

Well/ill posedness for the Euler-Korteweg-Poisson system and related problems

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Abstract

We consider a general Euler-Korteweg-Poisson system in R^3 , supplemented with the space periodic boundary conditions, where the quantum hydrodynamics equations and the classical fluid dynamics equations with capillarity are recovered as particular examples. We show that the system admits infinitely many global-in-time weak solutions for any sufficiently smooth initial data including the case of a vanishing initial density - the vacuum zones. Moreover, there is a vast family of initial data, for which the Cauchy problem possesses infinitely many dissipative weak solutions, i.e. the weak solutions satisfying the energy inequality. Finally, we establish the weak-strong uniqueness property in a class of solutions without vacuum. In this paper we show that, even in presence of a dispersive tensor, we have the same phenomena found by De Lellis and Székelyhidi.

Key words: Euler-Korteweg system, quantum hydrodynamics, weak solution, convex integration
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1 Introduction

A general EULER-KORTEWEG-POISSON SYSTEM describing the time evolution of the density $\varrho = \varrho(t, x)$ and the momentum $\mathbf{J} = \mathbf{J}(t, x)$ of an inviscid fluid can be written in the form:

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$$\partial_t \varrho + \operatorname{div}_x \mathbf{J} = 0, \quad (1.1)$$

$$\partial_t \mathbf{J} + \operatorname{div}_x \left(\frac{\mathbf{J} \times \mathbf{J}}{\varrho} \right) + \nabla_x p(\varrho) = -\alpha \mathbf{J} + \varrho \nabla_x \left(K(\varrho) \Delta_x \varrho + \frac{1}{2} K'(\varrho) |\nabla_x \varrho|^2 \right) + \varrho \nabla_x V, \quad (1.2)$$

$$\Delta_x V = \varrho - \bar{\varrho}, \quad (1.3)$$

where $K : (0, \infty) \rightarrow (0, \infty)$ is a smooth function, see Audiard [4], [3], Benzoni-Gavage et al. [6], [5]. In particular, taking $K = \bar{K} > 0$ a positive constant, we recover the standard equations of an *inviscid capillary fluid* (see Bresch et al. [7], Kotchote [18], [17]), while the choice $K(\varrho) = \frac{\hbar}{4\varrho}$ gives rise to the so-called *quantum fluid system* (see for instance Antonelli and Marcati [1], [2], Jüngel [15, Chapter 14] and the references therein). In the latter case, the equations (1.1 - 1.3), by using the Madelung transformations, may be formally seen as a description of the evolution of the momenta

$$\varrho = |\psi|^2, \quad \mathbf{J} = \hbar \Im[\bar{\psi} \nabla_x \psi], \quad (1.4)$$

where the wave function ψ , in the case $\alpha = 0$ and $\bar{\varrho} = 0$, is a solution of the following Schrödinger-Poisson system:

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2} \Delta_x \psi - V \psi + f(|\psi|^2) \psi, \quad \Delta V = |\psi|^2, \quad (1.5)$$

provided $p'(\varrho) = \varrho f'(\varrho)$.

For the sake of simplicity, we consider the system (1.1 - 1.3) supplemented with the spatially periodic boundary conditions, namely on the “flat” torus

$$\Omega = \mathbb{T}^3 \equiv \mathbb{R}^3 / \mathbb{Z}^3,$$

and with the initial state

$$\varrho(0, \cdot) = \varrho_0, \quad \mathbf{J}(0, \cdot) = \mathbf{J}_0. \quad (1.6)$$

In view of the applications to the quantum fluid models, we consider a general non-negative distribution of the density ϱ including the vacuum zones where $\varrho = 0$. We note that the Korteweg tensor can be written in the form

$$\begin{aligned} & \varrho \nabla_x \left(K(\varrho) \Delta_x \varrho + \frac{1}{2} K'(\varrho) |\nabla_x \varrho|^2 \right) \\ &= \operatorname{div}_x \left[\varrho \operatorname{div}_x \left(K(\varrho) \nabla_x \varrho \right) \mathbb{I} \right] + \frac{1}{2} \operatorname{div}_x \left[\left(K(\varrho) - \varrho K'(\varrho) \right) |\nabla_x \varrho|^2 \mathbb{I} \right] - \operatorname{div}_x \left[K(\varrho) \nabla_x \varrho \otimes \nabla_x \varrho \right]. \end{aligned}$$

Thus, introducing

$$\chi(\varrho) = \varrho K(\varrho), \quad (1.7)$$

we deduce that

$$\begin{aligned} & \varrho \nabla_x \left(K(\varrho) \Delta_x \varrho + \frac{1}{2} K'(\varrho) |\nabla_x \varrho|^2 \right) \\ &= \nabla_x \left(\chi(\varrho) \Delta_x \varrho \right) + \frac{1}{2} \nabla_x \left(\chi'(\varrho) |\nabla_x \varrho|^2 \right) - 4 \operatorname{div}_x \left(\chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho} \right) \equiv \operatorname{div}_x \mathcal{K}(\varrho, \nabla_x \varrho), \end{aligned} \quad (1.8)$$

$$\mathcal{K}(\varrho, \nabla_x \varrho) = \left[\chi(\varrho) \Delta_x \varrho + \frac{1}{2} \chi'(\varrho) |\nabla_x \varrho|^2 \right] \mathbb{I} - 4 \chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho}, \quad (1.9)$$

where the choice $\chi \equiv \hbar/4$ determines the quantum fluids while $\chi(\varrho) = \varrho$ corresponds to the capillary fluids with constant capillarity. Accordingly the choice of $\chi(\varrho)$ determines the role of the quadratic nonlinearities, in the case of the quantum fluids the term sensitive to the appearance of the vacuum, beyond the convective term, is then $\nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho}$.

The parameter $\alpha \geq 0$ in (1.2) represent a damping effect relevant in certain applications, in particular collision effects for quantum models for semiconductor devices. In what follows, we assume, for the sake of simplicity, that $\alpha = 1$. Strangely enough, the presence of damping makes the problem more difficult in view of the methods used in the present paper and due to the dispersive nature of the equations. We remark that the theory we develop below applies to the case $\alpha = 0$ as well, with only obvious modifications in the proofs.

1.1 Energy

The Euler-Korteweg-Poisson system (1.1-1.3) admits a natural energy density, namely

$$\begin{aligned} E(\varrho, \nabla_x \varrho, \mathbf{J}) &= \frac{1}{2} \frac{|\mathbf{J}|^2}{\varrho} + P(\varrho) + \frac{K(\varrho)}{2} |\nabla_x \varrho|^2 + \frac{1}{2} |\nabla_x V|^2 \\ &= \frac{1}{2} \frac{|\mathbf{J}|^2}{\varrho} + P(\varrho) + 2\chi(\varrho) |\nabla_x \sqrt{\varrho}|^2 + \frac{1}{2} |\nabla_x V|^2 \end{aligned} \quad (1.10)$$

where χ was introduced in (1.7) and

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} dz.$$

Indeed, taking the scalar product of the momentum equation (1.2) with \mathbf{J}/ϱ and using (1.1), (1.3), we obtain the energy balance

$$\frac{d}{dt} \int_\Omega E(\varrho, \nabla_x \varrho, \mathbf{J})(t, \cdot) dx + \int_\Omega \frac{|\mathbf{J}|^2}{\varrho}(t, \cdot) dx = 0. \quad (1.11)$$

In this paper, we focus on *bounded energy* (weak) solutions for which $E(\varrho, \nabla_x \varrho, \mathbf{J})$ is bounded on the whole physical space Ω and for any time $t \in [0, T]$. In particular, the momentum \mathbf{J} must vanish on the vacuum set where $\varrho = 0$.

1.2 Velocity Fields

As already pointed out several times, our goal is to consider the solutions that may contain vacuum zones. In the context of quantum hydrodynamics, the classical WKB formalism does not allow the definition of the velocity in the nodal regions, while the current measure $\mathbf{J}dx$ obtained via the Madelung transform can be differentiated in the sense of measure in ϱdx but the velocity field defined in this way is $L^1(\varrho dx)$ only. In the context of classical fluid mechanics, where vacuum is not permitted in the natural framework of applications of the model, it is customary to replace the momentum \mathbf{J} by $\varrho \mathbf{u}$, where \mathbf{u} is the macroscopic *velocity* of the fluid. We emphasize that the velocity \mathbf{u} has a physical interpretation only on the sets where $\varrho > 0$ and, in particular, it has no particular meaning on the vacuum. For these reasons, we avoid using the concept of velocity in the formulation of our problem and we are going to develop a self consistent theory in the (ϱ, \mathbf{J}) variables. The vacuum problem has been extensively discussed in [1], [2].

1.3 Weak solutions

Since the solutions of the problem (1.1-1.3), (1.6) may not be regular on the vacuum, quantum vortices may appear and moreover the hydrodynamic variables (ϱ, \mathbf{J}) may not have better regularity than the energy space, it seems natural to introduce the concept of *weak solution*.

Definition 1.1. *We say that*

$$\varrho \in C_{\text{weak}}([0, T]; L^2(\Omega)) \cap L^\infty((0, T) \times \Omega), \quad \mathbf{J} \in C_{\text{weak}}([0, T]; L^2(\Omega; \mathbb{R}^3)) \cap L^\infty((0, T) \times \Omega; \mathbb{R}^3)$$

is a bounded energy weak solution to the Euler-Korteweg-Poisson system (1.1-1.3), (1.6) if

$$\varrho(t, \cdot) > 0 \text{ a.a. in } \Omega \text{ for any } t \in (0, T), \quad \nabla_x \varrho \in L^\infty((0, T) \times \Omega), \quad (1.12)$$

$$\begin{aligned} E(\varrho, \nabla_x \varrho, \mathbf{J})(t, \cdot) &\leq \bar{E} \text{ for a.a. } t \in (0, T), \\ \varrho(0, \cdot) &= \varrho_0, \quad \mathbf{J}(0, \cdot) = \mathbf{J}_0, \end{aligned} \quad (1.13)$$

and the following integral identities

$$-\int_{\Omega} \varrho \varphi \, dx \Big|_{t=\tau_1}^{t=\tau_2} + \int_{\tau_1}^{\tau_2} \int_{\Omega} (\varrho \partial_t \varphi + \mathbf{J} \cdot \nabla_x \varphi) \, dx \, dt = 0, \quad (1.14)$$

$$\begin{aligned} &-\int_{\Omega} \mathbf{J} \cdot \varphi \, dx \Big|_{t=\tau_1}^{t=\tau_2} + \int_{\tau_1}^{\tau_2} \int_{\Omega} \left(\mathbf{J} \cdot \partial_t \varphi + \frac{\mathbf{J} \otimes \mathbf{J}}{\varrho} : \nabla_x \varphi + p(\varrho) \operatorname{div}_x \varphi \right) \, dx \, dt \\ &= \int_{\tau_1}^{\tau_2} \int_{\Omega} \left(-\nabla_x \varrho \cdot \nabla_x (\chi(\varrho) \operatorname{div}_x \varphi) + \frac{1}{2} \chi'(\varrho) |\nabla_x \varrho|^2 \operatorname{div}_x \varphi - 4\chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho} : \nabla_x \varphi \right) \, dx \\ &\quad + \int_{\tau_1}^{\tau_2} \int_{\Omega} (\mathbf{J} \cdot \varphi - \varrho \nabla_x \mathbf{V} \cdot \varphi) \, dx \, dt \end{aligned} \quad (1.15)$$

hold for any $0 \leq \tau_1 < \tau_2 \leq T$ and any test function $\varphi \in C_c^\infty([0, T] \times \Omega)$, $\varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^3)$, respectively, where the potential \mathbf{V} is the unique solution of the Poisson equation

$$\Delta \mathbf{V}(t, \cdot) = \varrho(t, \cdot) - \bar{\varrho}, \quad \int_{\Omega} \mathbf{V}(t, \cdot) \, dx = 0, \quad t \in [0, T], \quad \text{with } \bar{\varrho} = \frac{1}{|\Omega|} \int_{\Omega} \varrho_0 \, dx. \quad (1.16)$$

Remark 1.1. *In view of (1.12), the vacuum set where $\varrho = 0$ is of zero Lebesgue measure, in particular, all terms in the integral identities (1.14), (1.15) are well defined. This is in good agreement with the interpretation of ϱ as the density of a quantum fluid given by (1.4), (1.5) as the nodal zones of the Schrödinger equation are likely to be composed of “tiny” sets, see Kenig et al. [16], Seo [22].*

The present paper examines the well/ill posedness of the Euler-Korteweg-Poisson system in the class of weak solutions introduced above. Observe that for the particular choice $\varrho \equiv \bar{\varrho}$, the problem (1.1-1.3) reduces to the “damped” Euler system with *zero* pressure. In view of the recent ground-breaking results by DeLellis and Székelyhidi [12], [10], [11] based on the method of *convex integration*, such a system is ill-posed in the class of weak solutions, meaning it admits infinitely many solutions for *any* initial data.

Chiodaroli [8] obtained similar illposedness results for the compressible Euler system using a “non-constant” coefficient version of the method of [11]; later the method was further extended in [9] in order

to attack the more complex Euler-Fourier system. The main idea, elaborated in [9], is to consider the Helmholtz decomposition

$$\mathbf{J} = \mathbf{v} + \nabla_x \Psi, \operatorname{div}_x \mathbf{v} = 0,$$

to determine ϱ along with the acoustic potential Ψ , and to “solve” the momentum equation for \mathbf{J} as a “pressureless” Euler system with nonconstant coefficients. Adapting this approach to the present problem features an essential difficulty related to the presence of vacuum zones, where the equations become singular. To overcome this problem, we extend the technique of convex integration to problems with non-constant *singular* coefficients. In particular, we show a variant of the crucial *oscillatory increment* lemma on an arbitrary open set by means of a careful scale analysis of its original version in [11] and an application of Whitney covering lemma.

The solutions obtained by the method of convex integration suffer the well-known deficit that eliminates most of them as physically irrelevant: Although their energy remains bounded at any instant t including $t = 0$, they do not satisfy the total energy balance (1.11), not even as an inequality. In particular, the energy at any positive time may become strictly larger than that of the initial data. This motivates introducing the energy inequality

$$\int_{\Omega} E(\varrho, \nabla_x \varrho, \mathbf{J})(\tau, \cdot) \, dx + \int_0^\tau \int_{\Omega} \frac{|\mathbf{J}|^2}{\varrho} \, dx \, dt \leq \int_{\Omega} E(\varrho_0, \nabla_x \varrho_0, \mathbf{J}_0) \, dx \text{ for a.a. } \tau \in (0, T) \quad (1.17)$$

as a suitable *admissibility criterion* in the class of weak solutions. Indeed we show that the *dissipative* weak solutions, meaning the weak solutions satisfying (1.17), enjoy the weak-strong uniqueness property - they coincide with the strong solution emanating from the same initial data as long as the latter exists. This result will be a direct consequence of the method of relative entropies adapted from [13], [14].

Finally, we note that even the dissipative weak solution may fail to be unique, at least for certain (non-smooth) initial data. Such a result follows from a refined application of convex integration in the spirit of DeLellis and Székelyhidi [11].

The paper consists of two parts. In the first one, we discuss the problem of well/ill posedness of the Euler-Korteweg-Poisson system in the class of weak solutions. We start by stating the main result on the existence of infinitely many solutions in Section 2. In Section 3, we show how the method of convex integration can be adapted to the present setting and reduce the problem to *oscillatory lemma* proved in Section 4. The second part concerns the dissipative weak solutions introduced in Section 5. In Section 5.2, we show that the dissipative weak solutions possess the weak-strong uniqueness property. Finally, we discuss the ill posedness of the Euler-Korteweg-Poisson system in the class of dissipative weak solutions for particular initial data.

2 Well/ill posedness in the class of weak solutions

We start by introducing certain technical assumptions imposed on the structural properties of the functions $p = p(\varrho)$, $\chi = \chi(\varrho)$, specifically,

$$p \in C^1[0, \infty) \cap C^2(0, \infty), \quad p(0) = 0, \quad \chi \in C^2[0, \infty), \quad \chi > 0 \text{ in } (0, \infty). \quad (2.1)$$

Note that, in view of possible applications to the theory of quantum fluids, the pressure p need not be monotone, not even positive.

The main result of the first part of the paper reads:

Theorem 2.1. *Under the hypotheses (2.1), suppose that the initial data satisfy*

$$\varrho_0 = r_0^2, \quad r_0 \in C^2(\Omega), \quad \text{meas} \left\{ x \in \Omega \mid r_0(x) = 0 \right\} = 0, \quad (2.2)$$

$$\mathbf{J}_0 = \varrho_0 \mathbf{U}_0, \quad \mathbf{U}_0 \in C^3(\Omega; R^3). \quad (2.3)$$

Then the initial value problem (1.1-1.3), (1.6) admits infinitely many weak solutions in $(0, T)$ in the sense specified in Definition 1.1.

Remark 2.1. *It is easy to check that the initial data satisfying (2.2), (2.3) possess uniformly bounded energy $E(\varrho_0, \nabla_x \varrho_0, \mathbf{J}_0)$. The hypothesis (2.3) could be relaxed, the present form asserts the existence of the initial velocity \mathbf{U}_0 .*

The following two sections will be devoted to the proof of Theorem 2.1. We first extend the density ϱ to the whole time interval $[0, T]$ and then construct the desired weak solutions by the method of convex integration.

3 Convex integration

We start by extending the initial data ϱ_0, \mathbf{J}_0 as a suitable solution $[\varrho, \tilde{\mathbf{J}}]$ to the equation of continuity on the whole time interval $[0, T]$. The function $\varrho = \varrho(t, \cdot)$ will be the unique solution of the transport equation

$$\partial_t \varrho + \text{div}_x(\varrho[\mathbf{U}_0 - \mathbf{Z}]) = \partial_t \varrho + [\mathbf{U}_0 - \mathbf{Z}] \cdot \nabla_x \varrho + \varrho \text{div}_x \mathbf{U}_0 = 0, \quad \varrho(0, \cdot) = \varrho_0, \quad (3.1)$$

where the spatially homogeneous vector function $\mathbf{Z} = \mathbf{Z}(t)$ is chosen in such a way that

$$e^t \int_{\Omega} \varrho[\mathbf{U}_0 - \mathbf{Z}] \, dx = \int_{\Omega} \varrho_0 \mathbf{U}_0 \, dx \quad \text{for all } t \in [0, T], \quad (3.2)$$

in particular $\mathbf{Z}(0) = 0$.

Indeed, for any given $\mathbf{Z} \in C([0, T]; R^3)$, the Cauchy problem (3.1) admits a unique solution ϱ and we may define a mapping

$$\mathcal{T} : \mathbf{Z} \mapsto \left(\int_{\Omega} \varrho_0 \, dx \right)^{-1} \left(\int_{\Omega} \varrho \mathbf{U}_0 \, dx - e^{-t} \int_{\Omega} \varrho_0 \mathbf{U}_0 \, dx \right).$$

Clearly, the satisfaction of (3.2) corresponds to finding a fixed point of the mapping \mathcal{T} . To this end, it is enough to observe that the maximum of ϱ satisfying (3.1) is *independent* of \mathbf{Z} , and

$$\begin{aligned} \partial_t \mathcal{T}[\mathbf{Z}] &= \left(\int_{\Omega} \varrho_0 \, dx \right)^{-1} \left(\int_{\Omega} \partial_t \varrho \mathbf{U}_0 \, dx + e^{-t} \int_{\Omega} \varrho_0 \mathbf{U}_0 \, dx \right) \\ &= \left(\int_{\Omega} \varrho_0 \, dx \right)^{-1} \left(\int_{\Omega} \varrho \nabla_x \mathbf{U}_0 \cdot [\mathbf{U}_0 - \mathbf{Z}] \, dx + e^{-t} \int_{\Omega} \varrho_0 \mathbf{U}_0 \, dx \right); \end{aligned}$$

whence the existence of a fixed point \mathbf{Z} follows by a direct application of the Schauder theorem in a bounded ball of $C([0, T]; R^3)$.

Since \mathbf{U}_0 enjoys the regularity (2.3), we deduce that

- $\varrho(t, \cdot) \in C^2(\Omega)$ for any $t \in [0, T]$;
- $$\text{meas} \left\{ x \in \Omega \mid \varrho(t, x) = 0 \right\} = 0 \text{ for any } t \in [0, T]; \quad (3.3)$$

- for $\tilde{\mathbf{J}}(t, x) = \varrho(t, x) \left(\mathbf{U}_0(x) - \mathbf{Z}(t) \right)$ we have

$$e^t \int_{\Omega} \tilde{\mathbf{J}}(t, \cdot) \, dx = \int_{\Omega} \mathbf{J}_0 \, dx \text{ for any } t \in [0, T], \quad (3.4)$$

and

$$E(\varrho, \nabla_x \varrho, \tilde{\mathbf{J}})(t, \cdot) \leq \bar{E} \text{ for all } t \in [0, T]. \quad (3.5)$$

Remark 3.1. Let \mathbf{H} denote the standard Helmholtz projection onto the space of solenoidal functions. We have

$$\int_{\Omega} \tilde{\mathbf{J}} \, dx = \int_{\Omega} \mathbf{H}[\tilde{\mathbf{J}}] \, dx,$$

and (3.4) yields

$$\partial_t \int_{\Omega} \mathbf{H}[\tilde{\mathbf{J}}] \, dx + \int_{\Omega} \mathbf{H}[\tilde{\mathbf{J}}] \, dx = 0, \quad \int_{\Omega} \mathbf{H}[\tilde{\mathbf{J}}](0, \cdot) \, dx = \int_{\Omega} \mathbf{H}[\mathbf{J}_0] \, dx. \quad (3.6)$$

This relation is important in the construction of the so-called subsolutions introduced below.

3.1 Convex integration ansatz

The density ϱ being fixed through (3.1), we look for the flux \mathbf{J} in the form

$$\mathbf{J} = \mathbf{w} + \tilde{\mathbf{J}},$$

where

$$\mathbf{w} \in C_{\text{weak}}([0, T], L^2(\Omega; \mathbb{R}^3)) \cap L^\infty((0, T) \times \Omega; \mathbb{R}^3), \quad \text{div}_x \mathbf{w} = 0, \quad \mathbf{w}(0, \cdot) = 0. \quad (3.7)$$

In particular, the equation of continuity (1.14), together with the initial conditions (1.13), are satisfied.

In order to comply with (1.15), the function \mathbf{w} must be taken such that

$$\partial_t \left(\mathbf{w} + \tilde{\mathbf{J}} \right) + \text{div}_x \left(\frac{(\mathbf{w} + \tilde{\mathbf{J}}) \otimes (\mathbf{w} + \tilde{\mathbf{J}})}{\varrho} \right) + \nabla_x p(\varrho) + \left(\mathbf{w} + \tilde{\mathbf{J}} \right) = \quad (3.8)$$

$$\nabla_x \left(\chi(\varrho) \Delta_x \varrho \right) + \frac{1}{2} \nabla_x \left(\chi'(\varrho) |\nabla_x \varrho|^2 \right) - 4 \text{div}_x \left(\chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho} \right) + \varrho \nabla_x V$$

in the sense of distributions. Note that, in accordance with (3.3), (3.5), all quantities are bounded continuous functions on the (open) set where $\varrho > 0$, the complement of which in Ω is of zero measure.

For future analysis, it is convenient to rewrite (3.8) in a different form. We proceed in several steps:

Step 1

To begin, we write $\tilde{\mathbf{J}}$ in terms of the Helmholtz projection as

$$\tilde{\mathbf{J}} = \mathbf{H}[\tilde{\mathbf{J}}] + \nabla_x M;$$

whence, replacing $\mathbf{w} \approx \mathbf{w} + \mathbf{H}[\tilde{\mathbf{J}}]$, we convert (3.7), (3.8) to

$$\mathbf{w} \in C_{\text{weak}}([0, T], L^2(\Omega; R^3)) \cap L^\infty((0, T) \times \Omega; R^3), \operatorname{div}_x \mathbf{w} = 0, \mathbf{w}(0, \cdot) = \mathbf{H}[\mathbf{J}_0], \quad (3.9)$$

$$\begin{aligned} \partial_t \mathbf{w} + \operatorname{div}_x \left(\frac{(\mathbf{w} + \nabla_x M) \otimes (\mathbf{w} + \nabla_x M)}{\varrho} \right) + \mathbf{w} + \nabla_x (p(\varrho) + \partial_t M + M) = \\ \nabla_x (\chi(\varrho) \Delta_x \varrho) + \frac{1}{2} \nabla_x (\chi'(\varrho) |\nabla_x \varrho|^2) - 4 \operatorname{div}_x (\chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho}) + \varrho \nabla_x V. \end{aligned} \quad (3.10)$$

Step 2

Multiplying (3.10) by e^t and introducing a new quantity $\mathbf{v} = e^t \mathbf{w}$ we obtain

$$\mathbf{v} \in C_{\text{weak}}([0, T], L^2(\Omega; R^3)) \cap L^\infty((0, T) \times \Omega; R^3), \operatorname{div}_x \mathbf{v} = 0, \mathbf{v}(0, \cdot) = \mathbf{H}[\mathbf{J}_0], \quad (3.11)$$

$$\begin{aligned} \partial_t \mathbf{v} + \operatorname{div}_x \left(\frac{(\mathbf{v} + e^t \nabla_x M) \otimes (\mathbf{v} + e^t \nabla_x M)}{e^t \varrho} \right) + \nabla_x (e^t p(\varrho) + e^t \partial_t M + e^t M) = \\ \nabla_x (e^t \chi(\varrho) \Delta_x \varrho) + \frac{1}{2} \nabla_x (e^t \chi'(\varrho) |\nabla_x \varrho|^2) - 4 \operatorname{div}_x (e^t \chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho}) + e^t \varrho \nabla_x V, \end{aligned} \quad (3.12)$$

Step 3

Finally, writing

$$\varrho \nabla_x V = \operatorname{div}_x \left(\nabla_x V \otimes \nabla_x V - \frac{1}{3} |\nabla_x V|^2 \mathbb{I} \right) + \nabla_x \left(\bar{\varrho} V - \frac{1}{6} |\nabla_x V|^2 \right)$$

and introducing new quantities

$$r(t, x) = e^t \varrho(t, x),$$

$$\mathbf{h}(t, x) = e^t \nabla_x M(t, x),$$

$$\Pi(t, x) = e^t \left(p(\varrho) + \partial_t M + M - \chi(\varrho) \Delta_x \varrho - \frac{1}{2} \chi'(\varrho) |\nabla_x \varrho|^2 + \frac{4}{3} \chi(\varrho) |\nabla_x \sqrt{\varrho}|^2 - \bar{\varrho} V + \frac{1}{6} |\nabla_x V|^2 \right),$$

$$\mathbb{H}(t, x) = 4e^t \left(\chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho} - \frac{1}{3} \chi(\varrho) |\nabla_x \sqrt{\varrho}|^2 \mathbb{I} - \frac{1}{4} \nabla_x V \otimes \nabla_x V + \frac{1}{12} |\nabla_x V|^2 \mathbb{I} \right),$$

we obtain (3.12) in a concise form

$$\partial_t \mathbf{v} + \operatorname{div}_x \left(\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{r} + \mathbb{H} \right) + \nabla_x \Pi = 0, \quad (3.13)$$

where \mathbb{H} is a symmetric traceless tensor.

3.2 Subsolutions

Let

$$R_+ = \left\{ (t, x) \in (0, T) \times \Omega \mid r(t, x) > 0 \right\}$$

denote the set of positivity of the density ϱ . In accordance with (3.3), R_+ is an open set of full measure in $(0, T) \times \Omega$.

Following DeLellis and Székelyhidi [11], we introduce the set of subsolutions

$$X_{0,e} = \left\{ \mathbf{v} \in C_{\text{weak}}([0, T]; L^2(\Omega; R^3)) \mid \mathbf{v}(0, \cdot) = \mathbf{H}[\mathbf{J}_0] \right.$$

$$\left. \mathbf{v} \in C^1((0, T) \times \Omega; R^3), \partial_t \mathbf{v} + \text{div}_x \mathbb{U} = 0 \text{ in } (0, T) \times \Omega \text{ for a certain } \mathbb{U} \in C^1((0, T) \times \Omega; R_{\text{sym},0}^{3 \times 3}), \right.$$

$$\left. \frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{r} + \mathbb{H} - \mathbb{U} \right] < e \text{ in } R_+ \right\},$$

where $\lambda_{\max}[\mathbb{A}]$ stands for the maximal eigenvalue of a symmetric matrix \mathbb{A} .

Here, the functions \mathbf{h} , r , \mathbb{H} are the same as in (3.13), whereas the “energy” e is taken in the form

$$e(t, x) = \omega(t) - \frac{3}{2} \Pi(t, x), \tag{3.14}$$

where Π is the “pressure” in (3.13) while ω is a suitable spatially homogeneous function specified below. In accordance with (3.5), we have

$$\Pi \in L^\infty((0, T) \times \Omega), \quad \Pi \in C(R_+). \tag{3.15}$$

Finally, seeing that

$$\frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{r} + \mathbb{H} - \mathbb{U} \right] \geq \frac{1}{2} \frac{|\mathbf{v} + \mathbf{h}|^2}{r}$$

we introduce a non-positive functional

$$\mathcal{I}[\mathbf{v}] = \int_{R_+} \left(\frac{1}{2} \frac{|\mathbf{v} + \mathbf{h}|^2}{r} - e \right) dx dt = \int_0^T \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{v} + \mathbf{h}|^2}{r} - e \right) dx dt. \tag{3.16}$$

3.3 Proof of Theorem 2.1 by convex integration

The crucial ingredient of the proof of Theorem 2.1 is the following *oscillatory lemma*.

Lemma 3.1. *Let $U \subset R \times R^3$ be a bounded open set. Suppose that*

$$\mathbf{g} \in C(U; R^3), \quad \mathbb{W} \in C(U; R_{\text{sym},0}^{3 \times 3}), \quad e, r \in C(U), \quad r > 0, \quad e \leq \bar{e} \text{ in } U$$

are given such that

$$\frac{3}{2} \lambda_{\max} \left[\frac{\mathbf{g} \otimes \mathbf{g}}{r} - \mathbb{W} \right] < e \text{ in } U.$$

Then there exist sequences

$$\mathbf{w}_n \in C_c^\infty(U; R^3), \quad \mathbb{V}_n \in C_c^\infty(U; R_{\text{sym},0}^{3 \times 3}), \quad n = 0, 1, \dots$$

such that

$$\begin{aligned} \partial_t \mathbf{w}_n + \operatorname{div}_x \mathbb{V}_n &= 0, \quad \operatorname{div}_x \mathbf{w}_n = 0 \text{ in } R^3, \\ \frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{g} + \mathbf{w}_n) \otimes (\mathbf{g} + \mathbf{w}_n)}{r} - (\mathbb{W} + \mathbb{V}_n) \right] &< e \text{ in } U, \end{aligned}$$

and

$$\liminf_{n \rightarrow \infty} \int_U \frac{|\mathbf{w}_n|^2}{r} \, dx \, dt \geq c(\bar{e}) \int_U \left(e - \frac{1}{2} \frac{|\mathbf{g}|^2}{r} \right)^2 \, dx \, dt.$$

Remark 3.2. We point out that the functions \mathbf{g} , \mathbb{W} are continuous but not necessarily bounded on the open set U . Similarly, r need not be bounded below away from zero. Thus Lemma 3.1 can be interpreted as a singular version of similar results in [8], [11].

The proof of Lemma 3.1 will be given in Section 4. Taking this result for granted, the proof of Theorem 2.1 follows the arguments similar to [11]:

Step 1

First we observe that the set of subsolutions $X_{0,e}$ is non-empty, at least for a sufficiently large function ω in (3.14). To see this, it is enough to take

$$\mathbf{v} = e^t \mathbf{H}[\tilde{\mathbf{J}}].$$

As a consequence of (3.5) we get

$$\frac{|\mathbf{v} + \mathbf{h}|^2}{r} = e^t \frac{|\mathbf{H}[\tilde{\mathbf{J}}] + \nabla_x M|^2}{\varrho} = e^t \frac{|\tilde{\mathbf{J}}|^2}{\varrho} \leq \bar{E}.$$

Thus it is enough to find a suitable field $\mathbb{U} \in C^1((0, T) \times \Omega; R_{\text{sym},0}^{3 \times 3})$ such that

$$\operatorname{div}_x \mathbb{U} = -\partial_t \mathbf{v}.$$

This can be achieved by solving, for instance, the elliptic system

$$\operatorname{div}_x \left(\nabla_x \mathbf{w} + \nabla_x^t \mathbf{w} - \frac{2}{3} \operatorname{div}_x \mathbf{w} \mathbb{I} \right) = -\partial_t \mathbf{v}$$

since we have, by virtue of (3.6),

$$\int_{\Omega} \partial_t \mathbf{v} \, dx = \partial_t \int_{\Omega} e^t \mathbf{H}[\tilde{\mathbf{J}}] \, dx = 0.$$

As $\Pi(t, x)$ is bounded, we can choose ω in (3.14) so large that $\mathbf{v} \in X_{0,e}$.

Step 2

Applying *oscillatory lemma* (Lemma 3.1) with

$$U = R_+, \quad \mathbf{g} = \mathbf{h} + \mathbf{v}, \quad \mathbb{W} = \mathbb{H} - \mathbb{U}$$

we deduce that cardinality of the space $X_{0,e}$ is infinite.

Step 3

The last step leans on a sophisticated Baire category argument due to DeLellis and Székelyhidi [11]. Endowing $X_{0,e}$ with the metrizable (on $X_{0,e}$) topology of the space $C_{\text{weak}}([0, T]; L^2(\Omega; R^3))$, we deduce, by means of that the functional \mathcal{I} , defined through (3.16), admits infinitely many points of continuity on the closure of $X_{0,e}$ satisfying

$$\omega - \frac{3}{2}\Pi = e = \frac{1}{2} \frac{|\mathbf{v} + \mathbf{h}|^2}{r}, \quad \mathbb{U} = \mathbb{H} + \frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{r} - \frac{1}{3} \frac{|\mathbf{v} + \mathbf{h}|^2}{r}, \quad (3.17)$$

$$\partial_t \mathbf{v} + \text{div}_x \mathbb{U} = 0 \text{ in the distributional sense, } \mathbf{v}(0, \cdot) = \mathbf{H}[\mathbf{J}_0],$$

which is exactly equation (3.13), cf. [11].

We have proved Theorem 2.1.

4 Oscillatory lemma

Our goal in this section is to prove Lemma 3.1.

4.1 Basic result

We start with the following basic result due to DeLellis and Székelyhidi (cf. also Chiodaroli [8]).

Lemma 4.1. *Let*

$$Q = \left\{ (t, x) \mid t \in (0, 1), x \in (0, 1)^3 \right\}, \quad \mathbf{v} \in R^3, \quad \mathbb{U} \in R_{\text{sym},0}^{3 \times 3}, \quad e > 0$$

satisfying

$$\frac{3}{2} \lambda_{\max} [\mathbf{v} \otimes \mathbf{v} - \mathbb{U}] < e \leq \bar{e}. \quad (4.1)$$

Then there exists sequences

$$\mathbf{w}_n \in C_c^\infty(Q; R^3), \quad \mathbb{V}_n \in C_c^\infty(Q; R_{\text{sym},0}^{3 \times 3}), \quad n = 0, 1, \dots$$

such that

$$\begin{aligned} \partial_t \mathbf{w}_n + \text{div}_x \mathbb{V}_n &= 0, \quad \text{div}_x \mathbf{w}_n = 0 \text{ in } R^3, \\ \frac{3}{2} \lambda_{\max} [(\mathbf{v} + \mathbf{w}_n) \otimes (\mathbf{v} + \mathbf{w}_n) - (\mathbb{U} + \mathbb{V}_n)] &< e \leq \bar{e} \text{ in } Q. \end{aligned} \quad (4.2)$$

$$\mathbf{w}_n \rightarrow 0 \text{ weakly in } L^2(Q; R^3),$$

$$\liminf_{n \rightarrow \infty} \int_Q |\mathbf{w}_n|^2 \, dx \, dt \geq c(\bar{e}) \int_Q \left(e - \frac{1}{2} |\mathbf{v}|^2 \right)^2 \, dx \, dt \quad (4.3)$$

Remark 4.1. *It is important that the constant in (4.3) is independent of e , \mathbf{v} , and \mathbb{U} .*

4.2 Extending by scaling

Rescaling $\mathbf{w}_n \approx \mathbf{w}_n(t/L, x/L)$, $\mathbb{V}_n = \mathbb{V}_n(t/L, x/L)$ we can extend the validity of Lemma 3.1 to an arbitrary cube

$$Q_L = LQ = \left\{ (t, x) \mid t \in (0, L), x \in (0, L)^3 \right\}, \quad L > 0,$$

with the same constant $c(\bar{e})$ in (4.3).

Now, using additivity of the integral, we observe that Lemma 3.1 holds on any domain

$$Q_{T,L} = \left\{ (t, x) \mid t \in (0, T), x \in (0, L)^3 \right\}, \quad T, L > 0,$$

via a decomposition of $Q_{T,L}$ on a (finite) number of cubes.

Finally, introducing a new scaling

$$\mathbf{w}_n \approx \sqrt{r} \mathbf{w}(t/\sqrt{r}, x), \quad \mathbb{V}_n \approx \sqrt{r} \mathbb{V}_n(t/\sqrt{r}, x), \quad t \in (0, T), x \in (0, L)^3,$$

for a positive constant r , we conclude that the hypothesis (4.1) may be replaced by

$$\frac{3}{2} \lambda_{\max} \left[\frac{\mathbf{v} \otimes \mathbf{v}}{r} - \mathbb{U} \right] < e \leq \bar{e},$$

with the conclusion of Lemma 3.1 valid with the obvious changes

$$\frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \mathbf{w}_n) \otimes (\mathbf{v} + \mathbf{w}_n)}{r} - (\mathbb{U} + \mathbb{V}_n) \right] < e \leq \bar{e}, \quad (4.4)$$

$$\liminf_{n \rightarrow \infty} \int_{Q_{T,L}} \frac{|\mathbf{w}_n|^2}{r} \, dx \, dt \geq c(\bar{e}) \int_{Q_{T,L}} \left(e - \frac{1}{2} \frac{|\mathbf{v}|^2}{r} \right)^2 \, dx \, dt$$

in (4.2), (4.3), respectively.

4.3 Continuous perturbation

Our goal is to extend Lemma 3.1 to the case, where \mathbf{v} , \mathbb{U} , $r > 0$, and e are *continuous* functions on the (closed) cube \bar{Q}_L satisfying

$$\frac{3}{2} \lambda_{\max} \left[\frac{\mathbf{v} \otimes \mathbf{v}}{r} - \mathbb{U} \right] < e \leq \bar{e} \text{ in } \bar{Q}_L.$$

Let us point out that r , being continuous on \bar{Q}_L , is bounded below away from zero.

Now, choosing $\delta > 0$ small enough we decompose

$$\bar{Q}_L = \cup_{i=1}^m \bar{Q}^i, \quad Q^i \cap Q^j = \emptyset \text{ for } i \neq j,$$

where Q^i are cubes that can be taken small enough so that

$$\frac{3}{2} \lambda_{\max} \left[\frac{\mathbf{v}_i \otimes \mathbf{v}_i}{r_i} - \mathbb{U}_i \right] < e_i - \delta \text{ in } Q^i, \quad i = 1, \dots, m \quad (4.5)$$

for arbitrary constant quantities

$$\mathbf{v}_i = \mathbf{v}(t_{i,v}, x_{i,v}), \quad r_i = r(t_{i,r}, x_{i,r}), \quad \mathbb{U}_i = \mathbb{U}(t_{i,u}, r_{i,u}), \quad e_i = e(t_{i,r}, x_{i,r}), \quad (t_{i,\cdot}, x_{i,\cdot}) \in Q^i.$$

Moreover, by the same token, we may assume that

$$\left| \frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{v}_1 + \mathbf{w}) \otimes (\mathbf{v}_1 + \mathbf{w})}{r_1} - (\mathbb{U}_1 + \mathbb{V}) \right] - \frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{v}_2 + \mathbf{w}) \otimes (\mathbf{v}_2 + \mathbf{w})}{r_2} - (\mathbb{U}_2 + \mathbb{V}) \right] \right| < \frac{\delta}{2} \quad (4.6)$$

provided \mathbf{w}, \mathbb{V} are bounded and

$$\mathbf{v}_j = \mathbf{v}(t_{j,v}, x_{j,v}), \quad r_j = r(t_{j,r}, x_{j,r}), \quad \mathbb{U}_j = \mathbb{U}(t_{j,u}, r_{j,u}), \quad (t_{j,\cdot}, x_{j,\cdot}) \in Q^i, \quad j = 1, 2.$$

Thus, using (4.5), (4.6), together with the result for the constant coefficients shown above, we obtain the desired sequences $\{\mathbf{w}_n\}_{n=1}^\infty, \{\mathbb{V}_n\}_{n=1}^\infty$ satisfying

$$\liminf_{n \rightarrow \infty} \int_{Q_{T,L}} \frac{|\mathbf{w}_n|^2}{r} \, dx \, dt \geq c(\bar{\varepsilon}) \int_{Q_{T,L}} \left(e - \delta - \frac{1}{2} \frac{|\mathbf{v}|^2}{r} \right)^2 \, dx \, dt.$$

As $\delta > 0$ can be taken arbitrarily small, we conclude.

4.4 A decomposition lemma and the final result

To conclude, we make use of the standard Whitney decomposition lemma, see Stein [23]:

Lemma 4.2. *Let $U \subset \mathbb{R}^N$ be an arbitrary open set. Then there exists a countable family of (dyadic) open cubes Q^i such that*

$$U = \cup_{i=1}^\infty \bar{Q}^i, \quad Q^i \cap Q^j = \emptyset \text{ for } i \neq j,$$

and

$$\text{diam}[Q^i] \leq \text{dist}[Q^i, \partial U] \leq 4 \text{diam}[Q^i] \text{ for all } i = 1, \dots \quad (4.7)$$

Decomposing the domain U in Lemma 3.1 as in Lemma 4.2 and using the results of Section 4.3 on each cube Q^i , we complete the proof of Lemma 3.1. Note that, thanks to (4.7), the restriction of the continuous function r to Q^i is bounded below away from zero.

5 Dissipative weak solutions

A weak solution ϱ, \mathbf{J} is called *dissipative weak solution* if, in addition to the stipulations listed in Definition 1.1, it satisfies the energy inequality (1.17). It is worth revisiting the weak solutions constructed in the proof of Theorem 2.1 in the light of (1.17). We recall that the energy of these reads

$$E(\varrho, \nabla_x \varrho, \mathbf{J}) = \frac{1}{2} \frac{|\mathbf{J}|^2}{\varrho} + P(\varrho) + 2\chi(\varrho) |\nabla_x \sqrt{\varrho}|^2 + \frac{1}{2} |\nabla_x V|^2,$$

where, by virtue of (3.14), (3.17),

$$\frac{1}{2} \frac{|\mathbf{J}|^2}{\varrho} = e^{-t} \omega(t) - \frac{3}{2} \Pi(t, x).$$

The function $\omega(t)$ could be chosen arbitrary but large enough, here large means in terms of the initial data. Going back to the energy inequality (1.17) we get

$$\frac{d}{dt} \int_{\Omega} E(\varrho, \nabla_x \varrho, \mathbf{J}) \, dx + \int_{\Omega} \frac{|J|^2}{\varrho} \, dx = e^{-t} |\Omega| (\omega'(t) + \omega(t)) + h(t)$$

for a certain function h depending solely on the initial data. Thus, taking

$$\omega(t) = e^{-2t}M, \quad M = M(T) \text{ large enough,}$$

we obtain

$$\frac{d}{dt} \int_{\Omega} E(\varrho, \nabla_x \varrho, \mathbf{J}) \, dx + \int_{\Omega} \frac{|J|^2}{\varrho} \, dx = -e^{-3t} |\Omega| M + h(t) \leq 0 \text{ in } (0, T). \quad (5.1)$$

We conclude that the weak solutions in Theorem 2.1 can be constructed to satisfy the energy inequality in the *open* interval. On the other hand, in general, they are not expected to satisfy (1.17), meaning the energy balance may be violated at the initial time $t = 0$. We come back to this issue at the end of this section.

5.1 Relative energy (entropy) inequality

To simplify the forthcoming presentation, we restrict ourselves to the case of constant ‘‘capillarity’’ tensor $K = 1$ or, equivalently, $\chi(\varrho) = \varrho$. Motivated by the analysis in [13], we introduce *relative energy* functional

$$\begin{aligned} & \mathcal{E} \left(\varrho, \mathbf{J} \mid r, \mathbf{L} \right) \\ &= \int_{\Omega} \left[\frac{1}{2} \varrho \left| \frac{\mathbf{J}}{\varrho} - \frac{\mathbf{L}}{r} \right|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) + \frac{1}{2} |\nabla_x \varrho - \nabla_x r|^2 \right] \, dx, \end{aligned} \quad (5.2)$$

where $r > 0$, \mathbf{L} are smooth functions. Note that the expression

$$\frac{1}{2} \varrho \left| \frac{\mathbf{J}}{\varrho} - \frac{\mathbf{L}}{r} \right|^2 = \frac{1}{2} \frac{|\mathbf{J}|^2}{\varrho} - \mathbf{J} \cdot \frac{\mathbf{L}}{r} + \frac{1}{2} \varrho \frac{|\mathbf{L}|^2}{r^2}$$

makes sense for any finite energy weak solution to the Euler-Korteweg-Poisson system.

Similarly to [13], [14], we derive an inequality describing the time evolution of \mathcal{E} .

Step 1

Taking \mathbf{L}/r as a test function in the momentum balance (1.15) we obtain

$$\begin{aligned} & - \int_{\Omega} \mathbf{J} \cdot \frac{\mathbf{L}}{r} \, dx \Big|_{t=\tau_1}^{t=\tau_2} = - \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\mathbf{J} \cdot \partial_t \left(\frac{\mathbf{L}}{r} \right) + \frac{\mathbf{J} \otimes \mathbf{J}}{\varrho} : \nabla_x \left(\frac{\mathbf{L}}{r} \right) + p(\varrho) \operatorname{div}_x \left(\frac{\mathbf{L}}{r} \right) \right] \, dx \, dt \\ & - \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\nabla_x \varrho \cdot \nabla_x \left[\varrho \operatorname{div}_x \left(\frac{\mathbf{L}}{r} \right) \right] - \frac{1}{2} |\nabla_x \varrho|^2 \operatorname{div}_x \left(\frac{\mathbf{L}}{r} \right) + \nabla_x \varrho \otimes \nabla_x \varrho : \nabla_x \left(\frac{\mathbf{L}}{r} \right) \right] \, dx \\ & + \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\mathbf{J} \cdot \left(\frac{\mathbf{L}}{r} \right) - \varrho \nabla_x \mathbf{V} \cdot \left(\frac{\mathbf{L}}{r} \right) \right] \, dx \, dt. \end{aligned} \quad (5.3)$$

Step 2

Similarly, the choice $\varphi = \frac{1}{2} |\mathbf{L}|^2 / r$ in (1.14) gives rise to

$$\frac{1}{2} \int_{\Omega} \varrho \frac{|\mathbf{L}|^2}{r^2} \, dx \Big|_{t=\tau_1}^{t=\tau_2} = \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\varrho \partial_t \left(\frac{|\mathbf{L}|^2}{r^2} \right) + \mathbf{J} \cdot \nabla_x \left(\frac{|\mathbf{L}|^2}{r^2} \right) \right] \, dx \, dt. \quad (5.4)$$

Step 3

Taking $\varphi = P'(r)$ in (1.14) we get

$$-\int_{\Omega} \varrho P'(r) \, dx \Big|_{t=\tau_1}^{t=\tau_2} = -\int_{\tau_1}^{\tau_2} \int_{\Omega} [\varrho \partial_t P'(r) + \mathbf{J} \cdot \nabla_x P'(r)] \, dx \, dt. \quad (5.5)$$

Step 4

Finally, the test function $\varphi = \Delta_x r$ in (1.14) yields

$$\int_{\Omega} \nabla_x \varrho \cdot \nabla_x r \, dx \Big|_{t=\tau_1}^{t=\tau_2} + \int_{\tau_1}^{\tau_2} \int_{\Omega} (\varrho \partial_t \Delta_x r + \mathbf{J} \cdot \nabla_x \Delta_x r) \, dx \, dt = 0, \quad (5.6)$$

Step 5

Summing up (5.1), (5.3 - 5.5) we obtain the *relative energy inequality*

$$\begin{aligned} \mathcal{E} \left(\varrho, \mathbf{J} \mid r, \mathbf{L} \right) \Big|_{t=\tau_1}^{t=\tau_2} &= \int_{\Omega} \left[\frac{1}{2} \frac{|\mathbf{J}|^2}{\varrho} - \mathbf{J} \cdot \frac{\mathbf{L}}{r} + \frac{1}{2} \varrho \frac{|\mathbf{L}|^2}{r} + P(\varrho) - P'(r)\varrho + p(r) + \frac{1}{2} |\nabla_x \varrho - \nabla_x r|^2 \right] \, dx \Big|_{t=\tau_1}^{t=\tau_2} \\ &\leq -\frac{1}{2} \int_{\Omega} |\nabla_x V|^2 \, dx \Big|_{t=\tau_1}^{t=\tau_2} - \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{|\mathbf{J}|^2}{\varrho} \, dx \, dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} (\varrho \partial_t \Delta_x r + \mathbf{J} \cdot \nabla_x \Delta_x r - \partial_t r \Delta_x r) \, dx \, dt \\ &\quad - \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\mathbf{J} \cdot \partial_t \left(\frac{\mathbf{L}}{r} \right) + \frac{\mathbf{J} \otimes \mathbf{J}}{\varrho} : \nabla_x \left(\frac{\mathbf{L}}{r} \right) + p(\varrho) \operatorname{div}_x \left(\frac{\mathbf{L}}{r} \right) \right] \, dx \, dt \\ &\quad - \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\nabla_x \varrho \cdot \nabla_x \left[\varrho \operatorname{div}_x \left(\frac{\mathbf{L}}{r} \right) \right] - \frac{1}{2} |\nabla_x \varrho|^2 \operatorname{div}_x \left(\frac{\mathbf{L}}{r} \right) + \nabla_x \varrho \otimes \nabla_x \varrho : \nabla_x \left(\frac{\mathbf{L}}{r} \right) \right] \, dx \\ &\quad + \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\mathbf{J} \cdot \left(\frac{\mathbf{L}}{r} \right) - \varrho \nabla_x \mathbf{V} \cdot \left(\frac{\mathbf{L}}{r} \right) \right] \, dx \, dt + \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\varrho \partial_t \left(\frac{|\mathbf{L}|^2}{r^2} \right) + \mathbf{J} \cdot \nabla_x \left(\frac{|\mathbf{L}|^2}{r^2} \right) \right] \, dx \, dt \\ &\quad - \int_{\tau_1}^{\tau_2} \int_{\Omega} [\varrho \partial_t P'(r) + \mathbf{J} \cdot \nabla_x P'(r) - \partial_t p(r)] \, dx \, dt \end{aligned}$$

that can be written in a more concise form:

$$\begin{aligned} \mathcal{E} \left(\varrho, \mathbf{J} \mid r, \mathbf{L} \right) \Big|_{t=\tau_1}^{t=\tau_2} &+ \int_{\tau_1}^{\tau_2} \int_{\Omega} \left(\mathbf{J} - \varrho \frac{\mathbf{L}}{r} \right) \cdot \left(\frac{\mathbf{J}}{\varrho} - \frac{\mathbf{L}}{r} \right) \, dx \, dt \quad (5.7) \\ &\leq -\frac{1}{2} \int_{\Omega} |\nabla_x V|^2 \, dx \Big|_{t=\tau_1}^{t=\tau_2} - \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho \frac{\mathbf{L}}{r} \cdot \left(\frac{\mathbf{J}}{\varrho} - \frac{\mathbf{L}}{r} \right) \, dx \, dt \\ &+ \int_{\tau_1}^{\tau_2} \int_{\Omega} (\varrho \partial_t \Delta_x r + \mathbf{J} \cdot \nabla_x \Delta_x r - \partial_t r \Delta_x r) \, dx \, dt - \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho \nabla_x V \cdot \left(\frac{\mathbf{L}}{r} \right) \, dx \, dt \\ &\quad + \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\partial_t \left(\frac{\mathbf{L}}{r} \right) + \frac{\mathbf{J}}{\varrho} \cdot \nabla_x \left(\frac{\mathbf{L}}{r} \right) \right] \cdot \left(\frac{\varrho}{r} \mathbf{L} - \mathbf{J} \right) \, dx \, dt \\ &- \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\frac{1}{4} \nabla_x \varrho \cdot \nabla_x \left(\varrho \operatorname{div}_x \left(\frac{\mathbf{L}}{r} \right) \right) - \frac{1}{2} |\nabla_x \varrho|^2 \operatorname{div}_x \left(\frac{\mathbf{L}}{r} \right) + \nabla_x \varrho \otimes \nabla_x \varrho : \nabla_x \left(\frac{\mathbf{L}}{r} \right) \right] \, dx \\ &+ \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[(r - \varrho) \partial_t P'(r) + (\mathbf{L} - \mathbf{J}) \cdot \nabla_x P'(r) + (p(r) - p(\varrho)) \operatorname{div}_x \left(\frac{\mathbf{L}}{r} \right) \right] \, dx \, dt. \end{aligned}$$

Remark 5.1. *The relative energy inequality (5.7) holds for any smooth “test” functions r, \mathbf{L}, r bounded bellow away from zero. Alternatively, we may use (5.7) as a definition of a dissipative solution in the spirit of Lions [21] in the context of the incompressible Euler system.*

5.2 Weak-strong uniqueness

Supposing that the Euler-Korteweg-Poisson sytem admits a *smooth* solution $\tilde{\varrho} > 0, \tilde{\mathbf{J}}$, we set $r = \tilde{\varrho}, \mathbf{L} = \tilde{\mathbf{J}}$ as test functions in the relative energy inequality (5.7) to obtain

$$\begin{aligned}
& \mathcal{E} \left(\varrho, \mathbf{J} \mid \tilde{\varrho}, \tilde{\mathbf{J}} \right) \Big|_{t=\tau_1}^{t=\tau_2} \leq -\frac{1}{2} \int_{\Omega} |\nabla_x V|^2 dx \Big|_{t=\tau_1}^{t=\tau_2} \tag{5.8} \\
& + \int_{\tau_1}^{\tau_2} \int_{\Omega} (\varrho \partial_t \Delta_x \tilde{\varrho} + \mathbf{J} \cdot \nabla_x \Delta_x \tilde{\varrho} - \partial_t \tilde{\varrho} \Delta_x \tilde{\varrho}) dx dt \\
& + \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho \left(\frac{\mathbf{J}}{\varrho} - \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \cdot \nabla_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \cdot \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} - \frac{\mathbf{J}}{\varrho} \right) dx dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} (p(\tilde{\varrho}) - p'(\tilde{\varrho})(\tilde{\varrho} - \varrho) - p(\varrho)) \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) dx dt \\
& + \int_{\tau_1}^{\tau_2} \int_{\Omega} \left(\frac{\varrho}{\tilde{\varrho}} \tilde{\mathbf{J}} - \mathbf{J} \right) \cdot \left(\nabla_x \Delta_x \tilde{\varrho} + \nabla_x \Delta_x^{-1}(\tilde{\varrho} - \varrho) \right) dx dt - \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho \nabla_x V \cdot \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) dx dt \\
& - \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\nabla_x \varrho \cdot \nabla_x \left(\varrho \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \right) - \frac{1}{2} |\nabla_x \varrho|^2 \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) + \nabla_x \varrho \otimes \nabla_x \varrho : \nabla_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \right] dx \\
& + \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[(\tilde{\varrho} - \varrho) \partial_t P'(\tilde{\varrho}) + (\tilde{\varrho} - \varrho) \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \cdot \nabla_x P'(\tilde{\varrho}) + p'(\tilde{\varrho})(\tilde{\varrho} - \varrho) \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \right] dx dt,
\end{aligned}$$

where, furthermore,

$$(\tilde{\varrho} - \varrho) \partial_t P'(\tilde{\varrho}) + (\tilde{\varrho} - \varrho) \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \cdot \nabla_x P'(\tilde{\varrho}) + p'(\tilde{\varrho})(\tilde{\varrho} - \varrho) \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) = (\tilde{\varrho} - \varrho) P''(\tilde{\varrho}) (\partial_t \tilde{\varrho} + \operatorname{div}_x \tilde{\mathbf{J}}) = 0,$$

and

$$\begin{aligned}
& \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\left(\frac{\varrho}{\tilde{\varrho}} \tilde{\mathbf{J}} - \mathbf{J} \right) \cdot \nabla_x \Delta_x^{-1}(\tilde{\varrho} - \varrho) - \frac{\varrho}{\tilde{\varrho}} \tilde{\mathbf{J}} \cdot \nabla_x \Delta_x^{-1}(\varrho - \tilde{\varrho}) \right] dx dt \\
& = \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{\varrho}{\tilde{\varrho}} \tilde{\mathbf{J}} \cdot \nabla_x \Delta_x^{-1}(\tilde{\varrho} - \varrho) dx dt - \int_{\Omega} \varrho \Delta_x^{-1}(\tilde{\varrho} - \varrho) dx \Big|_{t=\tau_1}^{t=\tau_2} - \int_{\tau_1}^{\tau_2} \int_{\Omega} \Delta_x^{-1}(\varrho - \tilde{\varrho}) \operatorname{div}_x \tilde{\mathbf{J}} dx dt \\
& = \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) \tilde{\mathbf{J}} \cdot \nabla_x \Delta_x^{-1}(\tilde{\varrho} - \varrho) dx dt - \int_{\Omega} \varrho \Delta_x^{-1}(\tilde{\varrho} - \varrho) dx \Big|_{t=\tau_1}^{t=\tau_2} + \int_{\tau_1}^{\tau_2} \int_{\Omega} \partial_t \tilde{\varrho} \Delta_x^{-1}(\tilde{\varrho} - \varrho) dx dt.
\end{aligned}$$

Consequently, after a simple manipulation, the relation (5.8) reads

$$\begin{aligned}
& \mathcal{E} \left(\varrho, \mathbf{J} \mid \tilde{\varrho}, \tilde{\mathbf{J}} \right) \Big|_{t=\tau_1}^{t=\tau_2} \leq \frac{1}{2} \int_{\Omega} (\tilde{\varrho} - \varrho) \Delta_x^{-1}(\tilde{\varrho} - \varrho) dx \Big|_{t=\tau_1}^{t=\tau_2} \tag{5.9} \\
& + \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho \left(\frac{\mathbf{J}}{\varrho} - \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \cdot \nabla_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \cdot \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} - \frac{\mathbf{J}}{\varrho} \right) dx dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} (p(\tilde{\varrho}) - p'(\tilde{\varrho})(\tilde{\varrho} - \varrho) - p(\varrho)) \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) dx dt
\end{aligned}$$

$$\begin{aligned}
& + \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) \tilde{\mathbf{J}} \cdot \nabla_x \Delta_x^{-1} (\tilde{\varrho} - \varrho) \, dx \, dt \\
& + \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{\varrho}{\tilde{\varrho}} \tilde{\mathbf{J}} \cdot \nabla_x \Delta_x \tilde{\varrho} \, dx \, dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} (\varrho \partial_t \Delta_x \tilde{\varrho} - \partial_t \tilde{\varrho} \Delta_x \tilde{\varrho}) \, dx \, dt \\
& - \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\nabla_x \varrho \cdot \nabla_x \left(\varrho \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \right) - \frac{1}{2} |\nabla_x \varrho|^2 \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) + \nabla_x \varrho \otimes \nabla_x \varrho : \nabla_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \right] \, dx.
\end{aligned}$$

The next step is formal in the sense that it requires higher regularity of ϱ but the final relation can be justified by means of a density argument. We write

$$\begin{aligned}
& - \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[\nabla_x \varrho \cdot \nabla_x \left(\varrho \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \right) - \frac{1}{2} |\nabla_x \varrho|^2 \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) + \nabla_x \varrho \otimes \nabla_x \varrho : \nabla_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \right] \, dx \\
& = - \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{\varrho}{\tilde{\varrho}} \tilde{\mathbf{J}} \cdot \nabla_x \Delta_x \varrho \, dx \, dt,
\end{aligned}$$

and, consequently,

$$\begin{aligned}
& \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{\varrho}{\tilde{\varrho}} \tilde{\mathbf{J}} \cdot \nabla_x \Delta_x (\tilde{\varrho} - \varrho) \, dx \, dt \\
& = \int_{\tau_1}^{\tau_2} \int_{\Omega} \tilde{\mathbf{J}} \cdot \nabla_x \Delta_x (\tilde{\varrho} - \varrho) \, dx \, dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) \cdot \nabla_x \Delta_x (\varrho - \tilde{\varrho}) \, dx \, dt \\
& = \int_{\tau_1}^{\tau_2} \int_{\Omega} \partial_t \tilde{\varrho} \Delta_x (\tilde{\varrho} - \varrho) \, dx \, dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) \cdot \nabla_x \Delta_x (\varrho - \tilde{\varrho}) \, dx \, dt.
\end{aligned}$$

Thus, going back to (5.9) we obtain

$$\mathcal{E} \left(\varrho, \mathbf{J} \mid \tilde{\varrho}, \tilde{\mathbf{J}} \right) \Big|_{t=\tau_1}^{t=\tau_2} - \frac{1}{2} \int_{\Omega} (\tilde{\varrho} - \varrho) \Delta_x^{-1} (\tilde{\varrho} - \varrho) \, dx \Big|_{t=\tau_1}^{t=\tau_2} \quad (5.10)$$

$$\begin{aligned}
& \leq \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho \left(\frac{\mathbf{J}}{\varrho} - \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \cdot \nabla_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \cdot \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} - \frac{\mathbf{J}}{\varrho} \right) \, dx \, dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} (p(\tilde{\varrho}) - p'(\tilde{\varrho})(\tilde{\varrho} - \varrho) - p(\varrho)) \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \, dx \, dt \\
& + \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) \cdot \nabla_x \Delta_x^{-1} (\tilde{\varrho} - \varrho) \, dx \, dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) \cdot \nabla_x \Delta_x (\varrho - \tilde{\varrho}) \, dx \, dt.
\end{aligned}$$

Finally,

$$\begin{aligned}
& \int_{\Omega} \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) \cdot \nabla_x \Delta_x (\varrho - \tilde{\varrho}) \, dx = \\
& = \int_{\Omega} \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \cdot \operatorname{div}_x \left[(\varrho - \tilde{\varrho}) \Delta_x (\varrho - \tilde{\varrho}) + \frac{1}{2} |\nabla_x (\varrho - \tilde{\varrho})|^2 \mathbb{I} - \nabla_x (\varrho - \tilde{\varrho}) \otimes \nabla_x (\varrho - \tilde{\varrho}) \right] \, dx \, dt \\
& = \int_{\Omega} \nabla_x (\varrho - \tilde{\varrho}) \otimes \nabla_x (\varrho - \tilde{\varrho}) : \nabla_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \, dx - \frac{3}{2} \int_{\Omega} \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) |\nabla_x (\varrho - \tilde{\varrho})|^2 \, dx + \int_{\Omega} (\varrho - \tilde{\varrho}) \nabla_x \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \cdot \nabla_x (\varrho - \tilde{\varrho}) \, dx.
\end{aligned}$$

Thus the relation (5.10) takes its final form

$$\begin{aligned}
& \mathcal{E} \left(\varrho, \mathbf{J} \mid \tilde{\varrho}, \tilde{\mathbf{J}} \right) \Big|_{t=\tau_1}^{t=\tau_2} - \frac{1}{2} \int_{\Omega} (\tilde{\varrho} - \varrho) \Delta_x^{-1} (\tilde{\varrho} - \varrho) \, dx \Big|_{t=\tau_1}^{t=\tau_2} \\
& \leq \int_{\tau_1}^{\tau_2} \int_{\Omega} \varrho \left(\frac{\mathbf{J}}{\varrho} - \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \cdot \nabla_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \cdot \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} - \frac{\mathbf{J}}{\varrho} \right) \, dx \, dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} \left(p(\tilde{\varrho}) - p'(\tilde{\varrho})(\tilde{\varrho} - \varrho) - p(\varrho) \right) \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \, dx \, dt \\
& \quad + \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} (\varrho - \tilde{\varrho}) \cdot \nabla_x \Delta_x^{-1} (\tilde{\varrho} - \varrho) \, dx \, dt \\
& \int_{\Omega} \nabla_x (\varrho - \tilde{\varrho}) \otimes \nabla_x (\varrho - \tilde{\varrho}) : \nabla_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \, dx + \frac{3}{2} \int_{\Omega} \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) |\nabla_x (\varrho - \tilde{\varrho})|^2 \, dx + \int_{\Omega} (\varrho - \tilde{\varrho}) \nabla_x \operatorname{div}_x \left(\frac{\tilde{\mathbf{J}}}{\tilde{\varrho}} \right) \cdot \nabla_x (\varrho - \tilde{\varrho}) \, dx.
\end{aligned} \tag{5.11}$$

Applying Gronwall's lemma to (5.11) we deduce that

$$\mathcal{E} \left(\varrho, \mathbf{J} \mid \tilde{\varrho}, \tilde{\mathbf{J}} \right) (t, \cdot) = 0, \quad t \in [0, T] \text{ as soon as } \mathcal{E} \left(\varrho, \mathbf{J} \mid \tilde{\varrho}, \tilde{\mathbf{J}} \right) (0, \cdot) = 0.$$

We have shown the following *weak-strong uniqueness property* of the Euler-Korteweg-Poisson system

Theorem 5.1. *Let $K(\varrho) = \overline{K}$ be a positive constant. Let ϱ, \mathbf{J} be a dissipative weak solution to the Euler-Korteweg-Poisson system (1.1 - 1.3), (1.6), with in $(0, T) \times \Omega$ such that*

$$\varrho(t, x) \geq \underline{\varrho} > 0 \text{ for a.a. } (t, x) \in (0, T) \times \Omega.$$

Suppose that the problem (1.1 - 1.3), (1.6) admits a classical (strong) solution $\tilde{\varrho}, \tilde{\mathbf{J}}$ in $(0, T) \times \Omega$ emanating from the same initial data as ϱ, \mathbf{J} .

Then

$$\varrho \equiv \tilde{\varrho}, \quad \mathbf{J} \equiv \tilde{\mathbf{J}}.$$

Remark 5.2. *Having finished this paper, we have learned that more general results of the same type were obtained by Giesselmann, Lattanzio and Tzavaras [19].*

Remark 5.3. *The existence of local-in-time regular solutions and the existence of global-in-time regular solutions where the initial data are taken as small perturbations of subsonic steady states to the quantum hydrodynamics system was proved by Li and one of the authors in [20].*

5.3 Concluding remarks

Summarizing the previous discussion we may infer that:

- The Euler-Korteweg-Poisson system admits *infinitely many* weak solutions for *any* sufficiently smooth initial data. The solutions are defined on an arbitrary time interval $(0, T)$, where they satisfy the energy inequality, with a possible exception of the initial time $t = 0$.
- The dissipative weak solution satisfy the energy as well as the relative energy inequality in $(0, T)$. They coincide with the strong solution emanating from the same initial data as long as the latter exists. In other words, the strong solutions are unique in the class of weak solutions.

In the light of the above arguments, it may seem plausible to eliminate the majority of the “strange” weak solutions obtained by the method of convex integration by stipulating the energy (relative energy) inequality. Unfortunately, however, a nowadays straightforward modification of the method of convex integration yields the following result that can be proved, given the oscillatory lemma 3.1, by the arguments specified in [9].

Theorem 5.2. *Let ϱ_0 be given in the class specified in Theorem 2.1. Let $T > 0$ be an arbitrary positive time.*

Then there exists the initial distribution of the momentum

$$\mathbf{J}_0 \in L^\infty(\Omega; \mathbb{R}^3)$$

such that the Euler-Korteweg-Poisson system (1.1 - 1.3) admits infinitely many dissipative weak solutions ϱ, \mathbf{J} in $(0, T) \times \Omega$,

$$\varrho(0, \cdot) = \varrho_0, \mathbf{J}(0, \cdot) = \mathbf{J}_0.$$

Sketch of the proof.

We start with the ansatz $\varrho(0, \cdot) = \varrho_0, \mathbf{U}(0, \cdot) = \mathbf{U}_0 \equiv 0$, which yields $Z \equiv 0$ in (3.2), in particular, $\varrho(t, \cdot) = \varrho_0$ is the unique solution of (3.1) for any $t > 0$. As shown in Section 3, the technique of convex integration produces (infinitely many) weak solutions $[\tilde{\varrho} \equiv \varrho, \tilde{\mathbf{J}}]$ emanating from ϱ_0, \mathbf{U}_0 , satisfying the energy inequality (5.1) in the *open* interval $(0, T)$, where, in addition, the rate of total dissipation may be controlled by a suitable choice of the function h in (5.1). As ϱ is constant in time, the desired initial distribution of the momentum \mathbf{J}_0 can be taken as $\mathbf{J}_0 = \tilde{\mathbf{J}}(\tau, \cdot)$ at a suitable point $\tau \in (0, T)$ where the energy is continuous. Finally, the method of convex integration can be applied to produce infinitely many solutions for the initial data ϱ_0, \mathbf{J}_0 as soon as we show that the set of subsolutions $X_{0,\varepsilon}$ is non-empty. However, with Lemma 4.1 at hand, a suitable subsolution can be constructed exactly as in [9, Section 4.2].

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