which implies that $W \in L^\infty([0, +\infty), \mathcal{A}')$ and

$$W(t, y, k) \equiv T(y), \quad t \geqslant 0.$$

In light of proposition 5.3, this allows us to claim (5.13). The only thing yet to be shown, therefore, is (5.31).

5.5.1. Proof of (5.31). Letting $\widehat{W}^{(0)}_{\epsilon,\pm}=\widehat{W}_{\epsilon,\pm}(0)$ and $\widehat{U}^{(0)}_{\epsilon,\pm}=\widehat{U}_{\epsilon,\pm}(0)$ we obtain that for any $\lambda>0$ (see (10.5) of [4])

$$\begin{aligned}
\lambda\bar{w}_{\epsilon,+}-\widehat{W}^{(0)}_{\epsilon,+}&=-\mathrm{i}\delta_\epsilon\omega\bar{w}_{\epsilon,+}-\mathrm{i}\gamma R'p\bar{u}_{\epsilon,-}+\frac{\gamma}{\epsilon}\mathcal{L}(\bar{w}_{\epsilon,+}-\bar{u}_{\epsilon,+})+\epsilon\bar{r}^{(1)}_\epsilon,\\
\lambda\bar{u}_{\epsilon,+}-\widehat{U}^{(0)}_{\epsilon,+}&=\frac{2\bar{\omega}}{\epsilon}\bar{u}_{\epsilon,-}+\frac{\gamma}{\epsilon}\mathcal{L}\left[\bar{u}_{\epsilon,+}-\frac{1}{2}(\bar{w}_{\epsilon,+}+\bar{w}_{\epsilon,-})\right]+\epsilon\bar{r}^{(2)}_\epsilon,\\
\lambda\bar{u}_{\epsilon,-}-\widehat{U}^{(0)}_{\epsilon,-}&=-\frac{2\bar{\omega}}{\epsilon}\bar{u}_{\epsilon,+}-\frac{2\gamma}{\epsilon}R\bar{u}_{\epsilon,-}-\frac{\mathrm{i}\gamma R'p}{2}(\bar{w}_{\epsilon,-}-\bar{w}_{\epsilon,+})+\epsilon\bar{r}^{(3)}_\epsilon\\
\lambda\bar{w}_{\epsilon,-}-\widehat{W}^{(0)}_{\epsilon,-}&=\mathrm{i}\delta_\epsilon\omega\bar{w}_{\epsilon,-}+\mathrm{i}\gamma R'p\bar{u}_{\epsilon,-}+\frac{\gamma}{\epsilon}\mathcal{L}(\bar{w}_{\epsilon,-}-\bar{u}_{\epsilon,+})+\epsilon\bar{r}^{(4)}_\epsilon,
\end{aligned}\tag{5.32}$$

where

$$\begin{aligned}
\delta_\epsilon\omega(k,p)&:=\frac{1}{\epsilon}\left[\omega\left(k+\frac{\epsilon p}{2}\right)-\omega\left(k-\frac{\epsilon p}{2}\right)\right],\\
\bar{\omega}(k,p)&:=\frac{1}{2}\left[\omega\left(k+\frac{p}{2}\right)+\omega\left(k-\frac{p}{2}\right)\right],\\
\mathcal{L}w(k)&:=2\int_{\mathbb{T}}R(k,k')w(k')\mathrm{d}k'-2R(k)w(k),\quad w\in L^1(\mathbb{T}),
\end{aligned}\tag{5.33}$$

and

$$\begin{aligned}
R(k,k')&:=\frac{3}{4}\sum_{\iota\in\{-,+\}}\mathfrak{e}_\iota(k)\mathfrak{e}_{-\iota}(k'),\\
R(k)&:=\int_{\mathbb{T}}R(k,k')\mathrm{d}k',
\end{aligned}\tag{5.34}$$

with $\mathfrak{e}_\pm(k)$ given by (5.26). The remainder terms satisfy

$$\sup_{\epsilon\in(0,1]}\sup_{\lambda\in I,|p|\leqslant M}\sum_{j=1}^4\|\bar{r}^{(j)}_\epsilon(\lambda,p)\|_{L^2(\mathbb{T})}<+\infty\tag{5.35}$$

for any $M>0$, compact interval $I\subset(0,+\infty)$.

From the first equation of the system (5.32) we get

$$D^{(\epsilon)}\bar{w}_\epsilon=\epsilon\widehat{W}^{(0)}_\epsilon+\frac{3}{2}\gamma\sum_{\iota\in\{-,+\}}\mathfrak{e}_\iota w^{(-\iota)}_\epsilon+q_\epsilon,\tag{5.36}$$

where

$$D^{(\epsilon)}:=\epsilon\lambda+2\gamma R+\mathrm{i}\epsilon\delta_\epsilon\omega,\tag{5.37}$$

and $q_\epsilon:=\sum_{i=1}^3 q^{(i)}_\epsilon$, with

$$q^{(1)}_\epsilon:=\epsilon^2\bar{r}^{(1)}_\epsilon,\quad q^{(2)}_\epsilon:=-\gamma\mathcal{L}\bar{u}_{\epsilon,+},\quad q^{(3)}_\epsilon:=-\mathrm{i}\epsilon\gamma R'p\bar{u}_{\epsilon,-}.\tag{5.38}$$

Computing $\bar{w}_\epsilon$ from (5.36) and then multiplying both sides of the resulting equation scalarly by $\gamma\mathfrak{e}_\iota$, $\iota\in\{-,+\}$ we get the following system:

$$\gamma w_\epsilon^{(\iota)}\int_{\mathbb{T}}\left(1-\frac{3\gamma\mathfrak{e}_-\mathfrak{e}_+}{2D^{(\epsilon)}}\right)\mathrm{d}k-\frac{3\gamma^2}{2}w_\epsilon^{(-\iota)}\int_{\mathbb{T}}\frac{\mathfrak{e}_\iota^2}{D^{(\epsilon)}}\mathrm{d}k$$
$$=\gamma\epsilon\int_{\mathbb{T}}\frac{\mathfrak{e}_\iota\widehat{W}_\epsilon^{(0)}}{D^{(\epsilon)}}\mathrm{d}k+\gamma\int_{\mathbb{T}}\frac{\mathfrak{e}_\iota q_\epsilon}{D^{(\epsilon)}}\mathrm{d}k,\quad \iota\in\{-,+\}.$$

Adding the above equations corresponding to both values of ι sideways and then dividing both sides of the resulting equation by ε we obtain

$$a_w^{(\epsilon)}w_\epsilon^{(+)}-a_+^{(\epsilon)}(w_\epsilon^{(+)}-w_\epsilon^{(-)})=\frac{4\gamma}{3}\int_{\mathbb{T}}\frac{R\widehat{W}_\epsilon^{(0)}}{D^{(\epsilon)}}\mathrm{d}k+\frac{4\gamma}{3\epsilon}\int_{\mathbb{T}}\frac{Rq_\epsilon}{D^{(\epsilon)}}\mathrm{d}k, \tag{5.39}$$

where

$$a_w^{(\epsilon)}(\lambda,p):=\frac{4\gamma}{3}\int_{\mathbb{T}}\frac{\lambda+\mathrm{i}\delta_\epsilon\omega}{D^{(\epsilon)}}R\,\mathrm{d}k$$
$$a_+^{(\epsilon)}(\lambda,p):=\gamma\int_{\mathbb{T}}\frac{\lambda+\mathrm{i}\delta_\epsilon\omega}{D^{(\epsilon)}}\mathfrak{e}_+\,\mathrm{d}k. \tag{5.40}$$

Equality (5.31) is then a consequence of the following.

Proposition 5.5. *For any $J\in\mathcal{S}$ such that $J(y,k)\equiv J(y)$ and $\lambda>\lambda_0$ we have*

$$\lim_{\epsilon\to0+}\left(\int_{\mathbb{R}\times\mathbb{T}}\frac{2\gamma R}{D^{(\epsilon)}}\widehat{W}_\epsilon^{(0)}\hat{J}\mathrm{d}p\mathrm{d}k-\int_{\mathbb{R}\times\mathbb{T}}\hat{T}\hat{J}\mathrm{d}p\mathrm{d}k\right)=0, \tag{5.41}$$

In addition, for any $M>0$ and a compact interval $I\subset(0,+\infty)$

$$\lim_{\epsilon\to0+}\sup_{\lambda\in I,|p|\leqslant M}\left|a_+^{(\epsilon)}(\lambda,p)(w_\epsilon^{(+)}(\lambda,p)-w_\epsilon^{(-)}(\lambda,p))\right|=0, \tag{5.42}$$

$$\lim_{\epsilon\to0+}\sup_{\lambda\in I,|p|\leqslant M}\left|a_w^{(\epsilon)}(\lambda,p)-\frac{2\lambda}{3}\right|=0, \tag{5.43}$$

and

$$\lim_{\epsilon\to0+}\sup_{\lambda\in I,|p|\leqslant M}\left|\frac{4\gamma}{3\epsilon}\int_{\mathbb{T}}\frac{Rq_\epsilon}{D^{(\epsilon)}}\mathrm{d}k\right|=0. \tag{5.44}$$

Proof. Equality (5.41) follows from (2.27) and (5.23). Formulas (5.42) and (5.43) can be easily substantiated using the Lebesgue-dominated convergence theorem. To argue (5.44) it suffices only to prove that

$$\lim_{\epsilon\to0+}\sup_{\lambda\in I,|p|\leqslant M}\left|\frac{4\gamma}{3\epsilon}\int_{\mathbb{T}}\frac{Rq_\epsilon^{(j)}}{D^{(\epsilon)}}\mathrm{d}k\right|=0,\quad j=2,3. \tag{5.45}$$

Note that

$$\left|\frac{4\gamma}{3\epsilon}\int_{\mathbb{T}}\frac{Rq_\epsilon^{(3)}}{D^{(\epsilon)}}\mathrm{d}k\right|\leqslant\frac{4\gamma^2|p|\|R'\|_\infty}{3}\int_{\mathbb{T}}\frac{R|\bar{u}_{\epsilon,-}|}{|D^{(\epsilon)}|}\mathrm{d}k\leqslant C\int_{\mathbb{T}}|\bar{u}_{\epsilon,-}|\mathrm{d}k$$

and (5.45) for $j = 3$ follows from (5.28).

In the case $j = 2$, note that $\int_{\mathbb{T}} q_\epsilon^{(2)} \mathrm{d}k = 0$, therefore

$$\frac{4\gamma}{3\epsilon}\int_{\mathbb{T}} \frac{Rq_\epsilon^{(2)}}{D^{(\epsilon)}}\mathrm{d}k = \frac{2}{3}\int_{\mathbb{T}} \frac{(\lambda + \mathrm{i}\delta_\epsilon\omega)q_\epsilon^{(2)}}{D^{(\epsilon)}}\mathrm{d}k.$$

Since $k \mapsto \delta_\epsilon\omega(k,p)$ is odd for each p we obtain a latter integral equaling

$$\begin{aligned}
I_\epsilon &:= -\frac{2\gamma}{3}\int_{\mathrm{T}} \frac{\lambda(\epsilon\lambda + 2\gamma R) + \epsilon(\delta_\epsilon\omega)^2}{|D^{(\epsilon)}|^2}\mathcal{L}\bar{u}_{\epsilon,+}\mathrm{d}k \\
&= -\frac{\gamma}{2}\sum_{\iota\in\{-,+\}} u_{\epsilon,+}^{-\iota}\int_{\mathrm{T}} \frac{\mathfrak{e}_\iota}{|D^{(\epsilon)}|^2}\left[\lambda(\epsilon\lambda + 2\gamma R_\epsilon) + \epsilon(\delta_\epsilon\omega)^2\right]\mathrm{d}k \\
&+\frac{4\gamma}{3}\int_{\mathrm{T}} \frac{R\bar{u}_{\epsilon,+}}{|D^{(\epsilon)}|^2}\left[\lambda(\epsilon\lambda + 2\gamma R_\epsilon) + \epsilon(\delta_\epsilon\omega)^2\right]\mathrm{d}k, \qquad (5.46)
\end{aligned}$$

with

$$u_{\epsilon,\iota}^{(\iota')}(\lambda,p) := \langle \bar{u}_{\epsilon,\iota}(\lambda), \mathfrak{e}_{\iota'}\rangle_{L^2(\mathbb{T})}, \qquad \iota,\iota' \in \{-,+\}.$$

We therefore conclude that there exists a constant $C > 0$ such that

$$|I_\epsilon| \leqslant C\gamma\epsilon \sum_{\iota\in\{-,+\}} |u_{\epsilon,+}^{-\iota}| \int_{\mathbb{T}} \frac{\mathfrak{e}_\iota(\delta_\epsilon\omega)^2}{|D^{(\epsilon)}|^2}\mathrm{d}k + \gamma\epsilon\int_{\mathbb{T}} \frac{R|\bar{u}_{\epsilon,+}|(\delta_\epsilon\omega)^2}{|D^{(\epsilon)}|^2}\mathrm{d}k. \qquad (5.47)$$

Denote the terms appearing on the right-hand side by $I_\epsilon^{(1)}$ and $I_\epsilon^{(2)}$, respectively. Since $|D^{(\epsilon)}|^2 \geqslant 2\epsilon\lambda\gamma R$ we conclude that there exists $C > 0$ such that

$$I_\epsilon^{(1)} \leqslant C \sum_{\iota\in\{-,+\}} |u_{\epsilon,+}^{\iota}| \quad \text{and} \quad I_\epsilon^{(2)} \leqslant C\|\bar{u}_{\epsilon,+}\|_{L^1(\mathbb{T})}.$$

Therefore, (5.45) for $j = 2$ follows directly from (5.28). □

5.6. The limit of the energy functional at the superdiffusive time scale

We have shown in theorem 5.1 that the evolution of the temperature profile takes place on a scale longer than a hyperbolic one. In fact the right time-space scaling is given by $(t\epsilon^{-3/2}, x\epsilon^{-1})$ as it has been shown in theorem 3.1 of [4].

Theorem 5.6. *Suppose that the distribution of the initial configuration* $(\mathfrak{r},\mathfrak{p})$ *satisfies condition* (2.26). *Then, for any test function* $J \in C_0^\infty([0,+\infty)\times\mathbb{R})$ *we have:*

$$\lim_{\epsilon\to 0+} \epsilon\sum_x \int_0^{+\infty} J(t,\epsilon x)\mathbb{E}_\epsilon\mathfrak{e}_x\left(\frac{t}{\epsilon^{3/2}}\right)\mathrm{d}t = \int_0^{+\infty}\int_{\mathbb{R}} T(t,y)J(t,y)\mathrm{d}t\mathrm{d}y. \qquad (5.48)$$

Here, T(t, y) satisfies the fractional heat equation:

$$\partial_t T(t,y) = -\hat{c}|\Delta_y|^{3/4}T(t,y) \qquad (5.49)$$

with the initial condition $T(0, y) = T(y)$, *given by* (2.27) *and*

$$\hat{c} := \frac{[\alpha''(0)]^{3/4}}{2^{9/4}(3\gamma)^{1/2}}. \tag{5.50}$$

6. Proof of theorem 3.1 and corollary 3.2

Since the dynamics are linear, the solutions of (2.7) are of the form

$$\mathfrak{r}_x(t) = \mathfrak{r}'_x(t) + \mathfrak{r}''_x(t) \quad \text{and} \quad \mathfrak{p}_x(t) = \mathfrak{p}'_x(t) + \mathfrak{p}''_x(t), \qquad t \geqslant 0,\ x \in \mathbb{Z},$$

where $(\mathfrak{r}'(t), \mathfrak{p}'(t))$ and $(\mathfrak{r}''(t), \mathfrak{p}''(t))$ are the solutions corresponding to the initial configurations $(\mathfrak{r}'_x, \mathfrak{p}'_x)$ and $(\mathfrak{r}''_x, \mathfrak{p}''_x)$ under the dynamics (2.7), see section 3.1. The convergence of the $\mathfrak{r}$ and $\mathfrak{p}$ components of the vector $\mathfrak{w}(t)$ from corollary 3.2 are therefore a direct consequence of the conclusions of theorems 5.1 and 4.7.

The statements concerning the asymptotics of the thermal and phononic components of the energy functional contained in (3.9) and (3.10) also follow from the aforementioned theorems. To prove (3.11), also finishing the proof of corollary 3.2 in this way, note that

$$\mathfrak{e}_x\left(\frac{t}{\epsilon}\right) = \mathfrak{e}_{\mathrm{th},x}\left(\frac{t}{\epsilon}\right) + \mathfrak{e}_{\mathrm{ph},x}\left(\frac{t}{\epsilon}\right) + 2\mathfrak{e}_x\left(\mathfrak{r}'\left(\frac{t}{\epsilon}\right), \mathfrak{p}'\left(\frac{t}{\epsilon}\right); \mathfrak{r}''\left(\frac{t}{\epsilon}\right), \mathfrak{p}''\left(\frac{t}{\epsilon}\right)\right) \tag{6.1}$$

where $\mathfrak{e}_{\mathrm{th},x}(t)$ and $\mathfrak{e}_{\mathrm{ph},x}(t)$ are defined in (3.4). Given two configurations $(\mathfrak{r}^{(j)}, \mathfrak{p}^{(j)})$, $j = 1, 2$ the 'mixed' energy functional is defined as

$$\mathfrak{e}_x(\mathfrak{r}^{(1)}, \mathfrak{p}^{(1)}; \mathfrak{r}^{(2)}, \mathfrak{p}^{(2)}) := \frac{1}{2}\mathfrak{p}_x^{(1)}\mathfrak{p}_x^{(2)} - \frac{1}{4}\sum_{x'} \alpha_{x-x'}\mathfrak{q}^{(1)}_{x,x'}\mathfrak{q}^{(2)}_{x,x'}. \tag{6.2}$$

Here, $\mathfrak{q}^{(j)}_{x,x'}$ are computed from (2.9) for the respective configurations $\mathfrak{r}^{(j)}$, $j = 1, 2$. Formula (3.11) is a simple consequence of the following.

Lemma 6.1. *Under the assumptions of theorem 3.1 we have*

$$\lim_{\epsilon\to 0+} \epsilon \sum_x J(\epsilon x)\mathbb{E}_\epsilon \mathfrak{e}_x\left(\mathfrak{r}'\left(\frac{t}{\epsilon}\right), \mathfrak{p}'\left(\frac{t}{\epsilon}\right); \mathfrak{r}''\left(\frac{t}{\epsilon}\right), \mathfrak{p}''\left(\frac{t}{\epsilon}\right)\right) = 0 \tag{6.3}$$

for any $J \in C_0^\infty(\mathbb{R})$ and $t \geqslant 0$.

Proof. Let (see (2.20))

$$\delta_\epsilon \mathfrak{r}''_x\left(\frac{t}{\epsilon}\right) := \mathfrak{r}''_x\left(\frac{t}{\epsilon}\right) - r(t, \epsilon x), \qquad \delta_\epsilon \mathfrak{p}''\left(\frac{t}{\epsilon}\right) := \mathfrak{p}''\left(\frac{t}{\epsilon}\right) - p(t, \epsilon x).$$

Using the Cauchy–Schwartz inequality we can estimate

$$\begin{aligned} &\epsilon\left|\sum_x \mathbb{E}_\epsilon \mathfrak{e}_x\left(\mathfrak{r}'\left(\frac{t}{\epsilon}\right), \mathfrak{p}'\left(\frac{t}{\epsilon}\right); \delta_\epsilon\mathfrak{r}''\left(\frac{t}{\epsilon}\right), \delta_\epsilon\mathfrak{p}''\left(\frac{t}{\epsilon}\right)\right)\right| \\ &\leqslant \left\{\epsilon\sum_x \mathbb{E}_\epsilon \mathfrak{e}_x\left(\mathfrak{r}'\left(\frac{t}{\epsilon}\right), \mathfrak{p}'\left(\frac{t}{\epsilon}\right)\right)\right\}^{1/2}\left\{\epsilon\sum_x \mathbb{E}_\epsilon \mathfrak{e}_x\left(\delta_\epsilon\mathfrak{r}''\left(\frac{t}{\epsilon}\right), \delta_\epsilon\mathfrak{p}''\left(\frac{t}{\epsilon}\right)\right)\right\}^{1/2} \end{aligned}$$

The first factor on the right-hand side stays bounded, due to the conservation of energy properties of the dynamics, while the second one vanishes thanks to (4.38).

We can therefore conclude that

$$\lim_{\epsilon\to 0+}\epsilon\left|\sum_x \mathbb{E}_\epsilon \mathfrak{e}_x\left(\mathfrak{r}'\left(\frac{t}{\epsilon}\right),\mathfrak{p}'\left(\frac{t}{\epsilon}\right);\delta_\epsilon\mathfrak{r}''\left(\frac{t}{\epsilon}\right),\delta_\epsilon\mathfrak{p}''\left(\frac{t}{\epsilon}\right)\right)\right|=0, \tag{6.4}$$

for any $t \geqslant 0$, equality (6.3) would follow, provided we can show that

$$\lim_{\epsilon\to 0+}\epsilon\sum_x J(\epsilon x)\mathbb{E}_\epsilon \mathfrak{e}_x\left(\mathfrak{r}'\left(\frac{t}{\epsilon}\right),\mathfrak{p}'\left(\frac{t}{\epsilon}\right);r_\epsilon(t),p_\epsilon(t)\right)=0, \tag{6.5}$$

for any $J \in C_0^\infty(\mathbb{R})$ and $t \geqslant 0$, where

$$(r_\epsilon(t),p_\epsilon(t)) := (r(t,\epsilon x),p(t,\epsilon x))_{x\in\mathbb{Z}}.$$

The latter, however, is a direct consequence of (5.15). □

7. Proof of theorem 3.3

Using the notation from section 6 we can write the analog of (6.1) at the time scale $t/\epsilon^{3/2}$. Thanks to (4.40) for any $J \in C_0^\infty(\mathbb{R})$ we have

$$\lim_{\epsilon\to 0+}\sum_x J(\epsilon x)\mathrm{E}_\epsilon \mathfrak{e}_{\mathrm{ph},x}\left(\frac{t}{\epsilon^{3/2}}\right)=0. \tag{7.1}$$

The respective time–space weak limit of $\mathfrak{e}_{\mathrm{th},x}(t/\epsilon^{3/2})$ can be evaluated using theorem 3.3. Finally, to finish the proof of theorem 3.3 we need the following analog of lemma 6.1.

Lemma 7.1. *Under the assumptions of theorem 3.3 we have*

$$\lim_{\epsilon\to 0+}\epsilon\sum_x J(\epsilon x)\mathbb{E}_\epsilon \mathfrak{e}_x\left(\mathfrak{r}'\left(\frac{t}{\epsilon^{3/2}}\right),\mathfrak{p}'\left(\frac{t}{\epsilon^{3/2}}\right);\mathfrak{r}''\left(\frac{t}{\epsilon^{3/2}}\right),\mathfrak{p}''\left(\frac{t}{\epsilon^{3/2}}\right)\right)=0 \tag{7.2}$$

for any $J \in C_0^\infty(\mathbb{R})$

Proof. Using the Cauchy–Schwartz inequality we can estimate

$$\begin{aligned}&\epsilon\left|\sum_x J(\epsilon x)\mathbb{E}_\epsilon \mathfrak{e}_x\left(\mathfrak{r}'\left(\frac{t}{\epsilon^{3/2}}\right),\mathfrak{p}'\left(\frac{t}{\epsilon^{3/2}}\right);\mathfrak{r}''\left(\frac{t}{\epsilon^{3/2}}\right),\mathfrak{p}''\left(\frac{t}{\epsilon^{3/2}}\right)\right)\right|\\
&\leqslant\left\{\epsilon\|J\|_\infty\sum_x \mathbb{E}_\epsilon \mathfrak{e}_x\left(\mathfrak{r}'\left(\frac{t}{\epsilon^{3/2}}\right),\mathfrak{p}'\left(\frac{t}{\epsilon^{3/2}}\right)\right)\right\}^{1/2}\\
&\times\left\{\epsilon\sum_x |J(\epsilon x)|\mathbb{E}_\epsilon \mathfrak{e}_x\left(\mathfrak{r}''\left(\frac{t}{\epsilon^{3/2}}\right),\mathfrak{p}''\left(\frac{t}{\epsilon^{3/2}}\right)\right)\right\}^{1/2}.\end{aligned}$$

The first factor on the right-hand side stays bounded, due to the conservation of energy properties of the dynamics, while the second one vanishes thanks to (7.1). This ends the proof of the lemma. □

8. Proof of theorem 3.4

From the definition of the normal modes, see (3.17), we conclude that

$$d\hat{\mathfrak{f}}^{(\pm)}(t,k) = \Big\{ \pm(1-e^{-2i\pi k})\sqrt{\tau_1}\,\hat{\mathfrak{f}}^{(\pm)}(t,k) + D_\pm(1-e^{2i\pi k})^2\hat{\mathfrak{f}}^{(\pm)}(t,k) + O(\mathfrak{s}^3(k))\Big\}dt + dM_t(k), \tag{8.1}$$

where

$$dM_t(k) := 2i\gamma^{1/2}\int_{\mathbb{T}} r(k,k')\hat{\mathfrak{p}}(t,k-k')B(dt,dk'), \tag{8.2}$$

with $B(dt,dk)$ Gaussian white noise in (t, k), satisfying

$$B^*(dt,dk) = B(dt,-dk),$$
$$\mathbb{E}[B(dt,dk)B^*(dt',dk')] = \delta(t-t')\delta(k-k')$$

and $D_\pm$ are given by

$$D_\pm := \frac{3\gamma \pm \sqrt{\tau_1}}{2}. \tag{8.3}$$

Let

$$\hat{\bar{\mathfrak{f}}}_\epsilon^{(\pm)}(t,k) := \epsilon\mathbb{E}_\epsilon\hat{\mathfrak{f}}^{(\pm)}\left(\frac{t}{\epsilon},\epsilon k\right), \quad k\in\epsilon^{-1}\mathbb{T}.$$

It satisfies

$$\frac{d}{dt}\hat{\bar{\mathfrak{f}}}_\epsilon^{(\pm)}(t,k) = \pm\frac{1}{\epsilon}(1-e^{-2i\epsilon\pi k})\hat{\bar{\mathfrak{f}}}_\epsilon^{(\pm)}(0,k) + O(\epsilon). \tag{8.4}$$

An elementary stability theory for solutions of ordinary differential equations guarantees that for any $T, M > 0$ we have

$$\hat{\bar{\mathfrak{f}}}_\epsilon^{(\pm)}(t,k) = e^{\pm 2i\pi kt}(1+O(\epsilon))\hat{\bar{\mathfrak{f}}}_\epsilon^{(\pm)}(t,k), \quad |k|\leqslant M,\ |t|\leqslant T. \tag{8.5}$$

After a straightforward calculation we obtain that the left-hand side of (3.20) equals

$$\lim_{\epsilon\to 0+}\int_{\mathbb{R}}\hat{J}(p)\hat{\bar{\mathfrak{f}}}_\epsilon^{(\pm)}(t,-p)dp = \lim_{\epsilon\to 0+}\int_{\mathbb{R}}\hat{J}(p)e^{\mp 2i\pi pt}\hat{\bar{\mathfrak{f}}}_\epsilon^{(\pm)}(0,-p)dp,$$

which tends to the expression on the right-hand side of (3.20), as $\epsilon\to 0+$ for any $J\in C_0^\infty(\mathbb{R})$.

Let

$$\bar{\mathfrak{v}}_\epsilon^{(\iota)}(t,k) := \hat{\bar{\mathfrak{f}}}_\epsilon^{(\iota)}\left(\frac{t}{\epsilon},k\right)\exp\left\{-\iota(1-e^{-2i\pi\epsilon k})\frac{\sqrt{\tau_1}\,t}{\epsilon^2}\right\}, \quad \iota\in\{-,+\}$$

for $k\in\epsilon^{-1}\mathbb{T}$. For any $T, M > 0$ and $\iota\in\{-,+\}$ we have

$$\frac{d\bar{\mathfrak{v}}_\epsilon^{(\iota)}}{dt}(t,k) = \frac{D_\iota}{\epsilon^2}(1-e^{2i\pi\epsilon k})^2\bar{\mathfrak{v}}_\epsilon^{(\iota)}(t,k) + O(\epsilon),\ |k|\leqslant M,\ t\in[0,T]. \tag{8.6}$$

Using the elementary stability theory of ordinary differential equations again, we conclude that

$$\lim_{\epsilon\to 0+}\sup_{t\in[0,T],|k|\leqslant M}\left|\bar{\mathfrak{v}}_{\epsilon}^{(\iota)}(t,k)-\hat{\bar{\mathfrak{f}}}_{\epsilon}^{(\iota)}(0,k)\mathrm{e}^{-4\pi^2 tD_{\iota}|k|^2}\right|=0,\quad \iota\in\{-,+\}.$$

For any $J\in C_0^\infty(\mathbb{R})$ we can write that the expression under the limit on the left-hand side of (3.21) equals

$$\begin{aligned}&\lim_{\epsilon\to 0+}\int_{\mathbb{R}}\hat{J}(p)\exp\left\{-\iota 2\pi \mathrm{i}p\sqrt{\mathfrak{T}_1}\frac{t}{\epsilon}\right\}\hat{\bar{\mathfrak{f}}}_{\epsilon}^{(\iota)}\left(\frac{t}{\epsilon},-p\right)\mathrm{d}p\\
&=\lim_{\epsilon\to 0+}\int_{\mathbb{R}}\hat{J}(p)\exp\left\{\iota\sqrt{\mathfrak{T}_1}(1-\mathrm{e}^{-2\pi\mathrm{i}\epsilon p}-2\pi\mathrm{i}\epsilon p)\frac{t}{\epsilon^2}\right\}\bar{\mathfrak{v}}_{\epsilon}^{(\iota)}(t,-p)\mathrm{d}p\\
&=\lim_{\epsilon\to 0+}\int_{\mathbb{R}}\hat{J}(p)\exp\left\{2\iota\sqrt{\mathfrak{T}_1}\pi^2p^2t+O(\epsilon)\right\}\bar{\mathfrak{v}}_{\epsilon}^{(\iota)}(t,-p)\mathrm{d}p\end{aligned}$$

and the latest limit equals

$$\int_{\mathbb{R}}\hat{J}(p)\hat{f}^{(\iota)}(-p)\mathrm{e}^{-4\pi^2 tD|p|^2}\mathrm{d}p,$$

with $\hat{f}^{(\iota)}(p)$ the Fourier transform of $f^{(\iota)}(y)$, given by (3.19), which ends the proof of (3.21). □

9. Examples

In the final section we give examples of the initial data that are either the thermal or phononic types introduced in definitions 2.2 and 2.3. The examples are formulated in terms of the wave function.

9.1. Non-random initial data

Suppose that $\phi(x)$ is a function that belongs to $C_0^\infty(\mathbb{R})$. Let $a\geqslant 0$ and let μ_ϵ be δ-type measures on ℓ_2 concentrated at

$$\psi_x^{(\epsilon)}:=\epsilon^{(a-1)/2}\phi(\epsilon^a x),\quad x\in\mathbb{Z}. \tag{9.1}$$

We have

$$\hat{\psi}_\epsilon(k)=\epsilon^{(a-1)/2}\int_{\mathbb{R}}\hat{\phi}(p)\left[\sum_{x\in\mathbb{Z}}\exp\{2\pi\mathrm{i}x(\epsilon^a p-k)\}\right]\mathrm{d}p.$$

Using the Poisson summation formula (see e.g. page 566 of [5])

$$\sum_{x\in\mathbb{Z}}\exp\{\mathrm{i}bx\xi\}=\frac{2\pi}{|b|}\sum_{x\in\mathbb{Z}}\delta_0\left(\xi-\frac{2\pi}{b}x\right),\quad \xi\in\mathbb{R}, \tag{9.2}$$

(understood in the distribution sense) that holds for any $b\neq 0$, and the fact that $\hat{\phi}(k)$ is rapidly decaying, we conclude

$$\hat{\psi}_\epsilon(k)\approx\frac{1}{\epsilon^{(a+1)/2}}\hat{\phi}\left(\frac{k}{\epsilon^a}\right).$$

9.1.1. Case a = 1—macroscopic initial data. Note that then

$$\limsup_{\epsilon\to 0+} \int_{\mathbb{T}} \mathrm{d}k\ [\epsilon\left\langle |\hat{\psi}(k)|^2 \right\rangle_{\mu_\epsilon}] = \int_{\mathbb{R}} |\hat{\phi}(p)|^2 \mathrm{d}p < +\infty, \tag{9.3}$$

where $\hat{\phi}(p)$ is the Fourier transform of $\phi(y)$. The data is of the phononic type, in the sense of definition 2.3, with the macroscopic profile given by $\phi(y)$. On the other hand, condition (2.26) fails, as can be seen from the following computation:

$$\int_{\mathbb{T}} \mathrm{d}k\ [\epsilon\left\langle |\hat{\psi}(k)|^2 \right\rangle_{\mu_\epsilon}]^2 = \epsilon^2 \int_{\mathbb{T}} |\hat{\psi}_\epsilon(k)|^4 \mathrm{d}k \approx \frac{1}{\epsilon^2} \int_{-1/2}^{1/2} \left|\hat{\phi}\left(\frac{k}{\epsilon}\right)\right|^4 \mathrm{d}k \approx \frac{1}{\epsilon} \int_{\mathbb{R}} |\hat{\phi}(p)|^4 \mathrm{d}p.$$

9.1.2. Case a ∈ (0, 1)—oscillating (but not too fast) data. In this case, one can easily verify that condition (2.25) holds, but again condition (2.26) fails. However, the rate of the blow-up is slower than in the case of the macroscopic data. Indeed,

$$\begin{aligned} &\int_{\mathbb{T}} \mathrm{d}k\ [\epsilon\left\langle |\hat{\psi}(k)|^2 \right\rangle_{\mu_\epsilon}]^2 = \epsilon^2 \int_{\mathbb{T}} |\hat{\psi}_\epsilon(k)|^4 \mathrm{d}k \\ &\approx \frac{1}{\epsilon^a} \int_{-1/(2\epsilon^a)}^{1/(2\epsilon^a)} |\hat{\phi}(k)|^4 \mathrm{d}k \approx \frac{1}{\epsilon^a} \int_{\mathbb{R}} |\hat{\phi}(p)|^4 \mathrm{d}p. \end{aligned}$$

On the other hand, it is easy to verify that the macroscopic profile for the initial data vanishes but

$$K_0 = \lim_{\epsilon\to 0+} \epsilon \sum_x \left\langle |\psi_x^{(\epsilon)}|^2 \right\rangle_{\mu_\epsilon} = \int_{\mathbb{R}} |\hat{\phi}(k)|^2 \mathrm{d}k.$$

Hence the family $(\mu_\epsilon)_{\epsilon>0}$ is neither of phononic nor thermal type.

9.1.3. Case a = 0—microscopically oscillatory data. We have

$$\hat{\psi}_\epsilon(k) = \epsilon^{-1/2} \sum_{x\in\mathbb{Z}} \phi(x) \exp\{-2\pi \mathrm{i} x k\} = \epsilon^{-1/2} \tilde{\phi}(k),$$

where, the periodized Fourier transform of $\phi(y)$ is given by

$$\tilde{\phi}(k) = \sum_{x\in\mathbb{Z}} \hat{\phi}(x+k). \tag{9.4}$$

In this case condition (2.26) holds, since

$$\int_{\mathbb{T}} \mathrm{d}k \left[\epsilon\left\langle |\hat{\psi}(k)|^2 \right\rangle_{\mu_\epsilon}\right]^2 = \int_{\mathbb{T}} |\tilde{\phi}(k)|^4 \mathrm{d}k.$$

The data is of thermal type.

9.2. Random initial data

9.2.1. Modified stationary field. Assume that $(\eta_x)_{x\in\mathbb{Z}}$ is a zero mean, random stationary field such that $\mathbb{E}|\eta_0|^2 < +\infty$. We suppose that its covariance can be written as

$$r_x = \mathbb{E}(\eta_x \eta_0^*) = \int_{\mathbb{T}} \exp\{2\pi \mathrm{i} k x\} \hat{R}(k) \mathrm{d}k, \quad x \in \mathbb{Z}, \tag{9.5}$$

where $\hat{R} \in C(\mathbb{T})$ is non-negative. Given $a \geqslant 0$ and $\phi(x) \in C_0^\infty(\mathbb{R})$ define the wave function as

$$\psi_x^{(\epsilon)} := \epsilon^{(a-1)/2}\phi(\epsilon^a x)\eta_x, \quad x \in \mathbb{Z}. \tag{9.6}$$

One can easily check that condition (2.25) holds. We show that both micro- and macroscopically varying initial data satisfy condition (2.26).

For $a \in (0, 1]$ (the oscillatory case) we have

$$\hat{\psi}^{(\epsilon)}(k) \approx \frac{1}{\epsilon^{(a+1)/2}} \int_{\mathbb{T}} \hat{\phi}\left(\frac{k-\ell}{\epsilon^a}\right)\hat{\eta}(\mathrm{d}\ell), \tag{9.7}$$

where $\hat{\eta}(\mathrm{d}\ell)$ is the stochastic spectral measure corresponding to $(\eta_x)_{x\in\mathbb{Z}}$. Then, thanks to (9.7) we get

$$\begin{aligned}\int_{\mathbb{T}} \mathrm{d}k\ \{\epsilon\mathbb{E}|\hat{\psi}^{(\epsilon)}(k)|^2\}^2 &\approx \epsilon^{-2a}\int_{\mathbb{T}} \mathrm{d}k\left\{\int_{\mathbb{T}}\left|\hat{\phi}\left(\frac{k-\ell}{\epsilon^a}\right)\right|^2 \hat{R}(\ell)\mathrm{d}\ell\right\}^2\\ &= \epsilon^{-2a}\int_{\mathbb{T}} \mathrm{d}k\left\{\int_{\mathbb{T}^2}\left|\hat{\phi}\left(\frac{k-\ell}{\epsilon^a}\right)\right|^2\left|\hat{\phi}\left(\frac{k-\ell'}{\epsilon^a}\right)\right|^2 \hat{R}(\ell)\hat{R}(\ell')\mathrm{d}\ell\mathrm{d}\ell'\right\}.\end{aligned} \tag{9.8}$$

Changing variables $\tilde{\ell}' := \ell'/\epsilon^a$ and $\tilde{k} := k/\epsilon^a$ we get the last expression equaling

$$\int_{\mathbb{T}}\hat{R}(\ell)\mathrm{d}\ell\left\{\int_{-1/(2\epsilon^a)}^{1/(2\epsilon^a)}\int_{-1/(2\epsilon^a)}^{1/(2\epsilon^a)}\left|\hat{\phi}\left(k-\frac{\ell}{\epsilon^a}\right)\right|^2\left|\hat{\phi}(k-\ell')\right|^2\hat{R}(\epsilon^a\ell')\mathrm{d}k\mathrm{d}\ell'\right\},$$

which, as $\epsilon \to 0+$, tends to

$$\hat{R}(0)\int_{\mathbb{T}}\hat{R}(\ell)\mathrm{d}\ell\left\{\int_{\mathbb{R}}\left|\hat{\phi}(k)\right|^2\mathrm{d}k\right\}^2.$$

Thus, condition (2.26) is clearly satisfied by this family of fields. In the case $a = 0$ (microscopically varying the initial data) the condition is also valid, as then

$$\hat{\psi}^{(\epsilon)}(k) = \epsilon^{-1/2}\int_{\mathbb{T}}\tilde{\phi}(k-\ell)\hat{\eta}(\mathrm{d}\ell),$$

with $\tilde{\phi}(k)$ given by (9.4). As a result

$$\int_{\mathbb{T}} \mathrm{d}k\ \{\epsilon\mathbb{E}|\hat{\psi}^{(\epsilon)}(k)|^2\}^2 = \int_{\mathbb{T}}\mathrm{d}k\left\{\int_{\mathbb{T}}\left|\tilde{\phi}(k-\ell)\right|^2\hat{R}(\ell)\mathrm{d}\ell\right\}^2 < +\infty.$$

9.2.2. Locally stationary initial data. Assume that $(\mathfrak{r}_{x,\epsilon}, \mathfrak{p}_{x,\epsilon})_{x\in\mathbb{Z}}$, $\epsilon \in (0, 1]$ is a family of locally stationary random fields over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By the above, we mean the fields that satisfy the following:

(1) they are square integrable for each ε and there exist C_0^∞ functions $r, p : \mathbb{R} \to \mathbb{R}$, called the *mean profiles* satisfying

$$\mathfrak{r}''_{x,\epsilon} := \mathbb{E}\mathfrak{r}_{x,\epsilon} = r(\epsilon x), \qquad \mathfrak{p}''_{x,\epsilon} := \mathbb{E}\mathfrak{p}_x^{(\epsilon)} = p(\epsilon x),$$

(2) the covariance matrix of the field

$$\mathfrak{r}'_{x,\epsilon} := \mathfrak{r}_{x,\epsilon} - \mathfrak{r}''_{x,\epsilon}, \quad \mathfrak{p}'_{x,\epsilon} := \mathfrak{p}_{x,\epsilon} - \mathfrak{p}''_{x,\epsilon},$$

is given by

$$\mathbb{E}[\mathfrak{r}'_{x,\epsilon}\mathfrak{r}'_{x+x',\epsilon}] = C_{11}(\epsilon x, x'), \quad \mathbb{E}[\mathfrak{p}'_{x,\epsilon}\mathfrak{p}'_{x+x',\epsilon}] = C_{22}(\epsilon x, x'),$$

$$\mathbb{E}[\mathfrak{r}'_{x,\epsilon}\mathfrak{p}'_{x+x',\epsilon}] = C_{12}(\epsilon x, x'), \quad \mathbb{E}[\mathfrak{p}'_{x,\epsilon}\mathfrak{r}'_{x+x',\epsilon}] = C_{21}(\epsilon x, x'), \quad x, x' \in \mathbb{Z},$$

where $C_{ij} : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$, $i, j = 1, 2$ are functions that satisfy

$$C_* := \sum_{i=1}^{2}\sum_{x'}\left|\int_{\mathbb{R}} C_{ii}(x, x')\mathrm{d}x\right|^2 < +\infty. \tag{9.9}$$

The energy spectrum of the field $(\mathfrak{r}'_{x,\epsilon}, \mathfrak{p}'_{x,\epsilon})_{x\in\mathbb{Z}}$, see (3.3), equals

$$\mathfrak{w}'_\epsilon(k) = \sum_x C_{22}(\epsilon x, k) + \frac{\hat{\alpha}(k)}{4\mathfrak{s}^2(k)}\left(\sum_x C_{11}(\epsilon x, k)\right)$$

where

$$\hat{C}_{ij}(\epsilon x, k) = \sum_{x'} C_{ij}(\epsilon x, x')\exp\{-2\pi \mathrm{i} k x'\}, \quad (x, k) \in \mathbb{Z} \times \mathbb{T},\ i, j = 1, 2.$$

We can write

$$\epsilon^2\int_{\mathbb{T}}(\mathfrak{w}'_\epsilon)^2\mathrm{d}k \leqslant C\epsilon^2\sum_{j=1}^{2}\sum_{x_1,x_2}\int_{\mathbb{T}}\hat{C}_{jj}(\epsilon x_1, k)\hat{C}^*_{jj}(\epsilon x_2, k)\mathrm{d}k, \quad \epsilon \in (0, 1],$$

for some constant $C > 0$. By an application of the Plancherel identity we conclude that the right-hand side approximates CC_*, as $\epsilon \ll 1$. Thus, condition (3.1) holds in this case.

9.2.3. Local Gibbs measures. Another important example of random initial data is furnished by the local Gibbs measure. Given the profiles of temperature $\beta(\epsilon x)^{-1}$, momentum $p(\epsilon x)$, and tension $\tau(\epsilon x)$, where $\beta^{-1}, p, \tau \in C_0^\infty(\mathbb{R})$ and $\beta^{-1} \geqslant 0$ we define a product measure analogous to (1.6), in which the constant profiles are replaced by slowly varying functions. These measures are formally written as

$$\mathrm{d}\nu_{\lambda,\epsilon} := \prod_x \exp\{-\beta(\epsilon x)(\mathfrak{e}_x - p(\epsilon x)\mathfrak{p}_x - \tau(\epsilon x)\mathfrak{r}_x) - \mathcal{G}(\boldsymbol{\lambda}(\epsilon x))\}\mathrm{d}\mathfrak{r}_x\,\mathrm{d}\mathfrak{p}_x, \tag{9.10}$$

where $\boldsymbol{\lambda}(x) = (\beta(x), p(x), \tau(x))$ and $\mathcal{G}(\cdot)$ are appropriate Gibbs potentials that normalize the respective measure. In order to make the above 'definition' rigorous, one would have to consider the Gibbs measures in question as solutions of the respective DLR equations. We shall omit that issue by dealing with local Gibbs measures only, i.e. the case when $\beta(y)^{-1}$ is compactly supported so the relevant measure is defined on a finite dimensional space. On the sites where $\beta(\epsilon x)^{-1} = 0$, we let the corresponding exponential factor in (9.10) be a delta distribution concentrated at the point (0, 0). The corresponding profile of the volume stretch $r(\epsilon x)$ and temperature $\beta(\epsilon x)^{-1}$ is given by the analogs of relations (1.9) with r and u replaced by the respective slowly varying functions.

The natural decomposition in thermal and mechanical initial conditions is now given by

$$\mathfrak{r}_x = r(\epsilon x) + \mathfrak{r}'_x, \qquad \mathfrak{p}_x = p(\epsilon x) + \mathfrak{p}'_x$$

where $(\mathfrak{r}'_x, \mathfrak{p}'_x)_x$ are distributed by

$$\mathrm{d}\nu'_{\lambda,\epsilon} := \prod_x \exp\{-\beta(\epsilon x)\mathfrak{e}'_x - \mathcal{G}(\beta(\epsilon x),0,0)\}\mathrm{d}\mathfrak{r}'_x\,\mathrm{d}\mathfrak{p}'_x \tag{9.11}$$

and $\mathfrak{e}'_x$ is the energy at site x of the configuration $(\mathfrak{r}'_x, \mathfrak{p}'_x)_{x\in\mathbb{Z}}$. In this case we have

$$C_{22}(\epsilon x, x') = \delta_{x',0}\langle(\mathfrak{p}'_x)^2\rangle_{\nu'_{\lambda,\epsilon}} = \delta_{x',0}\beta^{-1}(\epsilon x). \tag{9.12}$$

9.2.4. Nearest neighbor interactions. Consider first the nearest neighbor case, i.e. when $\mathfrak{e}'_x = V(\mathfrak{r}'_x)$. Then,

$$C_{11}(\epsilon x, x') = \delta_{x',0}\langle(\mathfrak{r}'_x)^2\rangle_{\nu'_{\lambda,\epsilon}} = \delta_{x',0}\frac{\int_{\mathbb{R}} r^2 \mathrm{e}^{-\beta(\epsilon x)V(r)}\mathrm{d}r}{\int_{\mathbb{R}} \mathrm{e}^{-\beta(\epsilon x)V(r)}\mathrm{d}r}.$$

Let us assume that there exists $c_* > 0$, for which $c_* r^2 \leqslant V(r)$. Then,

$$\begin{aligned} 0 \leqslant C_{11}(\epsilon x, x') &\leqslant \frac{1}{c_*}\delta_{x',0}\frac{\int_{\mathbb{R}} V(r)\mathrm{e}^{-\beta(\epsilon x)V(r)}\mathrm{d}r}{\int_{\mathbb{R}} \mathrm{e}^{-\beta(\epsilon x)V(r)}\mathrm{d}r} \\ &= \frac{1}{c_*}\delta_{x',0}\left(\tilde{u}(\epsilon x) - \frac{1}{2}\beta^{-1}(\epsilon x)\right). \end{aligned}$$

Here, $\tilde{u}(\epsilon x) := u(0, \beta(\epsilon x))$ and $u(\tau, \beta)$ are the internal energy functions defined in (1.9), see e.g. (2.1.9) of [1]. Condition (9.9) is satisfied, once we assume that

$$\int_{\mathbb{R}} \beta^{-1}(y)\mathrm{d}y < +\infty \quad \text{and} \quad \int_{\mathbb{R}} \tilde{u}(y)\mathrm{d}y < +\infty.$$

9.2.5. Gaussian, local Gibbs measures. We assume that $\operatorname{supp}\beta^{-1} = [-K, K]$ and that $\beta^{-1}(y) > 0$ for $|y| < K$. In addition, we suppose that the sequence (α_x), besides satisfying conditions a1)-a3) and (2.1), is compactly supported—i.e. there exists a positive integer ℓ such that $\alpha_x = 0$ for $|x| > \ell$ and $\alpha_x < 0$ for $0 < |x| \leqslant \ell$ (obviously in light of (2.1) we have $\alpha_0 > 0$). In this case, the formal expression (9.11) is a probability measure on $\ell_2(\mathbb{Z})$ that equals (we omit writing primes)

$$\nu_{\lambda,\epsilon} := \mu_{\epsilon,K} \otimes \delta_{K^c}, \tag{9.13}$$

where δ_{K^c} is the Borel probability measure on the space of sequences $(\mathfrak{r}_x)_{|x|\geqslant\epsilon^{-1}K}$, concentrated on the sequence $\mathfrak{r}_x \equiv 0$, $|x| \geqslant K/\epsilon$. Measure $\mu_{\epsilon,K}$ is Gaussian on the Euclidean space corresponding to finite sequences $(\rho_x)_{|x|<\epsilon^{-1}K}$ whose characteristic functional equals

$$\exp\left\{-\frac{1}{2}\sum_{|x|,|x'|<\epsilon^{-1}K}\sigma_{x,x'}\rho_x\rho_{x'}\right\},$$

where $\Sigma := [\sigma_{x,y}]$ is the inverse of the symmetric matrix $S := [S_{x,y}]$ corresponding to the quadratic form

$$\sum_{|x|,|x'|<\epsilon^{-1}K} S_{x,x'}\mathfrak{r}_x\mathfrak{r}_{x'} = -\frac{1}{4}\sum_{|x|,|x'|<\epsilon^{-1}K}\beta(\epsilon x)\alpha_{x-x'}\mathfrak{q}^2_{x,x'},$$

where $\mathfrak{q}_{x,x'}$ is defined in (2.9). We claim that there exists $c_* > 0$ independent of ε and such that

$$\sum_{|x|,|x'|<\epsilon^{-1}K} S_{x,x'}\mathfrak{r}_x\mathfrak{r}_{x'} \geqslant c_* \sum_{|x|<\epsilon^{-1}K} \mathfrak{r}_x^2, \quad \forall\, (\mathfrak{r}_x)_{|x|<\epsilon^{-1}K}. \tag{9.14}$$

According to our assumptions, there exists $\beta_* > 0$ such that $\beta(y) \geqslant \beta_*$ for all $y \in \mathbb{R}$. We can therefore write

$$\begin{aligned}
&\inf\left\{\sum_{|x|,|x'|<\epsilon^{-1}K} S_{x,x'}\mathfrak{r}_x\mathfrak{r}_{x'} : \sum_{|x|<\epsilon^{-1}K} \mathfrak{r}_x^2 = 1\right\}\\
&\geqslant \frac{\beta_*}{4}\inf\left\{-\sum_{|x|,|x'|<\epsilon^{-1}K} \alpha_{x-x'}\mathfrak{q}_{x,x'}^2 : \sum_{|x|<\epsilon^{-1}K} \mathfrak{r}_x^2 = 1\right\}\\
&\geqslant \frac{\beta_*}{4}\inf\left\{-\sum_{x,x'} \alpha_{x-x'}\mathfrak{q}_{x,x'}^2 : \sum_{x} \mathfrak{r}_x^2 = 1\right\}.
\end{aligned} \tag{9.15}$$

One can easily verify the following identity (see (2.3))

$$-\frac{1}{2}\sum_{x,x'} \alpha_{x-x'}\mathfrak{q}_{x,x'}^2 = \tau_1 \int_{\mathbb{T}} \varphi(\mathfrak{s}^2(k))|\hat{\mathfrak{r}}(k)|^2\,\mathrm{d}k,$$

therefore the utmost left-hand side (9.15) can be estimated from below by

$$c_* := \frac{\tau_1\beta_*}{2}\inf\varphi.$$

Hence (9.14) holds with c_* as defined above.

We can write

$$\langle|\hat{\mathfrak{r}}(k)|^2\rangle_{\nu_{\lambda,\epsilon}} = \sum_{|x|,|x'|<\epsilon^{-1}K} \sigma_{x,x'}e_x(k)e_{x'}^*(k),$$

with $e_x(k) := \exp\{-2\pi \mathrm{i}xk\}$. Thanks to (9.14) we obtain

$$\langle|\hat{\mathfrak{r}}(k)|^2\rangle_{\nu_{\lambda,\epsilon}} \leqslant \frac{1}{c_*}\sum_{|x|\leqslant\epsilon^{-1}K} |e_x(k)|^2 = \frac{2(K+1)}{c_*\epsilon}, \quad \forall\, k \in \mathbb{T}. \tag{9.16}$$

Thanks to (9.12) we conclude that there exists $C > 0$ such that

$$\langle|\hat{\mathfrak{p}}(k)|^2\rangle_{\nu_{\lambda,\epsilon}} = \sum_x \langle \mathfrak{p}_x^2\rangle_{\nu_{\lambda,\epsilon}} = \sum_x \beta^{-1}(\epsilon x) \leqslant \frac{C}{\epsilon}, \quad \forall\, \epsilon \in (0,1]. \tag{9.17}$$

Combining (9.16) with (9.17) we conclude condition (2.26).

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Appendix. Proof of proposition 2.1

A simple calculation, using the fact that $\sum_x \alpha_x = 0$, shows that

$$\sum_x (\alpha * \mathfrak{q})_x \mathfrak{q}_x = -\frac{1}{2}\sum_{x,y}\alpha_{x-y}(\mathfrak{q}_x - \mathfrak{q}_y)^2 \tag{A.1}$$

for any $(\mathfrak{q}_x)$ such that $\mathfrak{r} := \nabla^*\mathfrak{q}$ is square summable.

Using (A.1.), the Plancherel identity and (2.3) we can write

$$\begin{aligned}\sum_x \mathfrak{e}_x(\mathfrak{r},\mathfrak{p}) &= \int_{\mathbb{T}}\left(\frac{|\hat{\mathfrak{p}}(k)|^2}{2} + \hat{\alpha}(k)|\hat{\mathfrak{q}}(k)|^2\right)\mathrm{d}k\\ &= \int_{\mathbb{T}}\left(\frac{|\hat{\mathfrak{p}}(k)|^2}{2} + \tau_1\varphi^2(\mathfrak{s}^2(k))|\hat{\mathfrak{r}}(k)|^2\right)\mathrm{d}k.\end{aligned} \tag{A.2}$$

Here $\hat{\mathfrak{r}}(k)$, $\hat{\mathfrak{p}}(k)$, $\hat{\mathfrak{q}}(k)$ are the Fourier transforms of $\mathfrak{r}_x$, $\mathfrak{p}_x$, $\hat{\mathfrak{q}}_x$ respectively. In light of the assumptions made about $\varphi(\cdot)$ it is clear that the utmost right-hand side is equivalent to $\sum_x(\mathfrak{r}_x^2 + \mathfrak{p}_x^2)$, so (2.16) follows.

Obviously only the lower bound in (2.17) requires a proof. Note that

$$\sum_x |\mathfrak{e}_x| \leqslant \frac{1}{2}\sum_x \mathfrak{p}_x^2 + \sum_{x,x'}|\alpha_{x-x'}|\mathfrak{q}_{x,x'}^2. \tag{A.3}$$

Here $\mathfrak{q}_{x,x'}$ is given by (2.9). By the Cauchy–Schwartz inequality for $x \geqslant x'$ we have

$$\mathfrak{q}_{x,x'}^2 \leqslant (x-x')\sum_{x'<x''\leqslant x}\mathfrak{r}_{x''}^2$$

and an analogous inequality also holds for $x < x'$. Therefore,

$$\begin{aligned}\sum_{x,x'}|\alpha_{x-x'}|\mathfrak{q}_{x,x'}^2 &\leqslant \sum_{x\geqslant x''>x'}|x-x'||\alpha_{x-x'}|\mathfrak{r}_{x''}^2\\ &+ \sum_{x'\geqslant x''>x}|x-x'||\alpha_{x-x'}|\mathfrak{r}_{x''}^2.\end{aligned}$$

Substituting $z := x - x'$ in the first summation and $z := x' - x$ in the second we get a right-hand side equaling

$$\sum_{z>0} z|\alpha_z|\sum_{x\geqslant x''>x-z}\mathfrak{r}_{x''}^2 + \sum_{z>0} z|\alpha_z|\sum_{x+z\geqslant x''>x}\mathfrak{r}_{x''}^2.$$

Denote the first and the second term by I and II respectively. We have

$$\begin{aligned}I &= \sum_{z>0} z|\alpha_z|\sum_{x''}\mathfrak{r}_{x''}^2\sum_{x''+z>x\geqslant x''}1\\ &\leqslant \left(\sum_{z>0} z^2|\alpha_z|\right)\sum_{x''}\mathfrak{r}_{x''}^2.\end{aligned}$$

On the other hand

$$II = \sum_{z>0} z|\alpha_z|\sum_{x''}\mathfrak{r}_{x''}^2\sum_{x+z\geqslant x''>x}1 \leqslant \left(\sum_{z>0} z^2|\alpha_z|\right)\sum_{x''}\mathfrak{r}_{x''}^2$$

and, as a result, we get

$$\sum_x |\mathfrak{e}_x| \leqslant C \sum_x (\mathfrak{p}_x^2 + \mathfrak{r}_x^2), \tag{A.4}$$

where $C := \max\left\{1/2, \sum_{z>0} z^2 |\alpha_z|\right\}$. This, combined with the estimate already shown (2.16), ends the proof of the lower bound in (2.17).

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