Gran Sasso Science Institute

**PHD PROGRAMME IN MATHEMATICS IN NATURAL, SOCIAL AND LIFE SCIENCES**

Cycle XXX

09/11/2018

# Asset price bubbles in financial networks

PHD CANDIDATE
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# Acknowledgements

First of all, I would like to thank a lot professors Francesca Biagini and Thilo Meyer-Brandis for their constant support and guidance through every stage of my PhD. Every time I have needed any suggestion, you were there for me despite your countless commitments, and this means a lot along such a long way with some, unavoidable, difficult moments. Your great mathematical expertise has been essential not only for my research but also for my professional growth. Every mistake I have done has been an opportunity for you to teach me something with a lot of patience, and I am extremely grateful for this. However, I think that one of the most important things in every working place is feeling a nice human atmosphere, because it's what makes you glad to spend five days per week there. Well, this has been the case since the first days I came to LMU three years ago, thanks to your kindness and ability to make me feeling comfortable.

Of course, this is also merit of all my colleagues of the Workgroup Financial Mathematics of the LMU during these three years, and I would like to thank you all. A special thought goes to the two of them who have been my office mates, Jacopo and Martin. Apart from the fact that you support the wrong football teams (in particular you, Jacopo) it has been and it is a pleasure to spend so many hours with you guys, laughing and chatting together during the breaks.

Of course, I also express all my gratitude to all the staff of GSSI for their support during these years. In particular, I thank prof. Pierangelo Marcati for his help and for the opportunity he was able to find for me here in Munich.

Ci tengo molto a ringraziare il prof. Andrea Pascucci di Bologna, che mi ha introdotto alla Finanza Matematica. Penso sia fondamentale trovare un primo "mentore" che trasmetta la passione per la materia e soprattutto che incoraggi a seguire una certa strada. Andrea Pascucci ha avuto questo ruolo prima durante il Corso di Laurea e poi durante la tesi e il master in Finanza Matematica, quando ho capito che avrei voluto provare a intraprendere la strada della ricerca. A questo proposito, non dimentico la chiacchierata col prof. Sergio Polidoro, sempre durante il master, quando mi ha detto chiaro e tondo che secondo lui ero portato per fare un dottorato e avrei dovuto perseguire questo obiettivo. Per me è stata una spinta fondamentale, e gliene sarò sempre molto grato.

Infine, il ringraziamento più importante va alla mia famiglia, genitori e fratelli, sempre un sostegno fondamentale nei momenti un po' più stressanti. Questo importante traguardo è davvero anche vostro.

# Abstract

Asset price bubbles are commonly defined as the difference between the market value of an asset and its fundamental value. In this work, we study the two main stages characterizing the evolution of an asset price bubble: a first phase when the bubble takes place and blows up, and a second period when the burst of the bubble can affect financial institutions, leading to a crisis. Networks play an essential role in our analysis: we study how an investors network influences the evolution of the bubble through trading effects via contagion between investors, and in which way a bubble alters the structure of a banking network due to preferential attachment mechanisms in investments between financial institutions.

The first part of the thesis is devoted to the analysis of the first phase of the evolution of a bubble. In our approach, we follow the so called martingale theory of bubbles, recently developed starting from the assumption of absence of arbitrage. However, we slightly move the focus of the analysis: whereas the classic martingale theory of bubble spotlights the fundamental price of an asset, assuming that the market price is exogenously given, we start from a constructive model given by Jarrow et al. [2012], where the fundamental value is exogenously given and the market price can deviate from it due to illiquidity effects. This makes it possible to directly model the fast increase of the market value commonly observed when a bubble takes place. We embed this model in the martingale theory of bubbles, finding a flow of equivalent local martingale measures for the market wealth of the asset such that the fundamental wealth is justified as the expectation of discounted future earnings.

In particular, we model the dynamics of the bubble as influenced by a contagion mechanism spreading among investors within a financial network. We show that the spread of contagion strongly depends on some characteristics of the network. In this way, the structure of the network affects the evolution of the bubble, through the impact of the trades on the price due to the illiquidity effects mentioned above.

On the other hand, we study the effects on the economy of the burst, and show how a bubble can influence the structure of a banking network. We consider a banking network represented by a system of stochastic differential equations coupled by their drift. We assume a core-periphery structure, and that the banks in the core hold a bubbly asset. The banks in the periphery have not direct access to the bubble, but can take initially advantage from its increase by investing on the banks in the core. Investments are modeled by the weight of the links, which is a function of the robustness of the banks. In this way, a preferential attachment mechanism towards the core takes place during the growth of the bubble. We then investigate how the bubble distort the shape

of the network, both for finite and infinitely large systems, assuming a non vanishing impact of the core on the periphery. Due to the influence of the bubble, the banks are no longer independent, and the law of large numbers cannot be directly applied at the limit. This results in a term in the drift of the diffusions which does not average out, and that increases systemic risk at the moment of the burst. We test this feature of the model by numerical simulations.

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# List of Symbols

$\mathcal{B}(\mathbb{R})$ ..... Borel $\sigma$-algebra on real numbers
$\beta = (\beta_t)_{t\geq 0}$ ..... size of the bubble
$\beta^Q = (\beta_t^Q)_{t\geq 0}$ ..... size of the $Q$-bubble
$D = (D_t)_{t\geq 0}$ ..... cumulative cash flow process generated by the asset
$\mathbb{E}[\cdot]$ ..... expectation under the *real world* probability measure $P$
$\mathbb{E}_Q[\cdot]$ ..... expectation under the pricing probability measure $Q$
$\mathbb{E}[\cdot|\cdot]$ ..... conditional expectation under the *real world* probability measure $P$
$\mathbb{E}_Q[\cdot|\cdot]$ ..... conditional expectation under the pricing probability measure $Q$
$\mathcal{E}(X)$ ..... stochastic exponential of a stochastic process $X$: $\mathcal{E}(X)_t = \exp\{X_t - X_0 - 1/2[X,X]_t\}$
$\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ ..... filtration, satisfying the usual hypothesis of completeness and right continuity
$\Lambda = (\Lambda_t)_{t\geq 0}$ ..... resiliency of the limit order book
$M = (M_t)_{t\geq 0}$ ..... illiquidity measure
$\mathcal{M}_{loc}(W)$ ..... the set of all equivalent local martingale measures for $W$
$\mathcal{M}_{UI}(W)$ ..... the subset of $Q \in \mathcal{M}_{loc}(W)$ for which $W$ is a uniformly integrable martingale
$\mathcal{M}_{NUI}(W)$ ..... the set given by $\mathcal{M}_{loc}(W) \setminus \mathcal{M}_{UI}(W)$
$\mathbb{N}$ ..... the set of natural numbers
$(p_k)_{k\in\mathbb{N}}$ ..... degree distribution of investors in a financial network
$P, Q, R$ ..... probability measures on the space $(\Omega, \mathcal{F})$
$(Q^t)_{t\in[0,T)}$ ..... flow of equivalent local martingale measures for the market wealth $W$
$\mathbb{R}$ ..... the set of real numbers
$\mathbb{R}^+$ ..... the set of positive real numbers
$\mathcal{R} = (R_t)_{t\geq 0}$ ..... flow in the space $\mathcal{M}_{loc}(W)$
$S = (S_t)_{t\geq 0}$ ..... market price of the asset
$\tilde{S} = (\tilde{S}_t)_{t\geq 0}$ ..... quoted (or marginal) price of the asset
$S^Q = (S_t^Q)_{t\geq 0}$ ..... fundamental price of the asset perceived under the measure $Q$
$T$ ..... random time representing the liquidation time of the asset
$W = (W_t)_{t\geq 0}$ ..... market wealth of the asset
$W^Q = (W_t^Q)_{t\geq 0}$ ..... fundamental wealth of the asset perceived under the measure $Q$
$X = (X_t)_{t\geq 0}$ ..... signed volume of market orders
$Z = (Z_t)_{t\in[0,T)}$ ..... density process of $Q$ with respect to $P$
$\rho^i = (\rho_t^i)_{t\geq 0}$ ..... financial robustness of a bank $i$ not holding the bubbly asset
$\bar{\rho}^{B,k} = (\bar{\rho}_t^{B,k})_{t\geq 0}$ ..... financial robustness of a bank $k$ holding the bubbly asset in the limit system

$\bar{\rho}^i = (\bar{\rho}^i_t)_{t\geq 0}$ ..... financial robustness of a bank $i$ not holding the bubbly asset in the limit system
$(\Omega, \mathcal{F}, P)$ ..... probability space
$(\Omega, \mathcal{F}, \mathbb{F}, P)$ ..... filtered probability space
$\mathbb{1}_{\{\cdot\}}$ ..... stochastic process that takes value 1 when the condition inside brackets is satisfied and 0 otherwise
$[X, Y]_t$ ..... quadratic variation of two stochastic processes $X$ and $Y$ at time $t$

# Chapter 1

# Introduction

## 1.1 Motivation

Financial bubbles are defined as deviations of the market price of an asset from its intrinsic value, usually computed as the expectation of the sum of the discounted future income generated by the asset. They are typically characterized by a sensational price increase followed by a crash, often leading to a financial crisis.
Such dramatic episodes have regularly taken place through the last centuries, from the so called Dutch tulip mania (1634-1637), until the very recent bitcoin bubble: among the most famous economic bubbles we mention for example the Mississippi bubble (1719-1721), originated by the rise and fall of the Compagnie des Indes, the South See bubble (1720), when the company's share price increased from about 120 pounds in January to 775 pounds in August, the Roaring Twenties stock market bubble in the US, the Japanese housing bubble (1970-1989), the Dot-Com bubble (1997-2002), the United States housing bubble of 2002-2006 and the China stock and property bubble (2003-2007).
The Tulip mania is the earliest bubble in recorded history, and one of the most striking examples of an extraordinary increase of the price of an asset, far away from its commonly perceived value. After having been introduced to Europe in the 16th century from the Ottoman Empire, tulip bulbs became a sort of fashionable status symbol among the Dutch population, giving rise soon to a speculative furor: in 1636 formal futures markets were created where contracts to buy bulbs at the end of the season were bought and sold, and at the peak of the bubble, the price of tulip bulbs increased twenty-fold from November 1636 to February 1637. The burst of the bubble took place when some prudent people decided to sell and crystallize their profits: as it often happens, a domino effect of progressively lower and lower prices took place as everyone tried to sell while not many were buying. The price began to dive, causing people to panic and sell regardless of losses.
Although some studies of the last decades assert that the prices of tulip bulbs were far more rational than was commonly perceived (see for example Garber [2000]), already in 1841 the Scottish journalist Charles Mackay named the mechanism behind the Tulip boom with the expression *Madness of crowd*: this term would had been used a lot of times again in the history of bubbles to describe the fever spreading among (often unskilled) investors making the prices increase.

Such an expression could be associated to the above mentioned Dot-Com bubble, a well known example of the diffusion of a speculative mania among investors taking place after a technologic innovation. At the end of the 1990s, the rapid development of the Internet stimulated a huge interest and excitement among population in the US: many investors were eager to invest, at any valuation, in any company that had one of the Internet-related prefixes or a ".com" suffix in its name. Due also to the low interest rates of 1998-99 that helped to increase the availability of funding (see Weinberger [2016]), an unprecedented amount of personal investing occurred during the boom (the press reported the phenomenon of people quitting their jobs to engage in full-time day trading, Kadlec [1999]), leading to to the biggest bubble of the 20th century.

A few examples can clearly illustrate the huge over-valuation of some little companies, that captured people's attention due to the internet mania. The book retailer *Books-a-Million* saw its stock price lift from around \$3 per share on November 25, 1998 to an intra-day high of \$47 five days later, before going down again to \$3 by 2000, while the price per share of the public company *e.Digital* rose from \$0.06 per share in January 1999 to a high of \$24.50 one year later. Another extreme example is the one of *Geeknet*, a provider of built-to-order Intel systems: the first day after its initial public offering, when the price was initially set at \$30 per share, ended at a valuation of \$239.25 per share.

After some time, the failure of many Internet firms to produce real earnings eventually caused a new pessimistic feeling among investors. This inversion in the mood of the market, together with other economic factors, made the bubble burst, with an extremely high speed. *InfoSpace*, whose stock reached at the peak a price $\$1,305$ per share, suffered such an intense loss that, by 2002, the price had declined to \$2 per share. Similarly, *VerticalNet* was valued at \$1.6 billion after its initial public offering, despite only having \$3.6 million in quarterly revenue, and *Lycos*, purchased by *Terra Networks* for \$12.5 billion in 2000, was sold in 2004 for \$95.4 million.

The consequences of such crashes on the economy are an extremely critical issue. During the 20th and the beginning of the 21st century, various bubbles led after the burst to financial crisis often extended far beyond the region where the bubble took place.

One of the most famous cases is given by the Roaring Twenties bubble. Propelled by new technologies that made possible mass production of consumer goods such as automobiles and radios, the bubble ended dramatically with the Stock Market Crash of October 29, 1929, giving rise to the Great Depression. This was the longest, deepest, and most widespread depression of the 20th century: between 1929 and 1932, worldwide gross domestic product fell by an estimated 15%, industrial production drop by $-46\%$ in the US and $-41\%$ in Germany, and unemployment increased more than six times in the US and more than two times in Germany and France.

Asian markets have shown several episodes of bubbles followed by serious financial crisis in the last decades. Two main examples are the Asian financial crisis in 1997, originated by the financial collapse of the Thai baht, which heavily involved the economies of countries as South Korea and Indonesia, and the Chinese stock bubble of 2007, whose burst caused a 9% loss of the

Shanghai Stock Exchange, followed by a drop of 416 points of the Dow Jones Industrial Average.
Another famous example is the Financial crisis of 2007-2008, triggered by the burst of the US housing bubble, which peaked in 2007.
From the examples described above, three main points of investigation about bubbles can be identified: which are the mechanisms that first trigger and then fuel the bubble, by which event the burst of the bubble can be determined, and which are the consequences on the economy of the burst of the bubble, that is, how the financial system reacts to the crash following the burst. The purpose of the present work is to provide an insight on the first and on the third question, examining the two main phases characterizing the boom and bust cycle typical of financial bubbles.

## 1.2 Overview of the thesis

The building up phase of a bubble is analyzed in Chapters 2, 3 and 4 of the thesis. In particular, Chapters 2 and 3 are devoted to the study of a coherent mathematical framework for bubbles. One of the main approaches in this sense is given by the martingale theory of bubbles as introduced by Cox and Hobson [2005] and Loewenstein and Willard [2000] and mainly developed in Jarrow and Protter [2009, 2011], Jarrow et al. [2007, 2010, 2011] and Biagini et al. [2014]. In this setting a $Q$-bubble is defined as the difference between the market wealth $W$ of a given financial asset and its fundamental wealth $W^F$, given by the expectation of future discounted dividends under an equivalent local martingale measure $Q$. As we specify in Chapter 2, defined in this way the bubble is a non-negative local martingale under $Q$, and it is strictly positive if and only if it is not a uniformly integrable martingale under $Q$. Since $W^F$, as an expectation, is a uniformly integrable martingale, the bubble is strictly positive if and only if $W$ is not a uniformly integrable martingale.
In Biagini et al. [2014], a continuous flow $\mathcal{R} = (\mathcal{R}_t)_{t\geq 0}$ in the space of martingale measures is considered, moving from an initial measure $Q$ under which $W$ is a uniformly integrable martingale (and then no bubble is perceived) to a measure $R$ under which $W$ is no more a uniformly integrable martingale (and then the bubble is fully perceived) via convex combinations of $Q$ and $R$, which put an increasing weight on $R$. Conditions are given on the flow $\mathcal{R}$ under which the bubble perceived under $R$ is an initial local submartingale that then turns into a supermartingale before it falls back to its initial value zero.
In Chapter 2, after having introduced the model, we enlarge the framework of Biagini et al. [2014] relaxing the conditions on $\mathcal{R}$: we show that under these new assumptions the bubble has a Doob-Meyer decomposition such that in general the local submartingale property is lost, but we are nonetheless able to find some specific examples under which the perceived bubble is still a local submartingale under $R$.
In order to investigate a bubble on the side of price formation, however, one should draw the attention from the fundamental value to the market value of the asset. The formation of asset price bubbles has been thoroughly investigated from an economical point of view in many contributions, see Abreu and Brunnermeier [2003], Allen and Gale [2000a], Choi and Douady [2011a,b],

DeLong et al. [1990], Earl et al. [2007], Föllmer [2005], Harrison and Kreps [1978], Kaizoji [2000], Miller [1977], Scheinkman and Xiong [2003, 2013], Tirole [1982], Xiong [2012], Zhuk [2013]. Different causes have been indicated as triggering factors for bubble birth, such as heterogenous beliefs between interacting agents (as in Föllmer [2005], Harrison and Kreps [1978], Scheinkman and Xiong [2003], Scheinkman and Xiong [2013], Xiong [2012], Zhuk [2013]), a breakdown of the dynamic stability of the financial system (see Choi and Douady [2011a], Choi and Douady [2011b]), the diffusion of new investment decision rules from a few expert investors to larger population of amateurs (see Earl et al. [2007]), the tendency of traders to choose the same behavior as the other traders' behavior as thoroughly as possible (see Kaizoji [2000]), the presence of short-selling constraints (see Miller [1977]). Moreover, in Biagini and Nedelcu [2015], the formation of a bubble is explained in the valuation of defaultable claims as caused by the trading activity of investors, overestimating the safety of the claim under certain circumstances.
On the other hand, an alternative model is given by Jarrow et al. [2012], where the market value is endogenously determined by the trading activity of investors, and studied through the analysis of the liquidity supply curve. In this setting a bubble is still defined as the difference between the market wealth $W$ and the fundamental wealth $W^F$, however it does not always coincide with the $Q$-bubble under a given equivalent martingale measure $Q$.
A natural question, which we address in Chapter 3, is then if it is possible to embed such a constructive model, where the fundamental price is exogenous and the market price endogenous, in the martingale theory of bubbles. In order to do this, one should determine a suitable flow of ELMMs for $W$ under which $W^F$ is justified from a fundamental point of view. In particular, given a liquidation time $T$ for the financial asset, we look for a flow $(Q^t)_{t\in[0,T)}$ of ELMMs for the market wealth $W$ such that the fundamental value of the asset is given as the expectation of the future discounted cash flow. Our main result is then that we can explicitly determine the form of such a flow of ELMMS in a liquidity driven model under very general assumptions. This require a consistent technical effort, mostly devoted to guarantee the martingale property of the chosen flows of (eventual) probability densities. In this way, we are able to directly connect the impact of the underlying macro-economic factors to the shift of the resulting pricing measure, which may change over time.
As an application of our method, in Chapter 4 we focus on the mechanism underlying the formation of a bubble in a financial network, and compute the generating flow of ELMMs. In particular, we study how the interaction of market participants in a financial network can produce the herding behavior and the speculative fever underlined in the historical examples given in Section 1.1, and ultimately affect asset price formation and the consequent birth of a bubble.
We provide numerical simulations to investigate how different networks generate different contagion mechanisms and lead to bubbles with different evolutions. In particular, it turns out that in more heterogenous networks (i.e. networks with a more right skewed degree distribution) contagion spread faster at the beginning, so that the bubble builds up faster and bursts sooner: the nodes with high degree, which in average get infected faster, contribute with an higher weight in the more right skewed distributions.

In Chapter 5 we focus on the second phase of the cycle of a bubble, as we study the consequences of the burst in a financial system. Specifically, we are interested in examining how the financial distress following a shock can propagate within a banking system: the main question is in which extent a loss suffered by a bank can propagate to other banks holding shares of the first one.
In particular, we analyze the so called *financial robustness* of banks, defined here, as done by Battiston et al. [2012] and Hull and White [2001], as an indicator of agent's creditworthiness or distance to default. For this purpose, we consider a banking network represented by a system of stochastic differential equations coupled by their drift. We model the weight of the links as a function of the robustness of the banks, so that the attractiveness of a node depends on its "fitness", as in the preferential attachment model of Bianconi and Barabàsi [2001]. We assume a core-periphery structure, and suppose that the banks of the core happen to hold a bubbly asset. We see that, due to the preferential attachment model introduced above, the bubble causes a distortion in the network: the banks holding the bubble have a bigger influence on the system, since they are attracting investments of other institutions due to their perceived robustness. For this reason, they are not only the most exposed to the crash deriving from the burst of the bubble, but also the biggest propagators of a possible shock.
Supposing that the number of banks holding the bubble remains constant, but that their impact on the periphery does not vanish when the total number of the banks goes to infinity, we also study the case of large networks: due to the influence of the banks in the core, the institutions of the system share a common stochastic source produced by the bubble, so that they are not pairwise independent, and the classic law of large numbers cannot be applied. We are anyway able to compute the system at the limit, and we see that the influence of the bubble, through the action of the banks in the core, ultimately results in a term which does not average out in the drift of the diffusions: because of this term, also the banks of the periphery are indirectly affected when the bubble bursts.
This results in a riskier system, as shown in simulations where we investigate how the burst of a bubble impacts the structure of the network, and ultimately the systemic risk.

# Chapter 2

# Martingale theory of bubbles

## 2.1 Motivation

A bubble takes place if there is a discrepancy between the market price of an asset and its fundamental value. The market price is the amount that the marginal buyer is willing to pay for the asset. In order to have a market which excludes arbitrage opportunities, one assumes that there exists a probability measure $Q$, equivalent to the so called *real world* probability measure $P$, that turns the market price process into a local martingale. Such a measure $Q$ is called an equivalent local martingale measure (ELMM). In this way, the problem of fair pricing of contingent claims is reduced to take expected values with respect to the measure $Q$, and the fundamental value of the asset is defined as the expected sum of future discounted dividends under $Q$.
The classical theory of mathematical finance only considers finite horizon models. In this framework, the presence of financial bubbles is excluded in an arbitrage-free market, since there is no difference between the market price, the arbitrage-free price, and the fundamental price, as stated by Harrison and Kreps [1978].
However, the modern theory of mathematical finance, introduced by the papers of Delbaen and Schachermayer [1994, 1998], where the correct formulation for the absence of arbitrage is formulated in full generality, goes beyond these limitations and permits the existence of bubbles in terms of strict local martingales. This analysis was first introduced in 2000 by Loewenstein and Willard [2000], in the framework of the "no free lunch with vanishing risk" hypothesis. In the work of Cox and Hobson [2005], a bubble is defined to be a price process which, when discounted, is a strict local martingale under the risk-neutral measure, and it can take place as a result of a feedback mechanism. The authors also show that, in the presence of a bubble, put-call parity fails, the price of an American call exceeds that of a European call and call prices are no longer increasing in maturity for a fixed strike.
An important contribution in the context of the martingale theory of bubbles is provided by the work of Jarrow et al. [2007], where bubbles are analyzed in the framework of a complete market, i.e. under the assumption that only an equivalent martingale measure $Q$ exists. The authors prove that an asset maturing at a stopping time $T$ can have three types of bubbles: a strictly positive bubble is always a local martingale, and in particular if $P(T < \infty) < 1$ it can also be a uniformly integrable martingale, if $P(T < \infty) = 1$ but $T$

is not bounded it can be a martingale, but not uniformly integrable, and if $T$ is bounded it is a strict local martingale. Nonetheless, since the bubbles of the first type are not interesting from an economic point of view (because they represent a permanent gap between an asset's fundamental value and its market price), a strictly positive bubble as it is commonly defined, is a local martingale under $Q$ but not a uniformly integrable martingale.
However, in the case of a complete market, since only one equivalent local martingale measure exists, just two possibilities are given: either no bubble appears at all, or a bubble is already present at the beginning. This is a strong modeling withdraw, since it contradicts economic intuition.
Thus, incomplete markets have been taken into consideration by the same authors in Jarrow et al. [2010]. Since more than one ELMMs exists, the fundamental value of the asset, and then the bubble, depends on the measure taken into consideration. The idea is that the market "chooses" different local martingale measures across time, giving rise to a shift of local martingale measures: this correspond to regime shifts in the underlying economic fundamentals (because of new technologies, changes in beliefs or risk aversion). In this way, the birth and the evolution of a bubble are determined by a flow of different ELMMs that gives rise to a corresponding shifting perception of the fundamental value of the asset. For example, it can be that at the beginning the bubble is a uniformly integrable martingale under the selected measure $Q_1$, and has therefore value zero, and then another ELMM $Q_2$ is chosen so that the bubble is no more uniformly integrable: in this way, the bubble takes place during the passage from $Q_1$ to $Q_2$. At every time, the martingale measure selected by the market is computed as the one consistent with the prices of derivatives, and the switches are supposed to happen only at given stopping times.
On the other hand, Biagini et al. [2014] consider the continuos case, i.e. they model the slow birth of a perceived bubble as given by a continuos shift in the space of ELMMs, and they give conditions under which the bubble starts as a local submartingale and vanishes continuously as a supermartingale. This property is particularly nice since it shows the two commonly observed phases of a bubble: an increase (in expectation) followed by a decrease (again, in expectation).
In this chapter, we first introduce the model and the results of Biagini et al. [2014], and then we relax the assumptions. It can be seen that, in general, the bubble is no more a local submartingale at the beginning of its life. However, we give some examples of flows for which the local submartingale property still holds.

## 2.2 The setting

Consider a risky asset generating an uncertain cumulative cash flow, modeled as a non-negative and adapted right-continuous process $D = (D_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ that satisfies the usual hypothesis [1].

[1] A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfies the usual hypothesis if $\mathcal{F}_0$ contains all the $P$-null sets of $\mathcal{F}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, i.e. $\mathcal{F}_t = \cap_{u>t}$, $0 \leq t < \infty$, see Protter [2005].

The market price of the asset is given by the non-negative, adapted càdlàg process $S = (S_t)_{t\geq 0}$, while the corresponding wealth process denoted $W = (W_t)_{t\geq 0}$ is defined by

$$W_t = S_t + D_t, \quad t \geq 0.$$

Denote by $\mathcal{M}_{loc}(W)$ the class of all probability measures $Q \approx P$ such that $W$ is a local martingale under $Q$, by $\mathcal{M}_{UI}(W)$ the class of measures $Q \approx P$ under which $W$ is an uniformly integrable martingale and set $\mathcal{M}_{NUI}(W) = \mathcal{M}_{loc}(W) \setminus \mathcal{M}_{UI}(W)$. It is assumed that $\mathcal{M}_{loc}(W) \neq 0$, so that arbitrages are excluded.
Take a measure $Q \in \mathcal{M}_{loc}(W)$. The fundamental price of the asset perceived under $Q$ is then defined as $S^Q = (S_t^Q)_{t\geq 0}$ with $S_t^Q := \mathbb{E}_Q[D_\infty - D_t|\mathcal{F}_t]$. On the other hand, the fundamental wealth perceived under $Q$ is $W^Q = (W_t^Q)_{t\geq 0}$ with $W_t^Q := \mathbb{E}_Q[D_\infty|\mathcal{F}_t] = S_t^Q + D_t$.
In this way, the bubble perceived under $Q$ is $\beta^Q = (\beta_t^Q)_{t\geq 0}$ with

$$\beta_t^Q = S_t - S_t^Q = W_t - W_t^Q, \quad t \geq 0.$$

The bubble is positive, as can be seen from Lemma 2.3 in Biagini et al. [2014], and in particular it is null under $Q \in \mathcal{M}_{UI}(W)$ and strictly positive under $R \in \mathcal{M}_{NUI}(W)$ (see Corollary 2.11 in Biagini et al. [2014]).
In order to capture the slow birth of a bubble starting from an initial value 0, in Biagini et al. [2014] the authors consider a flow $\mathcal{R} = (R_t)_{t\geq 0}$ in the space $\mathcal{M}_{loc}(W)$, describing a shifting system of predictions $(R_t[\cdot|\mathcal{F}_t])_{t\geq 0}$. In particular, they focus on a flow $\mathcal{R}$ that begins in $\mathcal{M}_{UI}(W)$ and then enters the class $\mathcal{M}_{NUI}(W)$, via adapted convex combinations. In this way, the birth of the bubble corresponds to the time when the flow enters in the subspace $\mathcal{M}_{NUI}(W)$.
Specifically, they take the flow of conditional distributions

$$R_t[\cdot|\mathcal{F}_t] = \xi_t R[\cdot|\mathcal{F}_t] + (1 - \xi_t)Q[\cdot|\mathcal{F}_t], \quad t \geq 0, \tag{2.1}$$

with $Q \in \mathcal{M}_{UI}(W)$, $R \in \mathcal{M}_{NUI}(W)$ and $\xi = (\xi_t)_{t\geq 0}$ adapted and càdlàg process with values in $[0, 1]$ and starting from $\xi_0 = 0$. Such a flow puts weight $\xi_t$ on the predictions coming from the martingale measure $R$ and the remaining weight on the prediction under $Q$.
Then the bubble perceived under the flow of ELMMs, called the $\mathcal{R}-$ bubble $\beta^{\mathcal{R}} = S - S^{\mathcal{R}}$, is given by

$$\beta_t^{\mathcal{R}} = \xi_t(S_t - S_t^R) = \xi_t \beta_t^R, \quad t \geq 0. \tag{2.2}$$

The model might have a microeconomic interpretation: the two martingale measures $Q$ and $R$ could be seen as the views of two financial "gurus", one optimistic and one pessimistic, each one of them having a group of followers. At the beginning, most of the investors follows the optimistic guru $Q$, but a trigger event can happen so that some of them move to $R$. Afterwards, due to contagion effects, more investors are attracted by the view of the pessimistic guru, so that the proportion between the two groups shifts towards $R$. The same does $\mathcal{R}$, the weighted average of $Q$ and $R$, depending on the present weights of the two groups.

## 2.3 The slow birth of the bubble as a local submartingale

We present first the results proved by Biagini et al. [2014]. As in Biagini et al. [2014], for the sake of simplicity we take the following

**Assumption 2.3.1.** *Suppose that the filtration is such that all martingales have continuous paths.*

**Remark 2.3.2.** *By Assumption 2.3.1 it can be seen, by stopping, that every local martingale has continuous paths. In particular, since it is a local martingale under $R$, $\beta^R$ has continuous paths. This does not hold in general for $\beta^{\mathcal{R}}$, that is in fact a local martingale under the flow $\mathcal{R}$.*
*From Itô's integration by parts, it holds*

$$\beta_t^{\mathcal{R}} = \xi_t \beta_t^R = \int_0^t \beta_s^R d\xi_s + \int_0^t \xi_{s-} d\beta_s^R + [\xi, \beta^R]_t, \quad t \geq 0,$$

*and it can be seen that $\beta^{\mathcal{R}}$ is continuous if and only if $\xi$ is continuous.*

We restate now Proposition 3.5 of Biagini et al. [2014].

**Proposition 2.3.3.** *If the process $\xi$ is increasing then the $\mathcal{R}-$bubble $\beta^{\mathcal{R}}$ is a local submartingale under $R$. If $\xi$ remains constant after some stopping time $\tau_1$, then after time $\tau_1$ the bubble $\beta^{\mathcal{R}}$ becomes a (positive) local martingale, and then a supermartingale.*

If $\xi$ is no longer increasing, suppose $\xi$ to be a special semimartingale with values in $[0, 1]$, having canonical decomposition

$$\xi = M^\xi + A^\xi, \tag{2.3}$$

with $M^\xi$ local R-martingale and $A^\xi$ predictable and finite variation.
Integration by parts for $\beta^{\mathcal{R}} = \xi \beta^R$ yields

$$d\beta_t^{\mathcal{R}} = (\xi_t d\beta_t^R + \beta_t^R dM_t^\xi) + dA_t^{\mathcal{R}}, \quad t \geq 0,$$

where

$$A_t^{\mathcal{R}} = \int_0^t \beta_s^R dA_s^\xi + [\xi, \beta^R]_t, \quad t \geq 0, \tag{2.4}$$

is a predictable finite variation process.
This case is examined in Proposition 3.6 of Biagini et al. [2014], that we report here:

**Proposition 2.3.4.** *Let the process $\xi$ be as in* (2.3)*. Then the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ is a local $R$-submartingale if and only if $A^{\mathcal{R}}$ in* (2.4) *is an increasing process. If $\xi$ is a submartingale, then the local $R$-submartingale property for $\beta^{\mathcal{R}}$ holds whenever the process $[\xi, \beta^R]$ is increasing.*

In particular, the authors in Biagini et al. [2014] take into consideration the case when the flow $\mathcal{R} = (R_t)_{t \geq 0}$ is of the form

$$R_t = (1 - \bar{\lambda}_t)Q + \bar{\lambda}_t R, \quad t \geq 0, \tag{2.5}$$

where $\bar{\lambda} = (\bar{\lambda}_t)_{t\geq 0}$ is a process taking values in $[0,1]$ and starting at $\bar{\lambda}_0 = 0$.

Our first goal is to prove the following Lemma and the subsequent Theorem in the case when $\bar{\lambda}$ is a continuous and adapted process. The same results are given in Biagini et al. [2014] supposing $\bar{\lambda}$ to be càdlàg and deterministic. The proofs are similar to the ones in Biagini et al. [2014], nonetheless we report them by completeness.

**Lemma 2.3.5.** *Suppose that the flow $\mathcal{R} = (R_t)_{t\geq 0}$ is of the form*

$$R_t = (1-\bar{\lambda}_t)Q + \bar{\lambda}_t R, \quad t \geq 0, \tag{2.6}$$

*where $(\bar{\lambda}_t)_{t\geq 0}$ is an adapted process taking values in $[0,1]$ and starting at $\bar{\lambda}_0 = 0$, and take the process $M^R = (M^R_t)_{t\geq 0}$ with*

$$M^R_t = \mathbb{E}_R\left[\frac{dQ}{dR}\Big|\mathcal{F}_t\right], \quad t \geq 0. \tag{2.7}$$

*Then the conditional distributions $R_t[\cdot|\mathcal{F}_t]$ are of the form (2.1) where the adapted process $\xi = (\xi_t)_{t\geq 0}$ is given by*

$$\xi_t = \frac{\bar{\lambda}_t}{\bar{\lambda}_t + (1-\bar{\lambda}_t)M^R_t}, \quad t \geq 0. \tag{2.8}$$

*Proof.* Take $Z$ positive and $\mathcal{F}$-measurable, and $A_t \in \mathcal{F}_t$ for $t \geq 0$: since $\bar{\lambda}$ is adapted, for $t \geq 0$ we have

$$\begin{aligned}
\mathbb{E}_{R_t}\left[Z\mathbb{1}_{A_t}\right] &= \mathbb{E}_R\left[\frac{dR_t}{dR}Z\mathbb{1}_{A_t}\right] = \mathbb{E}_R\left[\left(\bar{\lambda}_t + (1-\bar{\lambda}_t)M^R_\infty\right)Z\mathbb{1}_{A_t}\right] \\
&= \mathbb{E}_R\left[\mathbb{E}_R[\left(\bar{\lambda}_t + (1-\bar{\lambda}_t)M^R_\infty\right)Z|\mathcal{F}_t]\mathbb{1}_{A_t}\right] \\
&= \mathbb{E}_R\left[\left(\bar{\lambda}_t\mathbb{E}_R[Z|\mathcal{F}_t] + (1-\bar{\lambda}_t)\mathbb{E}_R[M^R_\infty Z|\mathcal{F}_t]\right)\mathbb{1}_{A_t}\right] \\
&= \mathbb{E}_R\left[\left(\bar{\lambda}_t\mathbb{E}_R[Z|\mathcal{F}_t] + (1-\bar{\lambda}_t)M^R_t\mathbb{E}_Q[Z|\mathcal{F}_t]\right)\mathbb{1}_{A_t}\right] \\
&= \mathbb{E}_{R_t}\left[\frac{dR}{dR_t}|_{\mathcal{F}_t}\left(\bar{\lambda}_t\mathbb{E}_R[Z|\mathcal{F}_t] + (1-\bar{\lambda}_t)M^R_t\mathbb{E}_Q[Z|\mathcal{F}_t]\right)\mathbb{1}_{A_t}\right].
\end{aligned} \tag{2.9}$$

Comparing (2.1) and (2.9) we have

$$\xi_t = \frac{\bar{\lambda}_t}{\bar{\lambda}_t + (1-\bar{\lambda}_t)M^R_t}, \quad t \geq 0,$$

since by (2.6) we have

$$\frac{dR_t}{dR}|_{\mathcal{F}_t} = \bar{\lambda}_t + (1-\bar{\lambda}_t)M^R_t, \quad t \geq 0. \qquad \square$$

**Theorem 2.3.6.** *Suppose $\bar{\lambda}$ adapted, continuos and increasing, and that $[W^R, M^R]$ is an increasing and continuous process. Then the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ is a local submartingale under $R$ with initial value $\beta^{\mathcal{R}}_0 = 0$. After time $\tau_1 = \inf\{t; \bar{\lambda}_t = 1\}$, $\beta^{\mathcal{R}}$ is a local martingale under $R$, and therefore an $R$-supermartingale.*

*Proof.* The behavior of the bubble after time $\tau_1$ follows directly from Proposition 2.3.3.
For what concerns the local submartingale property before $\tau_1$, let us prove first that $\xi$ is a submartingale under $R$.
We have $\xi_t = g(M_t^R, \bar{\lambda}_t)$, where

$$g(x,y) = \frac{y}{y + (1-y)x}$$

Observe that $g$ is convex and decreasing in $x$ and increasing in $y$.
We then have, by Jensen's inequality, that

$$\xi_s = g(\mathbb{E}_R[M_t^R|\mathcal{F}_s], \bar{\lambda}_s) \leq \mathbb{E}_R[g(M_t^R, \bar{\lambda}_s)|\mathcal{F}_s] \leq \mathbb{E}_R[g(M_t^R, \bar{\lambda}_t)|\mathcal{F}_s] = \mathbb{E}_R[\xi_t|\mathcal{F}_s],$$

$0 \leq s \leq t$, where the second inequality follows by the fact that $g(M^R, \bar{\lambda})$ is increasing in $\bar{\lambda}$ and $\bar{\lambda}$ is itself increasing.
We have then proved that $\xi$ is an R-submartingale, therefore by Proposition 2.3.4 it suffices to show that $[\xi, \beta^R]$ is increasing.
Applying Itô's formula to $\xi = g(M^R, \bar{\lambda})$ we obtain by Assumption 2.3.1 that

$$\xi_t = \int_0^t g_x(M_s^R, \bar{\lambda}_s) dM_s^R + \int_0^t g_y(M_s^R, \bar{\lambda}_s) d\bar{\lambda}_s + \frac{1}{2}\int_0^t g_{xx}(M_s^R, \bar{\lambda}_s) d[M^R, M^R]_s,$$

$t \geq 0$, where we have used the continuity of $M^R$ and $\bar{\lambda}$ and the fact that $\bar{\lambda}$ is quadratic pure jump, and therefore $[M^R, \bar{\lambda}]_t = [\bar{\lambda}, \bar{\lambda}]_t = 0$.
Moreover, the integrals

$$\int_0^t g_y(M_s^R, \bar{\lambda}_s) d\bar{\lambda}_s$$

(since $\bar{\lambda}$ is increasing) and

$$\int_0^t g_{xx}(M_s^R, \bar{\lambda}_s) d[M^R, M^R]_s$$

are quadratic pure jump, therefore we can write

$$\xi_t = \int_0^t g_x(M_s^R, \bar{\lambda}_s) dM_s^R + V_t, \quad t \geq 0,$$

where $V$ is a quadratic pure jump process.
Thus, since $\beta^R$ is continuous by Assumption 2.3.1, we have

$$d[\xi, \beta^R] = d\left[\int_0^t g_x(M_s^R, \bar{\lambda}_s) dM_s^R + V_t, \beta^R\right] = g_x(M_t^R, \bar{\lambda}_t) d[M^R, \beta^R]_t, \quad t \geq 0,$$

and $g_x(M_t^R, \bar{\lambda}_t)$ is negative because $g(x,y)$ is decreasing in $x$. Moreover, since $W$ is a local martingale under $Q$, $WM^R$ is a local martingale under $R$, and therefore $[M^R, W] = 0$.
Thus

$$[M^R, \beta^R] = [M^R, W - W^R] = -[W^R, M^R],$$

that is negative by assumption. Then $[\xi, \beta^R]$ is increasing, and the result follows. □

Under the hypothesis stated above, the bubble is described by a process first increasing in expectation and then decreasing in expectation, fitting the common thought of a bubble. Now we drop the monotonicity hypothesis on $\bar{\lambda}$, and we consider the case when it is a function $F$ of some process $c = (c_t)_{t\geq 0}$. One can think of $c$ as an economic characteristic that influences the opinion of investors, represented by $\bar{\lambda}$.

**Theorem 2.3.7.** *Assume that*

$$\bar{\lambda}_t = F(c_t), \quad t \geq 0,$$

*where $F \in C^2(\mathbb{R}^+)$ and $c = (c_t)_{t\geq 0}$ is a continuous and positive stochastic process satisfying*

$$dc_t = \rho_t^c dt + \sigma_t^c dB_t^c,$$

*with $B^c = (B_t^c)_{t\geq 0}$ standard Brownian motion, and $\rho^c = (\rho_t^c)_{t\geq 0}$, $\sigma^c = (\sigma_t^c)_{t\geq 0}$ adapted processes possibly depending on $c$.*

*Then the process $\xi$ defined in (2.8) has Doob-Meyer decomposition*

$$\xi_t = M_t^\xi + A_t^\xi, \quad t \geq 0, \tag{2.10}$$

*with*

$$M_t^\xi = \int_0^t g_x(M_s^R, \bar{\lambda}_s) dM_s^R + \int_0^t \sigma_s^c g_y(M_s^R, \bar{\lambda}_s) F'(c_s) dB_s^c, \quad t \geq 0, \tag{2.11}$$

*and*

$$\begin{aligned} A_t^\xi = &\frac{1}{2}\int_0^t g_{xx}(M_s^R, \bar{\lambda}_s) d[M^R, M^R]_s + \frac{1}{2}\int_0^t \sigma_s^c g_{xy}(M_s^R, \bar{\lambda}_s) F'(c_s) d[M^R, B^c]_s \\ &+ \int_0^t \left[ g_y(M_s^R, \bar{\lambda}_s) \left( \rho_s^c F'(c_s) + \frac{1}{2}(\sigma_s^c)^2 F''(c_s) \right) + \frac{1}{2} g_{yy}(M_s^R, \bar{\lambda}_s)(\sigma_s^c F'(c_s))^2 \right] ds, \end{aligned} \tag{2.12}$$

*$t \geq 0$, where $g(x, y) = \frac{y}{y+(1-y)x}$.*
*Suppose moreover that the function $F$ is monotone increasing, and that the process $c$ is a submartingale. Then $\bar{\lambda} = F(c)$ is an $R$-submartingale. In this case, if $[W^R, M^R]$ and $[\beta^R, B^c]$ are increasing processes then $[\xi, \beta^R]$ is also increasing.*

*Proof.* We first compute the Doob-Meyer decomposition. Since the process $\bar{\lambda} = F(c)$ is continuous, we have

$$\begin{aligned} \xi_t = &\int_0^t g_x(M_s^R, \bar{\lambda}_s) dM_s^R + \int_0^t g_y(M_s^R, \bar{\lambda}_s) d\bar{\lambda}_s + \frac{1}{2}\int_0^t g_{xx}(M_s^R, \bar{\lambda}_s) d[M^R, M^R]_s \\ &+ \frac{1}{2}\int_0^t g_{xy}(M_s^R, \bar{\lambda}_s) d[M^R, \bar{\lambda}]_s + \frac{1}{2}\int_0^t g_{yy}(M_s^R, \bar{\lambda}_s) d[\bar{\lambda}, \bar{\lambda}]_s \quad 0 \leq t. \end{aligned} \tag{2.13}$$

Applying Itô's formula to $\bar{\lambda}$, we obtain the expression

$$d\bar{\lambda}_t = \left( \rho_t^c F'(c_t) + \frac{1}{2}(\sigma_t^c)^2 F''(c_t) \right) dt + \sigma_t^c F'(c_t) dB_t^c, \quad t \geq 0, \tag{2.14}$$

and therefore we have

$$d[M^R, \bar{\lambda}]_t = \sigma_t^c F'(c_t) d[M^R, B^c]_t, \quad t \geq 0, \tag{2.15}$$

and

$$d[\bar{\lambda}, \bar{\lambda}]_t = (\sigma_t^c F'(c_t))^2 dt \quad t \geq 0. \tag{2.16}$$

Substituting (2.14), (2.15) and (2.16) in (2.13) we obtain (2.10)-(2.12). This is a Doob-Meyer decomposition since $A^\xi$ is continuous and therefore predictable.
Now we prove that $\bar{\lambda}$ is an R-submartingale: it holds

$$\bar{\lambda}_s = F(c_s) \leq F(\mathbb{E}_R[c_t|\mathcal{F}_s]) \leq \mathbb{E}_R[F(c_t)|\mathcal{F}_s] = \mathbb{E}_R[\bar{\lambda}_t|\mathcal{F}_s], \quad 0 \leq s \leq t,$$

where the first inequality comes from the hypothesis that $F$ is increasing and $c$ is an $R-$submartingale and the second one from the convexity of $F$.
Finally, by the continuity of $\beta^R$ and by (2.10)-(2.12), we obtain

$$[\xi, \beta^R]_t = \int_0^t g_x(M_s^R, \bar{\lambda}_s) d[M^R, \beta^R]_s + \int_0^t \sigma_s g_y(M_s^R, \bar{\lambda}_s) F'(c_s) d[B^c, \beta^R]_s, \quad t \geq 0,$$

since $A^\xi$ is quadratic pure jump. The first term is increasing by the same argument as in the proof of Theorem 2.3.6, whereas the second one is increasing if and only if $[B^c, \beta^R]$ is increasing, since $g(x, y)$ is increasing in $y$, $F$ is increasing and $\sigma$ is positive. □

Since $\bar{\lambda}$ is just a submartingale under $R$ and it is not increasing, it is not possible to prove that $\xi$ is also an $R$-submartingale, as required in Proposition 2.3.4 in order for the bubble to be a local submartingale. In the following, however, we give some possible explicit expressions for $\xi$ so that it is in fact a submartingale under $R$.
In particular, we take $\bar{\lambda}$ and $\xi$ as increasing functions of the perceived bubble $\beta^{\mathcal{R}}$ defined in (2.2). By doing this, we model a coupled interaction between the size of the bubble and the changes in the views of the market: when the bubble is big, a self reinforcing mechanism leads investor to converge their opinions towards the measure $R$, under which the presence of a bubble is fully confirmed.

## 2.4 $\bar{\lambda}$ as a function of the bubble $\beta^{\mathcal{R}}$

We start by modeling $\bar{\lambda}$ itself as a function of $\beta^{\mathcal{R}}$, considering therefore $\bar{\lambda}_t = H(\beta_t^{\mathcal{R}})$, $t \geq 0$, for some increasing function $H$.
By (2.8) we have

$$\xi_t = \frac{H(\beta_t^{\mathcal{R}})}{H(\beta_t^{\mathcal{R}}) + (1 - H(\beta_t^{\mathcal{R}}))M_t^R} = \frac{H(\beta_t^R \xi_t)}{H(\beta_t^R \xi_t) + (1 - H(\beta_t^R \xi_t))M_t^R}, \quad t \geq 0, \tag{2.17}$$

with $M^R$ in (2.7).
The function $H$ has to satisfy some features: in particular, one has to require that $\bar{\lambda}_t = H(\beta_t^{\mathcal{R}})$ belongs to $[0, 1]$ for all $t \geq 0$. Moreover, $H$ is supposed to be increasing and such that, for big values of the bubble, $\bar{\lambda}$ is close to 1: in

this way, when the bubble is very high, most part of the weight in the flow of martingale measures (2.6) is put towards the final measure $R$. We require therefore $\lim_{x\to\infty} H(x) = 1$. Analogously, since at time $t = 0$ we have $\beta_0^{\mathcal{R}} = 0$ and $\xi_0 = 0$, we also ask $H(0) = 0$.
As an example of a function that fulfills the properties above in the first variable, we consider $\bar{H} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, with $\bar{H}(x, y) = \frac{x}{x+y}$. This function is particularly interesting for our purposes because in this case, we can easily find an explicit expression for $\xi$.
Specifically, we take $\bar{\lambda} = \bar{H}(\beta^{\mathcal{R}}, Y)$, where $Y$ is a positive and continuous stochastic process.
Since $\beta^{\mathcal{R}} = \xi\beta^R$, we obtain the following

**Theorem 2.4.1.** *Let $\xi$ be as in (2.8) with $\bar{\lambda} = \frac{\xi\beta^R}{\xi\beta^R+Y}$, where $Y$ is a strictly positive stochastic process with continuos paths. Then we have*

$$\xi_t = \left(\frac{\beta_t^R - Y_t M_t^R}{\beta_t^R}\right) \mathbb{1}_{\{\beta_t^R \geq Y_t M_t^R\}}, \quad t \geq 0.$$

*Proof* From (2.17) we have

$$\xi_t = \frac{\xi_t\beta_t^R}{(\xi_t\beta_t^R + Y_t)\left(\frac{\xi_t\beta_t^R}{\xi_t\beta_t^R+Y_t} + \left(1 - \frac{\xi_t\beta_t^R}{\xi_t\beta_t^R+Y_t}\right) M_t^R\right)} = \frac{\xi_t\beta_t^R}{\xi_t\beta_t^R + Y_t M_t^R}, \quad t \geq 0,$$

where the denominator is a.s. strictly positive, since $\xi$ and $\beta^R$ are positive and $M^R$ and $Y$ are strictly positive. Therefore we find the equation

$$\xi_t(\xi_t\beta_t^R + Y_t M_t^R - \beta_t^R) = 0, \quad t \geq 0, \tag{2.18}$$

that has solutions $\xi_t = 0$ and $\xi_t = \frac{\beta_t^R - Y_t M_t^R}{\beta_t^R}$ if $\beta_t^R \neq 0$.
Since $\xi$ has to take values between 0 and 1, one can see the origin as an *absorbing boundary* and stop the process when it reaches zero, since $\xi_t = 0$ is a solution of (2.18). □

We consider now the bubble $\beta^{\mathcal{R}}$, that takes therefore the simple expression

$$\beta_t^{\mathcal{R}} = \left(\beta_t^R - Y_t M_t^R\right) \mathbb{1}_{\{\beta_t^R \geq Y_t M_t^R\}}, \quad t \geq 0.$$

**Lemma 2.4.2.** *Let a stochastic process $X$ defined in the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ to be a local martingale. Then the process $X\mathbb{1}_{\{X\geq 0\}}$ is a local submartingale.*

*Proof.* Let $\sigma$ be a localizing stopping time for $X$, that is, such that the stopped process $X^\sigma = (X_t^\sigma)_{t\geq 0}$ with $X_t^\sigma = X_{t\wedge\sigma}$ is a martingale.
Take $s > 0$ and $t \geq s$. Since $\mathbb{E}[X_{t\wedge\sigma}\mathbb{1}_{\{X_{t\wedge\sigma}<0\}}|\mathcal{F}_s] \leq 0$, it holds

$$\begin{aligned}\mathbb{E}[(X\mathbb{1}_{\{X\geq 0\}})_t^\sigma|\mathcal{F}_s] &= \mathbb{E}[X_{t\wedge\sigma}\mathbb{1}_{\{X_{t\wedge\sigma}\geq 0\}}|\mathcal{F}_s] = \mathbb{E}[X_{t\wedge\sigma}|\mathcal{F}_s] - \mathbb{E}[X_{t\wedge\sigma}\mathbb{1}_{\{X_{t\wedge\sigma}<0\}}|\mathcal{F}_s] \\ &\geq \mathbb{E}[X_{t\wedge\sigma}|\mathcal{F}_s] = X_{s\wedge\sigma} \geq X_{s\wedge\sigma}\mathbb{1}_{\{X_{s\wedge\sigma}\geq 0\}} = (X\mathbb{1}_{\{X\geq 0\}})_s^\sigma.\end{aligned}$$

We have then proved that if $\sigma$ is a localizing stopping time for $X$, then $(X\mathbb{1}_{\{X\geq 0\}})^\sigma$ is a submartingale. Therefore, $X\mathbb{1}_{\{X\geq 0\}}$ is a local submartingale with the same localizing sequence as $X$. □

Thus we obtain that $\beta^{\mathcal{R}}$ is a local submartingale under $R$ if $\beta^R - YM^R$ is a local martingale under $R$. Since $\beta^R$ and $M^R$ are local martingales under $R$, this is the case, for example, when $Y$ is a local martingale and is independent of $M^R$.

## 2.5 $\xi$ as a function of the bubble $\beta^{\mathcal{R}}$

We now model directly the process $\xi$ given in (2.1) without considering the flow $\bar{\lambda}$ in (2.6). In particular, we assume the process $\xi$ to be an explicit function of time, $\beta^R$ and $M^R$, i.e. we take $\xi_t = \psi(t, \beta^R_t, M^R_t)$, $t \geq 0$. In this way,

$$\beta^{\mathcal{R}}_t = \xi_t \beta^R_t = \psi(t, \beta^R_t, M^R_t)\beta^R_t, \quad t \geq 0.$$

We first apply Itô's formula to the function $\psi(t, x, y)$, and find

$$\begin{aligned} d\xi_t =& \partial_t \psi(t, \beta^R_t, M^R_t)dt + \partial_x \psi(t, \beta^R_t, M^R_t)d\beta^R_t + \partial_y \psi(t, \beta^R_t, M^R_t)dM^R_t \\ &+ \partial_{xy}\psi(t, \beta^R_t, M^R_t)d[\beta^R, M^R]_t, \quad t \geq 0. \end{aligned}$$

We thus write the Doob-Meyer decomposition for $\xi$ as

$$\xi_t = M^{\xi}_t + A^{\xi}_t, \quad t \geq 0,$$

where $M^R = (M^R_t)_{t\geq 0}$ with

$$M^{\xi}_t = \partial_x \psi(t, \beta^R_t, M^R_t)d\beta^R_t + \partial_y \psi(t, \beta^R_t, M^R_t)dM^R_t, \quad t \geq 0,$$

is a local martingale and $A^{\xi} = (A^{\xi}_t)_{t\geq 0}$ with

$$A^{\xi}_t = \partial_t \psi(t, \beta^R_t, M^R_t)dt + \frac{1}{2}\partial_{xy}\psi(t, \beta^R_t, M^R_t)d[\beta^R, M^R]_t, \quad t \geq 0,$$

is a continuous and finite variation process.
From Proposition 2.3.4 we have that $\xi$ is a local R-submartingale if and only if $A^{\mathcal{R}} = (A^{\mathcal{R}}_t)_{t\geq 0}$ is increasing, where

$$A^{\mathcal{R}}_t = \int_0^t \beta^R_s dA^{\xi}_s + [\xi, \beta^R]_t, \quad t \geq 0.$$

In our case,

$$\begin{aligned} A^{\mathcal{R}}_t =& \int_0^t \beta^R_s \partial_s \psi(s, \beta^R_s, M^R_s)ds + \frac{1}{2}\int_0^t \beta^R_s \partial_{xy}\psi(s, \beta^R_s, M^R_s)d[\beta^R, M^R]_s \\ &+ \int_0^t \partial_x \psi(s, \beta^R_s, M^R_s)d[\beta^R, \beta^R]_s + \int_0^t \partial_y \psi(s, \beta^R_s, M^R_s)d[\beta^R, M^R]_s \\ =& \int_0^t \beta^R_s \partial_s \psi(s, \beta^R_s, M^R_s)ds + \int_0^t \partial_x \psi(s, \beta^R_s, M^R_s)d[\beta^R, \beta^R]_s \\ &+ \int_0^t \left(\partial_y \psi(s, \beta^R_s, M^R_s) + \frac{1}{2}\beta^R_s \partial_{xy}\psi(s, \beta^R_s, M^R_s)\right) d[\beta^R, M^R]_s, \quad t \geq 0. \end{aligned} \tag{2.19}$$

Therefore, if $\xi$ does not depend on $M^R$ and it is an increasing function of the time $t$ and of $\beta^R$ it is a local submartingale under $R$.
We specify now some dynamics for $M^R$ and $\beta^R$: considering that $M^R$ is a martingale, for example, we can write

$$dM^R_t = \sigma_1(M^R_t)d\bar{B}^1_t, \quad t \geq 0, \tag{2.20}$$

where $\bar{B}^1$ is a standard Brownian motion and $\sigma_1$ is such that there exists a unique strong solution of (2.20) that is a positive martingale. In particular, a

standard result (see for example Protter [2013]) tells that $M^R$ is a martingale if and only if $\sigma_1$ is such that

$$\int_{\epsilon}^{\infty} \frac{x}{\sigma_1^2(x)} dx = \infty \tag{2.21}$$

for $\epsilon > 0$. Moreover, we suppose that $\beta^R$ satisfies

$$d\beta_t^R = \sigma_2(\beta_t^R) d(\rho \bar{B}_t^1 + \sqrt{1-\rho^2}\bar{B}_t^2), \quad t \geq 0, \tag{2.22}$$

where $\bar{B}^2$ is a standard Brownian motion that is independent on $\bar{B}^1$. Since $\beta^R$ is a strict local martingale under $R$, the function $\sigma_2$ is such that

$$\int_{\epsilon}^{\infty} \frac{x}{\sigma_2^2(x)} dx < \infty \tag{2.23}$$

for $\epsilon > 0$.
Inserting (2.20) and (2.22) in (2.19), we obtain

$$\begin{aligned} A_t^{\mathcal{R}} = &\int_0^t \beta_s^R \partial_s \psi(s, \beta_s^R, M_s^R) ds + \int_0^t \sigma_2^2(\beta_s^R) \partial_x \psi(s, \beta_s^R, M_s^R) ds \\ &+ \int_0^t \rho \sigma_1(M_s^R) \sigma_2(\beta_s^R) \left( \partial_y \psi(s, \beta_s^R, M_s^R) + \frac{1}{2} \beta_s^R \partial_{xy} \psi(s, \beta_s^R, M_s^R) \right) ds, \quad t \geq 0. \end{aligned}$$

Applying Proposition 2.3.4 we have therefore the following

**Theorem 2.5.1.** *Let $\xi_t = \psi(t, \beta_t^R, M_t^R)$, $t \geq 0$, with $M^R$ and $\beta^R$ satisfying the SDEs* (2.20) *and* (2.22) *respectively. Then $\beta^{\mathcal{R}} = \xi \beta^R$ is a local submartingale under the measure $R$ if and only if*

$$\begin{aligned} Z_t = &\beta_t^R \partial_t \psi(t, \beta_t^R, M_t^R) + \sigma_2^2(\beta_t^R) \partial_x \psi(t, \beta_t^R, M_t^R) \\ &+ \rho \sigma_1(M_t^R) \sigma_2(\beta_t^R) \left( \partial_y \psi(t, \beta_t^R, M_t^R) + \frac{1}{2} \beta_t^R \partial_{xy} \psi(t, \beta_t^R, M_t^R) \right) \geq 0, \end{aligned} \tag{2.24}$$

*for all $t \geq 0$.*

# Chapter 3

# Liquidity induced asset bubbles via flows of ELMMs

## 3.1 Motivation

The martingale theory of bubbles, as introduced by the papers of Loewenstein and Willard [2000] and Cox and Hobson [2005], and developed among the others by Jarrow et al. [2007], Jarrow et al. [2010] and Biagini et al. [2014], focuses on the concept of fundamental value: a bubble takes place if the fundamental value is not felt as fully justifying the price of the asset, i.e. if the market chooses an equivalent local martingale measure that reflects a *pessimistic* view.
Since the focus is on the fundamental value, the market price being exogenously given as a semimartingale, the martingale theory of bubbles as developed so far does not directly model the fast increase of the market price which is commonly observed. This is the goal of constructive models for bubbles. One interesting approach is given in this sense by the paper of Jarrow et al. [2012], where the asset's fundamental price process is defined exogenously and asset price bubbles are endogenously determined by market trading activity. Trades impact the market price of the asset through illiquidity effects, making it diverging from the fundamental price.
In particular, a stock is traded through a limit order book so that limit orders and market orders are possible. The firsts increase the supply of shares available for trade, therefore increasing liquidity, whereas market orders are executed against the best bid and ask limit orders, and they create a divergence from the fundamental value. This divergence can last longer than an instant, when the resiliency of the limit order book is weak and new market orders are added before the limit order book is restored by the injection of new limit orders. In this way, long-lasting bubbles can take place in the model. Nonetheless, they are not permanent since they can burst when the resilience is high or, in an example given in Jarrow et al. [2012], when the illiquidity goes to infinity.

In this scheme, the fundamental value of the asset is an exogenously given semimartingale, but it is not justified as the expectation of future cash flows. Our goal in this chapter is then to embed this model in the framework of the martingale theory of bubbles, i.e. to find a flow of ELMMs for the market value $W$ under which the fundamental value $W^F$ is justified as the expectation of

future incomes. In this way, the shift of the pricing measure is directly related to the trading activity of investors: whereas in Biagini et al. [2014] and in Jarrow et al. [2010] the flow of equivalent local martingale measures reflects a change on the perception of the fundamental value of an asset, here it is a result of a change in the views of the market that directly impacts the price. Note however that we do not obtain that $W^F$ is also a (local) martingale under each measure of the flow, as thoroughly discussed in Remark 3.4.1.

## 3.2 The liquidity model

In this section we introduce the liquidity model described in Jarrow et al. [2012]. It is derived from the paper of Roch [2011], where liquidity risk (i.e. the risk that takes place when trading activity has an impact on prices) is analyzed. As specified before, an asset is traded through a limit order book: an investor can decide either to place a limit order, that is, an order to buy or to sell the stock at a specific price when a second investor agrees with that price, or a market order, immediately executed against the existing limit orders.

The main point is that when buy market orders are executed they temporary erode the limit order book, so that if a transaction is big enough one ends up paying more than the initial price. The average price to pay per share for a transaction of size $x$ via a market order at time $t$ is given by

$$S_t(x) = \tilde{S}_t + M_t x, \tag{3.1}$$

where $\tilde{S}$ and $M$ are adapted processes. In particular, $\tilde{S}_t$ is the quoted (or marginal) price at time $t$, and it is defined as the price per share for a purchase or sale of an infinitesimal quantity of shares ($x = 0$) at time $t$.

The limit order book is described by a function $\rho_t(z)$ standing for the density of the number of shares offered at price $z$ at time $t$, so that the total number of shares offered between prices $z_1$ and $z_2$ is given by

$$\int_{z_1}^{z_2} \rho_t(z)dz.$$

As a result of a buy market order, limit orders in the limit order book are executed starting with the cheapest to the most expensive until the total number of shares ordered is reached. Therefore, an investor that buys $x$ shares at time $t$ thorough a market order has to pay a total number of dollars equal to

$$\int_{\tilde{S}_t}^{z_x} z\rho_t(z)dz,$$

where $z_x$ solves the equation

$$\int_{\tilde{S}_t}^{z_x} \rho_t(z)dz = x.$$

Since the supply curve in expression (3.1) is supposed to be linear, its limit order book density function equals $\rho_t(z) = \frac{1}{2M_t}$, so that $z_x = \tilde{S}_t + 2M_t x$. The total dollars paid per $x$ shares at time $t$ is thus

$$\frac{1}{2M_t}\int_{\tilde{S}_t}^{\tilde{S}_t+2M_t x} zdz = \tilde{S}_t x + M_t x^2 = xS_t(x). \tag{3.2}$$

This can be seen as the price impact of the trading of $x$ shares. In this sense, $M_t$ is a measure of illiquidity, as a larger $M_t$ brings a larger price impact.
In other words, the illiquidity causes a gap in the limit order book after a buy market order is placed: after the purchase of $\Delta X_t$ shares, the best ask price increases from $\tilde{S}_t$ to $S_t + 2M_t\Delta X_t$. In this way, immediately after the trade, the limit order book density function is 0 for prices between $\tilde{S}_t$ and $\tilde{S}_t + 2M_t\Delta X_t$.
Of course, this gap does not endure forever: the negative correlation between returns and the volume of incoming limit orders (see Rosenow and Weber [2005]) indicate that shortly after a market order, investors add new limit orders in the opposite direction. This is called resiliency effect, and it is modeled by assuming that the gap in the density function partially disappears immediately after a trade: a resiliency process $\Lambda = (\Lambda_t)_{t\geq 0}$ taking values between 0 and 1 is then introduced so that impact on the supply curve of a trade of size $\Delta X_t$ is shifted to $S_t + 2\Lambda_t M_t \Delta X_t$. When $\Lambda_t = 0$, there is full resiliency since the order book immediately recovers its previous shape after a market order of any size, whereas if $\Lambda_t = 1$ there is no resiliency and the gap remains.
In absence of other market orders, this price impact would decay to zero in the long term, as market prices return to fundamental values. However, if new market orders quickly enter before the price has time to decay again to the fundamental value, these short-term price variations may accumulate and result in a deviation from the fundamental wealth with a consequent bubble birth.
Taking inspiration from this construction, we introduce our setting in the next section.

## 3.3 The setting

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T > 0$ a random time on it, representing the maturity or liquidation time of the underlying risky asset as in the setting of Jarrow et al. [2010]. We assume that $(\Omega, \mathcal{F}, P)$ is endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual assumptions of completeness and right continuity.
On $(\Omega, \mathcal{F}, \mathbb{F}, P)$ we have $(B^1, B^2, B^3, B^4, N)$, where $B^i = (B^i_t)_{t\in[0,T]}$, $i = 1, 2, 3, 4$ are standard $\mathbb{F}$-Brownian motions and $N_t = \mathbb{1}_{\{\tau\leq t\}}$ is a jump process with $\tau$ totally inaccessible stopping time with intensity process $\pi = (\pi_t)_{t\in[0,T]}$. We assume that $(B^1, B^2, B^3, B^4, N)$ are independent processes.
We follow the approach of Jarrow et al. [2012] and study how trading activity may impact prices and generate the formation and bursting of speculative asset price bubbles. We consider a continuous time setting where a stock is traded through a limit order book. The *fundamental wealth or value* of the stock is given as a primitive of the model and represents the price process if market orders have no quantity impact on the price. The *market wealth* equals the fundamental value until market orders are executed. Market orders, which deplete or fill in the limit order book, produce a variation in the price over a small interval of time as described in Section 3.2.
More specifically, we consider a financial asset whose fundamental wealth $W^F = (W^F_t)_{t\in[0,T]}$ (associated to the cumulative dividend process $(D_t)_{t\in[0,T]}$

and to the liquidation value $F$ of the asset at time $T$) is given by

$$dW_t^F = W_t^F(adt + bdB_t^1), \quad 0 \leq t \leq T, \tag{3.3}$$

with $W_0^F > 0$, $a \in \mathbb{R}$ and $b > 0$.
We interpret $\tau$ as the time of birth of a bubble for this financial asset. The bubble follows the dynamics

$$d\beta_t = M_t\Lambda_t(-k\beta_t dt + 2dX_t + 2xW_t^F dN_t), \quad 0 \leq t < T, \tag{3.4}$$

where $X$ is the signed volume of market orders (buy market orders minus sell market orders), $x$ is the signed volume of market orders at $\tau$ and $M = (M_t)_{t\in[0,T]}$, $\Lambda = (\Lambda_t)_{t\in[0,T]}$ are respectively a measure of illiquidity and the resiliency. Moreover, in agreement with the approach of Jarrow et al. [2012], $k$ is the speed of decay, and it is strictly positive since the market price is supposed to go back to the fundamental value in the long term.

We put $\beta_\tau = 2x\Lambda_\tau M_\tau W_\tau^F$ for a given $x > 0$.

**Remark 3.3.1.** *As in Jarrow et al. [2012], we assume that the supply curve for the stock is linear, i.e. that the variation induced by a market order of size $y$ is proportional to $y$ via the stochastic coefficients $M$ and $\Lambda$. In this way $\frac{1}{2M_t\Lambda_t}$ gives the depth of the order book at time $t$, i.e. the size of the order required to change the price of an asset by one unit. This linearity assumption is better justified in the case of frequently traded and large volume stocks, see Blais and Protter [2010]. For less liquid companies, statistical analysis (see for example Cetin et al. [2006]) shows that the supply is at best piece-wise linear. For more details about the economical motivation of this setting, we refer to Jarrow et al. [2012].*

We consider that $X$ satisfies the following dynamics

$$\begin{aligned} &X_t = 0, && \text{for } 0 \leq t < \tau, \\ &dX_t = \mu_t dt + \sigma_t dB_t^2, && \text{for } \tau \leq t < T, \end{aligned} \tag{3.5}$$

where $\mu = (\mu_t)_{t\in[0,T]}$ and $\sigma = (\sigma_t)_{t\in[0,T]}$ are progressively measurable processes that a priori can also depend on $X$ itself or on the bubble $\beta$.
In Jarrow et al. [2012] the signed volume of market orders is modeled as in (3.5) with $\mu \equiv 0$ and $\sigma_t = \alpha\beta_t$. Here we introduce the drift $\mu$ in order to see the influence of the network on the size of the bubble, as we specify in Section 4.
Here the fundamental wealth process $W^F$ is exogenously given, while the market wealth process $W = (W_t)_{t\in[0,T]}$ is endogenously determined as

$$W_t = W_t^F + \beta_t, \quad 0 \leq t < T.$$

At liquidation time $T$ we have $W_T = W_T^F$: the asset is liquidated at time $T$ at the estimated firm's value, i.e. at the fundamental value. In particular we require in the sequel that there exists an equivalent local martingale measure for $W$ only on the open interval $[0, T)$, since around time $T$ the liquidation procedure is not subjected to market equilibrium mechanisms.

**Assumption 3.3.2.** *(i)* $\int_\tau^T \mu_s^2 ds < \infty$ *a.s.*

*(ii)* $\int_\tau^T \sigma_s^2 ds < \infty$ *a.s. and* $\int_\tau^T \frac{1}{\sigma_s^4} ds < \infty$ *a.s.*

*(iii)* $\mu$ *and* $\sigma$ *are Itô processes such that there exists a unique solution of* (3.5) *(see for example Theorem 7 in Protter [2005, chapter 5]);*

*(iv)* $M = (M_t)_{t\in[0,T]}$ *is a positive and adapted process that satisfies the dynamics*

$$dM_t = \mu^M(M_t)dt + \sigma^M(M_t)dB_t^3, \quad 0 \le t \le T,$$

*where* $\mu^M$ *and* $\sigma^M$ *are such that there exists a unique solution of* (3.6) *according to Theorem 7 in Protter [2005, chapter 5]. Moreover* $\int_a^b (\sigma^M)^{-4}(x)dx < \infty$ *for every* $a, b$ *such that* $0 < a < b < \infty$.

*(v)* $\Lambda = (\Lambda_t)_{t\in[0,T]}$ *satisfies the dynamics*

$$d\Lambda_t = \mu'(\Lambda_t)dt + \sigma'(\Lambda_t)dB_t^4, \quad 0 \le t \le T,$$

$\Lambda_0 \in (\lambda, 1)$, *with* $\mu'$, $\sigma'$ *that satisfy conditions Theorem 7 in Protter [2005, chapter 5]. Furthermore* $\mu'(\lambda) > 0$, $\mu'(1) < 0$, $\sigma'(1) = 0$, $\sigma'(\lambda) = 0$ *a.s., so that we obtain* $\lambda \le \Lambda_t \le 1$, *a.s. for all* $t \in [0, T]$.

*(vi)* $\pi = (\pi_t)_{t\in[0,T]}$ *is bounded, i.e.* $|\pi_t| \le \Pi < \infty$ *a.s for all* $t \in [0, T]$.

*(vii)* $T$ *is a bounded a.s. (possibly by a very large constant)* $\mathbb{F}$*-stopping time independent of* $(B^1, B^2, N)$ *such that* $\tau < T$ *a.s.*

*(viii) Either* $\sigma$ *does not depend on* $\beta$, *or* $\sigma_t = \alpha\beta_t$, $t \in [\tau, T]$, $\alpha > 0$. *In this latter case, we suppose* $\mu_t \ge 0$ *for every* $t \in [\tau, T]$.

Note that we assume $\tau < T$ and $T$ bounded a.s. for the sake of simplicity. The following results still hold without these conditions by imposing some integrability conditions on $T$. For example, it would be sufficient $T < \infty$ a.s., $\mathbb{E}_P[e^T|\mathcal{F}_t] < \infty$ and $\mathbb{E}_P[T - \tau|\mathcal{F}_t] > 0$ a.s. for $t \in [0, T]$.

**Proposition 3.3.3.** *From the hypothesis on* $M$ *it follows that* $\int_0^T M_s^\alpha ds < \infty$ *a.s. for all* $\alpha \in \mathbb{R}$.

*Proof.* Following the same argument as in Mijatovic and Urusov [2012b], we have that

$$\int_0^T M_s^\alpha ds = \int_0^T \frac{M_s^\alpha}{(\sigma^M)^2(M_s)} d[M, M]_s = \int_0^\infty \frac{x^\alpha}{(\sigma^M)^2(x)} L_T^x dx, \tag{3.6}$$

where $L_T^x$ is the local time at $T$ and the last equality follows by occupation time formula (see for example Corollary 1 in Protter [2005, chapter 4]).
Then the integral is finite since, by the fact that $0 < M_s < \infty$ a.s. for each $s \in [0, T]$, we have that the occupation time $L_T^a$ has compact support in $(0, \infty)$. □

**Remark 3.3.4.** *If* $\sigma$ *does not depend on* $\beta$, *the bubble has the following explicit expression:*

$$\beta_t = \beta_\tau e^{-k\int_\tau^t \Lambda_s M_s ds} + \int_\tau^t \mu_s \Lambda_s M_s e^{-k\int_s^t \Lambda_u M_u du} ds +$$

$$+\int_\tau^t \sigma_s \Lambda_s M_s e^{-k\int_s^t \Lambda_u M_u du} dB_s^2, \quad \tau \le t < T. \tag{3.7}$$

*On the other hand, if $\sigma_t = \alpha\beta_t$, we have that*

$$\beta_t = e^{\psi_t}\left(\beta_\tau + \int_\tau^t \Lambda_s M_s \mu_s e^{-\psi_s} ds\right), \quad \tau \le t \le T, \tag{3.8}$$

*where*

$$\psi_t = -2k\int_\tau^t \Lambda_s M_s ds - 2\alpha^2 \int_\tau^t \Lambda_s^2 M_s^2 ds + 2\alpha \int_\tau^t \Lambda_s M_s dB_s^2, \quad \tau \le t \le T. \tag{3.9}$$

*Note that the example given in Jarrow et al. [2012] is a particular case of* (3.8) *when $\mu \equiv 0$.*

We now prove that, if $\sigma$ does depend on $\beta$ and in particular $\sigma_t = \alpha\beta_t$ for all $t \in [\tau, T)$, then under the other hypothesis stated in Assumption 3.3.2 it holds $\int_\tau^T \sigma_s^2 ds < \infty$ a.s. and $\int_\tau^T \frac{1}{\sigma_s^4} ds < \infty$ a.s., i.e. point *(ii)* of Assumption 3.3.2 is satisfied.

**Proposition 3.3.5.** *Suppose that requirements (i), (iii), (iv), (v), (vi) and (vii) of Assumption 3.3.2 are satisfied and that $\mu_t \ge 0$ for all $t \in [\tau, T]$. Then the process $\sigma = (\sigma_t)_{t\in[\tau,T]}$ with $\sigma_t = \alpha\beta_t$ fulfills point (ii) of Assumption 3.3.2.*

*Proof.* Suppose by simplicity that $\tau = 0$, so that the bubble is born at time $t = 0$. We first prove that, a.s., $\int_0^T \frac{1}{\sigma_t^4} ds < \infty$. From (3.8), we have that

$$\begin{aligned}\int_0^T \frac{1}{\sigma_t^4} ds &= \alpha^{-4}\int_0^T \frac{1}{\beta_t^4} dt = \alpha^{-4}\int_0^T e^{-4\psi_t}\left(\beta_0 + \int_0^t \Lambda_s M_s \mu_s e^{-\psi_s} ds\right)^{-4} dt \\ &\le \alpha^{-4}\beta_0^{-4}\int_0^T e^{-4\psi_t} dt,\end{aligned}$$

since the processes $\mu$, $M$ and $\Lambda$ are positive. Then, since $\beta_0 > 0$ and $\alpha > 0$ by assumption, it suffices to prove that $\int_0^T e^{-4\psi_t} dt < \infty$ with probability 1. From (3.9) and from the fact that $\mu$, $M$ and $\Lambda$ are positive it follows

$$\begin{aligned}\int_0^T e^{-4\psi_t} dt &= \int_0^T \exp\left\{8k\int_0^t \Lambda_s M_s ds + 8\alpha^2 \int_0^t \Lambda_s^2 M_s^2 ds - 8\alpha\int_0^t \Lambda_s M_s dB_s^2\right\} dt \\ &\le \exp\left\{8k\int_0^T \Lambda_s M_s ds + 8\alpha^2\int_0^T \Lambda_s^2 M_s^2 ds\right\}\int_0^T \exp\left\{-8\alpha\int_0^t \Lambda_s M_s dB_s^2\right\} dt.\end{aligned} \tag{3.10}$$

The first term in (3.10) is finite a.s. by Proposition 3.3.3 and because $0 \le \Lambda_t \le 1$ for all $t \in [0, T]$ with probability 1.
By the properties of Brownian motion and by the hypothesis on $\alpha$ and $\Lambda$, proving that the right term is finite a.s. is equivalent to prove that

$$\int_0^T \exp\left\{\int_0^t M_s dB_s^2\right\} dt < \infty, \quad a.s.. \tag{3.11}$$

We have that

$$\int_0^T \exp\left\{\int_0^t M_s dB_s^2\right\} dt \leq T \sup_{0\leq t\leq T} \exp\left\{\int_0^t M_s dB_s^2\right\} = T \exp\left\{\sup_{0\leq t\leq T} \int_0^t M_s dB_s^2\right\}.$$

T is a.s. finite by Assumption 3.3.2, so it suffices to prove that $\sup_{0\leq t\leq T} \int_0^t M_s dB_s^2 < \infty$ with probability 1.
It is well known (see for example Theorem 42 in Protter [2005, chapter 2]) that there exists a standard Brownian motion $\bar{W} = (\bar{W}_s)_{s\geq 0}$ with filtration $\mathcal{G} = (\mathcal{G}_s)_{s\geq 0}$ such that

$$\int_0^t M_s dB_s^2 = \bar{W}_{\int_0^t M_s^2 ds}, \quad a.s., \quad 0 \leq t < \infty.$$

Therefore,

$$\sup_{0\leq t\leq T} \int_0^t M_s dB_s^2 = \sup_{0\leq t\leq T} \bar{W}_{\int_0^t M_s^2 ds} = \sup_{0\leq t\leq \int_0^T M_s^2 ds} \bar{W}_t \quad a.s.,$$

since $M$ is a.s. continuos by Assumption 3.3.2.
Since $\int_0^T M_s^2 ds < \infty$ a.s. by Proposition 3.3.3, and $\bar{W}$ is continuous a.s. on $\left[0, \int_0^T M_s^2 ds\right]$, we have

$$\sup_{0\leq t\leq \int_0^T M_s^2 ds} \bar{W}_t < \infty,$$

which implies that (3.11) holds and

$$\int_0^T e^{-4\psi_t} dt < \infty \quad a.s.. \tag{3.12}$$

Thus, $\int_0^T \frac{1}{\sigma_t^4} ds < \infty$ a.s. as well. We now show that $\int_\tau^T \sigma_s^2 ds < \infty$, with probability 1. Since it holds $0 \leq \Lambda_t \leq 1$ for all $t \in [0, T]$ a.s., have that

$$\begin{aligned}
\int_0^T \sigma_s^2 ds &= \alpha^2 \int_0^T \beta_s^2 ds = \alpha^2 \int_0^T e^{2\psi_t} \left(\beta_0 + \int_0^t \Lambda_s M_s \mu_s e^{-\psi_s} ds\right)^2 dt \\
&\leq \alpha^2 \left(\beta_0 + \int_0^T \Lambda_t M_t \mu_t e^{-\psi_t} dt\right)^2 \int_0^T e^{2\psi_t} dt \\
&\leq \alpha^2 \left(\beta_0 + \left(\int_0^T \mu_t^2 dt\right)^{1/2} \left(\int_0^T M_t^4 dt\right)^{1/4} \left(\int_0^T e^{-4\psi_t} dt\right)^{1/4}\right)^2 \int_0^T e^{2\psi_t} dt,
\end{aligned} \tag{3.13}$$

almost surely. The term in (3.13) is finite a.s. by point *(i)* of Assumption 3.3.2, by Proposition 3.3.3 and by (3.12). It remains to show that $\int_0^T e^{2\psi_t} dt < \infty$, a.s.. We have that

$$\begin{aligned}
\int_0^T e^{2\psi_t} dt &= \int_0^T \exp\left\{-4k \int_0^t \Lambda_s M_s ds - 4k\alpha^2 \int_0^t \Lambda_s^2 M_s^2 ds + 4\alpha \int_0^t \Lambda_s M_s dB_s^2\right\} dt \\
&\leq \int_0^T \exp\left\{4\alpha \int_0^t \Lambda_s M_s dB_s^2\right\} dt, \quad a.s.,
\end{aligned}$$

since $M$ and $\Lambda$ are a.s. positive. The integral above is finite by (3.11) because $\Lambda$ belongs to $[0, 1]$. □

## 3.4 Flow of equivalent local martingale measures

Let $\mathcal{M}_{loc}(W)$ be the space of equivalent local martingale measures for $W = (W_t)_{t\in[0,T)}$. Given $Q \in \mathcal{M}_{loc}(W)$, a $Q$-bubble $\beta^Q$ is defined as

$$\beta_t^Q = W_t - \mathbb{E}_Q[W_T|\mathcal{F}_t] \tag{3.14}$$

in the approach of Jarrow et al. [2007] and Jarrow et al. [2010]. In particular we have that the bubble introduced in (3.4) coincides with a $Q$-bubble if and only if

$$W_t^F = \mathbb{E}_Q[W_T|\mathcal{F}_t], \quad t \in [0,T) \tag{3.15}$$

for some $Q \in \mathcal{M}_{loc}(W)$.
This is of course not possible in our setting. However we can find a flow $(Q^t)_{t\in[0,T)} \subseteq \mathcal{M}_{loc}(W)$ such that

$$W_t^F = \mathbb{E}_{Q^t}[W_T|\mathcal{F}_t] = \mathbb{E}_{Q^t}[W_T^F|\mathcal{F}_t]. \tag{3.16}$$

In this way the bubble described in (3.4) is the result of the shift in the pricing measure induced by the change in the macro-economic and financial conditions in the market.

**Remark 3.4.1.** *We wish to point out the difference and relations between our constructive approach and the martingale theory of bubbles as Jarrow et al. [2007], Jarrow et al. [2010] and Biagini et al. [2014]. In our setting as well as under the approach of Jarrow et al. [2012], the bubble $\beta$ is defined as*

$$\beta_t = W_t - W_t^F, \tag{3.17}$$

*where $W^F$ is a primitive of the model. According to the martingale theory of bubbles as illustrated in Jarrow et al. [2007] and Jarrow et al. [2010], the market wealth $W$ is given a priori and for a given $Q \in \mathcal{M}_{loc}(W)$ the $Q$-bubble process $\beta^Q$ is defined as in* (3.14)*, which also implies that $\beta^Q$ is non-negative. The two definitions coincide if the fundamental wealth process $W^F$ in* (4.4) *is also a (local) $Q$-martingale for $Q \in \mathcal{M}_{loc}(W)$, i.e. if* (3.15) *holds, otherwise they differ.*
*In our setting as well as in Jarrow et al. [2012] (see Section 5), we have that $\mathcal{M}_{loc}(W) \cap \mathcal{M}_{loc}(W^F) = \emptyset$, so the bubble process cannot be a local martingale under any equivalent local martingale measure $Q \in \mathcal{M}_{loc}(W)$ for the wealth process $W$ and may also assume negative values. Hence the appearance of negative bubbles is not in contrast with arbitrage theory in our approach.*
*However, while in the martingale approach the model is automatically arbitrage-free because $\mathcal{M}_{loc}(W) \neq \emptyset$ is assumed a priori, in our "constructive" model for bubbles we need to explicitly exclude arbitrage possibilities. Since in Theorem 3.4.20 we show the existence of a flow $(Q^t)_{t\in[0,T)} \subseteq \mathcal{M}_{loc}(W)$, i.e. that $\mathcal{M}_{loc}(W) \neq \emptyset$, we obtain that our market model is arbitrage-free, see also Remark 3.4.21.*
*It is then a challenging question whether our constructive model can be included in the more fundamental view of the martingale theory of bubble of Jarrow et al. [2007] and Jarrow et al. [2010] by following Biagini et al. [2014]. To this purpose we investigate the existence of a flow $(Q^t)_{t\in[0,T)} \subseteq \mathcal{M}_{loc}(W)$ which can "fundamentally explain" the a-priori given fundamental wealth, i.e.*

*such that* (3.16) *holds. This is not in contrast with our comments above since now the measure* $Q^t$ *is not fixed all over the interval* $[0,T)$*, but it may change in time. In fact* (3.16) *does not imply that* $W^F$ *is a martingale under* $Q^t$ *over the interval* $[0,T)$ *because* (3.16) *holds t-wise and in general it is not true that*

$$W_s^F = \mathbb{E}_{Q^t}[W_T|\mathcal{F}_s]$$

*for* $s \neq t$, $s,t \in [0,T)$.

We now explicitly compute a flow $(Q^t)_{t\in[0,T)} \in \mathcal{M}_{loc}(W)$ justifying the existence of the bubble in (3.4) from a fundamental point of view.
Let $Q \in \mathcal{M}_{loc}(W)$. Then the density process $Z = (Z_t)_{t\in[0,T)}$ of $Q$ with respect to $P$ is given by

$$Z_t = \frac{dQ}{dP}|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^{\cdot} \alpha_s^1 dB_s^1 + \int_0^{\cdot} \alpha_s^2 dB_s^2 + \int_0^{\cdot} \alpha_s^3 d\tilde{N}_s + \int_0^{\cdot} \alpha_s^4 dB_s^3 + \int_0^{\cdot} \alpha_s^5 dB_s^4 + L_t\right)_t,$$

$0 \leq t < T$, where $\tilde{N}_t = N_t - \int_0^{t\wedge\tau} n_s ds$, $t \in [0,T)$, $L$ is a martingale strongly orthogonal to $(B^1, B^2, B^3, B^4, N)$ and the processes $\alpha^i$, $i = 1,\ldots,5$ are such that for $0 \leq s < T$ the following equality holds:

$$W_s^F(a+b\alpha_s^1)+2\Lambda_s M_s\left(\mu_s + \sigma_s\alpha_s^2 - k\beta_s\right)\mathbb{1}_{\{s\geq\tau\}}+2\pi_s x W_s^F \Lambda_s M_s(\alpha_s^3+1)\mathbb{1}_{\{s<\tau\}} = 0. \tag{3.18}$$

Since (3.18) does not involve $\alpha^4$, $\alpha^5$ or $L$, we put $\alpha^4 \equiv \alpha^5 \equiv L \equiv 0$.
We can split (3.18) as

$$b\alpha_s^1 = -a - 2\pi_s x\Lambda_s M_s(\alpha_s^3 + 1) \quad \text{for } s < \tau \tag{3.19}$$

and

$$b\alpha_s^1 = -a + \frac{2\Lambda_s M_s}{W_s^F}\left(k\beta_s - \mu_s - \sigma_s\alpha_s^2\right) \quad \text{for } s \geq \tau. \tag{3.20}$$

We look for a flow of the form

$$Z_{t,s} = \frac{dQ^t}{dP}|_{\mathcal{F}_s} = \mathcal{E}\left(\int_0^{\cdot} \alpha_u^{t,1} dB_u^1 + \int_0^{\cdot} \alpha_u^{t,2} dB_u^2 + \int_0^{\cdot} \alpha_u^{t,3} d\tilde{N}_u\right)_s, \quad s \in [0,T), \tag{3.21}$$

since (3.18) does not involve conditions on $\alpha^{t,4}$, $\alpha^{t,5}$ and $\alpha^{t,6}$. In particular, we note that the laws of $M$, $\Lambda$ and $T$ are invariant under this change of measure. If $\alpha^{t,1}$, $\alpha^{t,2}$ and $\alpha^{t,3}$ satisfy (3.19) and (3.20), the fundamental process under $Q^t$ is given by

$$\frac{dW_s^F}{W_s^F} = \tilde{\mu}_s^t ds + b d\tilde{B}_s^t, \quad 0 \leq s \leq T, \tag{3.22}$$

where $\tilde{B}^t$ denote the $Q^t$-standard Brownian motion given by

$$\tilde{B}_s^t = B_s^1 - \int_0^s \alpha_u^{t,1} du, \quad 0 \leq s \leq T,$$

and

$$\tilde{\mu}_s^t = \begin{cases} -2\pi_s x\Lambda_s M_s(\alpha_s^{t,3}+1) & \text{for } s < \tau, \\ \frac{2\Lambda_s M_s}{W_s^F}\left(k\beta_s - \mu_s - \sigma_s\alpha_s^{t,2}\right) & \text{for } s \geq \tau. \end{cases} \tag{3.23}$$

If the condition

$$\mathbb{E}_{Q^t}\left[\int_t^T (W_s^F)^2 ds\right] < \infty \tag{3.24}$$

is satisfied, we have that (3.16) is equivalent to

$$\mathbb{E}_{Q^t}\left[\int_t^T W_s^F \tilde{\mu}_s^t ds \Big| \mathcal{F}_t\right] = 0,$$

that is

$$0 = \mathbb{E}_{Q^t}\left[\int_t^\tau W_s^F \pi_s x \Lambda_s M_s(\alpha_s^{t,3}+1)ds + \int_\tau^T \Lambda_s M_s\left(k\beta_s - \mu_s - \sigma_s\alpha_s^{t,2}\right) ds \Big| \mathcal{F}_t\right] \tag{3.25}$$

for $t < \tau$ and

$$\mathbb{E}_{Q^t}\left[\int_t^T \Lambda_s M_s\left(k\beta_s - \mu_s - \sigma_s\alpha_s^{t,2}\right) ds \Big| \mathcal{F}_t\right] = 0 \tag{3.26}$$

for $t \geq \tau$.
To show the existence of the flow $(Q^t)_{t\in[0,T)} \subseteq \mathcal{M}_{loc}(W)$, we choose $\alpha^{t,2}$ and $\alpha^{t,3}$ so that the integrals inside the conditional expectation in (3.25) and (3.26) are zero almost surely. We show later on that a posteriori this choice ensures as well that (3.24) holds.
For $t \geq \tau$, let

$$\alpha_s^{t,2} = \frac{1}{\Lambda_s M_s \sigma_s}\left(s - \frac{\mathbb{E}[T|\mathcal{F}_t]+t}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t] - \mathbb{E}[T^2|\mathcal{F}_t]}{2(\mathbb{E}[T|\mathcal{F}_t]-t)}\right) + \frac{k\beta_s}{\sigma_s} - \frac{\mu_s}{\sigma_s}, \quad t \leq s < T.$$

Note that such $\alpha_s^{t,2}$ is well defined since from Assumption 3.3.2 it holds $\Lambda_s > 0$, $M_s > 0$, $\sigma_s > 0$ a.s. for every $s \in [0,T]$.
With this choice we have on $\{T > t\}$ that

$$\begin{aligned}
&\mathbb{E}_{Q^t}\left[\int_t^T \Lambda_s M_s\left(k\beta_s - \mu_s - \sigma_s\alpha_s^{t,2}\right) ds \Big| \mathcal{F}_t\right] \\
=&\mathbb{E}_{Q^t}\left[\int_t^T \left(s - \frac{\mathbb{E}[T|\mathcal{F}_t]+t}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t] - \mathbb{E}[T^2|\mathcal{F}_t]}{2(\mathbb{E}[T|\mathcal{F}_t]-t)}\right) ds \Big| \mathcal{F}_t\right] \\
=&\mathbb{E}_{Q^t}\left[\left(\frac{T^2-t^2}{2} - (T-t)\frac{\mathbb{E}[T|\mathcal{F}_t]+t}{2} + (T-t)\frac{\mathbb{E}^2[T|\mathcal{F}_t] - \mathbb{E}[T^2|\mathcal{F}_t]}{2(\mathbb{E}[T|\mathcal{F}_t]-t)}\right) \Big| \mathcal{F}_t\right] \\
=&\frac{\mathbb{E}[T^2|\mathcal{F}_t]-t^2}{2} - \frac{(\mathbb{E}[T|\mathcal{F}_t]-t)(\mathbb{E}[T|\mathcal{F}_t]+t)}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t] - \mathbb{E}[T^2|\mathcal{F}_t]}{2} = 0,
\end{aligned} \tag{3.27}$$

since by Assumption 3.3.2 the law of $T$ does not change under $Q^t$.
For $t < \tau$ define

$$C_{t,\tau} := \int_t^\tau W_s^F \pi_s x \Lambda_s M_s(\alpha_s^{t,3}+1)ds$$

and choose $\alpha_s^{t,2}$ to be such that

$$\mathbb{E}_{Q^t}\left[\int_\tau^T \Lambda_s M_s\left(k\beta_s - \mu_s - \sigma_s\alpha_s^{t,2}\right) ds \Big| \mathcal{F}_t\right] = -\mathbb{E}_{Q^t}\left[C_{t,\tau}|\mathcal{F}_t\right],$$

i.e.

$$\alpha_s^{t,2} = \frac{1}{\Lambda_s M_s \sigma_s}\Big(s - \frac{\mathbb{E}_{Q^t}[C_{t,\tau}|\mathcal{F}_t]}{\mathbb{E}[T-\tau|\mathcal{F}_t]} - \frac{\mathbb{E}[T+\tau|\mathcal{F}_t]}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t] - \mathbb{E}[T^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau|\mathcal{F}_t]} - \frac{\mathbb{E}^2[\tau|\mathcal{F}_t] - \mathbb{E}[\tau^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau|\mathcal{F}_t]}\Big) + \frac{k\beta_s}{\sigma_s} - \frac{\mu_s}{\sigma_s}, \quad t \le s < T,$$

so that

$$\begin{aligned}
&\mathbb{E}_{Q^t}\left[\int_\tau^T \Lambda_s M_s \left(k\beta_s - \mu_s - \sigma_s \alpha_s^{t,2}\right) ds \Big| \mathcal{F}_t\right] \\
=&\mathbb{E}_{Q^t}\Big[\int_\tau^T \Big(s - \frac{\mathbb{E}_{Q^t}[C_{t,\tau}|\mathcal{F}_t]}{\mathbb{E}[T-\tau|\mathcal{F}_t]} - \frac{\mathbb{E}[T+\tau|\mathcal{F}_t]}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t] - \mathbb{E}[T^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau|\mathcal{F}_t]} \\
&\quad - \frac{\mathbb{E}^2[\tau|\mathcal{F}_t] - \mathbb{E}[\tau^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau|\mathcal{F}_t]}\Big) ds \Big| \mathcal{F}_t\Big] \\
=&\frac{\mathbb{E}[T^2-\tau^2|\mathcal{F}_t]}{2} - \mathbb{E}_{Q^t}[C_{t,\tau}|\mathcal{F}_t] - \frac{\mathbb{E}[T-\tau|\mathcal{F}_t]\mathbb{E}[T+\tau|\mathcal{F}_t]}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t] - \mathbb{E}[T^2|\mathcal{F}_t]}{2} \\
&\quad - \frac{\mathbb{E}^2[\tau|\mathcal{F}_t] - \mathbb{E}[\tau^2|\mathcal{F}_t]}{2} \\
=&-\mathbb{E}_{Q^t}[C_{t,\tau}|\mathcal{F}_t],
\end{aligned}$$

and then (3.25) holds.
For $s < t \vee \tau$ we set $\alpha_s^{t,2} = 0$.
Summarizing:

$$\alpha_s^{t,2} = \begin{cases} 0 & \text{for } s < \tau \vee t, \\ \frac{1}{\Lambda_s M_s \sigma_s}(s - \eta_{t,\tau}) + \frac{k\beta_s}{\sigma_s} - \frac{\mu_s}{\sigma_s} & \text{for } s \ge \tau \vee t, \end{cases} \tag{3.28}$$

where

$$\eta_{t,\tau} = \frac{\mathbb{E}_{Q^t}[\int_{t\wedge\tau}^\tau W_s^F \pi_s x \Lambda_s M_s(\alpha_s^{t,3}+1)ds|\mathcal{F}_t]}{2\mathbb{E}[T-\tau\vee t|\mathcal{F}_t]} - \frac{\mathbb{E}[T+\tau\vee t|\mathcal{F}_t]}{2} + \frac{\mathbb{E}^2[T|\mathcal{F}_t] - \mathbb{E}[T^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau\vee t|\mathcal{F}_t]} - \frac{\mathbb{E}^2[\tau\vee t|\mathcal{F}_t] - \mathbb{E}[(\tau\vee t)^2|\mathcal{F}_t]}{2\mathbb{E}[T-\tau\vee t|\mathcal{F}_t]}. \tag{3.29}$$

**Remark 3.4.2.** *Note that from Assumption 3.3.2 and from the fact that the integral in* (3.29) *is bounded, we have that $\eta_{t,\tau}$ is finite and $\mathcal{F}_t$-measurable, and that moreover $\mathbb{E}[\eta_{t,\tau}^\alpha] < \infty$ for all $\alpha \in \mathbb{R}$.*

Choosing

$$\alpha_s^{t,3} = \begin{cases} 0 & \text{for } s < t \text{ or } s \ge \tau, \\ \frac{1}{(M_s+1)(W_s^F+1)} - 1 & \text{for } t \le s < \tau, \end{cases} \tag{3.30}$$

and

$$\alpha_s^{t,1} = \begin{cases} 0 & \text{for } s < t, \\ -\frac{a}{b} - \frac{2}{b}\pi_s \Lambda_s \frac{M_s}{M_s+1}\frac{1}{W_s^F+1} & \text{for } t \le s < \tau, \\ -\frac{a}{b} - \frac{2}{bW_s^F}(s - \eta_{t,\tau}) & \text{for } s \ge \tau \vee t. \end{cases} \tag{3.31}$$

we have that (3.25) and (3.26) hold.

Now we give the following

**Proposition 3.4.3.** *Let $\alpha^{t,1}$, $\alpha^{t,2}$ and $\alpha^{t,3}$ be as in* (3.28)-(3.31). *Then*

$$\mathbb{E}_{Q^t}\left[\int_t^T (W_s^F)^2 ds\right] < \infty, \quad t \in [0,T].$$

*Proof.* From (3.23) and from the expressions of $\alpha^{t,1}$, $\alpha^{t,2}$ and $\alpha^{t,3}$ in (3.28)-(3.31) we have that

$$\tilde{\mu}_s^t = \begin{cases} -2\pi_s x \Lambda_s \frac{M_s}{M_s+1}\frac{1}{W_s^F+1} & \text{for } s < \tau, \\ \frac{1}{W_s^F}(\eta_{t,\tau} - s) & \text{for } s \geq \tau, \end{cases}$$

where $\eta_{t,\tau}$ is given in (3.29). Then from (3.22) it holds that under $Q^t$

$$\begin{aligned} dW_s^F =& \psi_s ds + bW_s^F d\tilde{B}_s^t && \text{for } s < \tau, \\ dW_s^F =& (\eta_{t,\tau} - s)ds + bW_s^F d\tilde{B}_s^t && \text{for } s \geq \tau, \end{aligned}$$

where $\psi_s = -2\pi_s x \Lambda_s \frac{M_s}{M_s+1}\frac{1}{W_s^F+1}$.
Thus we have

$$W_s^F = \begin{cases} e^{b\tilde{B}_s^t - \frac{b^2}{2}s}\int_0^s \psi_u e^{-b\tilde{B}_u^t + \frac{b^2}{2}u} du & \text{for } s < \tau, \\ e^{b\tilde{B}_s^t - \frac{b^2}{2}s}\int_0^s (\eta_{t,\tau} - u) e^{-b\tilde{B}_u^t + \frac{b^2}{2}u} du & \text{for } s \geq \tau. \end{cases}$$

Then

$$\begin{aligned} &\mathbb{E}_{Q^t}\left[\int_t^T (W_s^F)^2 ds\right] \\ =&\mathbb{E}_{Q^t}\Big[\int_{t\wedge\tau}^{\tau}\left(\int_0^s \psi_u e^{b(\tilde{B}_s^t - \tilde{B}_u^t) - \frac{b^2}{2}(s-u)} du\right)^2 ds \\ &+ \int_{\tau}^{T}\left(\int_0^s (\eta_{t,\tau} - u) e^{b(\tilde{B}_s^t - \tilde{B}_u^t) - \frac{b^2}{2}(s-u)} du\right)^2 ds\Big] \\ \leq&\mathbb{E}_{Q^t}\Big[4\Pi^2 x^2 \int_{t\wedge\tau}^{\tau}\left(\int_0^s e^{b(\tilde{B}_s^t - \tilde{B}_u^t) - \frac{b^2}{2}(s-u)} du\right)^2 ds \\ &+ (|\eta_{t,\tau}| + T)^2 \int_{\tau}^{T}\left(\int_0^s e^{b(\tilde{B}_s^t - \tilde{B}_u^t) - \frac{b^2}{2}(s-u)} du\right)^2 ds\Big] \\ \leq&\left(4\Pi^2 x^2 + \mathbb{E}\left[(|\eta_{t,\tau}| + T)^4\right]^{\frac{1}{2}}\right)\mathbb{E}_{Q^t}\left[\int_{t\wedge\tau}^{T}\left(\int_0^s e^{b(\tilde{B}_s^t - \tilde{B}_u^t) - \frac{b^2}{2}(s-u)} du\right)^4 ds\right]^{\frac{1}{2}}. \end{aligned}$$

Since $T$ is bounded and the first term is finite by Remark 3.4.2, it remains to prove

$$\mathbb{E}_{Q^t}\left[\int_{t\wedge\tau}^{T}\left(\int_0^s e^{b(\tilde{B}_s^t - \tilde{B}_u^t) - \frac{b^2}{2}(s-u)} du\right)^4 ds\right] < \infty. \tag{3.32}$$

We have that

$$\begin{aligned} &\mathbb{E}_{Q^t}\left[\int_{t\wedge\tau}^{T}\left(\int_0^s e^{b(\tilde{B}_s^t - \tilde{B}_u^t) - \frac{b^2}{2}(s-u)} du\right)^4 ds\right] = \mathbb{E}_{Q^t}\left[\int_{t\wedge\tau}^{T}\left(\int_0^s e^{b\tilde{B}_{s-u}^t - \frac{b^2}{2}(s-u)} du\right)^4 ds\right] \\ &= \mathbb{E}_{Q^t}\left[\int_{t\wedge\tau}^{T}\left(\int_0^s e^{b\tilde{B}_r^t - \frac{b^2}{2}r} dr\right)^4 ds\right] \leq \mathbb{E}_{Q^t}\left[(T - t\wedge\tau)^2\right]^{\frac{1}{2}}\mathbb{E}_{Q^t}\left[\left(\int_0^T e^{b\tilde{B}_r^t - \frac{b^2}{2}r} dr\right)^8\right]^{\frac{1}{2}}. \end{aligned}$$

The first term is finite by Assumption 3.3.2 on $T$ and $\tau$, whereas

$$\mathbb{E}_{Q^t}\left[\left(\int_0^T e^{b\tilde{B}_r^t-\frac{b^2}{2}r}dr\right)^8\right] \leq \mathbb{E}_{Q^t}\left[\int_0^T e^{8b\tilde{B}_r^t-4b^2r}dr\right] = \int_0^T \mathbb{E}_{Q^t}\left[e^{8b\tilde{B}_r^t-4b^2r}\right]dr < \infty.$$

Then (3.32) holds and we have the result. □
We have therefore proved that, if we take $\alpha^{t,1}$, $\alpha^{t,2}$ and $\alpha^{t,3}$ as in (3.28)-(3.31), then (3.19), (3.20) and (3.16) are satisfied.
From now on we denote $Z_{t,s} := \frac{dQ^t}{dP}|_{\mathcal{F}_s}$ for all $s \geq t$, and $Z_{t,s} = 1$ for $s < t$.
Note that we have not yet used the hypothesis on $\mu$ and $\sigma$ of Assumption 3.3.2 to derive (3.21). From now on we will need them to prove that $(Z_{t,s})_{s\in[t,T)}$ is a true martingale for each $t \in [0,T)$, i.e. that each $Q^t$, $t \in [0,T)$, in (3.21) belongs to $\in \mathcal{M}_{loc}(W)$.

**Remark 3.4.4.** *By Assumption 3.3.2, as proved in Proposition 3.3.3, we exclude that the integral $\int_0^\cdot M_s^2 ds$ can explode in finite time. This is a difference with respect to Jarrow et al. [2012], where the bubble bursts (i.e. $\beta_t = 0$) at $\inf\{s|\int_0^s M_u^2 du = +\infty\}$.*
*In our model, however, the bubble can be zero, and also negative, even if the liquidity is not zero: it can be seen that this can happen when the drift $\mu$ of the signed volume of market orders in* (3.7) *becomes negative. In this approach, therefore, whether or not the bubble is positive depends more on the attitude of the investors than on the liquidity. In Section 4 we propose an example to show how contagion between traders in financial networks can determine the value of $\mu$.*

From now on, we fix $t \in [0,T)$. We begin the analysis by noticing that, since $[B^1, N] \equiv [B^2, N] \equiv 0$,

$$\begin{aligned} Z_{t,s} =& \mathcal{E}\left(\int_0^s \alpha_u^{t,1}dB_u^1 + \int_0^s \alpha_u^{t,2}dB_u^2 + \int_0^s \alpha_u^{t,3}d\tilde{N}_u\right) \\ =& \mathcal{E}\left(\int_0^s \alpha_u^{t,1}dB_u^1 + \int_0^s \alpha_u^{t,2}dB_u^2\right)\mathcal{E}\left(\int_0^s \alpha_u^{t,3}d\tilde{N}_s\right) \end{aligned}$$

for $s \in [0,T)$.
Moreover

$$\begin{aligned} \mathcal{E}\left(\int_0^s \alpha_u^{t,3}d\tilde{N}_u\right) \leq& \exp\left\{\int_0^s \left[\alpha_u^{t,3} - \frac{1}{2}(\alpha_u^{t,3})^2\right]dN_u - \int_0^s \alpha_u^{t,3}\pi_u du\right\}\cdot \\ & \cdot \prod_{0\leq u\leq s}(1+\Delta(\alpha_u^{t,3}N_u))\exp\{\Delta(\alpha_u^{t,3}N_u) + \frac{1}{2}\Delta(\alpha_u^{t,3}N_u)^2\} \\ \leq& 2\exp\left\{\frac{3}{2} + \int_0^s \left[|\alpha_u^{t,3}| + \frac{1}{2}|\alpha_u^{t,3}|^2\right]dN_u + \int_0^s |\alpha_u^{t,3}|\pi_u du\right\} \\ \leq& 2e^{3+T\Pi}, \end{aligned}$$

since by (3.30) it holds $|\alpha_s^{t,3}| \leq 1$.
Then, taking $(\bar{Z}_{t,s})_{s\in[0,T)}$ with

$$\bar{Z}_{t,s} = \mathcal{E}\left(\int_0^s \alpha_u^{t,1}dB_u^1 + \int_0^s \alpha_u^{t,2}dB_u^2\right) \tag{3.33}$$

we have

$$Z_{t,s} \leq 2e^{3+T\Pi}\bar{Z}_{t,s}. \tag{3.34}$$

We give the following

**Lemma 3.4.5.** *Let $X$, $Y$ be two positive stochastic processes such that $Y_t \leq X_t$ a.s. $\forall t \geq 0$, and let $X$ be of class $DL$[1]. Then $Y$ is of class $DL$ as well.*

*Proof.* By Theorem 11 in Protter [2005, chapter 1] we have that a family of random variables $(U_\alpha)_{\alpha \in A}$ is uniformly integrable if and only if there exists a function $G$ defined on $[0, \infty)$, positive, increasing and convex, such that $\lim_{x\to\infty} \frac{G(x)}{x} = +\infty$ and $\sup_\alpha \mathbb{E}[G \circ |U_\alpha|] < \infty$. Fix now $t \geq 0$, and call $J_t = \{\tau : \tau \leq t \text{ stopping time}\}$, $U_X^t = \{X_\tau : \tau \in J_t\}$ and $U_Y^t = \{Y_\tau : \tau \in J_t\}$. Since by hypothesis $U_X^t$ is uniformly integrable, there exists a function $G$ that satisfies the properties stated before. We have that

$$G(Y_\tau) \leq G(X_\tau), \quad a.s. \text{ for } \tau \in J_t,$$

and then that

$$\mathbb{E}[G(Y_\tau)] \leq \mathbb{E}[G(X_\tau)], \quad \tau \in J_t.$$

Thus

$$\sup_{\tau \in J_t} \mathbb{E}[G(Y_\tau)] \leq \sup_{\tau \in J_t} \mathbb{E}[G(X_\tau)] < \infty.$$

Therefore $U_Y^t$ is uniformly integrable and $Y$ is of class $DL$. □
We have then the following

**Proposition 3.4.6.** *$(Z_{t,s})_{s\in[0,T)}$ in (3.21) is a martingale if $(\bar{Z}_{t,s})_{s\in[0,T)}$ is a martingale.*

*Proof.* Since a local martingale is a true martingale if and only if it is of class $DL$, see Proposition 1.7 in Revuz and Yor [1999, chapter 4], we have that if $\bar{Z}$ is a true martingale then $2e^{3+T\Pi}\bar{Z}$, being a martingale as well, is of class $DL$. Thus, by Lemma 3.4.5 and by (3.34), $Z$ is of class $DL$, and therefore by Proposition 1.7 in Revuz and Yor [1999, chapter 4] it is a true martingale. □

From now on, therefore, we will check the martingale property for $(\bar{Z}_{t,s})_{s\in[0,T]}$ in (3.33).
We note that the Novikov condition is not satisfied since for example the integrand $\alpha^{t,1}$ contains the term $\frac{1}{W^F}$ with

$$W_s^F = \exp\left(\left(\mu - \sigma^2/2\right)s + \sigma B_s^1\right),$$

and it can be seen that the expectation of the double exponential of the Brownian motion under $P$ is not finite. The same problem arises with the terms $\frac{1}{M_s}$ and $\frac{1}{\sigma_s}$ in (3.28).
Therefore, since the other terms are strictly positive and bounded, we can not use Novikov condition.

To prove that $\bar{Z}$ is a martingale we rely on some results provided by Mijatovic and Urusov [2012a], by Blei and Engelbert [2009] and by Wong and Heyde [2004]. We first need some preliminaries.
Consider the state space $J = (l, r)$, $-\infty \leq l < r \leq \infty$ and a $J$-valued diffusion $Y = (Y_s)_{s\in[0,T)}$ on some filtered probability space, governed by the SDE

$$dY_s = \mu_Y(Y_s)ds + \sigma_Y(Y_s)dB_s, \quad 0 \leq s < T, \tag{3.35}$$

[1] A stochastic process $X$ is of class $DL$ if, for each $t \geq 0$, $\{X_\tau : \tau \leq t \text{ stopping time}\}$ is uniformly integrable.

with $Y_0 = x_0 \in J$, $W$ Brownian motion and deterministic functions $\mu_Y(\cdot)$ and $\sigma_Y(\cdot)$, that from now on we will simply denote by $\mu_Y$ and $\sigma_Y$, such that

$$\sigma_Y(x) \neq 0 \quad \forall x \in J \tag{3.36}$$

and

$$\frac{1}{\sigma_Y^2},\ \frac{\mu_Y}{\sigma_Y^2} \in L^1_{loc}(J), \tag{3.37}$$

where $L^1_{loc}(J)$ denotes the class of locally integrable functions on $J$, i.e. the measurable functions $(J, \mathcal{B}(J)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that are integrable on compact subsets of $J$.
Consider the stochastic exponential

$$\mathcal{E}\left(\int_0^s f(Y_u)dB_u\right), \quad 0 \leq s < T, \tag{3.38}$$

with $f(\cdot)$ such that

$$\frac{f^2}{\sigma_Y^2} \in L^1_{loc}(J) \tag{3.39}$$

and the auxiliary $J$-valued diffusion $\tilde{Y}$ governed by the SDE

$$d\tilde{Y}_s = \left(\mu_Y(\tilde{Y}_s) + f(\tilde{Y}_s)\sigma_Y(\tilde{Y}_s)\right)ds + \sigma_Y(\tilde{Y}_s)d\tilde{B}_s, \quad 0 \leq s < T, \tag{3.40}$$

where $\tilde{B}$ is a Brownian motion on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.
Put $\bar{J} = [l, r]$ and, fixing an arbitrary $c \in J$, define

$$\rho(x) := \exp\left\{-\int_c^x \frac{2\mu_Y}{\sigma_Y^2}(y)dy\right\}, \quad x \in J, \tag{3.41}$$

$$\tilde{\rho}(x) := \rho(x)\exp\left\{-\int_c^x \frac{2f}{\sigma_Y}(y)dy\right\}, \quad x \in J, \tag{3.42}$$

$$s(x) := \int_c^x \rho(y)dy, \quad x \in \bar{J}, \tag{3.43}$$

$$\tilde{s}(x) := \int_c^x \tilde{\rho}(y)dy, \quad x \in \bar{J}. \tag{3.44}$$

Denote $\rho = \rho(\cdot)$, $s = s(\cdot)$, $s(r) = \lim_{x\to r^-} s(x)$, $s(l) = \lim_{x\to l^+} s(x)$, and analogously for $\tilde{s}(\cdot)$ and $\tilde{\rho}(\cdot)$.
Recall that by Feller's test for explosions $\tilde{Y}$ exits its state space with positive probability at the boundary point $r$ if and only if

$$\tilde{s}(r) < \infty \quad \text{and} \quad \frac{\tilde{s}(r) - \tilde{s}}{\tilde{\rho}\sigma_Y^2} \in L^1_{loc}(r-), \tag{3.45}$$

where $L^1_{loc}(r-) := \{g | g : (J, \mathcal{B}(J)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\int_x^r g(y)dy < \infty$ for some $x \in J\}$. Similarly, $\tilde{Y}$ exits its state space with positive probability at the boundary point $l$ if and only if

$$\tilde{s}(l) > -\infty \quad \text{and} \quad \frac{\tilde{s} - \tilde{s}(l)}{\tilde{\rho}\sigma_Y^2} \in L^1_{loc}(l+), \tag{3.46}$$

where $L^1_{loc}(l+) := \{g | g : (J, \mathcal{B}(J)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\int_l^x g(y)dy < \infty$ for some $x \in J\}$ Moreover, the endpoint $r$ of $J$ is said to be *good* if

$$s(r) < \infty \quad \text{and} \quad \frac{(s(r) - s)f^2}{\rho\sigma_Y^2} \in L^1_{loc}(r-), \tag{3.47}$$

or equivalently (see Mijatovic and Urusov [2012a]) if

$$\tilde{s}(r) < \infty \quad \text{and} \quad \frac{(\tilde{s}(r) - \tilde{s})f^2}{\tilde{\rho}\sigma_Y^2} \in L^1_{loc}(r-). \tag{3.48}$$

Similarly, the endpoint $l$ of $J$ is said to be *good* if

$$s(l) > -\infty \quad \text{and} \quad \frac{(s - s(l))f^2}{\rho\sigma_Y^2} \in L^1_{loc}(l+), \tag{3.49}$$

or equivalently if

$$\tilde{s}(l) > -\infty \quad \text{and} \quad \frac{(\tilde{s} - \tilde{s}(l))f^2}{\tilde{\rho}\sigma_Y^2} \in L^1_{loc}(l+). \tag{3.50}$$

We recall here Theorem 2.1 in Mijatovic and Urusov [2012a].

**Theorem 3.4.7.** *Let the functions $\mu_Y$, $\sigma_Y$, and $f$ satisfy conditions* (3.36), (3.37) *and* (3.39), *and let $Y$ be a solution of the SDE* (3.35).
*Then the Doléans exponential given by* (3.38) *is a martingale for any $T < \infty$ if and only if both of the following requirements are satisfied:*

*(a) condition* (3.45) *does not hold or conditions* (3.47)-(3.48) *hold;*

*(b) condition* (3.46) *does not hold or conditions* (3.49)-(3.50) *hold.*

We now obtain the following

**Proposition 3.4.8.** *Let $\bar{S} = (\bar{S}_s)_{s\in[0,T)}$ be a geometric Brownian motion*

$$d\bar{S}_s = \mu_0\bar{S}_s ds + \sigma_0\bar{S}_s dB_s, \quad 0 \le s < T, \tag{3.51}$$

*where $B$ is a Brownian motion, $\mu_0 \in \mathbb{R}$ and $\sigma_0 > 0$.*
*Then the process*

$$\mathcal{E}\left(\int_0^s (\bar{S}_u)^{-1} dB_u\right), \quad 0 \le s < T,$$

*is a martingale.*

*Proof.* We show that the requirements of Theorem 3.4.7 hold for $Y = \bar{S}$, with $\mu_Y(x) = \mu_0 x$, $\sigma_Y(x) = \sigma_0 x$ and $f(x) = x^{-1}$. Note that $\mu_Y$, $\sigma_Y$ and $f$ satisfy conditions (3.36), (3.37) and (3.39) with $J = (0, \infty)$. Then, taking $c = 1$ for the functions (3.41)-(3.44) and first assuming $\frac{2\mu_0}{\sigma_0^2} \neq 1$, we have

$$\rho(x) = \exp\left\{-\int_1^x \frac{2\mu_Y}{\sigma_Y^2}(y)dy\right\} = x^{\frac{-2\mu_0}{\sigma_0^2}}, \tag{3.52}$$

$$\tilde{\rho}(x) = \rho(x)\exp\left\{-\int_1^x \frac{2f}{\sigma_Y}(y)dy\right\} = x^{\frac{-2\mu_0}{\sigma_0^2}} \exp\left(\frac{2}{\sigma_0}\left(\frac{1}{x} - 1\right)\right), \tag{3.53}$$

$$s(x) = \int_1^x \rho(y)dy = \frac{\sigma_0^2}{2\mu_0 - \sigma_0^2}(1 - x^{-\gamma_0}), \tag{3.54}$$

$$\tilde{s}(x) = \int_1^x \tilde{\rho}(y)dy = e^{-\frac{2}{\sigma_0}} \left(-\frac{2}{\sigma_0}\right)^{-\gamma_0} \left[\bar{\Gamma}\left(\gamma_0, -\frac{2}{x\sigma_0}\right) - \bar{\Gamma}\left(\gamma_0, -\frac{2}{\sigma_0}\right)\right], \tag{3.55}$$

with $\gamma_0 = \frac{2\mu_0}{\sigma_0^2} - 1$ and where $\bar{\Gamma}(a, z) = \int_z^\infty e^{-t}t^{a-1}dt$, $a \in \mathbb{R}^+$, $z \in \mathbb{R}$, is the incomplete Gamma function extended to all $\mathbb{R}$.
Note that in (3.55) we have that

$$\begin{aligned}
\tilde{s}(x) &= e^{-\frac{2}{\sigma_0}} \left(-\frac{2}{\sigma_0}\right)^{-\gamma_0} \left[\bar{\Gamma}\left(\gamma_0, -\frac{2}{x\sigma_0}\right) - \bar{\Gamma}\left(\gamma_0, -\frac{2}{\sigma_0}\right)\right] \\
&= e^{-\frac{2}{\sigma_0}} \left(\frac{2}{\sigma_0}\right)^{-\gamma_0} (-1)^{-\gamma_0} \int_{-\frac{2}{x\sigma_0}}^{-\frac{2}{\sigma_0}} e^{-t}(-1)^{\gamma_0 - 1}|t|^{\gamma_0 - 1}dt \\
&= -e^{-\frac{2}{\sigma_0}} \left(\frac{2}{\sigma_0}\right)^{-\gamma_0} \int_{-\frac{2}{x\sigma_0}}^{-\frac{2}{\sigma_0}} e^{-t}|t|^{\gamma_0 - 1}dt \in \mathbb{R}.
\end{aligned} \tag{3.56}$$

We obtain that:

- in $l = 0$ we have

$$\tilde{s}(0) = -e^{-\frac{2}{\sigma_0}} \left(\frac{2}{\sigma_0}\right)^{-\gamma_0} \int_{-\infty}^{-\frac{2}{\sigma_0}} e^{-t}|t|^{\gamma_0 - 1}dt = -\infty,$$

  thus condition (3.46) does not hold and the first requirement of (b) in Theorem 3.4.7 is fulfilled;

- if $\gamma_0 < 0$ we have

$$\tilde{s}(\infty) = e^{-\frac{2}{\sigma_0}} \left(\frac{2}{\sigma_0}\right)^{-\gamma_0} \int_{-\frac{2}{\sigma_0}}^{0} e^{-t}|t|^{\gamma_0 - 1}dt = \infty$$

  then condition (3.45) does not hold and the first requirement of (a) in Theorem 3.4.7 is fulfilled;

- if $\gamma_0 > 0$ then $s(\infty) = \frac{\sigma_0^2}{2\mu_0 - \sigma_0^2} = C < \infty$, and condition (3.47) holds since

$$\frac{s(r) - s}{\rho\sigma_0^2} = C\frac{x^{-\gamma_0}x^{\frac{2\mu_0}{\sigma_0^2}}}{x^4} = \frac{1}{x^3}.$$

  Therefore the second requirement of (a) in Theorem 3.4.7 is fulfilled.

So we have that if $\gamma_0 \neq 0$ the requirements of Theorem 3.4.7 are satisfied, and thus $Z$ is a martingale.
In the case $\gamma_0 = 0$, i.e. $\mu_0 = \frac{\sigma_0^2}{2}$, we have that the process $\bar{S} = (\bar{S}_u)_{u\in[0,T)}$ in (3.51) takes the form $\bar{S}_u = e^{\sigma_0 B_u}$, $0 \leq u < T$. We can thus apply the results of Theorem 3.4.7 taking $J = (-\infty, \infty)$, $\mu_Y \equiv 0$, $\sigma_Y \equiv 1$, $f(x) = e^{-\sigma_0 x}$ and $c = 0$ in (3.41)-(3.44). We have

$$\rho(x) = \exp\left\{-\int_0^x \frac{2\mu_Y}{\sigma_Y^2}(y)dy\right\} = 1,$$

$$\tilde{\rho}(x) = \rho(x) \exp\left\{ - \int_0^x \frac{2f}{\sigma_Y}(y)dy \right\} = \exp\left(2(e^{-\sigma_0 x} - 1)/\sigma_0\right), \tag{3.57}$$
$$s(x) = \int_0^x \rho(y)dy = x$$
$$\tilde{s}(x) = \int_0^x \tilde{\rho}(y)dy = \frac{1}{\sigma_0} e^{-\frac{2}{\sigma_0}} \left( Ei\left(2/\sigma_0\right) - Ei\left(2e^{-\sigma_0 x}/\sigma_0\right)\right),$$

where $Ei(z) = -\int_{-z}^{\infty} \frac{e^{-u}}{u} du$ is the exponential integral function that satisfies $\lim_{z\to\infty} Ei(z) = \infty$ and $\lim_{z\to 0} Ei(z) = -\infty$. Therefore $\tilde{s}(\infty) = \infty$ and $\tilde{s}(-\infty) = -\infty$, then the first requirements of (a) and (b) of Theorem 3.4.7 are both satisfied and $Z$ is a martingale. □

Then we have immediately

**Corollary 3.4.9.** *Under Assumptions 3.3.2, the process*

$$\mathcal{E}\left( \int_\tau^s \frac{1}{W_u^F} dB_u^1 \right), \quad \tau \le s < T, \tag{3.58}$$

*is a martingale for every fixed $T < \infty$.*

To prove that Corollary 3.4.9 also implies that $\mathcal{E}\left(\int_\tau^s \alpha_u^{t,1} dB_u^1\right)$ is a martingale, we need Theorem 4.1 in Blei and Engelbert [2009], that we report here as a Proposition.

**Proposition 3.4.10.** *Let $H$ be a continuous local martingale. Then $\mathcal{E}(H)$ is a martingale if and only if*

$$\lim_{n\to\infty} Q_s(\{A_s < n\}) = 1 \quad \forall s \ge 0, \tag{3.59}$$

*where $A_s = [H, H]_s$ and $dQ_s = \mathcal{E}(H_{T_s})dP$, with $T_s := \inf\{u \ge 0 : A_u > s\}$.*

We now give

**Proposition 3.4.11.** *In the setting of Section 3.3, the process*

$$\mathcal{E}\left( \int_0^s |\alpha_u^{t,1}| dB_u^1 \right), \quad 0 \le s < T,$$

*with $\alpha^{t,1}$ in (3.31), is a martingale for each $t \in [0, T)$.*

*Proof.* For $s < \tau$ we have

$$|\alpha_s^{t,1}| = \frac{a}{b} + \frac{2}{b}\pi_s \Lambda_s \frac{M_s}{M_s + 1} \frac{1}{W_s^F + 1} \le \frac{a}{b} + \frac{2}{b}\Pi,$$

then $\mathcal{E}\left(\int_0^\cdot |\alpha_u^{t,1}| dB_u^1\right)$ is a martingale up to time $\tau$ as it satisfies Novikov condition, since

$$\mathbb{E}\left[ \exp\left( \int_0^\tau (\alpha_s^{t,1})^2 ds \right) \right] \le \mathbb{E}\left[ \exp(c^2 \tau) \right]$$

with $c = \frac{a}{b} + \frac{2}{b}\Pi$.
Consider now $s \ge \tau$. In this case,

$$|\alpha_s^{t,1}| = \frac{a}{b} + \frac{2}{bW_s^F}(s + \eta_{t,\tau}) \le \frac{a}{b} + \frac{2}{bW_s^F}(T + \eta_{t,\tau}), \tag{3.60}$$

with $\eta_{t,\tau}$ in (3.29). From Proposition 3.4.10 and Corollary 3.4.9 we have

$$\lim_{n\to\infty} Q_s\left(\left\{\int_0^s \frac{1}{(W_u^F)^2}du < n\right\}\right) = 1 \quad \forall s \geq 0,$$

where $dQ_s = \mathcal{E}\left(\int_0^{T_s} \frac{1}{(W_u^F)^2}du\right)dP$, with $T_s := \inf\left\{u \geq 0 : \int_0^u \frac{1}{(W_r^F)^2}dr > s\right\}$. From (3.60), we have that (3.59) holds also for $\int_\tau^\cdot |\alpha_u^{t,1}|dB_u^1$, and then the result follows. □

We can now give

**Proposition 3.4.12.** *Under Assumption 3.3.2, the process* $(Z_{t,s}^1)_{s\in[0,T)}$ *with*

$$Z_{t,s}^1 = \mathcal{E}\left(\int_0^s \alpha_u^{t,1}dB_u^1\right), \quad 0 \leq s < T, \tag{3.61}$$

*with* $\alpha^{t,1}$ *in* (3.31)*, is a martingale for each* $t \in [0,T)$*.*

The proof follows by Proposition 3.4.11 and by the following

**Lemma 3.4.13.** *Consider* $H_s = \int_0^s Y_u dB_u$ *and* $\bar{H}_s = \int_0^s |Y_u|dB_u$*,* $0 \leq s < T$*, where* $Y$ *is a stochastic process such that the stochastic integral is well defined. Then* $\mathcal{E}(H)$ *is a martingale if and only if* $\mathcal{E}(\bar{H})$ *is a martingale.*

*Proof.* We use again Proposition 3.4.10. Since $[H,H]_s = \int_0^s Y_u^2 du = \int_0^s |Y_u|^2 du = [\bar{H},\bar{H}]_s$, property (3.59) holds for $H$ if and only if it holds for $\bar{H}$. Hence we have the result. □

We now want to prove that

$$\mathcal{E}\left(\int_0^s \alpha_u^{t,2}dB_u^2\right), \quad 0 \leq s < T, \tag{3.62}$$

with $\alpha^{t,2}$ in (3.28), is a martingale as well, supposing for the sake of simplicity $\tau = 0$ since $\alpha_s^{t,2} = 0$ for $s \leq \tau$.

To this purpose, we reformulate a result by Wong and Heyde [2004], which Mijatović and Urusov [2011] have criticised not to hold in general. However, in our context we are able to prove that the required condition is satisfied in our formulation, as we show in the sequel.

Consider a continuous $\mathbb{F}$-progressively measurable $d$-dimensional process $H = (H_s)_{s\in[0,T)}$ of the form

$$H_s = \xi(B.,s)\zeta_s + \eta_s, \tag{3.63}$$

where $\xi \in C_0(\mathbb{R}^{d+1},\mathbb{R}^d)$, $B$ is a $d$-dimensional progressively measurable Brownian motion and $\zeta,\eta$ are $d$-dimensional stochastic processes independent of $B$. Here the product between $\xi$ and $\zeta$ is intended componentwise.

Define

$$\tau_N^{M_H} = \inf\left(s \in [0,T) : M_H(t) := \int_0^t \|H_u\|^2 du \geq N\right),$$

with the convention that $\inf\emptyset = \infty$, and then

$$\tau^{M_H} = \lim_{N\to\infty} \tau_N^{M_H}. \tag{3.64}$$

Then we have the following

**Proposition 3.4.14.** *Let $H$ be a $\mathbb{F}$-progressively measurable $d$-dimensional process as in* (3.63)*, stopped at the explosion time $\tau^{M_H}$ in* (3.64)*. Assume that the equation*

$$Y_s = \xi\left(B_\cdot + \int_0^\cdot Y_u du, s\right)\zeta_s + \eta_s, \quad s \in [0,T), \tag{3.65}$$

*admits a unique solution $Y = (Y_s)_{s\in[0,\tau^{M_Y})}$, where $\tau^{M_Y}$ is the explosion time*

$$\tau^{M_Y} = \lim_{N\to\infty} \tau_N^{M_Y},$$

*where*

$$\tau_N^{M_Y} = \inf\left(s \in [0,T) : M_Y(s) := \int_0^s \|Y_u\|^2 du \geq N\right). \tag{3.66}$$

*Thus, the stochastic exponential $Z^H = (Z_s^H)_{s\in[0,T)}$ with $Z_s^H = \mathcal{E}\left(\int_0^s H_u dW_u\right)$ satisfies*

$$P(\tau^{M_Y} > T) \leq \mathbb{E}[Z_T^H].$$

*Hence $Z^H$ is a (true) martingale if $P(\tau^{M_Y} > T) = 1$.*

**Remark 3.4.15.** *The proof of Proposition 3.4.14 is analogous to the one of Theorem 1 in Wong and Heyde [2004], where the authors consider the process $\tilde{H}_s = \xi(B_\cdot, s)$, which corresponds in our case to choose $\zeta_t = 1$, $\eta_t = 0$ for all $t \in [0,T)$. In Wong and Heyde [2004], the authors state that equation* (3.65) *always has a solution. However, the proof of this statement is not correct, as noted by Mijatović and Urusov [2011]. This is the reason why here we* suppose *that* (3.65) *admits a solution. We prove in Propositions 3.4.16 and 3.4.18 that this $Y$ exists and is unique.*

*Proof.* Define $Y_t^N = Y_{t\wedge\tau_N^{M_Y}}$, with $t_N^{M_Y}$ in (3.66). Let $\tilde{\mathcal{F}}_T$ be a $\sigma$-algebra on $\tilde{\Omega} := C_0\left((0,T),\mathbb{R}^d\right)$. Consider a set $A \in \tilde{\mathcal{F}}_T$, and define the strict subset of A

$$D_N^A = \left\{ y \in C_0\left((0,T),\mathbb{R}^d\right) : \quad \int_0^T \|y(u)\|^2 du < N \right\} \cap A.$$

It holds

$$\begin{aligned} P(Y^N \in D_N^A) &= P\left(Y^N \in A,\ \int_0^T \|Y_u^N\|^2 du < N\right) = P(Y^N \in A,\ \tau_N^{M_Y} > T) \\ &= P(Y \in A,\ \tau_N^{M_Y} > T), \end{aligned} \tag{3.67}$$

since $Y = Y^N$ on the set $\{\tau_N^{M_Y} > T\}$.
Take now the process $Z^{-Y^N} = (Z_t^{-Y^N})_{t\in[0,T)}$, so that $Z_t^{-Y^N} := Z^{-Y}_{t\wedge\tau_N^{M_Y}} = \mathcal{E}\left(-\int_0^{t\wedge\tau_N^{M_Y}} Y_s dB_s\right)$, $t \in [0,T)$. Since $Y^N$ is bounded, it is possible to define a new measure $Q_N^Y \sim P$ by

$$Q_N^Y(Y^N \in A) = \mathbb{E}[Z_T^{-Y^N} \mathbb{1}_{\{Y^N\in A\}}],$$

for all $A \in \tilde{\mathcal{F}}_T$. Under this measure, Girsanov theorem implies that $(B_t^{Q_N})_{0\leq t\leq T}$ with

$$B_t^{Q_N^Y} = B_t + \int_0^t Y_u^N du, \quad 0 \leq t < T,$$

is a $Q_N^Y$-Brownian motion in $\mathbb{R}^d$. On the set $\{T \leq \tau_N^{M_H}\}$ we have $Y_t = Y_t^N$ for all $t \in [0,T)$, and then

$$Y_t = \xi\left(B_\cdot^{Q_N^Y}, t\right)\zeta_t + \eta_t, \quad 0 \leq t < T. \tag{3.68}$$

By (3.63) and (3.68) we have

$$\begin{aligned} P(Y^N \in D_N^A) =& \mathbb{E}_{Q_N^Y}[Z_T^{Y^N}\mathbb{1}_{\{Y^N \in D_N^A\}}] = \mathbb{E}_{Q_N^Y}[Z_T^{Y^N}\mathbb{1}_{\{Y^N \in A, \tau_N^{M_Y} > T\}}] \\ =& \mathbb{E}_{Q_N^Y}[Z_T^{Y}\mathbb{1}_{\{Y \in A, \tau_N^{M_Y} > T\}}] = \mathbb{E}_P[Z_T^{H}\mathbb{1}_{\{H \in A, \tau_N^{M_H} > T\}}]. \end{aligned} \tag{3.69}$$

Putting together (3.67) and (3.69), we have

$$P(Y \in A,\ \tau_N^{M_Y} > T) = \mathbb{E}_P[Z_T^H \mathbb{1}_{\{H \in A, \tau_N^{M_H} > T\}}] \leq \mathbb{E}_P[Z_T^H]. \tag{3.70}$$

We can apply Lebesgue's dominated convergence theorem and pass to the limit as $N \to \infty$ the left hand side of (3.70). It follows

$$P(\tau^{M_Y} > T) \leq \mathbb{E}_P[Z_T^H],$$

as required. □

We now prove that in our setting the assumption of existence (and uniqueness) of a solution of (3.65) is satisfied when the process $H$ is the bubble $\beta$ defined in (3.7).

**Proposition 3.4.16.** *Let $\beta$ be the bubble as in* (3.7)*, i.e., in the case when $\sigma$ does not depend on $\beta$. Then there exists a unique solution to the equation* (3.65) *when $H = \beta$ in* (3.63)*.*

*Proof.* If we rewrite $\beta$ in the form (3.63), we obtain that

$$\xi(B_\cdot^2, s) = \int_0^s \sigma_u \Lambda_u M_u e^{-k\int_u^s \Lambda_r M_r dr} dB_u^2, \quad 0 \leq s < T,$$

so that equation (3.65) has the form

$$\begin{aligned} Y_s =& \beta_0 e^{-k\int_0^s \Lambda_u M_u du} + \int_0^s \mu_u \Lambda_u M_u e^{-k\int_u^s \Lambda_r M_r dr} du + \int_0^s \sigma_u \Lambda_u M_u e^{-k\int_u^s \Lambda_r M_r dr} dB_u^2 \\ &+ \int_0^s \sigma_u \Lambda_u M_u e^{-k\int_u^s \Lambda_r M_r dr} Y_u du, \quad 0 \leq s < T. \end{aligned} \tag{3.71}$$

Differentiating both sides, we obtain the SDE

$$dY_s = \Lambda_s M_s \left[((\sigma_s - k)Y_s + \mu_s)ds + \sigma_s dB_s^2\right], \quad 0 \leq s < T, \tag{3.72}$$

$Y_0 = \beta_0$. By Theorem 7 in Chapter V.3 in Protter [2005], (3.72) has a unique strong solution, which is the unique solution of (3.71). □

We can now apply Proposition 3.4.14 in order to prove the following

**Proposition 3.4.17.** *Let $\beta$ be the bubble as defined in* (3.7)*. Under Assumption 3.3.2, the Doléans exponential*

$$\mathcal{E}\left(\int_0^s \beta_u dB_u^2\right), \quad 0 \leq s < T,$$

*is a martingale.*

*Proof.* By Proposition 3.4.16, we can apply the result of Proposition 3.4.14. The solution of (3.72) is given by

$$\begin{aligned}\bar{Y}_s =&\beta_0 e^{\int_0^s(-k+\sigma_u)\Lambda_u M_u ds} + \int_0^s \mu_u \Lambda_u M_u e^{\int_u^s(-k+\sigma_r)\Lambda_r M_r dr} du \\ &+ \int_0^s \sigma_u \Lambda_u M_u e^{\int_u^s(-k+\sigma_r)\Lambda_r M_r dr} dB_u^2, \quad 0 \le s < T. \end{aligned} \tag{3.73}$$

We first prove that $\bar{Y}_s < \infty$ for each $s \in [0,T)$. We have $\int_u^s(-k+\sigma_r)\Lambda_r M_r dr < \infty$ a.s. for each $s \in [0,T)$ by the hypothesis on $\sigma$ and $\Lambda$ in Assumption 3.3.2 and by Proposition 3.3.3.
Thus by Theorem 2.4 of Mijatovic and Urusov [2012b] and by the fact that $T$ is bounded, we obtain

$$\int_0^T e^{\alpha \int_u^s(-k+\sigma_r)\Lambda_r M_r dr} du < \infty \tag{3.74}$$

for all $\alpha \in \mathbb{R}$, and then by the hypothesis on $\mu$ in Assumption 3.3.2, and again by Proposition 3.3.3, we have

$$\int_0^s \mu_u \Lambda_u M_u e^{\int_u^s(-k+\sigma_r)\Lambda_r M_r dr} du < \infty, \quad 0 \le s < T.$$

By (3.74) and by Assumption 3.3.2 it follows that the stochastic integral in (3.73) does not explode before $T$, so we have that $\bar{Y}_s < \infty$ for each $s \in [0,T)$. We prove that this implies $\int_0^T \bar{Y}_s^2 ds < \infty$. By the expression of $\bar{Y}$ in (3.73) we have

$$\int_0^T \bar{Y}_s^2 ds = \int_0^T \bar{Y}_s^2 \frac{1}{M_s^2 \Lambda_s^2 \sigma_s^2} d[\bar{Y}, \bar{Y}]_s$$

(by the Kunita-Watanabe inequality)

$$\le \left( \int_0^T \bar{Y}_s^4 d[\bar{Y}, \bar{Y}]_s \right)^{1/2} \left( \int_0^T \frac{1}{M_s^4 \Lambda_s^4 \sigma_s^4} d[\bar{Y}, \bar{Y}]_s \right)^{1/2}$$

(by the occupation time formula)

$$= \left( \int_{-\infty}^{\infty} a^4 L_T^a da \right)^{1/2} \left( \int_0^T \frac{1}{M_s^2 \Lambda_s^2 \sigma_s^2} ds \right)^{1/2} < \infty : \tag{3.75}$$

the first integral is finite because the local time $L_T^a$ has bounded support in $(-\infty, \infty)$, since $\bar{Y}$ does not explode before $T$, and the second one is finite by Assumption 3.3.2 and Proposition 3.3.3. Then the result follows by Proposition 3.4.14. □

We can now exploit Proposition 3.4.17 to prove that $\mathcal{E}\left(\int_0^s \alpha_u^{t,2} dB_u^2\right)$ is a martingale. Before doing this, we give the following

**Proposition 3.4.18.** *Let $(\alpha_s^{t,2})_{s\in[0,T)}$ be the process defined in* (3.28)*, with $\beta$ as in* (3.7) *and $\sigma$ independent of $\beta$. Then there exists a unique solution to the equation* (3.65) *when $H = \alpha^{t,2}$ in* (3.63)*.*

*Proof.* Since $\alpha_s^{t,2} = 0$ for $s \leq t$ and

$$\alpha_s^{t,2} = \frac{s - \eta_{t,0}}{\Lambda_s M_s \sigma_s} - \frac{\mu_s}{\sigma_s} + k\frac{\beta_s}{\sigma_s}, \quad t \leq s < T,$$

equation (3.65) takes the form[2]

$$Y_s = \alpha_s^{t,2} + \frac{k}{\sigma_s}\int_t^s \sigma_u \Lambda_u M_u Y_u e^{-k\int_u^s \Lambda_r M_r dr} du, \quad t \leq s < T. \tag{3.76}$$

Differentiating (3.76), we obtain for $t \leq s < T$

$$\begin{aligned} dY_s &= d\alpha_s^{t,2} + k\int_t^s \sigma_u \Lambda_u M_u Y_u e^{-k\int_u^s \Lambda_r M_r dr} du \cdot d\left(\frac{1}{\sigma_s}\right) + \frac{k}{\sigma_s} d\left(\int_t^s \sigma_u \Lambda_u M_u Y_u e^{-k\int_u^s \Lambda_r M_r dr} du\right) \\ &= d\alpha_s^{t,2} + (Y_s - \alpha_s^{t,2})\sigma_s d\left(\frac{1}{\sigma_s}\right) + k\Lambda_s M_s \alpha_s^{t,2} ds, \end{aligned} \tag{3.77}$$

$Y_t = \alpha_0^{t,2}$. Theorem 7 in Chapter V.3 in Protter [2005] together with Assumption 3.3.2 imply that (3.77) has a unique strong solution, which is solution of (3.76). □

**Proposition 3.4.19.** *Under Assumption 3.3.2, the process* $(Z_{t,s}^2)_{s\in[0,T)}$ *with*

$$Z_{t,s}^2 = \mathcal{E}\left(\int_0^s \alpha_u^{t,2} dB_u^2\right), \quad 0 \leq s < T, \tag{3.78}$$

*with* $\alpha^{t,2}$ *in* (3.28) *is a martingale for each* $t \in [0, T)$.

*Proof.* We first consider the case when $\sigma$ does not depend on $\beta$. Initially, we prove that $\mathcal{E}\left(\int_0^s \sigma_u \alpha_u^{t,2} dB_u^2\right)$ is a martingale. We have that $\sigma_s \alpha_s^{t,2} = 0$ for $s \leq t$ and

$$\sigma_s \alpha_s^{t,2} = \frac{s - \eta_{t,0}}{\Lambda_s M_s} - \mu_s + k\beta_s, \quad t \leq s < T, \tag{3.79}$$

so that equation (3.65) takes the form

$$Y_s = \sigma_s \alpha_s^{t,2} + k\int_t^s \sigma_u \Lambda_u M_u e^{-k\int_u^s \Lambda_r M_r dr} Y_u du, \quad t \leq s < T. \tag{3.80}$$

Differentiating both sides of (3.80), we obtain

$$dY_s = d(\sigma_s \alpha_s^{t,2}) + k\Lambda_s M_s \left((\sigma_s - 1)Y_s + \sigma_s \alpha_s^{t,2}\right) ds, \quad t \leq s < T,$$

which has a unique strong solution by Theorem 7 in Chapter V.3 in Protter [2005]. Then we can apply Proposition 3.4.14 since also equation (3.80) admits a (unique) solution, that we call $\tilde{Y}$.

We have

$$\begin{aligned} \tilde{Y}_s =& \sigma_s \alpha_s^{t,2} + k\int_t^s \sigma_u \Lambda_u M_u e^{-k\int_u^s \Lambda_r M_r dr} \tilde{Y}_u du \\ =& \frac{s - \eta_{t,0}}{\Lambda_s M_s} - \mu_s + k\beta_s + k\int_t^s \sigma_u \Lambda_u M_u e^{-k\int_u^s \Lambda_r M_r dr} \hat{Y}_u du \end{aligned}$$

[2] It is straightforward to adapt the results of Proposition 3.4.14 to the case when the initial time is $t > 0$.

$$
\begin{aligned}
&+ k\int_t^s \sigma_u\Lambda_u M_u e^{-k\int_u^s \Lambda_r M_r dr}(\tilde{Y}_u - \hat{Y}_u)du,\\
=&\frac{s-\eta_{t,0}}{\Lambda_s M_s} - \mu_s + \hat{Y}_s + k\int_t^s \sigma_u\Lambda_u M_u e^{-k\int_u^s \Lambda_r M_r dr}(\tilde{Y}_u - \hat{Y}_u)du, \quad t \le s < T,
\end{aligned}
$$

where $\hat{Y} = k\bar{Y}$ with $\bar{Y}$ in (3.73). Consequently, for $\bar{D} = \tilde{Y} - \hat{Y}$, it holds

$$
\bar{D}_s = \frac{s-\eta_{t,0}}{\Lambda_s M_s} - \mu_s + k\int_t^s \sigma_u\Lambda_u M_u e^{-k\int_u^s \Lambda_r M_r dr}\bar{D}_u du, \quad t \le s < T,
$$

and then

$$
\begin{aligned}
\bar{D}_s =&\frac{s-\eta_{t,0}}{\Lambda_s M_s} - \mu_s + k\int_t^s \left(\frac{u-\eta_{t,0}}{\Lambda_u} - \mu_u M_u\right)\sigma_u\Lambda_u e^{k\int_u^s \Lambda_r M_r(\sigma_r - 1)dr}du\\
\le&\frac{\eta_{t,0}+T}{\lambda M_s} + \mu_s + k\int_t^s \left(\frac{\eta_{t,0}+T}{\lambda} + \mu_u M_u\right)\sigma_u\Lambda_u e^{k\int_u^s \Lambda_r M_r(\sigma_r - 1)dr}du, \quad t \le s < T.
\end{aligned}
$$

By Assumption 3.3.2 and by Proposition 3.3.3, with the same argument as in the proof of Proposition 3.4.17, we have that

$$
\int_t^T \bar{D}_s^2 ds = \int_t^T |\tilde{Y}_s - \hat{Y}_s|^2 ds < \infty.
$$

Then, since by Proposition 3.4.17 we have $\int_t^T |\hat{Y}_s|^2 ds < \infty$, we obtain

$$
\int_t^T |\tilde{Y}_s|^2 ds < \infty. \tag{3.81}
$$

Now we prove that $\mathcal{E}\left(\int_0^s \alpha_u^{t,2} dB_u^2\right)$ is a martingale. By Proposition 3.4.18 can apply the results of Proposition 3.4.14. Let $Y$ be the (unique) solution of (3.76). Then it holds

$$
\begin{aligned}
Y_s =&\frac{1}{\sigma_s}\left(\sigma_s\alpha_s^{t,2} + k\int_t^s \sigma_u\Lambda_u M_u\tilde{Y}_u e^{-k\int_u^s \Lambda_r M_r dr}du\right) + \frac{1}{\sigma_s}k\int_t^s \sigma_u\Lambda_u M_u(Y_u - \tilde{Y}_u)e^{-k\int_u^s \Lambda_r M_r dr}du\\
=&\frac{1}{\sigma_s}\left(\tilde{Y}_s + k\int_t^s \sigma_u\Lambda_u M_u(Y_u - \tilde{Y}_u)e^{-k\int_u^s \Lambda_r M_r dr}du\right), \quad t \le s < T.
\end{aligned}
$$

We have

$$
\sigma_s Y_s - \tilde{Y}_s = \Psi_s + k\int_t^s \Lambda_u M_u(\sigma_u Y_u - \tilde{Y}_u)e^{-k\int_u^s \Lambda_r M_r dr}du, \quad t \le s < T,
$$

where $(\Psi_s)_{s\in[t,T)}$ is given by

$$
\Psi_s = k\int_t^s \Lambda_u M_u(\tilde{Y}_u - \sigma_u\tilde{Y}_u)e^{-k\int_u^s \Lambda_r M_r dr}du, \quad t \le s < T. \tag{3.82}
$$

It follows that $D_s = \sigma_s Y_s - \tilde{Y}_s$ satisfies

$$
dD_s = d\Psi_s + k\Lambda_s M_s\Psi_s ds, \quad t \le s < T,
$$

and so that it takes the form

$$
D_s = \Psi_s + k\int_t^s \Lambda_u M_u\Psi_u du, \quad t \le s < T.
$$

Since by Assumption 3.3.2 the process $\Psi$ in (3.82) does not explode before $T$, $D_s = \sigma_s Y_s - \tilde{Y}_s < \infty$ a.s. for each $s \in [t, T]$.
Thus, with the same argument as in the proof of Proposition 3.4.17 it can be proved that

$$\int_t^T |\sigma_s Y_s - \tilde{Y}_s|^2 ds < \infty.$$

By (3.81) we then have

$$\int_t^T |\sigma_s Y_s|^2 ds < \infty.$$

Then by the integrability hypothesis on $\frac{1}{\sigma^4}$ in (ii) of Assumption 3.3.2 it holds

$$\int_t^T |Y_s|^2 ds < \infty.$$

The result then follows by Proposition 3.4.14.
In the case $\sigma_t = \alpha\beta_t$, it holds

$$\alpha_s^{t,2} = \frac{1}{\Lambda_s M_s \sigma_s}(s - \eta_{t,\tau}) - \frac{\mu_s}{\sigma_s} + \frac{k}{\alpha}, \quad t \le s < T,$$

with $\sigma_s = \alpha\beta_s$.

We first prove that $\mathcal{E}\left(\int_0^\cdot |\alpha_u^{t,2}| dB_u^2\right)$ is a martingale. By the expression of $\beta$ in (3.8)-(3.9), and since the bubble is positive by the assumption $\mu_t \ge 0$ $\forall t \in [0, T)$, we have that equation (3.65) has the form

$$Y_s = |\alpha_s^{t,2}| e^{-2\alpha \int_0^s \Lambda_u M_u Y_u du} + \frac{k}{\alpha}, \quad 0 \le s \le T. \tag{3.83}$$

Differentiating (3.65), we find the SDE

$$dY_s = \frac{d|\alpha_s^{t,2}|}{|\alpha_s^{t,2}|} Y_s - 2\alpha \Lambda_s M_s Y_s^2 ds, \quad 0 \le s \le T, \tag{3.84}$$

$Y_0 = |\alpha_0^{t,2}| + \frac{k}{\alpha}$, so that we have only locally Lipschitz coefficients. However, Theorem 38 in Chapter V.7 in Protter [2005] ensures the existence of a unique solution of (3.84) up to an explosion time $\tau^Y$. Therefore, the assumption of Proposition 3.4.14 is satisfied.

Moreover, since $\alpha, \Lambda$ and $M$ are positive, and $Y_0$ is positive as well, by (3.83) we have

$$Y_s = |\alpha_s^{t,2}| + \frac{k}{\alpha} \quad 0 \le s < T,$$

and $\int_0^T |Y_s|^2 ds < \infty$ by Assumption 3.3.2 and by Proposition 3.3.5. The result then follows by Lemma 3.4.13. □

We are now ready to state the main result of the Section:

**Theorem 3.4.20.** *Under Assumption 3.3.2, $Q^t$ defined in* (3.21) *belongs to $\mathcal{M}_{loc}(W)$ for each $t \in [0, T)$.*

*Proof* The proof follows by the fact that taking $\alpha^{t,1}$ and $\alpha^{t,2}$ as in (3.31) and (3.28), with $\mu_t$, $\sigma_t$, $M$, $\Lambda$ and $\pi$ satisfying Assumption 3.3.2, then $(\bar{Z}_{t,s})_{s\in[0,T)}$ with

$$\bar{Z}_{t,s} = \mathcal{E}\left(\int_0^s \alpha_u^{t,1} dB_u^1 + \int_0^s \alpha_u^{t,2} dB_u^2\right)$$

is a martingale with respect to time $s$.
This follows immediately from Proposition 3.4.12 and Proposition 3.4.19: $(Z^1_{t,s})_{s\in[0,T)}$ in (3.61) and $(Z^2_{t,s})_{s\in[0,T)}$ in (3.78) are martingales, so by Proposition 3.4.14 we know that $H^1 = \alpha^{t,1}$ and $H^2 = \alpha^{t,2}$ are such that the associated processes $Y^1$ and $Y^2$ defined in Proposition 3.4.14 do not explode before $T$. Taking now $H = (H^1, H^2)$, the associated process $Y = (Y^1, Y^2)$ does not explode before $T$ as well, and this concludes the proof. □

**Remark 3.4.21.** *Theorem 3.4.20 shows that our constructive model can be included in the more fundamental view of the martingale theory of bubble of Jarrow et al. [2007] and Jarrow et al. [2010]. To this purpose we need to admit the possibility of shifting pricing views over time as suggested in Biagini et al. [2014]. However we emphasize that our definition of bubble and the models proposed in Section 3.3 and further investigated in Section 4 are independent of any choice of $Q \in \mathcal{M}_{loc}(W)$. This can be seen as an advantage of this framework since the definition of $Q$-bubble could arise some criticisms (see Guasoni and Rasonyi [2015]).*
*Note that Theorem 3.4.20 also implies that $\mathcal{M}_{loc}(W) \neq \emptyset$, hence that our market model is arbitrage-free on $[0, T)$.*

# Chapter 4

# Liquidity induced bubbles in a network

## 4.1 Motivation

Different contributions show how contagion between investors and herding behavior may play an essential role when a bubble grows up: euphoria and exuberance can propagate among market participants, due to exchanges of ideas (see Lux [1995]) or to the fact that investors may be attracted by the short period earnings of acquaintances investing in the bubbly asset, as observed by Bayer et al. [2016], where analyzing data from the housing bubble in L. A. in the 2000s the authors note a strong contagion between neighbors.
Several works in the last years have been focusing on how some properties of a network, like mean degree or degree heterogeneity, can influence the contagion of failures and losses between banks during a financial crisis (see for example Acemoglu et al. [2012], Allen and Gale [2000b], Amini et al. [2016], Cont et al. [2013], Gai and Kapadia [2010], Newman et al. [2001], Watts [2002], Watts and Strogatz [1998]). Some investigation has been proposed about how bubbles are generated at the microeconomic level by the interaction of market participants (see among others Lux [1995], Scheinkman and Xiong [2003], Scheinkman and Xiong [2013], Tirole [1982], Zhuk [2013]). However, only a few studies have been devoted to understand how the structure of a given financial network can influence the spread of contagion between investors that generates a bubble. In Lux [1995], for example, the author models the bubble as caused by a self-organizing process of infection between traders, expressed by a system of PDEs, leading to equilibrium prices that deviate from the fundamental value. Nonetheless, he considers a world in which everybody is connected with everybody, so that the network structure does not enter into play. We also cite the works of Battiston [2015], where it is shown how bubbles can have an impact on the structure of a banking network, and Bouchard et al. [2015], where the authors describe the passage from a well-connected network with high global confidence to a poorly connected network with low global confidence, producing a boom and bust cycle.
Our approach is however quite different. We consider an *information* network of $N$ investors who may be influenced by the trading activity of their neighborhoods. In particular, we assume that the number $N$ of traders in the network is big enough to guarantee that our hypothesis on the linearity

of the supply curve holds. Then, we model the trading contagion mechanism between agents taking place from the time $\tau$ of the birth of the bubble, via the dynamics of the signed volume of market orders introduced in Chapter 3. The evolution of the signed volume impacts in turn the market price of the asset and then the bubble as described in Chapter 3. Investors may place a buy market order on the bubbly asset because they imitate neighbors in the network that have successfully bought the asset as well, eventually leading to some self-exciting herding effect.
We refer to some literature about information networks (see among others Ozsoylev and Walden [2011], Ozsoylev et al. [2014], Walden [2014]) where investors share information with neighbors so that, as in Ozsoylev et al. [2014], two traders linked together buy or sell the same stock at a similar point in time.
Our analysis is based on some epidemiological studies, which describe how diseases spread in social networks, or how computer viruses spread from computer to computer. In particular, we here focus on the SIS (susceptible-infected-susceptible) model, studied for example by Pastor-Satorras and Vespignani [2001a,b] to analyze virus diffusion in a population. In the SIS model, every node of the network represents an individual and each link is a connection along which the infection can spread to others. Individuals can only exist in two discrete states, namely, susceptible, or "healthy", and infected. At every time step, each susceptible node is infected with rate $\lambda$ by an infected node, if there is a connection between the two. At the same time, infected nodes are cured and become again susceptible with rate $\delta$. In this way, individuals run stochastically through the cycle susceptible $\rightarrow$ infected $\rightarrow$ susceptible.
In the next section, we adapt this model to our financial framework.

## 4.2 The model

As in Chapter 3, we call $\tau$ the first time when the signed volume of market orders $X$ become strictly positive, determine the birth of the bubble. We reinterpret virus diffusion as trading contagion and consider as a first step in our model building process the following stochastic version of the SIS model for the contagion evolution of the fraction $(\rho_t^k)_{\tau\leq t\leq T}$ of traders of degree $0 \leq k \leq N$ (i.e. traders with information channels to $k$ other traders) that has bought the asset before or at time $t$:

$$d\rho_t^k = \left(-\delta\rho_t^k + \lambda k m_t(1-\rho_t^k)\right)dt + \bar{\sigma}_t^k(\rho_t^k)^\alpha(1-\rho_t^k)^\alpha dB_t^2, \quad \tau \leq t \leq T, \quad 0 < \rho_\tau^k < 1. \tag{4.1}$$

Here $m_t$ is the probability that an individual at the end of an edge has bought the asset before or at time $t$, $\lambda$ is the rate of buying contagion, $\delta$ is the rate of selling, $\bar{\sigma}^k = (\bar{\sigma}_t^k)_{\tau\leq t\leq T}$, $k = 1, \cdots, N$, are progressively measurable processes, which we assume bounded from above and away from zero, and $\alpha \geq 1$. Then the evolution (4.1) guarantees that $0 \leq \rho^k \leq 1$.
To determine the probability $m_t$, we observe that by Bayes rule, and since for any given node $v$ it holds

$$P(\text{meet } v | deg(v) = k) = \frac{k}{\sum_j j q_j}$$

where $q_j$ is the number of nodes with degree $j$, we have that

$$P(deg(v)=k|\text{meet } v)=\frac{P(\text{meet } v|deg(v)=k)P(deg(v)=k)}{P(\text{meet } v)}=\frac{k}{\frac{1}{N}\sum_j jq_j}p_k=\frac{kp_k}{z},$$

where $z:=\frac{1}{N}\sum_j jq_j$ is the average degree and $p_k=P(deg(v)=k)=q_k/N$. Therefore we have

$$m_t=\sum_k P(deg(v)=k|\text{meet } v)\rho_t^k=\frac{1}{z}\sum_k kp_k\rho_t^k,\quad \tau\le t<T. \tag{4.2}$$

Given the contagion evolution of the fraction $\rho^k$, we model the average signed volume of market orders of an agent of degree $k$ by $X_t^k=\theta_t^k\rho_t^k$, where the size of market orders $(\theta_t^k)_{\tau\le t\le T}$ of a trader of degree $k$ that buys the asset is given by a positive continuous process with dynamics

$$d\theta_t^k=\mu_t^k dt+\sigma_t^k dB_t^2,\quad \tau\le t<T,\quad 0<\theta_\tau^k, \tag{4.3}$$

where for all $k=1,\cdots,N$, $(\mu_t^k)_{\tau\le t\le T}$ is an adapted continuous process, and $(\sigma_t^k)_{\tau\le t\le T}$ is a positive adapted continuous process.[1] Since we have $d[\rho^k,\theta^k]_t=\bar\sigma_t^k\sigma_t^k(\rho_t^k)^\alpha(1-\rho_t^k)^\alpha dt$, by Itô's formula it holds

$$\begin{aligned}dX_t^k&=\theta_t^k d\rho_t^k+\rho_t^k d\theta_t^k+d[\rho^k,\theta^k]_t \\ &=\left(-\delta X_t^k+\lambda k m_t(\theta_t^k-X_t^k)+\rho_t^k\mu_t^k+\bar\sigma_t^k\sigma_t^k(\rho_t^k)^\alpha(1-\rho_t^k)^\alpha\right)dt \\ &\quad+\left(\theta_t^k\bar\sigma_t^k(\rho_t^k)^\alpha(1-\rho_t^k)^\alpha+\rho_t^k\sigma_t^k\right)dB_t^2.\end{aligned} \tag{4.4}$$

Finally, we obtain that the signed volume of total market orders is given by $X_t=\sum_{k=0}^N q_k X_t^k$, where $q_k$ is the number of investors of degree $k$. From (4.4) we thus obtain

$$dX_t=\left(-\delta X_t+\lambda m_t(\theta_t-n_t)+\eta_t\right)dt+\bar\Sigma_t dB_t^2, \tag{4.5}$$

with

$$n_t=\sum_k kq_kX_t^k,\qquad \theta_t=\sum_k kq_k\theta_t^k,\qquad \eta_t=\sum_k kq_k\left(\rho_t^k\mu_t^k+\bar\sigma_t^k\sigma_t^k(\rho_t^k)^\alpha(1-\rho_t^k)^\alpha\right) \tag{4.6}$$

and

$$\bar\Sigma_t=\sum_k q_k\left(\bar\sigma_t^k\theta_t^k(\rho_t^k)^\alpha(1-\rho_t^k)^\alpha+\rho_t^k\sigma_t^k\right) \tag{4.7}$$

We are thus in the framework[2] of Section 3.3, with

$$\mu_t=-\delta X_t+\lambda m_t(\theta_t-n_t)+\eta_t \tag{4.8}$$

and $\sigma_t=\bar\Sigma_t$, leading to the following SDE for the bubble $\beta$:

$$d\beta_t=\Lambda_t M_t\left[-k\beta_t+2\left(-\delta X_t+\lambda m_t(\theta_t-n_t)+\eta_t\right)\right]dt+2\Lambda_t M_t\bar\Sigma_t dB_t^2 \tag{4.9}$$

[1] Note that the following analysis still holds under different integrability and measurability conditions on $\bar\sigma$ and $\sigma^k,\mu^k$.

[2] The assumption that $\theta^k$ is driven by the same Brownian motion of $\rho^k$ allows to show the existence of the flow by using directly the results of Section 3.3, but it can be easily relaxed, letting $\theta^k$ depend also on an additional Brownian motion $B^\theta$ independent of $B^2$, as we do in Section 4.3.

for $\tau \leq t < T$, with explicit solution

$$\beta_t = \beta_\tau e^{-k\int_\tau^t \Lambda_s M_s ds} + \int_\tau^t \left(-\delta X_s + \lambda m_s(\theta_s - n_s) + \eta_s\right) \Lambda_s M_s e^{-k\int_s^t \Lambda_u M_u du} ds + \\ + \int_\tau^t \bar{\Sigma}_s \Lambda_s M_s e^{-k\int_s^t \Lambda_u M_u du} dB_s^2, \quad \tau \leq t < T. \tag{4.10}$$

**Remark 4.2.1.** *Setting $\mu^j \equiv \bar{\sigma}^j \equiv \sigma^j \equiv 0$ for all $0 \leq j \leq N$ in* (4.1) *and* (4.3) *respectively, we can identify the driving deterministic contagion evolution for the signed volume of market orders as implied by the SIS network model approach:*

$$dX_t = \left(-\delta X_t + \lambda m_t(\theta_t - n_t)\right) dt. \tag{4.11}$$

**Remark 4.2.2.** *In the next Subsection 4.3 we consider two different types of network topologies in order to see how the characteristics of the network influence the dynamics of the expected fraction of buyers through $n_t$. In the first one we have a connectivity distribution which is very peaked at the average value $z$ and decaying exponentially fast for $k \gg z$ and $k \ll z$. Examples of this kind of networks are random graph models Erdős and Rényi [1960] and the small-world model of Watts and Strogatz [1998]. In the second one the degree distribution is more right skewed, following for example a power law, as in the Barabási and Albert preferential attachment model Barabási and Albert [1999]. From* (4.11) *and* (4.6) *we can see that the expected contagion between buyers will spread faster in the second kind of network, since the distribution puts more weight on the nodes with higher degree, resulting in a bigger value of $n_t$ in* (4.6).

We conclude the introduction of the model by showing a sufficient condition under which the above bubble specification can be represented by a flow of local martingale measures as analyzed in the general framework of the previous sections, i.e. that there exists a flow $Q^t \in \mathcal{M}_{loc}(W)$ with Radon-Nykodim derivative process

$$Z_{t,s} = \frac{dQ^t}{dP}|_{\mathcal{F}_s} = \mathcal{E}\left(\int_0^\cdot \alpha_u^{t,1} dB_u^1 + \int_0^\cdot \alpha_u^{t,2} dB_u^2 + \int_0^\cdot \alpha_u^{t,3} d\tilde{N}_u\right)_s, \quad s \in [0,T), \tag{4.12}$$

such that

$$W_t^F = \mathbb{E}_{Q^t}[W_T^F|\mathcal{F}_t], \quad 0 \leq t \leq T.$$

Taking $\alpha^{t,1}$, $\alpha^{t,2}$ and $\alpha^{t,3}$ in (3.31), (3.28) and (3.30) respectively we only need to show that that $Z$ in (4.12) is in fact a martingale.
First we prove that, under certain conditions on the parameters, it holds $0 < \rho^k < 1$ for the processes $(\rho_t^k)_{t\leq 0}$ in (4.1).
We rely on the results given in Mijatovic and Urusov [2012b]. In particular, consider a process $Y = (Y_t)_{t\geq 0}$ satisfying

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t, \quad t \geq 0, \tag{4.13}$$

with $W$ standard Brownian motion. Call $J = (\ell, r)$ the state space of $Y$, with $-\infty \leq \ell < r \leq \infty$, and $\zeta$ its exit time. Define

$$\rho(x) := \exp\left\{-\int_c^x \frac{2\mu_Y}{\sigma_Y^2}(y)dy\right\}, \quad x \in J, \tag{4.14}$$

and

$$s(x) := \int_c^x \rho(y)dy, \quad x \in \bar{J}, \tag{4.15}$$

with $c \in J$. Then (see Proposition 2.4 of Mijatovic and Urusov [2012b]) we have that $\zeta = \infty$ if $s(\ell) = -\infty$ and $s(r) = +\infty$.

We now give the following

**Lemma 4.2.3.** *Suppose that $\bar{\sigma}$ is bounded from above and away from zero, and that one of the following conditions holds:*

- *$\alpha > 1$, and $\exists k$ such that*

$$\frac{1}{z}\lambda p_k k^2 - \delta > 0. \tag{4.16}$$

- *$\alpha = 1$, and $\exists k$ such that*

$$\frac{1}{z}\lambda p_k k^2 - \delta > \frac{1}{2}. \tag{4.17}$$

*Then the process $(\rho_t^k)_{t\leq 0}$ in (4.1) is such that $0 < \rho_t^k < 1$ a.s. for all $t \geq 0$.*

*Proof.* Take $k$ satisfying (4.16) (or (4.17) if $\alpha = 1$), and suppose by the sake of simplicity $\rho^j \equiv 0$ for $j \neq k$ [3]. Then the process $(\rho_t^k)_{t\geq 0}$ satisfies

$$d\rho_t^k = \left(-\delta\rho_t^k + \frac{1}{z}\lambda p_k k^2 \rho_t^k(1-\rho_t^k)\right)dt + \bar{\sigma}_t(\rho_t^k)^\alpha(1-\rho_t^k)^\alpha dB_t^2, \quad t \geq 0. \tag{4.18}$$

Since $\bar{\sigma}$ is bounded from above and from below, we can apply the results in Mijatovic and Urusov [2012b] taking

$$\mu_Y(x) = -\delta x + A_k x(1-x), \qquad \sigma_Y(x) = x^\alpha(1-x)^\alpha, \tag{4.19}$$

where we have set $A_k = \frac{1}{z}\lambda p_k k^2$. We want to prove that $s(0) = -\infty$ and $s(1) = +\infty$, where $s$ is defined in (4.15).
If $\alpha > 1$, consider

$$c_1 < \frac{A_k - \delta}{A_k}, \tag{4.20}$$

and

$$c_2 > \frac{A_k - \delta}{A_k}. \tag{4.21}$$

If $\alpha = 1$, take $c_2$ as in (4.21) and

$$c_1 \leq \frac{1}{A_k}\left(A_k - \delta - \frac{1}{2}\right). \tag{4.22}$$

By (4.16), (4.17) and since $\delta > 0$ we have $0 < c_1 < c_2 < 1$. We choose $c \in (c_1, c_2)$, so that we have

$$0 < c_1 < c < c_2 < 1. \tag{4.23}$$

[3] The following result holds a fortiori if $\rho_t^j > 0$, $j \neq k$.

We then obtain

$$s(0) = \lim_{x\to 0}\int_c^x \rho(y)dy = -\lim_{x\to 0}\int_x^c \rho(y)dy = -\lim_{x\to 0}\int_x^{c_1}\rho(y)dy - \int_{c_1}^c \rho(y)dy,$$

where by (4.14), (4.19) and (4.23) it holds $0 < \int_{c_1}^c \rho(y)dy < +\infty$.
Analogously,

$$s(1) = \lim_{x\to 1}\int_c^x \rho(y)dy = \int_c^{c_2}\rho(y)dy + \lim_{x\to 1}\int_{c_2}^x \rho(y)dy,$$

where $0 < \int_c^{c_2}\rho(y)dy < \infty$.
It follows that

$$s(0) = -\infty \iff \lim_{x\to 0}\int_x^{c_1}\rho(y)dy = +\infty, \tag{4.24}$$

and

$$s(1) = +\infty \iff \lim_{x\to 1}\int_{c_2}^x \rho(y)dy = +\infty. \tag{4.25}$$

We now compute

$$\begin{aligned}
\rho(x) &= \exp\left\{-\int_c^x \frac{2\mu_Y}{\sigma_Y^2}(y)dy\right\}\\
&= \exp\left\{2\int_c^x \frac{\delta y - A_k y(1-y)}{y^{2\alpha}(1-y)^{2\alpha}}dy\right\}\\
&= \exp\left\{2\int_c^x y^{1-2\alpha}(1-y)^{-2\alpha}\left(\delta - A_k(1-y)\right)dy\right\}\\
&= \exp\left\{2\int_c^x y^{1-2\alpha}(1-y)^{-2\alpha}\left(\delta - A_k + A_k y\right)dy\right\}.
\end{aligned}$$

Let us consider first the case $x < c_1 < c$, for which we have

$$\begin{aligned}
\rho(x) &= \exp\left\{2\int_x^c y^{1-2\alpha}(1-y)^{-2\alpha}\left(A_k - \delta - A_k y\right)dy\right\}\\
&= K_{c_1,c}\exp\left\{2\int_x^{c_1} y^{1-2\alpha}(1-y)^{-2\alpha}\left(A_k - \delta - A_k y\right)dy\right\}\\
&\geq K_{c_1,c}\exp\left\{2\left(A_k - \delta - c_1 A_k\right)\int_x^{c_1} y^{1-2\alpha}(1-y)^{-2\alpha}dy\right\}\\
&\geq K_{c_1,c}\exp\left\{2\left(A_k - \delta - c_1 A_k\right)\int_x^{c_1} y^{1-2\alpha}dy\right\},
\end{aligned} \tag{4.26}$$

where

$$K_{c_1,c} = \exp\left\{\int_{c_1}^c y^{1-2\alpha}(1-y)^{-2\alpha}\left(A_k - \delta - A_k y\right)dy\right\}.$$

If $\alpha > 1$, we then have that

$$\rho(x) \geq K_{c_1,c}K_{c_1}\exp\left\{\frac{A_k - \delta - c_1 A_k}{\alpha - 1}x^{2-2\alpha}\right\},$$

with

$$K_{c_1} = \exp\left\{\frac{A_k - \delta - c_1 A_k}{1-\alpha} c_1^{2-2\alpha}\right\}.$$

Thus

$$\lim_{x\to 0}\int_x^{c_1} \rho(y)dy \geq K_{c_1,c}K_{c_1} \lim_{x\to 0}\int_x^{c_1} \exp\left\{\frac{A_k - \delta - c_1 A_k}{\alpha - 1} y^{2-2\alpha}\right\} dy = +\infty, \tag{4.27}$$

since $\alpha > 1$ and by (4.20). [4]
If $\alpha = 1$, from (4.26) it follows

$$\begin{aligned}\rho(x) &\geq K_{c_1,c} \exp\left\{2(A_k - \delta - c_1 A_k)\left(\log(c_1) - \log(x)\right)\right\} \\ &= K_{c_1,c} c_1^{2(A_k-\delta-c_1A_k)} x^{-2(A_k-\delta-c_1A_k)}.\end{aligned}$$

In this case

$$\lim_{x\to 0}\int_x^{c_1} \rho(y)dy \geq K_{c_1,c} c_1^{2(A_k-\delta-c_1A_k)} \lim_{x\to 0}\int_x^{c_1} y^{-2(A_k-\delta-c_1A_k)} dy = +\infty, \tag{4.28}$$

since by (4.22) we have that $2(A_k - \delta - c_1 A_k) \geq 1$.
In the case $c < c_2 < x$, on the other hand, it holds

$$\begin{aligned}\rho(x) &= \exp\left\{2\int_c^x y^{1-2\alpha}(1-y)^{-2\alpha}\left(\delta - A_k + A_k y\right) dy\right\} \\ &= \bar{K}_{c_2,c} \exp\left\{2\int_{c_2}^x y^{1-2\alpha}(1-y)^{-2\alpha}\left(\delta - A_k + A_k y\right) dy\right\} \\ &\geq \bar{K}_{c_2,c} \exp\left\{2\left(\delta - A_k + c_2 A_k\right)\int_{c_2}^x y^{1-2\alpha}(1-y)^{-2\alpha} dy\right\} \\ &\geq \bar{K}_{c_2,c} \exp\left\{2\left(\delta - A_k + c_2 A_k\right)\int_{c_2}^x (1-y)^{-2\alpha} dy\right\} \\ &\geq \bar{K}_{c_2,c}\bar{K}_{c_2} \exp\left\{\frac{2(\delta - A_k + c_2 A_k)}{2\alpha - 1}(1-x)^{-2\alpha+1}\right\},\end{aligned}$$

where

$$\bar{K}_{c_2,c} = \exp\left\{2\int_c^{c_2} y^{1-2\alpha}(1-y)^{-2\alpha}\left(\delta - A_k + A_k y\right) dy\right\}$$

and

$$\bar{K}_{c_2} = \exp\left\{-\frac{2(\delta - A_k + c_2 A_k)}{2\alpha - 1} c_2^{-2\alpha+1}\right\}.$$

Therefore,

$$\lim_{x\to 1}\int_{c_2}^x \rho(y)dy \geq \bar{K}_{c_1,c}\bar{K}_{c_1} \lim_{x\to 1}\int_{c_2}^x \exp\left\{\frac{2(\delta - A_k + c_2 A_k)}{2\alpha - 1}(1-y)^{-2\alpha+1}\right\} dy = +\infty, \tag{4.29}$$

by (4.21) and since $\alpha \geq 1$ (see the footnote above).
The limits (4.27)-(4.29) together with (4.24) and (4.25) imply the result. □

[4] It can be easily seen that $\lim_{x\to 0}\int_x^{c_1} e^{ay^{-b}} dy = \infty$ for all $a, b > 0$: let $n \in \mathbb{N}$ such that $nb > 1$, then $\lim_{x\to 0}\int_x^{c_1} e^{ay^{-b}} dy \geq \frac{a^n}{n!}\lim_{x\to 0}\int_x^{c_1} y^{-nb} dy = +\infty$.

We now prove that from Lemma 4.2.3 it follows that

$$\int_0^t \frac{1}{\bar{\sigma}_s^4(\rho_s^k)^{4\alpha}(1-\rho_s^k)^{4\alpha}}ds < \infty \quad a.s., \quad t < \infty, \tag{4.30}$$

and use (4.30) to prove that Assumption 2.2 holds. Call

$$\eta_D = \zeta \wedge \inf\{t \geq 0 : Y_t \in D\}, \tag{4.31}$$

where $Y$ is the process in (4.13), $\zeta$ is the exit time from the state space $J = (\ell, r)$ and

$$D = \left\{ x \in J : \frac{f}{\sigma_Y} \notin L^1_{loc}(x) \right\}, \tag{4.32}$$

with

$$L^1_{loc}(x) = \left\{ F : J \to \mathbb{R} \text{ s.t. } \exists \epsilon > 0 : \int_{x-\epsilon}^{x+\epsilon} |F(x)|dx < \infty \right\}.$$

Suppose that

$$\sigma_Y(x) \neq 0 \quad \forall x \in J, \qquad \frac{1}{\sigma_Y^2}, \frac{\mu}{\sigma_Y^2} \in L^1_{loc}(J). \tag{4.33}$$

Then by Theorem 2.6 of Mijatovic and Urusov [2012b] we have

$$\int_0^t f(Y_y)dy < \infty \quad a.s., \quad t \in [0, \eta_D).$$

We can now give the following

**Proposition 4.2.4.** *Suppose the hypothesis of Lemma 4.2.3 hold. Then the process* $(\rho_t^k)_{t\leq 0}$ *in* (4.18) *is such that*

$$\int_0^t \frac{1}{\bar{\sigma}_t^4(\rho_s^k)^{4\alpha}(1-\rho_s^k)^{4\alpha}}ds < \infty \quad a.s., \quad t < \infty.$$

*Proof.* We have

$$\mu_Y(x) = \left(\frac{1}{z}\lambda p_k k^2 - \delta\right)x - \frac{1}{z}\lambda p_k k^2 x^2, \qquad \sigma_Y(x) = x^\alpha(1-x)^\alpha, \qquad f(x) = x^{-4\alpha}(1-x)^{-4\alpha}.$$

Then condition (4.33) is fulfilled. We prove that $D = \emptyset$ in (4.32), i.e. that all $x \in (0,1)$ are such that $\frac{f}{\sigma_Y} \in L^1_{loc}(x)$.
Take first $0 < x < 1/2$ and set $\epsilon = x/2$. In this way, $x - \epsilon > 0$ and

$$\int_{x-\epsilon}^{x+\epsilon} \frac{f(y)}{\sigma^2(y)}dx = \int_{x-\epsilon}^{x+\epsilon} y^{-6\alpha}(1-y)^{-6\alpha}dy < \infty.$$

Analogously, for $1/2 \leq x < 1$ take $\epsilon = (1-x)/2$, so that $x + \epsilon < 1$ and

$$\int_{x-\epsilon}^{x+\epsilon} \frac{f(y)}{\sigma^2(y)}dx < \infty.$$

Then $D = \emptyset$, thus $\eta_D = \zeta$ in (4.31). By Lemma 4.2.3 it follows that $\zeta = \infty$. Then we have

$$\int_0^t f(\rho_s^k)ds = \int_0^t \frac{1}{(\rho_s^k)^{4\alpha}(1-\rho_s^k)^{4\alpha}}ds < \infty \quad a.s., \quad t < \infty.$$

The thesis follows since $\bar{\sigma}$ is bounded away from zero by hypothesis. □

**Proposition 4.2.5.** *Assume that $\alpha \geq 1$ and that there exists a $\bar{k} \in 1, \cdots, N$ that satisfies the hypothesis of Lemma 4.2.3 and such that $\theta_t^{\bar{k}} > \epsilon$ a.s. for all $t \in [\tau, T]$, where $\epsilon > 0$. Then for each $t \in [0, T]$, $(Z_{t,u})_{u \in [0,T)}$ is a $(P, \mathcal{F})$-martingale.*

*Proof.* We show that $\mu$ and $\Sigma$ in (4.8) and (4.7) satisfy Assumption 3.3.2. We have $\int_\tau^T \mu_s^2 ds < \infty$ since $m, \bar{\sigma}^j$ are bounded and $\sigma^j, \mu^j, X, \theta, n$ are continuous processes for $j \in 1, \cdots, N$. Analogously one can show $\int_\tau^T \bar{\Sigma}_s^2 ds < \infty$. Finally by using that $\sigma^{\bar{k}}, \rho^{\bar{k}} \geq 0$ and that $\theta^{\bar{k}}, \bar{\sigma}^{\bar{k}}$ are bounded away from zero, it is easy to see that

$$\int_\tau^T \frac{1}{\bar{\Sigma}_s^4} ds \leq \frac{C}{q_{\bar{k}}^4} \int_\tau^T \frac{1}{(\rho_s^{\bar{k}})^{4\alpha}(1 - \rho_s^{\bar{k}})^{4\alpha}} ds \tag{4.34}$$

for some constant $C$. The integral on the right side of (4.34) is finite by Lemma 4.2.3. □

## 4.3 Analysis of the model

We now comment on our model and specify how the evolution of the bubble described in (4.9) depends on the involved parameters as well as on the structure of the network.
The evolution of the bubble is characterized by two different phases: in the first one the bubble builds up, since the quick increase of the signed volume of market orders $X$ dominates in equation (3.4). However, after a while the processes $\rho^k$ in equation (4.1) tend towards an equilibrium in which the drift of $\rho^k$ vanishes. When this drift's component (and also the contribution of $\eta$ in (4.5)) is sufficiently small, the mean reverting term of equation (3.4) starts to dominate, leading to the burst of the bubble and to the second phase, i.e. the decrease of the bubble towards zero.
In the ascending phase, assuming first for illustration purposes the process $(\theta_t)_{t \geq \tau}$ to be constantly equal to $\theta > 0$ and $\bar{\sigma}^j = 0$ for all $0 \leq j \leq N$, the essential force of the bubble is given by the deterministic contagion mechanism (4.11) driving the signed volume of market orders $X$ in (4.5). The contagion accelerates to a maximum and then slows down. In this way $X$ evolves along an "S" shape as shown in Figure 4.1 growing towards an equilibrium/maximum that is increasing in the volume term $\theta$ and the contagion rate $\lambda$ and decreasing in the recovery rate $\delta$. Further, the speed at which $X$ grows towards the maximum is increasing in $\lambda$ and decreasing in $\delta$. However, since both the length and the maximum of observed speculation bubbles are highly uncertain, we randomize this mechanism by letting $\theta$ be a stochastic process. The impact of a random volume term $\theta$ will be to modify the "S" pattern by allowing the bubble to slow down or pick up in a random way until it reaches a random maximum. In the bursting phase, the dynamics of the bubble will be dominated by the mean reverting factor $k$, which drives the bubble down.
We now focus on the impact of the choice of the underlying network on the dynamics of the bubble. We compare two different cases, an Erdős-Rényi

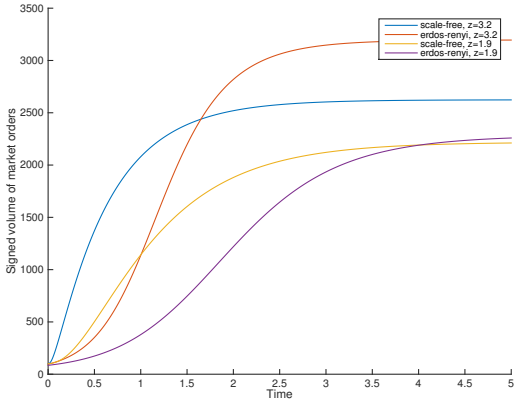


Figure 4.1: Deterministic fraction of buyers for different networks, with $\lambda = 1$, $\delta = 1$.

network with Poisson degree distribution

$$p_j = \frac{e^{-\tilde{\lambda}}\tilde{\lambda}^j}{j!}, \quad j \in \mathbb{N}, \quad \tilde{\lambda} \in \mathbb{R},$$

and a scale-free network with a power law distribution

$$p_j \sim j^{-\gamma}, \quad 2 < \gamma < 3, \quad j \in \mathbb{N}. \tag{4.35}$$

The Erdős-Rényi network has a degree distribution which is very peaked around the mean degree $z$, whereas the scale-free one, that is well-known to better represent real world information networks (see Ozsoylev et al. [2014]), has a much larger right tale, which allows for a more heterogeneous degree distribution with some nodes being very connected and others less (core-periphery structure).
For simplicity, we consider the following deterministic specifications: we set $\bar{\sigma}^j = 0$ for all $0 \leq j \leq N$ and assume the processes $(M_t)_{t\geq\tau}$, $(\Lambda_t)_{t\geq\tau}$ and $(\theta_t)_{t\geq\tau}$ to be constantly equal to $M = \Lambda = \theta = 1$. Further, we choose $\tau = 0$.
We take two different values of $\gamma$ in (4.35)), i.e. $\gamma_1 = 2.2$ and $\gamma_2 = 2.5$, obtaining therefore a more connected network (with $z = z_1 \sim 3.2$) and a less connected one (with $z = z_2 \sim 1.9$). We consider as well two Erdős-Rényi networks with $z = z_1 \sim 3.2$ and $z = z_2 \sim 1.9$, respectively. We take the distribution $p_j$ up to a maximum degree that corresponds to a network with 5000 nodes, see 3.3.2 of Newman [2003].
In Figure 4.1 we illustrate the trajectories of $X$ for the four different networks taking $\lambda = 1$, $\delta = 1$. One can note that both the mean degree and the degree heterogeneity play a key role in the evolution of $X$: in particular, both of them are positively correlated with the speed of the increase. It can also be seen that in the Erdős-Rényi network, i.e. in the less right skewed one, as well as in the less connected networks, the fraction reaches its equilibrium later in time.
We then focus on the behavior of the bubble and consider a mean reversion level $k = 0.4$ in (4.10). In Figure 4.2 and Figure 4.3 we show the maximum reached by the bubble as a function of $\lambda$ and $\delta$ respectively, whereas in Figure 4.4 and Figure 4.5 we plot the time needed to reach the maximum, again as a

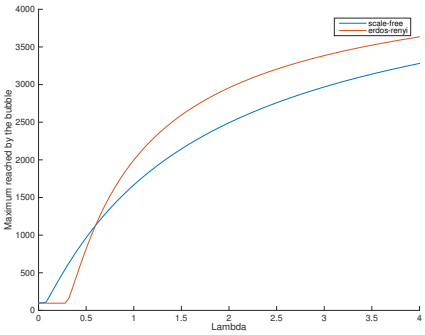


Figure 4.2: Maximum value of the bubble as a function of $\lambda$ with $\delta = 1$, $k = 0.4$.

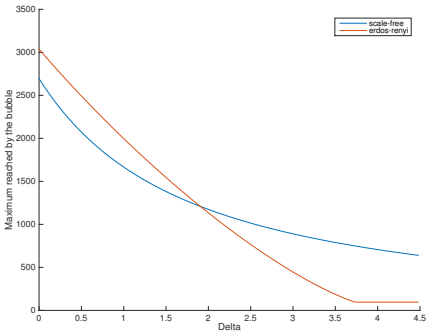


Figure 4.3: Maximum value of the bubble as a function of $\delta$ with $\lambda = 1$, $k = 0.4$.

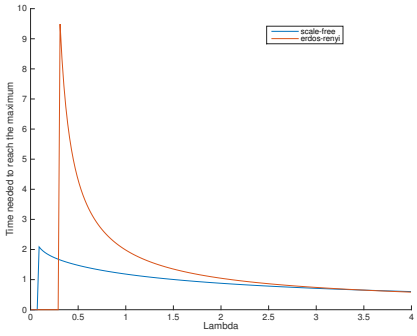


Figure 4.4: Time to the maximum as a function of $\lambda$ with $\delta = 1$, $k = 0.4$.

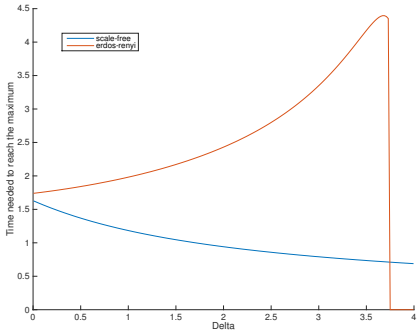


Figure 4.5: Time to the maximum as a function of $\delta$ with $\lambda = 1$, $k = 0.4$.

function of $\lambda$ and $\delta$ respectively.

Figure 4.5 shows that the time to the maximum is decreasing in $\delta$ in the scale-free network and increasing in $\delta$ in the Erdős-Rényi one, i.e. the two networks give rise to different behaviors. It can be seen that for small $\lambda$ and big $\delta$ the maximum is higher in the case of the scale-free network, whereas the opposite holds for big $\lambda$ and small $\delta$. On the other hand, the time needed by the bubble to attain the maximum is always higher in the case of the Erdős-Rényi network.

In Figure 4.6 and in Figure 4.7 we plot the average velocity of the growth of the bubble from time 0 to the time of the maximum, as a function of $\lambda$ and $\delta$ respectively: it is increasing with respect to $\lambda$ and decreasing with respect to $\delta$. Moreover, for fixed $\lambda = 1$, the velocity in the scale-free network is higher for not too small $\delta$, whereas for fixed $\delta = 1$, it is higher in the Erdős-Rényi network for big values of $\lambda$.

We also analyze the influence of $k$ on the bubble's evolution: figures 4.9, 4.10, 4.10 show the maximum, the time needed to reach it and the average velocity of the increase of the bubble respectively, as functions of $k$ for $\lambda = \delta = 1$. We see that, as it could be forecasted, both the maximum and the time to the maximum are decreasing in $k$. In particular, the time to the maximum is decreasing because for large $k$ the mean reverting term starts to dominate

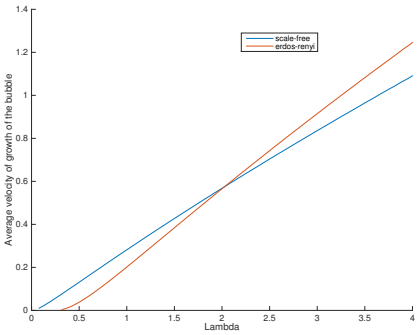


Figure 4.6: Average velocity of growth as a function of $\lambda$, with $\delta = 1$, $k = 0.4$.

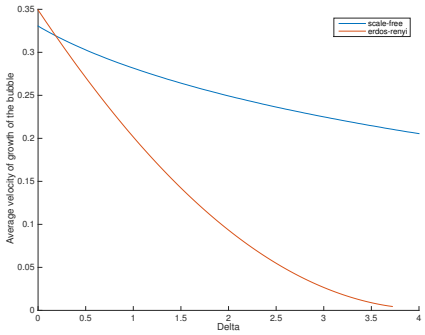


Figure 4.7: Average velocity of growth as a function of $\delta$, with $\lambda = 1$, $k = 0.4$.

sooner. On the other hand, as one can see from Figure 4.10, the velocity of the growth of the bubble is not monotone: for small $k$, the maximum is reached very late in time, when the increase of the bubble is already decelerating because the drift of $X$ approaches zero, so that the mean velocity decreases. There is one value of $k$ for which the average velocity is maximum, before it starts to decrease for large values of $k$ because of the decelerating effect of the mean reverting term.

In our analysis up to this point, we have taken the process $\theta = (\theta_t)_{t \geq \tau}$ to be constant. We now show the influence of the process $\theta$ on the dynamics of the bubble assuming that it satisfy the SDE

$$d\theta_t = \sigma^\theta \theta_t dB_t^3, \quad \tau \leq t < T,$$

where $\sigma^\theta = 0.4$, taking $\delta = 0.2$, $\lambda = 0.4$, $\Lambda = 0.5$, $k = 1$, $\bar{\sigma}^j = 0.1$ for all $0 \leq j \leq N$, $\tau = 0$, $T = 7$, $M = 1$, $\theta_0 = 3$. See for example Figure 4.12 and Figure 4.11 for the case of a scale-free network with mean degree $z = 3.2$.
The influence of the process $\theta$ on the bubble is apparent. If $\theta$ has an increase from its initial value, the bubble bursts relatively late, see the yellow dynamics: in this sense, the growth of $\theta$ can postpone the burst of the bubble. The other trajectories evolve similarly to each other up to the point where the corresponding processes $\theta$ differ. In the blue case, $\theta$ decreases and the bubble bursts soon. For the red dynamics, $\theta$ increases, making the bubble growing more.
We conclude the section illustrating the impact of the structure of the network by showing three trajectories of the bubble in Figure 4.13 for the the scale-free case and in Figure 4.14 for the Erdős-Rényi one. We choose $\delta = 0.2$, $\lambda = 0.3$, $\Lambda = 0.5$, $k = 1$, $\bar{\sigma}^j = 0.2$ for all $0 \leq j \leq N$, $\tau = 0$, $T = 3$, $M = 1$, $\theta_0 = 3$ and $\sigma^\theta = 0.2$. We can see that the bubble builds up faster in the scale-free network, but at the same time the trajectories have a steeper decrease, and therefore the effect of the burst of the bubble is more dramatic. On the other hand, Figure 4.12 shows that a quick decrease of $\theta$ can also lead to a quick burst, and then to an hard landing.

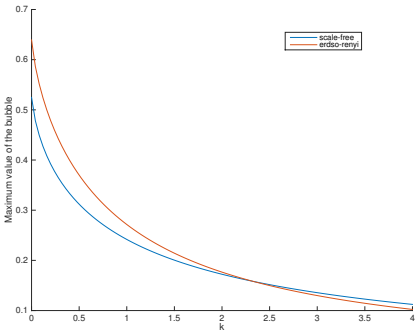


Figure 4.8: Maximum value of the bubble, $\lambda = 1$, $\delta = 1$, $\theta = 1$.

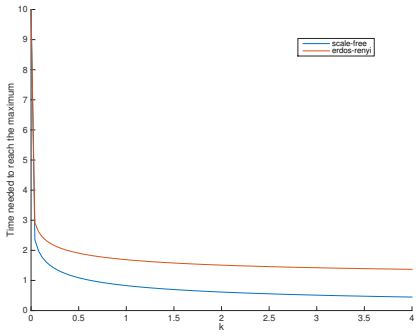


Figure 4.9: Time to the maximum, $\lambda = 1$, $\delta = 1$, $\theta = 1$

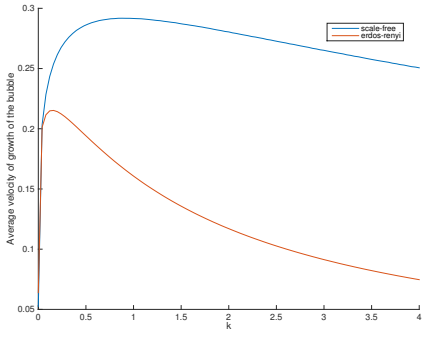


Figure 4.10: Average velocity of growth of the bubble, $\delta = 1$, $\lambda = 1$, $\theta = 1$.

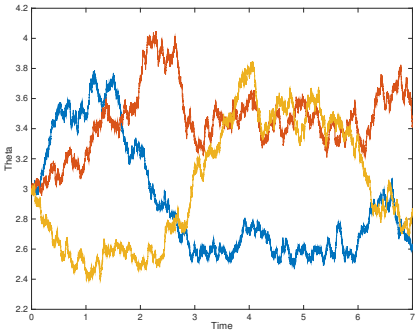


Figure 4.11: Example of trajectories of $\theta$ with $\mu^\theta = 0$, $\sigma^\theta = 0.2$.

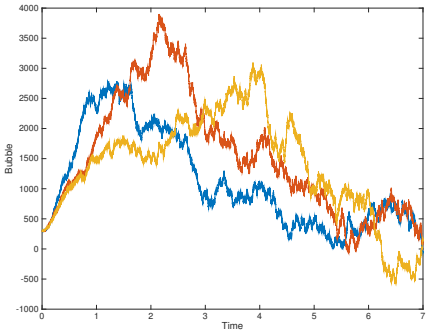


Figure 4.12: Corresponding trajectories of the bubble in a scale-free network with mean degree $z = 3.2$.

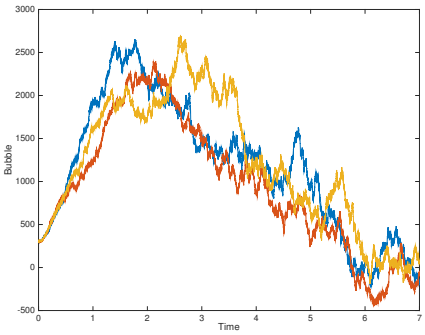


Figure 4.13: Example of trajectories for a bubble in a scale-free network.

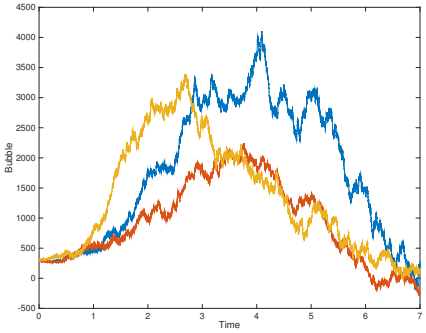


Figure 4.14: Example of trajectories for a bubble in an Erdős-Rényi network.

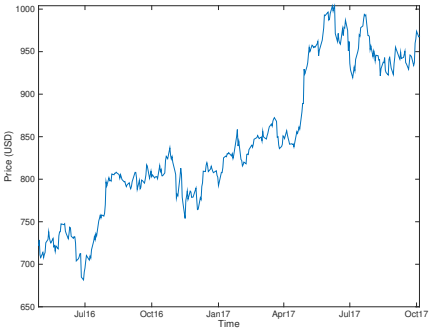


Figure 4.15: Price in USD of Alphabet Inc, June 2016 - October 2017.

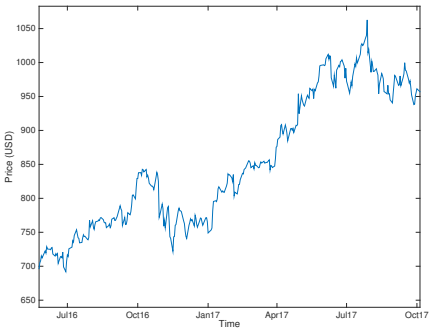


Figure 4.16: Price in USD of Amazon.com Inc, July 2016 - October 2017.

### 4.3.1 Model testing on real data

In this subsection we test some features of our model on real data. Since we were not able to find tick by tick data for the signed volume of market orders of well-known bubbles of the past such as for example for the *dot com bubble*, we consider the asset prices Alphabet Inc (NASDAQ:GOOG) and Amazon.com Inc (NASDAQ:AMZN). For these stocks we could obtain tick by tick data for the signed volume of their market orders starting from the first months of 2016. These companies, as it can be seen also by the prices reported in Figure 4.15 and in Figure 4.16, have experienced in the last years a boom, which has brought many economists to theorize the presence of a new tech bubble, after the *dot com mania* of the late 1990s (see for example Bercovici [2017], Ozimek [2017], Seria [2017], Sharma [2017]). Even if prices today are not as widely overvalued as in 1999, there are some evidences of a resurgent tech mania among investors.

We consider the realized signed volume of market orders since 2016. As shown in Figures 4.17 and 4.18, the signed volume tends to increase over time, for both Alphabet and Amazon. This behavior indicates the tendency of traders to invest on these companies, contributing to the increase of the price in line with our model.

Our aim is to investigate whether typical trading behaviour in a bubble environment is captured in our model. In particular, since we deal with a relatively small time window of a potential bubble, we calibrate the coefficients of the deterministic component for $X$ in (4.11), underlying the signed volume of market orders on the observed data for Amazon and Google by employing a quadratic regression. In doing so, we further assume $\bar{\sigma}^j = 0$ for $j \in 1, \cdots, N$, the process $(\theta_t)_{t \geq \tau}$ to be constant and that all the nodes of the network have the same degree $d = 3$, i.e. that the degree distribution of our network is a Dirac delta centered in $d = 3$.
In Figure 4.17 and Figure 4.18 we can observe the "S" behavior discussed in Section 4.3. We remark that since we perform a local analysis by considering a specific short time window with constant $\theta$, this behaviour cannot be directly interpreted as indication for a decreasing phase of the bubble. In the next time window the signed volume may start to grow steeply again, due to the impact of a stochastic $\theta$. In this case the curve describing the evolution of the signed volume would also grow for a longer time, distorting the "S" shape as illustrated in Figure 4.11 and in Figure 4.12.
We can conclude that the analysis shows the flexibility of our model and its capacity of

1. describing both the increasing and the descending phase of the bubble;
2. capturing the impact of signed volume market orders on bubbles' formation and burst;
3. taking into account the underlying network structure in the contagion process of a bubble's evolution;
4. describing typical features of a bubble's behavior like steep increase and hard landing.

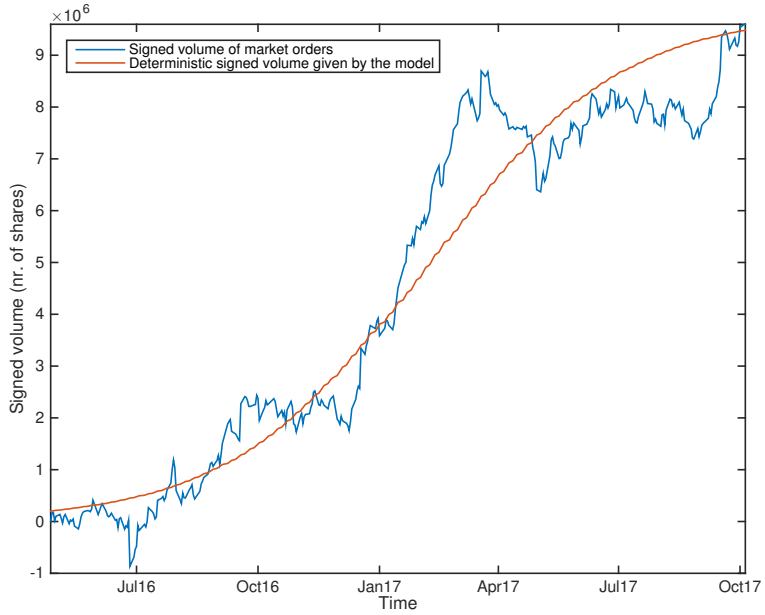


Figure 4.17: Realized signed volume of market orders and deterministic signed volume given by equation (4.11), Alphabet Inc, June 2016 - October 2017.

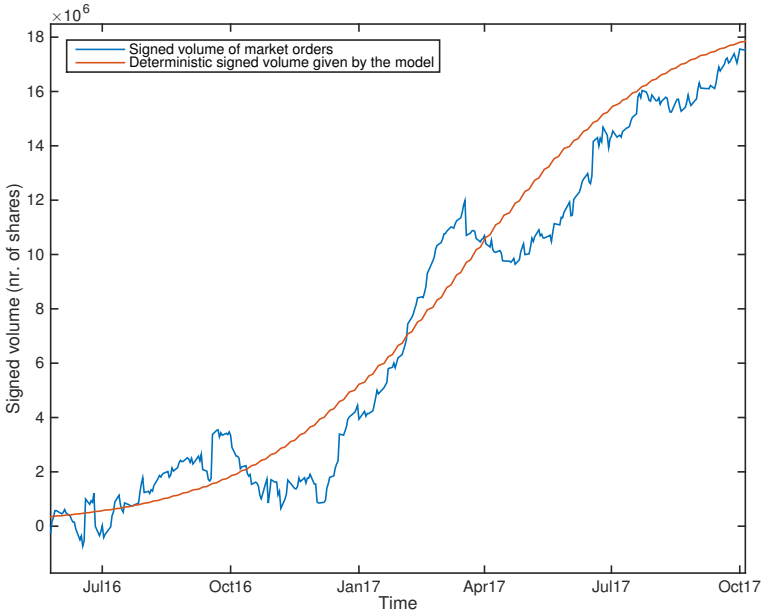


Figure 4.18: Realized signed volume of market orders and deterministic signed volume given by equation (4.11), Amazon.com Inc, June 2016 - October 2017.

# Chapter 5

# Financial asset bubbles in banking networks

## 5.1 Introduction

Contagion within a banking system and possible default propagation have become central topics in the last decades due to a number of financial crisis. It is then of fundamental importance to investigate how financial distress can propagate in a network of financial institutions, and develope new quantitative methods to deal with these topics.
One stream of research aims at extending the traditional regulatory framework of monetary risk measures, that quantify the risk of financial institutions based on a stand alone basis, to multivariate systemic risk measures that take as a primitive the whole financial system. For an overview about this topic, see Biagini et al. [2017, 2018], Bisias et al. [2012], Chen et al. [2013], Drapeau et al. [2016], Feinstein et al. [2017], Hoffmann et al. [2016a,b], Kromer et al. [2016] and references therein.
Other studies investigate potential default cascades due to various contagion effects in the setting of explicit network models for the financial system. In the seminal work of Eisenberg and Noe [2001] and its many extensions (see e.g. Hurd [2016] and references therein) cascade processes in static, deterministic network models are analized by computing endogenously determined clearing/equilibrium payment vectors. Within the framework of random graph theory, cascade processes are studied in large financial random networks by means of law-of-large number effects in Amini and Minca [2016], Amini et al. [2012, 2016], Detering et al. [2015, 2017, 2018] and Hurd [2016], and in finite random networks by Elliott et al. [2014], Gai and Kapadia [2010].
Our approach follows the stream of mean field models of interacting systems of diffusions, first proposed by the influential papers of McKean [1966a,b]. In recent years, this framework has been applied to the study of systemic risk in large financial networks, whose dynamic evolution is studied by means of a system of interacting diffusions. The SDEs stand e.g. for the wealth, monetary reserves, or other more general indicators for the health of financial institutions. The links between the nodes in the network represent investments or loans, so that the SDEs are tied together through a term in the drift that implies the structure of the network. A first simple model in this direction is given in Fouque and Sun [2013], where a system of SDEs is proposed with

dynamics

$$dX_t^i = \frac{\lambda}{n}\sum_{j=1}^{n}(X_t^j - X_t^i)dt + \sigma dW_t^i, \quad 0 \le t < \infty, \tag{5.1}$$

where $W = (W_t^1, \dots, W_t^n)_{t\ge 0}$ is a standard $n$-dimensional Brownian motion and $\lambda, \sigma > 0$. Here, the $X^i$ stand for the log-monetary reserves of banks, and the drift terms $\lambda(X_t^j - X_t^i)$ represent the connections between banks in the network. In this case, the borrowing and lending rate $\lambda$ is supposed to be the same for every couple of banks. When the network size $n$ grows towards infinity, it is a well-know result (see Sznitman [1991]) that due to law-of-large-number effects the diffusions in (5.1) converge towards their mean-field limit

$$d\bar{Y}_t^i = \lambda\left(\mathbb{E}[\bar{Y}_t] - \bar{Y}_t^i\right)dt + \sigma dW_t^i, \quad 0 \le t < \infty.$$

Thus, for large networks propagation of chaos applies and the evolution of the $X^i$ asymptotically de-couples due to averaging effects, which allows to asymptotically describe the complex system by a representative particle evolution. The simple model in (5.1) to study systemic risk has been generalized in various ways in a number of articles, see e.g. Carmona et al. [2015, 2016] where mean-field games are considered, Fouque and Ichiba [2013] where the probability distributions of multiple default times is approximated, Garnier et al. [2013a,b] and Battiston et al. [2012] where a tradeoff between individual and systemic risk in a banking network is described, and Chong and Klüppelberg [2015], Kley et al. [2015] where partial mean-field limits are studied.
The main objective of our contribution is to extend the model in (5.1) so that the effect of a financial speculation bubble on the structure of the network and the evolving systemic risk can be studied. It is a common understanding that bubbles are intimately connected with financial crises, and many historical crises indeed originated after the burst of a bubble (e.g. the Great Depression of the 1930s and the financial crisis of 2007-2008). This causality is investigated for example in Brunnermeier [2008] and statistically confirmed in Brunnermeier and Schnabel [2015]. However, it seems that literature on mathematical models that deal with this question is very scarce.
We take a first step towards filling this gap and model by a system of coupled stochastic differential equations the so called *financial robustness* of banks, defined here, as done by Battiston et al. [2012] and Hull and White [2001], as an indicator of agent's creditworthiness or distance to default. We suppose that a group of banks, constituting the core of the network, have access to assets affected by a bubble, and hold them. Other banks in the periphery have not direct access to those assets, and can make profits from the increase of their value only investing money on the banks of the core, as in Battiston [2015]. In particular, we assume that every bank invests money on other institutions at time $t$ depending on their robustness at time $t - \delta$, where $\delta > 0$ is a delay meaning that the banks do not immediately react to the changes in the system. As a result of our assumption, a preferential attachment mechanism takes place where the weights of links towards a node does not depend on its degree but on its "fitness". Due to this behavior, the bubble breaks the homogeneity of the network: the banks holding the bubble attract more investment and become more systemic.
We then study the behavior of the system when the number of nodes gets large.

More precisely, we let the number of periphery banks go towards infinity, but keep the number of core banks holding the bubble constant. We assume moreover that the impact of the banks of the core on the system does not vanish when the total number of banks goes to infinity. The main implication of this assumption is that the banks in the periphery are no more independent, since are all impacted by a stochastic term, so that the classic law of large numbers does not apply anymore. Our main result then determines a partial mean-field limit for the system where the drift of the diffusions contains an additional term, that represents the impact of the bubble in the network: by this term, all the banks are directly or indirectly affected when the bubble bursts. In other words, at the moment of the burst the most systemic banks are also the most prone ones to be hit by the shock. This effect is amplified by the impossibility to immediately disinvest when the robustness of some banks decreases due to the delay $\delta$.
One important feature of our model is the presence of a number of banks, sometimes called *large players*, which are significantly more central and important than the others in the network. This core-periphery structure of our banking system is confirmed by empirical studies about networks of financial institutions in Austria, see Boss et al. [2004], in Brazil (Cont and Moussa [2010-03]), in Germany (Craig and von Peter [2014]), and for the interbank payment flows within the US Fedwire service (May et al. [2008], Soramaki et al. [2007]), where large banks are disproportionately connected to small banks. Core-periphery structured networks are also studied by Chong and Klüppelberg [2015] and Kley et al. [2015]. In particular, our setting is similar to the one of Chong and Klüppelberg [2015], where heterogeneous systems of diffusions governed by Lévy processes are introduced within a core-periphery structured network, where some particles have a non-vanishing influence when the system becomes large. However, the scopes of the two approaches are still quite different. The motivation of our work is to study the impact of a bubble on a financial network, and for this reason we consider coefficients which are possibly nonlinear functions of the value of the diffusions. In Chong and Klüppelberg [2015], where no bubble is present, the setting is more general, since no particular focus is put on preferential attachment mechanisms, although coefficients are supposed to be linear functionals (an extension to nonlinear Lipschitz coefficients is possible taking more restrictive hypothesis on the structure of the network) and don't take into account the delay. On the other hand, in Chong and Klüppelberg [2015] the authors prove a law of large numbers type theorem with explicit bounds on the mean squared error and give a large deviation result, whereas we only prove a (partial) mean-field limit result without computing bounds on the error. Moreover, differently than in our approach, in Chong and Klüppelberg [2015] it is supposed that every particle in the system is only affected by a negligible number (w.r.t to the size of the network) of periphery particles, or alternatively a single core particle affects a negligible number of periphery particles. The content of this chapter is mostly developed in Biagini et al. [In preparation].

## 5.2 The model

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space endowed with a $(m+n+2)$-dimensional Brownian motion $\bar{W} = (W_t^1, \dots, W_t^n, W_t^{B,1}, \dots, W_t^{B,m}, B_t^1, B_t^2)_{t\geq 0}$, $m, n \in \mathbb{N}$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is the natural filtration of $\bar{W}$. We consider a network of $m+n$ banks, consisting of $m$ banks holding a bubbly asset in their portfolio (also referred to as *core*), and $n$ banks that do not directly hold the bubbly asset (also referred to as *periphery*).
By following a similar approach as in Kley et al. [2015], we model the robustness of the banks in the system. This coefficient dynamically evolves and represents a measure of how healthy a bank remains in stress situations. Let $\rho^{i,n} = (\rho_t^{i,n})_{t\geq 0}$, $i = 1, \dots n$, and $\rho^{k,B} = (\rho_t^{k,B})_{t\geq 0}$, $k = 1, \dots, m$, be the robustness of banks not holding and holding the bubble, respectively. We assume that they satisfy the following system of stochastic differential delay equations (SDDEs) for $t \geq \delta$, $\delta > 0$,

$$\begin{aligned} d\rho_t^{i,n} = &\left( \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} f^P(\rho_{t-\delta}^{j,n} - A_{t-\delta}^n)(\rho_t^{j,n} - A_t^n) + \frac{1}{m} \sum_{k=1}^{m} f^B(\rho_{t-\delta}^{k,B} - A_{t-\delta}^n)\ (\rho_t^{k,B} - A_t^n) \right) \\ &+ \lambda(A_t^n - \rho_t^{i,n})dt + \sigma_1 dW_t^i, \end{aligned} \tag{5.2}$$

$$\begin{aligned} d\rho_t^{k,B} = &\left( \frac{1}{n} \sum_{i=1}^{n} f^P(\rho_{t-\delta}^{i,n} - A_{t-\delta}^n)(\rho_t^{i,n} - A_t^n) + \frac{1}{m-1} \sum_{\ell=1, \ell\neq k}^{m} f^B(\rho_{t-\delta}^{\ell,B} - A_{t-\delta}^n)(\rho_t^{\ell,n} - A_t^n) \right) dt \\ &+ \lambda(A_t^n - \rho_t^{k,B})dt + \sigma_2 dW_t^{k,B} + d\beta_t, \end{aligned} \tag{5.3}$$

where $\lambda > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$ and

$$A_t^n = \frac{1}{m+n} \left( \sum_{r=1}^{n} \rho_t^{r,n} + \sum_{h=1}^{m} \rho_t^{h,B} \right), \quad t \geq \delta, \tag{5.4}$$

is the mean of the robustness of all the banks in the network at time $t$. For $t \in [0, \delta)$, we assume that $(\rho_s^{i,n})_{s\in[0,\delta)}$, $(\rho_s^{k,B})_{s\in[0,\delta)}$, $i = 1, \dots, n$, $k = 1, \dots, m$, satisfy (5.2)-(5.3) with $\delta = 0$, by following the approach of Mao [2007]. We also suppose that $\rho_0^{i,n} = \rho_0 > 0$ for all $i = 1, \dots, n$.
The process $\beta = (\beta_t)_{t\geq 0}$ in (5.3) represents the influence of the asset price bubble on the robustness of core banks and has dynamics

$$d\beta_t = \mu_t dt + \sigma_B dB_t^1, \quad t \geq 0, \tag{5.5}$$

where $\sigma_B > 0$ and $\mu$ is an adapted process satisfying

$$d\mu_t = \tilde{b}(\mu_t)dt + \tilde{\sigma}(\mu_t)dB_t^2, \quad t \geq 0, \tag{5.6}$$

where $\tilde{b}$, $\tilde{\sigma}$ fulfill the usual Lipschitz and sublinear growth conditions such that there exists a unique solution of (5.6) , satisfying

$$\int_0^t \mathbb{E}[|\mu_s|^2]ds < \infty, \qquad 0 \leq t < \infty. \tag{5.7}$$

Later on in Section 5.4 we will specify a concrete model for the bubbly evolution in (5.5).

The interdependencies of the banks' robustness and corresponding contagion effects are specified through the drifts in (5.2) and (5.3). The term $\lambda(A_t^n - \rho_t^{i,n})$ represents an attraction of the individual robustness towards the average robustness of the system with rate $\lambda$ as in the classical mean-field model (5.1). In addition to the homogeneous average term, we introduce the terms of type $f^P(\rho_{t-\delta}^{j,n} - A_{t-\delta}^n)(\rho_t^{j,n} - A_t^n)$ and $f^B(\rho_{t-\delta}^{k,B} - A_{t-\delta}^n)$ $(\rho_t^{k,B} - A_t^n)$ that represent a robustness-dependent evolution of the network connectivity: for typically positive and increasing $f^B$ and $f^P$, bank $i$ is the more connected to bank $j$ the higher bank $j$'s robustness is above the average. In this way, the evolution of the bubble alters the connectivity structure of the network according to a model of preferential attachment. Moreover, the propensity of a node $i$ to attract future links not only depends on the current level of robustness of $i$, but also on the robustness of the banks already connected to $i$. This induces a form of *preferential preferential attachment*, which creates a strong clustering effect. This change in network structure then comes along with an increasing systemic risk and instability in case the bubble burst, as noted by Battiston [2015]. Further we introduce the delay $\delta > 0$ to reflect the fact that the bank i's investment decisions does not immediately react to changes in bank j's robustness. Note that when there are no bubble banks and $f^P = \lambda$, the system (5.2)-(5.3) collapses to the basis mean-field model in (5.1).
We assume the following hypothesis on $f^B$ and $f^P$.

**Assumption 5.2.1.** *The functions $f^B, f^P : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ are measurable, Lipschitz continuous and such that also the functions $F^B(x) := xf^B(x)$, $F^P(x) := xf^P(x)$, $x \in \mathbb{R}$, are Lipschitz continuous, i.e.*

$$|f^\ell(x) - f^\ell(y)| \leq K_1|x - y|, \quad x, y \in \mathbb{R}, \quad \ell = B, P, \tag{5.8}$$

*and*

$$|xf^\ell(x) - yf^\ell(y)| \leq K_2|x - y|, \quad x, y \in \mathbb{R}, \quad \ell = B, P, \tag{5.9}$$

*with $0 < K_1, K_2 < \infty$.*

Note that (5.9) implies that $f^B$ and $f^P$ are bounded, since if $f(x)x$ is Lipschitz then

$$|f(x)x| = |f(x)x - f(0) \cdot 0| \leq K_2|x|. \tag{5.10}$$

**Example 5.2.2.** *We have that $f(x) = 1 + 2\arctan(x)/\pi$ satisfies Assumption 5.2.1: $f$ takes values in $[0, 2]$, and both $f$ and $F(x) = xf(x)$ are Lipschitz, because they have bounded derivative.*
*In particular, $f$ is increasing, so that if $\rho_t^j > \rho_t^i$ then the link towards $j$ is bigger then the link towards $i$. If the robustness $\rho_t^j$ of bank $j$ is equal to the average $A_t^n$ in (5.4), then the link towards bank $j$ has weight $f(0) = 1$, if $\rho_t^j > A_t^n$ the link has weight bigger than 1 and if $\rho_t^j < A_t^n$ the link has weight less than 1. If all the banks have the same robustness, we have an homogenous network, where all the links have weight equal to 1.*
*Furthermore, any constant function clearly satisfies Assumption 5.2.1. For such a choice, we have a static and homogenous network.*

**Proposition 5.2.3.** *Under Assumption 5.2.1, for every $\delta \geq 0$ there exists a unique strong solution for the system of SDEs (5.2)-(5.3). Moreover, it holds*

$$\sup_{0 \leq s \leq t} \mathbb{E}[|\rho_s^i|^2] < \infty, \quad 0 < t < \infty, \quad i = 1, \ldots, n, \tag{5.11}$$

$$\sup_{0\leq s\leq t}\mathbb{E}[|\rho_s^{k,B}|^2]<\infty,\quad 0<t<\infty,\quad k=1,\dots,m. \tag{5.12}$$

*Proof.* Suppose by simplicity $\lambda=1$. We start by proving existence and uniqueness of the strong solution of (5.2)-(5.3) when $\delta=0$. In this case we can write the system of SDEs given by (5.2),(5.3) and (5.6) as an $(m+n+1)$-dimensional SDE

$$dX_t=b(X_t)dt+\sigma(X_t)d\bar{W}_t,\quad t\geq 0, \tag{5.13}$$

where

$$b(x)=\begin{pmatrix} \frac{1}{n-1}\sum_{j=2}^{n}f^P(x_j-\bar{x})(x_j-\bar{x})+\frac{1}{m}\sum_{k=n+1}^{m+n}f^B(x_k-\bar{x})(x_k-\bar{x})+\bar{x}-x_1,\\ \vdots\\ \frac{1}{n-1}\sum_{j=1}^{n-1}f^P(x_j-\bar{x})(x_j-\bar{x})+\frac{1}{m}\sum_{k=n+1}^{m+n}f^B(x_k-\bar{x})(x_k-\bar{x})+\bar{x}-x_n\\ \frac{1}{n}\sum_{j=1}^{n}f^P(x_j-\bar{x})(x_j-\bar{x})+\frac{1}{m-1}\sum_{k=n+2}^{m+n}f^B(x_k-\bar{x})(x_k-\bar{x})+\bar{x}-x_{n+1}\\ \vdots\\ \frac{1}{n}\sum_{j=1}^{n}f^P(x_j-\bar{x})(x_j-\bar{x})+\frac{1}{m-1}\sum_{k=n+1}^{m+n-1}f^B(x_k-\bar{x})(x_k-\bar{x})+\bar{x}-x_{m+n}\\ \tilde{b}(x_{m+2}) \end{pmatrix}, \tag{5.14}$$

with $\bar{x}=\frac{1}{m+n}\sum_{i=1}^{m+n}x_i$. Here $\sigma(x)$ is a $(n+m+1)\times(n+m+1)$ block matrix of the form

$$\sigma(x)=\begin{pmatrix}\Sigma_1(x) & 0 & 0\\ 0 & \Sigma_2(x) & 0\\ 0 & 0 & \tilde{\sigma}(x_{m+2})\end{pmatrix}, \tag{5.15}$$

where $\Sigma_1(x)$ is a $n\times n$ diagonal matrix with diagonal $(\sigma_1,\dots,\sigma_1)$ and $\Sigma_2(x)$ is the $m\times(m+1)$ matrix

$$\Sigma_2(x)=\begin{pmatrix}\sigma_2 & 0 & \dots & 0 & \sigma_B\\ 0 & \sigma_2 & \dots & 0 & \sigma_B\\ \vdots & \vdots & \ddots & 0 & \sigma_B\\ 0 & 0 & \dots & \sigma_2 & \sigma_B\end{pmatrix}.$$

We use Theorem 9.11 in Pascucci [2011] to prove existence and uniqueness of the strong solution of (5.13), and that the second moments of the solution are finite. To this purpose, we show that $b(\cdot)$ and $\sigma(\cdot)$ defined in (5.14) and (5.15), respectively, are Lipschitz continuous in $x$ and that there exists some $C$ such that

$$\|\sigma(x)\|^2+\|b(x)\|^2\leq C(1+\|x\|^2).$$

We begin by proving the first condition. The Lipschitz property clearly holds for $\sigma(\cdot)$, since $\tilde{\sigma}(\cdot)$ is Lipschitz by hypothesis. Given $x=(x_1,\dots,x_{m+n}), x'=(x'_1,\dots,x'_{m+n})\in\mathbb{R}^{m+n}$, we show that there exists a constant $\bar{K}\in(0,\infty)$ such that

$$\|b(x)-b(x')\|\leq\bar{K}\|x-x'\|.$$

For the first entry of (5.14) we have

$$|b_1(x) - b_1(x')| \leq \frac{1}{n-1}\sum_{j=2}^{n} |f^P(x_j - \bar{x})(x_j - \bar{x}) - f^P(x'_j - \bar{x}')(x'_j - \bar{x}')|$$
$$+\frac{1}{m}\sum_{k=n+1}^{m+n} |f^B(x_k - \bar{x})(x_k - \bar{x}) - f^B(x'_k - \bar{x}')(x'_k - \bar{x}')| + |\bar{x} - \bar{x}'| + |x_1 - x'_1|,$$

and by Assumption 5.2.1 we have

$$\begin{aligned}
|b_1(x) - b_1(x')| &\leq K_2 \frac{1}{n-1}\sum_{j=2}^{n} |(x_j - \bar{x}) - (x'_j - \bar{x}')| + K_2 \frac{1}{m}\sum_{k=n+1}^{m+n} |(x_k - \bar{x}) - (x'_k - \bar{x}')| \\
&\quad + |\bar{x} - \bar{x}'| + |x_1 - x'_1| \\
&\leq K_2 \left( \frac{1}{n-1}\sum_{j=2}^{n} |x_j - x'_j| + \frac{1}{m}\sum_{k=n+1}^{m+n} |x_k - x'_k| \right) + (2K_2 + 1)|\bar{x} - \bar{x}'| + |x_1 - x'_1| \\
&\leq K_2 \left( \frac{1}{n-1}\sum_{j=2}^{n} |x_j - x'_j| + \frac{1}{m}\sum_{k=n+1}^{m+n} |x_k - x'_k| \right) + \frac{2K_2 + 1}{m+n}\sum_{i=2}^{m+n} |x_i - x'_i| \\
&\quad + |x_1 - x'_1|.
\end{aligned}$$

Then, since for $z_1, \ldots, z_N \in \mathbb{R}$ it holds $\left(\sum_{i=1}^{N} |z_i|\right)^2 \leq N \sum_{i=1}^{N} |z_i|^2$, we have

$$\begin{aligned}
|b_1(x) - b_1(x')|^2 &\leq 6(m+n)\left( (K_2)^2 \left( \frac{1}{(n-1)^2}\sum_{j=2}^{n} |x_j - x'_j|^2 + \frac{1}{m^2}\sum_{k=n+1}^{m+n} |x_k - x'_k|^2 \right) \right) \\
&\quad + 6(m+n)\left( (2K_2 + 1)^2 \frac{1}{(m+n)^2}\sum_{i=1}^{m+n} |x_i - x'_i|^2 + |x_1 - x'_1|^2 \right) \\
&\leq C_1 \|x - x'\|^2,
\end{aligned}$$

for a suitable constant $C_1 > 0$. Similarly,

$$|b_i(x) - b_i(x')| \leq C_i \|x - x'\|^2, \quad 2 \leq i \leq m+n,$$

for a suitable constant $C_i > 0$, whereas

$$|b_{m+2}(x) - b_{m+2}(x')| = |\tilde{b}(x_{m+2}) - \tilde{b}(x'_{m+2})| \leq K_\mu |x_{m+2} - x'_{m+2}|,$$

where $K_\mu$ is the Lipschitz constant for the function $\tilde{b}(\cdot)$ in (5.6). Then we obtain

$$\|b(x) - b(x')\|^2 = \sum_{i=1}^{m+n+1} |b_i(x) - b_i(x')|^2 \leq \left( \sum_{i=1}^{m+n+1} C_i + K_\mu^2 \right) \|x - x'\|^2. \tag{5.16}$$

The second condition, i.e.

$$\|\sigma(x)\|^2 + \|b(x)\|^2 \leq C(1 + \|x\|^2), \tag{5.17}$$

for some $C > 0$, holds because of Assumption 5.2.1 and the hypothesis on $\tilde{\sigma}(\cdot)$. Inequalities (5.11) and (5.12) then follow by Theorem 9.11 in Pascucci [2011],

and in particular by estimation (A.2) in the Appendix.
When $\delta > 0$, equation (5.13) becomes

$$dX_t = \bar{b}(X_t, X_{t-\delta})dt + \bar{\sigma}(X_t, X_{t-\delta})dW_t, \quad t \geq \delta, \tag{5.18}$$

where $\bar{\sigma}(x, y) = \sigma(x)$ as in (5.15) and

$$b(x,y) = \begin{pmatrix} \frac{1}{n-1}\sum_{j=2}^{n} f^P(y_j - \bar{y})(x_j - \bar{x}) + \frac{1}{m}\sum_{k=n+1}^{m+n} f^B(y_k - \bar{y})(x_k - \bar{x}) + \bar{x} - x_1, \\ \vdots \\ \frac{1}{n-1}\sum_{j=1}^{n-1} f^P(y_j - \bar{y})(x_j - \bar{x}) + \frac{1}{m}\sum_{k=n+1}^{m+n} f^B(y_k - \bar{y})(x_k - \bar{x}) + \bar{x} - x_n \\ \frac{1}{n}\sum_{j=1}^{n} f^P(y_j - \bar{y})(x_j - \bar{x}) + \frac{1}{m-1}\sum_{k=n+2}^{m+n} f^B(y_k - \bar{y})(x_k - \bar{x}) + \bar{x} - x_{n+1} \\ \vdots \\ y\frac{1}{n}\sum_{j=1}^{n} f^P(y_j - \bar{y})(x_j - \bar{x}) + \frac{1}{m-1}\sum_{k=n+1}^{m+n-1} f^B(y_k - \bar{y})(x_k - \bar{x}) + \bar{x} - x_{m+n} \\ \tilde{b}(x_{m+2}) \end{pmatrix}.$$

By Theorem 3.1 in Mao [2007, chapter 5], to prove existence and uniqueness of the solution it suffices to show that the linear growth condition

$$\|\bar{b}(x,y)\|^2 \leq C(1 + \|x\|^2 + \|y\|^2) \tag{5.19}$$

holds and that $\bar{b}$ is Lipschitz in the variable $x$ uniformly in $y$, i.e. that there exists a constant $\tilde{K} \in (0, \infty)$ such that

$$\|\bar{b}(x,y) - \bar{b}(x',y)\|^2 \leq \tilde{K}\|x - x'\|^2 \tag{5.20}$$

for all $y \in \mathbb{R}$, $x, x' \in \mathbb{R}^{m+n}$. Property (5.19) can be proven by computations similar to the ones used for showing (5.37). For the Lipschitz condition we have

$$\begin{aligned} |\bar{b}_1(x,y) - \bar{b}_1(x',y)| \leq & \frac{1}{n-1}\sum_{j=2}^{n} |f^P(y_j - \bar{y})||(x_j - \bar{x}) - (x'_j - \bar{x}')| \\ & + \frac{1}{m}\sum_{k=n+1}^{m+n} |f^B(y_k - \bar{y})||(x_k - \bar{x}) - (x'_k - \bar{x}')| + |\bar{x} - \bar{x}'| + |x_1 - x'_1|. \end{aligned}$$

Hence, as $f^B$ and $f^P$ are bounded by $K_2$, the computations to show (5.20) are identical to the ones for (5.16).
In order to prove (5.11) and (5.12), we apply the same argument used in the proof of Theorem 3.1 in Mao [2007, chapter 5]: on $[0, \delta]$ we have by hypothesis a classic stochastic differential equation, and by inequality (9.15) in Theorem 9.11 in Pascucci [2011] it holds

$$\mathbb{E}[\sup_{0 \leq s \leq \delta} \|X_s\|^2] < \infty. \tag{5.21}$$

On the interval $[\delta, 2\delta]$, we can write equation (5.18) as

$$dX_t = \bar{b}(X_t, \xi_t)dt + \bar{\sigma}(X_t, \xi_t)dW_t, \quad \delta \leq t \leq 2\delta,$$

where $\xi_t = X_{t-\delta}$. Once the solution on $[0, \delta]$ is known, this is again a classic SDE (without delay) with initial value $X_\delta = \xi_0$, so that by Theorem 9.11 in Pascucci [2011], there exists a constant $C_{2\delta} > 0$ such that

$$\mathbb{E}[\sup_{\delta \leq s \leq 2\delta} \|X_s\|^2] \leq C_{2\delta}\left(1 + \mathbb{E}[\|X_\delta\|^2]\right) e^{2\delta C_{2\delta}}, \tag{5.22}$$

which is finite by (5.21). Repeating this argument on the interval $[2\delta, 3\delta]$, we obtain

$$\mathbb{E}[\sup_{2\delta\le s\le 3\delta}\|X_s\|^2] \le C_{3\delta}\left(1+\mathbb{E}[\|X_{2\delta}\|^2]\right)e^{3\delta C_{3\delta}} \le C_{3\delta}\left(1+\mathbb{E}[\sup_{\delta\le s\le 2\delta}\|X_s\|^2]\right)e^{3\delta C_{3\delta}} < \infty$$

by (5.22). Recursively we have

$$\mathbb{E}[\sup_{(k-1)\delta\le s\le k\delta}\|X_s\|^2] < \infty.$$

Then,

$$\sup_{0\le s\le t}\mathbb{E}[\|X_s\|^2] = \sup_{s\in[\bar{k}\delta,(\bar{k}+1)\delta]}\mathbb{E}[\|X_s\|^2] < \infty, \tag{5.23}$$

for some $\bar{k}$ with $[\bar{k}\delta,(\bar{k}+1)\delta] \subseteq [0,t]$. □

## 5.3 Mean field limit

We now study a mean field limit for the system of banks (5.2)-(5.3) for large $n$.
Define the processes $\tilde{\rho}^i = (\tilde{\rho}^i_t)_{t\ge 0}$, $i = 1,\dots,n$, $\bar{\rho}^{k,B} = (\bar{\rho}^{k,B}_t)_{t\ge 0}$, $k = 1,\dots,m$, and $\nu = (\nu_t)_{t\ge 0}$ as the solutions of the following system of SDEs for $t \ge \delta$:

$$d\tilde{\rho}^i_t = -\lambda\tilde{\rho}^i_t dt + \sigma_1 dW^i_t, \tag{5.24}$$

$$d\nu_t = \left(\varphi(t,t-\delta) + \frac{1}{m}\sum_{k=1}^m f^B\left(\bar{\rho}^{k,B}_{t-\delta} - \nu_{t-\delta} - \mathbb{E}[\tilde{\rho}^i_{t-\delta}]\right)\left(\bar{\rho}^{k,B}_t - \nu_t - \mathbb{E}[\tilde{\rho}^i_t]\right) + \lambda\mathbb{E}[\tilde{\rho}^i_t]\right)dt, \tag{5.25}$$

$$\begin{aligned} d\bar{\rho}^{k,B}_t = &\left(\varphi(t,t-\delta) + \frac{1}{m-1}\sum_{\ell=1,\ell\ne k}^m f^B\left(\bar{\rho}^{\ell,B}_{t-\delta} - \nu_{t-\delta} - \mathbb{E}[\tilde{\rho}^i_{t-\delta}]\right)\left(\bar{\rho}^{\ell,B}_t - \nu_t - \mathbb{E}[\tilde{\rho}^i_t]\right)\right)dt \\ &+ \left(\mu_t + \lambda(\mathbb{E}[\tilde{\rho}^i_t] + \nu_t - \bar{\rho}^{k,B}_t)\right)dt + \sigma_2 dW^{k,B}_t + \sigma_B dB^1_t, \end{aligned} \tag{5.26}$$

with

$$\varphi(t,t-\delta) := \mathbb{E}\left[f^P\left(\tilde{\rho}^i_{t-\delta} - \mathbb{E}[\tilde{\rho}^i_{t-\delta}]\right)\left(\tilde{\rho}^i_t - \mathbb{E}[\tilde{\rho}^i_t]\right)\right], \quad t \ge \delta. \tag{5.27}$$

For $t \in [0,\delta]$ we assume that $(\tilde{\rho}_t)_{0\le t\le\delta}$, $(\nu_t)_{0\le t\le\delta}$ and $(\bar{\rho}^{k,B}_t)_{0\le t\le\delta}$ satisfy (5.24)-(5.26) for $\delta = 0$, with initial conditions $\tilde{\rho}^i_0 = \rho_0 \in \mathbb{R}$, $\nu_0 = 0$, $\bar{\rho}^{k,B}_0 = \rho^{k,B}_0 \in \mathbb{R}$.
Note that in equation (5.25) the expression of $\varphi$ is independent of the choice of $\tilde{\rho}^i$ since $\tilde{\rho}^i$, $i = 1,\dots,n$, are identically distributed. For the same reason, the process $\nu$ in (5.25) does not depend on $\tilde{\rho}^i$.
Set

$$\bar{\rho}^i := \tilde{\rho}^i + \nu, \quad i = 1,\dots,n. \tag{5.28}$$

In particular,

$$\begin{aligned} \bar{\rho}^i_t = &\bar{\rho}^i_\delta + \int_\delta^t \left(\varphi(s,s-\delta) + \frac{1}{m}\sum_{k=1}^m f^B(\bar{\rho}^{k,B}_{s-\delta} - \nu_{s-\delta} - \mathbb{E}[\tilde{\rho}^i_{s-\delta}])\left(\bar{\rho}^{k,B}_s - \nu_s - \mathbb{E}[\tilde{\rho}^i_s]\right) + \lambda(\mathbb{E}[\tilde{\rho}^i_s] - \tilde{\rho}^i_s)\right)ds \\ &+ \sigma_1 W^i_s, \quad t \ge \delta. \end{aligned} \tag{5.29}$$

**Proposition 5.3.1.** *Under Assumption 5.2.1, for every $\delta \geq 0$ there exists a unique strong solution of the system of SDEs (5.24)-(5.26). In particular, it holds*

$$\sup_{0\leq s\leq t} \mathbb{E}[|\nu_s|^2] < \infty, \quad 0 < t < \infty, \tag{5.30}$$

$$\sup_{0\leq s\leq t} \mathbb{E}[|\rho_s^{k,B}|^2] < \infty, \quad 0 < t < \infty, \quad k = 1, \ldots, m. \tag{5.31}$$

*Proof.* For the sake of simplicity we take $\lambda = 1$. It is well known that (5.24) admits a unique strong solution. As before, we start by proving existence and uniqueness of the strong solution of (5.25)-(5.26) when $\delta = 0$. The system given by (5.25), (5.26) and (5.6) can be written as an $(m+2)$-dimensional SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t \geq 0, \tag{5.32}$$

where $W = (W_t^{B,1}, \ldots, W_t^{B,m}, B_t^1, B_t^2)_{t\geq 0}$, and

$$b(t,x) = \begin{pmatrix} \varphi(t) + \frac{1}{m}\sum_{k=1}^{m} f^B(x_k - x_1 - \psi(t))(x_k - x_1 - \psi(t)) + \psi(t), \\ \varphi(t) + \frac{1}{m-1}\sum_{\ell=3}^{m+1} f^B(x_\ell - x_1 - \psi(t))(x_\ell - x_1 - \psi(t)) + x_1 + x_{m+2} - x_2 + \psi(t), \\ \vdots \\ \varphi(t) + \frac{1}{m-1}\sum_{\ell=2}^{m} f^B(x_\ell - x_1 - \psi(t))(x_\ell - x_1 - \psi(t)) + x_1 + x_{m+2} - x_{m+1} + \psi(t), \\ \tilde{b}(x_{m+2}) \end{pmatrix} \tag{5.33}$$

with $\psi(t) = \mathbb{E}[\tilde{\rho}_t^i]$ and

$$\varphi(t) := \mathbb{E}\left[f^P\left(\tilde{\rho}_t^i - \mathbb{E}[\tilde{\rho}_t^i]\right)\left(\tilde{\rho}_t^i - \mathbb{E}[\tilde{\rho}_t^i]\right)\right], \quad t \geq 0. \tag{5.34}$$

The $(m+2)\times(m+2)$ matrix $\sigma(x)$ has the form

$$\sigma(t,x) = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 0 \\ \sigma_2 & 0 & \ldots & 0 & \sigma_B & 0 \\ 0 & \sigma_2 & \ldots & 0 & \sigma_B & 0 \\ \vdots & \vdots & \ddots & 0 & \sigma_B & 0 \\ 0 & 0 & \ldots & \sigma_2 & \sigma_B & 0 \\ 0 & 0 & \ldots & 0 & 0 & \tilde{\sigma}(x_{m+2}) \end{pmatrix}. \tag{5.35}$$

As before, we rely on Theorem 9.11 in Pascucci [2011]. We have to show that $b$ and $\sigma$ defined in (5.33) and (5.35) respectively are Lipschitz continuous in $x$ uniformly in $t$ and that for each constant $T > 0$ there exists some $\tilde{C}$ such that for all $t \in [0, T]$ it holds

$$\|\sigma(t,x)\|^2 + \|b(t,x)\|^2 \leq \tilde{C}(1 + \|x\|^2).$$

We begin by proving the first condition. The Lipschitz property clearly holds for $\sigma$, since $\tilde{\sigma}$ is Lipschitz by hypothesis. Take now $x = (x_1, \ldots, x_{m+2})$, $x' = (x'_1, \ldots, x'_{m+2})$. We have that

$$\begin{aligned} &|b_1(t,x) - b_1(t,x')| \\ &\quad \leq \frac{1}{m}\sum_{k=1}^{m} |f^B\left(x_k - x_1 - \psi(t)\right)\left(x_k - x_1 - \psi(t)\right) - f^B\left(x'_k - x'_1 - \psi(t)\right)\left(x'_k - x'_1 - \psi(t)\right)| \end{aligned}$$

$$\leq K_2 \frac{1}{m} \sum_{k=1}^{m} |(x_k - x_1 - \psi(t)) - (x'_k - x'_1 - \psi(t))|$$
$$= K_2 \left( \frac{1}{m} \sum_{k=1}^{m} |x_k - x'_k| + |x_1 - x'_1| \right).$$

Similarly, for $k = 2, \ldots, m+1$ we have

$$|b_k(t,x) - b_k(t,x')| \leq K_2 \left( \frac{1}{m-1} \sum_{\ell=1, \ell \neq k}^{m} |x_\ell - x'_\ell| + |x_1 - x'_1| + |x_k - x'_k| \right).$$

With computations as in the proof of Proposition 5.2.3 we obtain that for $t \geq 0$ it holds

$$\|b(t,x) - b(t,x')\|^2 \leq \bar{C} \|x - x'\|^2, \tag{5.36}$$

for some appropriate $\bar{C}$.
We now show the second condition, i.e. that for $t \in [0,T]$ it holds

$$\|\sigma(t,x)\|^2 + \|b(t,x)\|^2 \leq \tilde{C}(1 + \|x\|^2), \tag{5.37}$$

for some $\tilde{C} > 0$. By (5.35) we can focus only on $\|b(t,x)\|$. The computations are here the same as in Proposition 5.2.3, but we have to estimate the term $\varphi(t)$ from (5.34). Since

$$|\varphi(t)| = \left| \mathbb{E}\left[ f^P \left( \tilde{\rho}_t^i - \mathbb{E}[\tilde{\rho}_t^i] \right) \left( \tilde{\rho}_t^i - \mathbb{E}[\tilde{\rho}_t^i] \right) \right] \right| \leq K_2 \mathbb{E}[|\tilde{\rho}_t^i - \mathbb{E}[\tilde{\rho}_t^i]|],$$
$$\leq K_2 \left( \mathbb{E}[|\tilde{\rho}_t^i - \mathbb{E}[\tilde{\rho}_t^i]|^2] \right)^{1/2} \leq K_2 \left( \frac{\sigma_1^2}{2} (1 - e^{-2t}) \right)^{1/2} \leq K_2 \frac{\sigma_1}{\sqrt{2}},$$

(5.37) follows by the proof of Proposition 5.2.3.
Inequalities (5.30) and (5.31) follow since, by Theorem 9.11 in Pascucci [2011], (5.36) and (5.37) guarantee that the second moments of the solution of (5.32) are finite.
The proof for the case $\delta > 0$, based on Theorem 3.1 in Mao [2007, chapter 5], is analogous to the one of Proposition 5.2.3. □

Denote $|x - y|_t^* = \sup_{s \leq t} |x_s - y_s|$. We have the following

**Theorem 5.3.2.** *Fix $i \in \mathbb{N}$. Under Assumption 5.2.1, for any $t \in [0, \infty)$ and $\delta \geq 0$ it holds*

$$\lim_{n \to \infty} \left( \mathbb{E}\left[ |\rho^{i,n} - \bar{\rho}^i|_t^* \right] + \mathbb{E}[|\rho^{k,B} - \bar{\rho}^{k,B}|_t^*] \right) = 0, \quad k = 1, \ldots, m,$$

*where $\rho^{i,n}$, $\bar{\rho}^i$, $\rho^{k,B}$, $\bar{\rho}^{k,B}$ are defined in (5.2), (5.29), (5.3), (5.26) respectively.*

Before proving Theorem 5.3.2, we give the following

**Proposition 5.3.3.** *Under Assumption 5.2.1, for $0 \leq \delta < \infty$, it holds*

$$\lim_{n \to \infty} \int_0^\delta \mathbb{E}\left[ \left| \frac{1}{n} \sum_{i=1}^{n} f^P(\bar{\rho}_s^i - \bar{A}_s^n)(\bar{\rho}_s^i - \bar{A}_s^n) - \mathbb{E}\left[ f^P \left( \tilde{\rho}_s^i - \mathbb{E}[\tilde{\rho}_s^i] \right) \left( \tilde{\rho}_s^i - \mathbb{E}[\tilde{\rho}_s^i] \right) \right] \right| \right] ds = 0, \tag{5.38}$$

*and*

$$\lim_{n\to\infty}\int_\delta^t \mathbb{E}\left[\Big|\frac{1}{n}\sum_{i=1}^n f^P(\bar\rho^i_{s-\delta}-\bar A^n_{s-\delta})(\bar\rho^i_s-\bar A^n_s)-\mathbb{E}\left[f^P\left(\tilde\rho^i_{s-\delta}-\mathbb{E}[\tilde\rho^i_{s-\delta}]\right)\left(\tilde\rho^i_s-\mathbb{E}[\tilde\rho^i_s]\right)\right]\Big|\right]ds=0,$$

*for* $0\le\delta\le t<\infty$, *where* $\tilde\rho^i$ *and* $\bar\rho^i$ *satisfy (5.24) and (5.29), respectively, and*

$$\bar A^n_t=\frac{1}{m+n}\left(\sum_{r=1}^n\bar\rho^r_t+\sum_{h=1}^m\bar\rho^{h,B}_t\right),\quad t\ge 0. \tag{5.39}$$

*Proof.* We limit ourselves to prove the second limit, since the first one follows as a particular case. Let us write, for $t\ge\delta>0$,

$$\begin{aligned}
&\mathbb{E}\left[\Big|\frac{1}{n}\sum_{i=1}^n f^P(\bar\rho^i_{t-\delta}-\bar A^n_{t-\delta})(\bar\rho^i_t-\bar A^n_t)-\mathbb{E}\left[f^P\left(\tilde\rho^i_{t-\delta}-\mathbb{E}[\tilde\rho^i_{t-\delta}]\right)\left(\tilde\rho^i_t-\mathbb{E}[\tilde\rho^i_t]\right)\right]\Big|\right]\\
&\quad\le\frac{1}{n}\sum_{i=1}^n\mathbb{E}\left[\Big|f^P(\bar\rho^i_{t-\delta}-\bar A^n_{t-\delta})(\bar\rho^i_t-\bar A^n_t)-f^P(\tilde\rho^i_{t-\delta}-\mathbb{E}[\tilde\rho^i_{t-\delta}])(\tilde\rho^i_t-\mathbb{E}[\tilde\rho^i_t])\Big|\right]\\
&\qquad+\mathbb{E}\left[\Big|\frac{1}{n}\sum_{i=1}^n f^P(\tilde\rho^i_{t-\delta}-\mathbb{E}[\tilde\rho^i_{t-\delta}])(\tilde\rho^i_t-\mathbb{E}[\tilde\rho^i_t])-\mathbb{E}\left[f^P\left(\tilde\rho^i_{t-\delta}-\mathbb{E}[\tilde\rho^i_{t-\delta}]\right)\left(\tilde\rho^i_t-\mathbb{E}[\tilde\rho^i_t]\right)\right]\Big|\right],
\end{aligned}$$

since $\bar\rho^i$, $i=1,\dots,n$ are identically distributed and the same holds for $\tilde\rho^i$, $i=1,\dots,n$.
By (5.28) we have that

$$\bar A^n_t=\frac{1}{m+n}\left(\sum_{r=1}^n\bar\rho^r_t+\sum_{h=1}^m\bar\rho^{h,B}_t\right)=\frac{1}{m+n}\left(n\nu_t+\sum_{r=1}^n\tilde\rho^r_t+\sum_{h=1}^m\bar\rho^{h,B}_t\right),$$

so that

$$\lim_{n\to\infty}\bar A^n_t=\nu_t+\lim_{n\to\infty}\frac{1}{m+n}\sum_{r=1}^n\tilde\rho^r_t=\nu_t+\mathbb{E}[\tilde\rho^i_t],\quad a.s.,$$

by (5.12) and the law of large numbers, as $\tilde\rho^i$, $i=1,\dots,n$, are independent and identically distributed. Then we have

$$\begin{aligned}
\lim_{n\to\infty}f^P(\bar\rho^i_{t-\delta}-\bar A^n_{t-\delta})(\bar\rho^i_t-\bar A^n_t)&=f^P\left(\nu_{t-\delta}+\tilde\rho^i_{t-\delta}-(\nu_{t-\delta}+\mathbb{E}[\tilde\rho^i_{t-\delta}])\right)\left(\nu_t+\tilde\rho^i_t-(\nu_t+\mathbb{E}[\tilde\rho^i_t])\right)\\
&=f^P\left(\tilde\rho^i_{t-\delta}-\mathbb{E}[\tilde\rho^i_{t-\delta}]\right)\left(\tilde\rho^i_t-\mathbb{E}[\tilde\rho^i_t]\right)\quad a.s.
\end{aligned}\tag{5.40}$$

We now prove that the family of random variables $\{\frac{1}{n}\sum_{i=1}^n f^P(\bar\rho^i_{s-\delta}-\bar A^n_{s-\delta})(\bar\rho^i_s-\bar A^n_s)\}_{n\in\mathbb{N}}$ is uniformly integrable for every $s\in[\delta,t]$, so that convergence almost surely implies convergence in $L^1$.
By point (iii) of Theorem 11 in Protter [2005, chapter 1] it is enough to prove that for every $s\in[\delta,t]$,

$$\sup_n\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n f^P(\bar\rho^i_{s-\delta}-\bar A^n_{s-\delta})(\bar\rho^i_s-\bar A^n_s)\right)^2\right]<\infty. \tag{5.41}$$

For every $s\in[\delta,t]$, we have that

$$\mathbb{E}\left[\Big(\frac{1}{n}\sum_{i=1}^n f^P(\bar\rho^i_{s-\delta}-\bar A^n_{s-\delta})(\bar\rho^i_s-\bar A^n_s)\Big)^2\right]\le(K_2)^2\mathbb{E}\left[\Big(\frac{1}{n}\sum_{i=1}^n|\bar\rho^i_s-\bar A^n_s|\Big)^2\right]$$

$$
\begin{aligned}
&\leq (K_2)^2\mathbb{E}\Big[\Big((1-n/(m+n))|\nu_s| + \big|\tilde{\rho}^i_s\big| + \frac{1}{m+n}\sum_{r=1}^{n}|\tilde{\rho}^r_s| + \frac{1}{m+n}\sum_{h=1}^{m}\big|\bar{\rho}^{h,B}_s\big|\Big)^2\Big]\\
&\leq (K_2)^2\mathbb{E}\Big[\Big(|\nu_s| + \big|\tilde{\rho}^i_s\big| + \frac{1}{n}\sum_{r=1}^{n}|\tilde{\rho}^r_s| + \frac{1}{m}\sum_{h=1}^{m}\big|\bar{\rho}^{h,B}_s\big|\Big)^2\Big]\\
&\leq 4(K_2)^2\Big(\mathbb{E}\Big[|\nu_s|^2 + |\tilde{\rho}^i_s|^2 + \sum_{k=1}^{m}|\bar{\rho}^{k,B}_s|^2\Big] + \mathbb{E}\Big[\Big(\frac{1}{n}\sum_{r=1}^{n}|\tilde{\rho}^r_s|\Big)^2\Big]\Big)\\
&\leq 4(K_2)^2\Big(\mathbb{E}\Big[|\nu_s|^2 + |\tilde{\rho}^i_s|^2 + \sum_{k=1}^{m}|\bar{\rho}^{k,B}_s|^2\Big] + \frac{1}{n}\mathbb{E}\Big[\sum_{r=1}^{n}|\tilde{\rho}^r_s|^2\Big]\Big).\\
&\leq 4(K_2)^2\Big(\mathbb{E}\Big[|\nu_s|^2 + |\tilde{\rho}^i_s|^2 + \sum_{k=1}^{m}|\bar{\rho}^{k,B}_s|^2\Big] + \mathbb{E}[|\tilde{\rho}^i_s|^2]\Big) < \infty,
\end{aligned}
$$

by (5.30) and (5.31) and because $\mathbb{E}|\tilde{\rho}^i_s|^2] < \infty$. Hence, $\{\frac{1}{n}\sum_{i=1}^{n} f^P(\bar{\rho}^i_{s-\delta} - \bar{A}^n_{s-\delta})(\bar{\rho}^i_s - \bar{A}^n_s)\}_{n\in\mathbb{N}}$ is uniformly integrable and we obtain therefore by (5.40) that

$$
\lim_{n\to\infty}\mathbb{E}\Big[\Big|f^P(\bar{\rho}^i_{t-\delta} - \bar{A}^n_{t-\delta})(\bar{\rho}^i_t - \bar{A}^n_t) - f^P(\tilde{\rho}^i_{t-\delta} - \mathbb{E}[\tilde{\rho}^i_{t-\delta}])(\tilde{\rho}^i_t - \mathbb{E}[\tilde{\rho}^i_t])\Big|\Big] = 0.
$$

Moreover, for $\delta \leq s \leq t$ it holds

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|f^P(\bar{\rho}^i_{t-\delta} - \bar{A}^n_{t-\delta})(\bar{\rho}^i_t - \bar{A}^n_t) - f^P(\tilde{\rho}^i_{t-\delta} - \mathbb{E}[\tilde{\rho}^i_{t-\delta}])(\tilde{\rho}^i_t - \mathbb{E}[\tilde{\rho}^i_t])\Big|\Big]\\
&\quad\leq K_1(\mathbb{E}[|\bar{\rho}^i_t - \bar{A}^n_t|] + \mathbb{E}[|\tilde{\rho}^i_t - \mathbb{E}[\tilde{\rho}^i_t|]),
\end{aligned}
$$

where the second term belongs to $L^1([\delta,t])$ and does not depend on $n$. On the other hand, we have

$$
\begin{aligned}
\int_0^t \mathbb{E}[|\bar{\rho}^i_s - \bar{A}^n_s|]ds &\leq \int_0^t \mathbb{E}\left[|\tilde{\rho}^i_s| + (1-n/(m+n))\,|\nu_s| + \frac{1}{m+n}\sum_{r=1}^{n}|\tilde{\rho}^r_s| + \frac{1}{m+n}\sum_{h=1}^{m}|\bar{\rho}^{h,B}_s|\right]ds\\
&\leq \int_0^t \mathbb{E}\left[2|\bar{\rho}^i_s| + |\nu_s| + |\bar{\rho}^{h,B}_s|\right]ds\\
&\leq t\sup_{0\leq s\leq t}\mathbb{E}\left[2|\tilde{\rho}^i_s| + |\nu_s| + |\bar{\rho}^{h,B}_s|\right] < \infty, \qquad (5.42)
\end{aligned}
$$

by (5.30) and (5.31). We can then apply the dominated convergence theorem to obtain, for $t \in [\delta,\infty)$,

$$
\lim_{n\to\infty}\int_\delta^t \mathbb{E}\Big[\Big|f^P(\bar{\rho}^i_{s-\delta} - \bar{A}^n_{s-\delta})(\bar{\rho}^i_s - \bar{A}^n_s) - f^P(\tilde{\rho}^i_{s-\delta} - \mathbb{E}[\tilde{\rho}^i_{s-\delta}])(\tilde{\rho}^i_s - \mathbb{E}[\tilde{\rho}^i_s])\Big|\Big]ds = 0, \quad t \geq \delta. \qquad (5.43)
$$

It remains to show that for $t \geq \delta$ it holds

$$
\lim_{n\to\infty}\int_\delta^t \mathbb{E}\Big[\Big|\frac{1}{n}\sum_{i=1}^{n} f^P(\tilde{\rho}^i_{s-\delta} - \mathbb{E}[\tilde{\rho}^i_{s-\delta}])(\tilde{\rho}^i_s - \mathbb{E}[\tilde{\rho}^i_s]) - \mathbb{E}\left[f^P\left(\tilde{\rho}^i_{s-\delta} - \mathbb{E}[\tilde{\rho}^i_{s-\delta}]\right)\left(\tilde{\rho}^i_s - \mathbb{E}[\tilde{\rho}^i_s]\right)\right]\Big|\Big]ds = 0. \qquad (5.44)
$$

Since $\tilde{\rho}^i$, $i = 1,\dots,n$, are independent and identically distributed, we have that, for $\delta \leq s \leq t$,

$$
\lim_{n\to\infty}\mathbb{E}\Big[\Big|\frac{1}{n}\sum_{i=1}^{n} f^P(\tilde{\rho}^i_{s-\delta} - \mathbb{E}[\tilde{\rho}^i_{s-\delta}])(\tilde{\rho}^i_s - \mathbb{E}[\tilde{\rho}^i_s]) - \mathbb{E}\left[f^P\left(\tilde{\rho}^i_{s-\delta} - \mathbb{E}[\tilde{\rho}^i_{s-\delta}]\right)\left(\tilde{\rho}^i_s - \mathbb{E}[\tilde{\rho}^i_s]\right)\right]\Big|\Big] = 0.
$$

Then limit (5.44) follows by the dominated convergence theorem, by Assumption 5.2.1 and since the Ornstein-Uhlenbeck process has finite moments, see the computations in (5.42). □

*Proof of Theorem 5.3.2.* We suppose by simplicity $\lambda = 1$ and we proceed by steps, starting from the case when $0 \leq t < \delta$, i.e. when there is no delay in equations (5.2)-(5.3) and (5.25)-(5.26).
*First step: case* $0 \leq t < \delta$.
For every $i = 1, \dots, n$ and $t \in [0, \delta)$, we have

$$\rho_t^{i,n} - \bar{\rho}_t^i = \int_0^t \Delta_s^n ds,$$

where

$$\begin{aligned}\Delta_s^n =& \frac{1}{n-1} \sum_{j=1, j\neq i}^n f^P(\rho_s^{j,n} - A_s^n)(\rho_s^{j,n} - A_s^n) - \mathbb{E}\left[f^P\left(\tilde{\rho}_s^i - \mathbb{E}[\tilde{\rho}_s^i]\right)\left(\tilde{\rho}_s^i - \mathbb{E}[\tilde{\rho}_s^i]\right)\right] \\ &+ \frac{1}{m}\sum_{k=1}^m \Big(f^B(\rho_s^{k,B} - A_s^n)(\rho_s^{k,B} - A_s^n) - f^B(\bar{\rho}_s^{k,B} - \nu_s^n - \mathbb{E}[\tilde{\rho}_s^i])(\bar{\rho}_s^{k,B} - \nu_s - \mathbb{E}[\tilde{\rho}_s^i])\Big) \\ &- (\rho_s^{i,n} - \bar{\rho}_s^i) + (A_s^n - \bar{A}_s^n) + (\bar{A}_s^n - \mathbb{E}[\tilde{\rho}_s^i] - \nu_s).\end{aligned}$$

Thus

$$|\rho^{i,n} - \bar{\rho}^i|_t^* = \sup_{s\leq t}\left|\int_0^s \Delta_u^n du\right| \leq \sup_{s\leq t}\int_0^s |\Delta_u^n|\, du = \int_0^t |\Delta_u^n|\, du.$$

Therefore, for every $i = 1, \dots, n$ and $t \geq 0$, we have

$$\begin{aligned}&\mathbb{E}[|\rho^{i,n} - \bar{\rho}^i|_t^*] \leq \mathbb{E}\left[\int_0^t |\Delta_s^n| ds\right] \\ &\leq \int_0^t \mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j=1,j\neq i}^n (f^P(\rho_s^{j,n} - A_s^n)(\rho_s^{j,n} - A_s^n) - f^P(\bar{\rho}_s^j - \bar{A}_s^n)(\bar{\rho}_s^j - \bar{A}_s^n))\Big|\Big] ds \\ &\quad + \int_0^t \mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j=1,j\neq i}^n f^P(\bar{\rho}_s^j - \bar{A}_s^n)(\bar{\rho}_s^j - \bar{A}_s^n) - \mathbb{E}\left[f^P\left(\tilde{\rho}_s^i - \mathbb{E}[\tilde{\rho}_s^i]\right)\left(\tilde{\rho}_s^i - \mathbb{E}[\tilde{\rho}_s^i]\right)\right]\Big|\Big] ds \\ &\quad + \int_0^t \mathbb{E}\Big[\Big|\frac{1}{m}\sum_{k=1}^m \left(f^B(\rho_s^{k,B} - A_s^n)(\rho_s^{k,B} - A_s^n) - f^B(\bar{\rho}_s^{k,B} - \bar{A}_s^n)(\bar{\rho}_s^{k,B} - \bar{A}_s^n)\right)\Big|\Big] ds \\ &\quad + \int_0^t \mathbb{E}\Big[\frac{1}{m}\sum_{k=1}^m \left|f^B(\bar{\rho}_s^{k,B} - \bar{A}_s^n)(\bar{\rho}_s^{k,B} - \bar{A}_s^n) - f^B(\bar{\rho}_s^{k,B} - \nu_s^n - \mathbb{E}[\tilde{\rho}_s^i])(\bar{\rho}_s^{k,B} - \nu_s - \mathbb{E}[\tilde{\rho}_s^i])\right|\Big] ds \\ &\quad + \int_0^t \mathbb{E}[|\rho_s^{i,n} - \bar{\rho}_s^i|] ds + \int_0^t \mathbb{E}[|A_s^n - \bar{A}_s^n|] ds + \int_0^t \mathbb{E}\left[|\bar{A}_s^n - \mathbb{E}[\tilde{\rho}_s^i] - \nu_s|\right] ds.\end{aligned} \tag{5.45}$$

By (5.9) it holds

$$\begin{aligned}&\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j=1,j\neq i}^n (f^P(\rho_s^{j,n} - A_s^n)(\rho_s^{j,n} - A_s^n) - f^P(\bar{\rho}_s^j - \bar{A}_s^n)(\bar{\rho}_s^j - \bar{A}_s^n))\Big|\Big] ds \\ &\quad \leq \frac{1}{n-1}\sum_{j=1,j\neq i}^n \int_0^t \mathbb{E}\Big[\Big|f^P(\rho_s^{j,n} - A_s^n)(\rho_s^{j,n} - A_s^n) - f^P(\bar{\rho}_s^j - \bar{A}_s^n)(\bar{\rho}_s^j - \bar{A}_s^n)\Big|\Big] ds\end{aligned}$$

$$\leq K_2 \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} \int_0^t \mathbb{E}\Big[\Big|(\rho_s^{j,n} - A_s^n) - (\bar{\rho}_s^j - \bar{A}_s^n)\Big|\Big] ds$$
$$\leq K_2 \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} \int_0^t \mathbb{E}\left[\left|\rho_s^{j,n} - \bar{\rho}_s^j\right| + \left|A_s^n - \bar{A}_s^n\right|\right] ds$$
$$= K_2 \int_0^t \mathbb{E}\left[\left|\rho_s^{i,n} - \bar{\rho}_s^i\right|\right] ds + K_2 \int_0^t \mathbb{E}\left[\left|A_s^n - \bar{A}_s^n\right|\right] ds, \quad t \geq 0. \tag{5.46}$$

By (5.4) and (5.39) we have that

$$\int_0^t \mathbb{E}\left[\left|A_s^n - \bar{A}_s^n\right|\right] ds \leq \int_0^t \mathbb{E}\Big[\frac{1}{m+n} \sum_{r=1}^{n} |\rho_s^{r,n} - \bar{\rho}_s^r|\Big] ds + \int_0^t \mathbb{E}\Big[\frac{1}{m+n} \sum_{k=1}^{m} \left|\rho_s^{h,B} - \bar{\rho}_s^{h,B}\right|\Big] ds$$
$$\leq \int_0^t \mathbb{E}\left[\left|\rho_s^{i,n} - \bar{\rho}_s^i\right|\right] ds + \int_0^t \mathbb{E}\left[\left|\rho_s^{k,B} - \bar{\rho}_s^{k,B}\right|\right] ds, \quad t \geq 0, \tag{5.47}$$

because all $\rho^i$, $i = 1, \ldots, n$, and $\rho^{k,B}$, $k = 1, \ldots, m$, are identically distributed, respectively.
We can conclude by (5.46) and (5.47) that

$$\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n-1} \sum_{j=1, j\neq i}^{n} (f^P(\rho_s^{j,n} - A_s^n)(\rho_s^{j,n} - A_s^n) - f^P(\bar{\rho}_s^j - \bar{A}_s^n)(\bar{\rho}_s^j - \bar{A}_s^n))\Big|\Big] ds$$
$$\leq 2K_2 \int_0^t \mathbb{E}\left[\left|\rho_s^{i,n} - \bar{\rho}_s^i\right|\right] ds + K_2 \int_0^t \mathbb{E}\left[\left|\rho_s^{k,B} - \bar{\rho}_s^{k,B}\right|\right] ds$$
$$\leq 2K_2 \int_0^t \mathbb{E}\left[\left|\rho^{i,n} - \bar{\rho}^i\right|_s^*\right] ds + K_2 \int_0^t \mathbb{E}\left[\left|\rho^{k,B} - \bar{\rho}^{k,B}\right|_s^*\right] ds, \quad t \geq 0. \tag{5.48}$$

Similarly,

$$\int_0^t \mathbb{E}\Big[\Big|\frac{1}{m} \sum_{k=1}^{m} \left(f^B(\rho_s^{k,B} - A_s^n)(\rho_s^{k,B} - A_s^n) - f^B(\bar{\rho}_s^{k,B} - \bar{A}_s^n)(\bar{\rho}_s^{k,B} - \bar{A}_s^n)\right)\Big|\Big] ds$$
$$\leq K_2 \int_0^t \mathbb{E}\left[\left|\rho^{i,n} - \bar{\rho}^i\right|_s^*\right] ds + 2K_2 \int_0^t \mathbb{E}\left[\left|\rho^{k,B} - \bar{\rho}^{k,B}\right|_s^*\right] ds \quad t \geq 0. \tag{5.49}$$

From (5.45), (5.47), (5.48) and (5.49) we have that for $t \geq 0$ it holds

$$\mathbb{E}[|\rho^{i,n} - \bar{\rho}^i|_t^*]$$
$$\leq (3K_2 + 2) \int_0^t \mathbb{E}\left[\left|\rho^{i,n} - \bar{\rho}^i\right|_s^*\right] ds + (3K_2 + 1) \int_0^t \mathbb{E}\left[\left|\rho^{k,B} - \bar{\rho}^{k,B}\right|_s^*\right] ds$$
$$+ \int_0^t \mathbb{E}\Big[\Big|f^B(\bar{\rho}_s^{k,B} - \bar{A}_s^n)(\bar{\rho}_s^{k,B} - \bar{A}_s^n) - f^B(\bar{\rho}_s^{k,B} - \nu_s^n - \mathbb{E}[\tilde{\rho}_s^i])(\bar{\rho}_s^{k,B} - \nu_s - \mathbb{E}[\tilde{\rho}_s^i])\Big|\Big] ds$$
$$+ \int_0^t \mathbb{E}\Big[\Big|\frac{1}{n-1} \sum_{j=1, j\neq i}^{n} f^P(\bar{\rho}_s^j - \bar{A}_s^n)(\bar{\rho}_s^j - \bar{A}_s^n) - \mathbb{E}\left[f^P\left(\tilde{\rho}_s^i - \mathbb{E}[\tilde{\rho}_s^i]\right)\left(\tilde{\rho}_s^i - \mathbb{E}[\tilde{\rho}_s^i]\right)\right]\Big|\Big] ds$$
$$+ \int_0^t \mathbb{E}\left[\left|\bar{A}_s^n - \mathbb{E}[\tilde{\rho}_s^i] - \nu_s\right|\right] ds, \quad t \geq 0. \tag{5.50}$$

Proceeding as before, we find

$$\mathbb{E}[|\rho^{k,B} - \bar{\rho}^{k,B}|_t^*]$$
$$\leq (3K_2 + 1) \int_0^t \mathbb{E}\left[\left|\rho^{i,n} - \bar{\rho}^i\right|_s^*\right] ds + (3K_2 + 2) \int_0^t \mathbb{E}\left[\left|\rho^{k,B} - \bar{\rho}^{k,B}\right|_s^*\right] ds$$

$$+\int_0^t \mathbb{E}\Big[\Big|f^B(\bar\rho_s^{k,B}-\bar A_s^n)(\bar\rho_s^{k,B}-\bar A_s^n)-f^B(\bar\rho_s^{k,B}-\nu_s^n-\mathbb{E}[\tilde\rho_s^i])(\bar\rho_s^{k,B}-\nu_s-\mathbb{E}[\tilde\rho_s^i])\Big|\Big]ds$$
$$+\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n}\sum_{i=1}^n f^P(\bar\rho_s^i-\bar A_s^n)(\bar\rho_s^i-\bar A_s^n)-\mathbb{E}\left[f^P\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]\Big|\Big]ds$$
$$+\int_0^t \mathbb{E}\left[|\bar A_s^n-\nu_s-\mathbb{E}[\tilde\nu_s^i|]\right]ds, \tag{5.51}$$

so that, summing up (5.50) and (5.51), we have

$$\begin{aligned}
&\mathbb{E}[|\rho^{i,n}-\bar\rho^i|_t^*]+\mathbb{E}[|\rho^{k,B}-\bar\rho^{k,B}|_t^*]\\
&\le (6K_2+3)\int_0^t \mathbb{E}\left[\left|\rho^{i,n}-\bar\rho^i\right|_s^*\right]ds+(6K_2+3)\int_0^t \mathbb{E}\left[\left|\rho^{k,B}-\bar\rho^{k,B}\right|_s^*\right]ds\\
&\quad+2\int_0^t \mathbb{E}\Big[\Big|f^B(\bar\rho_s^{k,B}-\bar A_s^n)(\bar\rho_s^{k,B}-\bar A_s^n)-f^B(\bar\rho_s^{k,B}-\nu_s^n-\mathbb{E}[\tilde\rho_s^i])(\bar\rho_s^{k,B}-\nu_s-\mathbb{E}[\tilde\rho_s^i])\Big|\Big]ds\\
&\quad+\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j=1,j\neq i}^n f^P(\bar\rho_s^j-\bar A_s^n)(\bar\rho_s^j-\bar A_s^n)-\mathbb{E}\left[f^P\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]\Big|\Big]ds\\
&\quad+\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n}\sum_{i=1}^n f^P(\bar\rho_s^i-\bar A_s^n)(\bar\rho_s^i-\bar A_s^n)-\mathbb{E}\left[f^P\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]\Big|\Big]ds\\
&\quad+2\int_0^t \mathbb{E}\left[|\bar A_s^n-\nu_s-\mathbb{E}[\tilde\nu_s^i|]\right]ds,\quad t\ge 0.
\end{aligned}\tag{5.52}$$

We can now apply Gronwall's Lemma and obtain

$$\begin{aligned}
&\mathbb{E}[|\rho^{i,n}-\bar\rho^i|_t^*]+\mathbb{E}[|\rho_t^{k,B}-\bar\rho_t^{k,B}|_s^*]\\
&\le e^{(6K_2+3)t}\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j=1,j\neq i}^n f^P(\bar\rho_s^j-\bar A_s^n)(\bar\rho_s^j-\bar A_s^n)-\mathbb{E}\left[f^P\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]\Big|\Big]ds\\
&\quad+e^{(6K_2+3)t}\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n}\sum_{i=1}^n f^P(\bar\rho_s^i-\bar A_s^n)(\bar\rho_s^i-\bar A_s^n)-\mathbb{E}\left[f^P\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]\Big|\Big]ds\\
&\quad+2e^{(6K_2+3)t}\int_0^t \mathbb{E}\Big[\Big|f^B(\bar\rho_s^{k,B}-\bar A_s^n)(\bar\rho_s^{k,B}-\bar A_s^n)-f^B(\bar\rho_s^{k,B}-\nu_s-\mathbb{E}[\tilde\rho_s^i])(\bar\rho_s^{k,B}-\nu_s-\mathbb{E}[\tilde\rho_s^i])\Big|\Big]ds\\
&\quad+2e^{(6K_2+3)t}\int_0^t \mathbb{E}\left[|\bar A_s^n-\nu_s-\mathbb{E}[\tilde\nu_s^i|]\right]ds,\quad t\ge 0.
\end{aligned}\tag{5.53}$$

We can write

$$\begin{aligned}
&\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j=1,j\neq i}^n f^P(\bar\rho_s^j-\bar A_s^n)(\bar\rho_s^j-\bar A_s^n)-\mathbb{E}\left[f^P\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]\Big|\Big]ds\\
&\le\left(\frac{1}{n-1}-\frac{1}{n}\right)\int_0^t \mathbb{E}\Big[\Big|\sum_{j=1,j\neq i}^n f^P(\bar\rho_s^j-\bar A_s^n)(\bar\rho_s^j-\bar A_s^n)\Big|\Big]ds\\
&\quad+\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n}\sum_{i=1}^n f^P(\bar\rho_s^i-\bar A_s^n)(\bar\rho_s^i-\bar A_s^n)-\mathbb{E}\left[f^P\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]\Big|\Big]ds\\
&\quad+\frac{1}{n}\int_0^t \mathbb{E}\left[f^P\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]ds\quad t\ge 0,
\end{aligned}$$

with

$$\left(\frac{1}{n-1}-\frac{1}{n}\right)\int_0^t \mathbb{E}\Big[\Big|\sum_{j=1,j\neq i}^n f^P(\bar\rho_s^i-\bar A_s^n)(\bar\rho_s^i-\bar A_s^n)\Big|\Big]ds$$

$$\leq \frac{1}{n(n-1)} \int_0^t \sum_{j=1, j\neq i}^{n} \mathbb{E}[|f^P(\bar{\rho}_s^i - \bar{A}_s^n)(\bar{\rho}_s^i - \bar{A}_s^n)|]ds$$
$$= \frac{1}{n} \int_0^t \mathbb{E}[|f^P(\bar{\rho}_s^i - \bar{A}_s^n)(\bar{\rho}_s^i - \bar{A}_s^n)|]ds \leq \frac{K_2}{n} \int_0^t \mathbb{E}[|\bar{\rho}_s^i - \bar{A}_s^n|]ds, \quad t \geq 0,$$

where the last term tends to zero when $n \to \infty$ by (5.42).
Since it can be shown, for $t \geq 0$, that

$$\lim_{n\to\infty} \int_0^t \mathbb{E}\Big[ \big| f^B(\bar{\rho}_s^{k,B} - \bar{A}_s^n)(\bar{\rho}_s^{k,B} - \bar{A}_s^n) - f^B(\bar{\rho}_s^{k,B} - \nu_s - \mathbb{E}[\tilde{\rho}_s^i])(\bar{\rho}_s^{k,B} - \nu_s - \mathbb{E}[\tilde{\rho}_s^i]) \big| \Big] ds = 0,$$

and

$$\lim_{n\to\infty} \int_0^t \mathbb{E}\left[ |\bar{A}_s^n - \nu_s - \mathbb{E}[\tilde{\nu}_s^i]| \right] ds = 0, \quad t \geq 0,$$

with the same proof as for (5.43), then by (5.38) we obtain the result for $t \in [0, \delta)$.

*Second step: case $t \in [\delta, 2\delta)$.*
For every $i = 1, \ldots, n$ and $t \geq \delta$, we have

$$|\rho_t^{i,n} - \bar{\rho}_t^i| \leq \left| \int_0^\delta (\rho_s^{i,n} - \bar{\rho}_s^i) ds + \int_\delta^t \Delta_s^{\delta,n} ds \right|,$$

where

$$\begin{aligned}
\Delta_s^{\delta,n} =& \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} f^P(\rho_{s-\delta}^{j,n} - A_{s-\delta}^n)(\rho_s^{j,n} - A_s^n) - \mathbb{E}\left[ f^P\left(\tilde{\rho}_{t-\delta}^i - \mathbb{E}[\tilde{\rho}_{t-\delta}^i]\right)\left(\tilde{\rho}_t^i - \mathbb{E}[\tilde{\rho}_t^i]\right) \right] \\
&+ \frac{1}{m} \sum_{k=1}^{m} \Big( f^B(\rho_{s-\delta}^{k,B} - A_{s-\delta}^n)(\rho_s^{k,B} - A_s^n) - f^B(\bar{\rho}_{s-\delta}^{k,B} - \nu_{s-\delta} - \mathbb{E}[\tilde{\rho}_{s-\delta}^i])(\bar{\rho}_s^{k,B} - \nu_s - \mathbb{E}[\tilde{\rho}_s^i]) \Big) \\
&- (\rho_s^{i,n} - \bar{\rho}_s^i) + (A_s^n - \bar{A}_s^n) + (\bar{A}_s^n - \mathbb{E}[\tilde{\rho}_s^i] - \nu_s|).
\end{aligned}$$

Thus

$$\begin{aligned}
|\rho^{i,n} - \bar{\rho}^i|_t^* &= \sup_{s\leq t} \left| \int_0^\delta (\rho_u^{i,n} - \bar{\rho}_u^i) du + \int_\delta^s \Delta_u^{\delta,n} du \right| \leq \int_0^\delta |\rho_u^{i,n} - \bar{\rho}_u^i| du + \sup_{\delta\leq s\leq t} \int_\delta^s \left| \Delta_u^{\delta,n} \right| du \\
&= \int_0^\delta |\rho_u^{i,n} - \bar{\rho}_u^i| du + \int_\delta^t \left| \Delta_u^{\delta,n} \right| du, \quad \delta \leq t. \qquad (5.54)
\end{aligned}$$

For every $i = 1, \ldots, n$, we have

$$\begin{aligned}
&\mathbb{E}\left[ \int_\delta^t |\Delta_s^{\delta,n}| ds \right] \\
&\leq \int_\delta^t \mathbb{E}\Big[ \Big| \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} (f^P(\rho_{s-\delta}^{j,n} - A_{s-\delta}^n)(\rho_s^{j,n} - A_s^n) - f^P(\bar{\rho}_{s-\delta}^j - \bar{A}_{s-\delta}^n)(\bar{\rho}_s^j - \bar{A}_s^n)) \Big| \Big] ds \\
&\quad + \int_\delta^t \mathbb{E}\Big[ \Big| \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} f^P(\bar{\rho}_{s-\delta}^j - \bar{A}_{s-\delta}^n)(\bar{\rho}_s^j - \bar{A}_s^n) - \mathbb{E}\left[ f^P\left(\tilde{\rho}_{s-\delta}^i - \mathbb{E}[\tilde{\rho}_{s-\delta}^i]\right)\left(\tilde{\rho}_s^i - \mathbb{E}[\tilde{\rho}_s^i]\right) \right] \Big| \Big] ds \\
&\quad + \int_\delta^t \mathbb{E}\Big[ \Big| \frac{1}{m} \sum_{k=1}^{m} \Big( f^B(\rho_{s-\delta}^{k,B} - A_{s-\delta}^n)(\rho_s^{k,B} - A_s^n) - f^B(\bar{\rho}_{s-\delta}^{k,B} - \bar{A}_{s-\delta}^n)(\bar{\rho}_s^{k,B} - \bar{A}_s^n) \Big) \Big| \Big] ds \\
&\quad + \int_\delta^t \mathbb{E}\Big[ \frac{1}{m} \sum_{k=1}^{m} \Big| f^B(\bar{\rho}_{s-\delta}^{k,B} - \bar{A}_{s-\delta}^n)(\bar{\rho}_s^{k,B} - \bar{A}_s^n) - f^B(\bar{\rho}_{s-\delta}^{k,B} - \nu_{s-\delta} - \mathbb{E}[\tilde{\rho}_{s-\delta}^i])(\bar{\rho}_s^{k,B} - \nu_s - \mathbb{E}[\tilde{\rho}_s^i]) \Big| \Big] ds
\end{aligned}$$

$$+\int_\delta^t \mathbb{E}[|\rho_s^{i,n}-\bar\rho_s^i|]ds+\int_0^t \mathbb{E}[|A_s^n-\bar A_s^n|]ds+\int_0^t \mathbb{E}\left[|\bar A_s^n-\mathbb{E}[\tilde\rho_s^i]-\nu_s|\right]ds,\quad \delta\le t. \tag{5.55}$$

By (5.9) it holds

$$\begin{aligned}
&\int_\delta^t \mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j=1,j\neq i}^n (f^P(\rho_{s-\delta}^{j,n}-A_{s-\delta}^n)(\rho_s^{j,n}-A_s^n)-f^P(\bar\rho_{s-\delta}^j-\bar A_{s-\delta}^n)(\bar\rho_s^j-\bar A_s^n))\Big|\Big]ds\\
&\le \frac{1}{n-1}\sum_{j=1,j\neq i}^n \int_\delta^t \mathbb{E}\Big[\Big|f^P(\rho_{s-\delta}^{j,n}-A_{s-\delta}^n)\left((\rho_s^{j,n}-A_s^n)+(\bar\rho_s^j-\bar A_s^n)\right)\Big|\Big]ds\\
&\quad+\frac{1}{n-1}\sum_{j=1,j\neq i}^n \int_\delta^t \mathbb{E}\Big[\Big|(\bar\rho_s^j-\bar A_s^n)\left(f^P(\rho_{s-\delta}^{j,n}-A_{s-\delta}^n)-f^P(\bar\rho_{s-\delta}^j-\bar A_{s-\delta}^n)\right)\Big|\Big]ds\\
&\le K_2\int_\delta^t \mathbb{E}[|\rho_s^{i,n}-\bar\rho_s^i|]ds+K_2\int_\delta^t \mathbb{E}[|A_s^n-\bar A_s^n|ds\\
&\quad+\int_\delta^t \mathbb{E}\left[\left|\bar\rho_s^i-\bar A_s^n\right|\left|f^P(\rho_{s-\delta}^{i,n}-A_{s-\delta}^n)-f^P(\bar\rho_{s-\delta}^i-\bar A_{s-\delta}^n)\right|\right]ds.
\end{aligned}\tag{5.56}$$

We have that for $\delta\le t$

$$\begin{aligned}
&\int_\delta^t \mathbb{E}\left[\left|\bar\rho_s^i-\bar A_s^n\right|\left|f^P(\rho_{s-\delta}^{i,n}-A_{s-\delta}^n)-f^P(\bar\rho_{s-\delta}^i-\bar A_{s-\delta}^n)\right|\right]ds\\
&\le\int_\delta^t \left(\mathbb{E}[|\bar\rho_s^i-\bar A_s^n|^2]ds\right)^{1/2}\left(\mathbb{E}\left[\left|f^P(\rho_{s-\delta}^{i,n}-A_s^n)-f^P(\bar\rho_{s-\delta}^i-\bar A_s^n)\right|^2\right]\right)^{1/2}ds\\
&\le\left(\int_\delta^t \mathbb{E}\left[\left|\bar\rho_s^i-\bar A_s^n\right|^2\right]ds\right)^{1/2}\left(\int_\delta^t \mathbb{E}\left[\left|f^P(\rho_{s-\delta}^{i,n}-A_{s-\delta}^n)-f^P(\bar\rho_{s-\delta}^i-\bar A_{s-\delta}^n)\right|^2\right]ds\right)^{1/2}\\
&\le\left(\int_\delta^t \mathbb{E}\left[\left|\bar\rho_s^i-\bar A_s^n\right|^2\right]ds\right)^{1/2}\left(\int_\delta^t \mathbb{E}\left[\left|f^P(\rho_{s-\delta}^{i,n}-A_{s-\delta}^n)^2-f^P(\bar\rho_{s-\delta}^i-\bar A_{s-\delta}^n)^2\right|\right]ds\right)^{1/2}\\
&\le\sqrt{2K_2}\left(\int_\delta^t \mathbb{E}\left[\left|\bar\rho_s^i-\bar A_s^n\right|^2\right]ds\right)^{1/2}\left(\int_\delta^t \mathbb{E}\left[\left|f^P(\rho_{s-\delta}^{i,n}-A_{s-\delta}^n)-f^P(\bar\rho_{s-\delta}^i-\bar A_{s-\delta}^n)\right|\right]ds\right)^{1/2}\\
&\le\sqrt{2K_1K_2}\left(\int_\delta^t \mathbb{E}\left[\left|\bar\rho_s^i-\bar A_s^n\right|^2\right]ds\right)^{1/2}\left(\int_\delta^t \mathbb{E}\left[\left|\rho_{s-\delta}^{i,n}-\bar\rho_{s-\delta}^i\right|+\left|A_{s-\delta}^n-\bar A_{s-\delta}^n\right|\right]ds\right)^{1/2},
\end{aligned}$$

where we have used that $|a-b|^2\le|a^2-b^2|$ for $a,b\in\mathbb{R}^+$.
Then, setting $G_1^n(t):=\left(\int_\delta^t \mathbb{E}\left[\left|\bar\rho_s^i-\bar A_s^n\right|^2\right]ds\right)^{1/2}$, by (5.56) we have

$$\begin{aligned}
&\int_\delta^t \mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j=1,j\neq i}^n (f^P(\rho_{s-\delta}^{j,n}-A_{s-\delta}^n)(\rho_s^{j,n}-A_s^n)-f^P(\bar\rho_{s-\delta}^j-\bar A_{s-\delta}^n)(\bar\rho_s^j-\bar A_s^n))\Big|\Big]ds\\
&\le K_2\int_\delta^t \mathbb{E}[|\rho_s^{i,n}-\bar\rho_s^i|]ds+K_2\int_\delta^t \mathbb{E}[|A_s^n-\bar A_s^n|ds\\
&\quad+\sqrt{2K_1K_2}G_1^n(t)\left(\int_\delta^t \mathbb{E}\left[\left|\rho_{s-\delta}^{i,n}-\bar\rho_{s-\delta}^i\right|+\left|A_{s-\delta}^n-\bar A_{s-\delta}^n\right|\right]ds\right)^{1/2},\quad \delta\le t.
\end{aligned}\tag{5.57}$$

Since

$$\int_\delta^t \mathbb{E}\left[\left|\rho_{s-\delta}^{i,n}-\bar\rho_{s-\delta}^{i}\right|+\left|A_{s-\delta}^{n}-\bar A_{s-\delta}^{n}\right|\right]ds=\mathbb{E}\left[\int_\delta^t\left(\left|\rho_{s-\delta}^{i,n}-\bar\rho_{s-\delta}^{i}\right|+\left|A_{s-\delta}^{n}-\bar A_{s-\delta}^{n}\right|\right)ds\right]$$
$$=\mathbb{E}\left[\int_0^{t-\delta}\left(\left|\rho_{u}^{i,n}-\bar\rho_{u}^{i}\right|+\left|A_{u}^{n}-\bar A_{u}^{n}\right|\right)du\right]\le\int_0^\delta\mathbb{E}[\left|\rho_{u}^{i,n}-\bar\rho_{u}^{i}\right|+\left|A_{u}^{n}-\bar A_{u}^{n}\right|]du,\quad \delta\le t<2\delta,$$

we can rewrite (5.57) as

$$\int_\delta^t\mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j=1,j\neq i}^{n}(f^P(\rho_{s-\delta}^{j,n}-A_{s-\delta}^{n})(\rho_s^{j,n}-A_s^n)-f^P(\bar\rho_{s-\delta}^{j}-\bar A_{s-\delta}^{n})(\bar\rho_s^{j}-\bar A_s^n))\Big|\Big]ds$$
$$\le K_2\int_\delta^t\mathbb{E}[|\rho_s^{i,n}-\bar\rho_s^i|]ds+K_2\int_\delta^t\mathbb{E}[|A_s^n-\bar A_s^n|ds$$
$$+\sqrt{2K_1K_2}G_1^n(t)\left(\int_0^\delta\mathbb{E}\left[\left|\rho_s^{i,n}-\bar\rho_s^i\right|+\left|A_s^n-\bar A_s^n\right|\right]ds\right)^{1/2},\quad \delta\le t. \quad (5.58)$$

Similarly,

$$\int_0^t\mathbb{E}\Big[\Big|\frac{1}{m}\sum_{k=1}^{m}\left(f^B(\rho_{s-\delta}^{k,B}-A_{s-\delta}^{n})(\rho_s^{k,B}-A_s^n)-f^B(\bar\rho_{s-\delta}^{k,B}-\bar A_{s-\delta}^{n})(\bar\rho_s^{k,B}-\bar A_s^n)\right)\Big|\Big]ds$$
$$\le K_2\int_\delta^t\mathbb{E}[|\rho_s^{k,B}-\bar\rho_s^{k,B}|]ds+K_2\int_\delta^t\mathbb{E}[|A_s^n-\bar A_s^n|ds$$
$$+\sqrt{2K_1K_2}G_2^n(t)\left(\int_0^\delta\mathbb{E}\left[\left|\rho_s^{k,B}-\bar\rho_s^{k,B}\right|+\left|A_s^n-\bar A_s^n\right|\right]ds\right)^{1/2},\quad \delta\le t. \quad (5.59)$$

with $G_2^n(t):=\left(\int_\delta^t\mathbb{E}\left[\left|\bar\rho_s^{k,B}-\bar A_s^n\right|^2\right]ds\right)^{1/2}$.

From (5.47), (5.54), (5.55), (5.58) and (5.59) we obtain

$$\mathbb{E}[|\rho^{i,n}-\bar\rho^i|_t^*]$$
$$\le(3K_2+2)\int_\delta^t\mathbb{E}\left[\left|\rho^{i,n}-\bar\rho^i\right|_s^*\right]ds+(3K_2+1)\int_\delta^t\mathbb{E}[|\rho^{k,B}-\bar\rho^{k,B}|_s^*]ds$$
$$+\sqrt{2K_1K_2}G_1^n(t)\left(\int_0^\delta\mathbb{E}\left[\left|\rho_s^{i,n}-\bar\rho_s^i\right|+\left|A_s^n-\bar A_s^n\right|\right]ds\right)^{1/2}$$
$$+\sqrt{2K_1K_2}G_2^n(t)\left(\int_0^\delta\mathbb{E}\left[\left|\rho_s^{k,B}-\bar\rho_s^{k,B}\right|+\left|A_s^n-\bar A_s^n\right|\right]ds\right)^{1/2}$$
$$+\int_0^t\mathbb{E}\left[\left|f^B(\bar\rho_{s-\delta}^{k,B}-\bar A_{s-\delta}^n)(\bar\rho_s^{k,B}-\bar A_s^n)-f^B(\bar\rho_{s-\delta}^{k,B}-\nu_{s-\delta}-\mathbb{E}[\tilde\rho_{s-\delta}^i])(\bar\rho_s^{k,B}-\nu_s-\mathbb{E}[\tilde\rho_s^i])\right|\right]ds$$
$$+\int_0^t\mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j=1,j\neq i}^{n}f^P(\bar\rho_{s-\delta}^{j}-\bar A_{s-\delta}^{n})(\bar\rho_s^{j}-\bar A_s^n)-\mathbb{E}\left[f^P\left(\tilde\rho_{s-\delta}^i-\mathbb{E}[\tilde\rho_{s-\delta}^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]\Big|\Big]ds$$
$$+\int_0^\delta\mathbb{E}[|\rho_s^{i,n}-\bar\rho_s^i|]ds+\int_0^t\mathbb{E}\left[|\bar A_s^n-\mathbb{E}[\tilde\rho_s^i]-\nu_s|\right]ds,\quad \delta\le t<2\delta. \quad (5.60)$$

At the same way, by (5.3) and (5.26) we have

$$\mathbb{E}[|\rho^{k,B}-\bar\rho^{k,B}|_t^*]$$
$$\le(3K_2+1)\int_\delta^t\mathbb{E}\left[\left|\rho^{i,n}-\bar\rho^i\right|_s^*\right]ds+(3K_2+2)\int_\delta^t\mathbb{E}[|\rho^{k,B}-\bar\rho^{k,B}|_s^*]ds$$

$$
\begin{aligned}
&+\sqrt{2K_1K_2}G_1^n(t)\left(\int_0^\delta \mathbb{E}\left[\left|\rho_s^{i,n}-\bar\rho_s^i\right|+\left|A_s^n-\bar A_s^n\right|\right]ds\right)^{1/2}\\
&+\sqrt{2K_1K_2}G_2^n(t)\left(\int_0^\delta \mathbb{E}\left[\left|\rho_s^{k,B}-\bar\rho_s^{k,B}\right|+\left|A_s^n-\bar A_s^n\right|\right]ds\right)^{1/2}\\
&+\int_0^t \mathbb{E}\Big[\Big|f^B(\bar\rho_{s-\delta}^{k,B}-\bar A_{s-\delta}^n)(\bar\rho_s^{k,B}-\bar A_s^n)-f^B(\bar\rho_{s-\delta}^{k,B}-\nu_{s-\delta}-\mathbb{E}[\tilde\rho_{s-\delta}^i])(\bar\rho_s^{k,B}-\nu_s-\mathbb{E}[\tilde\rho_s^i])\Big|\Big]ds\\
&+\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n}\sum_{i=1}^n f^P(\bar\rho_{s-\delta}^i-\bar A_{s-\delta}^n)(\bar\rho_s^i-\bar A_s^n)-\mathbb{E}\left[f^P\left(\tilde\rho_{s-\delta}^i-\mathbb{E}[\tilde\rho_{s-\delta}^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]\Big|\Big]ds\\
&+\int_0^\delta \mathbb{E}[|\rho_s^{k,B}-\bar\rho_s^{k,B}|]ds+\int_0^t \mathbb{E}\left[|\bar A_s^n-\nu_s-\mathbb{E}[\tilde\nu_s^i|]\right]ds,\quad \delta\le t<2\delta.
\end{aligned}
\tag{5.61}
$$

Summing up (5.60) and (5.61) we find

$$
\begin{aligned}
&\mathbb{E}[|\rho^{i,n}-\bar\rho^i|_t^*]+\mathbb{E}[|\rho^{k,B}-\bar\rho^{k,B}|_t^*]\\
&\le(6K_2+3)\int_0^t\left(\mathbb{E}[|\rho^{i,n}-\bar\rho^i|_s^*]+\mathbb{E}[|\rho^{k,B}-\bar\rho^{k,B}|_s^*]\right)ds\\
&+\sqrt{2K_1K_2}(G_1^n(t)+G_2^n(t))\left(\int_0^\delta\left(\mathbb{E}[|\rho_s^{i,n}-\bar\rho_s^i|]+\mathbb{E}[|\rho_s^{k,B}-\bar\rho_s^{k,B}|]+\mathbb{E}\left[\left|A_s^n-\bar A_s^n\right|\right]\right)ds\right)^{1/2}
\end{aligned}
\tag{5.62}
$$

$$
\begin{aligned}
&+\int_0^t \mathbb{E}\Big[\Big|f^B(\bar\rho_{s-\delta}^{k,B}-\bar A_{s-\delta}^n)(\bar\rho_s^{k,B}-\bar A_s^n)-f^B(\bar\rho_{s-\delta}^{k,B}-\nu_{s-\delta}-\mathbb{E}[\tilde\rho_{s-\delta}^i])(\bar\rho_s^{k,B}-\nu_s-\mathbb{E}[\tilde\rho_s^i])\Big|\Big]ds\\
&+\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n-1}\sum_{j=1,j\neq i}^n f^P(\bar\rho_{s-\delta}^j-\bar A_{s-\delta}^n)(\bar\rho_s^j-\bar A_s^n)-\mathbb{E}\left[f^P\left(\tilde\rho_{s-\delta}^i-\mathbb{E}[\tilde\rho_{s-\delta}^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]\Big|\Big]ds\\
&+\int_0^t \mathbb{E}\Big[\Big|\frac{1}{n}\sum_{i=1}^n f^P(\bar\rho_{s-\delta}^i-\bar A_{s-\delta}^n)(\bar\rho_s^i-\bar A_s^n)-\mathbb{E}\left[f^P\left(\tilde\rho_{s-\delta}^i-\mathbb{E}[\tilde\rho_{s-\delta}^i]\right)\left(\tilde\rho_s^i-\mathbb{E}[\tilde\rho_s^i]\right)\right]\Big|\Big]ds\\
&+2\int_0^t \mathbb{E}\left[|\bar A_s^n-\nu_s-\mathbb{E}[\tilde\nu_s^i|]\right]ds,\quad \delta\le t<2\delta.
\end{aligned}
\tag{5.63}
$$

With the same computations used in the first step of the proof, we show that the last four terms of (5.63) converge to zero when $n\to\infty$ by the proof of Proposition 5.3.3. The term in (5.62) also goes to zero when $n\to\infty$, by the first step of the proof and because $\lim_{n\to\infty}[G_1^n(t)+G_2^n(t)]<\infty$, by (5.42). Then applying Gronwall's Lemma to (5.63) we prove the result for $t\in[\delta,2\delta)$. The result then follows by proceeding in the same way for all the steps $t\in[k\delta,(k+1)\delta)$, $k\ge 2$. □

## 5.4 Numerical analysis

We now study by numerical simulations how the system described in Section 5.3 reacts to the growth and the burst of a bubble. In particular, we investigate how a bank not holding the bubbly asset can be affected by a bubble burst through contagion mechanisms. We first consider the case of (5.2)-(5.3), i.e. of a network with a finite number of banks, and then we analyze the limit system (5.24)-(5.26).

The bubble has the dynamics specified in Chapter 3, i.e. it solves (5.5) with

$$
\mu_t=M_t\Lambda_t(-k\beta_t+2\bar\mu_t),\qquad \sigma_t=2\bar\sigma M_t\Lambda_t,\quad t\ge 0,
$$

where $M = (M_t)_{t\in[0,T]}$, $\Lambda = (\Lambda_t)_{t\in[0,T]}$ are respectively a measure of illiquidity and the so called *resiliency*, $\bar{\mu} = (\bar{\mu}_t)_{t\geq 0}$ is the drift of the signed volume of market orders (buy market orders minus sell market orders) and $\bar{\sigma} > 0$. Here, the illiquidity $M$ is supposed to be a geometric Brownian motion, i.e.

$$dM_t = M_t(\mu^M dt + \sigma^M dB_t^3), \quad t \geq 0.$$

with $\mu^M \in \mathbb{R}$ and $\sigma^M > 0$. We choose the same function $f$ for both core and periphery banks in (5.2)-(5.3), i.e. $f^B = f^P = f$. In particular, we take $f(x) = 1 + 2\arctan(x)/\pi$, as in Example 5.2.2.

### 5.4.1 Risk analysis for the finite case

We first focus on the system (5.2)-(5.3). We investigate how the first bank reacts when banks holding the bubble are in trouble. Specifically, we here introduce and compute the risk measure

$$Risk_\alpha^i = -\sup_{x\in\mathbb{R}}\left\{\left[\frac{1}{N_s}\sum_{k=1}^{N_s}\mathbb{1}_{\left\{(\rho^{i,n,k}_{\tau_k+\Delta}-\rho^{i,n,k}_{\tau_k})/\rho^{i,n,k}_{\tau_k}\leq x\right\}}\right]\leq\alpha\right\}, \tag{5.64}$$

with $\alpha > 0$, where $N_s$ is the number of simulations of the processes in (5.2)-(5.3), $\tau_k$ is the value at the $k$-th simulation of the bursting time $\tau$ of the bubble, and $\rho_t^{i,n,k}$ is the value of $\rho_t^{i,n}$ computed in the $k$-th simulation.
The risk measure $Risk_\alpha^i$ as defined in (5.64) is analogous to the CoVar of a bank without the bubble with respect to a bank with the bubble (for a definition of CoVar see e.g. Biagini et al. [2017] and Brunnermeier and Oehmke [2013]). Note that, since the banks not holding the bubble are identically distributed, we only compute the risk for one bank.
From now on, we set $\alpha = 0.05$ in (5.64). We perform $N_s = 10000$ simulations of $Risk_{0.05}^1$ in the case when there are $n = 6$ banks not holding the bubble and $m = 2$ banks holding it. We consider different values of $\lambda$ and of the delay $\delta$. The results are given in Table 5.1:

| | $\delta = 0$ | $\delta = 0.025$ | $\delta = 0.05$ | $\delta = 0.075$ | $\delta = 0.1$ | $\delta = 0.2$ | $\delta = 0.3$ |
|---|---|---|---|---|---|---|---|
| $\lambda = 0.5$ | 0.283 | 0.390 | 0.451 | 0.716 | 0.925 | 0.916 | 0.901 |
| $\lambda = 1$ | 0.281 | 0.385 | 0.434 | 0.661 | 0.886 | 0.879 | 0.875 |
| $\lambda = 2$ | 0.280 | 0.377 | 0.422 | 0.641 | 0.851 | 0.824 | 0.819 |

Table 5.1: $Risk_{0.05}^1$ in the case when the robustness is given by (5.2)-(5.3), with parameters $\sigma_1 = \sigma_2 = 0.2$, $\Delta = 0.1$, $\rho_0^{i,6} = \rho_0^{k,B} = 0.5$, $i = 1,\dots,6$, $k = 1, 2$.

As expected, the risk is bigger for large delays, since a large delay means that the banks without the bubble are not able to quickly disinvest, when other institutions holding the bubble are in trouble. However, for delays larger than 0.1, the risk is still big but it decreases. This depends on the fact that we check the robustness of the banks at time $\tau + 0.1$: at this time, when $\delta = 0.2, 0.3$, $f$ is smaller than in the case $\delta = 0.1$ because banks are cross investing on each other according to a value of the robustness, which is realized much before the bubble's burst.

Moreover, the risk is decreasing with $\lambda$. Indeed, it follows by (5.2) that $\rho^{i,n}$ reverts to

$$A_t^n+\frac{1}{\lambda}\left(\frac{1}{n}\sum_{i=1}^{n} f(\rho_{t-\delta}^{i,n}-A_{t-\delta}^n)(\rho_t^{i,n}-A_t^n)+\frac{1}{m-1}\sum_{\ell=1,\ell\neq k}^{m} f(\rho_{t-\delta}^{\ell,B}-A_{t-\delta}^n)(\rho_t^{\ell,n}-A_t^n)\right),$$

so that for large $\lambda$ the term involving the network, and then the direct effects of the banks holding the bubbly asset, is less significative.

We now consider (5.2)-(5.3) when $\beta$ is replaced by $\bar{\beta}$, where

$$d\bar{\beta}_t=\begin{cases}0 \quad \text{for } t\leq\tau,\\ \frac{\rho_\tau^{1,\bar{\beta}}}{\rho_\tau^{1,\beta}}d\beta_t \quad \text{for } t\geq\tau,\end{cases} \tag{5.65}$$

where $\rho^{1,\beta}$ is the robustness of bank 1 when there is a bubble in the network, and $\rho^{1,\bar{\beta}}$ is the robustness of bank 1 when there is no bubble. In this way we model the case when the banks that used to hold the bubbly asset are subject at time $\tau$ to the same (relative) shock, but without having experienced the growth of the bubble. The results are given in Table 5.2, for the same parameters as in Table 5.1.

| | $\delta=0$ | $\delta=0.025$ | $\delta=0.05$ | $\delta=0.075$ | $\delta=0.1$ | $\delta=0.2$ | $\delta=0.3$ |
|---|---|---|---|---|---|---|---|
| $\lambda=0.5$ | 0.281 | 0.383 | 0.388 | 0.415 | 0.505 | 0.499 | 0.494 |
| $\lambda=1$ | 0.280 | 0.381 | 0.385 | 0.403 | 0.502 | 0.494 | 0.492 |
| $\lambda=2$ | 0.280 | 0.371 | 0.380 | 0.399 | 0.500 | 0.490 | 0.489 |

Table 5.2: $Risk_{0.05}^1$ in the case when the robustness is given by (5.2)-(5.3) with no bubble in the system, but with the same shock at time $\tau$, for parameters $\sigma_1=\sigma_2=0.2$, $\Delta=0.1$, $\rho_0^{i,6}=\rho_0^{k,B}=0.5$, $i=1,\dots,6$, $k=1,2$.

We note that for $\delta=0$ there is not any significant difference with the case when there is a bubble in the system, since the banks are able to disinvest immediately at the time when the shock hits the banks with the bubble. Anyway, this difference increases with the delay. When the delay is big, the banks with no bubble are much more in trouble in the first case, i.e when they are attached to banks holding the bubbly asset.

We can then conclude that the increase of the value of the bubbly asset can put the network in trouble, because it makes the system more centralized on the riskier banks, due to the preferential attachment mechanism implied by (5.2)-(5.3).

This can also be seen by considering a static network, i.e. by taking $f^B=f^P=1$ in (5.2)-(5.3). In this case, we obtain the following values of the risk for different values of $\lambda$:

| $\lambda=0.5$ | $\lambda=1$ | $\lambda=2$ |
|---|---|---|
| 0.670 | 0.626 | 0.599 |

Table 5.3: $Risk_{0.05}^1$ with $\Delta=0.1$ in the case of a static network, with $f^B=f^P=1$ and with parameters $\sigma_1=\sigma_2=0.2$, $\Delta=0.1$, $\rho_0^{i,6}=\rho_0^{k,B}=0.5$, $i=1,\dots,6$, $k=1,2$..

Note that in this case the delay plays no role since it only affects the dynamics through $f^B$ and $f^P$. Comparing this result with Table 5.1, one can see that when $\delta$ in (5.2)-(5.3) is small, then the fact that banks are able to quickly disinvest makes the system safer than in the case of a static network. On the other hand, for big values of $\delta$, a centralized network towards the banks holding the bubble and the impossibility to disinvest quickly after the burst give rise to a more dangerous system than in the static case.

### 5.4.2 Risk analysis for the mean field limit

We now consider the case of the limit system (5.24)-(5.26). We compute

$$Risk^1_{0.05} = -\sup_{x\in\mathbb{R}}\left\{\left[\frac{1}{N_s}\sum_{k=1}^{N_s}\mathbb{1}_{\left\{(\bar\rho^{1,k}_{\tau_k+\Delta}-\bar\rho^{1,k}_{\tau_k})/\bar\rho^{1,k}_{\tau_k}\le x\right\}}\right]\le 0.05\right\}, \tag{5.66}$$

where $N_s$ and $\tau_k$ are the number of simulations and the time of the burst of the bubble in the $k$-th simulation, respectively, and $\bar\rho^{1,k}_t$ is the value of $\bar\rho^1_t$ computed in the $k$-th simulation.
As before, we consider $m = 2$ banks holding the bubble and we make $N_s = 10000$ simulations of (5.24)-(5.26) taking different values of $\lambda$ and $\delta$.
We compute $\phi(t, t-\delta) = \mathbb{E}\left[f\left(\tilde\rho^i_{t-\delta} - \mathbb{E}[\tilde\rho^i_{t-\delta}]\right)\left(\tilde\rho^i_t - \mathbb{E}[\tilde\rho^i_t]\right)\right]$ in (5.25) and (5.26) via Monte Carlo simulations of the trajectories of the Ornstein-Uhlenbeck process in (5.24). Note that $\mathbb{E}[\tilde\rho^i_t] = \rho_0 e^{-\lambda t}$. The results are gathered in Table 5.4.

| | $\delta=0$ | $\delta=0.025$ | $\delta=0.05$ | $\delta=0.075$ | $\delta=0.1$ | $\delta=0.2$ | $\delta=0.3$ |
|---|---|---|---|---|---|---|---|
| $\lambda=0.5$ | 0.305 | 0.367 | 0.563 | 0.908 | 1.281 | 1.251 | 1.226 |
| $\lambda=1$ | 0.302 | 0.360 | 0.521 | 0.765 | 1.170 | 1.125 | 1.117 |
| $\lambda=2$ | 0.302 | 0.356 | 0.503 | 0.647 | 0.908 | 0.907 | 0.877 |

Table 5.4: $Risk^1_{0.05}$ with $\Delta = 0.1$ of the mean field limit (5.24)-(5.26), with parameters $\sigma_1 = \sigma_2 = 0.2$, $\rho_0^{k,B} = 0.5$, $k = 1, 2$.

As before, the risk is increasing with the delay until $\delta = 0.1$ and decreasing with $\lambda$, since $\bar\rho^i_t$ reverts to

$$\frac{1}{\lambda}\left(\varphi(t,t-\delta) + \frac{1}{m}\sum_{k=1}^{m} f\left(\bar\rho^{k,B}_{t-\delta} - \nu_{t-\delta} - \mathbb{E}[\tilde\rho^i_{t-\delta}]\right)\left(\bar\rho^{k,B}_t - \nu_t - \mathbb{E}[\tilde\rho^i_t]\right)\right) + \mathbb{E}[\tilde\rho^i_t] - \tilde\rho^i_t,$$

so that a large $\lambda$ diminishes the influence of the banks holding the bubbly asset.
We can also see that the risk is bigger at the limit by comparing (5.2) and (5.29): since $\nu_{t-\delta} + \mathbb{E}[\tilde\rho^i_t] < A^n_{t-\delta}$, because the first term is the average robustness of banks not holding the bubble, the argument of $f$ is bigger in (5.29). This leads to a bigger weight multiplying the loss at the moment of the burst at the limit.
In Table 5.5, we report the results for the case when $\beta$ is replaced by $\bar\beta$ as in (5.65), i.e. when there is no bubble in the network.
As before, it can be seen that, when the delay is large enough, the preferential

| | $\delta = 0$ | $\delta = 0.025$ | $\delta = 0.05$ | $\delta = 0.075$ | $\delta = 0.1$ | $\delta = 0.2$ | $\delta = 0.3$ |
|---|---|---|---|---|---|---|---|
| $\lambda = 0.5$ | 0.303 | 0.355 | 0.468 | 0.682 | 0.698 | 0.659 | 0.645 |
| $\lambda = 1$ | 0.302 | 0.347 | 0.410 | 0.528 | 0.640 | 0.628 | 0.627 |
| $\lambda = 2$ | 0.300 | 0.340 | 0.395 | 0.455 | 0.612 | 0.561 | 0.550 |

Table 5.5: $Risk^1_{0.05}$ with $\Delta = 0.1$ in the mean field limit (5.24)-(5.26) with no bubble, with parameters $\sigma_1 = \sigma_2 = 0.2$, $\rho_0^{k,B} = 0.5$, $k = 1, 2$.

attachment mechanism, that takes place during the ascending phase of the bubble, creates a network more exposed to systemic risk at the time of the shock. If we consider a a static network, with $f^B = f^P = 1$, the results, shown in Table 5.6, agree with the ones obtained in the case of the finite network: for small delays the dynamic network is less exposed to systemic risk with respect to the static one, whereas when the delay increases and the banks in the dynamic network are slower in disinvesting, the risk is bigger than for the static network.

| $\lambda = 0.5$ | $\lambda = 1$ | $\lambda = 2$ |
|---|---|---|
| 1.001 | 0.910 | 0.866 |

Table 5.6: $Risk^1_{0.05}$ with $\Delta = 0.1$ in the case of a static network with $f^B = f^P = 1$ in the mean field limit, with parameters $\sigma_1 = \sigma_2 = 0.2$, $\Delta = 0.1$, $\rho_0^{k,B} = 0.5$, $k = 1, 2$.

# Appendix A

# Existence and uniqueness theorems

For the reader's convenience we report here the results, which we have used here to prove existence and uniqueness of a strong solution of a system of stochastic differential equations (SDEs) and of stochastic differential delay equations (SDDEs). These theorems also guarantee the finiteness of the second moments of the strong solution.
In the following, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, and $B_t = (B_t^1, \dots, B_t^m)_{t\geq 0}$, be an $m$-dimensional $\mathbb{F}$-Brownian motion defined on $(\Omega, \mathcal{F}, P)$.
We begin by the following existence and uniqueness result for a system of SDEs, given in Theorem 9.11 in Pascucci [2011].

**Theorem A.0.1.** *Let $X_0$ be an $\mathcal{F}_{t_0}$-measurable $\mathbb{R}^d$-valued random variable such that $\mathbb{E}[X_0^2] < \infty$. Consider the $d$-dimensional stochastic differential equation of Itô type*

$$dX_t = f(t, X_t)dt + g(t, X_t)dB_t, \quad t_0 \leq t \leq T, \tag{A.1}$$

*with $X_{t_0} = X_0$, where $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are both Borel measurable.*
*Assume that there there exist two positive constants $K_1$ and $K_2$ such that:*

1. *(Lipschitz condition) for all $x, y \in \mathbb{R}^d$ and $t \in [t_0, T]$,*

$$\|f(t,x) - f(t,y)\|^2 + \|g(t,x) - g(t,y)\|^2 \leq K_1\|x - y\|^2;$$

2. *(Linear growth condition) for all $(t, x) \in [t_0, T] \times \mathbb{R}^d$,*

$$\|f(t,x)\|^2 + \|g(t,x)\|^2 \leq K_2(1 + \|x\|^2).$$

*Then there exists a unique solution $X = (X_t)_{x\in[t_0,T]}$ to equation (A.1) and it holds*

$$\mathbb{E}\left[\sup_{t_0\leq s\leq t} \|X_s\|^2\right] \leq C(1 + \mathbb{E}\left[\|X_0\|^2\right])e^{Ct}, \quad t \in [t_0, T], \tag{A.2}$$

*where $C$ is a constant depending on $K_2$ and $T$ only.*

We now recall Theorem 3.1 in Mao [2007, chapter 5], that provides the existence and uniqueness results for SDDEs.

**Theorem A.0.2.** *Let $F : [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $G : [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Borel-measurable. Consider the delay equation*

$$dX_t = F(t, X_t, X_{t-\tau})dt + G(t, X_t, X_{t-\tau})dB_t, \tag{A.3}$$

*with initial data $\{X_s : t_0 - \tau \leq s \leq t_0\}$, such that $X_s$ is $\mathcal{F}_{t_0}$-measurable for all $s \in [t_0 - \tau, t_0]$ and $\mathbb{E}[\|X_s\|^2] < \infty$ for all $s \in [t_0 - \tau, t_0]$.*
*Assume that there exists two positive constants $\tilde{K_1}$ and $\tilde{K_2}$ such that*

1. *(Linear growth condition) for all $(t, x, y) \in [t_0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\|F(t, x, y)\|^2 + \|G(t, x, y)\|^2 \leq \tilde{K_1}(1 + \|x\|^2 + \|y\|^2);$$

2. *(Lipschitz condition on $x$) for all $t \in [t_0, T]$, $y \in \mathbb{R}^d$ and $x, \bar{x} \in \mathbb{R}^d$,*

$$\|F(t, x, y) - F(t, \bar{x}, y)\|^2 + \|G(t, x, y) - G(t, \bar{x}, y)\|^2 \leq \tilde{K_2}\|x - \bar{x}\|^2.$$

*Then there exists a unique solution $X = (X_t)_{t\in[t_0,T]}$ to equation (A.3).*

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