Gran Sasso Science Institute

MATHEMATICS OF NATURAL, SOCIAL AND LIFE SCIENCES
DOCTORAL PROGRAMME

Cycle XXXI - AY 2015/2018

Flows of irregular vector fields and applications to Transport and Euler equations

PHD CANDIDATE

Gennaro Ciampa

PhD Thesis Submitted
July 3, 2019

ADVISORS

Prof. Dr. Gianluca Crippa
Universität Basel

Dr. Stefano Spirito
Università dell'Aquila

GRAN SASSO SCIENCE INSTITUTE
SCHOOL OF ADVANCED STUDIES
Scuola Universitaria Superiore

ma per seguir virtute e canoscenza
SISSA
Scuola Internazionale Superiore di Studi Avanzati

GRAN SASSO SCIENCE INSTITUTE
SCHOOL OF ADVANCED STUDIES
Scuola Universitaria Superiore

SISSA
Scuola Internazionale Superiore di Studi Avanzati

Flows of irregular vector fields and applications to Transport and Euler equations

PHD CANDIDATE
Gennaro Ciampa

ADVISORS
Prof. Dr. Gianluca Crippa
Dr. Stefano Spirito

PhD Thesis Submitted
July 3, 2019

Gran Sasso Science Institute

MATHEMATICS OF NATURAL, SOCIAL AND LIFE SCIENCES DOCTORAL PROGRAMME

Cycle XXXI - AY 2015/2018

Ringraziamenti

Questa tesi rappresenta la fine di un percorso ricco di gioie e soddisfazioni ed è per questo che mi sento in dovere di ringraziare diverse persone.

I primi a cui rivolgo i miei più sinceri ringraziamenti sono i miei advisor, Gianluca e Stefano, per avermi seguito durante questi anni, facendomi crescere come matematico e per avermi introdotto in un'area di ricerca tanto interessante quanto stimolante.

Ringrazio il Prof. Piero Marcati: sono orgoglioso di essere stato uno studente del GSSI. Credo che da nessun'altra parte avrei avuto l'opportunità di imparare e viaggiare tanto.

A L'Aquila ho potuto conoscere e stringere legami importanti con tante persone. Tra questi voglio ringraziare in particolare Ahmed, Anna Paola, Cosimo, Luigi, Michele, Paolo, Raffaele Folino e Raffaele Scandone. Ma la mia più profonda gratitudine va a Lars e Roberto, colleghi coinquilini e amici. Grazie a tutti voi per i momenti che abbiamo condiviso! Mi mancheranno un sacco le serate al Boss seguite dai Fernettoni e le acquette al Pantasima.

Pur facendo parte dell'esperienza aquilana, sento di dover ringraziare separatamente Antonio Esposito: con lui ho avuto il piacere di condividere l'esperienza universitaria e di dottorato. Questi anni non sarebbero stati gli stessi senza di te.

Vorrei poi ringraziare Renato Lucà, punto di riferimento nei miei mesi a Basilea. È un onore ereditare il tuo posto in ufficio, bomber!

Ringrazio gli amici di Montefalcione per avermi sempre fatto evadere da quelle che potevano essere le mie ansie e le mie preoccupazioni. In particolare Giuseppe, Ivan e Raul, sempre presenti nel momento del bisogno. Siete troppi per nominarvi tutti ma per voi, ragazzi, non ci sono parole.

Un ringraziamento speciale va poi a Gaetano per il suo continuo incoraggiamento e per gli ottimi consigli che ha saputo darmi negli anni. Metà mentore e metà fratello maggiore, non ti sarò mai grato abbastanza. Ricorda che però avanzi sempre due schiaffi!

Grazie a Stefania, la persona che più mi ha dovuto sopportare nei momenti negativi del dottorato, dandomi serenità credendo sempre in me. Ti auguro il meglio.

Infine vorrei ringraziare i miei genitori per la loro continua fiducia e per la libertà che da sempre mi hanno lasciato in tutte le mie decisioni. Spero di poterli ripagare nel migliore dei modi un giorno.

Concludo con un pensiero per nonno Peppo e nonno Gennaro. Avrei voluto condividere questo momento con loro, sono sempre stati per me il simbolo del lavoro e del sacrificio. Sono sicuro che sarebbero stati orgogliosi di me e spero che possano festeggiare insieme, ovunque essi siano.

Contents

Introduction

The transport equation

In many models coming from Mathematical Physics there are evolutionary phenomena described by the *transport equation*, which is the following PDE

$$\begin{cases} \partial_t u + b \cdot \nabla u = 0, \\ u(0, \cdot) = u_0, \end{cases} \tag{TE}$$

where $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is the unknown, $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a given vector field, and $u_0 : \mathbb{R}^d \to \mathbb{R}$ is a given initial configuration. In most of the applications, u can be thought as a density of particles which is advected by the velocity field b. The transport equation is strictly connected for both theoretical and physical reasons to the following system of ODE

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X(t,x) = b(t, X(t,x)), \\ X(0,x) = x, \end{cases} \tag{ODE}$$

where $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the so-called *flow* of b and it represents the position at time t of a particle which moves accordingly to b starting from a point x. The Cauchy-Lipschitz theory gives existence and uniqueness of smooth solutions of (ODE), provided the vector field is Lipschitz in space uniformly in time. Moreover, a well-posedness theory for smooth solutions of the transport equation can be proved by exploiting the connection between (ODE) and (TE) given by the theory of characteristics. In fact, a solution u of (TE) is given by the formula

$$u(t,x) = u_0(X^{-1}(t, \cdot)(x)), \tag{0.0.1}$$

where $X^{-1}(t, \cdot)$ denotes the inverse map in space at a fixed time t, and a simple estimate of the difference of two possible solutions of (TE) starting with the same initial datum implies that such u is also the unique solution of (TE). Mainly due to the applications to other kind of PDEs (fluid dynamics, conservation laws, Vlasov-Poisson,...) the setting of smooth vector fields is too restrictive and a theory in weaker regularity settings has been developed in the last decades. While the existence of weak solutions is obtained by standard approximation arguments requiring only integrability hypothesis on b, the uniqueness is much more difficult and requires additional regularity assumptions on the vector field. In this direction the first result is due to DiPerna and Lions in [29] where they proved the uniqueness of distributional solutions of (TE) with Sobolev vector field with bounded divergence; later on this result was improved by Ambrosio [6] to the case of BV vector fields with bounded divergence. More recently, Bianchini and Bonicatto in [9] have shown uniqueness in the case of a nearly incompressible BV vector field, without assumptions on the divergence, giving a positive answer to the Bressan's compactness conjecture, see [14]. In [29, 6] the key point is a weak version of the chain rule for distributional solutions of (TE) which is encoded in the definition of *renormalized* solutions: loosely speaking, a solution u is renormalized if many other non-linear functions of u satisfy the transport equation. This is a strong property and in general implies uniqueness and stability. The proof of the renormalization property in [29, 6] is based on a regularization scheme of the equation and the regularity of the vector field comes into play in order to get the convergence to zero of an error term. Moreover, the well-posedness of the transport equation transfers to (ODE) by exploiting the connection between them. More recently, a purely Lagrangian approach for the well-posedness of (ODE) was developed in [21]. With a suitable notation of flow and by showing some *a priori* estimates, the authors are able to prove the well-posedness of (ODE) under Sobolev regularity of the vector field. This approach was then generalized in [13] to vector fields having derivatives given by singular integrals of L^1 functions which is an interesting class because of the applications to the 2D Euler equations.

The two-dimensional Euler equations

The motion of an incompressible inviscid two-dimensional fluid is modeled by the equations

$$\begin{cases} \partial_t v + (v \cdot \nabla)\, v + \nabla p = 0, \\ \operatorname{div} v = 0, \\ v(0,\cdot) = v_0, \end{cases} \tag{IE}$$

where $v_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is a given initial divergence-free velocity field and the unknowns are the velocity field $v : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ and the scalar pressure $p : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$. The incompressibility of the fluid is encoded in the divergence-free condition, while the first equation is the conservation of the momentum. In two dimensions the vorticity $\omega := \partial_{x_1} v_2 - \partial_{x_2} v_1$ plays a special role since it is a scalar and satisfies the transport equation

$$\partial_t \omega + v \cdot \nabla \omega = 0. \tag{0.0.2}$$

The non-linearity (and the non-locality) of the equations (IE) are in the coupling between the velocity and the vorticity given by the Biot-Savart law

$$v = K * \omega, \qquad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}.$$

For smooth initial data existence and uniqueness of classical solutions of (IE) was proved first locally in time in [31] and then globally in [44]. In their seminal paper [30], DiPerna and Majda addressed the problem of the existence of (IE) under the assumption of vortex-sheet initial vorticity, that is $\omega_0 \in \mathcal{M} \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$. Precisely, they gave the definition of an approximate solution sequence of the two-dimensional incompressible Euler equations and they showed that this kind of approximate solutions converge to measure-valued solutions. Moreover, they gave three different examples of approximation methods that satisfy their definition:

(ES) Approximation by exact smooth solutions of (IE);

(VV) Vanishing viscosity from the two-dimensional Navier-Stokes equations;

(VB) Vortex blob approximation.

Moreover, for initial vorticities $\omega_0 \in L^1 \cap L^p(\mathbb{R}^2)$ with $1 < p \leq \infty$ they proved global existence of weak solutions of (IE) obtained trough the methods (ES) and (VV), while for weak solutions constructed by (VB) the same result was obtained by Beale in [10]. In these existence results the fact that ω_0 is more integrable than L^1 allows to use Sobolev embeddings which guarantee the strong convergence in L^2 of an approximate solution sequence. However, in the case ω_0 is in L^1 or it is a measure with distinguished sign, it happens that concentrations may occur for sequences whose limit satisfies the weak formulation of the equations. This is a purely 2D phenomenon known as *concentration-cancellation*, and it was studied in [27, 43]. The uniqueness of weak solutions in the class considered in [30] is still an open problem, contrary to the case $p = \infty$ which has been proved by Yudovič [45].

In this work we give particular attention to two physical properties of the solutions of (IE). The first one is the *Lagrangian* property, which means that the vorticity is transported by the flow of the velocity; the second one is the conservation of the *kinetic energy*, which mathematically traduces in the conservation of the L^2 norm of the solution. These properties are naturally satisfied by smooth solutions, while under low integrability assumptions on the vorticity they (in principle) strongly depend from the approximation scheme that produce the solutions.

Concerning the Lagrangian property, in [33] it has been observed that when $\omega_0 \in L^p(\mathbb{R}^2)$, with $p \geq 2$, any weak solution of the Euler equations in vorticity form is renormalized in the sense of DiPerna and Lions and admits a representation formula in terms of the flow of the velocity. Moreover, when $\omega_0 \in L^p(\mathbb{R}^2)$ with $1 < p < 2$, all solutions obtained as limit of (ES) are Lagrangian as a consequence of the stability theorem in [29]. The case of weak solutions produced by (ES) with L^1-initial vorticity is considered in [12].

Regarding the vanishing viscosity limit, in [25] it has been proved that solutions $\omega_0 \in L^p(\mathbb{R}^2)$ obtained via (VV) are Lagrangian if $1 < p < 2$, while the case $p = 1$ is considered in [23]. Note that the Lagrangian property is non-trivial even at the linear level for the transport equation. In fact, in [38, 39, 37] the authors show via convex-integration techniques that there exist solutions of the linear transport equation which are not Lagrangian, if the integrability of ∇b and of u are much below the threshold provided by the DiPerna-Lions' theory. In particular,

for the 2D Euler equations we are in the situation described in [37] when we assume low integrability conditions on the initial vorticity, namely $\omega_0 \in L^p(\mathbb{R}^2)$ with $1 \le p < 4/3$.
Regarding the conservation of the kinetic energy, in [16] the authors consider the 2D Euler equations on the two-dimensional flat torus $\mathbb{T}^2$ and prove that *all* weak solutions satisfy the energy conservation if the vorticity $\omega \in L^\infty((0,T); L^p(\mathbb{T}^2))$ with $p \ge 3/2$. The proof is based on a mollification argument and the exponent $p = 3/2$ is required in order to have weak continuity of a commutator term in the energy balance. The authors also give an example of the sharpness of the exponent $p = 3/2$ in their argument. Moreover, they show that if $\omega \in L^\infty((0,T); L^p(\mathbb{T}^2))$, with $1 < p < 3/2$, solutions constructed by (ES) and (VV) conserve the energy.

Overview of our results

In this work we collect some recent results about the theory of flows of rough vector field with particular interest in the applications to the linear transport equation and the two-dimensional Euler equations. The main original results of the thesis are two and they are presented in Chapter 2 and 4.
In Chapter 2 we analyze the selection problem for weak solutions of the transport equation with rough vector fields. We are interested in possible selections criteria for a unique distributional solutions of the transport equation. For example, it may happen that the selection is given by the choice of the approximation scheme. In particular, in [18] we answer in the negative the question whether solutions of the equation with a regularized vector field converge to a unique limit. To this aim, we give a new example of a vector field which admits infinitely many flows. Then we construct a smooth approximating sequence of the vector field for which the corresponding solutions have sub-sequences converging to different solutions of the limit equation. Moreover, we give some heuristics and ideas on the selection problem for a sub-class of flows as done in [17].
In Chapter 4 we consider weak solutions of the two-dimensional Euler equations and we study the validity the Lagrangian property and the conservation of the energy. In particular, in [19] we study weak solutions obtained by the vortex-blob approximation (VB): this approximation method is the prototype of several important numerical schemes and is based on the idea of approximating the vorticity with a finite number of cores which evolve according to the velocity of the fluid. In order to prove the Lagrangian property, we first prove in Chapter 3 that the sequence of approximate vorticity constructed via (VB) is equi-integrable. Then, exploiting the stability theorems for Lagrangian solutions of the linear transport equation contained in [13, 21], we use an estimate on the L^p distance between the approximate vorticity obtained by vortex-blob approximation and the solution of a linear transport equation where the advecting term is the approximate velocity field obtained by the vortex-blob approximation. In particular, the equi-integrability of the approximate vorticity will also allow us to improve the existence result of Beale in [10] to the case of initial vorticity $\omega_0 \in L^1 \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$. To prove the

conservation of the energy we use a modified version of the *Serfati identity* [5, 41] to get the global convergence in L^2 of the approximate velocity together with a precise blow-up estimate for the velocity. We also prove a local balance of the energy when $\omega_0 \in L^p(\mathbb{R}^2)$ with $p \geq 6/5$. These results in a certain sense completes the picture of the canonical approximations studied by DiPerna and Majda in [30].

Plan of the thesis

The thesis is divided as follows

- In Chapter 1 we start by recalling the theory of smooth solutions of the transport equation. Then, we present the DiPerna-Lions-Ambrosio's theory of renormalized solutions and the Lagrangian approach derived by the a priori estimates on the flow. Finally, we list several counter-examples to the uniqueness in the case of less regular vector fields.

- In Chapter 2 we discuss the selection problem of distributional solutions of the transport equation. We give the motivations of the problem and we prove, by constructing an explicit example, the non selection of solutions obtained as limit of exact smooth solutions of an approximating system. Moreover, we give some heuristics ideas on what can happen if we want to select a specific subclass of solutions.

- In Chapter 3 we recall some classical results about the existence and uniqueness of solutions of the 2D Euler equations. Moreover, we prove the existence of weak solutions by using three different approximation methods studied in [30] under several integrability assumptions on the initial vorticity.

- In Chapter 4 we deal with the Lagrangian property and the conservation of the energy for weak solutions of the two dimensional Euler equations. In particular, given an initial vorticity in $L^p(\mathbb{R}^2)$, we prove that the three approximation methods considered in [30] produce in the limit Lagrangian solution if $p \geq 1$ and these solutions are also conservative if $p > 1$.

Notations

We will denote by $L^p(\mathbb{R}^d)$ the standard Lebesgue spaces and with $\|\cdot\|_{L^p}$ their norm. We will use the notation $\|\cdot\|_{L^p(A)}$ when the norm is computed on a subset $A \subset \mathbb{R}^d$. Moreover, $L^p_c(\mathbb{R}^d)$ denotes the space of L^p functions defined on $\mathbb{R}^d$ with compact support. The Sobolev space of L^p functions with distributional derivatives of first order in L^p is denoted by $W^{1,p}(\mathbb{R}^d)$. The spaces $L^p_{\text{loc}}(\mathbb{R}^d), W^{1,p}_{\text{loc}}(\mathbb{R}^d)$ denote the space of functions which are locally in $L^p(\mathbb{R}^d), W^{1,p}(\mathbb{R}^d)$ respectively. We will denote by $H^1(\mathbb{R}^d)$ the space $W^{1,2}(\mathbb{R}^d)$ and by $H^{-1}(\mathbb{R}^d)$ its dual space. Moreover, we will say that a function u is in $H^{-1}_{\text{loc}}(\mathbb{R}^d)$ if $\rho u \in H^{-1}(\mathbb{R}^d)$ for every function $\rho \in C^\infty_c(\mathbb{R}^d)$. We also denote by $\mathcal{M}(\mathbb{R}^d)$ the space of finite Radon measures on $\mathbb{R}^d$. We denote by $L^p((0,T); L^q(\mathbb{R}^d))$ the space of all measurable functions u defined on $[0,T] \times \mathbb{R}^d$ such that

$$\|u\|_{L^p((0,T);L^q(\mathbb{R}^d))} := \left(\int_0^T \|u(t,\cdot)\|^p_{L^q} \mathrm{d}t \right)^{\frac{1}{p}} < \infty,$$

for all $1 \le p < \infty$, and

$$\|u\|_{L^\infty((0,T);L^q(\mathbb{R}^d))} := \operatorname*{ess\,sup}_{t\in[0,T]} \|u(t,\cdot)\|_{L^q} < \infty,$$

and analogously for the spaces $L^p((0,T); W^{1,q}(\mathbb{R}^d))$. We denote by B_R the ball of radius $R > 0$ and center the origin in $\mathbb{R}^d$, by $\mathscr{L}^d$ the standard Lebesgue measure in $\mathbb{R}^d$, and for $f : \mathbb{R}^d \to \mathbb{R}^d$ we consider the push-forward measure of $\mathscr{L}^d$ defined by

$$f_\# \mathscr{L}^d(A) = \mathscr{L}^d(f^{-1}(A)), \qquad \text{for all Borel sets } A \subseteq \mathbb{R}^d.$$

Finally, it is useful to denote with $\star$ the following variant of the convolution

$$v \star w = \sum_{i=1}^{2} v_i * w_i \qquad \text{if } v, w \text{ are vector fields in } \mathbb{R}^2, \tag{0.0.3}$$

$$A \star B = \sum_{i,j=1}^{2} A_{ij} * B_{ij} \quad \text{if } A, B \text{ are matrix-valued functions in } \mathbb{R}^2. \tag{0.0.4}$$

With the notations above it is easy to check that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a scalar function and $v : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field, then

$$f * \operatorname{curl} v = \nabla^\perp f \star v,$$

$$\nabla^\perp f \star \operatorname{div}(v \otimes v) = \nabla \nabla^\perp f \star (v \otimes v).$$

CHAPTER 1

Flows and transport equations

This chapter is an overview of the theory of the transport equation and of flows of irregular vector fields. In Section 1.1 we start by setting the equations and we briefly recall some classical results for Lipschitz vector fields. In Section 1.2 we present the theory developed by DiPerna and Lions in [29] for weak solutions of the transport equation with Sobolev vector fields and the improvement to BV vector fields done by Ambrosio in [6]. In Section 1.3 we present the more recent well-posedness theory of regular Lagrangian flows of a rough vector field developed in [21, 13] which is based on quantitative estimates on the flow. Finally, in Section 1.4 we will give some examples of vector fields for which non-uniqueness of flows and of solutions of the transport equation holds.

1.1 Smooth solutions

Given a vector field $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ we consider the system of ordinary differential equations

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X(t,x) = b(t, X(t,x)), \\ X(0,x) = x. \end{cases} \tag{ODE}$$

We refer to a function X that solves (ODE) as the *flow* of b. Provided that the vector field b is sufficiently smooth, mainly in the space variable, the Cauchy-Lipschitz theory gives the existence of a unique solution X of (ODE).

Theorem 1.1.1. (Cauchy − Lipschitz) *Let $b : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be a continuous vector field and assume that is Lipschitz continuous in the space variable, uniformly in time. Then for every $x \in \mathbb{R}^d$ there exists $0 < T(x) < T$ and a unique maximal solution $X(\cdot, x)$ of* (ODE) *defined on a maximal time interval $[0, T(x))$. Moreover, the map $X(t, \cdot)$ is locally Lipschitz on its domain.*

The proof of the previous theorem is a very classical result based on the Banach-Caccioppoli fixed point Theorem. It remains valid if the vector field is only summable in time, that is

$b \in L^1((0,T);\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^d))$. Moreover, it is possible to prove uniqueness of flows under some milder regularity assumptions on b than Lipschitz, for instance the so-called *one side Lipschitz condition* or the *Osgood condition.*

Let $u_0 : \mathbb{R}^d \to \mathbb{R}$ be a smooth function; we can consider then the Cauchy problem for the *linear transport equation* related to b and u_0 which is

$$\begin{cases} \partial_t u + b \cdot \nabla u = 0, \\ u(0,\cdot) = u_0. \end{cases} \tag{TE}$$

The equation (TE) is strictly connected to (ODE) via the theory of characteristics. In fact, if X is a flow of b and u is a smooth solution of (TE), a simple computation shows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} u(t, X(t,x)) &= \partial_t u(t, X(t,x)) + \frac{\mathrm{d}}{\mathrm{d}t} X(t,x) \cdot \nabla u(t, X(t,x)) \\ &= \partial_t u(t, X(t,x)) + b(t, X(t,x)) \cdot \nabla u(t, X(t,x)) = 0, \end{aligned}$$

which means that u is constant along the characteristics of b. It can be easily checked that the function u given by the formula

$$u(t,x) = u_0(X^{-1}(t,\cdot)(x)) \tag{1.1.1}$$

is a solution of (TE) and since every smooth solution has to be constant along the characteristics of b, Theorem 1.1.1 implies that (1.1.1) is also the unique solution of (TE), provided that b is globally Lipschitz in space.

We finish this section with a very classical example of non-uniqueness when the vector field fails to be Lipschitz in space.

Example 1. *Let consider the one dimensional vector field* $b(t,x) = \sqrt{|x|}$*: it is a continuous autonomous vector field but b is not Lipschitz. The ordinary differential equation*

$$\begin{cases} \dot\gamma(t) = b(\gamma(t)), \\ \gamma(0) = 0 \end{cases}$$

admits infinitely many solutions γ^τ *defined as*

$$\gamma^\tau(t) = \begin{cases} 0 & \text{if } t \le \tau, \\ \frac{1}{4}(t-\tau)^2 & \text{if } t > \tau \end{cases} \tag{1.1.2}$$

for every choice of the parameter $\tau \in [0, +\infty)$*. Roughly speaking, the parameter* τ *represents the amount of time the solution remains at rest in the origin.*

1.2 Weak solutions of the transport equation

In this section we deal with the well-posedness of the transport equation in the case of weakly differentiable vector fields. In particular, we will briefly present the theory of renormalized solutions developed by DiPerna and Lions in [29] for Sobolev vector fields and the improvement of Ambrosio [6] to vector fields with bounded variation. We will always assume that the vector field is divergence-free but all definitions and results can be extended to the case of bounded divergence with suitable changes. We take advantage of this additional hypothesis in view of the applications to the incompressible Euler equations that will come later.
We start by giving the definition of distributional solution of the transport equation (TE).

Definition 1.2.1. (Distributional solution) *Let $b \in L^1((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^d))$ be a divergence-free vector field and $u_0 \in L^q(\mathbb{R}^d)$ for p, q such that $\frac{1}{p} + \frac{1}{q} \leq 1$. A function $u \in L^\infty((0,T); L^q(\mathbb{R}^d))$ is a distributional solution of* (TE) *if for any $\varphi \in C^\infty_c([0,T) \times \mathbb{R}^d)$ the following equality holds:*

$$\int_0^T \int_{\mathbb{R}^d} u(\partial_t \varphi + b \cdot \nabla \varphi) \mathrm{d}x \mathrm{d}t + \int_{\mathbb{R}^d} u_0 \varphi_{|t=0} \mathrm{d}x = 0.$$

Note that the definition of distributional solutions requires that the product $ub \in L^1_{\mathrm{loc}}$: this is in general not true in several applications, as in the case of the 2D Euler equations. We will come back to this point later.
The existence of distributional solutions is very easy to prove and it relies on a simple approximation argument as shown in the following.

Theorem 1.2.1. *Let $b \in L^1((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^d))$ be a divergence-free vector field and $u_0 \in L^q(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} \leq 1$. Then there exists at least one distributional solution $u \in L^\infty((0,T); L^q(\mathbb{R}^d))$ of* (TE).

Proof. Let ρ_ε be a convolution kernel on $\mathbb{R}^d$ and φ_ε be a convolution kernel on $\mathbb{R}^{1+d}$. We define $u_0^\varepsilon = u_0 * \rho_\varepsilon$ and $b_\varepsilon = b * \varphi_\varepsilon$. Let u^ε be the unique solution to the regularized transport equation

$$\begin{cases} \partial_t u^\varepsilon(t,x) + b_\varepsilon(t,x) \cdot \nabla u^\varepsilon(t,x) = 0, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x). \end{cases} \tag{1.2.1}$$

There exists a unique solution u^ε of (1.2.1) which is given by $u^\varepsilon(t,x) = u_0^\varepsilon((X^\varepsilon(t,\cdot))^{-1}(x))$, where X^ε is the unique flow of b_ε. The sequence u^ε is equi-bounded in $L^\infty((0,T); L^q(\mathbb{R}^d))$: for $q = \infty$ it is a direct consequence of the explicit formula of u^ε, while for $1 \leq q < \infty$ exploiting the divergence-free condition it is easy to prove the conservation of the L^q-norm

$$\|u^\varepsilon(t,\cdot)\|_{L^q(\mathbb{R}^d)} = \|u_0^\varepsilon\|_{L^q(\mathbb{R}^d)} \leq \|u_0\|_{L^q(\mathbb{R}^d)}.$$

For $q > 1$ by standard compactness arguments we can extract a sub-sequence which converges weakly-* to a function $u \in L^\infty((0,T); L^q(\mathbb{R}^d))$ and it is immediate to deduce that u is a distributional solution of (TE) thanks to the linearity of the equation. For $q = 1$ we need a further approximation in order to conclude: in this case $p = \infty$ and we can not rely on the strong convergence of the approximating vector field. It is not difficult to prove that the sequence u^ε is equi-integrable in space uniformly in time, which implies that there exists $u \in L^\infty((0,T); L^1(\mathbb{R}^d))$ such that

$$u^\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty((0,T); L^1(\mathbb{R}^d)).$$

Let $u_0^n \in \mathcal{D}(\mathbb{R}^d)$ which converges to u_0 in $L^1(\mathbb{R}^d)$ and we consider $u_0^{n,\varepsilon} = u_0^n * \rho_\varepsilon$. Let $u^{n,\varepsilon}$ the unique solution of (TE) with initial datum $u_0^{n,\varepsilon}$ and vector field b_ε, we have that

$$\sup_{t\in[0,T]} \|u^{n,\varepsilon}(t,\cdot)\|_{L^p} \le C(n,p),$$

$$\sup_{t\in[0,T]} \|u^\varepsilon(t,\cdot) - u^{n,\varepsilon}(t,\cdot)\|_{L^1} \le C\|u_0 - u_0^{n,\varepsilon}\|_{L^1} \le C\|u_0 - u_0^n\|_{L^1}.$$

With the use of the previous inequalities it is possible to justify the limit in the integral formulation and this shows that u is a distributional solution of (TE). □

While the existence of distributional solutions is rather easy to prove, the uniqueness problem is much harder and require additional assumptions on the vector field.
The first proof of uniqueness of distributional solutions for rough vector fields was given by DiPerna and Lions in [29]. Their idea was very simple and consists in justifying the following formal computation: multiplying the equation (TE) by $2\,u(t,x)$ we have that

$$\partial_t |u|^2 + b \cdot \nabla |u|^2 = 0.$$

Then integrating in space we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |u(t,x)|^2 \mathrm{d}x = 0,$$

which means that

$$\int_{\mathbb{R}^d} |u(t,x)|^2 \mathrm{d}x = \int_{\mathbb{R}^d} |u_0(x)|^2 \mathrm{d}x.$$

Considering as initial datum the function $u_0 = 0$, the linearity of the equation implies the uniqueness of solutions.
In the previous computation we improperly use the chain rule for the products

$$2u(t,x)\partial_t u(t,x) \qquad 2u(t,x)\nabla u(t,x),$$

which is in general not true when we deal with non-smooth functions. In order to deal with solutions which satisfy the chain rule in a distributional sense, the authors in [29] introduced the following class of solutions:

Definition 1.2.2. (Renormalized solution) *Let $b \in L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^d))$ be a divergence-free vector field and $u_0 \in L^q(\mathbb{R}^d)$ with $q \geq 1$. We say that a function $u \in L^\infty((0,T); L^q(\mathbb{R}^d))$ is a renormalized solution of (TE) if for any $\beta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, vanishing in a neighborhood of 0,*

$$\iint \beta(u)\,(\partial_t\varphi + b\cdot\nabla\varphi)\,\mathrm{d}x\mathrm{d}t + \int \beta(u_0)\varphi_{|t=0}\mathrm{d}x = 0,$$

for any $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$.

It is worth to note that when u and b satisfy the integrability hypothesis of the Definition 1.2.1, renormalized solutions are distributional solutions. In [29], a key point is to prove the opposite for Sobolev vector fields.

Theorem 1.2.2. *Let $b \in L^1((0,T); W^{1,p}_{\text{loc}}(\mathbb{R}^d))$ be a given divergence-free vector field. Let $u \in L^\infty((0,T); L^q(\mathbb{R}^d))$ a distributional solution of* (TE) *with initial datum $u_0 \in L^q(\mathbb{R}^d)$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then u is a renormalized solution of* (TE).

Proof. Let consider a standard mollifier φ_δ and define the function $u^\delta = u * \varphi_\delta$: it satisfies the following equation

$$\begin{cases} \partial_t u^\delta + b\cdot\nabla u^\delta = r^\delta, \\ u^\delta(0,\cdot) = u_0^\delta, \end{cases} \tag{1.2.2}$$

where the right hand side is defined as

$$r^\delta := b\cdot\nabla u^\delta - (b\cdot\nabla u) * \varphi_\delta.$$

Since u^δ is smooth with respect to x, from the equation (1.2.2) we can infer that $\partial_t u^\delta$ is integrable in time and therefore, for every fixed $\delta > 0$, $u^\delta \in W^{1,1}_{\text{loc}}((0,T)\times\mathbb{R}^d)$. Multiplying the equation by $\beta'(u^\delta)$, by Stampacchia chain rule for Sobolev space we obtain that

$$\partial_t\beta(u^\delta) + b\cdot\nabla\beta(u^\delta) = \beta'(u^\delta)r^\delta.$$

For the left hand side we trivially have that

$$\lim_{\delta\to 0}\partial_t\beta(u^\delta) + b\cdot\nabla\beta(u^\delta) = \partial_t\beta(u) + b\cdot\nabla\beta(u) \qquad \text{in } \mathcal{D}'(\mathbb{R}^d),$$

and since β' is bounded, in order to conclude it is enough to prove that r^δ converges to 0 in $L^1_{\text{loc}}((0,T)\times\mathbb{R}^d)$. We compute

$$\begin{aligned} r^\delta &= b\cdot\nabla u^\delta - (b\cdot\nabla u)*\varphi_\delta \\ &= b(t,x)\cdot\int_{\mathbb{R}^d} u(t,y)\nabla\varphi_\delta(x-y)\mathrm{d}y - \int_{\mathbb{R}^d}\operatorname{div}(ub)(t,y)\varphi_\delta(x-y)\mathrm{d}y \\ &= \int_{\mathbb{R}^d} u(t,y)\,(b(t,x)-b(t,y))\cdot\nabla\varphi_\delta(x-y)\mathrm{d}y \\ &= \int_{\mathbb{R}^d} u(t,x+\delta z)\frac{b(t,x)-b(t,x+\delta z)}{\delta}\cdot\nabla\varphi(z)\mathrm{d}z. \end{aligned}$$

In the previous computations we have used the divergence-free condition on b, the change of variables $y := x + \delta z$, and the fact that $\nabla\varphi$ is odd. It is well known that the translation operator is strongly continuous in L^q and moreover, by standard theory of Sobolev spaces, we have that

$$\frac{b(t,x) - b(t,x+\delta z)}{\delta} \to \nabla b \cdot z \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^d), \text{ as } \delta \to 0. \tag{1.2.3}$$

Then, we deduce that r^δ converges strongly in L^1_{loc} to

$$\begin{aligned} u(t,x)\int_{\mathbb{R}^d}(\nabla b(t,x)\cdot z)\cdot\nabla\varphi(z)\mathrm{d}z &= u(t,x)\int_{\mathbb{R}^d}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}b_i(t,x)z_j\frac{\partial}{\partial z_i}\varphi(z)\mathrm{d}z \\ &= u(t,x)\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}b_i(t,x)\int_{\mathbb{R}^d}z_j\frac{\partial}{\partial z_i}\varphi(z)\mathrm{d}z \\ &= -u(t,x)\,\mathrm{div}\,b(t,x) = 0, \end{aligned}$$

since $\int_{\mathbb{R}^d} z_j \frac{\partial}{\partial z_i}\varphi(z)\mathrm{d}z = \delta_{ij}$. □

Before continuing, we introduce some growth conditions for the vector field which will always be assumed in the following.

(R1) The vector field b admits a decomposition

$$\frac{|b(t,x)|}{1+|x|} = b_1(t,x) + b_2(t,x),$$

where $b_1 \in L^1((0,T);L^1(\mathbb{R}^d))$ and $b_2(t,x) \in L^1((0,T);L^\infty(\mathbb{R}^d))$.

We can now prove the uniqueness of distributional solutions of the transport equation with Sobolev vector fields. Since (TE) is a linear PDE, it is enough to prove the following.

Theorem 1.2.3. *Let $b \in L^1((0,T);W^{1,p}_{\text{loc}}(\mathbb{R}^d))$ be a divergence-free vector field which verifies the growth condition (R1). Let $u \in L^\infty((0,T);L^q(\mathbb{R}^d))$, with $\frac{1}{p}+\frac{1}{q}=1$, be a distributional solution of* (TE) *with initial condition $u_0 = 0$. Then, $u = 0$.*

Proof. By Theorem 1.2.2 we now that u^δ satisfies

$$\begin{cases} \partial_t\beta(u^\delta) + b\cdot\nabla\beta(u^\delta) = \beta'(u^\delta)r^\delta, \\ u^\delta(0,\cdot) = 0. \end{cases} \tag{1.2.4}$$

Let $\chi \in C^\infty(\mathbb{R}^d)$ a cut-off function such that

$$\chi \geq 0, \quad \text{supp}\,\chi \subseteq B_2, \quad \chi = 1 \text{ on } B_1,$$

and define $\chi_R = \chi\left(\frac{\cdot}{R}\right)$. Multiplying equation (1.2.4) by χ_R and integrating on $[0,t]\times\mathbb{R}^d$ we obtain

$$\int_{\mathbb{R}^d}\beta(u^\delta)\chi_R\mathrm{d}x = \int_0^t\int_{\mathbb{R}^d}\beta(u^\delta)b\cdot\nabla\chi_R\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\mathbb{R}^d}\beta'(u^\delta)r^\delta\chi_R\mathrm{d}x\mathrm{d}s,$$

and letting $\delta \to 0$ we have that

$$\int_{\mathbb{R}^d} \beta(u)\chi_R \mathrm{d}x = \int_0^t \int_{\mathbb{R}^d} \beta(u) b \cdot \nabla \chi_R \mathrm{d}x\mathrm{d}s.$$

Let $M > 0$ and consider the function $\beta_M(t) = (|t| \wedge M)^q$. Since β_M is Lipschitz but not C^1, by a further approximation arguments we obtain that

$$\int_{\mathbb{R}^d} (|u| \wedge M)^q \chi_R \mathrm{d}x = \int_0^t \int_{\mathbb{R}^d} (|u| \wedge M)^q b \cdot \nabla \chi_R \mathrm{d}x\mathrm{d}s.$$

For the decomposition (R1) we can estimate the right hand side as follows

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} (|u| \wedge M)^q b \cdot \nabla \chi_R \mathrm{d}x\mathrm{d}s &\le \int_0^T \int_{R<|x|<2R} (|u| \wedge M)^q \frac{|b|}{R} \mathrm{d}x\mathrm{d}s \\ &\le C \int_0^T \int_{|x|>R} (|u| \wedge M)^q \frac{|b|}{1+|x|} \mathrm{d}x\mathrm{d}s \\ &\le M^q \int_0^T \int_{|x|>R} |b_1| \mathrm{d}x\mathrm{d}s + \int_0^T \|b_2(s,\cdot)\|_{L^\infty} \int_{|x|>R} (|u| \wedge M)^q \mathrm{d}x\mathrm{d}s.\end{aligned}$$

Since $(|u| \wedge M)^q \in L^\infty((0,T); L^1 \cap L^\infty(\mathbb{R}^d))$, letting $R \to \infty$ we conclude that, whenever $q < \infty$, it holds $|u| \wedge M = 0$ a.e. and we conclude letting $M \to \infty$.
In the case $q = \infty$ we need to deal with a duality argument. First of all, note that it is enough to prove that

$$\int_0^T \int_{\mathbb{R}^d} u\chi = 0, \qquad \text{for all } \chi \in C_c^\infty((0,T) \times \mathbb{R}^d).$$

In order to do so, consider the solution of the following backward problem

$$\begin{cases} -\partial_t \phi - b \cdot \nabla \phi = \chi, \\ \phi(T,\cdot) = 0. \end{cases} \tag{1.2.5}$$

The first part of the proof gives the existence of a unique solution $\phi \in L^\infty((0,T); L^1 \cap L^\infty(\mathbb{R}^d))$. By Theorem 1.2.2 we have that ϕ^δ and u^δ satisfy the following

$$\begin{cases} -\partial_t \phi^\delta - b \cdot \nabla \phi^\delta = \chi + \psi^\delta, \\ \phi^\delta(T,\cdot) = 0, \end{cases} \qquad \begin{cases} \partial_t u^\delta + b \cdot \nabla u^\delta = r^\delta, \\ u^\delta(0,\cdot) = 0, \end{cases}$$

where $r^\delta, \psi^\delta \to 0$ in $L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$. Multiplying (1.2.2) by $\phi^\delta \chi_R$ and using the equation for ϕ^δ we find

$$-\int_0^T \int_{\mathbb{R}^d} u^\delta(\chi + \psi^\delta)\chi_R + r^\delta \phi^\delta \chi_R \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} u^\delta \phi^\delta b \cdot \nabla \chi_R \mathrm{d}x\mathrm{d}t = 0.$$

Letting $\delta \to 0$, we deduce that

$$\left| \int_0^T \int_{\mathbb{R}^d} u\chi \mathrm{d}x\mathrm{d}t \right| \le \int_0^T \int_{R<|x|<2R} |u||\chi| \frac{|b|}{1+|x|} \mathrm{d}x\mathrm{d}t,$$

and we can conclude letting $R \to \infty$, since $|u||\chi| \in L^\infty((0,T); L^1 \cap L^\infty(\mathbb{R}^d))$. □

An immediate consequence of Theorem 1.2.2 and Theorem 1.2.3 is the following stability result.

Theorem 1.2.4. *Let b_n be a sequence of smooth divergence-free vector fields converging in $L^1((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^d))$ to a vector field $b \in L^1((0,T);W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d))$, with $p \geq 1$, and satisfying the growth condition (R1). Assume that for some decomposition*

$$\frac{|b_n(t,x)|}{1+|x|} = \tilde{b}_{n,1}(t,x) + \tilde{b}_{n,2}(t,x)$$

we have

$$\|\tilde{b}_{n,1}\|_{L^1((0,T);L^1(\mathbb{R}^d))} + \|\tilde{b}_{n,2}\|_{L^1((0,T);L^\infty(\mathbb{R}^d))} \leq C$$

for all $n \in \mathbb{N}$ and for some positive constant C. Let $u_0 \in L^q(\mathbb{R}^d)$ and let u_n be the weak solution of the Cauchy problem (TE) *with initial datum u_0 and vector field b_n. Then,*

$$u_n \to u \qquad \text{in } C([0,T];L^q(\mathbb{R}^d))$$

where u is the unique solution of (TE) *with initial datum u_0 and vector field b.*

The proof is rather technical and is based on a classical weak-strong convergence argument

$$u_n \rightharpoonup u, \quad \beta(u_n) \rightharpoonup \beta(u) \qquad \Longrightarrow \qquad u_n \to u,$$

provided that β is strictly convex, for instance $\beta(t) = t^2$.
We have proved that in the case of the Sobolev vector fields the renormalization property implies the well-posedness. By proving the renormalization property for bounded distributional solutions in the case of BV vector fields, Ambrosio [6] was able to improve the result of [29]. Note that the equivalence between renormalized and distributional solutions depends on the convergence of the commutator r^δ. As shown in the proof of Theorem 1.2.2, the Sobolev regularity comes into play only in the convergence (1.2.3) which is not true for BV vector fields. Consider $b \in L^1((0,T);BV(\mathbb{R}^d))$ a divergence-free vector field. The gradient ∇b is a finite Radon measure and it can be decomposed as

$$\nabla b = \nabla^a b + \nabla^s b,$$

where $\nabla^a b, \nabla^s b$ are respectively the absolutely continuous part and the singular part of the measure ∇b. Furthermore, the difference quotient can be decomposed as

$$\frac{b(t,x) - b(t,x+\delta z)}{\delta} = b_1^\delta(t,x,z) + b_2^\delta(t,x,z),$$

where

$$b_1^\delta(t,x,z) \to \nabla^a b(t,x) z \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^d), \text{ as } \delta \to 0,$$

$$\limsup_{\delta \to 0} \int_K |b_2^\delta(t,x,z)| \mathrm{d}x \leq |\nabla^s b(t,x) z|(K) \quad \text{for every compact set } K \subset \mathbb{R}^d.$$

We can split the commutator as $r^\delta = r_1^\delta + r_2^\delta$, involving respectively b_1^δ and b_2^δ: r_1^δ can be treated as in the Sobolev setting, while r_2^δ depends strongly on the anisotropic structure of $\nabla^s b$. At this point it is clear that the improvement from Sobolev to BV lies in treatment of $\nabla^s b$. It can be done utilizing Alberti's rank-one Theorem: roughly speaking this theorem selects the bad directions of $\nabla^s b$ allowing to choose a regularization kernel which takes care of these directions and then gives the desired convergence.
The theorem proved by Ambrosio in [6] is the following:

Theorem 1.2.5. *Let $b \in L^1((0,T); BV_{\mathrm{loc}}(\mathbb{R}^d))$ be a divergence-free vector field which satisfies the growth condition (R1). Let $u_0 \in L^\infty(\mathbb{R}^d)$, then there exists a unique solution $u \in L^\infty((0,T); L^\infty(\mathbb{R}^d))$ of* (TE) *with initial datum u_0 and vector field b. Moreover, let b_n be a sequence of smooth divergence-free vector fields converging in $L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$ to b and assume that for some decomposition*

$$\frac{|b_n(t,x)|}{1+|x|} = \tilde{b}_{n,1}(t,x) + \tilde{b}_{n,2}(t,x)$$

we have

$$\|\tilde{b}_{n,1}\|_{L^1((0,T);L^1(\mathbb{R}^d))} + \|\tilde{b}_{n,2}\|_{L^1((0,T);L^\infty(\mathbb{R}^d))} \leq C$$

for all $n \in \mathbb{N}$ and for some positive constant C. Let u_n be the weak solution of the Cauchy problem (TE) *with initial datum u_0 and vector field b_n. Then,*

$$u_n \to u \qquad \text{in } L^\infty((0,T); L^\infty(\mathbb{R}^d) - w*) \cap L^\infty((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d)).$$

We finish this section with a duality formula for transport equations proved in [29] which will be crucial in the applications to the 2D Euler equations in Chapter 3.

Lemma 1.2.6. *Let $b \in L^1((0,T); W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d))$ be a divergence-free vector field which satisfies assumption (R1) and $u \in L^\infty((0,T); L^p(\mathbb{R}^d))$ be a renormalized solution of* (TE)*. Let $\phi \in L^\infty((0,T); L^q(\mathbb{R}^d))$ with $1/p + 1/q = 1$ be a renormalized solution of the backward problem*

$$\begin{cases} -\partial_t \phi - b \cdot \nabla \phi = \chi, \\ \phi(T,\cdot) = \phi_T, \end{cases} \tag{1.2.6}$$

with $\phi_T \in L^q(\mathbb{R}^d)$ and $\chi \in L^1((0,T); L^q(\mathbb{R}^d))$. Then, the following formula holds

$$\int_0^T \int_{\mathbb{R}^d} u(t,x)\chi(t,x)\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^d} \phi(0,x)u_0(x)\mathrm{d}x - \int_{\mathbb{R}^d} \phi_T(x)u(T,x)\mathrm{d}x. \tag{1.2.7}$$

Proof. Let ρ_δ be a standard mollifier and define $\phi^\delta = \phi * \rho_\delta, u^\delta * \rho_\delta, \chi^\delta = \chi * \rho_\delta$. By Theorem 1.2.2 we have that ϕ^δ and u^δ satisfy the following

$$\begin{cases} -\partial_t \phi^\delta - b \cdot \nabla \phi^\delta = \chi + \psi^\delta, \\ \phi^\delta(T,\cdot) = \phi_T^\delta, \end{cases} \qquad \begin{cases} \partial_t u^\delta + b \cdot \nabla u^\delta = r^\delta, \\ u^\delta(0,\cdot) = u_0^\delta, \end{cases}$$

where $r^\delta, \psi^\delta \to 0$ in $L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^d))$. Let f_R be a cut-off function, multiplying (1.2.2) by $\phi^\delta f_R$ and using the equation (1.2.6) we find

$$\int_0^T \int_{\mathbb{R}^d} u^\delta \chi f_R \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^d} \phi^\delta(0,x) u_0^\delta(x) f_R(x)\mathrm{d}x - \int_{\mathbb{R}^d} \phi_T^\delta(x) u^\delta(T,x) f_R(x)\mathrm{d}x$$
$$+ \int_{\mathbb{R}^d} r^\delta \phi^\delta f_R \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} u^\delta \psi^\delta f_R \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} u^\delta \phi^\delta b \cdot \nabla f_R \mathrm{d}x\mathrm{d}t,$$

and we can conclude the proof by letting $\delta \to 0$, then $R \to \infty$. □

1.3 The Lagrangian approach

In this section we summarize the Lagrangian theory of the transport equation, which relies only on a priori estimates of the flow. First of all we introduce a suitable notion of flow in the non-smooth setting, namely the *regular Lagrangian flow*. Then, we prove some estimates on the difference of two flows which will allow to get a well-posedness theory for both the regular Lagrangian flows and the Lagrangian solutions of the transport equation. In particular we will be able to recover the results of DiPerna-Lions and also to deal with vector fields which derivatives can be expressed as singular integral of L^1 functions. Note that this latter case is out of the class of [29, 6] and will be very important for the applications to the 2D Euler equations.
We start by giving the definition of regular Lagrangian flows.

Definition 1.3.1. *Let $b \in L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^d))$ be a divergence-free vector field. We say that $X : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ is a regular Lagrangian flow associated to b if*

1. *for a.e. $x \in \mathbb{R}^d$ the map $t \mapsto X(t,x)$ is an absolutely continuous integral solution of the ordinary differential equation*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(t,x) = b(t, X(t,x)), \\ X(0,x) = x, \end{cases} \tag{1.3.1}$$

2. *the push-forward measure $X(t,\cdot)_\# \mathscr{L}^d$ satisfies*

$$X(t,\cdot)_\# \mathscr{L}^d = \mathscr{L}^d. \tag{1.3.2}$$

The condition (1.3.2) is equivalent to require that for all positive measurable functions φ we have that

$$\int_{\mathbb{R}^d} \varphi(X(t,x))\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x.$$

It is worth to note that when the vector field is not divergence-free, the condition (1.3.2) can be replaced by

$$X(t, \cdot)_{\#}\mathscr{L}^d \leq L\mathscr{L}^d, \tag{1.3.3}$$

where $L > 0$ is a constant independent of time which is called *compressibility constant.*

Example 2. *Consider the square root vector field in Example 1. The unique solution of the ODE that verifies* (1.3.3) *is the one that exits immediately from the origin, corresponding in* (1.1.2) *to* $\tau = 0$. *Consider the Lipschitz approximation of* $b(\gamma) = \sqrt{|\gamma|}$ *given by*

$$b_\varepsilon(\gamma) = \begin{cases} \sqrt{|\gamma|} & \text{if } -\infty < \gamma \leq -\varepsilon^2, \\ \varepsilon & \text{if } -\varepsilon^2 \leq \gamma \leq \lambda_\varepsilon - \varepsilon^2, \\ \sqrt{\gamma - \lambda_\varepsilon + 2\varepsilon^2} & \text{if } \lambda_\varepsilon - \varepsilon^2 \leq \gamma < +\infty, \end{cases} \tag{1.3.4}$$

with $\lambda_\varepsilon - \varepsilon^2 > 0$. *Let* $c \in \mathbb{R}$ *and consider the solution of the approximating ODE*

$$\begin{cases} \dot\gamma_\varepsilon(t) = b_\varepsilon(\gamma_\varepsilon(t)), \\ \gamma_\varepsilon(0) = -c^2, \end{cases}$$

which is given by

$$\gamma_\varepsilon(t) = \begin{cases} -\left(c - \frac{1}{2}t\right)^2 & \text{if } 0 \leq t \leq t_\varepsilon := 2(c-\varepsilon), \\ -\varepsilon^2 + \varepsilon(t - t_\varepsilon) & \text{if } t_\varepsilon \leq t \leq t_\varepsilon + \frac{\lambda_\varepsilon}{\varepsilon}, \\ \lambda_\varepsilon - 2\varepsilon^2 + \frac{1}{4}(t - t_\varepsilon - \frac{\lambda_\varepsilon}{\varepsilon})^2 & \text{if } t_\varepsilon + \frac{\lambda_\varepsilon}{\varepsilon} \leq t < +\infty. \end{cases} \tag{1.3.5}$$

Choosing $\lambda_\varepsilon = \varepsilon\tau$, *with* $\tau > 0$, *by this approximation we can select in the limit the solutions such that, once they reach the origin, they do not move exactly for an amount of time* τ. *Consider instead the approximation* $b_\varepsilon(\gamma) = \sqrt{\varepsilon + |\gamma|}$: *the solution of the approximating ODE starting at* $-c^2$ *is given by*

$$\gamma_\varepsilon(t) = \varepsilon - \frac{1}{4}\left(t - 2\sqrt{\varepsilon + c^2}\right)^2.$$

Then, the previous approximation selects in the limit the solutions that move immediately away from the singularity at $\gamma = 0$, *which is given by*

$$\gamma(t) = -\frac{1}{4}\left(t - 2c\right)^2. \tag{1.3.6}$$

Among all the possibilities, this family of solutions $\gamma(t,x)$ *is singled out by the property that* $\gamma(t,\cdot)_{\#}\mathscr{L}^1$ *is absolutely continuous with respect to* $\mathscr{L}^1$, *so that no concentrations of trajectories occurs at the origin. In fact, for fixed* t *we have that*

$$\{x : \gamma(t,x) = 0\} = \left\{-\frac{t^2}{4}\right\}$$

and then it follows that

$$\gamma(t,\cdot)_{\#}\mathscr{L}^1(\{0\}) = \mathscr{L}^1\left(\{x : \gamma(t,x) = 0\}\right) = 0,$$

integrating in time and using Fubini's theorem we get

$$0 = \int_{\mathbb{R}} \mathscr{L}^1\left(\{t : \gamma(t,x) = 0\}\right) \mathrm{d}x.$$

Hence, for $\mathscr{L}^1 -$ a.e. x, the trajectory $\gamma(\cdot,x)$ does not stay in the origin for a strictly positive set of times and then γ is a regular Lagrangian flow.

1.3.1 Estimate on the super-levels

Since we are dealing with vector fields satisfying the growth condition (R1), we have to take care of the fact that, according to Definition 1.3.1, a regular Lagrangian flow is not in general locally integrable. For this reason we need a control on the measure of the set of initial data such that the corresponding trajectories exit from a fixed ball at some time. First of all, we define the sub-levels of a flow X as

$$G_\lambda := \{x \in \mathbb{R}^d : |X(t,x)| \leq \lambda \text{ for almost every } t \in [0,T]\}.$$

The following Lemma gives an estimate on the super-level of a regular Lagrangian flow assuming that the vector field verifies the growth condition (R1).

Lemma 1.3.1. *Let $b : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a divergence-free vector field which admits a decomposition as in (R1) and let X be a regular Lagrangian flow relative to b. Then $\forall\ r,\ \lambda > 0$*

$$\mathscr{L}^d(B_r \setminus G_\lambda) \leq g(r,\lambda),$$

where the function g depends on $\|b_1\|_{L^\infty((0,T);L^1(\mathbb{R}^d))}$ and $\|b_2\|_{L^\infty((0,T);L^\infty(\mathbb{R}^d))}$, and satisfies $g(r,\lambda) \to 0$ for fixed r and $\lambda \to \infty$.

Proof. Let $\beta_\varepsilon(z) = \log\left(1 + \sqrt{\varepsilon^2 + |z|^2}\right)$ for some $\varepsilon > 0$. Then, we have that

$$\partial_s \beta_\varepsilon(X(s,x)) = \beta_\varepsilon'(X(s,x))b(s,X(s,x))$$

and by (R1) we get that $\partial_s\beta_\varepsilon(X(s,x)) \in L^1((t,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$. So for $\mathscr{L}^d$-a.e $x \in \mathbb{R}^d$, we have that $\partial_s\beta_\varepsilon(X(\cdot,x)) \in L^1(t,T)$. This implies that for such x the function $\beta_\varepsilon(X(s,x))$ coincides almost everywhere with an absolutely continuous function $\Xi_\varepsilon(\cdot,x)$ in $[t,T]$. So for $s \in [t,T]$ we have that

$$\Xi_\varepsilon(s,x) = \Xi_\varepsilon(t,x) + \int_t^s \beta_\varepsilon'(X(\tau,x))b(\tau,X(\tau,x))\mathrm{d}\tau,$$

thus

$$\text{for a.e. } x \in \mathbb{R}^d, \quad \text{for a.e. } s \in (t,T) \quad \beta_\varepsilon(s,x) = \Xi_\varepsilon(t,x) + \int_t^s \beta_\varepsilon'(X(\tau,x))b(\tau,X(\tau,x))\mathrm{d}\tau. \tag{1.3.7}$$

But since the integral on the right hand side belongs to $C([t,T];L^1_{\text{loc}}(\mathbb{R}^d))$, and $\beta_\varepsilon(X) \in C([t,T];L^0_{\text{loc}}(\mathbb{R}^d)) \cap \mathscr{B}([t,T];L^1_{\text{loc}}(\mathbb{R}^d))$, we have that $\Xi(t,\cdot) \in L^1_{\text{loc}}(\mathbb{R}^d)$ and (1.3.7) is valid for all s i.e.

$$\text{for all } s \in (t,T), \text{for a.e. } x \in \mathbb{R}^d, \qquad \beta_\varepsilon(s,x) = \beta_\varepsilon(t,x) + \int_t^s \beta'_\varepsilon(X(\tau,x))b(\tau,X(\tau,x))\mathrm{d}\tau.$$

Since $|\beta'(z)| \leq \frac{1}{1+|z|}$, for all s and a.e. x we have

$$\beta_\varepsilon(s,x) \leq \beta_\varepsilon(t,x) + \int_t^s \frac{|b(\tau,X(\tau,x))|}{1+|X(\tau,x)|}\mathrm{d}\tau.$$

Letting $\varepsilon \to 0$ this yields that for all s and a.e. x,

$$\log(1+|X(s,x)|) \leq \log(1+|x|) + \int_t^s \frac{|b(\tau,X(\tau,x))|}{1+|X(\tau,x)|}\mathrm{d}\tau.$$

Integrating for $x \in B_r$ we have

$$\int_{B_r} \log\left(\frac{1+|X(s,x)|}{1+|x|}\right)\mathrm{d}x \leq \|b_1\|_{L^1((0;T)L^1(B_r)} + \mathscr{L}^d(B_r)\|b_2\|_\infty. \tag{1.3.8}$$

Moreover, a simple lower bound gives that

$$\int_{B_r} \log\left(\frac{1+|X(s,x)|}{1+|x|}\right)\mathrm{d}x \geq \mathscr{L}^d(B_r \setminus G_\lambda)\log(1+\lambda) - \mathscr{L}^d(B_r)\log(1+r), \tag{1.3.9}$$

and putting together (1.3.8) and (1.3.9) we have

$$\mathscr{L}^d(B_r \setminus G_\lambda) \leq \frac{\|b_1\|_{L^1((0;T)L^1(B_r)} + \mathscr{L}^d(B_r)\|b_2\|_\infty + \mathscr{L}^d(B_r)\log(1+r)}{\log(1+\lambda)}, \tag{1.3.10}$$

which gives the result. □

1.3.2 A priori estimates on the flow

In this section we prove the a priori estimates discussed previously. We first introduce a functional that measures an integral distance between two flows $X, \tilde{X}$ respectively of two different vector fields $b, \tilde{b}$. Then we prove bounds on this functional under several regularity assumptions on the vector fields, which will allow to get the well-posedness of (ODE).
Let $\delta > 0$ be a small parameter, we define

$$\Phi_\delta(t) := \int_{B_r \cap G_\lambda \cap \tilde{G}_\lambda} \log\left(1 + \frac{|X(t,x) - \tilde{X}(t,x)|}{\delta}\right)\mathrm{d}x, \tag{1.3.11}$$

where $G_\lambda, \tilde{G}_\lambda$ are respectively the sub-levels of $X, \tilde{X}$. Note that $\Phi_\delta(0) = 0$. We can find a non-trivial lower bound for Φ_δ: let $\gamma > 0$ then

$$\begin{aligned}\Phi_\delta(t) &\geq \int_{B_r \cap \{|X(t,x)-\tilde{X}(t,x)|>\gamma\} \cap G_\lambda \cap \tilde{G}_\lambda} \log\left(1+\frac{\gamma}{\delta}\right)\mathrm{d}x \\ &\geq \left(\mathscr{L}^d\left(B_r \cap \{|X(t,x) - \tilde{X}(t,x)| > \gamma\}\right) - \mathscr{L}^d(B_r \setminus G_\lambda) - \mathscr{L}^d(B_r \setminus \tilde{G}_\lambda)\right)\log\left(1+\frac{\gamma}{\delta}\right),\end{aligned}$$

which leads to

$$\mathscr{L}^d\left(B_r \cap \{|X(t,x)-\tilde{X}(t,x)|>\gamma\}\right) \leq \frac{\Phi_\delta(t)}{\log\left(1+\frac{\gamma}{\delta}\right)} + \mathscr{L}^d(B_r \setminus G_\lambda) + \mathscr{L}^d(B_r \setminus \tilde{G}_\lambda). \quad (1.3.12)$$

In view of Lemma 1.3.1 together with an appropriate upper bound on Φ_δ, the above inequality gives many (quantitative) information concerning the well-posedness of (ODE). To find such bounds, we differentiate Φ_δ obtaining

$$\begin{aligned}\Phi'_\delta(t) &\leq \int_{B_r\cap G_\lambda\cap\tilde{G}_\lambda} \frac{|b(t,X(t,x))-\tilde{b}(t,\tilde{X}(t,x))}{\delta+|X(t,x)-\tilde{X}(t,x)|}\mathrm{d}x \\ &\leq \underbrace{\int_{B_r\cap G_\lambda\cap\tilde{G}_\lambda} \frac{|b(t,X(t,x))-b(t,\tilde{X}(t,x))}{|X(t,x)-\tilde{X}(t,x)|}\mathrm{d}x}_{(I)} + \underbrace{\int_{B_r\cap\tilde{G}_\lambda} \frac{|b(t,\tilde{X}(t,x))-\tilde{b}(t,\tilde{X}(t,x))}{\delta}\mathrm{d}x}_{(II)}\end{aligned} \quad (1.3.13)$$

We estimate the terms on the right hand side separately. First of all, $\tilde{b}$ is divergence-free and changing variables in (II) we have that

$$(II) \leq \frac{1}{\delta}\int_{B_\lambda} |b(t,y)-\tilde{b}(t,y)|\mathrm{d}y. \quad (1.3.14)$$

In order to estimate (I) we will assume different hypothesis on the regularity in space of the vector field b.

The vector field b satisfies

(R2a) $b \in L^1((0,T);W^{1,p}(\mathbb{R}^d))$ **for some** $p>1$**.**

In this case, by lemma A.5.2 and A.3.2 we can estimate (I) in the following way

$$\begin{aligned}(I) &\leq \int_{B_r\cap G_\lambda} |M_{2\lambda}Db(t,X(t,x))|\mathrm{d}x + \int_{B_r\cap\tilde{G}_\lambda} |M_{2\lambda}Db(t,\tilde{X}(t,x))|\mathrm{d}x \\ &\leq 2\int_{B_\lambda} |M_{2\lambda}Db(t,y)|\mathrm{d}y \leq C\lambda^{d\frac{p-1}{p}}\|M_{2\lambda}Db(t,\cdot)\|_{L^p(B_\lambda)} \\ &\leq C\lambda^{d\frac{p-1}{p}}\|Db(t,\cdot)\|_{L^p(B_{3\lambda})},\end{aligned}$$

where $M_{2\lambda}Db$ is the local maximal function of Db, defined as in Definition A.3.1. Integrating in time in (1.3.13) we have that

$$\Phi_\delta(t) \leq \frac{1}{\delta}\|b-\tilde{b}\|_{L^1((0,T)\times B_\lambda)} + C\lambda^{d\frac{p-1}{p}}\|Db\|_{L^1((0,T);L^p(B_{3\lambda}))}, \quad (1.3.15)$$

and substituting in (1.3.12) we get

$$\begin{aligned}\mathscr{L}^d\left(B_r \cap \{|X(t,x)-\tilde{X}(t,x)|>\gamma\}\right) &\leq \frac{\|b-\tilde{b}\|_{L^1((0,T)\times B_\lambda)}}{\delta\log\left(1+\frac{\gamma}{\delta}\right)} + \frac{C\lambda^{d\frac{p-1}{p}}}{\log\left(1+\frac{\gamma}{\delta}\right)}\|Db\|_{L^1((0,T);L^p(B_{3\lambda}))} \\ &\quad + \mathscr{L}^d(B_r\setminus G_\lambda) + \mathscr{L}^d(B_r\setminus\tilde{G}_\lambda),\end{aligned} \quad (1.3.16)$$

where λ, δ are free parameters.

The vector field b satisfies

(R2b) $b \in L^1((0,T); W^{1,1}(\mathbb{R}^d))$,

(R3) $b \in L^p_{\text{loc}}((0,T) \times \mathbb{R}^d)$ **for some** $p > 1$.

In this case we need a more careful estimate on the term

$$\int_{B_r \cap G_\lambda \cap \tilde{G}_\lambda} \frac{|b(t, X(t,x)) - b(t, \tilde{X}(t,x))}{\delta + |X(t,x) - \tilde{X}(t,x)|} \mathrm{d}x. \tag{1.3.17}$$

First of all, since $\nabla b \in L^1((0,T); L^1(\mathbb{R}^d))$, for every fixed $\varepsilon > 0$ there exist a constant C_ε and a set with finite measure A_ε such that ∇b can be splitted as

$$\nabla b = g_1^\varepsilon + g_2^\varepsilon$$

where the functions $g_1^\varepsilon, g_2^\varepsilon$ satisfy the following

$$\|g_1^\varepsilon\|_{L^1((0,T);L^1(\mathbb{R}^d))} \leq \varepsilon, \qquad \text{supp } g_2^\varepsilon \subset A_\varepsilon, \qquad \|g_2^\varepsilon\|_{L^1((0,T);L^2(\mathbb{R}^d))} \leq C_\varepsilon.$$

In (1.3.17) we do not drop the δ dependence as done in (1.3.13) but we estimate in the following way

$$\begin{aligned}
&\int_{B_r \cap G_\lambda \cap \tilde{G}_\lambda} \frac{|b(t, X(t,x)) - b(t, \tilde{X}(t,x))|}{\delta + |X(t,x) - \tilde{X}(t,x)|} \mathrm{d}x \\
&\qquad \leq \int_{B_r \cap G_\lambda \cap \tilde{G}_\lambda} \underbrace{\min\left\{ \frac{|b(t, X(t,x))| + |b(t, \tilde{X}(t,x))|}{\delta}; g_1^\varepsilon(t, X(t,x)) + g_1^\varepsilon(t, \tilde{X}(t,x)) \right\}}_{\varphi(t,x)} \mathrm{d}x \\
&\qquad + \int_{B_r \cap G_\lambda \cap \tilde{G}_\lambda} g_2^\varepsilon(t, X(t,x)) + g_2^\varepsilon(t, \tilde{X}(t,x)) \mathrm{d}x.
\end{aligned}$$

Since $g_2^\varepsilon \in L^2$ we easily find that

$$\int_0^T \int_{B_r \cap G_\lambda \cap \tilde{G}_\lambda} g_2^\varepsilon(t, X(t,x)) + g_2^\varepsilon(t, \tilde{X}(t,x)) \mathrm{d}x\mathrm{d}t \leq C\left(T\mathscr{L}^d(B_\lambda)\right)^{\frac{1}{2}} \|g_2^\varepsilon\|_{L^2} \leq C\left(T\mathscr{L}^d(B_\lambda)\right)^{\frac{1}{2}} C_\varepsilon. \tag{1.3.18}$$

For the function φ we have that

$$\|\varphi\|_{L^p\left((0,T)\times(B_r \cap G_\lambda \cap \tilde{G}_\lambda)\right)} \leq \frac{C}{\delta} \|b\|_{L^p((0,T)\times B_\lambda)},$$

while

$$|||\varphi|||_{M^1((0,T)\times(B_r \cap G_\lambda \cap \tilde{G}_\lambda))} \leq 2|||g_1^\varepsilon|||_{M^1((0,T)\times B_\lambda)} \leq C\|g_1^\varepsilon\|_{L^1((0,T)\times\mathbb{R}^d)}.$$

Then, using the interpolation Lemma A.2.1 we have

$$\|\varphi\|_{L^1((0,T)\times(B_r\cap G_\lambda\cap\tilde{G}_\lambda)} \le C\frac{p}{p-1}\|g_1^\varepsilon\|_{L^1((0,T)\times\mathbb{R}^d)}\left[1+\log\left(\frac{\|b\|_{L^p((0,T)\times B_\lambda)}}{\|g_1^\varepsilon\|_{L^1((0,T)\times\mathbb{R}^d)}}\frac{\left(T\mathscr{L}^d(B_r)\right)^{1-\frac{1}{p}}}{\delta}\right)\right]. \tag{1.3.19}$$

Substituting (1.3.18) and (1.3.19) in (1.3.12) we have that

$$\begin{aligned}\mathscr{L}^d\left(B_r\cap\{|X(t,x)-\tilde{X}(t,x)|>\gamma\}\right) &\le \frac{\|b-\tilde{b}\|_{L^1((0,T)\times B_\lambda)}}{\delta\log\left(1+\frac{\gamma}{\delta}\right)}+\frac{\left(T\mathscr{L}^d(B_\lambda)\right)^{\frac{1}{2}}C_\varepsilon}{\log\left(1+\frac{\gamma}{\delta}\right)}\\ &+C\frac{p}{p-1}\frac{\varepsilon}{\log\left(1+\frac{\gamma}{\delta}\right)}\left[1+\log\left(\frac{\|b\|_{L^p((0,T)\times B_\lambda)}}{\varepsilon}\frac{\left(T\mathscr{L}^d(B_r)\right)^{1-\frac{1}{p}}}{\delta}\right)\right]\\ &+\mathscr{L}^d(B_r\setminus G_\lambda)+\mathscr{L}^d(B_r\setminus\tilde{G}_\lambda),\end{aligned} \tag{1.3.20}$$

where in the estimate (1.3.19) we use that the function $z\in[0,\infty)\mapsto z\left[1+\log\left(\frac{C}{z}\right)\right]\in[0,\infty)$ is non-decreasing.

The vector field b satisfies

(R2c) For every $i,j=1,...,n$ we have

$$\partial_j b^i=\sum_{k=1}^m S^i_{jk}g^i_{jk} \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^d),$$

where S^i_{jk} are singular operator of fundamental type in $\mathbb{R}^d$ (acting as operators in $\mathbb{R}^d$ independently on time) and the functions $g^i_{jk}\in L^1((0,T)\times\mathbb{R}^d)$ for every $i,j=1,...,n$ and every $k=1,...,m$,

(R3) $b\in L^p_{\mathrm{loc}}((0,T)\times\mathbb{R}^d)$ for some $p>1$.

Following the proof for the $W^{1,1}$ case, it is possible to show that the same estimate as in (1.3.20) holds, as long as we change how to deal with the different quotients. In this case we can apply Lemma A.5.3 for almost every t which gives the existence of a function $U(t,\cdot)\in M^1(\mathbb{R}^d)$ such that

$$\begin{aligned}&\int_{B_r\cap G_\lambda\cap\tilde{G}_\lambda}\frac{|b(t,X(t,x))-b(t,\tilde{X}(t,x))|}{\delta+|X(t,x)-\tilde{X}(t,x)|}\mathrm{d}x\\ &\qquad\le\int_{B_r\cap G_\lambda\cap\tilde{G}_\lambda}\min\left\{\frac{|b(t,X(t,x))|+|b(t,\tilde{X}(t,x))|}{\delta};U(t,X(t,x))+U(t,\tilde{X}(t,x))\right\}\mathrm{d}x.\end{aligned} \tag{1.3.21}$$

Fix $\varepsilon>0$ fixed, the family g_{jk} is equi-integrable in $L^1((0,T)\times\mathbb{R}^d)$ since it is finite. This gives the existence of a constant C_ε and a set A_ε with finite measure, such that for every $j=1,...,d$

and $k = 1, ..., m$, we have a decomposition

$$g_{jk}(t,x) = g^1_{jk}(t,x) + g^2_{jk}(t,x),$$

such that

$$\|g^1_{jk}\|_{L^1((0,T)\times\mathbb{R}^d)} \leq \varepsilon, \qquad \text{supp } g^2_{jk} \subset A_\varepsilon, \qquad \|g^2_{jk}\|_{L^2((0,T)\times\mathbb{R}^d)} \leq C_\varepsilon. \tag{1.3.22}$$

By using (1.3.22) we can decompose the function U as

$$\begin{aligned} U &= \sum_{j=1}^{d}\sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j},\xi\in\mathbb{S}^{d-1}\}}\left(S_{jk}g_{jk}\right) \\ &\leq \underbrace{\sum_{j=1}^{d}\sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j},\xi\in\mathbb{S}^{d-1}\}}\left(S_{jk}g^1_{jk}\right)}_{U_1} + \underbrace{\sum_{j=1}^{d}\sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j},\xi\in\mathbb{S}^{d-1}\}}\left(S_{jk}g^2_{jk}\right)}_{U_2}, \end{aligned} \tag{1.3.23}$$

and by Theorem A.4.4 we have the following estimates

$$|||U_1|||_{M^1((0,T)\times\mathbb{R}^d)} \leq P_1\|g_1\|_{L^1((0,T)\times\mathbb{R}^d)}, \tag{1.3.24}$$

$$\|U_2\|_{L^2((0,T)\times\mathbb{R}^d)} \leq P_2\|g_2\|_{L^2((0,T)\times\mathbb{R}^d)}. \tag{1.3.25}$$

Substituting (1.3.23) in (1.3.21), by using the estimates (1.3.24),(1.3.25) and exploiting the interpolation lemma A.2.1 we can find that the estimate (1.3.20) holds true also in this case.

1.3.3 Well-posedness results

In this section we state the well-posedness theorems which follow from the a priori estimate shown above. We start by stating the result concerning the regular Lagrangian flows.

Theorem 1.3.2. *Let b a divergence-free vector field which satisfies assumptions (R1), (R2a)/(R2b)/(R2c), (R3). Then, there exists a unique regular Lagrangian flow X of b. Moreover, let b_n be a sequence of smooth divergence-free vector fields converging in $L^1((0,T);L^1_{\text{loc}}(\mathbb{R}^d))$ to b and assume that for some decomposition*

$$\frac{|b_n(t,x)|}{1+|x|} = \tilde{b}_{n,1}(t,x) + \tilde{b}_{n,2}(t,x)$$

we have

$$\|\tilde{b}_{n,1}\|_{L^1((0,T);L^1(\mathbb{R}^d))} + \|\tilde{b}_{n,2}\|_{L^1((0,T);L^\infty(\mathbb{R}^d))} \leq C$$

for all $n \in \mathbb{N}$ and for some constant C. Let X_n and X be the regular Lagrangian flows associated respectively to b_n and b. Then, $X_n \to X$ locally in measure uniformly in time; that is, for every compact set $K \subset \mathbb{R}^d$

$$\sup_{[0,T]}\int_K 1 \wedge |X_n(t,x) - X(t,x)|\mathrm{d}x \to 0 \quad \text{as } n \to +\infty.$$

Theorem 1.3.2 allows to construct a pure Lagrangian theory of the transport equation. In fact we can define a Lagrangian solution of the transport equation (TE) as follows.

Definition 1.3.2. (Lagrangian solutions) *Let b be a divergence-free vector field which satisfies (R1), (R3) and at least one of (R2a)/(R2b)/(R2c). Let $u_0 \in L^1(\mathbb{R}^d)$ be given, a function u is called a* Lagrangian solution *of* (TE) *if $u \in L^\infty((0,T); L^1(\mathbb{R}^d))$ and there exists an invertible regular Lagrangian flow X associated to b such that*

$$u(t,x) = u_0(X^{-1}(t,\cdot)(x))$$

for all $t \in (0,T)$ and a.e. $x \in \mathbb{R}^d$, where $X^{-1}(t,\cdot)$ denotes the inverse map in space at a fixed time t.

As a consequence of Theorem 1.3.2, Lagrangian solutions of (TE) are well-defined since there always exists a unique flow. Moreover they are stable under approximations as shown in the following proposition.

Proposition 1.3.3. *Let b_ε, b be divergence free vector fields satisfying assumptions (R1), (R2a)/(R2b)/(R2c), and (R3). Assume that $b_\varepsilon \to b$ in $L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^d))$ and that for some decomposition $\dfrac{|b_\varepsilon(t,x)|}{1+|x|} = b_{\varepsilon,1}(t,x) + b_{\varepsilon,2}(t,x)$ we have that*

$$\|b_{\varepsilon,1}\|_{L^\infty((0,T);L^1(\mathbb{R}^d))} + \|b_{\varepsilon,2}\|_{L^\infty((0,T);L^\infty(\mathbb{R}^d))} \leq C.$$

Consider a Lagrangian solution u^ε of (TE) *with coefficient b_ε and initial datum $u_0^\varepsilon \in L^q(\mathbb{R}^d)$, as well as u associated to b and $u_0 \in L^q(\mathbb{R}^d)$. Let $1 \leq q < \infty$, then*

(i) if $u_0^\varepsilon \rightharpoonup u_0$ in L^q, then $u^\varepsilon \rightharpoonup u$ in $C([0,T]; L^q(\mathbb{R}^d))$,

(ii) if $u_0^\varepsilon \to u_0$ in L^q, then $u^\varepsilon \to u$ in $C([0,T]; L^q(\mathbb{R}^d))$.

It is worth to note that a consequence of the stability of Lagrangian solutions is that they are also renormalized solutions. However, even if Lagrangian solutions are unique this does not implies that distributional solutions are unique. It may happen that there exist several weak solutions of (TE) with only one associated to a flow: this is the case when b is Sobolev but the integrability of ∇b does not "match" with the integrability of u and we will return on it in the next section.

1.4 Counterexamples to the well-posedness

In this section we present some examples of non-uniqueness of solutions of both (TE) and (ODE). We start by constructing an example due to Depauw [27] of a vector field whose regularity is very close to the one of Ambrosio's theorem and for which non-uniqueness of solutions holds. Then we present an example of non-uniqueness of (ODE) due to DiPerna-Lions [29] involving a vector field with fractional Sobolev regularity. In section 1.4.3 we will briefly recall some results on the well-posedness in two dimension and as a consequence of these results how is possible to construct a vector field of Hölder regularity $C^{0,\alpha}$ for which non-uniqueness of solutions of (TE) holds. Finally, we will comment the results of Modena and Székelyhidi about the non-uniqueness of solutions of the transport equation with Sobolev vector fields which are not integrable enough to be in the class of well-posedness of DiPerna and Lions.

1.4.1 Depauw's counterexample

Following the idea of Aizenmann [1], Depauw in [28] gave a counterexample to the uniqueness of bounded distributional solutions of the transport equation. He constructs a bounded divergence-free vector field $a : [0,1] \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ for which there exists a non-trivial solution of

$$\begin{cases} \partial_t w + a \cdot \nabla w = 0, \\ w(0, \cdot) = 0. \end{cases} \tag{1.4.1}$$

In order to define a we first consider the vector field $b : [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}^2$ defined as follows

$$b(x_1, x_2) = \begin{cases} (0, 4x_1) & \text{if } 0 < |x_2| < |x_1| < \frac{1}{4}, \\ (-4x_2, 0) & \text{if } 0 < |x_2| < |x_1| < \frac{1}{4}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.4.2}$$

and then we extend b periodically on the whole $\mathbb{R}^2$. The vector field $a(t,x)$ is then defined as follows

$$a(t,x) = \begin{cases} 0 & \text{if } t < 0 \text{ or } t > 1, \\ b(2^j x) & \text{if } t \in I_j := 2^{-j}\left(\frac{1}{2}, 1\right) \text{ for some } j \in \mathbb{N}, \end{cases} \tag{1.4.3}$$

and also define $c(t,x) = a(1-t,x)$. It follows immediately that a and c are bounded and divergence-free. Moreover, $a \in L^1([\varepsilon,1]; BV_{\text{loc}}(\mathbb{R}^2))$ for any $\varepsilon > 0$ but not for $\varepsilon = 0$. We want to describe the flow of c: we denote with $X^c(t_1, t_0, x)$ the solution at time t_1 of

$$\begin{cases} \dot{\psi}(t,x) = c(t, \psi(t,x)), \\ \psi(t_0, x) = x. \end{cases}$$

Since c is piecewise smooth on $[t_0, t_1] \times \mathbb{R}^2$, X^c is well-defined and in particular it is measure preserving and verifies the semi-group property.
Consider now the $\mathbb{Z}^2$-periodic function u_0 defined as

$$u_0(x_1, x_2) = \text{sgn}(x_1 x_2) \quad \text{for } (x_1, x_2) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right],$$

and let $u_j(x) := u_0(2^j x)$. By the semi-group property of X^c we have that

$$u_0(X^c(0, 1-2^j, x)) = u_j(x),$$

therefore the function $u(t,x) = u_0(X^c(0,t,x))$ is a bounded distributional solution of

$$\begin{cases} \partial_t u + \operatorname{div}(cu) = 0, \\ u(0,\cdot) = u_0. \end{cases}$$

Note that u converges weakly-* (but not strongly) to 0 as $t \to 1$. Now define the function $w(t,x) = u(1-t,x)$: w is a non-trivial bounded distributional solution of (1.4.1) since

$$|w| = 1 \qquad \mathscr{L}^3 - a.e. \text{ in } [0,1] \times \mathbb{R}^2,$$

and w satisfies

$$\int_0^1 \int_{\mathbb{R}^2} (\partial\phi + a \cdot \nabla\phi)\, w \mathrm{d}x\mathrm{d}t = 0.$$

Remark 1.4.1. *Depauw's example has been revisited in [4] allowing to construct a vector field $b \in L^1([\varepsilon,T];\mathrm{Lip}(\mathbb{R}^2))$ for all $\varepsilon > 0$ but not for $\varepsilon = 0$, such that the uniqueness of solutions of* (TE) *fails.*

1.4.2 A divergence-free vector field with fractional Sobolev regularity

Define the two dimensional vector field $b = (b_1, b_2)$ as

$$\begin{cases} b_1(x,y) = -\operatorname{sgn}(y)\left(\dfrac{x}{|y|^2}\chi_{\{|x|\le|y|\}} + \operatorname{sgn}(x)\ \chi_{\{|x|>|y|\}}\right), \\ b_2(x,y) = -\left(\dfrac{1}{|y|}\chi_{\{|x|\le|y|\}} + \chi_{\{|x|>|y|\}}\right). \end{cases} \tag{1.4.4}$$

The vector field $b \in W^{s,1}_{\mathrm{loc}}(\mathbb{R}^2)$ for all $s \in [0,1)$, $\operatorname{div} b = 0$ in the sense of distributions, $b \in L^p + L^\infty$ for all $p \in [1,2)$. We can define two different regular Lagrangian flows of b that preserve the Lebesgue measure. In particular, they are different on the set $\{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$ and are defined as follows

$$\begin{cases} X_1(t,x,y) = \dfrac{x}{y}\sqrt{|y^2 - 2t|}, \\ X_2(t,y) = \sigma\sqrt{|y^2 - 2t|}, \end{cases}$$

and

$$\begin{cases} \tilde{X}_1(t,x,y) = \sigma\dfrac{x}{y}\sqrt{|y^2 - 2t|}, \\ \tilde{X}_2(t,y) = \sigma\sqrt{|y^2 - 2t|}, \end{cases}$$

where $\sigma = 1$ if $t \le y^2/2$ and $\sigma = -1$ if $t > y^2/2$.

The nonuniqueness of the flows has the following geometric interpretation: consider the trapezium $\mathcal{T}$ in the half plane $\{y > 0\}$ as in Figure 1.2, then there exists a time t^* such that

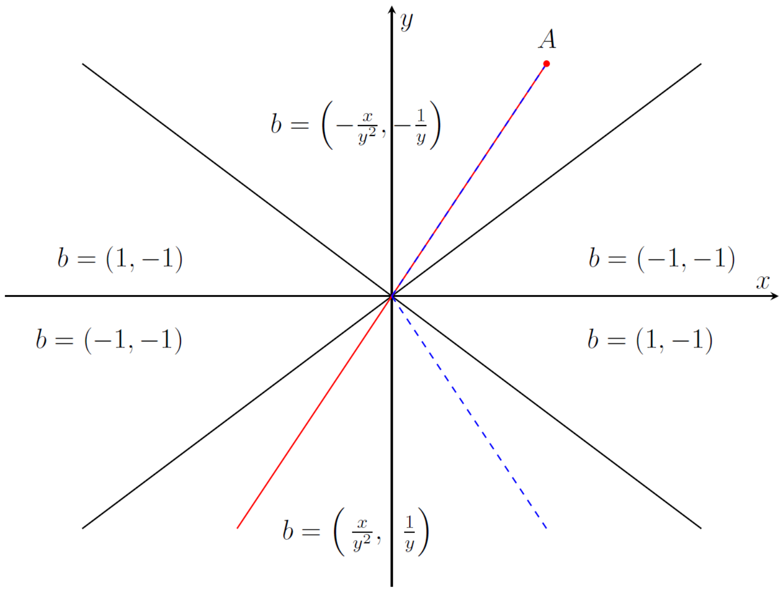

Figure 1.1: The blue line is a characteristic of the flow X while the red line is a characteristic of the flow $\tilde{X}$.

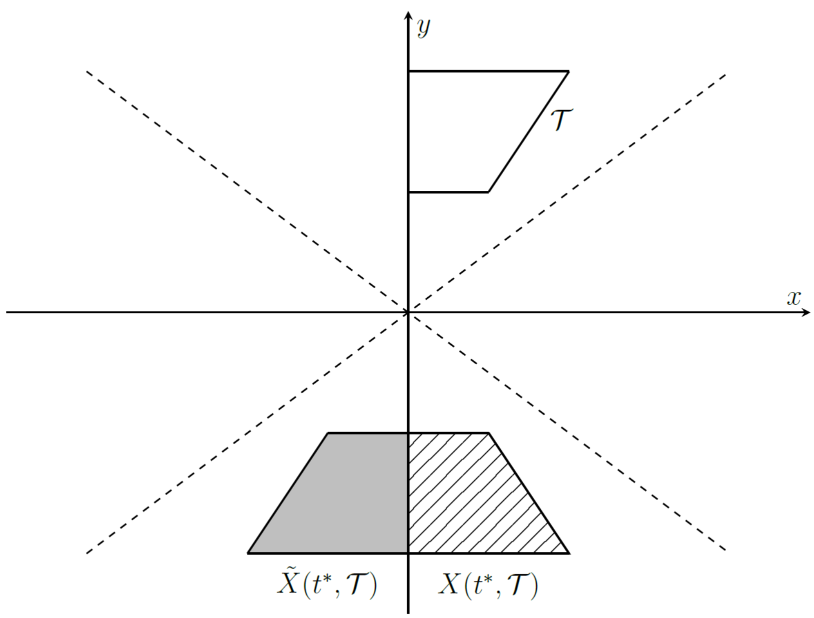

Figure 1.2: Action of the flows on the trapezium $\mathcal{T}$.

- the region filled with diagonal lines is $X(t^*, \mathcal{T})$ and it is symmetric to $\mathcal{T}$ with respect to $\{y = 0\}$;
- the grey region is $\tilde{X}(t^*, \mathcal{T})$ and it is symmetric to $\mathcal{T}$ with respect to $(0, 0)$.

1.4.3 $C^{0,\alpha}$ vector fields

In [2] the authors characterize the two-dimensional autonomous, divergence-free compactly supported vector fields such that the Cauchy problem for the transport equation (TE) has a unique bounded distributional solution for every bounded initial datum. In particular, the two-dimensional case takes advantage of the following fact: since $\operatorname{div} b = 0$ and $\mathbb{R}^2$ is simply connected there exists a compactly supported Lipschitz function H such that

$$b(x) = \nabla^{\perp} H(x) \qquad \text{for almost every } x \in \mathbb{R}^2, \tag{1.4.5}$$

where H is called the *Hamiltonian* associated to b. Since H is constant along the characteristics of b and using the fact that the level sets are invariant under the action of the flow, the transport equation can be reduced to a one dimensional equation on the level sets of H. Indeed, for a.e. $h \in \mathbb{R}$, every connected component C of $E_h = H^{-1}(h)$ is a simple Lipschitz curve which admits a parametrization $\gamma : I \to C$, where $I \subset \mathbb{R}$ is an interval. Under the change of variables $x = \gamma(s)$, the equation on C becomes

$$\partial_t \left(\hat{u}(1+\lambda)\right) + \partial_s \hat{u} = 0 \qquad \text{in } \mathcal{D}'((0,T) \times I), \tag{1.4.6}$$

where λ is a suitable singular measure on I and $\hat{u} = u \circ \gamma$. It is possible to prove that (1.4.6), and as a consequence (TE), admits a unique bounded solution if and only if $\lambda = 0$ for every non trivial connected component C of a.e. level sets E_h. Furthermore, it can be shown that this is equivalent to the following condition on H, known as *weak Sard property*:

$$H_{\#}\mathscr{L}^2 \llcorner_{S \cap E^\star} \perp \mathscr{L}^1, \tag{1.4.7}$$

where $E^\star$ is the union of all connected components with positive lengths of all level sets of H. The main result in [2] is the following:

Theorem 1.4.2. *Let H and b defined as above. Then the following statements are equivalent:*

- *if $u : [0,T) \times \mathbb{R}^2 \to \mathbb{R}$ is a bounded solution of*

$$\begin{cases} \partial_t u + b \cdot \nabla u = 0, \\ u(0,\cdot) = 0, \end{cases} \tag{1.4.8}$$

  then $u(t,x) = 0$ for a.e. $(t,x) \in [0,T) \times \mathbb{R}^2$;

- *H satisfies the weak Sard property.*

As a consequence of the previous theorem, since there exists an Hamiltonian that does not satisfies the weak Sard property, the following result holds.

Corollary 1.4.3. *There exists a divergence-free autonomous vector field on the plane which belongs to $C^{0,\alpha}(\mathbb{R}^2)$ for every $\alpha < 1$, and for which the Cauchy problem* (1.4.8) *has a non trivial bounded solution.*

1.4.4 Non-uniqueness for Sobolev vector fields

It is worth to note that in the previous counterexamples the Eulerian non-uniqueness is obtained as a consequence of the Lagrangian non-uniqueness. In fact, we have constructed vector fields for which the ODE admits more than one flow and then with these flows we can produce many solutions of the PDE. So it is natural to look for solutions of the PDE which are not transported by a flow. In a series of papers, Modena and Székelyhidi in [38, 39] and then Modena and Sattig in [37], proved the following

Theorem 1.4.4. *Let $p, p', \tilde{p} \in [1, \infty)$ such that*

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{p} + \frac{1}{\tilde{p}} > 1 + \frac{1}{d}.$$

Then there are infinitely many incompressible vector fields satisfying

$$u \in C([0,T]; L^{p'}(\mathbb{T}^d)) \cap C([0,T]; W^{1,\tilde{p}}(\mathbb{T}^d))$$

for which uniqueness of distributional solutions to the transport equation (TE) *fails in the class of densities*

$$\rho \in C([0,T]; L^p(\mathbb{T}^d)).$$

Moreover, if $p = 1$ it holds $u \in C([0,T] \times \mathbb{T}^d)$.

In the previous theorem the non-uniqueness is purely Eulerian in the following sense. Consider an initial datum $\rho_0 \in L^p(\mathbb{T}^d)$ and construct a vector field u as in Theorem 1.4.4: since $u \in C([0,T]; W^{1,\tilde{p}}(\mathbb{T}^d))$ the theory developed in the Section 1.3 holds so that u possess a unique regular Lagrangian flow X. This means that the function $\rho(t,x) = \rho_0(X^{-1}(t,\cdot)(x))$ is the unique Lagrangian solution of

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0, \\ \rho(0,\cdot) = \rho_0. \end{cases} \tag{1.4.9}$$

However, in view of Theorem 1.4.4, ρ is not the unique distributional solution (1.4.9): in particular all the other solutions are not transported by the flow and then the Eulerian and the Lagrangian point of view are not equivalent.

We remark that the uniqueness of distributional solutions is still not known when the exponents are in the following range

$$1 < \frac{1}{p} + \frac{1}{\tilde{p}} < 1 + \frac{1}{d}.$$

CHAPTER 2

Smooth approximation is not a selection principle for the transport equation

In this chapter we analyze the problem of possible selection of a unique weak solution of the transport equation with rough vector field. The chapter is divided as follows: in Section 2.1 we give a precise setting of the problem. In Section 2.2 we give a new example of a vector field which admits infinitely many flows and in Section 2.3 we construct a smooth approximating sequence of the vector field for which the corresponding solutions converge to a prescribed solution of the limit problem. Then in Section 2.4 we prove our main theorems and finally in Section 2.5 we give some ideas of possible extensions.

2.1 The problem of selection

In this section we want to give a precise setting to the selection problem for weak solutions of the transport equation. As already stressed in the previous chapter, the existence of solutions of (TE) is an easy issue and does not require any regularity assumption on the vector field. In fact, going back to the proof of Theorem 1.2.1, it is enough that the vector field is locally integrable. The argument of the proof can be summarized as follows:

(i) consider a sequence of smooth vector fields b_ε which converge to b in L^1_{loc};

(ii) for every fixed $\varepsilon > 0$ there exists a unique solution u^ε to the approximating equation (1.2.1);

(iii) a uniform bound on $\|u^\varepsilon\|_{L^\infty}$ guarantees that there exists a weak-* limit u as $\varepsilon \to 0$ and the linearity of the equation is enough to assert that u is a weak solution.

Of course, as for all compactness arguments, we have no information on the uniqueness since there is a passage to subsequences. Therefore, a very natural question is the following:

(Q1) *Does the approximation procedure obtained by smoothing the vector field select a unique solution of* (TE)*?*

For several PDEs, selection principles or admissibility criteria are needed when the regularity of weak solutions is not enough to guarantee uniqueness. For example, this is the case for scalar conservation laws: if we consider weak solutions satisfying in addition the entropy inequality it is possible to prove uniqueness. In the context of the incompressible Euler equations general admissibility criteria, that can be satisfied by only one weak solution, are not known when the initial datum $u_0 \in L^2$. Contrary to the case of scalar conservation laws, criteria based on an energy inequality are known not to select a unique solution, as proved in [26]. Another natural approach would be to consider weak solutions of Euler equations obtained as limit of Navier-Stokes equations. In this regard, in [8] the authors prove that for shear-flow solutions of the Euler equations, the vanishing viscosity limit of Leray weak solutions of the Navier-Stokes equations selects a unique solution. On the other hand, the recent result in [15] shows that the limit of weak solutions of Navier-Stokes equations, which are not Leray weak solutions, does not select a unique solution. Therefore it is fair to say that there is not a clear picture of selection principles in fluid dynamics. Our result shows that, already for the linear transport equation, the very natural approximation procedure of smoothing the vector field, which in analogy with fluid-mechanics might be called Leray-type approximation, does not select a unique solution.

It is worth pointing out that, differently from the non-uniqueness examples obtained via convex integration, the approximation for the linear transport equation that we construct in this chapter is explicit and consists of functions u^ε which are the unique exact solutions of (1.2.1). In this spirit, the problem of selection for bounded solutions can also be posed for other types of approximations which guarantee uniqueness at the approximate level, such as

(Q2) *Does the approximation procedure obtained by smoothing the vector field via a convolution with a suitably chosen mollifier select a unique solution of* (TE)*?*

(Q3) *Does the approximation procedure obtained by vanishing viscosity limit of*

$$\partial_t u^\varepsilon + b \cdot \nabla u^\varepsilon = \varepsilon \Delta u^\varepsilon \tag{2.1.1}$$

select a unique solution of (TE)*?*

Unfortunately we are not able to provide an answer to the two questions above with the techniques of this work. Nevertheless if one looks to a slightly different version of (Q3), considering u^ε as the solution of

$$\partial_t u^\varepsilon + b_\varepsilon \cdot \nabla u^\varepsilon = \varepsilon \Delta u^\varepsilon, \tag{2.1.2}$$

in which we also regularize the vector field, an easy corollary of our main theorem exploiting a diagonal argument shows that there exists a vector field b and a smooth approximation b_ε

for which the selection of a unique solution as limit of solutions of (2.1.2) does not hold.
We point out that in [7] the authors study the behaviour of solutions of (TE) in dimension one, constructed as limit of two different approximations: (i) u^ε_{visc} solution of (2.1.1); (ii) u^ε_{stoc} solution of (TE) with a multiplicative noise of the form $\varepsilon \nabla u \circ \frac{\mathrm{d}W(t)}{\mathrm{d}t}$ at the right hand side, where $W(t)$ is a one-dimensional Brownian motion. In particular, they show that in the limit the sequences u^ε_{visc} and u^ε_{stoc} converge to two different solutions of the limit equation.

2.2 The vector field and its flows

In this section we introduce the vector field b, which will be the limit of the approximation that we are going to construct in the next section. More precisely, we look for a vector field for which the uniqueness of the flow fails. We would like to use the example of DiPerna-Lions presented in Section 1.4.1 to give a negative answer to (Q1). It is not a problem to construct a smooth approximation of (1.4.4) which gives X in the limit. Instead, it is not clear to us how to get $\tilde{X}$ in the limit: we are not able to construct an approximation b_ε of (1.4.4) avoiding intersections of trajectories for fixed ε. In order to avoid this topological problem, we rather work in three space dimensions.
We now introduce the vector field

$$b(x,y,z) = \begin{cases} \left(-\mathrm{sgn}(z)\dfrac{x}{|z|^2}, -\mathrm{sgn}(z)\dfrac{y}{|z|^2}, -\dfrac{2}{|z|} \right) & \text{if } x \in P, \\ (0,0,0) & \text{otherwise,} \end{cases} \tag{2.2.1}$$

where $P \subset \mathbb{R}^3$ denotes the set

$$P = P^+ \cup P^- = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq z\} \cup \{\ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq -z\},$$

being the union of two symmetric paraboloids. The vector field $b \in L^p_{\mathrm{loc}}(\mathbb{R}^3)$ for all $p \in \left[1, \frac{4}{3}\right]$ and it can be directly checked that $\operatorname{div} b = 0$ in the sense of distributions on the whole $\mathbb{R}^3$, in particular b is tangent to ∂P. Moreover the vector field b satisfies the growth conditions of Theorem 1.2.5. Observe that this vector field does not belong to any Sobolev space $W^{1,p}(\mathbb{R}^3)$ or to $BV(\mathbb{R}^3)$. However, similarly to the vector field in Example 1.4.4, we have that $b \in W^{s,1}_{\mathrm{loc}}(\mathbb{R}^3)$ for $0 < s < 1/2$.
We can easily define infinitely many different regular Lagrangian flows of b. Since we are considering flows defined almost everywhere, we need to define them only on $\mathbb{R}^3 \setminus \{0\}$. We start for $\mathbf{x} \in \mathbb{R}^3 \setminus P$: in this region the vector field is identically 0 so that we define a flow X simply as

$$X(t, \mathbf{x}) = \mathbf{x} \qquad \forall t \geq 0.$$

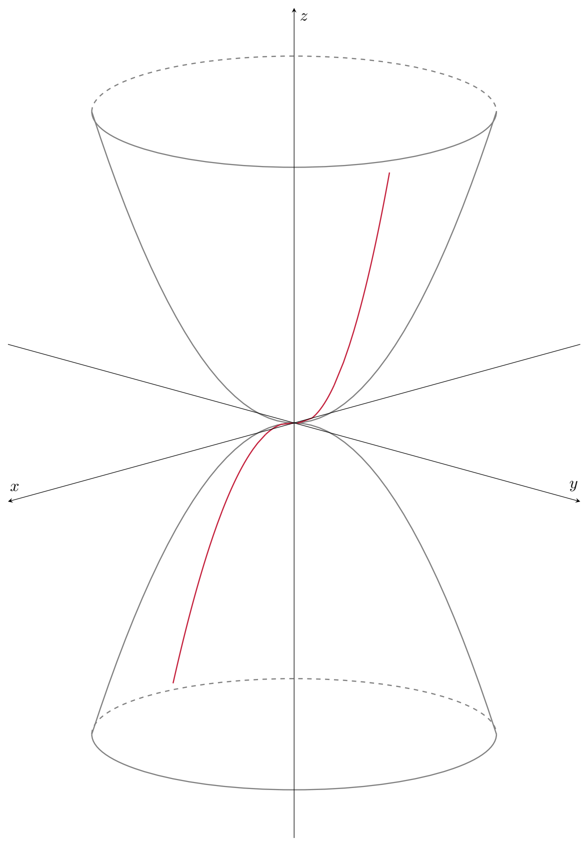

Figure 2.1: An example of flow X^Θ

If $\mathbf{x} = (x, y, z) \in P^-$ we define

$$\begin{cases} X_1(t, x, z) = \dfrac{x}{\sqrt{-z}} \sqrt[4]{z^2 + 4t} \\ X_2(t, y, z) = \dfrac{y}{\sqrt{-z}} \sqrt[4]{z^2 + 4t} \qquad \forall\, t \geq 0. \\ X_3(t, z) = -\sqrt{z^2 + 4t} \end{cases} \tag{2.2.2}$$

Finally, when $\mathbf{x} = (x, y, z) \in P^+$ define the flow as

$$\begin{cases} X_1(t, x, z) = \dfrac{x}{\sqrt{z}} \sqrt[4]{z^2 - 4t} \\ X_2(t, y, z) = \dfrac{y}{\sqrt{z}} \sqrt[4]{z^2 - 4t} \qquad \text{for } t \in \left[0, \dfrac{z^2}{4}\right]. \\ X_3(t, z) = \sqrt{z^2 - 4t} \end{cases} \tag{2.2.3}$$

At time $t = \frac{z^2}{4}$ the trajectories reach the origin. A formal computation shows that the quantity

$$\frac{X_1^2(t, \mathbf{x}) + X_2^2(t, \mathbf{x})}{|X_3(t, \mathbf{x})|} = \frac{x^2 + y^2}{|z|}$$

is conserved by solutions of (3.2). This suggest to define the flow as

$$\begin{cases} X_1(t,x,z) = \frac{x}{\sqrt{z}}\sqrt[4]{4t-z^2}\cos\Theta - \frac{y}{\sqrt{z}}\sqrt[4]{4t-z^2}\sin\Theta \\ X_2(t,y,z) = \frac{x}{\sqrt{z}}\sqrt[4]{4t-z^2}\sin\Theta + \frac{y}{\sqrt{z}}\sqrt[4]{4t-z^2}\cos\Theta \qquad t \geq \frac{z^2}{4}. \\ X_3(t,z) = -\sqrt{4t-z^2} \end{cases} \tag{2.2.4}$$

where $\Theta \in (0, 2\pi]$ is arbitrary. An easy computation shows that X, defined as above, is a regular Lagrangian flow of b for every $\Theta \in (0, 2\pi]$. We call those kind of solutions X^Θ, where Θ represents a rotation in the xy plane. Heuristically, we can define this kind of flows as a consequence of the fact that the trajectories, once they reach the origin, can come out arbitrarily. The lack of uniqueness is a consequence of the fact that all the solutions can be extended in infinitely many ways once they reach the origin. This reproduces the same mechanism of Example 1.4.4, although in this case the additional dimension allows for more flexibility and for an easy and explicit description of the nonunique flows. Actually there are other possible ways to define regular Lagrangian flows of b; we will give a more in-depth discussion on that in Section 2.5.

2.3 Construction of the approximating sequence

In this section we provide an approximation b_ε of the vector field b such that, for a fixed $\Theta \in (0, 2\pi]$, the sequence X^ε of flows of b_ε converges to X^Θ. Our strategy is to approximate the vector field b close to the origin forcing the trajectories to rotate very fast along a given helix. In order to do this, we first smooth the union of the two paraboloids in the origin, see Figure 2.2. Then, we choose the rotation velocity in the cylinder C_ε to be proportional to the height of C_ε. Precisely, the smaller the height of the cylinder, the faster the velocity of rotation of the characteristics. In order to get a smooth transition for the vector field between the truncated paraboloids $P_\varepsilon^+, P_\varepsilon^-$ and the cylinder, we then consider two transitions zones $T_\varepsilon^+, T_\varepsilon^-$ (see again Figure 2.2). Finally, we define the region $\tilde{P}^\varepsilon$ as

$$\tilde{P}^\varepsilon = P_\varepsilon^+ \cup T_\varepsilon^+ \cup C_\varepsilon \cup T_\varepsilon^- \cup P_\varepsilon^-.$$

The main properties of the sequence of approximating vector fields $\{b_\varepsilon\}_\varepsilon$ that we will construct are described in the following proposition.

Proposition 2.3.1. *Let b be the vector field in* (2.2.1). *Given $\Theta \in (0, 2\pi]$ there exists a sequence of vector fields b_ε such that*

1. *b_ε converges to b in $L^1_{\text{loc}}(\mathbb{R}^3)$;*

2. $\operatorname{div} b_\varepsilon = 0$ *in the sense of distributions, in particular b_ε is tangent to $\partial\tilde{P}^\varepsilon$;*

3. *the flow* X^ε *of* b_ε *converges uniformly to* X^Θ*;*

4. $b_\varepsilon \in \mathrm{Lip}(\tilde{P}^\varepsilon)$, b_ε *is identically* 0 *on* $\mathbb{R}^3 \setminus \tilde{P}^\varepsilon$*, and* $b_\varepsilon \in BV_{\mathrm{loc}}(\mathbb{R}^3)$*;*

5. $\frac{b_\varepsilon}{1+|x|} = b_{1,\varepsilon} + b_{2,\varepsilon}$*, with* $b_{1,\varepsilon} \in L^1(\mathbb{R}^3)$ *and* $b_{2,\varepsilon} \in L^\infty(\mathbb{R}^3)$*.*

Proof. We divide the proof in the following steps.

<u>*Step 1*</u> *Construction of the approximation.*

For any $\varepsilon > 0$ we define:

$$b_\varepsilon(x,y,z) = \begin{cases} \left(-\frac{x}{|z|^2}, -\frac{y}{|z|^2}, -\frac{2}{|z|}\right) & \text{in } P_\varepsilon^+, \\ (b_1(x,y,z), b_2(x,y,z), b_3(z)) & \text{in } T_\varepsilon^+, \\ \left(-\frac{y}{\beta^2\varepsilon^2}, \frac{x}{\beta^2\varepsilon^2}, -\frac{27}{16\beta\varepsilon}\right) & \text{in } C_\varepsilon, \\ (\bar{b}_1(x,y,z), \bar{b}_2(x,y,z), \bar{b}_3(z)) & \text{in } T_\varepsilon^-, \\ \left(\frac{x}{|z|^2}, \frac{y}{|z|^2}, -\frac{2}{|z|}\right) & \text{in } P_\varepsilon^-, \\ (0,0,0) & \text{otherwise.} \end{cases} \tag{2.3.1}$$

In the above formula $(b_1(x,y,z), b_2(x,y,z), b_3(z))$ and $\left(\bar{b}_1(x,y,z), \bar{b}_2(x,y,z), \bar{b}_3(z)\right)$ will be defined in the following, while $\alpha, \beta, \gamma, \eta \in \mathbb{R}_+$ and

$$\begin{aligned} P_\varepsilon^+ &:= \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \le z,\ z \ge \alpha\varepsilon\}, \\ T_\varepsilon^+ &:= \left\{(x,y,z) \in \mathbb{R}^3 : \beta\varepsilon + \beta\varepsilon\sqrt{\frac{27(x^2+y^2) - 32\beta\varepsilon}{27(x^2+y^2)}} \le z,\ z \in [\beta\varepsilon, \alpha\varepsilon]\right\}, \\ C_\varepsilon &:= \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \le \frac{32}{27}\beta\varepsilon, z \in [-\gamma\varepsilon, \beta\varepsilon]\right\}, \\ T_\varepsilon^- &:= \left\{(x,y,z) \in \mathbb{R}^3 : -\gamma\varepsilon - \gamma\varepsilon\sqrt{\frac{27(x^2+y^2) - 32\gamma\varepsilon}{27(x^2+y^2)}} \le -z,\ z \in [-\eta\varepsilon, -\gamma\varepsilon]\right\}, \\ P_\varepsilon^- &:= \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \le -z,\ z \le -\eta\varepsilon\}. \end{aligned}$$

In the regions T_ε^+ and T_ε^-, here referred to as *transition zones*, we combine the effects of rotation and dilation for the first two components. This means that in T_ε^+ we define b_1 and b_2 by interpolating linearly in the z variable the value of $b_\varepsilon(x,y,\alpha\varepsilon)$ and $b_\varepsilon(x,y,\beta\varepsilon)$. The third component and the geometry of the regions are defined in order to have that b_ε is a divergence-free vector field: this means that we obtain b_3 and the definition of T_ε^+ by solving the equations

$$\partial_x b_1 + \partial_y b_2 + \partial_z b_3 = 0,$$

$$b_\varepsilon|_{\partial T_\varepsilon^+} \cdot n = 0,$$

where n denotes the unit exterior normal to ∂T_ε^+. With the same idea we define $T_\varepsilon^-, \bar{b}_1, \bar{b}_2, \bar{b}_3$.. For $(x,y,z) \in T_\varepsilon^+$, the vector field b_ε is defined as:

$$b_1(x,y,z) = \frac{z-\beta\varepsilon}{\varepsilon}\frac{x}{\alpha^2\varepsilon^2(\beta-\alpha)} - \frac{z-\alpha\varepsilon}{\varepsilon}\frac{y}{\beta^2\varepsilon^2(\beta-\alpha)},$$

$$b_2(x,y,z) = \frac{z-\alpha\varepsilon}{\varepsilon}\frac{x}{\beta^2\varepsilon^2(\beta-\alpha)} + \frac{z-\beta\varepsilon}{\varepsilon}\frac{y}{\alpha^2\varepsilon^2(\beta-\alpha)},$$

$$b_3(z) = \frac{2}{\alpha^2\varepsilon^3(\beta-\alpha)}\left(\beta\varepsilon z - \frac{z^2}{2}\right).$$

Instead, for $(x,y,z) \in T_\varepsilon^-$, the vector field b_ε is defined as:

$$\bar{b}_1(x,y,z) = \frac{z+\gamma\varepsilon}{\varepsilon}\frac{x}{\eta^2\varepsilon^2(\gamma-\eta)} + \frac{z+\eta\varepsilon}{\varepsilon}\frac{y}{\beta^2\varepsilon^2(\gamma-\eta)},$$

$$\bar{b}_2(x,y,z) = -\frac{z+\eta\varepsilon}{\varepsilon}\frac{x}{\beta^2\varepsilon^2(\gamma-\eta)} + \frac{z+\gamma\varepsilon}{\varepsilon}\frac{y}{\eta^2\varepsilon^2(\gamma-\eta)},$$

$$\bar{b}_3(z) = -\frac{2}{\eta^2\varepsilon^3(\gamma-\eta)}\left(\frac{z^2}{2} + \gamma\varepsilon z\right).$$

Moreover, in order to connect the various regions, the parameters are chosen so that:

$$4\beta = 3\alpha,\ 4\gamma = 3\eta,\ \beta = \gamma.$$

We remark that β is the only free parameter, representing the half height of the cylinder, and it will be chosen later in the proof. The vector fields b_ε and b differ only in the set $A_\varepsilon := T_\varepsilon^+ \cup C_\varepsilon \cup T_\varepsilon^-$. Since $\mathscr{L}^3(A_\varepsilon) = C\varepsilon^2$, $\|b_\varepsilon\|_\infty \leq C\varepsilon^{-3/2}$, by the trivial estimate

$$\int_{A_\varepsilon} |b_\varepsilon - b|\ \mathrm{d}x \leq \int_{A_\varepsilon} |b_\varepsilon|\mathrm{d}x + \int_{A_\varepsilon} |b|\mathrm{d}x,$$

we get the L^1_{loc} convergence of b_ε to b.

<u>Step 2</u> *Construction of the characteristics.*

We now compute the characteristics of the vector field b_ε for $\mathbf{x} \in P_\varepsilon^+$, as it is the region of interest for the non-uniqueness. Similar computations allow to compute the characteristics in the whole $\mathbb{R}^3$ and so we omit them. Consider the following system of ordinary differential equations

$$\begin{cases} \dot{X}^\varepsilon(t,\mathbf{x}) = b_\varepsilon(X^\varepsilon(t,\mathbf{x})) \\ X^\varepsilon(0,\mathbf{x}) = \mathbf{x} \end{cases} \qquad \mathbf{x} \in P_\varepsilon^+. \tag{2.3.2}$$

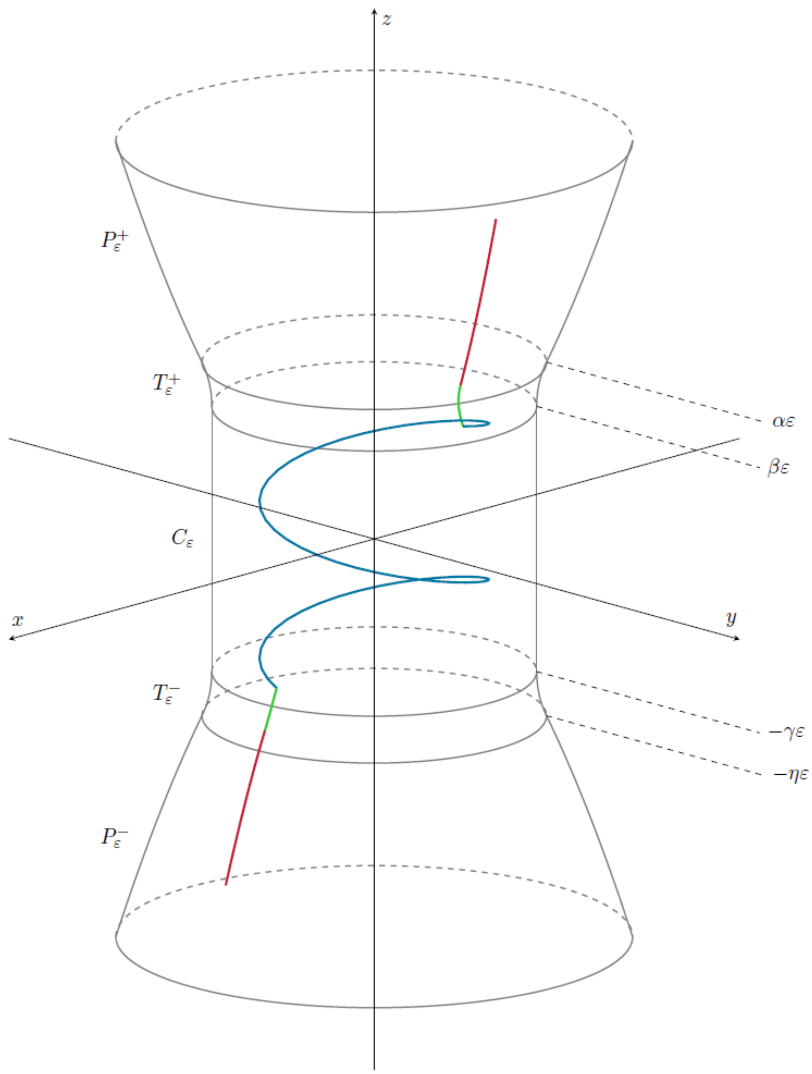

Figure 2.2: The flow X^ε is represented by different colors according to the region in which it is located. In the limit, it converges to the one of Figure 2.1.

Since b_ε is smooth on P_ε^+, (2.3.2) has a unique solution given by:

$$\begin{cases} X_1^\varepsilon(t,x,z) = \frac{x}{\sqrt{z}}\sqrt[4]{z^2-4t} \\ X_2^\varepsilon(t,y,z) = \frac{y}{\sqrt{z}}\sqrt[4]{z^2-4t} \\ X_3^\varepsilon(t,z) = \sqrt{z^2-4t} \end{cases} \quad t \in \left[0, t_1^\varepsilon := \frac{z^2-\alpha^2\varepsilon^2}{4}\right].$$

At $t = t_1^\varepsilon$, we have $X_3^\varepsilon(t_1^\varepsilon, z) = \alpha\varepsilon$ and the equations change. Specifically we have for the third component

$$\begin{cases} \dot{X}_3^\varepsilon(t,z) = -\frac{27}{8\beta^3\varepsilon^3}\left(\beta\varepsilon X_3^\varepsilon - \frac{(X_3^\varepsilon)^2}{2}\right), \\ X_3^\varepsilon(t_1^\varepsilon, z) = \alpha\varepsilon. \end{cases}$$

The solution is

$$X_3^\varepsilon = \frac{4\beta\varepsilon}{2+\exp\left(\frac{27}{8\beta^2\varepsilon^2}(t-t_1^\varepsilon)\right)}, \tag{2.3.3}$$

up to the time $t_2^\varepsilon := t_1^\varepsilon + \frac{8\beta^2\varepsilon^2}{27}\ln(2)$, when $X_3^\varepsilon(t_2^\varepsilon, z) = \beta\varepsilon$. Substituting (2.3.3) in the first two equations, we can rewrite them in the form

$$\begin{cases} \dot{X}_1^\varepsilon = A(t)X_1^\varepsilon - B(t)X_2^\varepsilon, \\ \dot{X}_2^\varepsilon = B(t)X_1^\varepsilon + A(t)X_2^\varepsilon, \\ X_1^\varepsilon(t_1^\varepsilon, x, z) = \frac{x}{\sqrt{z}}\sqrt{\alpha\varepsilon}, \\ X_2^\varepsilon(t_1^\varepsilon, y, z) = \frac{y}{\sqrt{z}}\sqrt{\alpha\varepsilon}, \end{cases} \tag{2.3.4}$$

where the coefficients $A(t), B(t)$ are defined as

$$A(t) = -\frac{27}{16}\frac{X_3^\varepsilon - \beta\varepsilon}{\beta^3\varepsilon^3},$$

$$B(t) = -\frac{3X_3^\varepsilon - 4\beta\varepsilon}{\beta^3\varepsilon^3}.$$

Multiplying the first equation by X_1^ε and the second one by X_2^ε, adding the two equations and setting $\varphi_\varepsilon = (X_1^\varepsilon)^2 + (X_2^\varepsilon)^2$, we obtain that

$$\begin{cases} \dot{\varphi}_\varepsilon = 2A(t)\varphi_\varepsilon, \\ \varphi_\varepsilon(t_1^\varepsilon) = \frac{x^2+y^2}{z}\alpha\varepsilon, \end{cases}$$

yielding

$$\varphi_\varepsilon(t) = \frac{x^2+y^2}{z}\alpha\varepsilon e^{-\frac{27}{8\beta^2\varepsilon^2}(t-t_1^\varepsilon)}\left(\frac{2+\exp\left(\frac{27}{8\beta^2\varepsilon^2}(t-t_1^\varepsilon)\right)}{3}\right)^2. \tag{2.3.5}$$

Because $X_1^\varepsilon = \sqrt{\varphi_\varepsilon}\cos\theta$ and $X_2^\varepsilon = \sqrt{\varphi_\varepsilon}\sin\theta$, substituting these expressions in the equations (2.3.4) we get

$$\dot{\theta}(t) = B(t),$$

and then

$$\theta(t) = \theta_0 + \frac{2}{\beta^2\varepsilon^2}(t - t_\varepsilon^1) - \frac{16}{9}\ln\left(\frac{2+\exp\left(\frac{27}{8\beta^2\varepsilon^2}(t-t_1^\varepsilon)\right)}{3}\right), \tag{2.3.6}$$

where

$$\theta_0 = \begin{cases} \pi & \text{if } y=0, x<0, \\ 2\arctan\left(\frac{y}{x+\sqrt{x^2+y^2}}\right) & \text{otherwise.} \end{cases}$$

During the passage in the first transition zone, the trajectory rotates by an angle

$$\bar{\theta} = \theta(t_2^\varepsilon) - \theta_0 = \frac{16}{27}\ln(2) + \frac{16}{9}\ln\left(\frac{3}{4}\right).$$

At time t_2^ε the flow enters the cylinder and the system becomes

$$\begin{cases} \dot{X}_1^\varepsilon = -\frac{X_2^\varepsilon}{\varepsilon^2}, \\ \dot{X}_2^\varepsilon = \frac{X_1^\varepsilon}{\varepsilon^2}, \\ \dot{X}_3^\varepsilon = -\frac{27}{16\beta\varepsilon}. \end{cases}$$

Then, the solution can be extended as

$$\begin{cases} X_1^\varepsilon(t) = X_1^\varepsilon(t_2^\varepsilon)\cos\left(\frac{t-t_2^\varepsilon}{\varepsilon^2}\right) - X_2^\varepsilon(t_2^\varepsilon)\sin\left(\frac{t-t_2^\varepsilon}{\varepsilon^2}\right), \\ X_2^\varepsilon(t) = X_1^\varepsilon(t_2^\varepsilon)\sin\left(\frac{t-t_2^\varepsilon}{\varepsilon^2}\right) + X_2^\varepsilon(t_2^\varepsilon)\cos\left(\frac{t-t_2^\varepsilon}{\varepsilon^2}\right), \\ X_3^\varepsilon(t) = \beta\varepsilon - \frac{27}{16\beta\varepsilon}(t-t_2^\varepsilon), \end{cases} \tag{2.3.7}$$

up to the time $t_3^\varepsilon := t_2^\varepsilon + \frac{32}{27}\beta^2\varepsilon^2$ when $X_3^\varepsilon(t_3^\varepsilon) = -\beta\varepsilon$. Then during the time $t_3^\varepsilon - t_2^\varepsilon$ the trajectory rotates with respect to $X_1^\varepsilon(t_2^\varepsilon), X_2^\varepsilon(t_2^\varepsilon)$ of an angle

$$\frac{t_3^\varepsilon - t_2^\varepsilon}{\varepsilon^2} = \frac{32}{27}\beta^2.$$

Following the same steps as before, the solution of the system in the second transition zone is

$$\begin{cases} X_1^\varepsilon = \sqrt{\rho}\cos\phi, \\ X_2^\varepsilon = \sqrt{\rho}\sin\phi, \\ X_3^\varepsilon = -\frac{2\beta\varepsilon}{1+\exp\left(-\frac{27}{8\beta^2\varepsilon^2}(t-t_3^\varepsilon,)\right)}, \end{cases} \tag{2.3.8}$$

where

$$\rho(t) = \rho(t_3^\varepsilon)e^{\frac{27}{8\beta^2\varepsilon^2}(t-t_3^\varepsilon)}\left(\frac{1+\exp\left(-\frac{27}{8\beta^2\varepsilon^2}(t-t_3^\varepsilon)\right)}{2}\right)^2,$$

$$\phi(t) = \phi_0 - \frac{16}{9}\ln\left(\frac{1+\exp\left(-\frac{27}{8\beta^2\varepsilon^2}(t-t_3^\varepsilon)\right)}{2}\right) - \frac{2}{\beta^2\varepsilon^2}(t-t_3^\varepsilon),$$

up to the time $t_4^\varepsilon := t_3^\varepsilon + \frac{8\beta^2\varepsilon^2}{27}\ln(2)$, where $\phi_0 = \theta_0 + \bar{\theta} + \frac{32}{27}\beta^2$. Then, note that $\phi(t_4^\varepsilon) = -\theta(t_2^\varepsilon) = -\bar{\theta}$ and the new initial datum for the ODE system is

$$\left(\sqrt{\frac{x^2+y^2}{z}}\sqrt{\frac{4}{3}\beta\varepsilon}\cos\left(\theta_0 + \frac{32}{27}\beta^2\right),\ \sqrt{\frac{x^2+y^2}{z}}\sqrt{\frac{4}{3}\beta\varepsilon}\sin\left(\theta_0 + \frac{32}{27}\beta^2\right),\ -\frac{4}{3}\beta\varepsilon\right).$$

For time larger than t_4^ε, the flow continues as

$$\begin{cases} X_1^\varepsilon = \sqrt{\frac{x^2+y^2}{z}}\sqrt[4]{4(t-t_4^\varepsilon)+\frac{16}{9}\beta^2\varepsilon^2}\cos\left(\theta_0+\frac{32}{27}\beta^2\right), \\ X_2^\varepsilon = \sqrt{\frac{x^2+y^2}{z}}\sqrt[4]{4(t-t_4^\varepsilon)+\frac{16}{9}\beta^2\varepsilon^2}\sin\left(\theta_0+\frac{32}{27}\beta^2\right), \\ X_3^\varepsilon = -\sqrt{4(t-t_4^\varepsilon)+\frac{16}{9}\beta^2\varepsilon^2}. \end{cases} \tag{2.3.9}$$

In conclusion, to find the solution X^Θ in the limit, we have to choose the parameter β as

$$\beta = \sqrt{\frac{27}{32}\Theta}.$$

<u>*Step 3*</u> *Convergence of the flows.*

In this step we prove the convergence of the flows.
First we know that

$$X^\varepsilon(t,\mathbf{x}) = X^\Theta(t,\mathbf{x}) \qquad \forall \mathbf{x}\in P_\varepsilon^- \cup \left(\mathbb{R}^3 \setminus \tilde{P}^\varepsilon\right),\ \forall t\in[0,T].$$

We prove only the convergence for $x\in P_\varepsilon^+$, since the same argument works in $T_\varepsilon^+\cup C_\varepsilon\cup T_\varepsilon^-$. First of all, we have that

$$X^\varepsilon(t,\mathbf{x}) = X^\Theta(t,\mathbf{x}) \qquad \forall t\in[0,t_1^\varepsilon],\ \forall \mathbf{x}\in P_\varepsilon^+.$$

Then the trajectories X^ε and X^Θ enter the approximated region and exit from it after a different amount of time, namely

$$\Delta t_\varepsilon = (2+\ln 2)\frac{16}{27}\beta^2\varepsilon^2, \qquad \Delta t_\Theta = \frac{8}{9}\beta^2\varepsilon^2.$$

Since for $t\in[t_1^\varepsilon, t_1^\varepsilon+\Delta t_\Theta]$ both $X^\varepsilon(t,\mathbf{x})$ and $X^\Theta(t,\mathbf{x})$ are in $T_\varepsilon^+\cup C_\varepsilon\cup T_\varepsilon^-$, we have

$$|X^\varepsilon(t,\mathbf{x}) - X^\Theta(t,\mathbf{x})| \le C\sqrt{\varepsilon}, \qquad \forall t\in[t_1^\varepsilon, t_1^\varepsilon+\Delta t_\Theta],\ \forall \mathbf{x}\in P_\varepsilon^+.$$

For $t\in[t_1^\varepsilon+\Delta t_\Theta, t_1^\varepsilon+\Delta t_\varepsilon]$ the flow X^ε is still in C_ε while X^Θ lies in P_ε^-. Since

$$X_3^\Theta(t_4^\varepsilon, z) = -\frac{4}{3}\beta\varepsilon\sqrt{\frac{5+4\ln 2}{3}},$$

we have that

$$|X^\varepsilon(t,\mathbf{x}) - X^\Theta(t,\mathbf{x})| \le C\sqrt{\varepsilon}, \qquad \forall t\in[t_1^\varepsilon, t_4^\varepsilon],\ \forall \mathbf{x}\in P_\varepsilon^+.$$

Then, for $t\ge t_4^\varepsilon$, the flow X^ε exit the approximated region at the same point as the flow X^Θ and it can be written as

$$X^\varepsilon(t,\mathbf{x}) = X^\Theta(t-\Delta_\varepsilon,\mathbf{x}),$$

where $\Delta_\varepsilon = O(\varepsilon^2)$ is such that $t_4^\varepsilon = \frac{z^2}{4}+\Delta_\varepsilon$. So for $t\ge t_4^\varepsilon$, we estimate the difference $X^\varepsilon - X^\Theta$ component by component:

- for the third component we have

$$\begin{aligned}
|X_3^\varepsilon(t,\mathbf{x}) - X_3^\Theta(t,\mathbf{x})| &= \left|X_3^\Theta(t-\Delta_\varepsilon, z) - X_3^\Theta(t,z)\right| \\
&= \left|\sqrt{4t-z^2} - \sqrt{4(t-\Delta_\varepsilon)-z^2}\right| \\
&= \frac{4\Delta_\varepsilon}{|\sqrt{4t-z^2} + \sqrt{4(t-\Delta_\varepsilon)-z^2}|} \\
&\le \frac{4\Delta_\varepsilon}{\sqrt{4t-z^2}} \le 2\sqrt{\Delta_\varepsilon} \le C\varepsilon,
\end{aligned}$$

- for $i \in \{1,2\}$ we have

$$\begin{aligned}
|X_i^\varepsilon(t,\mathbf{x}) - X_i^\Theta(t,\mathbf{x})| &= |X_i^\Theta(t-\Delta_\varepsilon,\mathbf{x}) - X_i^\Theta(t,\mathbf{x})| \\
&\le \sqrt{\frac{x^2+y^2}{z}}\left|\sqrt[4]{4t-z^2} - \sqrt[4]{4(t-\Delta_\varepsilon)-z^2}\right| \\
&\le \left|\frac{\sqrt{4t-z^2} - \sqrt{4(t-\Delta_\varepsilon)-z^2}}{\sqrt[4]{4t-z^2} + \sqrt[4]{4(t-\Delta_\varepsilon)-z^2}}\right| \\
&\le \frac{4\Delta_\varepsilon}{\sqrt[4]{4t-z^2}\sqrt{4t-z^2}} \le C\sqrt[4]{\Delta_\varepsilon} \le C\sqrt{\varepsilon}.
\end{aligned}$$

Note that in the previous estimate we have used the condition $x^2+y^2 \le z$. In conclusion, we have

$$\sup_{t\in[0,T]}\sup_{\mathbf{x}\in\mathbb{R}^3} |X^\varepsilon(t,\mathbf{x}) - X^\Theta(t,\mathbf{x})| < C\sqrt{\varepsilon},$$

which gives the desired convergence.

Step 4 *Regularity of the approximation.*

In this step we check the regularity of b_ε. It is easy to verify that b_ε is locally bounded and

$$\|\nabla b_\varepsilon\|_\infty \le C\varepsilon^{-5/2},$$

inside $\tilde{P}_\varepsilon$ up to the boundary, so b_ε is Lipschitz inside $\tilde{P}_\varepsilon$ for fixed ε. Furthermore $b_\varepsilon = 0$ in $\mathbb{R}^3 \setminus \tilde{P}^\varepsilon$ and the jump across the surface $\partial\tilde{P}^\varepsilon$ is controlled by $C\varepsilon^{-5/2}$ implying $b_\varepsilon \in BV_{\text{loc}}(\mathbb{R}^3)$. We can easily prove that $\operatorname{div} b_\varepsilon = 0$ inside $\tilde{P}_\varepsilon$ and that b_ε is tangent to $\partial\tilde{P}_\varepsilon$, hence it is divergence-free in the sense of distributions in the whole space. The growth condition follows easily from the fact that the limit vector field b verifies it. □

2.4 Non selection of the approximating sequence

In this section we give the proofs of the main theorems mentioned in the introduction. First of all, we prove a non selection theorem for the flows.

Theorem 2.4.1. *There exists a divergence free vector field $b \in L^p_{\text{loc}}(\mathbb{R}^3)$ with $p \in [1, \frac{4}{3}]$, and a sequence of divergence-free vector fields $b_n \in C^\infty(\mathbb{R}^3)$ such that $b_n \to b$ strongly in $L^p_{\text{loc}}(\mathbb{R}^3)$ and the uniquely defined sequence X^n of flows of b_n does not converge, but has at least two different subsequences along converging in $L^\infty((0,T); L^1_{\text{loc}}(\mathbb{R}^3))$ to two different flows.*

Proof. Let b be the vector field defined in (2.2.1) and let $\Theta, \Phi \in (0, 2\pi]$ with $\Theta \neq \Phi$. From Proposition 2.3.1 there exist $b^\Theta_\varepsilon, b^\Phi_\varepsilon \in BV_{\text{loc}}(\mathbb{R}^3)$ and $X^\Theta, X^\Phi \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}^3))$ with the following properties. First, it holds that

$$\begin{aligned} b^\Theta_\varepsilon &\longrightarrow b \text{ in } L^1_{\text{loc}}(\mathbb{R}^3), \text{ as } \varepsilon \to 0, \\ b^\Phi_\varepsilon &\longrightarrow b \text{ in } L^1_{\text{loc}}(\mathbb{R}^3), \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.4.1}$$

Moreover, by denoting with $X^\Theta_\varepsilon, X^\Phi_\varepsilon$ the unique regular Lagrangian flows of $b^\Theta_\varepsilon, b^\Phi_\varepsilon$, it holds that

$$\begin{aligned} X^\Theta_\varepsilon &\longrightarrow X^\Theta \text{ in } L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^3)), \text{ as } \varepsilon \to 0, \\ X^\Phi_\varepsilon &\longrightarrow X^\Phi \text{ in } L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^3)), \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.4.2}$$

Let $b^\Theta_{\varepsilon,l}, b^\Phi_{\varepsilon,k} \in C^\infty(\mathbb{R}^3)$ be mollifications of $b^\Theta_\varepsilon, b^\Phi_\varepsilon$. Since $b^\Theta_\varepsilon, b^\Phi_\varepsilon$ are in $BV_{\text{loc}}(\mathbb{R}^3)$ for fixed ε, by using Theorem 1.3.3 it follows that, for $\varepsilon > 0$ fixed

$$\begin{aligned} X^\Theta_{\varepsilon,l} &\longrightarrow X^\Theta_\varepsilon \text{ in } L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^3)), \text{ as } l \to \infty, \\ X^\Phi_{\varepsilon,k} &\longrightarrow X^\Phi_\varepsilon \text{ in } L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^3)), \text{ as } k \to \infty, \end{aligned} \tag{2.4.3}$$

where $X^\Theta_{\varepsilon,l}, X^\Phi_{\varepsilon,k}$ denote the smooth flows of $b^\Theta_{\varepsilon,l}, b^\Phi_{\varepsilon,k}$ respectively. By using (2.4.2), (2.4.3) and a simple diagonal argument there $\varepsilon_i, l_i, \varepsilon_j, k_j$ with $i, j \in \mathbb{N}$ such that

$$\begin{aligned} X^\Theta_{\varepsilon_i,l_i} &\longrightarrow X^\Theta \text{ in } L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^3)), \text{ as } i \to \infty, \\ X^\Phi_{\varepsilon_j,k_j} &\longrightarrow X^\Phi \text{ in } L^1((0,T); L^1_{\text{loc}}(\mathbb{R}^3)), \text{ as } j \to \infty. \end{aligned}$$

Finally, since both $b^\Theta_{\varepsilon,l}, b^\Phi_{\varepsilon,k}$ strongly converge in $L^1_{\text{loc}}(\mathbb{R}^3)$ to b, by merging $b^\Theta_{\varepsilon_i,l_i}, b^\Phi_{\varepsilon_j,k_j}$ and appropriately renaming the indexes we can infer that there exists $\{b_n\}_n$ as claimed in the statement of the theorem. □

We now move to the proof of the analougus theorem for the transport equation. Consider the set

$$\mathcal{I} = \left\{ f \in C^\infty(\mathbb{R}^3) : f = f(r, \theta, z) \text{ with } f(r, \theta_1, z) \neq f(r, \theta_2, z) \text{ if } \theta_1 \neq \theta_2 \right\},$$

where (r, θ, z) denotes the cilindrical coordinates in $\mathbb{R}^3$. The theorem is the following.

Theorem 2.4.2. *There exist an autonomous divergence-free vector field $b \in L^p_{\text{loc}}(\mathbb{R}^3)$ with $p \in [1, \frac{4}{3}]$, and a sequence of divergence-free vector fields $b_n \in C^\infty(\mathbb{R}^3)$ converging to b strongly in $L^p_{\text{loc}}(\mathbb{R}^3)$ such that the following happens. Let $u_0 \in \mathcal{I} \cap L^\infty(\mathbb{R}^3)$ be a given inital datum, then there exist subsequences n_i and n_j such that the sequences u_{n_i} and u_{n_j}, solutions of* (1.2.1)*, converge in $L^\infty((0,T); L^\infty(\mathbb{R}^3) - w*) \cap L^\infty((0,T); L^1_{\text{loc}}(\mathbb{R}^3))$ to two different limits, which are bounded distributional solutions of* (TE).

Proof. Let b be the vector field defined in (2.2.1) and let $\Theta, \Phi \in (0, 2\pi]$ with $\Theta \neq \Phi$. From Proposition 2.3.1 there exist $b_\varepsilon^\Theta, b_\varepsilon^\Phi \in BV_{\text{loc}}(\mathbb{R}^3)$ such that

$$\begin{aligned} &b_\varepsilon^\Theta \longrightarrow b \text{ in } L^1_{\text{loc}}(\mathbb{R}^3), \text{ as } \varepsilon \to 0, \\ &b_\varepsilon^\Phi \longrightarrow b \text{ in } L^1_{\text{loc}}(\mathbb{R}^3), \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.4.4}$$

Let $u_0 \in \mathcal{I} \cap L^\infty(\mathbb{R}^3)$ and consider the Cauchy problems

$$\begin{cases} \partial_t u_\Theta^\varepsilon + b_\varepsilon^\Theta \cdot \nabla u_\Theta^\varepsilon = 0, \\ u_\Theta^\varepsilon|_{t=0} = u_0, \end{cases} \tag{2.4.5}$$

$$\begin{cases} \partial_t u_\Phi^\varepsilon + b_\varepsilon^\Phi \cdot \nabla u_\Phi^\varepsilon = 0, \\ u_\Phi^\varepsilon|_{t=0} = u_0. \end{cases} \tag{2.4.6}$$

Since $b_\varepsilon^\Theta, b_\varepsilon^\Phi$ verify the hypothesis of Theorem 1.2.5 for every fixed ε, the solutions of 2.4.5 and 2.4.6 are unique and they are given respectively by the formulas

$$\begin{aligned} &u_\Theta^\varepsilon(t,x) = u_0((X_\varepsilon^\Theta(t,\cdot)^{-1}(x)), \\ &u_\Phi^\varepsilon(t,x) = u_0((X_\varepsilon^\Phi(t,\cdot)^{-1}(x)), \end{aligned} \tag{2.4.7}$$

where $X_\varepsilon^\Theta, X_\varepsilon^\Phi$ are the unique regular Lagrangian flows of $b_\varepsilon^\Theta, b_\varepsilon^\Phi$. Then u_Θ^ε converge uniformly on compact sets to $u_\Theta := u_0((X^\Theta)^{-1})$, since

$$\begin{aligned} &\sup_{t\in[0,T]} \sup_{B_R} |u_\Theta^\varepsilon(t,\cdot) - u_\Theta(t,\cdot)| \\ &\qquad = \sup_{t\in[0,T]} \sup_{B_R} |u_0((X_\varepsilon^\Theta(t,\cdot)^{-1}) - u_0((X^\Theta(t,\cdot)^{-1})| \\ &\qquad \leq \|\nabla u_0\|_{L^\infty(B_R)} \sup_{t\in[0,T]} \sup_{\mathbb{R}^3} |X_\varepsilon^\Theta(t,\cdot)^{-1} - X^\Theta(t,\cdot)^{-1}| \\ &\qquad \leq C\sqrt{\varepsilon}. \end{aligned}$$

Here B_R is a closed ball of radius $R > 0$, $u_0 \in C^\infty(\mathbb{R}^3)$ so it is Lipschitz on compact sets and the backward flow $X_\varepsilon^\Theta(t,\cdot)^{-1}$ converges uniformly to $X^\Theta(t,\cdot)^{-1}$ with the same rate of convergence of the forward flows. The convergence of $X_\varepsilon^\Theta(t,\cdot)^{-1}$ towards $X^\Theta(t,\cdot)^{-1}$ is an easy consequence of (3) in Proposition 2.3.1 and so we omit the details. The same convergence holds for u_Φ^ε towards $u_\Phi := u_0((X^\Phi)^{-1})$.

Let $b_{\varepsilon,l}^\Theta, b_{\varepsilon,k}^\Phi \in C^\infty(\mathbb{R}^3)$ be regularizations of $b_\varepsilon^\Theta, b_\varepsilon^\Phi$. Since $b_\varepsilon^\Theta, b_\varepsilon^\Phi$ are in $BV_{\text{loc}}(\mathbb{R}^3)$ for fixed $\varepsilon > 0$, using Theorem 1.2.5 it follows that

$$\begin{aligned} &u_\Theta^{\varepsilon,l} \longrightarrow u_\Theta^\varepsilon \text{ in } L^\infty([0,T]; L^\infty(\mathbb{R}^3) - w*) \cap L^\infty([0,T]; L^1_{\text{loc}}(\mathbb{R}^3)), \text{ as } l \to \infty, \\ &u_\Phi^{\varepsilon,k} \longrightarrow u_\Phi^\varepsilon \text{ in } L^\infty([0,T]; L^\infty(\mathbb{R}^3) - w*) \cap L^\infty([0,T]; L^1_{\text{loc}}(\mathbb{R}^3)), \text{ as } k \to \infty. \end{aligned}$$

Arguing as in the proof of Theorem 4.2.9, by a diagonal argument we can infer that there exist $\varepsilon_i, l_i, \varepsilon_j, k_j$ with $i, j \in \mathbb{N}$ such that

$$\begin{aligned} &u_\Theta^{\varepsilon_i,l_i} \longrightarrow u_\Theta \text{ in } L^\infty([0,T]; L^\infty(\mathbb{R}^3) - w*) \cap L^\infty([0,T]; L^1_{\text{loc}}(\mathbb{R}^3)), \text{ as } i \to \infty, \\ &u_\Phi^{\varepsilon_j,k_j} \longrightarrow u_\Phi \text{ in } L^\infty([0,T]; L^\infty(\mathbb{R}^3) - w*) \cap L^\infty([0,T]; L^1_{\text{loc}}(\mathbb{R}^3)), \text{ as } j \to \infty. \end{aligned}$$

Since both $b^{\Theta}_{\varepsilon,l}, b^{\Phi}_{\varepsilon,k}$ strongly converge in $L^1_{\text{loc}}(\mathbb{R}^3)$ to b, by merging $b^{\Theta}_{\varepsilon_i,l_i}, b^{\Phi}_{\varepsilon_j,k_j}$ and appropriately renaming the indexes we can infer that there exists $\{b_n\}_n$ as claimed in the statement of the theorem. Indeed, considering an initial datum $u_0 \in \mathcal{I} \cap L^\infty(\mathbb{R}^3)$, it holds that $u_\Theta \neq u_\Phi$. Thus, they are actually two different solutions of (TE). □

2.5 Selection of a sub-class of flows

Even if Theorem 4.2.9 shows that a smooth approximation does not select in general a unique solution of (ODE), our construction somehow selects solutions of the form X^{Θ}. Indeed, note that another possible way to define a flow for $\mathbf{x} \in P^+$ is the following:

$$\begin{cases} X_1(t,r,\theta,z) = \dfrac{r}{\sqrt{z}} \sqrt[4]{z^2-4t} \;\cos\theta \\ X_2(t,r,\theta,z) = \dfrac{r}{\sqrt{z}} \sqrt[4]{z^2-4t} \;\sin\theta \\ X_3(t,z) = \sqrt{z^2-4t} \end{cases} \qquad \text{for } t \in \left[0, \frac{z^2}{4}\right], \tag{2.5.1}$$

and

$$\begin{cases} X_1(t,r,\theta,z) = \dfrac{r}{\sqrt{z}} \sqrt[4]{4t-z^2} \;\cos\psi(\theta) \\ X_2(t,r,\theta,z) = \dfrac{r}{\sqrt{z}} \sqrt[4]{4t-z^2} \;\sin\psi(\theta) \\ X_3(t,z) = -\sqrt{4t-z^2} \end{cases} \qquad \text{for } t \geq \frac{z^2}{4}, \tag{2.5.2}$$

where the map $\psi : [0,2\pi] \to [0,2\pi]$ is arbitrary and (r,θ,z) denote the cylindrical coordinates in $\mathbb{R}^3$. It is easy to check that the map in (2.5.1),(2.5.2) is a solution of the ODE relative to b; we call X_ψ such a map. It will turn out to be useful the flow on $P \setminus \{0\}$ and not only in $\mathring{P}$ although we deal with functions defined almost everywhere with respect to the 3D Lebesgue measure. The reason for that lies in the fact that for our purpose we will compute X on $\partial P \setminus \{0\}$; this would not make sense without a suitable definition of X on the boundary of P. Such definition is made accordingly to the everywhere definition of b.

We prove now that solutions of the form X_ψ are regular Lagrangian flows of b whenever ψ is a measure preserving map. Before doing this, note that the map X_ψ associated to $\psi(\theta) = \alpha$, where $\alpha \in (0, 2\pi]$ is fixed, is a solution of the ODE but it does not preserve the 3D Lebesgue measure and then it is not a regular Lagrangian flow.

Now we recall the definition of a measure preserving map on the unit circle.

Definition 2.5.1. *Let $\psi : \mathbb{S}^1 \to \mathbb{S}^1$ be a measurable map, where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the unit circle with the 1D Lebesgue measure. The map ψ is called measure preserving if*

$$\psi_{\#}\mathscr{L}^1 = \mathscr{L}^1.$$

We identify $\mathbb{S}^1 \sim [0, 2\pi]$ and we define the set $\mathscr{M}$ as

$$\mathscr{M} := \{\psi : [0, 2\pi] \to [0, 2\pi] : \psi \text{ satisfies Definition 2.5.1}\}.$$

Moreover, define the maps

$$I_\pm : \theta \in [0, 2\pi] \to (\cos\theta, \sin\theta, \pm 1) \in \mathbb{R}^3.$$

Proposition 2.5.1. *Given a regular Lagrangian flow X there exists $\psi \in \mathscr{M}$ such that $X = X_\psi$. Viceversa given $\psi \in \mathscr{M}$ there exists a unique regular Lagrangian flow X such that $X = X_\psi$.*

Proof. Consider a regular Lagrangian flow $X(t, \mathbf{x})$ and define

$$\psi(\theta) = I_-^{-1}\left(X\left(\frac{1}{2}, I_+(\theta)\right)\right) \qquad \theta \in [0, 2\pi].$$

We need to show that such a map preserves the 1D Lebesgue measure: consider a Borel set $E \subseteq [0, 2\pi]$ and define $\mathbf{E}$ as the set

$$\mathbf{E} = \{(\rho, \theta, z) : \theta \in E, \rho \in [0, \sqrt{z}], z \in [-1, 0]\}.$$

A straightforward computation shows that

$$X^{-1}\left(\frac{1}{2}, \cdot\right)(\mathbf{E}) = \{(\rho, \theta, z) : \theta \in \psi^{-1}(E), \rho \in [0, \sqrt{z}], z \in [1, \sqrt{2}]\},$$

and

$$\mathscr{L}^1(\psi^{-1}(E)) = 4\mathscr{L}^3\left(X^{-1}\left(\frac{1}{2}, \cdot\right)(\mathbf{E})\right) = 4\mathscr{L}^3(\mathbf{E}) = \mathscr{L}^1(E), \tag{2.5.3}$$

hence ψ is measure preserving.
We now prove the other implication. Consider a measure preserving map ψ, a point $\mathbf{x} \in \mathbb{S}^1 \times \{1\}$ and solve the system

$$\begin{cases} \dot{X}(t, \mathbf{x}) = b(X(t, \mathbf{x})), \\ X(0, \mathbf{x}) = \mathbf{x}, \\ X\left(\frac{1}{2}, \mathbf{x}\right) = I_-\left(\psi(I_+^{-1}(\mathbf{x}))\right). \end{cases} \tag{2.5.4}$$

It is easy to see that (2.5.4) admits a unique solution X_ψ. We have to prove that X_ψ is measure preserving. A computation like (2.5.3) shows that $X_\psi(t, \mathbf{E})_\# \mathscr{L}^3 = \mathscr{L}^3(\mathbf{E})$ for all sets $\mathbf{E}$ of the form

$$\mathbf{E} = \{(r, \theta, z) : \theta \in E_1, r \in [0, \sqrt{z}], z \in E_2\}, \tag{2.5.5}$$

where $E_1 \subset [0, 2\pi], E_2 \subset \mathbb{R}$. Sets of the form (2.5.5) are a basis for the Borel σ-algebra, hence X_ψ preserve the 3D Lebesgue measure on Borel sets. Since X_ψ maps null sets into null sets, it follows that it is a regular Lagrangian flow. □

Consider the maps

$$\psi_1(\theta) = \begin{cases} \theta & \text{if } \theta \in [0,\pi), \\ 3\pi - \theta & \text{if } \theta \in [\pi, 2\pi], \end{cases}$$

and

$$\psi_2(\theta) = \begin{cases} 2\theta & \text{if } \theta \in [0,\pi), \\ 2(\theta - \pi) & \text{if } \theta \in [\pi, 2\pi]. \end{cases}$$

The map ψ_1 leaves half a circle fixed and flips the other half, while the map ψ_2 rotates twice around $\mathbb{S}^1$. Since the strategy of the proof of Theorem 4.2.9 produces in the limit only solutions of the form X^Θ, we wonder if it is possible to obtain, as limit of a suitable approximation, the flows X_{ψ_1}, X_{ψ_2} associated to ψ_1, ψ_2 as in the proof of Proposition 3.2. This is a concrete example of the following general question:

(Q4) *Does the approximation procedure obtained by smoothing the vector field select a subset of the flows of b?*

The strategy of [17] selects the regular Lagrangian flows corresponding to measure preserving map of the form $\psi(\theta) = \theta + \Theta \mod 2\pi$. These flows are in a sense “better" than the others for the following reasons:

- the flows X^Θ self intersect only in the origin, while this is not true for X_{ψ_2}, which is not even a.e. invertible;
- the Jacobian of X^Θ does not change sign, while this is the case for X_{ψ_1}.

Consider a general smooth approximation b_ε of the vector field b; the corresponding Cauchy problem admits a uniquely defined sequence of flows X^ε and one can ask to which X_ψ the sequence X^ε may converge. It is not clear to us if it is possible to construct an approximation of b in such a way that the approximated flow converge to X_{ψ_1} or X_{ψ_2}, especially if we want to approximate b only close to the singularity at the origin. We can however provide some heuristics motivating why it is not trivial to exclude the possibility of getting X_{ψ_1} in the limit just by arguing on the base of "topological obstructions". In fact, we can approximate the flow X_{ψ_1} with maps X^ε of the form:

$$X^\varepsilon(t,\mathbf{x}) = \begin{cases} X(t,\mathbf{x}) & \text{for } 0 \le t \le t_1^\varepsilon := \frac{z^2-\varepsilon^2}{4}, \\ \frac{t-t_1^\varepsilon}{t_2^\varepsilon - t_1^\varepsilon} I_-\left(\psi_1(I_+^{-1}(\mathbf{x}))\right) + \frac{t_2^\varepsilon - t}{t_2^\varepsilon - t_1^\varepsilon} X(t_1^\varepsilon, \mathbf{x}) & \text{for } t_1^\varepsilon \le t \le t_2^\varepsilon := \frac{z^2}{4} + \frac{\varepsilon^2}{4}, \\ X\left(t - t_2^\varepsilon, I_-\left(\psi_1(I_+^{-1}(\mathbf{x}))\right)\right) & \text{for } t_2^\varepsilon \le t < \infty, \end{cases}$$

where $\mathbf{x} \in P^+$. Each X^ε is a well-defined map, which is however not a flow of a vector field. Therefore, this does not answer our question. However, this example tells us that an answer in the positive to our question could not just rely on topological properties of the approximating flows.

CHAPTER 3

The two-dimensional Euler equations

In this chapter we give an overview of the theory of the 2D incompressible Euler equations. We start by setting the problem in Section 3.1 and by discussing the existence and the uniqueness of smooth solutions. In Section 3.2 we prove the existence of weak solutions under several integrability assumptions on the initial vorticity; moreover, we prove the uniqueness of such solutions when the vorticity is bounded. Finally, in Section 3.3 and Section 3.4 we prove the existence of weak solutions obtained respectively as the vanishing viscosity limit of the two dimensional Navier-Stokes equations and as the limit of the vortex-blob approximation.

3.1 A preliminary analysis and well-posedness of smooth solutions

The 2D Euler equations model the motion of an incompressible inviscid two-dimensionale fluid and are given by

$$\begin{cases} \partial_t v + (v \cdot \nabla)\, v + \nabla p = 0, \\ \operatorname{div} v = 0, \\ v(0, \cdot) = v_0, \end{cases} \tag{IE}$$

where the velocity field $v : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ and the scalar pressure $p : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are the unknowns, while $v_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is a given initial velocity field. The symbol $(v \cdot \nabla)$ is the differential operator given by

$$(v \cdot \nabla)u = v_1\, \partial_{x_1} u + v_2\, \partial_{x_2} u,$$

and by elementary vector calculus, since v is divergence-free, $(v \cdot \nabla)v = \operatorname{div}(v \otimes v)$. In two dimensions a very special role is played by the vorticity, which is defined as

$$\omega := \operatorname{curl}\, v = \partial_{x_1} v_2 - \partial_{x_2} v_1. \tag{3.1.1}$$

Note that the vorticity is a scalar quantity and that system (IE) can be rewritten in terms of ω as

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \\ v = K * \omega, \\ \omega(0, \cdot) = \omega_0, \end{cases} \tag{VE}$$

where $\omega_0 = \operatorname{curl} v_0$ and K is defined as

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} = \frac{1}{2\pi} \frac{(-x_2, x_1)}{|x|^2}.$$

The coupling between the velocity and the vorticity given by the formula

$$v = K * \omega,$$

is known as Biot-Savart law and it is an alternative way to express (3.1.1). We summarize here some properties of the Biot-Savart kernel K and their consequences for the velocity field v, which will be largely used in the following.

- $K \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for any $1 \le p < 2$;
- $K \in L^q(\mathbb{R}^2 \setminus B_r)$ for every $r > 0$ and for any $q > 2$;
- K has distributional derivative given by the following singular kernel:

$$\partial_{x_j} K_i(x) = \frac{1}{2\pi} \partial_{x_j} \left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)_i \qquad \text{for } i, j = 1, 2;$$

- the Fourier transform of $\partial_{x_j} K_i$ is bounded and it is given by

$$\widehat{\partial_{x_j} K_i}(\xi) = \frac{1}{2\pi} \xi_j \left(\frac{-\xi_2}{|\xi|^2}, \frac{\xi_1}{|\xi|^2} \right) \in L^\infty(\mathbb{R}^2).$$

The kernel $\partial_{x_j} K_i$ satisfies Definition A.4.2, thus by Theorem A.4.1 the associated singular integral operator has an extension for L^p with $1 < p < \infty$. For $u \in L^1(\mathbb{R}^2)$ the kernel $\partial_{x_j} K_i$ defines a tempered distribution $S_{ij} \in \mathcal{S}'(\mathbb{R}^2)$ via the formula

$$\langle S_{ij} u, \varphi \rangle = \langle u, \tilde{S}_{ij} \varphi \rangle \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^2),$$

where $\tilde{S}_{ij}$ is the singular integral operator associated to $\partial_{x_j} K_i(-x)$. Indeed, for $\varphi \in \mathcal{S}(\mathbb{R}^2)$, we have $\tilde{S}_{ij} \varphi \in H^q(\mathbb{R}^2) \subset C_0(\mathbb{R}^2)$ for $q > 1$ and then $S_{ij} : L^1(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$ is well-defined. For $i, j = 1, 2$ we have that

$$(\nabla v(t, x))_{ij} = \partial_{x_j} v_i(t, x) = S_{ij} \omega(t, x) \qquad \text{in } \mathcal{S}'((0, T) \times \mathbb{R}^2). \tag{3.1.2}$$

Then, for $1 < p < \infty$ the Calderón-Zygmund theorem gives the estimate

$$\|\nabla v\|_{L^\infty((0,T); L^p(\mathbb{R}^2))} \le C(p) \|\omega\|_{L^\infty((0,T); L^p(\mathbb{R}^2))}. \tag{3.1.3}$$

In particular, we have a rough estimate for the constant $C(p)$ in (3.1.3) which is the following

$$C(p) \leq \begin{cases} \frac{C}{p-1} & \text{for } 1 < p < 2, \\ c\,p & \text{for } 2 \leq p < \infty. \end{cases} \tag{3.1.4}$$

Moreover, if the vorticity $\omega \in L^\infty((0,T); L^1(\mathbb{R}^2))$, then the velocity satisfies the following growth condition

$$v \in L^\infty((0,T); L^1(\mathbb{R}^2)) + L^\infty((0,T); L^\infty(\mathbb{R}^2)), \tag{3.1.5}$$

as a consequence of Young's inequality and the integrability properties of K. Finally, the kernel K defines a compact operator.

Lemma 3.1.1. *Let K be the 2D Biot-Savart kernel and denote by $\tau_a K(x) = K(x-a)$. Then for any $1 < r < 2$ and all $a \in \mathbb{R}^2$*

$$\|\tau_a K - K\|_{L^r} \leq C(r)|a|^\alpha, \tag{3.1.6}$$

where $\alpha = 2/r - 1$. Moreover choosing p, q such that

$$1 + \frac{1}{q} - \frac{1}{p} > \frac{1}{2}, \tag{3.1.7}$$

if $\{u^\varepsilon\} \subset L^p(\mathbb{R}^2)$ is uniformly bounded in ε, then the sequence $K * u^\varepsilon$ is relatively sequentally compact in $L^q_{\text{loc}}(\mathbb{R}^2)$.

Proof. We start by proving (3.1.6). Fix $a \in \mathbb{R}^2$ with $a \neq 0$. For $|x| > 2|a|$, we have for all $0 \leq \theta \leq 1$, $|x + \theta a| > |x| - \theta|a| > \frac{|x|}{2}$, thus we have that

$$|\tau_a K(x) - K(x)| \leq |a| \sup_{0\leq\theta\leq 1} |\nabla K(x + \theta a| \leq |a| \sup_{0\leq\theta\leq 1} \frac{C}{|x+\theta a|^2} \leq C\frac{|a|}{|x|^2}.$$

Then we estimate

$$\int_{|x|>2|a|} |\tau_a K(x) - K(x)|^r \mathrm{d}x \leq C \int_{|x|>2|a|} \frac{|a|^r}{|x|^2 r}\mathrm{d}x = C(r)|a|^{2-r}. \tag{3.1.8}$$

Next, for $|x| \leq 2|a|$ we have

$$|\tau_a K(x) - K(x)| \leq \frac{1}{|x+a|} + \frac{1}{|x|},$$

and then

$$\begin{aligned} \int_{|x|\leq 2|a|} |\tau_a K(x) - K(x)|^r \mathrm{d}x &\leq \int_{|x|\leq 2|a|} \frac{1}{|x+a|^r} + \frac{1}{|x|^r}\mathrm{d}x \\ &\leq 2\int_{|x|\leq 3|a|} \frac{1}{|x|^r}\mathrm{d}x = C(r)|a|^{2-r}. \end{aligned} \tag{3.1.9}$$

Combining (3.1.8) and (3.1.9) we get (3.1.6).
To prove the compactness we want to verify the hypotesis of the Fréchet-Kolmogorov theorem. Let u^ε be a bounded sequence in $L^p(\mathbb{R}^2)$. We want to prove that

$$\lim_{a\to 0} \|\tau_a(K * u^\varepsilon) - (K * u^\varepsilon)\|_{L^q} = 0 \quad \text{uniformly in } \varepsilon.$$

Thanks to the properties of the convolution, we have that

$$\begin{aligned}\|\tau_a(K * u^\varepsilon) - (K * u^\varepsilon)\|_{L^q} &= \|(\tau_a K - K) * u^\varepsilon\|_{L^q} \\ &\le \|\tau_a K - K\|_{L^r}\|u^\varepsilon\|_{L^p} \\ &\le C(r)|a|^{\frac{2}{r}-1},\end{aligned}$$

which concludes the proof since our choice of p, q implies that $1 < r < 2$. □

3.1.1 Smooth solutions

The local existence of classical solutions of (IE) with smooth initial data was proved for the first time in the twenties by Lichtenstein in [31], while the global in time existence was proved by Wolibner in [44] in 1933. In the last century there have been many contributions in this direction and the crucial point in these proof is the propagation of the L^∞ norm of the initial vorticity. Let V^m be the subspace of the Sobolev space H^m of divergence-free vector fields.

Theorem 3.1.2. *Let $m \ge 2$ and $v_0 \in V^m(\mathbb{R}^2)$. Let v be a solution of* (IE) *in the space $C([0,T^*); V^m(\mathbb{R}^2)) \cap C^1([0,T); V^{m-1}(\mathbb{R}^2))$ where T^* is the maximal time of existence of v in the previous class. Then, if $T^* < \infty$ it holds that*

$$\int_0^{T^*} \|\omega(t,\cdot)\|_{L^\infty} \mathrm{d}t = \infty.$$

The previous theorem is known as the Beale-Kato-Majda criterion and since the vorticity solves the equation (VE), it follows that $\|\omega(t,\cdot)\|_{L^\infty} \le \|\omega_0\|_{L^\infty}$ and then Theorem 3.1.2 implies the global in time existence of smooth solutions. However, it is too restrictive to require that $v_0 \in V^m(\mathbb{R}^2)$ since it means that v has to be globally square integrable but this is not really necessary in order to define smooth solutions. To this end, the global existence of smooth solutions without requiring $v_0 \in L^2(\mathbb{R}^2)$ was given in [30] and the result reads as follows

Theorem 3.1.3. *Let $v_0 \in L^2_{\mathrm{loc}} \cap C^\infty(\mathbb{R}^2)$ be such that* $\operatorname{curl} v_0 = \omega_0 \in C_c^\infty(\mathbb{R}^2)$. *Then there exists a unique smooth solution of the Cauchy problem* (IE) *in $C([0,\infty) \times \mathbb{R}^2)$.*

Smooth solutions enjoy two very natural properties: the first one is that they are *Lagrangian*, namely they solve the equivalent formulation of (IE) given by the following system of ODE

$$\begin{cases} \dot{X}(t,x) = v(t, X(t,x)) \\ v(t,x) = (K * \omega)(t,x) \\ \omega(t,x) = \omega^0(X^{-1}(t,\cdot)(x)) \\ X(0,x) = x \end{cases} \qquad \text{for } t \in [0,T] \text{ and } x \in \mathbb{R}^2. \tag{3.1.10}$$

The property of being Lagrangian is a relevant physical information since it gives the equivalence between the Lagrangian and the Eulerian description of the motion of the fluid.
The second property is that smooth solutions conserve the *kinetic energy*: if $v \in C^1([0,T]\times\mathbb{R}^2)$ and $v_0 \in L^2(\mathbb{R}^2)$, multiplying the equation by v and integrating in space we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} |v(t,x)|^2 \mathrm{d}x + \int_{\mathbb{R}^2} v(t,x)\cdot\nabla\left(|v(t,x)|^2 + p(t,x)\right)\mathrm{d}x = 0,$$

and by using the divergence-free condition we get

$$\|v(t)\|_{L^2} = \|v_0\|_{L^2} \qquad \text{for any } t\in[0,T]. \tag{3.1.11}$$

3.2 Weak solutions

In this section we deal with weak solutions of (IE); we start by giving the definition

Definition 3.2.1. *A weak solution of (IE) is a vector valued function* $v \in L^\infty((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^2))$ *which satisfies:*

1. *for all test functions* $\Phi \in C_0^\infty((0,T)\times\mathbb{R}^2)$ *with* $\operatorname{div}\Phi = 0$,
$$\iint \left(\partial_t\Phi\cdot v + \nabla\Phi : v\otimes v\right)\mathrm{d}x\mathrm{d}t = 0; \tag{3.2.1}$$
2. $\operatorname{div} v = 0$ *in the sense of distributions;*
3. $v \in \mathrm{Lip}([0,T); H^{-L}_{\mathrm{loc}}(\mathbb{R}^2))$ *for some* $L > 0$ *and* $v(0,x) = v_0(x)$.

We will say that v vanishes uniformly as $|x|\to\infty$ if it admits a decomposition

$$v(t,x) = \bar{v}(x) + \tilde{v}(t,x), \tag{3.2.2}$$

where $\bar{v}$ is an exact steady smooth solution of the 2D Euler equations and $\tilde{v}$ satisfies

$$\operatorname{div}\tilde{v} = 0, \qquad \operatorname{curl}\tilde{v} = \tilde{\omega} \in L^1(\mathbb{R}^2), \qquad \int_{\mathbb{R}^2}\tilde{\omega} = 0,$$

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^2}|\tilde{v}(t,x)|^2\mathrm{d}x \le C.$$

In [30], DiPerna and Majda developed a rigorous framework for the study of approximate solution sequences for the 2D Euler equations with L^2_{loc} initial datum. The precise definition of an approximate solution sequence is the following.

Definition 3.2.2. *A sequence of smooth velocity fields* v^ε *vanishing uniformly as* $|x|\to\infty$ *with vorticity* $\operatorname{curl} v^\varepsilon = \omega^\varepsilon \in C([0,T]; L^1(\mathbb{R}^2))$ *is an approximate solution sequence for the 2D Euler equations provided that* *(i)* v^ε *has uniformly bounded local kinetic energy and* v^ε *is incompressible, i.e., for each* $R > 0$ *and* $T > 0$*, there exists* $C(R) > 0$ *such that*

$$\max_{t\in[0,T]} \int_{B_R} |v^\varepsilon(t,x)|^2 \mathrm{d}x \le C(R), \qquad \operatorname{div} v^\varepsilon = 0;$$

(ii) the vorticity ω^ε *is uniformly bounded in* L^1*, i.e., for every* $T > 0$*,*

$$\max_{t\in[0,T]} \int_{\mathbb{R}^2} |\omega^\varepsilon(t,x)| \mathrm{d}x \le C;$$

(iii) for some $L > 0$*,* v^ε *is uniformly Lipschitz in the negative Sobolev space* $H^{-L}_{\mathrm{loc}}(\mathbb{R}^2)$*;*

(iv) v^ε *is weakly consistent with the 2D Euler equations, i.e.*

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}^2} \left(\partial_t \Phi \cdot v^\varepsilon + \nabla\Phi : v^\varepsilon \otimes v^\varepsilon\right) \mathrm{d}x\mathrm{d}t = 0, \tag{3.2.3}$$

for every $\Phi \in C^\infty_c((0,T)\times\mathbb{R}^2)$ *with* $\operatorname{div}\Phi = 0$.

The previous definition deserves some comments:

- the condition that v^ε is uniformly Lipschitz in $H^{-L}_{\mathrm{loc}}(\mathbb{R}^2)$ means that $v^\varepsilon \in \mathrm{Lip}([0,T]; H^{-L}_{\mathrm{loc}}(\mathbb{R}^2))$ and, for every $\rho \in C^\infty_c(\mathbb{R}^2)$ and $T > 0$, there exists a constant C such that

$$\|\rho v^\varepsilon(t_1,\cdot) - \rho v^\varepsilon(t_2,\cdot)\|_{H^{-L}(\mathbb{R}^2)} \le C|t_1 - t_2|,$$

  for all $0 \le t_1, t_2 \le T$. This technical condition gives an interpretation to the weak sense in which the initial datum is assumed at time $t = 0$ in the limit process;

- the requirement that v^ε vanishes uniformly as $|x| \to \infty$ means that for every $\varepsilon > 0$ v^ε has a decomposition as in (3.2.2) and the estimates are ε-independent.

For any approximate solution sequence the following compactness theorem holds.

Theorem 3.2.1. *Let* $v_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ *be a divergence-free vector field vanishing uniformly as* $|x| \to \infty$ *and let* $\omega_0 = \operatorname{curl} v_0 \in \mathcal{M} \cap H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$*. Let* v^ε *be an approximate solution sequence with initial data* v_0^ε *such that* $v_0^\varepsilon \to v_0$ *in* $L^2_{\mathrm{loc}}(\mathbb{R}^2)$*. Then, there exists a subsequence* v^ε *and a vector field* $v \in L^\infty((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^2)) \cap \mathrm{Lip}([0,T]; H^{-L}_{\mathrm{loc}}(\mathbb{R}^2))$ *which vanishes uniformly as* $|x| \to \infty$ *with the following properties:*

- $v(0,\cdot) = v_0$*,*
- $v^\varepsilon \to v$ *in* $L^p((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^2))$ *for every* $1 \le p < 2$*,*
- $v^\varepsilon \overset{*}{\rightharpoonup} v$ *in* $L^\infty((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^2))$*,*
- $\omega^\varepsilon \overset{*}{\rightharpoonup} \omega$ *in* $L^\infty((0,T); \mathcal{M}(\mathbb{R}^2))$*,*

- $\omega^\varepsilon \to \omega$ *in* $C([0,T]; H^{-L-1}_{\text{loc}}(\mathbb{R}^2))$.

In [30] the authors give three different examples of approximate solutions sequences which verifies Definition 3.2.2; the methods are the following

(ES) Approximation by exact smooth solutions of (IE);

(VV) Vanishing viscosity from the two-dimensional Navier-Stokes equations;

(VB) Vortex blob approximation.

Theorem 3.2.2. *Let* $v_0 \in L^2_{\text{loc}}(\mathbb{R}^2)$ *such that* $\omega_0 \in \mathcal{M} \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$. *Let* v^ε *a sequence constructed through one of the methods (ES), (VV), and (VB). Then* v^ε *satisfies Definition 3.2.2.*

From Theorem 3.2.1 we know that any given approximate solution sequence $(\omega^\varepsilon, v^\varepsilon)$ admits a limit $(\omega, v) \in L^\infty((0,T); \mathcal{M}(\mathbb{R}^2)) \times L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^2))$ and in particular $v^\varepsilon \to v$ strongly in L^1_{loc}. This convergence is not enough to pass to the limit in the non-linear term $v^\varepsilon \otimes v^\varepsilon$, so that in general we can not expect that v is a weak solution of (IE) accordingly to the Definition 3.2.1. In fact, even if the L^1 converges avoids the presence of oscillations, concentrations still may occur passing to the limit in the non-linear term $v^\varepsilon \otimes v^\varepsilon$. It turns out that if in addition we assume that $\omega_0 \in L^p(\mathbb{R}^2)$ for some $p > 1$, the approximating velocity field converges strongly in L^2_{loc} which eventually implies that the limit v is a weak solution of (IE).

Theorem 3.2.3. (DiPerna, Majda) *Let* $v_0 \in L^2_{\text{loc}}(\mathbb{R}^2)$ *be a divergence-free vector field and let* $\omega_0 \in L^1 \cap L^p(\mathbb{R}^2)$ *for some* $p > 1$. *Then, there exists a weak solution* $v \in L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^2))$ *of* (IE) *with initial datum* v_0.

Proof. Let $\rho \in C^\infty_c(\mathbb{R}^2)$ be a standard mollifier and let $\rho_\varepsilon(x) = \varepsilon^{-2}\rho\left(\frac{x}{\varepsilon}\right)$. Define $v_0^\varepsilon = v_0 * \rho_\varepsilon$ and $\omega_0^\varepsilon = \omega_0 * \rho_\varepsilon$ and let v^ε be the unique smooth solution of (IE) with initial datum v_0^ε and $\operatorname{curl} v^\varepsilon = \omega^\varepsilon$. Since ω^ε solves the equation (VE) with initial datum ω_0^ε and v^ε is divergence-free we have the following estimate

$$\sup_{t\in[0,T]} \|\omega^\varepsilon(t,\cdot)\|_{L^p} = \|\omega_0^\varepsilon\|_{L^p} \le \|\omega_0\|_{L^p}. \tag{3.2.4}$$

By Theorem 3.2.2 we also know that

$$\{v^\varepsilon\} \subset \text{Lip}([0,T]; W^{-s,L}(\mathbb{R}^2)),$$

uniformly in ε for some $s, L > 0$. Then, by Lemma 3.1.1, with the choice of $q = 2$ and $1 < r < 2$ such that $1 + 1/2 = 1/p + 1/r$, and applying Aubin-Lions' lemma we deduce that $v^\varepsilon \to v$ in $L^2((0,T); L^2_{\text{loc}}(\mathbb{R}^2))$. Writing the nonlinear term as

$$\nabla\Phi : v^\varepsilon \otimes v^\varepsilon - \nabla\Phi : v \otimes v = \nabla\Phi : v^\varepsilon \otimes (v^\varepsilon - v) + \nabla\Phi : (v^\varepsilon - v) \otimes v^\varepsilon,$$

it is clear that

$$\lim_{\varepsilon\to0} \int_0^T \int_{\mathbb{R}^2} (\partial_t\Phi \cdot v^\varepsilon + \nabla\Phi : v^\varepsilon \otimes v^\varepsilon)\, \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbb{R}^2} (\partial_t\Phi \cdot v + \nabla\Phi : v \otimes v)\, \mathrm{d}x\mathrm{d}t,$$

which concludes the proof. □

The L^2_{loc} convergence of the approximate velocity field is sufficient but not necessary to conclude that the limit of an approximate solution sequence is a weak solution of (IE). In fact, exploiting the special structure of the non-linear term in 2D, Delort in [27] proved the existence of weak solutions when the initial vorticity is a vortex-sheet, namely $\omega_0 \in \mathcal{M}_+ \cap H^{-1}(\mathbb{R}^2)$, and later on, following the same idea of Delort, Vecchi and Wu in [43] proved the same existence result for $\omega_0 \in L^1 \cap H^{-1}(\mathbb{R}^2)$ without any sign condition. The main step of their proofs is the following: since we use test functions $\Phi \in C_c^\infty((0,T) \times \mathbb{R}^2)$ with $\operatorname{div} \Phi = 0$, this is equivalent to considering $\Phi = \nabla^\perp \eta$ with $\eta \in C_c^\infty((0,T) \times \mathbb{R}^2)$. Substituting in the integral formulation we obtain

$$\int_0^T \int_{\mathbb{R}^2} (\eta_{tx_2} v_1 - \eta_{tx_1} v_2)\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathbb{R}^2} \eta_{x_1 x_2}(v_2^2 - v_1^2)(\eta_{x_2 x_2} - \eta_{x_1 x_1})(v_1 v_2)\mathrm{d}x\mathrm{d}t = 0. \quad (3.2.5)$$

Thus, if v^ε is an approximating solution sequence, it is sufficient to show that the quantities $(v_2^\varepsilon)^2 - (v_1^\varepsilon)^2, v_1^\varepsilon v_2^\varepsilon$ converge in the sense of the distributions to $v_2^2 - v_1^2, v_1 v_2$. By a rotation of $\pi/4$, $(v_2^\varepsilon)^2 - (v_1^\varepsilon)^2$ becomes $v_1^\varepsilon v_2^\varepsilon$, then by the rotational invariance of the Euler equations we only need to prove that

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^2} \phi(t,x) v_1^\varepsilon(t,x) v_2^\varepsilon(t,x)\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbb{R}^2} \phi(t,x) v_1(t,x) v_2(t,x)\mathrm{d}x\mathrm{d}t, \quad (3.2.6)$$

for every $\phi \in C_c^\infty((0,T) \times \mathbb{R}^2)$. Trough a standard density argument, there is no loss of generality to verify (3.2.6) for test functions

$$\phi(t,x) = \psi(t)\varphi(x) \qquad \text{with } \psi \in C_c^\infty(\mathbb{R}_+),\ \varphi \in C_c^\infty(\mathbb{R}^2).$$

Then, we use the Biot-Savart law to rewrite

$$\int_0^T \int_{\mathbb{R}^2} v_1^\varepsilon(t,x) v_2^\varepsilon(t,x) \phi(t,x)\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(t)\omega^\varepsilon(t,x)\omega^\varepsilon(t,y) H_\varphi(x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t, \quad (3.2.7)$$

where

$$H_\varphi(x,y) = C \text{ p.v.} \int_{\mathbb{R}^2} \frac{x_1 - z_1}{|x-z|^2} \frac{y_2 - z_2}{|y-z|^2} \varphi(z)\mathrm{d}z,$$

for some constant $C > 0$. As shown in [27, Proposition 1.2.3], the function $H_\phi \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, is continuous outside the diagonal, and goes to 0 at infinity. We recall the Proposition here for completness.

Proposition 3.2.4. *The function H_φ is a bounded function on $\mathbb{R}^2 \times \mathbb{R}^2$, continuous on the complement of the diagonal, and tends to 0 at infinity. In fact*

$$H_\varphi(x,y) = \frac{1}{2}\left(\varphi(x) + \varphi(y)\right) h(x-y) + r(x,y), \quad (3.2.8)$$

where $h(x) = \dfrac{x_1 x_2}{4\pi |x|^2}$ and $r(x,y)$ is a bounded continuous function tending to zero at infinity.

Besides the structure of the 2D non-linear term, in Delort's proof is crucial that the approximate vorticity satisfies a uniform decay of the maximal vorticity function. We recall the following proposition from [34]

Proposition 3.2.5. (Uniform Decay of the Vorticity Maximal Function) *Let v^ε an approximate solution sequence such that and* $\operatorname{curl} v_0^\varepsilon = \omega_0^\varepsilon \geq 0$. *Then, the vorticity maximal function satisfies the following a priori uniform decay rate*

$$\max_{0<\varepsilon<\varepsilon_0} \max_{\substack{t\in[0,T] \\ x_0\in\mathbb{R}^2}} \int_{|x-x_0|<R_0} \omega^\varepsilon(t,x)\mathrm{d}x \leq C(T,R_0)\,|\log(2R)|^{-\frac{1}{2}} \quad \text{for } R<\frac{1}{2}, \tag{3.2.9}$$

where $C(T,R_0)$ is a constant depending on T,R_0. The same estimate also applies to the limiting vorticity measure $\omega = \operatorname{curl} v$, in fact

$$\max_{\substack{t\in[0,T] \\ x_0\in\mathbb{R}^2}} \int_{|x-x_0|<R_0} \mathrm{d}\omega(t,x) \leq C(T,R_0)\,|\log(4R)|^{-\frac{1}{2}} \quad \text{for } R<\frac{1}{4}. \tag{3.2.10}$$

In particular, the bounds (3.2.10),(3.2.9) do not hold if the vorticity does not have a distinguished sign. However, the equi-integrability of ω^ε would also imply a uniform decay of the vorticity maximal function, allowing the phenomenon of concentration-cancellation for v^ε. This is the idea behind the proof of the existence of weak solutions in the case $\omega_0 \in L^1(\mathbb{R}^2)$ given by Vecchi and Wu in [43]. We are now in position to prove the theorems of Delort and Vecchi-Wu; we collect them in one theorem since the proof are very similar.

Theorem 3.2.6. (Delort, Vecchi – Wu) *Let $v_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ be a divergence-free vector field and let $\omega_0 \in \mathcal{M}_+ \cap H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$ or $\omega_0 \in L^1 \cap H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$. Then, there exists a weak solution $v \in L^\infty((0,T);L^2_{\mathrm{loc}}(\mathbb{R}^2))$ of* (IE) *with initial datum v_0.*

Proof. We divide the proof in several steps.

Step 1

We assume for simplicity that the support of ω_0 is compact. Let $\rho \in C_c^\infty(\mathbb{R}^2)$ be a standard mollifier and let $\rho_\varepsilon(x) = \varepsilon^{-2}\rho\left(\frac{x}{\varepsilon}\right)$. Define $v_0^\varepsilon = v_0 * \rho_\varepsilon$ and $\omega_0^\varepsilon = \omega_0 * \rho_\varepsilon$ and let v^ε be the unique smooth solution of (IE) with initial datum v_0^ε and $\operatorname{curl} v^\varepsilon = \omega^\varepsilon$. By Theorem 3.2.1 we know that there exist $v \in L^\infty((0,T);L^2_{\mathrm{loc}}(\mathbb{R}^2))$ and $\omega \in L^\infty((0,T);\mathcal{M}(\mathbb{R}^2))$ such that, up to subsequences, the following convergences hold

$$\begin{aligned} v^\varepsilon &\overset{*}{\rightharpoonup} v \quad \text{in } L^\infty((0,T);L^2_{\mathrm{loc}}(\mathbb{R}^2)), \\ \omega^\varepsilon &\overset{*}{\rightharpoonup} \omega \quad \text{in } L^\infty((0,T);\mathcal{M}(\mathbb{R}^2)). \end{aligned} \tag{3.2.11}$$

Moreover, ω^ε is given by the formula

$$\omega^\varepsilon(t,x) = \omega_0^\varepsilon((X^\varepsilon)^{-1}(t,\cdot)(x)) \tag{3.2.12}$$

where X^ε is the flow of v^ε. Since v^ε is divergence-free, it follows that X^ε is measure-preserving and from this consideration it is straightforward to prove that the sequence ω^ε defined in (3.2.12) is equi-integrable in L^1 if $\omega_0 \in L^1(\mathbb{R}^2)$. In this case, the weak limit $\omega \in L^1(\mathbb{R}^2)$ and instead of the convergence (3.2.11) we have

$$\omega^\varepsilon \overset{*}{\rightharpoonup} \omega \quad \text{in } L^\infty((0,T); L^1(\mathbb{R}^2)), \tag{3.2.13}$$

where the weak convergence in L^1 is in duality with the space $L^\infty(\mathbb{R}^2)$.

Step 2

Let $\rho \in C_c^\infty(\mathbb{R}^2)$ be a fixed positive cut-off function such that $\rho(x) = 1$ for $|x| \leq 1$ and $\rho(x) = 0$ for $|x| > 2$. We rewrite (3.2.7) as

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(t)\omega^\varepsilon(t,x)\omega^\varepsilon(t,y)H_\varphi(x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\qquad = \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(t)\rho\left(\frac{|x-y|}{\delta}\right)\omega^\varepsilon(t,x)\omega^\varepsilon(t,y)H_\varphi(x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\qquad + \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(t)\left(1-\rho\left(\frac{|x-y|}{\delta}\right)\right)\omega^\varepsilon(t,x)\omega^\varepsilon(t,y)H_\varphi(x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t.
\end{aligned}$$

For any fixed $0 < \delta < 1$, by Proposition 3.2.4 $\psi(t)\left(1-\rho\left(\frac{|x-y|}{\delta}\right)\right)H_\varphi(x,y)$ is a continuous function tending to zero at infinity, and from (3.2.11) it follows that

$$\omega^\varepsilon(t,x)\otimes\omega^\varepsilon(t,y) \overset{*}{\rightharpoonup} \mathrm{d}\omega(t,x)\otimes\mathrm{d}\omega(t,y) \qquad \text{in } L^\infty((0,T);\mathcal{M}(\mathbb{R}^2)). \tag{3.2.14}$$

Combining Proposition 3.2.4 and (3.2.14) we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(t)\left(1-\rho\left(\frac{|x-y|}{\delta}\right)\right)\omega^\varepsilon(t,x)\omega^\varepsilon(t,y)H_\varphi(x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\qquad = \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(t)\left(1-\rho\left(\frac{|x-y|}{\delta}\right)\right)H_\varphi(x,y)\mathrm{d}\omega(t,x)\otimes\mathrm{d}\omega(t,y)\mathrm{d}t.
\end{aligned} \tag{3.2.15}$$

In particular, if $\omega_0 \in L^1$ the limit measure is

$$\mathrm{d}\omega(t,x)\otimes\mathrm{d}\omega(t,y) = \omega(t,x)\omega(t,y)\mathrm{d}x\mathrm{d}y.$$

Next, we need the uniform decay of the vorticity maximal function in order to take care of the remaining term. For $\omega_0 \in \mathcal{M}_+$ we compute

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi(t)\rho\left(\frac{|x-y|}{\delta}\right)\omega^\varepsilon(t,x)\omega^\varepsilon(t,y)H_\varphi(x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\right| \\
&\qquad \leq C(\varphi,\psi)\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho\left(\frac{|x-y|}{\delta}\right)\omega^\varepsilon(t,x)\omega^\varepsilon(t,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t && (3.2.16)\\
&\qquad \leq C(\varphi,\psi,R_0,T)\left(\log\left(\frac{1}{2\delta}\right)\right)^{-\frac{1}{2}}\int_0^T\int_{\mathbb{R}^2}\omega^\varepsilon(t,x)\mathrm{d}x\mathrm{d}t && (3.2.17)\\
&\qquad \leq C\left(\log\left(\frac{1}{2\delta}\right)\right)^{-\frac{1}{2}}. && (3.2.18)
\end{aligned}$$

We have used Proposition 3.2.4 and the fact that the vorticity is nonnegative to obtain (3.2.16), while the uniform decay on the maximal vorticity function has been applied to get (3.2.17), and finally the uniform control on the mass of the measures ω^ε guarantees (3.2.18). The same uniform estimate applies with ω^ε replaced by the limiting vorticity measure ω. Thus, (3.2.15) and (3.2.18) guarantee that

$$\left|\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^2}\phi(t,x)v_1^\varepsilon(t,x)v_2^\varepsilon(t,x)\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbb{R}^2}\phi(t,x)v_1(t,x)v_2(t,x)\mathrm{d}x\mathrm{d}t\right|\le C\left(\log\left(\frac{1}{4\delta}\right)\right)^{-\frac{1}{2}}.$$

Since δ can be choosen arbitrarly small, v is a weak solution of the 2D Euler equations with the required intial datum v_0.
In the case $\omega_0 \in L^1$, by the equi-integrability of the approximating sequence ω^ε, in (3.2.16) we can estimate putting the absolute values inside the integrals and we make this quantity as small as we want without using the explicit decay rate as in Proposition 3.2.5. The proof is now complete. □

The uniqueness of weak solutions of the two-dimensional Euler equations in the class of [30] is an outstanding open problem and it is known to hold only for bounded vorticities. This was proved by Yudovič in [45] and it is the content of the following theorem.

Theorem 3.2.7. *Let $v_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ be a divergence-free vector field and let $\omega_0 \in L^1\cap L^\infty(\mathbb{R}^2)$. Then, the weak solution $(v,\omega)\in L^\infty((0,T);L^2_{\mathrm{loc}}(\mathbb{R}^2))\cap L^\infty((0,T);L^\infty(\mathbb{R}^2))$ of* (IE) *with initial datum v_0 is unique.*

Proof. For simplicity we assume that $\operatorname{supp}\omega_0\subset B_R$ for some $R>0$. Let v be a solution of (IE) with initial datum v_0. Decomposing K as in we have that v is uniformly bounded since

$$\sup_{t\in[0,T]}\|v(t,\cdot)\|_{L^\infty}\le\|K_1\|_{L^1}\|\omega_0\|_{L^\infty}+\|K_2\|_{L^\infty}\|\omega_0\|_{L^1},$$

and then, since the vorticity is transported by the flow of v, the support of ω has finite propagation speed. This means that there exists a uniformly increasing bounded function $R(t)$ such that

$$\operatorname{supp}\omega(t,\cdot)\subset B_{R(t)}.$$

Let v_1, v_2 two solutions of (IE) with the same initial vorticity ω_0 and define $w = v_1 - v_2$. By using the asymptotic espansion of K_2 we have that

$$v_i(t,x)=\frac{c}{|x|}\int_{\mathbb{R}^2}\omega(t,y)\mathrm{d}y+\mathcal{O}\left(|x|^{-2}\right),\qquad \text{for } |x|>2R_i(t),$$

and then setting $\max_i R_i(t) := \tilde{R}(t)$ we obtain

$$w(t,x)=\mathcal{O}\left(|x|^{-2}\right),\qquad \text{for } |x|>2\tilde{R}(t),$$

which implies that w has globally finite kinetic energy. Note that w satisfies the following equation

$$\partial_t w + w \cdot \nabla v_2 + v_1 \cdot \nabla w + \nabla(p_1 - p_2) = 0, \tag{3.2.19}$$

in the sense of distributions. Define $E(t)$ as

$$E(t) = \int_{\mathbb{R}^2} |w(t,x)|^2 \mathrm{d}x,$$

by taking the L^2 inner product of (3.2.19) w and integrating by parts we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \int_{\mathbb{R}^2} (w(t,x) \cdot \nabla v_2(t,x)) w(t,x) \mathrm{d}x = 0.$$

The Hölder inequality implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &\le 2\int_{\mathbb{R}^2} w^2(t,x)|\nabla v_2(t,x)|\mathrm{d}x \\ &\le 2\|\nabla v_2(t,\cdot)\|_{L^p}\left(|w(t,x)|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}} \\ &\le 2\|\nabla v_2(t,\cdot)\|_{L^p}\left(\|w(t,\cdot)\|_{L^\infty}^{\frac{2}{p-1}}\int_{\mathbb{R}^2}|w(t,x)|^2\mathrm{d}x\right)^{\frac{p-1}{p}}.\end{aligned}$$

By the estimates (3.1.3) and (3.1.4) together with the fact that the velocities are bounded, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le pME(t)^{1-\frac{1}{p}}, \tag{3.2.20}$$

where M is a constant which depends on $\|\omega_0\|_{L^\infty}$. We want to conclude by showing that $E(t) = 0$ for all $t > 0$. Because $E(0) = 0$, $E(t) = 0$ is a trivial solution of (3.2.20). However, this inequality does not have a unique solution but it has a maximal solution $\bar{E}$: this means that any solution $E(t)$ of the inequality (3.2.20) satisfies $E(t) \le \bar{E}(t)$ for any $t > 0$. In particular, the maximal solution is given by $\bar{E}(t) = (Mt)^p$. Now we consider an interval $[0, T^*]$ such that $MT^* < \frac{1}{2}$ and we have that

$$E(t) < \left(\frac{1}{2}\right)^p \to 0 \qquad \text{as } p \to \infty,$$

so $E(t) = 0$ for every $t \in [0, T^*]$. Repeating this arguments for a partition of $[0, T]$, we deduce that $E(t) = 0$ for every $t \in [0, T]$ which implies that $v_1 = v_2$ almost everywhere. □

3.3 The vanishing viscosity limit

In this section we prove the existence of weak solutions of the 2D Euler equation obtained as the vanishing viscosity limit of the 2D Navier-Stokes equations, namely

$$\begin{cases}\partial_t v^\nu + v^\nu \cdot \nabla v^\nu + \nabla p^\nu = \nu\Delta v^\nu, \\ \operatorname{div} v^\nu = 0, \\ v^\nu(0,\cdot) = v_0^\nu.\end{cases} \tag{3.3.1}$$

The Cauchy problem for the Navier-Stokes equation in vorticity form is the following

$$\begin{cases} \partial_t \omega^\nu + v^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu, \\ v^\nu = K * \omega^\nu, \\ \omega^\nu(0, \cdot) = \omega_0^\nu. \end{cases} \tag{3.3.2}$$

Theorem 3.3.1. *For any smooth initial datum ω_0^ν there exists a unique smooth solution of* (3.3.2)*.*

The proof of the previous theorem can be found in [35]. In particular, let $v_0 \in L^2_{\mathrm{loc}}$ such that $\omega_0 \in L^p_c(\mathbb{R}^2)$ and we consider (3.3.2) with $\omega_0^\nu = \omega_0 * \rho_\nu$, where ρ_ν is a standard mollifier. With this choice of initial datum, we know that there exists a unique solution v^ν, ω^ν respectively of (3.3.1) and (3.3.2). By Theorem 3.2.2 we have that Theorem 3.2.1 holds for (v^ν, ω^ν) and we want now prove some a priori bound on the L^p norm of ω^ν in order to prove that the limit v of v^ν is actually a weak solution of the 2D Euler equations.

Lemma 3.3.2. *Let $\omega_0 \in L^p_c(\mathbb{R}^2)$ with $1 \leq p < \infty$, and define $\omega_0^\nu = \omega_0 * \rho_\nu$, where ρ_ν is a standard mollifier. Then, the unique solution ω^ν of* (3.3.2) *satisfies the following bound*

$$\sup_{t \in (0,T)} \|\omega^\nu(t, \cdot)\|_{L^p} \leq \|\omega_0\|_{L^p}. \tag{3.3.3}$$

Proof. Since ω^ν is smooth, by multiply the vorticity equation (3.3.2) by $p\omega^\nu |\omega^\nu|^{p-1}$ we get that

$$\partial_t |\omega^\nu|^p + v^\nu \cdot \nabla |\omega^\nu|^p = \nu p \Delta \omega^\nu \omega^\nu |\omega^\nu|^{p-1}.$$

By integrating the previous equation and using that v^ν is divergence-free we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} |\omega^\nu(t,x)|^p \mathrm{d}x &= \nu \int_{\mathbb{R}^2} \Delta \omega^\nu(t,x) \omega^\nu(t,x) |\omega^\nu(t,x)|^{p-1} \mathrm{d}x \\ &= -\nu \int_{\mathbb{R}^2} |\nabla \omega^\nu(t,x)|^2 |\omega^\nu(t,x)|^{p-1} \mathrm{d}x \\ &\quad - \nu p(p-1) \int_{\mathbb{R}^2} |\nabla \omega^\nu(t,x)|^2 |\omega^\nu(t,x)|^p \mathrm{d}x \leq 0, \end{aligned}$$

and then (3.3.3) follows by standard properties of the convolutions. □

By following the proof of Theorem 3.2.3, by using the previous uniform estimates it is easy to prove the following

Theorem 3.3.3. *Let $v_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ be a divergence-free vector field and let $\omega_0 \in L^p_c(\mathbb{R}^2)$ for some $p > 1$. Let ρ^ν be a standard mollifier and define $\omega_0^\nu = \omega_0 * \rho_\nu, v_0^\nu = v_0 * \rho_\nu$ Then, there exists $v \in L^\infty((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^2))$ which is a weak solution of* (IE) *with initial datum v_0 such that*

$$v^\nu \to v \qquad \text{in } L^2((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^2)).$$

3.4 The vortex-blob method

In this section we describe the vortex-blob approximation and we prove the convergence to classical weak solutions of (IE). Consider an initial vorticity $\omega_0 \in L^p_c(\mathbb{R}^2)$ with $1 \leq p \leq \infty$. Let $\varepsilon \in (0,1)$, we consider two small parameters in $(0,1)$, which later will be chosen as functions of ε, denoted by $\delta(\varepsilon)$ and $h(\varepsilon)$.
First of all, we consider the lattice

$$\Lambda_h := \{\alpha_i \in \mathbb{Z} \times \mathbb{Z} : \alpha_i = h(i_1, i_2), \text{ where } i_1, i_2 \in \mathbb{Z}\},$$

and define R_i the square with sides of lenght h parallel to the coordinate axis and centered at $\alpha_i \in \Lambda_h$. Let j_δ be a standard mollifier and define

$$\omega_0^\varepsilon := \omega_0 * j_{\delta(\varepsilon)}. \tag{3.4.1}$$

For any $\delta \in (0,1)$ the support of ω_0^ε is contained in a fixed compact set in $\mathbb{R}^2$, then it can be tiled by a finite number $N(\varepsilon)$ of squares R_i. Define the quantities

$$\Gamma_i^\varepsilon = \int_{R_i} \omega_0^\varepsilon(x) \, \mathrm{d}x, \qquad \text{for } i = 1, ..., N(\varepsilon).$$

Let φ_ε be another mollifier, we define the approximate vorticity to be

$$\omega^\varepsilon(t,x) = \sum_{i=1}^{N(\varepsilon)} \Gamma_i^\varepsilon \varphi_\varepsilon(x - X_i^\varepsilon(t)), \tag{3.4.2}$$

where $\{X_i^\varepsilon(t)\}_{i=1}^{N(\varepsilon)}$ is a solution of the O.D.E. system

$$\begin{cases} \dot{X}_i^\varepsilon(t) = v^\varepsilon(t, X_i^\varepsilon(t)), \\ X_i^\varepsilon(0) = \alpha_i, \end{cases} \tag{3.4.3}$$

with v^ε defined as

$$v^\varepsilon(t,x) = K * \omega^\varepsilon(t,x) = \sum_{i=1}^{N(\varepsilon)} \Gamma_i^\varepsilon K_\varepsilon(x - X_i^\varepsilon(t)), \tag{3.4.4}$$

where $K_\varepsilon = K * \varphi_\varepsilon$. Note that, since δ and h are ε-dependent, we only use the superscript ε. The ordinary differential equations (3.4.3) are known as the *vortex-blob approximation.*
It is not difficult to show the bound

$$\sup_{t\in[0,T]} (\|v^\varepsilon(t,\cdot)\|_{L^\infty} + \|\nabla\, v^\varepsilon(t,\cdot)\|_{L^\infty}) \leq \frac{C}{\varepsilon^2}, \tag{3.4.5}$$

see [30]. From (3.4.5) it follows that, for every fixed $\varepsilon > 0$, there exists a unique smooth solution $\{X_i^\varepsilon(t)\}_{i=1}^{N(\varepsilon)}$ of the ODE system (3.4.3), which implies that v^ε and ω^ε are well-defined smooth functions. Note that v^ε and ω^ε are not exact solutions of the Euler equations because of the presence of an error term, due to the fact that each blob is rigidly translated by the flow. Precisely, the approximate vorticity ω^ε satisfies the following equation

$$\partial_t \omega^\varepsilon + v^\varepsilon \cdot \nabla \omega^\varepsilon = E_\varepsilon, \tag{3.4.6}$$

where by a direct computation the error term is given by

$$E_\varepsilon(t,x) := \sum_{i=1}^{N(\varepsilon)} [v^\varepsilon(t,x) - v^\varepsilon(t, X_i^\varepsilon(t)] \cdot \nabla \varphi_\varepsilon(x - X_i^\varepsilon(t))\Gamma_i^\varepsilon. \tag{3.4.7}$$

Concerning the approximate velocity v^ε, consider the quantity

$$w^\varepsilon = \partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla)\, v^\varepsilon.$$

Since w^ε satisfies the system

$$\begin{cases} \operatorname{curl} w^\varepsilon = E_\varepsilon, \\ \operatorname{div} w^\varepsilon = \operatorname{div}\operatorname{div}(v^\varepsilon \otimes v^\varepsilon), \end{cases} \tag{3.4.8}$$

we derive that there exists a function p^ε such that

$$-\Delta p^\varepsilon = \operatorname{div}\operatorname{div}(v^\varepsilon \otimes v^\varepsilon),$$

and

$$w^\varepsilon = -\nabla p^\varepsilon + K * E_\varepsilon.$$

Then, the velocity given by the vortex-blob approximation verifies the following equations

$$\begin{cases} \partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla)\, v^\varepsilon + \nabla p^\varepsilon = K * E_\varepsilon, \\ \operatorname{div} v^\varepsilon = 0. \end{cases} \tag{3.4.9}$$

Since v^ε is divergence-free, E_ε can be rewritten as $E_\varepsilon(t,x) = \operatorname{div} F_\varepsilon(t,x)$ where

$$F_\varepsilon(t,x) := \sum_{i=1}^{N(\varepsilon)} [v^\varepsilon(t,x) - v^\varepsilon(t, X_i^\varepsilon(t)]\, \varphi_\varepsilon(x - X_i^\varepsilon(t))\Gamma_i^\varepsilon. \tag{3.4.10}$$

3.4.1 A priori estimates and Beale's theorem

We prove now some a priori estimates on ω^ε, v^ε, and the error term F_ε, taken from [10]. First of all, we introduce the following auxiliary problem. Let $\bar{\omega}^\varepsilon$ be the solution of the linear transport equation with vector field v^ε, that is

$$\begin{cases} \partial_t \bar{\omega}^\varepsilon + v^\varepsilon \cdot \nabla \bar{\omega}^\varepsilon = 0, \\ \bar{\omega}^\varepsilon(0,\cdot) = \omega_0^\varepsilon. \end{cases} \tag{3.4.11}$$

Since v^ε satisfies (3.4.5), there exists a unique smooth solution $\bar{\omega}^\varepsilon$, which is given by the formula

$$\bar{\omega}^\varepsilon(t,x) = \omega_0^\varepsilon((X^\varepsilon)^{-1}(t,\cdot)(x)), \tag{3.4.12}$$

where X^ε is the flow of v^ε, that is,

$$\begin{cases} \dot{X}^\varepsilon(t,x) = v^\varepsilon(t, X^\varepsilon(t,x)), \\ X^\varepsilon(0,x) = x. \end{cases} \tag{3.4.13}$$

Moreover, since $\operatorname{div} v^\varepsilon = 0$, we have

$$\|\bar{\omega}^\varepsilon(t,\cdot)\|_{L^p} = \|\omega_0^\varepsilon\|_{L^p} \leq \|\omega_0\|_{L^p}.$$

We will use $\bar{\omega}^\varepsilon$ in order to prove uniform L^p-bounds on ω^ε. Before doing that, note that ω^ε can be seen as a discretization of $\varphi_\varepsilon * \bar{\omega}^\varepsilon$, since a change of variables gives

$$\varphi_\varepsilon * \bar{\omega}^\varepsilon(t,x) = \int_{\mathbb{R}^2} \varphi_\varepsilon(x-z)\bar{\omega}^\varepsilon(t,z)\mathrm{d}z = \int_{\mathbb{R}^2} \varphi_\varepsilon(x - X^\varepsilon(t,y))\omega_0^\varepsilon(y)\mathrm{d}y, \tag{3.4.14}$$

compare with (3.4.2). We now give a lemma which is, loosely speaking, an estimate on the L^p norms of the error we commit substituting the integral in (3.4.14) with the sum in (3.4.2). The following estimate is new for $1 \leq p < \infty$, while the case $p = \infty$ has been proved in [10].

Lemma 3.4.1. *Let $\omega_0 \in L^1(\mathbb{R}^2)$ and let $h = h(\varepsilon)$ be chosen as*

$$h(\varepsilon) = \frac{\varepsilon^4}{\exp\left(C_1 \varepsilon^{-2} \|\omega_0\|_{L^1} T\right)}, \tag{3.4.15}$$

where $C_1 > 0$ is a positive constant. Then, the estimate

$$\sup_{0 \leq t \leq T} \|\omega^\varepsilon - \varphi_\varepsilon * \bar{\omega}^\varepsilon\|_{L^p} \leq C\varepsilon^{1+\frac{2}{p}}, \tag{3.4.16}$$

holds for all $1 \leq p \leq \infty$, where $C > 0$ is a positive constant which does not depend on ε.

Proof. We start by proving the inequality (3.4.16) in the case $p = 1$. By using the definitions of ω^ε and $\bar{\omega}^\varepsilon$ we have that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\omega^\varepsilon(t,x) - \varphi_\varepsilon * \bar{\omega}^\varepsilon(t,x)| \,\mathrm{d}x \\
&= \int_{\mathbb{R}^2} \left| \sum_i \Gamma_i^\varepsilon \varphi_\varepsilon(x - X_i^\varepsilon(t)) - \int_{\mathbb{R}^2} \varphi_\varepsilon(x-z)\bar{\omega}^\varepsilon(t,z)dz \right| \mathrm{d}x \\
&= \int_{\mathbb{R}^2} \left| \sum_i \int_{R_i} \omega_0^\varepsilon(y)\mathrm{d}y \varphi_\varepsilon(x - X_i^\varepsilon(t)) - \int_{\mathbb{R}^2} \varphi_\varepsilon(x-z)\omega_0^\varepsilon((X^\varepsilon)^{-1}(t,\cdot)(z))dz \right| \mathrm{d}x \\
&= \int_{\mathbb{R}^2} \left| \sum_i \int_{R_i} \omega_0^\varepsilon(y)\varphi_\varepsilon(x - X_i^\varepsilon(t))\mathrm{d}y - \int_{\mathbb{R}^2} \varphi_\varepsilon(x - X^\varepsilon(t,y))\omega_0^\varepsilon(y)\mathrm{d}y \right| \mathrm{d}x \\
&= \int_{\mathbb{R}^2} \left| \sum_i \int_{R_i} \omega_0^\varepsilon(y) \left[\varphi_\varepsilon(x - X_i^\varepsilon(t)) - \varphi_\varepsilon(x - X^\varepsilon(t,y))\right] \mathrm{d}y \right| \mathrm{d}x \\
&\leq \int_{\mathbb{R}^2} \sum_i \int_{R_i} |\omega_0^\varepsilon(y)| \, |\underbrace{\varphi_\varepsilon(x - X_i^\varepsilon(t)) - \varphi_\varepsilon(x - X^\varepsilon(t,y))}_{(*)}| \mathrm{d}y\mathrm{d}x. \qquad (3.4.17)
\end{aligned}$$

For $(*)$ we have the following estimate

$$\begin{aligned}
&\varphi_\varepsilon(x - X_i^\varepsilon(t)) - \varphi_\varepsilon(x - X^\varepsilon(t,y)) \\
&= \int_0^1 \nabla\varphi_\varepsilon(x - X^\varepsilon(t,y) + s(X^\varepsilon(t,y) - X_i^\varepsilon(t)))\mathrm{d}s \, (X^\varepsilon(t,y) - X_i^\varepsilon(t)) \\
&= \varepsilon^{-3} \int_0^1 \nabla\varphi \left(\frac{s(x - X_i^\varepsilon(t)) + (1-s)(x - X^\varepsilon(t,y))}{\varepsilon} \right) \mathrm{d}s \, (X^\varepsilon(t,y) - X_i^\varepsilon(t)) \, .
\end{aligned}$$

So, for any $y \in R_i$ we have that

$$\left| X^\varepsilon(t,y) - X_i^\varepsilon(t) \right| \leq C \,\mathrm{Lip}(X^\varepsilon(t,\cdot)) \, h,$$

where $\mathrm{Lip}(X^\varepsilon(t,\cdot))$ is the Lipschitz constant of the flow $X^\varepsilon(t,\cdot)$, which is bounded by

$$\mathrm{Lip}(X^\varepsilon(t,\cdot)) \leq \exp\left(C\varepsilon^{-2} \|\omega_0\|_{L^1} T \right), \qquad (3.4.18)$$

as a consequence of (3.4.5). Then, rescaling in the x variable in (3.4.17) we have

$$\|\omega^\varepsilon(t,\cdot) - \varphi_\varepsilon * \bar{\omega}^\varepsilon(t,\cdot)\|_{L^1} \leq h\,\varepsilon^{-1} \,\mathrm{Lip}(X^\varepsilon(t,\cdot)) \, \|\nabla\varphi\|_{L^1} \|\omega_0\|_{L^1}.$$

Choosing the function h as in (3.4.15) we get (3.4.16) for $p = 1$.
For $p = \infty$ we can argue in a similar way; by the same computations as in (3.4.17) we have that

$$\begin{aligned}
&|\omega^\varepsilon(t,x) - \varphi_\varepsilon * \bar{\omega}^\varepsilon(t,x)| \\
&\leq \sum_{i=1}^{N(\varepsilon)} \int_{R_i} |\omega_0^\varepsilon(y)| |\underbrace{\varphi_\varepsilon(x - X_i^\varepsilon(t)) - \varphi_\varepsilon(x - X^\varepsilon(t,y))}_{(*)}| \mathrm{d}y,
\end{aligned}$$

and we can estimate $(*)$ as before, so that

$$\|\omega^\varepsilon(t,\cdot)-\varphi_\varepsilon*\bar{\omega}^\varepsilon(t,\cdot)\|_{L^\infty}\leq h\,\varepsilon^{-3}\,\mathrm{Lip}(X^\varepsilon(t,\cdot))\,\|\nabla\varphi\|_{L^\infty}\|\omega_0\|_{L^1},$$

and choosing h as in (3.4.15) we get the result.
Finally, by interpolating we have

$$\|\omega^\varepsilon-\varphi_\varepsilon*\bar{\omega}^\varepsilon\|_{L^p}\leq\|\omega^\varepsilon-\varphi_\varepsilon*\bar{\omega}^\varepsilon\|_{L^1}^{\frac{1}{p}}\|\omega^\varepsilon-\varphi_\varepsilon*\bar{\omega}^\varepsilon\|_{L^\infty}^{1-\frac{1}{p}}\leq C\varepsilon^{1+\frac{2}{p}},$$

and this concludes the proof. □

We are now in the position to prove the uniform L^p bound on ω^ε.

Lemma 3.4.2. *Let $\omega_0\in L^p_c(\mathbb{R}^2)$. Then, the approximate vorticities ω^ε defined in* (3.4.2) *satisfy the following*

$$\sup_{0\leq t\leq T}\|\omega^\varepsilon(t,\cdot)\|_{L^p}\leq C\left(\|\omega_0\|_{L^p}+\|\omega_0\|_{L^1}^{\frac{1}{p}}\right),$$

for $1\leq p<\infty$, and

$$\sup_{0\leq t\leq T}\|\omega^\varepsilon(t,\cdot)\|_{L^\infty}\leq C\|\omega_0\|_{L^\infty}.$$

Proof. First of all, the case $p=1$ follows directly from the definition of ω^ε since

$$\|\omega^\varepsilon(t,\cdot)\|_{L^1}\leq\sum_i^{N(\varepsilon)}|\Gamma_i^\varepsilon|\int\varphi_\varepsilon(x-X_i^\varepsilon(t))\mathrm{d}x\leq\|\omega_0\|_{L^1}.$$

Let consider now $1<p<\infty$ and let $A(t)$ and $B(t)$ be the sets

$$A(t):=\{x\in\mathbb{R}^2:|\omega^\varepsilon(t,x)|>1\},$$

$$B(t):=\{x\in\mathbb{R}^2:|\omega^\varepsilon(t,x)|\leq 1\}.$$

By Chebishev inequality

$$\mathscr{L}^2(A(t))\leq C\|\omega_0\|_{L^1},$$

uniformly in time. Let $\bar{\omega}^\varepsilon$ the solution of (3.4.11), we have that

$$\begin{aligned}\|\omega^\varepsilon(t,\cdot)\|_{L^p(A(t))}&\leq\|\varphi_\varepsilon*\bar{\omega}^\varepsilon(t,\cdot)\|_{L^p(A(t))}+\|\omega^\varepsilon(t,\cdot)-\varphi_\varepsilon*\bar{\omega}^\varepsilon(t,\cdot)\|_{L^p(A(t))}\\&\leq C\|\omega_0\|_{L^p}+\mathscr{L}^2(A(t))^{\frac{1}{p}}\|\omega^\varepsilon-\varphi_\varepsilon*\bar{\omega}^\varepsilon\|_{L^\infty}\\&\leq C\left(\|\omega_0\|_{L^p}+\varepsilon\|\omega_0\|_{L^1}^{\frac{1}{p}}\right).\end{aligned}$$

On the other hand, for the set $B(t)$ we have

$$\int_B|\omega^\varepsilon(t,x)|^p\mathrm{d}x\leq\int_{\mathbb{R}^2}|\omega^\varepsilon(t,x)|\mathrm{d}x\leq C\|\omega_0\|_{L^1},$$

since $|\omega^\varepsilon(t,x)|^p\leq|\omega^\varepsilon(t,x)|$ on $B(t)$. Combining the previous estimates, since $\varepsilon<1$, taking the supremum in time we have the result.
Finally, the case $p=\infty$ follows from the triangle inequality and (3.4.16). □

We give now a convergence result for the error term F_ε.

Lemma 3.4.3. *Let $\omega_0 \in L^p_c(\mathbb{R}^2)$ with $p \geq 1$, then the quantity F_ε defined in* (3.4.10) *satisfies*

$$\sup_{t\in[0,T]} \|F_\varepsilon(t,\cdot)\|_{L^1} \to 0, \qquad \text{as } \varepsilon \to 0. \tag{3.4.19}$$

In particular, for $1 < p < 2$ we have the following bound

$$\|F_\varepsilon(t,\cdot)\|_{L^1} \leq C\delta^{-\beta}\varepsilon^{\frac{1}{3}}\|\omega_0\|_{L^1}, \tag{3.4.20}$$

where $\beta = 2\left(\frac{1}{p} - \frac{1}{3}\right)$ and $\delta(\varepsilon) = \varepsilon^\sigma$ with $0 < \sigma < \frac{1}{4}$. Moreover, choosing $h(\varepsilon) = C_1\varepsilon^6 \exp\left(-C_0\varepsilon^{-2}\right)$ where C_1, C_0 are positive constants, we have that F_ε satisfies the following additional bound

$$\|F_\varepsilon(t,\cdot)\|_{L^2} \leq C\delta^{-\beta}\varepsilon^{\frac{7}{3}}\|\omega_0\|_{L^1},$$

which goes to 0 choosing δ as above and $0 < \sigma < \frac{1}{7}$.

Proof. We divide the proof in several steps.

<u>Step 1</u> $\quad L^p$ bound, $1 \leq p < 2$.

First of all, from the definition of F_ε it follows that

$$\|F_\varepsilon(t,\cdot)\|_{L^1} \leq \sum_{i=1}^{N(\varepsilon)} \|F^i_\varepsilon(t,\cdot)\|_{L^1}\Gamma^\varepsilon_i, \tag{3.4.21}$$

where

$$\begin{aligned} F^i_\varepsilon(t,\cdot) &= \left[v^\varepsilon(t,x) - v^\varepsilon(t,X^\varepsilon_i(t))\right]\varphi_\varepsilon(x - X^\varepsilon_i(t)) \\ &= \int_0^1 \nabla v^\varepsilon(t, sx + (1-s)X^\varepsilon_i(t)) \cdot (x - X^\varepsilon_i(t))\varphi_\varepsilon(x - X^\varepsilon_i(t))\mathrm{d}s \\ &= \int_0^1 f^i_\varepsilon(t,x,s)\mathrm{d}s. \end{aligned}$$

Define $\mu_\varepsilon(z) = z\varphi_\varepsilon(z)$, so that

$$f^i_\varepsilon(t,x,s) = \nabla v^\varepsilon(t, sx + (1-s)X^\varepsilon_i(t))\mu_\varepsilon(x - X^\varepsilon_i(t)),$$

then $\mu_\varepsilon(z) = \varepsilon^{-1}\mu(z/\varepsilon)$ and we have

$$\|\mu_\varepsilon\|_{L^r} \leq C\varepsilon^{-1+\frac{2}{r}},$$

which is small if $r < 2$. We choose $r = \frac{3}{2}$ and we want to bound ∇v^ε in L^3: we have that

$$\|\nabla v^\varepsilon(t,\cdot)\|_{L^3} \leq C\delta^{-\beta},$$

where $\beta = 2\left(\frac{1}{p} - \frac{1}{3}\right)$. Thus, after a rescaling in the L^3 norm of

$$\nabla v^\varepsilon(t, sx + (1-s)X_i^\varepsilon(t)),$$

as a function of x, by Hölder inequality we have

$$\|f_\varepsilon^i(t,\cdot,s)\|_{L^1} \leq Cs^{-\frac{2}{3}}\delta^{-\beta}\varepsilon^{\frac{1}{3}},$$

and integrating in s we have

$$\|F_\varepsilon^i(t,\cdot)\|_{L^1} \leq C\delta^{-\beta}\varepsilon^{\frac{1}{3}}. \tag{3.4.22}$$

Combining (3.4.22) with (3.4.21) we get the result.

Step 2 L^p bound, $p \geq 2$.

In this case we may carry out the estimate for F_ε^i in (3.4.22) differently so that there is no dependence from δ. In fact, applying the Hölder inequality to $f_\varepsilon^i = \nabla v^\varepsilon \cdot \mu_\varepsilon$ we can estimate ∇v^ε in L^p and μ_ε in L^r with r dual to p. Since ∇v^ε is bounded in L^p, (3.4.22) is replaced by

$$\|F_\varepsilon^i(t,\cdot)\|_{L^1} \leq C\|\mu_\varepsilon\|_{L^r} \leq C\varepsilon^{1-\frac{2}{p}},$$

which is small. In particular in this case our argument applies without the initial smoothing by j_δ.

Step 3 L^2 *bound.*

To get the L^2 bound, we need a refinement of the bound on F_ε^i. Define the quantities $\gamma_i = h^{-2}\Gamma_i^\varepsilon$, we have that

$$\sum_{i=1}^{N(\varepsilon)} \gamma_i^2 h^2 = \sum_{i=1}^{N(\varepsilon)} h^{-2}\left(\int_{R_i} \omega_0^\varepsilon(x)\mathrm{d}x\right)^2 \leq \sum_{i=1}^{N(\varepsilon)} h^{-2}h^2 \int_{R_i} |\omega_o^\varepsilon(x)|^2 \mathrm{d}x = \|\omega_0^\varepsilon\|_{L^2}^2, \tag{3.4.23}$$

and by Young convolution's inequality we may estimate

$$\|\omega_0^\varepsilon\|_{L^2} \leq \delta^{-1}\|\omega_0\|_{L^1}. \tag{3.4.24}$$

By Step 1, there exists a constant $M := C\delta^{-\frac{4}{3}}\varepsilon^{\frac{1}{3}} > 0$ such that for every $i \in \{1, ..., N(\varepsilon)\}$ we have that

$$\|F_\varepsilon^i(t,\cdot)\|_{L^1} \leq M. \tag{3.4.25}$$

We prove now the following bound

$$\sum_{i=1}^{N(\varepsilon)} |F_\varepsilon^i(t,x)|h^2 \leq M, \qquad \text{for every } x, \tag{3.4.26}$$

since (3.4.26) together with (3.4.25) will give that

$$\|F_\varepsilon(t,\cdot)\|_{L^2}^2 \le M^2 \sum_{i=1}^{N(\varepsilon)} \gamma_i^2 h^2. \tag{3.4.27}$$

Define the function $g(\alpha) = [v^\varepsilon(t,x) - v^\varepsilon(t, X^\varepsilon(t,\alpha))]\varphi_\varepsilon(x - X^\varepsilon(t,\alpha))$, then

$$\sum_{i=1}^{N(\varepsilon)} |F_\varepsilon^i(t,x)|h^2 = \sum_{i=1}^{N(\varepsilon)} |g(ih)|h^2,$$

which is a discretization of

$$\int_{\mathbb{R}^2} |g(\alpha)|\mathrm{d}\alpha = \int_{\mathbb{R}^2} |v^\varepsilon(t,x) - v^\varepsilon(t, X^\varepsilon(t,\alpha))|\varphi_\varepsilon(x - X^\varepsilon(t,\alpha))\mathrm{d}\alpha. \tag{3.4.28}$$

Arguing in a similar way as in the proof of Lemma 3.4.1 we have that the discretization error can be bounded by

$$\begin{aligned} Ch\left(\varepsilon^{-1} + \varepsilon^{-2}\|\omega_0\|_{L^1} \exp\left(C_1\varepsilon^{-2}\|\omega_0\|_{L^1}T\right)\right)\left(\varepsilon^{-3}\exp\left(C_1\varepsilon^{-2}\|\omega_0\|_{L^1}T\right)\right) \\ \le Ch\varepsilon^{-5}\|\omega_0\|_{L^1}\exp\left(2C_1\varepsilon^{-2}\|\omega_0\|_{L^1}T\right), \end{aligned}$$

which is small due to our choice of $h(\varepsilon)$. We have now reduced (3.4.27) to prove that (3.4.28) is bounded by M at each x. But this can be done the same way as estimating (3.4.25) with $(X^\varepsilon(t,\alpha), x)$ instead of $(x, X_i^\varepsilon(t))$ and this shows that (3.4.28) is bounded by M. In view of the discretization estimate, (3.4.26) is now verified. Combining (3.4.23),(3.4.24),(3.4.27) we have

$$\|F_\varepsilon(t,\cdot)\|_{L^2} \le C\delta^{-\frac{7}{3}}\varepsilon^{\frac{1}{3}},$$

and this concludes the proof. □

It is worth to note that the dependence on p of the bound in (3.4.20) is due to the fact that, in order to obtain the convergence in (3.4.19), for $1 \le p < 2$ we need to regularize the initial vorticity, while is not needed for $p > 2$. We introduce the following definition.

Definition 3.4.1. *Let $v \in L^\infty((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^2))$ and $v_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. We say that v is a* VB-solution *of the 2D incompressible Euler equations with initial datum v_0 if*

- *v is a weak solution of* (IE)
- *there exists an approximate sequence v^ε constructed with the vortex-blob method such that, as $\varepsilon \to 0$ along a subsequence,*

$$\begin{aligned} v^\varepsilon &\overset{*}{\rightharpoonup} v && \text{in } L^\infty((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^2)), \\ v^\varepsilon(0,\cdot) &\to v_0 && \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^2). \end{aligned}$$

The uniform bound of Lemma 3.4.2 together with Lemma 3.1.1 give the following compactness lemma for the vortex-blob method.

Lemma 3.4.4. *Let* $\omega_0 \in L^1_c(\mathbb{R}^2) \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$ *or* $\omega_0 \in L^p_c(\mathbb{R}^2)$ *for* $p > 1$ *and let* $\{(\omega^\varepsilon, v^\varepsilon)\}_\varepsilon$ *be the approximate vorticity and velocity constructed by the vortex-blob approximation. Then there exists*

$$\omega \in L^\infty((0,T); L^p(\mathbb{R}^2)), \qquad v \in L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^2)),$$

such that up to subsequences the following hold true

(i) if $p > 1$*, then* v *satisfies (R2a) and*

$$v^\varepsilon \to v \qquad \text{in} \;\; L^2((0,T); L^2_{\text{loc}}(\mathbb{R}^2)),$$

(ii) if $p = 1$*, then* v *satisfies (R2b) and for every* $1 \le q < 2$

$$v^\varepsilon \to v \qquad \text{in} \;\; L^q((0,T); L^q_{\text{loc}}(\mathbb{R}^2)),$$

(iii) $\omega^\varepsilon \overset{*}{\rightharpoonup} \omega \qquad$ *in* $\;L^\infty((0,T); L^p(\mathbb{R}^2))$.

Proof. We divide the proof in several steps.

Step 1   *Convergence of the vorticity.*

By Lemma 3.4.2, we have that the approximate vorticity satisfies

$$\sup_\varepsilon \sup_{t\in[0,T]} \int_{\mathbb{R}^2} |\omega^\varepsilon(t,x)|^p \mathrm{d}x \le C. \tag{3.4.29}$$

Moreover, when $p = 1$ we also have by Lemma 3.4.6 that $\{\omega^\varepsilon\}$ is equi-integrable. Then, there exists $\omega \in L^\infty((0,T); L^p(\mathbb{R}^2))$ such that

$$\omega^\varepsilon \overset{*}{\rightharpoonup} \omega \quad \text{in } L^\infty((0,T); L^p(\mathbb{R}^2)). \tag{3.4.30}$$

Step 2   *Convergence of the velocity.*

The approximate velocity v^ε satisfies the following uniform bound

$$\sup_\varepsilon \sup_{t\in[0,T]} \int_{B_R} |v^\varepsilon(t,x)|^2 \mathrm{d}x \le C(R), \tag{3.4.31}$$

as a consequence of Young's inequality in the case $p > 1$ and of Proposition 3.4.7 for $p = 1$. From the bound (3.4.31) it follows that there exists $v \in L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^2))$ such that

$$v^\varepsilon \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^2)). \tag{3.4.32}$$

In addition, by Theorem 3.2.2 we also have that for some $s, r > 0$

$$\{v^\varepsilon\} \subset \mathrm{Lip}([0,T]; W^{-s,r}(\mathbb{R}^2)),$$

uniformly in ε. Then, thanks to Aubin-Lions' Lemma together with Lemma 3.1.1, for $p > 1$ we can upgrade the convergence (3.4.32) to

$$v^\varepsilon \to v \quad \text{in } L^2((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^2)),$$

while for $p = 1$ we have

$$v^\varepsilon \to v \quad \text{in } L^q((0,T); L^q_{\mathrm{loc}}(\mathbb{R}^2)),$$

for any $1 \le q < 2$, and this concludes the proof. □

By putting together Lemma 3.4.4 and 3.4.3 it is easy to prove Beale's theorem [10], where he showed the existence of VB-solutions when the initial vorticity is in L^p, with $p > 1$, and compactly supported. In detail:

Theorem 3.4.5. *Let $v_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ and assume that the vorticity $\omega_0 = \mathrm{curl}\, v_0 \in L^p_c(\mathbb{R}^2)$ for some $p > 1$. Let ω^ε be given by the vortex-blob approximation with parameters chosen so that $\delta(\varepsilon) = \varepsilon^\sigma$ for some $0 < \sigma < 1/4$, and $h(\varepsilon) \le C\varepsilon^4 \exp(-C_0\varepsilon^{-2})$ for some constants C_0, C. Then up to subsequences, v^ε converges strongly in $L^2((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^2))$ to a classical weak solution of the Euler equations with initial velocity v_0.*

3.4.2 The L^1 case

We consider now the case of initial vorticities $\omega_0 \in L^1_c(\mathbb{R}^2)$. In particular, we prove the equi-integrability of the sequence of approximate vorticities $\{\omega^\varepsilon\}$ given by the vortex-blob method and this will be crucial in the extension of Beale's result to the case $p = 1$. Moreover, the fact that ω^ε is equi-integrable will also be fundamental for the applications of the linear theory discussed in Chapter 1 to the 2D Euler equations. We start by showing the equi-integrability of ω^ε in the following (up to our knowledge original) lemma.

Lemma 3.4.6. *Let $\{\omega_0^\varepsilon\}_\varepsilon \subset L^1_c$ be an equi-integrable sequence. Then the approximate vorticities ω^ε as in (3.4.2) are equi-integrable in $L^1((0,T) \times \mathbb{R}^2)$.*

Proof. We divide the proof in several steps.

<u>Step 1</u> The sequence $\{\bar\omega^\varepsilon\}_\varepsilon$ is equi-integrable.

We start by proving the equi-integrability of the sequence $\bar\omega^\varepsilon$ on small sets; we have that

$$\int_A |\bar\omega^\varepsilon(t,x)|\mathrm{d}x = \int_A |\omega_0^\varepsilon((X^\varepsilon)^{-1}(t,\cdot)(x))|\mathrm{d}x = \int_{X^\varepsilon(t,A)} |\omega_0^\varepsilon(y)|\mathrm{d}y.$$

Since v^ε is divergence-free we have that $\mathscr{L}^2(X^\varepsilon(t,A)) = \mathscr{L}^2(A)$, so the measure of the set $X^\varepsilon(t,A)$ is independent from t and ε and then the equi-integrability of ω_0^ε gives the result. We move now to the proof of the equi-integrability at infinity; we have that

$$\int_{\mathbb{R}^2\setminus B_r} |\bar{\omega}^\varepsilon(t,x)|\mathrm{d}x = \int_{\mathbb{R}^2\setminus B_r} |\omega_0^\varepsilon((X^\varepsilon)^{-1}(t,x))|\mathrm{d}x = \\ = \int_{\{y\in B_R:|X^\varepsilon(t,y)|>r\}} |\omega_0^\varepsilon(y)|\mathrm{d}y,$$

where supp $\omega_0^\varepsilon \subseteq B_R$. By Lemma 1.3.1, the measure of the set

$$\{y \in B_R : |X^\varepsilon(t,y)| > r\},$$

can be made arbitrary small for r big enough, independently from ε and t. Then by the equi-integrability of ω_0^ε the claim of the first step follows.

Step 2 The sequence $\{\varphi_\varepsilon * \bar{\omega}^\varepsilon\}_\varepsilon$ is equi-integrable.

We start by proving the equi-integrability of $\varphi_\varepsilon * \bar{\omega}^\varepsilon$ on small sets. Since the initial datum ω_0^ε has compact support (uniformly in ε) and converges strongly, therefore weakly, to ω_0 in L^1, De la Vallé-Poussin's theorem provides the existence of a function G positive, increasing and superlinear such that

$$\sup_\varepsilon \int_{\mathbb{R}^2} G(|\omega_0^\varepsilon(x)|)\mathrm{d}x < \infty.$$

Then, for $\varphi_\varepsilon * \bar{\omega}^\varepsilon$ we have that

$$\int_{\mathbb{R}^2} G(|\varphi_\varepsilon * \bar{\omega}^\varepsilon|(t,x))\mathrm{d}x = \int_{\mathbb{R}^2} G\left(\left|\int \varphi_\varepsilon(x-y)\bar{\omega}^\varepsilon(t,y)\mathrm{d}y\right|\right)\mathrm{d}x \tag{3.4.33}$$

$$\leq \int_{\mathbb{R}^2} G\left(\int \varphi_\varepsilon(x-y)|\bar{\omega}^\varepsilon(t,y)|\mathrm{d}y\right)\mathrm{d}x \tag{3.4.34}$$

$$\leq \int\int \varphi_\varepsilon(x-y)G(|\bar{\omega}^\varepsilon(t,y)|)\mathrm{d}y\mathrm{d}x \tag{3.4.35}$$

$$= \int_{\mathbb{R}^2} G(|\bar{\omega}^\varepsilon(t,y)|)\int \varphi_\varepsilon(x-y)\mathrm{d}x\mathrm{d}y$$

$$= \int_{\mathbb{R}^2} G(|\bar{\omega}^\varepsilon(t,y)|)\mathrm{d}y. \tag{3.4.36}$$

Note that in (3.4.34) we have taken the modulus inside the integral and used that G is increasing, while in (3.4.35) we used Jensen's inequality since G is convex and $\varphi_\varepsilon(x-\cdot)\mathrm{d}y$ is a probability measure. Multiplying (3.4.12) by $G'(|\bar{\omega}^\varepsilon|)$, integrating in space and using the divergence-free condition of v^ε, from the equi-integrability of ω_0^ε it follows that

$$\sup_\varepsilon \sup_{t\in[0,T]} \int_{\mathbb{R}^2} G(|\bar{\omega}^\varepsilon(t,x)|)\mathrm{d}x \leq \sup_\varepsilon \int_{\mathbb{R}^2} G(|\omega_0^\varepsilon(x)|)\mathrm{d}x < \infty. \tag{3.4.37}$$

Then, taking the supremum in time and in ε in (3.4.33) and estimating (3.4.36) with (3.4.37) shows the equi-integrability on small sets. The equi-integrability at infinity is an immediate consequence of that of $\bar{\omega}^\varepsilon$.

Step 3 The sequence $\{\omega^\varepsilon\}_\varepsilon$ is equi-integrable.

We start by proving equi-integrability on small sets. We can compute

$$\begin{aligned}\int_A |\omega^\varepsilon(t,x)|\mathrm{d}x &\le \int_A |\omega^\varepsilon(t,x) - \varphi_\varepsilon * \bar{\omega}^\varepsilon(t,x)|\mathrm{d}x + \int_A |\varphi_\varepsilon * \bar{\omega}^\varepsilon(t,x)|\mathrm{d}x \\ &\le \|\omega^\varepsilon - \varphi_\varepsilon * \bar{\omega}^\varepsilon\|_{L^\infty}\mathscr{L}^2(A) + \int_A |\varphi_\varepsilon * \bar{\omega}^\varepsilon(t,x)|\mathrm{d}x.\end{aligned}$$

Fix $\eta > 0$. The first term can be estimate using $\|\omega^\varepsilon - \varphi_\varepsilon * \bar{\omega}^\varepsilon\|_\infty \le C\varepsilon \le C$ and choosing $\gamma_1 < \dfrac{\eta}{2C}$ so that for $\mathscr{L}^2(A) < \gamma_1$

$$\|\omega^\varepsilon - \varphi_\varepsilon * \bar{\omega}^\varepsilon\|_\infty \mathscr{L}^2(A) \le C\gamma_1 \le \frac{\eta}{2}.$$

For the second term we use the equi-integrability of $\varphi_\varepsilon * \bar{\omega}^\varepsilon$. There exists γ_2 such that

$$\int_A |\varphi_\varepsilon * \bar{\omega}^\varepsilon|\mathrm{d}x < \frac{\eta}{2},$$

if $\mathscr{L}^2(A) \le \gamma_2$. So taking $\gamma = \min(\gamma_1, \gamma_2)$, assuming $\mathscr{L}^2(A) \le \gamma$ and then taking the supremum in time, the equi-integrability on small sets is proven.
We prove now the equi-integrability at infinity. Fix $\eta > 0$ and decompose

$$\int_{B_R^c} |\omega^\varepsilon(t,x)|\mathrm{d}x \le \int_{B_R^c} |\omega^\varepsilon(t,x) - \varphi_\varepsilon * \bar{\omega}^\varepsilon(t,x)|\mathrm{d}x + \int_{B_R^c} |\varphi * \bar{\omega}^\varepsilon(t,x)|\mathrm{d}x.$$

Since $\varphi * \bar{\omega}^\varepsilon$ is equi-integrable there exists $R_1 > 0$ such that for every $R > R_1$

$$\int_{B_R^c} |\varphi_\varepsilon * \bar{\omega}^\varepsilon(t,x)|\mathrm{d}x \le \frac{\eta}{2},$$

and by (3.4.16), if we consider $\varepsilon \le \bar{\varepsilon} := \sqrt[3]{\frac{\eta}{2C}}$ we obtain

$$\int_{B_R^c} |\omega^\varepsilon(t,x)|\mathrm{d}x \le C\varepsilon^3 + \frac{\eta}{2} \le \eta.$$

For $\varepsilon > \bar{\varepsilon}$ we do not use estimate (3.4.16) but we focus our attention on the flows X_i^ε. From the definition of ω^ε we know

$$\int_{B_R^c} |\omega^\varepsilon(t,x)|\mathrm{d}x \le \sum_i |\Gamma_i^\varepsilon| \int_{B_R^c} \varphi_\varepsilon(x - X_i^\varepsilon(t))\mathrm{d}x. \tag{3.4.38}$$

For the flows $X_i^\varepsilon(t)$ we have that for a given finite time T

$$|X_i^\varepsilon(t)| \le |\alpha_i| + \int_0^T |v^\varepsilon(\tau, X_i^\varepsilon(\tau))|\mathrm{d}\tau. \tag{3.4.39}$$

Since $\alpha_i \in \text{supp}\ \omega_0^\varepsilon$ which is compact, we have $|\alpha_i| \le \tilde{R}$. Decompose the Biot-Savart Kernel as $K = K\chi_{B_1} + K\chi_{B_1^c} := K_1 + K_2$ where $K_1 \in L^1$ and $K_2 \in L^\infty$. Using Young inequality for convolutions we get

$$\int_0^T |v^\varepsilon(\tau, X_i^\varepsilon(\tau))| \mathrm{d}\tau \le T\left(\|K_1\|_{L^1}\|\omega^\varepsilon\|_{L^\infty} + \|K_2\|_{L^\infty}\|\omega_0\|_{L^1}\right)$$

and

$$|\omega^\varepsilon(t,x)| \le \frac{1}{\varepsilon^2}\sum_i |\Gamma_i^\varepsilon| \le \frac{1}{\bar{\varepsilon}^2}\|\omega_0\|_{L^1}.$$

Then by (3.4.39) we have that

$$|X_i^\varepsilon(t)| \le \tilde{R} + T\left(\|K_1\|_{L^1}\frac{1}{\bar{\varepsilon}^2} + \|K_2\|_{L^\infty}\right)\|\omega_0\|_{L^1}.$$

Defining $R_2 > 0$ as

$$R_2 = \tilde{R} + T\left(\|K_1\|_{L^1}\frac{1}{\bar{\varepsilon}^2} + \|K_2\|_\infty\right)\|\omega_0\|_{L^1} + 2,$$

we have that for $|x| > R_2$

$$|x - X_i^\varepsilon(t)| \ge R_2 - \tilde{R} - T\left(\|K_1\|_{L^1}\frac{1}{\bar{\varepsilon}^2} + \|K_2\|_{L^\infty}\right)\|\omega_0\|_{L^1} > 1,$$

so that in (3.4.38) we integrate out of the support of φ_ε and the integral therefore vanishes. Setting $R = \max(R_1, R_2)$ and taking the supremum in ε and t we have the result since $\bar{\varepsilon}$ depends only on η. □

If we assume in addition that the initial vorticity $\omega_0 \in H^{-1}_{\text{loc}}(\mathbb{R}^2)$, then the initial velocity v_0 is locally square integrable; this is the content of the following proposition.

Proposition 3.4.7. *Let $\omega_0 \in L^1_c \cap H^{-1}_{\text{loc}}(\mathbb{R}^2)$ and v^ε defined as in* (3.4.4). *Then $v^\varepsilon \in L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^2))$ and*

$$\sup_{t\in[0,T]} \|v^\varepsilon(t,\cdot)\|_{L^2(B_R)} \le C(R).$$

Proof. First of all, we decompose $v^\varepsilon(t,x) = \tilde{v}^\varepsilon(t,x) + \bar{v}(x)$ where $\bar{v}$ is a smooth steady solution of the 2D Euler equations and $\tilde{v}^\varepsilon(t,x)$ is at each time in L^2 with zero total circulation. To do this, consider the same mollifier φ as in the definition of ω^ε in (3.4.2) and set

$$\Gamma = -\int_{\mathbb{R}^2} \omega_0(x)\mathrm{d}x, \qquad \bar{\omega}(x) = \Gamma\varphi(x),$$

$$\bar{v}(x) = K * \bar{\omega}, \qquad \tilde{v}^\varepsilon = v^\varepsilon - \bar{v}, \qquad \tilde{\omega}^\varepsilon = \omega^\varepsilon - \bar{\omega}.$$

Then, $\tilde{v}^\varepsilon$ solves the following equation

$$\partial_t \tilde{v}^\varepsilon + (v^\varepsilon\cdot\nabla)\,\tilde{v}^\varepsilon + (\tilde{v}^\varepsilon\cdot\nabla)\,\bar{v} + \nabla p^\varepsilon = K * (\operatorname{div} F_\varepsilon)\,. \tag{3.4.40}$$

Multiplying (3.4.40) by $\tilde{v}^\varepsilon$ and integrating over $\mathbb{R}^2$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{v}^\varepsilon(t,\cdot)\|_{L^2}^2 \leq \|\nabla\bar{v}\|_{L^\infty}\|\tilde{v}^\varepsilon(t,\cdot)\|_{L^2}^2 + \|K*(\operatorname{div}F_\varepsilon)\|_{L^2}\|\tilde{v}^\varepsilon(t,\cdot)\|_{L^2}. \tag{3.4.41}$$

Since $F \mapsto (K * \operatorname{div} F)$ is a bounded operator in L^2 we get

$$\|\tilde{v}^\varepsilon(t,\cdot)\|_{L^2} \leq C(T)\|\tilde{v}_0^\varepsilon\|_{L^2}, \qquad \text{for all } 0 \leq t \leq T.$$

In order to conclude, it is enough to prove that $\|\tilde{v}_0^\varepsilon\|_{L^2}$ is finite. Note that

$$\tilde{v}_0^\varepsilon(x) = \sum_{i=1}^{N(\varepsilon)} \Gamma_i^\varepsilon K_\varepsilon(x-\alpha_i) - \Gamma K * \varphi(x).$$

The previous sum is a discretization of the integral

$$\int_{\mathbb{R}^2} K_\varepsilon(x-\alpha)\omega_0^\varepsilon(\alpha)\mathrm{d}\alpha = (K*\varphi_\varepsilon)*(j_\delta*\omega_0),$$

which is by hypothesis bounded in L^2. Since in the definition of $\tilde{v}_0$ the kernel φ is chosen to be the same as in (3.4.3), the discretization error can be pointwise bounded by $h\|\omega_0\|_{L^1}\|\nabla K_\varepsilon\|_{L^\infty} = Ch\varepsilon^{-2}$, which is small by our choice of $h(\varepsilon)$. It follows that v_0^ε, and therefore $\tilde{v}_0^\varepsilon$ is uniformly bounded in $L^2(B_{2R})$ where $R > 0$ is such that $\operatorname{supp}\omega_0^\varepsilon \subseteq B_R$. For $|x| > 2R$, K_ε is just K and then

$$\tilde{v}_0^\varepsilon(x) = \sum_{i=1}^{N(\varepsilon)} \Gamma_i^\varepsilon \left(K(x-\alpha_i) - K(x)\right),$$

and it is easy to see that it is bounded by

$$\sum_{i=1}^{N(\varepsilon)} C|x|^{-2}|\Gamma_i^\varepsilon| \leq \|\omega_0\|_{L^1}|x|^{-2},$$

thus it is bounded in $L^2(B_{2R}^c)$. □

The equi-integrability of the vorticity ω^ε constructed via the vortex-blob method guarantees the phenomenon of concentration-cancellations as in Theorem 3.2.6. This fact together with the consistency of the method implies the existence of VB-solutions in the case of L_c^1 initial vorticity. In particular with Lemma 3.4.6 we improve the result of [10] to the case $\omega_0 \in L_c^1 \cap H_{\mathrm{loc}}^{-1}(\mathbb{R}^2)$ and this is the content of the following theorem.

Theorem 3.4.8. *Let $v_0 \in L_{\mathrm{loc}}^2(\mathbb{R}^2)$ and assume that the vorticity $\operatorname{curl} v_0 = \omega_0 \in L_c^1(\mathbb{R}^2) \cap H_{\mathrm{loc}}^{-1}(\mathbb{R}^2)$. Let ω^ε be given by the vortex-blob approximation with the parameters chosen so that $\delta(\varepsilon) = \varepsilon^\sigma$ for some $0 < \sigma < \frac{1}{7}$, and $h(\varepsilon) \leq C\varepsilon^6 \exp(-C_0\varepsilon^{-2})$ for certain C_0, C. Then there exists a subsequence of v^ε which converges strongly in $L^q((0,T); L_{\mathrm{loc}}^q(\mathbb{R}^2))$ for any $1 \leq q < 2$ and weakly in $L^\infty((0,T); L_{\mathrm{loc}}^2(\mathbb{R}^2))$ to a classical weak solution v of the Euler equations with initial velocity v_0.*

We omit the proof since it is the same as the one of Theorem 3.2.6.

CHAPTER 4

Lagrangian and conservative solutions of the 2D Euler equations

In this chapter we are going to deal with the Lagrangian property and the conservation of the energy for weak solutions of the 2D Euler equations. In particular, in Section 4.1 we prove the existence of distributional, Lagrangian and renormalized solutions of (VE) and we discuss the equivalence of the definitions under different integrability assumptions on the initial vorticity. Then, in Sections 4.1.1 and 4.1.2 we prove that the solutions that are obtained via the vanishing viscosity limit and the vortex-blob method are Lagrangian and renormalized. In Section 4.2 we discuss the conservation of the energy for weak solutions of (IE). We prove that all weak solutions are conservative if $\omega_0 \in L^p_c(\mathbb{R}^2)$ with $p \geq 3/2$ and then we prove that all the three methods of constructing approximate solutions sequence discussed previously generate conservative solutions in the case $1 < p < 3/2$.

4.1 Lagrangian and renormalized solutions

In this section we deal with the existence of Lagrangian and renormalized solutions to the 2D Euler equations in a non-smooth context. In Chapter 3 we have proved the existence of weak solutions of the Euler equations (IE) but nothing was said about the validity of the vorticity equation (VE) when we assume only an L^p control on the initial vorticity ω_0. Here we want to apply the theory developed in Chapter 1 to study a weak formulation of (VE). In particular, note that we can give a distributional meaning to the equation only if the product $v\omega \in L^1_{\text{loc}}(\mathbb{R}^2)$ which is not true with low integrability assumptions on ω. In fact, by the Calderón-Zygmund theorem and Sobolev embeddings it is easy to see that if $\omega \in L^p(\mathbb{R}^2)$ with $p \geq 4/3$ then automatically the product $v\omega \in L^1_{\text{loc}}(\mathbb{R}^2)$ but for $1 \leq p < 4/3$ this might not be the case. For this reason when $1 \leq p < 4/3$ the notions of Lagrangian and renormalized solutions seem to be necessary. We recall the definitions of Lagrangian and renormalized solutions, noticing that they are the same as in the linear case.

Definition 4.1.1. *Let* $\omega \in C([0,T];L^p(\mathbb{R}^2))$ *with* $1 \le p \le \infty$. *The pair* (v,ω) *is a renormalized solution of* (VE) *if for any* $\beta \in C^1 \cap L^\infty(\mathbb{R})$ *it holds*

$$\int_0^T \int_{\mathbb{R}^2} \beta(\omega)(\partial_t \varphi + v \cdot \nabla \varphi)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^2} \beta(\omega_0)\varphi(0,x)\mathrm{d}x = 0,$$

for any $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^2)$, *and*

$$v(t,x) = K * \omega(t,x) \qquad a.e. \text{ in } (0,T) \times \mathbb{R}^2.$$

Definition 4.1.2. *Let* $\omega \in C([0,T];L^p(\mathbb{R}^2))$ *with* $1 \le p \le \infty$. *The pair* (v,ω) *is a Lagrangian solution of* (VE) *if*

- $\omega(t,x) = \omega_0(X^{-1}(t,\cdot)(x))$ *for all* $t \in [0,T]$ *and a.e.* $x \in \mathbb{R}^2$;
- $v(t,x) = K * \omega(t,x)$ *a.e. in* $(0,T) \times \mathbb{R}^2$;
- X *is a regular Lagrangian flow of* v.

It is worth to note that Lagrangian solutions as in Definition 4.1.2 are also renormalized as in the sense of Definition 4.1.1. In fact, let ω be a Lagrangian solution and let X be the regular Lagrangian flow of v as in Definition 4.1.2. Since X is absolutely continuous in time, for all $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^2)$ and for all $\beta \in C^1 \cap L^\infty(\mathbb{R})$ we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \varphi(t,x)\beta(\omega(t,x))\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \varphi(t,x)\beta(\omega_0(X^{-1}(t,\cdot)(x)))\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \varphi(t,X(t,x))\beta(\omega_0(x))\mathrm{d}x \\ &= \int_{\mathbb{R}^2} \beta(\omega_0(x))\left(\partial_t\varphi(t,X(t,x)) + v(t,X(t,x)) \cdot \nabla\varphi(t,X(t,x))\right)\mathrm{d}x,\end{aligned}$$

where we have used that v is divergence-free and a change of variables. By integrating in time and then changing variables we have the result. We prove now a theorem on the equivalence of the definitions of solutions of (VE) under several integrability assumptions on ω_0.

Theorem 4.1.1. *Let* $\omega_0 \in L^p_c(\mathbb{R}^2)$ *with* $p > 1$. *Then, the following statements holds*

(1) If $p \ge 4/3$, *there exists* $(v,\omega) \in L^\infty((0,T);W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)) \times L^\infty((0,T);L^p(\mathbb{R}^2))$ *which is a distributional solution of* (VE).

(2) If $p > 1$, *there exists* $(v,\omega) \in L^\infty((0,T);W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)) \times L^\infty((0,T);L^p(\mathbb{R}^2))$ *which is a renormalized solution of* (VE).

(3) If $p \ge 2$, *every distributional solution* $(v,\omega) \in L^\infty((0,T);W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)) \times L^\infty((0,T);L^p(\mathbb{R}^2))$ *is a renormalized solution.*

(4) If $p > 1$, *every renormalized solution* $(v,\omega) \in L^\infty((0,T);W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)) \times L^\infty((0,T);L^p(\mathbb{R}^2))$ *is a Lagrangian solution.*

Proof. We divide the proof in several steps.

Step 1 *Statement (1).*

Let ρ^ε a standard mollifier and define $\omega_0^\varepsilon = \omega_0 * \rho^\varepsilon$. Moreover, by standard properties of mollifiers $v_0^\varepsilon = K * \omega_0^\varepsilon \in L^2_{\mathrm{loc}} \cap C^\infty(\mathbb{R}^2)$. Let v^ε the unique smooth solution of (IE) with initial datum v_0^ε, then $\omega^\varepsilon = \operatorname{curl} v^\varepsilon$ is a solution of (VE) with initial datum ω_0^ε and by using the equation we have the uniform bound

$$\sup_{t\in[0,T]} \|\omega(t,\cdot)\|_{L^p} \leq \|\omega_0^\varepsilon\|_{L^p} \leq \|\omega_0\|_{L^p}.$$

As a consequence of the previous uniform bound, by using standard compactness argument we have that up to subsequences not relabeled

$$\omega^\nu \overset{*}{\rightharpoonup} \omega \qquad \text{in } L^\infty((0,T);L^p(\mathbb{R}^2)), \tag{4.1.1}$$

$$v^\varepsilon \overset{*}{\rightharpoonup} v \qquad \text{in } L^\infty((0,T);W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)). \tag{4.1.2}$$

In particular, by Sobolev embedding theorems v^ε is uninformly bounded in $L^\infty((0,T);L^2_{\mathrm{loc}}(\mathbb{R}^2))$ and since v^ε solves the equations (IE) it follows that $\partial_t v^\varepsilon$ is uniformly bounded in $L^\infty((0,T);\mathcal{D}(\mathbb{R}^2))$. Then, we can apply Aubin-Lions' Lemma and we have that for any $1 \leq q < 2p/(2-p)$

$$v^\varepsilon \to v \qquad \text{in } L^q((0,T);L^q_{\mathrm{loc}}(\mathbb{R}^2)). \tag{4.1.3}$$

By using (4.1.2) and (4.1.3) it is straightforward to pass to the limit in the integral formulation of the equation (VE) since for $p \geq 4/3$ the Hölder conjugate $p' = p/(p-1) < 2p/(2-p)$. In order to conclude we need to prove the Biot-Savart law for the limit v. For $\eta \in C_c^\infty((0,T)\times\mathbb{R}^2)$, thanks to the convergences above we have that

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0} \iint (v^\varepsilon - K*\omega^\varepsilon)\eta \mathrm{d}x\mathrm{d}t = \lim_{\varepsilon\to 0} \iint v^\varepsilon\eta - \omega^\varepsilon(K*\eta)\mathrm{d}x\mathrm{d}t \\ &= \iint v\eta - \omega(K*\eta)\mathrm{d}x\mathrm{d}t = \iint (v - K*\omega)\eta \mathrm{d}x\mathrm{d}t, \end{aligned}$$

so that $v = K * \omega$ for almost every $(t,x) \in [0,T]\times\mathbb{R}^2$.

Step 2 *Statement (2).*

We consider again the sequence of smooth solutions $(v^\varepsilon, \omega^\varepsilon)$ as in Step 1. Since v satisfies the growth condition (3.1.5) we can apply the stability theorem for renormalized solution (Theorem 1.2.4), so that the convergence (4.1.1) can be upgraded to

$$\omega^\varepsilon \to \omega \qquad \text{in } C([0,T];L^p(\mathbb{R}^2)), \tag{4.1.4}$$

which implies that ω is a renormalized solution.

Step 3 *Statement (3).*

We restrict now to $p \geq 2$. Since $v \in L^\infty((0,T); W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2))$ it follows that $v \in L^\infty((0,T); W^{1,q}_{\mathrm{loc}}(\mathbb{R}^2))$ with $q = p/(p-1)$ and the conclusion follows from Theorem 1.2.4 since in this case every distributional solution is renormalized.

Step 4 *Statement (4).*

Let $\omega_0 \in L^p(\mathbb{R}^2)$ and let (v, ω) be a renormalized solution of (VE). Then, (v, ω) lies in the space $L^\infty((0,T); W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)) \times C([0,T]); L^p(\mathbb{R}^2))$ and by Theorem 1.3.2 we can consider X the unique regular Lagrangian flow of v. So, the function $\bar{\omega}(t,x) = \omega_0(X^{-1}(t,\cdot)(x))$ is a Lagrangian and then a renormalized solution of the linear problem

$$\begin{cases} \partial_t \bar{\omega} + v \cdot \nabla \bar{\omega} = 0, \\ \bar{\omega}(0,\cdot) = \omega_0. \end{cases} \tag{4.1.5}$$

It follows that for every $\beta \in C^1 \cap L^\infty(\mathbb{R})$ the following initial value problem is satisfied

$$\begin{cases} \partial_t \left(\beta(\bar{\omega}) - \beta(\omega)\right) + v \cdot \nabla \left(\beta(\bar{\omega}) - \beta(\omega)\right) = 0, \\ \left(\beta(\bar{\omega}) - \beta(\omega)\right)_{|t=0} = 0. \end{cases} \tag{4.1.6}$$

Then, the function $\gamma = \beta(\bar{\omega}) - \beta(\omega)$ is bounded in $L^\infty((0,T); L^\infty(\mathbb{R}^2))$ and it is the unique distributional solution of

$$\begin{cases} \partial_t \gamma + v \cdot \nabla \gamma = 0, \\ \gamma(0,\cdot) = 0. \end{cases} \tag{4.1.7}$$

So $\gamma = 0$ and from this it follows that $\bar{\omega} = \omega$. Then (v, ω) is a Lagrangian solution. □

Note that the proof of Statement 4 in Theorem 4.1.1 implicitly gives the following result

Corollary 4.1.2. *Let $\omega_0 \in L^p_c(\mathbb{R}^2)$ with $1 < p < 2$. Let ρ_ε be a standard mollifier and define $\omega_0^\varepsilon = \omega_0 * \rho_\varepsilon$. Let ω^ε be the unique smooth solution of* (VE) *with initial datum ω_0^ε, then there exists*

$$(v, \omega) \in L^\infty((0,T); W^{1,p}(\mathbb{R}^2)) \times L^\infty((0,T); L^p(\mathbb{R}^2)),$$

such that, up to subsequences,

$$\omega^\nu \stackrel{*}{\rightharpoonup} \omega \qquad \text{in } L^\infty((0,T); L^p(\mathbb{R}^2)),$$

$$v^\nu \to v \qquad \text{in } L^p((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^2)).$$

Moreover, (v, ω) is a renormalized solution of (VE).

In the case $p = 1$, v is no longer in a Sobolev space, but its derivative can be expressed as singular integral of an L^1 function. Since a sequence ω^ε constructed as an exact smooth solution of (VE) is equi-integrable, by using Proposition 1.3.3 it is not difficult to prove that solutions obtained via (ES) are Lagrangian when $p = 1$, see [12]. Since no uniqueness results are known for unbounded vorticity, it is interesting to understand whether solutions obtained by other approximation methods are Lagrangian. From Statement (3) of Theorem 4.1.1 this is non-trivial only in the case $1 \leq p < 2$; for this reasons we are going to deal in the following with the approximations (VV) and (VB).

4.1.1 The vanishing viscosity limit

We now prove that solutions obtained as vanishing viscosity limit of the 2D Navier-Stokes equations are Lagrangian and renormalized, see [25].

Theorem 4.1.3. *Let $\omega_0 \in L^p_c(\mathbb{R}^2)$ with $1 < p < 2$. Let $\{\omega_0^\nu\} \subset L^p_c(\mathbb{R}^2)$ be a sequence of smooth functions converging strongly to ω_0 in $L^p(\mathbb{R}^2)$ and (v^ν, ω^ν) be the unique smooth solution of* (3.3.2)*. Then, there exists*

$$(v, \omega) \in L^\infty((0,T); W^{1,p}(\mathbb{R}^2)) \times L^\infty((0,T); L^p(\mathbb{R}^2)),$$

such that, up to subsequences,

$$\omega^\nu \overset{*}{\rightharpoonup} \omega \qquad \text{in } L^\infty((0,T); L^p(\mathbb{R}^2)),$$

$$v^\nu \to v \qquad \text{in } L^p((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^2)).$$

Moreover, (v, ω) is a renormalized solution of (VE)*.*

Proof. We divide the proof in several steps.

Step 1 *Existence of the limit.*

Since ω_0^ν is uniformly bounded in $L^p_c(\mathbb{R}^2)$ and converge strongly to ω_0 in $L^p(\mathbb{R}^2)$ we have that

$$\omega_0^\nu \to \omega_0 \qquad \text{in } L^r(\mathbb{R}^2) \text{ for any } r \in [1, p]. \tag{4.1.8}$$

By using Theorem 3.3.1, we get that for any $\nu > 0$ there exists a unique solution ω^ν of (3.3.2) such that for any $r \in [1, p]$ we have

$$\{\omega^\nu\} \subset L^\infty((0,T); L^r(\mathbb{R}^2)),$$

$$\{v^\nu\} \subset L^\infty((0,T); W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)),$$

so that there exists $v \in L^\infty((0,T); W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2))$ such that, up to subsequences not relabeled

$$v^\nu \to v \qquad \text{in } L^p((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^2)) \tag{4.1.9}$$

and there exists $\omega \in L^\infty((0,T); L^1 \cap L^p(\mathbb{R}^2))$ such that

$$\omega^\nu \stackrel{*}{\rightharpoonup} \omega \qquad \text{in } L^\infty((0,T); L^p(\mathbb{R}^2)). \tag{4.1.10}$$

Moreover, it follows that (v, ω) is such that

$$v = K * \omega \qquad \text{a.e. in } (0,T) \times \mathbb{R}^2,$$

and the velocity field satisfies (R1) since

$$v \in L^\infty((0,T); L^1(\mathbb{R}^2)) + L^\infty((0,T); L^\infty(\mathbb{R}^2)).$$

Step 2 *A dual problem.*

Let us introduce the following backward transport-diffusion problem

$$\begin{cases} -\partial_t \phi^\nu - \operatorname{div}(\phi^\nu v^\nu) - \nu \Delta \phi^\nu = \chi, \\ \phi^\nu(T, 0) = 0, \end{cases} \tag{4.1.11}$$

where $\chi \in \mathcal{D}((0,T) \times \mathbb{R}^2)$ and $\{v^\nu\}$ is the subsequence chosen in (4.1.9). By standard energy estimates, for every fixed $\nu > 0$ there exists a unique smooth solution ϕ^ν of (4.1.11). We want to prove that the sequence $\{\phi^\nu\}$ converges to the unique distributional solution of

$$\begin{cases} -\partial_t \phi - \operatorname{div}(\phi v) = \chi, \\ \phi(T, 0) = 0. \end{cases} \tag{4.1.12}$$

Indeed, by multiplying the equation by $2\phi^\nu$ we get

$$-\frac{\mathrm{d}}{\mathrm{d}t} \|\phi^\nu\|^2_{L^2} + 2\nu \|\nabla \phi^\nu\|^2_{L^2} = 2 \int_{\mathbb{R}^2} \chi \phi^\nu \mathrm{d}x.$$

Then, by Cauchy-Schwartz inequality, the fact that χ is smooth and taking into account that the time is reversed we have by Gronwall Lemma that uniformly with respect to ν

$$\phi^\nu \in L^\infty((0,T); L^2(\mathbb{R}^2)), \qquad \sqrt{\nu} \nabla \phi^\nu \in L^2((0,T); L^2(\mathbb{R}^2)). \tag{4.1.13}$$

Moreover, by multiplying by $q\phi^\nu |\phi^\nu|^{q-2}$ for any $q \in [2, \infty)$ we get

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} |\phi^\nu|^q \mathrm{d}x + \nu q(q-1) \int_{\mathbb{R}^2} |\nabla \phi^\nu|^2 |\phi^\nu|^{q-2} \mathrm{d}x = q \int_{\mathbb{R}^2} \chi \phi^\nu |\phi^\nu|^{q-2} \mathrm{d}x.$$

By using Hölder inequality, Gronwall's Lemma and the smoothness of χ we get that for every $q \in [2, \infty)$

$$\phi^\nu \in L^\infty((0,T); L^q(\mathbb{R}^2)), \tag{4.1.14}$$

uniformly with respect to ν. Finally, by using (4.1.14) and the fact that ϕ^ν solves (4.1.11) we can improve the convergence in time, namely up to subsequences we have that

$$\phi^\nu \to \phi \qquad \text{in } C([0,T]; L^q_{\text{weak}}(\mathbb{R}^2)), \tag{4.1.15}$$

for any $q < \infty$. Let $\psi \in \mathcal{D}((0,T)\times\mathbb{R}^2)$, by multiplying (4.1.11) by ψ and integrating by parts we have that

$$\int_0^T \int_{\mathbb{R}^2} [\phi^\nu \partial_t \psi + \nu \nabla \phi^\nu \nabla \psi + (v^\nu \cdot \nabla \psi)\phi^\nu - \chi\psi]\, \mathrm{d}x\mathrm{d}t = 0.$$

By using (4.1.9),(4.1.13), and (4.1.14), we get that the limit ϕ satisfies

$$\int_0^T \int_{\mathbb{R}^2} [\phi \partial_t \psi + (v \cdot \nabla \psi)\phi - \chi\psi]\, \mathrm{d}x\mathrm{d}t = 0.$$

Since ϕ^ν is uniformly bounded in $L^\infty((0,T); L^q(\mathbb{R}^2))$ for any $q > 2$, we can choose $q = p/(p-1)$ and by using Theorem 1.2.3, ϕ is the unique renormalized solution of (4.1.11). Then, by the uniqueness of the limit, ϕ^ν converges to ϕ along the whole subsequence chosen in (4.1.9).

<u>Step 3</u> *A duality formula.*

Multiply (3.3.2) by ϕ^ν, after integrates by parts we get

$$\int_0^T \int_{\mathbb{R}^2} -\partial\phi^\nu\omega^\nu - \nu\Delta\phi^\nu\omega^\nu - \operatorname{div}(\phi^\nu v^\nu)\omega^\nu \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^2} \phi(0,x)^\nu \omega_0^\nu(x)\mathrm{d}x,$$

and by using (4.1.11) we have

$$\int_0^T \int_{\mathbb{R}^2} \omega^\nu \chi \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^2} \phi(0,x)^\nu \omega_0^\nu(x)\mathrm{d}x.$$

By using the convergences in (4.1.8), (4.1.10), and (4.1.15), we can pass to the limit in the previous equation and we obtain that

$$\int_0^T \int_{\mathbb{R}^2} \omega \chi \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^2} \phi(0,x) \omega_0(x)\mathrm{d}x, \tag{4.1.16}$$

where ϕ is the unique renormalized solution of (1.2.6).

<u>Step 4</u> *Renormalization.*

By using Theorem 1.2.3 there exists a unique renormalized solution $\bar{\omega}$ of the transport equation with vector field v, the limit obtained in (4.1.9), with initial datum ω_0, namely

$$\begin{cases} \partial_t \bar{\omega} + v \cdot \nabla \bar{\omega} = 0, \\ \bar{\omega}(0,\cdot) = \omega_0. \end{cases} \tag{4.1.17}$$

By Theorem 1.2.6, $\bar{\omega}$ satisfies

$$\int_0^T \int_{\mathbb{R}^2} \bar{\omega}\chi \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^2} \phi(0,x)\omega_0(x)\mathrm{d}x, \tag{4.1.18}$$

where ϕ is the unique renormalized solution of. By taking the difference of (4.1.16) and (4.1.18) we get that

$$\int_0^T \int_{\mathbb{R}^2} (\bar{\omega} - \omega)\chi \mathrm{d}x\mathrm{d}t = 0.$$

Note that is crucial that the backward problem has a unique distributional solution in the class $L^\infty((0,T); L^q(\mathbb{R}^2))$. By varying $\chi \in C_c^\infty((0,T) \times \mathbb{R}^2)$ we get that

$$\omega = \bar{\omega} \qquad \text{a.e. in } (0,T) \times \mathbb{R}^2,$$

and then ω is renormalized since it agrees almost everywhere with the renormalized solution $\bar{\omega}$. □

Theorem 4.1.3 has been extended in [23] to the case $p = 1$; precisely, the following theorem holds.

Theorem 4.1.4. *For initial vorticities in L^1, solutions of* (IE) *obtained as vanishing viscosity limit of the 2D Navier-Stokes equations are renormalized and also Lagrangian solutions of* (VE).

It is important to note that in the proof of Theorem 4.1.4 it is crucial to prove that the linear transport equation admits a unique renormalized solution (which is also Lagrangian) in the space $L^\infty((0,T); L^1 \cap L^\infty(\mathbb{R}^2))$ when the vector field satisfies the regularity assumption (R2c), [23, Theorem 1.2], which is out of the class of DiPerna-Lions [29].

4.1.2 The vortex-blob method

In this section we prove that VB-solutions satisfy the 2D Euler equations in the Lagrangian and renormalized sense. In order to prove that VB-solutions are Lagrangian we will not rely on a duality argument, as done for the vanishing viscosity limit, but we will make a crucial use of the estimate in Lemma 3.4.1.

Theorem 4.1.5. *Let v be a VB-solution. Then, v satisfies the Euler equations in the sense of Lagrangian and renormalized solutions.*

Proof. We divide the proof in two steps.

Step 1 *Representation formula and additional regularity of v.*

Let $(\omega^\varepsilon, v^\varepsilon)$ be a sequence constructed via the vortex blob method which converges to (ω, v) as in Lemma 3.4.4. We want to prove that $v = K * \omega$ a.e. in $(0,T)\times\mathbb{R}^2$. For $\eta \in C_c^\infty((0,T)\times\mathbb{R}^2)$ we have

$$0 = \lim_{\varepsilon\to 0}\iint (v^\varepsilon - K * \omega^\varepsilon)\eta \mathrm{d}x\mathrm{d}t = \lim_{\varepsilon\to 0}\iint v^\varepsilon\eta - \omega^\varepsilon(K*\eta)\mathrm{d}x\mathrm{d}t$$
$$= \iint v\eta - \omega(K*\eta)\mathrm{d}x\mathrm{d}t = \iint (v - K*\omega)\eta\mathrm{d}x\mathrm{d}t,$$

where we have used the fact that $K*\eta \in L^q$ for every $1 \le q \le \infty$. By varing $\eta \in C_c^\infty((0,T)\times\mathbb{R}^2)$ we have the result. Moreover, the gradient of v can be written as

$$(\nabla v)_{ij} = S^i_j \omega \qquad \text{in } \mathcal{S}'(\mathbb{R}^2), \quad i,j = 1,2,$$

where each S^i_j is a singular integral operator of fundamental type with kernel the distributional derivative $\partial_{x_j} K_i$. Hence v satisfies hypothesis (R2b) if $\omega_0 \in L^1$ since $\partial_{x_j}K_i$ define singular integral operators of fundamental type. In the case $p > 1$, by standard Calderòn-Zygmund theory on singular integrals we have the estimate

$$\sup_{t\in[0,T]}\|\nabla v(t,\cdot)\|_{L^p} \le \sup_{t\in[0,T]}\|\omega(t,\cdot)\|_{L^p} \le C\|\omega_0\|_{L^p}$$

and then v satisfies (R2a).

Step 2 *Lagrangian property of the solution.*

Let $(\omega^\varepsilon, v^\varepsilon)$ be chosen as in the previous step and consider the auxiliary problem (3.4.11). By Theorem 1.3.3 we have the existence of $\bar\omega \in C([0,T];L^p(\mathbb{R}^2))$ such that

$$\bar\omega^\varepsilon \to \bar\omega \quad \text{in } C([0,T];L^p(\mathbb{R}^2))$$

where $\bar\omega(t,x) = \omega_0(X^{-1}(t,\cdot)(x))$ and X is the unique regular Lagrangian flow of v. In particular, in the case $p=1$ the equi-integrability of the vortex-blob method proved in Lemma 3.4.6 is crucial in order to apply Theorem 1.3.3. We want to prove that $\omega = \bar\omega$ almost everywhere and this will be enough to conclude. Let $\chi \in C_c^\infty((0,T)\times\mathbb{R}^2)$ and compute

$$\int_0^T\int_{\mathbb{R}^2}(\omega - \bar\omega)\chi\mathrm{d}x\mathrm{d}t = \lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^2}(\omega^\varepsilon - \bar\omega^\varepsilon)\chi\mathrm{d}x\mathrm{d}t$$
$$= \lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^2}(\omega^\varepsilon - \varphi_\varepsilon * \bar\omega^\varepsilon)\chi\mathrm{d}x\mathrm{d}t + \lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^2}(\varphi_\varepsilon * \bar\omega^\varepsilon - \bar\omega^\varepsilon)\chi\mathrm{d}x\mathrm{d}t.$$

By estimate (3.4.16) and standard properties of the convolution, it is easy to check that the previous sum goes to 0 as $\varepsilon \to 0$, and by varying $\chi \in C_c^\infty((0,T)\times\mathbb{R}^2)$ we have that $\omega = \bar\omega$ a.e. in $(0,T)\times\mathbb{R}^2$. □

Remark 4.1.6. *Note that the Step 2 of the proof of Theorem 4.1.5 together with the estimate in Lemma 3.4.16 give that the convergence in* (3.4.30) *of the approximate vorticity* ω^ε *towards the Lagrangian solution* ω *is actually strong.*

4.2 Conservative solutions

In this section we discuss the conservation of the kinetic energy for weak solutions of the 2D Euler equations. We start by giving the definition of conservative solution.

Definition 4.2.1. *Let $v \in C([0,T]; L^2(\mathbb{R}^2))$ be a weak solution of* (IE) *with initial datum $v_0 \in L^2(\mathbb{R}^2)$. We call v a* conservative weak solution *if*

$$\|v(t,\cdot)\|_{L^2} = \|v_0\|_{L^2} \qquad \forall t \in [0,T].$$

First of all, note that in the previous definition we are dealing with initial data which are globally square integrable in space, which is equivalent to require that the vorticity has zero mean value. This is the content of the following proposition, which can be found in [35, Prop. 3.3]:

Proposition 4.2.1. *An incompressible velocity field in $\mathbb{R}^2$ with vorticity of compact support has finite kinetic energy if and only if the vorticity has zero mean value, that is*

$$\int_{\mathbb{R}^2} |v(t,x)|^2 \mathrm{d}x < \infty \iff \int_{\mathbb{R}^2} \omega(t,x) \mathrm{d}x = 0. \tag{4.2.1}$$

We now prove that every weak solution is conservative provided that $p \geq 3/2$. The proof of the following theorem is taken from [16].

Theorem 4.2.2. *Let $v \in C([0,T]; L^2(\mathbb{R}^2))$ be a weak solution of* (IE) *such that the corresponding vorticity $\omega \in L^\infty((0,T); L^1 \cap L^{\frac{3}{2}}(\mathbb{R}^2))$. Then v is conservative. Moreover, the following local balance of the energy holds*

$$\partial_t \left(\frac{|v|^2}{2} \right) + \operatorname{div} \left(v \left(\frac{|v|^2}{2} + p \right) \right) = 0 \qquad \textit{in } \mathcal{D}'(\mathbb{R}^2). \tag{4.2.2}$$

Proof. Let $\rho \in C_c^\infty(\mathbb{R}^2)$ be a standard mollifier and let $\rho_\varepsilon(x) = \varepsilon^{-2} \rho\left(\frac{x}{\varepsilon}\right)$. Define $v^\varepsilon = v * \rho_\varepsilon$ and $\omega^\varepsilon = \omega * \rho_\varepsilon$, we compute

$$\partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla) v^\varepsilon + \nabla p^\varepsilon = \mathcal{R}^\varepsilon, \tag{4.2.3}$$

whith $\mathcal{R}^\varepsilon = (v^\varepsilon \cdot \nabla) v^\varepsilon - \rho_\varepsilon * [(v \cdot \nabla) v]$. Since $v^\varepsilon, p^\varepsilon, \mathcal{R}^\varepsilon$ are smooth in space, $\partial_t v^\varepsilon$ is smooth in space a.e. in time and from (4.2.3) we deduce that v^ε is Lipschitz in time. Hence, we multiply the equation (4.2.3) by v^ε and we get

$$\partial_t \left(\frac{|v^\varepsilon|^2}{2} \right) + \operatorname{div} \left(v^\varepsilon \left(\frac{|v^\varepsilon|^2}{2} + p^\varepsilon \right) \right) = v^\varepsilon \cdot \mathcal{R}^\varepsilon. \tag{4.2.4}$$

In order to conclude it is enough to prove the following convergences:

(A) $\partial_t \left(\frac{|v^\varepsilon|^2}{2} \right) \to \partial_t \left(\frac{|v|^2}{2} \right)$ in the sense of distributions;

(B) $\operatorname{div}\left(v^\varepsilon\left(\frac{|v^\varepsilon|^2}{2}+p^\varepsilon\right)\right) \to \operatorname{div}\left(v\left(\frac{|v|^2}{2}+p\right)\right)$ in the sense of distributions;

(C) $v^\varepsilon \cdot \mathcal{R}^\varepsilon \to 0$ strongly in $L^\infty((0,T);L^1(\mathbb{R}^2))$.

First, we note that since $\omega \in L^\infty((0,T);L^{\frac{3}{2}}(\mathbb{R}^2))$, by Calderòn-Zygmund we have $\nabla v \in L^\infty((0,T);L^{\frac{3}{2}}(\mathbb{R}^2))$ and by the Sobolev embedding theorem $v \in L^\infty((0,T);L^6(\mathbb{R}^2))$. To establish item (A), we show that

$$\frac{|v^\varepsilon|^2}{2} \to \frac{|v|^2}{2}$$

in $L^\infty((0,T);L^{\frac{6}{5}}_{\mathrm{loc}}(\mathbb{R}^2))$. Indeed,

$$\begin{aligned}\left\||v^\varepsilon|^2-|v|^2\right\|_{L^\infty((0,T);L^{\frac{6}{5}}(B_R))} &= \|(v^\varepsilon-v)(v^\varepsilon+v)\|_{L^\infty((0,T);L^{\frac{6}{5}}(B_R))}\\ &\le \|v^\varepsilon-v\|_{L^\infty((0,T);L^6(\mathbb{R}^2))}\|v^\varepsilon+v\|_{L^\infty((0,T);L^{\frac{3}{2}}(B_R))},\end{aligned}$$

which goes to 0 as $\varepsilon \to 0$. We deal now with item (B); it is enough to show that

$$v^\varepsilon\left(\frac{|v^\varepsilon|^2}{2}+p^\varepsilon\right) \to v\left(\frac{|v|^2}{2}+p\right) \text{ in } L^\infty((0,T);L^1_{\mathrm{loc}}(\mathbb{R}^2)).$$

To show this we write:

$$\begin{aligned}v^\varepsilon\left(\frac{|v^\varepsilon|^2}{2}+p^\varepsilon\right)-v\left(\frac{|v|^2}{2}+p\right) &= (v^\varepsilon-v)\frac{|v^\varepsilon|^2}{2}+v\left(\frac{|v^\varepsilon|^2}{2}-\frac{|v|^2}{2}\right)\\ &\quad+(v^\varepsilon-v)p^\varepsilon+v(p^\varepsilon-p).\end{aligned}$$

Since $-\Delta p = \operatorname{div}\operatorname{div}(v\otimes v)$, $p \in L^\infty((0,T);L^3_{\mathrm{loc}}(\mathbb{R}^2))$ and so we estimate (B) as follows

$$\begin{aligned}&\left\|v^\varepsilon\left(\frac{|v^\varepsilon|^2}{2}+p^\varepsilon\right)-v\left(\frac{|v|^2}{2}+p\right)\right\|_{L^\infty((0,T);L^1(B_R))}\\ &\le \|v^\varepsilon-v\|_{L^\infty((0,T);L^3(B_R))}\left\|\frac{|v^\varepsilon|^2}{2}\right\|_{L^\infty((0,T);L^{\frac{3}{2}}(B_R))}+\|v^\varepsilon-v\|_{L^\infty((0,T);L^{\frac{3}{2}}(B_R))}\|p^\varepsilon\|_{L^\infty((0,T);L^3(B_R))}\\ &+\|v\|_{L^\infty((0,T);L^6(\mathbb{R}^2))}\left\|\frac{|v^\varepsilon|^2}{2}-\frac{|v|^2}{2}\right\|_{L^\infty((0,T);L^{\frac{6}{5}}(\mathbb{R}^2))}+\|v\|_{L^\infty((0,T);L^{\frac{3}{2}}(B_R))}\|p^\varepsilon-p\|_{L^\infty((0,T);L^3(B_R))},\end{aligned}$$

which goes to 0 as $\varepsilon \to 0$. Finally, to prove item (C) it is enough to show that $\mathcal{R}^\varepsilon \to 0$ in $L^\infty((0,T);L^{\frac{6}{5}}(\mathbb{R}^2))$ since v^ε is bounded in $L^\infty((0,T);L^6(\mathbb{R}^2))$.

$$\begin{aligned}\|\mathcal{R}^\varepsilon\|_{L^\infty((0,T);L^{\frac{6}{5}}(\mathbb{R}^2)} &= \|(v^\varepsilon\cdot\nabla)v^\varepsilon-\rho_\varepsilon*[(v\cdot\nabla)v]\,\|_{L^\infty((0,T);L^{\frac{6}{5}}(\mathbb{R}^2))}\\ &\le \|(v^\varepsilon\cdot\nabla)(v^\varepsilon-v)\|_{L^\infty((0,T);L^{\frac{6}{5}}(\mathbb{R}^2))}+\|(v^\varepsilon-v)\nabla v\|_{L^\infty((0,T);L^{\frac{6}{5}}(\mathbb{R}^2))}\\ &\quad+\|(v\cdot\nabla)v-\rho_\varepsilon*[(v\cdot\nabla)v]\,\|_{L^\infty((0,T);L^{\frac{6}{5}}(\mathbb{R}^2))}\\ &\le \|v^\varepsilon\|_{L^\infty((0,T);L^6(\mathbb{R}^2))}\|\nabla v^\varepsilon-\nabla v\|_{L^\infty((0,T);L^{\frac{3}{2}}(\mathbb{R}^2))}\\ &\quad+\|v^\varepsilon-v\|_{L^\infty((0,T);L^6(\mathbb{R}^2))}\|\nabla v\|_{L^\infty((0,T);L^{\frac{3}{2}}(\mathbb{R}^2))}\\ &\quad+\|(v\cdot\nabla)v-\rho_\varepsilon*[(v\cdot\nabla)v]\,\|_{L^\infty((0,T);L^{\frac{6}{5}}(\mathbb{R}^2))}\end{aligned}$$

which goes to 0 as $\varepsilon \to 0$, and this concludes the proof. □

The criticality of the exponent $p = 3/2$ in the proof above was required only for the weak continuity of the stress term $v^\varepsilon \mathcal{R}^\varepsilon$ in the energy balance. In fact the weak continuity of all the other terms still work provided $p > 6/5$. It is not known if there exist weak solutions with vorticity in some L^p with $1 < p < 3/2$ which are not conservative. For this reason it is interesting to study if solutions constructed via (ES), (VV), and (VB) are conservative for $p < 3/2$.

4.2.1 Limit of exact smooth solutions

We start by studying (ES); in the next lemma we prove that the velocity field v^ε converges globally in L^2 towards v in the same spirit as it is done in [19]. Since the conservation of the energy is non-trivial only in the case $1 < p < 3/2$, we give the proofs of the following theorems under this assumption.

Lemma 4.2.3. *Let $\omega_0 \in L^p_c(\mathbb{R}^2)$, with $1 < p < 3/2$, which verifies* (4.2.1). *Let $v \in C([0,T]; L^2(\mathbb{R}^2))$ be a solution of* (IE) *constructed as limit of a sequence $\{v^n\}_n$ of exact smooth solutions of* (IE)*, such that $v^n_0 \to v_0$ in L^2. Then, up to a subsequence not relabelled the velocity field v^n satisfies the following convergence*

$$v^n \to v \qquad \text{in } C([0,T]; L^2(\mathbb{R}^2)). \tag{4.2.5}$$

Proof. According to Statement (4) in Theorem 4.1.1, up to a subsequence not relabelled, there exists $\omega \in C([0,T]; L^p(\mathbb{R}^2))$ such that

$$\omega^n \to \omega \quad \text{in } C([0,T]; L^p(\mathbb{R}^2)).$$

Moreover, both v and v^n are in $L^\infty((0,T); L^2(\mathbb{R}^2))$. In order to prove the convergence stated in (4.2.26), we will prove that v^n is a Cauchy sequence in $C([0,T]; L^2(\mathbb{R}^2))$. We divide the proof in several steps.

<u>Step 1</u> *The Serfati identity.*

In this step we derive a formula for the approximate velocity v^n in the same spirit of the Serfati identity derived in [5, 41].

Let $a \in C^\infty_c(\mathbb{R}^2)$ be a smooth function such that $a(x) = 1$ if $|x| < 1$ and $a(x) = 0$ for $|x| > 2$. Differentiating in time the Biot-Savart formula we obtain that for $i = 1, 2$

$$\begin{aligned} \partial_s v^n_i(s,x) &= K_i * (\partial_s \omega^n)(s,x) \\ &= (aK_i) * (\partial_s \omega^n)(s,x) + [(1-a)K_i] * (\partial_s \omega^n)(s,x). \end{aligned} \tag{4.2.6}$$

Now we use the equation (IE) for ω^n obtaining

$$\partial_s \omega^n = -v^n \cdot \nabla \omega^n,$$

and substituting in (4.2.27) we obtain

$$\partial_s v_i^n = (aK_i) * (\partial_s \omega^n) - [(1-a)K_i] * (v^n \cdot \nabla \omega^n). \tag{4.2.7}$$

By integrating by parts we also have that

$$[(1-a)K_i] * (v^n \cdot \nabla \omega^n) = \Big(\nabla \nabla^\perp [(1-a)K_i]\Big) \star (v^n \otimes v^n), \tag{4.2.8}$$

where the notation $\star$ was already introduced in (0.0.3) and (0.0.4). Substituting (4.2.8) in (4.2.6) and integrating in time we have that v^n satisfies the following formula:

$$\begin{aligned} v_i^n(t,x) &= v_i^n(0,x) + (aK_i) * (\omega^n(t,\cdot) - \omega^n(0,\cdot))(x) \\ &\quad - \int_0^t \Big(\nabla \nabla^\perp [(1-a)K_i]\Big) \star (v^n(s,\cdot) \otimes v^n(s,\cdot))(x) \mathrm{d}s. \end{aligned} \tag{4.2.9}$$

<u>Step 2</u> *v^n is a Cauchy sequence in $C([0,T]; L^2(\mathbb{R}^2))$.*

Using formula (4.2.9) we can prove that v^n is a Cauchy sequence. We consider v^n, v^m with $n, m \in \mathbb{N}$. By linearity of the convolution we have that $v^n - v^m$ satisfies the following

$$\begin{aligned} v_i^n(t,x) - v_i^m(t,x) &= \underbrace{v_i^n(0,x) - v_i^m(0,x)}_{(I)} \\ &+ \underbrace{(aK_i) * (\omega^n(t,\cdot) - \omega^m(t,\cdot))(x)}_{(II)} + \underbrace{(aK_i) * (\omega^m(0,\cdot) - \omega^n(0,\cdot))(x)}_{(III)} \\ &- \int_0^t \underbrace{\Big(\nabla \nabla^\perp [(1-a)K_i]\Big) \star (v^n(s,\cdot) \otimes v^n(s,\cdot) - v^m(s,\cdot) \otimes v^m(s,\cdot))(x)}_{(IV)} \mathrm{d}s. \end{aligned} \tag{4.2.10}$$

In order to estimate $\|v^n(t,\cdot) - v^m(t,\cdot)\|_{L^2}$ we estimate separately the L^2 norms of the terms on the right hand side of (4.2.10). We start by estimating (I): given $\eta > 0$, since the initial datum $v^n(0,\cdot)$ converges in L^2 to v_0, we have that there exists N_1 such that

$$\|v^n(0,\cdot) - v^m(0,\cdot)\|_{L^2} < \eta \qquad \text{for any } n, m > N_1. \tag{4.2.11}$$

We deal now with $(II), (III)$: if $\omega_0 \in L^p_c(\mathbb{R}^2)$ with $1 < p < 2$, by Young's convolution inequality we have that

$$\|(aK) * (\omega^n(t,\cdot) - \omega^m(t,\cdot))\|_{L^2} \le \|aK\|_{L^q} \|\omega^n(t,\cdot) - \omega^m(t,\cdot)\|_{L^p},$$

where $1 < q < 2$ is such that $1 + 1/2 = 1/p + 1/q$, while for $p \ge 2$

$$\|(aK) * (\omega^n(t,\cdot) - \omega^m(t,\cdot))\|_{L^2} \le \|aK\|_{L^1} \|\omega^n(t,\cdot) - \omega^m(t,\cdot)\|_{L^2}.$$

Notice that $\|aK\|_{L^q} \le \|K\|_{L^q(B_2)}$ and $K \in L^q_{\text{loc}}(\mathbb{R}^2)$ for any $1 \le q < 2$. Moreover, by the strong convergence of ω^n in $C((0,T); L^p(\mathbb{R}^2))$ and the bound $\{\omega^n\}_n \subset L^\infty((0,T); L^1 \cap L^p(\mathbb{R}^2))$ we conclude that there exists N_2 such that

$$\|(aK) * (\omega^n(t,\cdot) - \omega^m(t,\cdot))\|_{L^2} + \|(aK) * (\omega^n(0,\cdot) - \omega^m(0,\cdot))\|_{L^2} < C\eta, \tag{4.2.12}$$

for any $n, m > N_2$. We deal now with (IV): by Young's convolution inequality we have that

$$\begin{aligned} &\|\nabla\nabla^\perp[(1-a)K] \star (v^n(s,\cdot) \otimes v^n(s,\cdot) - v^m(s,\cdot) \otimes v^m(s,\cdot)\|_{L^2} \\ &\qquad \le \|\nabla\nabla^\perp[(1-a)K]\|_{L^2} \underbrace{\|v^n(s,\cdot) \otimes v^n(s,\cdot) - v^m(s,\cdot) \otimes v^m(s,\cdot)\|_{L^1}}_{(IV*)}. \end{aligned} \tag{4.2.13}$$

We add and subtract $v^n(s,\cdot) \otimes v^m(s,\cdot)$ in $(IV*)$ and by Hölder inequality we have

$$\begin{aligned} &\|v^n(s,\cdot) \otimes v^n(s,\cdot) - v^m(s,\cdot) \otimes v^m(s,\cdot)\|_{L^1} \\ &\qquad \le (\|v^n(t,\cdot)\|_{L^2} + \|v^m(t,\cdot)\|_{L^2}) \|v^n(s,\cdot) - v^m(s,\cdot)\|_{L^2}. \end{aligned}$$

For the first factor in (4.2.13) we have that

$$\nabla\nabla^\perp[(1-a)K_i] = -(\nabla\nabla^\perp a)K_i - \nabla^\perp a \nabla K_i - \nabla a \nabla^\perp K_i + (1-a)\nabla\nabla^\perp K_i,$$

and it is easy to see that each term on the right hand side has uniformly bounded L^2 norm. Then we have that

$$\begin{aligned} &\int_0^t \|\nabla\nabla^\perp[(1-a)K] \star (v^n(s,\cdot) \otimes v^n(s,\cdot) - v^m(s,\cdot) \otimes v^m(s,\cdot)\|_{L^2} \mathrm{d}s \\ &\qquad \le C\|v_0\|_{L^2} \int_0^t \|v^n(s,\cdot) - v^m(s,\cdot)\|_{L^2} \mathrm{d}s. \end{aligned} \tag{4.2.14}$$

Then, putting together (4.2.11),(4.2.12), and (4.2.14) we obtain that for all $n, m > N := \max\{N_1, N_2\}$

$$\|v^n(t,\cdot) - v^m(t,\cdot)\|_{L^2} \le C\left(\eta + \int_0^t \|v^n(s,\cdot) - v^m(s,\cdot)\|_{L^2} \mathrm{d}s\right), \tag{4.2.15}$$

and by Gronwall's lemma

$$\|v^n(t,\cdot) - v^m(t,\cdot)\|_{L^2} \le C(T)\eta. \tag{4.2.16}$$

Taking the supremum in time in (4.2.16) we have the result. □

The conservation of the energy for (ES) is an immediate consequence of the previous lemma.

Theorem 4.2.4. *Let $\omega_0 \in L^p_c(\mathbb{R}^2)$, with $1 < p < 3/2$, which verifies* (4.2.1). *Let $v \in C([0,T]; L^2(\mathbb{R}^2))$ be a solution of* (IE) *constructed as limit of a sequence $\{v^\varepsilon\}_\varepsilon$ of exact smooth solutions of* (IE)*, such that $v_0^\varepsilon \to v_0$ in L^2. Then, v is conservative.*

Proof. Since v^ε are exact smooth solutions of (IE) we have that for every $t \in [0,T]$

$$\|v^\varepsilon(t,\cdot)\|_{L^2} = \|v_0^\varepsilon\|_{L^2},$$

so thanks to Lemma 4.2.3 we have the result. □

4.2.2 The vanishing viscosity limit

We now discuss the conservation of the energy for (VV); we consider the case of the two-dimensional torus, $\mathbb{T}^2$, as in [16].

Definition 4.2.2. *Let $v \in C([0,T];L^2(\mathbb{T}^2))$. We say that v is a physically realizable weak solution of the 2D incompressible Euler equations with intial datum $v_0 \in L^2(\mathbb{T}^2)$ if the following conditions hold:*

(1) v is a weak solution of (IE)*;*

(2) there exists a family of solutions of the incompressible Navier-Stokes equations with viscosity $\nu > 0$, $\{v^\nu\}$, such that

(a) $v^\nu \overset{*}{\rightharpoonup} v$ in $L^\infty((0,T);L^2(\mathbb{T}^2))$, as $\nu \to 0$,

(b) $v^\nu(0,\cdot) = v_0^\nu \to v_0$ in $L^2(\mathbb{T}^2)$, as $\nu \to 0$.

In order to prove the conservation of the kinetic energy, we need to prove that the following convergence lemma.

Lemma 4.2.5. *Let $v \in C([0,T];L^2(\mathbb{T}^2))$ be a physically realizable weak solution of the 2D incompressible Euler equations and let $\{v^\nu\}$ be a family of solutions to the 2D Navier-Stokes equations with viscosity ν as in Definition 4.2.2. Suppose that $v_0 \in L^2(\mathbb{T}^2)$ is such that* $\operatorname{curl} v_0 = \omega_0 \in L^p(\mathbb{T}^2)$ *for some $p > 1$. Then, for all $0 \le t \le T$,*

$$\lim_{\nu \to 0} \|v^\nu(t,\cdot)\|_{L^2}^2 = \|v(t,\cdot)\|_{L^2}^2. \tag{4.2.17}$$

Proof. By Lemma 3.3.2 and the Calderòn-Zygmund theorem it easily follows that $\{v^\nu\}$ lies in a bounded subset of $L^\infty((0,T);W^{1,p}(\mathbb{T}^2))$. In addition, using the Navier-Stokes equation it is easy to show that v^ν is bounded in $\mathrm{Lip}([0,T];H^{-s}(\mathbb{T}^2))$ for some possibly large $s > 0$. Since $W^{1,p}(\mathbb{T}^2)$ is compactly embedded into $L^2(\mathbb{T}^2)$, by Aubin-Lions' lemma we deduce that $\{v^\nu\}$ lies in a compact subset of $C([0,T];L^2(\mathbb{T}^2))$. Since $v^\nu \overset{*}{\rightharpoonup} v$ in $L^\infty((0,T);L^2(\mathbb{T}^2))$ it follows, by uniqueness of the limits, that

$$v^\nu \to v \qquad \text{in } C([0,T];L^2(\mathbb{T}^2)).$$

This is enough to conclude the proof. □

We remark that we assume that $1 < p < 3/2$ otherwise the conclusion follows from Theorem 4.2.2. We now prove one of the main results of [16].

Theorem 4.2.6. *Let v a physically realizable weak solution of the 2D Euler equations with initial datum $v_0 \in L^2(\mathbb{T}^2)$ such that* $\operatorname{curl} v_0 = \omega_0 \in L^p_c(\mathbb{T}^2)$*, for some $1 < p < 3/2$. Then v is conservative.*

Proof. Let v^ν be a family of solutions of the Navier-Stokes equations with viscosity ν as in Definition 4.2.2. Let $\omega^\nu = \operatorname{curl} v^\nu$, the vorticity equation writes

$$\partial_t \omega^\nu + v^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu. \tag{4.2.18}$$

Multiplying the vorticity equation by ω^ν and integrating over $\mathbb{T}^2$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\omega^\nu(t,\cdot)\|_{L^2}^2 = -2\nu \|\nabla \omega^\nu(t,\cdot)\|_{L^2}^2. \tag{4.2.19}$$

By using the Gagliardo-Nieremberg inequality we have that

$$\|\omega^\nu(t,\cdot)\|_{L^2} \leq \|\nabla \omega^\nu(t,\cdot)\|_{L^2}^{1-\frac{p}{2}} \|\omega^\nu(t,\cdot)\|_{L^p}^{\frac{p}{2}}, \tag{4.2.20}$$

and it follows that

$$-2\nu \|\nabla \omega^\nu(t,\cdot)\|_{L^2} \leq -2\nu \|\omega^\nu(t,\cdot)\|_{L^2}^{\frac{4}{2-p}} \|\omega^\nu(t,\cdot)\|_{L^p}^{-\frac{2p}{2-p}}. \tag{4.2.21}$$

We multiply (4.2.18) by $|\omega^\nu|^{p-2}\omega^\nu$ and integrating on $\mathbb{T}^2$ we also get

$$\|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0^\nu\|_{L^p},$$

and substituting in (4.2.21) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\omega^\nu(t,\cdot)\|_{L^2}^2 \leq -2\nu \|\omega^\nu(t,\cdot)\|_{L^2}^{\frac{4}{2-p}} \|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}. \tag{4.2.22}$$

Write $y(t) = \|\omega^\nu(t,\cdot)\|_{L^2}^2$ and $C_0 = |\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}$. Then, integrating in time in (4.2.22), starting from $\delta > 0$, we obtain

$$y(t)^{-\frac{p}{2-p}} - y(\delta)^{-\frac{p}{2-p}} \geq \frac{2\nu p C_0}{2-p}(t-\delta).$$

Taking the limit for $\delta \to 0$ and considering the fact that $\lim_{\delta\to 0} \|\omega^\nu(\delta,\cdot)\|_{L^2} = +\infty$ we find that

$$\|\omega^\nu(t,\cdot)\|_{L^2}^2 \leq \left(\frac{2\nu p C_0 t}{2-p}\right)^{-\frac{2-p}{p}}. \tag{4.2.23}$$

Since solutions of the 2D Navier-Stokes equations satisfy the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \|v^\nu(t,\cdot)\|_{L^2}^2 = -2\nu \|\nabla v^\nu(t,\cdot)\|_{L^2}^2,$$

and rewriting the right hand side in terms of the vorticity we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|v^\nu(t,\cdot)\|_{L^2}^2 = -2\nu \|\omega^\nu(t,\cdot)\|_{L^2}^2. \tag{4.2.24}$$

Hence, integrating in time in (4.2.24) and using (4.2.23) we deduce that

$$\begin{aligned} 0 \geq \|v^\nu(t,\cdot)\|_{L^2}^2 - \|v_0^\nu\|_{L^2}^2 &\geq -2\nu \int_0^t \left(\frac{2\nu p C_0 s}{2-p}\right)^{-\frac{2-p}{p}} \mathrm{d}s \\ &= -2\nu \left(\frac{2\nu p C_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}, \end{aligned}$$

that is

$$0 \geq \|v^\nu(t,\cdot)\|_{L^2}^2 - \|v_0^\nu\|_{L^2}^2 \geq -(2\nu)^{\frac{2(p-1)}{p}} \left(\frac{p\,C_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}. \tag{4.2.25}$$

Now, since $p > 1$ the right hand side of (4.2.25) vanishes as $\nu \to 0$. Therefore,

$$\lim_{\nu\to 0} \|v^\nu(t,\cdot)\|_{L^2}^2 - \|v_0^\nu\|_{L^2}^2 = 0,$$

and by Lemma 4.2.5 we can conclude. □

4.2.3 Limit of the vortex-blob method

In this section we prove the conservation of the kinetic energy for VB-solutions. First of all, we prove that the hypothesis (4.2.1) guarantees that the approximate velocity given by the vortex-blob method is globally square integrable in space.

Lemma 4.2.7. *Let* $\omega_0 \in L^1_c(\mathbb{R}^2)$ *which verifies* (4.2.1). *Then the velocity field* v^ε *given by* (3.4.4) *verifies the following uniform bound*

$$\sup_{t\in[0,T]} \|v^\varepsilon(t,\cdot)\|_{L^2} \leq C,$$

provided that $\delta(\varepsilon) = \varepsilon^\sigma$ *with* $0 < \sigma < 1/7$.

Proof. Multiply the equation (3.4.9) by v^ε; integrating over the whole plane and using the notation $\star$ introduced in (0.0.3), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|v^\varepsilon(s,\cdot)\|_{L^2}^2 &= \int_{\mathbb{R}^2} (K * E_\varepsilon)\cdot v^\varepsilon \mathrm{d}x = -\int_{\mathbb{R}^2} E_\varepsilon (K \star v^\varepsilon)\mathrm{d}x \\ &= \int_{\mathbb{R}^2} F_\varepsilon \cdot \nabla(K\star v^\varepsilon)\mathrm{d}x = -\int_{\mathbb{R}^2} (\nabla K \star F_\varepsilon)\cdot v^\varepsilon \mathrm{d}x \\ &\leq \|\nabla K \star F_\varepsilon(s,\cdot)\|_{L^2}\|v^\varepsilon(s,\cdot)\|_{L^2} \leq \|F_\varepsilon(s,\cdot)\|_{L^2}\|v^\varepsilon(s,\cdot)\|_{L^2},\end{aligned}$$

which means that

$$\frac{\mathrm{d}}{\mathrm{d}s}\|v^\varepsilon(s,\cdot)\|_{L^2} \leq \|F_\varepsilon(s,\cdot)\|_{L^2}.$$

Integrating in time we have that

$$\|v^\varepsilon(t,\cdot)\|_{L^2} \leq \int_0^T \|F_\varepsilon(s,\cdot)\|_{L^2}\mathrm{d}s + \|v^\varepsilon(0,\cdot)\|_{L^2}.$$

Note that $v^\varepsilon(0,\cdot) = K * \omega^\varepsilon(0,\cdot)$ verifies the hypothesis of Proposition 4.2.1 and, since the support of $\omega^\varepsilon(0,\cdot)$ is uniformly bounded in ε, we have that

$$\|v^\varepsilon(0,\cdot)\|_{L^2} \leq C,$$

where the constant C is independent from ε. We omit the details of the proof of the previous inequality since it can be done with the same computations of the bound of the L^2 norm of $\tilde{v}_0^\varepsilon$ in Proposition 3.4.7. This fact together with Lemma 3.4.3 gives the result. □

We can now move to prove the global convergence of v^ε towards v in L^2.

Lemma 4.2.8. *Let $\omega_0 \in L^p_c(\mathbb{R}^2)$, with $1 < p < 3/2$, which verifies (4.2.1). Let v be a VB-solution and $\{v^\varepsilon\}_\varepsilon$ as in Definition 3.4.1. Then, up to a subsequence not relabelled the velocity field v^ε satisfies the following convergence*

$$v^\varepsilon \to v \qquad \text{in } C([0,T];L^2(\mathbb{R}^2)). \tag{4.2.26}$$

Proof. According to Lemma 3.4.4 and Remark 4.1.6, up to a subsequence not relabelled, there exists $\omega \in C([0,T];L^p(\mathbb{R}^2))$ such that

$$\omega^\varepsilon \to \omega \quad \text{in } C([0,T];L^p(\mathbb{R}^2)).$$

Moreover, by Lemma 4.2.7 both v and v^ε are in $L^\infty((0,T);L^2(\mathbb{R}^2))$. In order to prove the convergence stated in (4.2.26), we will prove that v^ε is a Cauchy sequence in $C([0,T];L^2(\mathbb{R}^2))$. Let $\{\varepsilon_n\}_n$ be any infinitesimal sequence. We denote v^n, ω^n the velocity field and the vorticity given by the vortex-blob approximation. We divide the proof in several steps.

<u>Step 1</u> *A Serfati identity for the vortex-blob approximation.*

This step is very similar to Step 1 in Lemma 4.2.3 in which we derive the Serfati formula for the approximate velocity v^n.
Let $a \in C^\infty_c(\mathbb{R}^2)$ be a smooth function such that $a(x) = 1$ if $|x| < 1$ and $a(x) = 0$ for $|x| > 2$. Differentiating in time the Biot-Savart formula we obtain that for $i = 1,2$

$$\begin{aligned}\partial_s v^n_i(s,x) &= K_i * (\partial_s\omega^n)(s,x)\\ &= (aK_i) * (\partial_s\omega^n)(s,x) + [(1-a)K_i] * (\partial_s\omega^n)(s,x).\end{aligned} \tag{4.2.27}$$

Now we use the equation (3.4.6) for ω^n obtaining

$$\partial_s\omega^n = -v^n \cdot \nabla\omega^n + E_n,$$

and substituting in (4.2.27) we obtain

$$\partial_s v^n_i = (aK_i) * (\partial_s\omega^n) - [(1-a)K_i] * (v^n \cdot \nabla\omega^n) + [(1-a)K_i] * E_n. \tag{4.2.28}$$

Since $E_n = \operatorname{div} F_n$ and by the identity

$$v^n \cdot \nabla\omega^n = \operatorname{curl}(v^n \cdot \nabla v^n) = \operatorname{curl}\operatorname{div}(v^n \otimes v^n),$$

we obtain that

$$[(1-a)K_i] * (v^n \cdot \nabla\omega^n) = \Big(\nabla\nabla^\perp[(1-a)K_i]\Big) \star (v^n \otimes v^n), \tag{4.2.29}$$

$$[(1-a)K_i] * E_n = (\nabla[(1-a)K_i]) \star F_n, \tag{4.2.30}$$

where the notation $\star$ was introduced in (0.0.3) and (0.0.4). Substituting the expressions (4.2.29) and (4.2.30) in (4.2.27) and integrating in time we have that v^n satisfies the following formula:

$$\begin{aligned} v_i^n(t,x) = v_i^n(0,x) &+ (aK_i) * (\omega^n(t,\cdot) - \omega^n(0,\cdot))(x) \\ &- \int_0^t \Big(\nabla\nabla^\perp[(1-a)K_i]\Big) \star (v^n(s,\cdot)\otimes v^n(s,\cdot))(x)\mathrm{d}s \\ &+ \int_0^t ((\nabla[(1-a)K_i]) \star F_n(s,\cdot))(x)\mathrm{d}s. \end{aligned} \tag{4.2.31}$$

<u>Step 2</u> *v^n is a Cauchy sequence in $C([0,T];L^2(\mathbb{R}^2))$.*

Using formula (4.2.31) we can prove that v^n is a Cauchy sequence. We consider v^n, v^m with $n, m \in \mathbb{N}$. By linearity of the convolution we have that $v^n - v^m$ satisfies the following

$$\begin{aligned} v_i^n(t,x) - v_i^m(t,x) = &\underbrace{v_i^n(0,x) - v_i^m(0,x)}_{(I)} \\ &+ \underbrace{(aK_i) * (\omega^n(t,\cdot) - \omega^m(t,\cdot))(x)}_{(II)} + \underbrace{(aK_i) * (\omega^m(0,\cdot) - \omega^n(0,\cdot))(x)}_{(III)} \\ &- \int_0^t \underbrace{\Big(\nabla\nabla^\perp[(1-a)K_i]\Big) \star (v^n(s,\cdot)\otimes v^n(s,\cdot) - v^m(s,\cdot)\otimes v^m(s,\cdot))(x)}_{(IV)}\,\mathrm{d}s \\ &+ \int_0^t \underbrace{((\nabla[(1-a)K_i]) \star (F_n(s,\cdot) - F_m(s,\cdot))(x)}_{(V)}\,\mathrm{d}s. \end{aligned} \tag{4.2.32}$$

In order to estimate $\|v^n(t,\cdot) - v^m(t,\cdot)\|_{L^2}$ we estimate separately the L^2 norms of the terms on the right hand side of (4.2.32). The estimates on $(I), (II), (III), (IV)$ can be done as in Lemma 4.2.3: given $\eta > 0$, since the initial datum $v^n(0,\cdot)$ converges in L^2 to v_0, we have that there exists N_1 such that

$$\|v^n(0,\cdot) - v^m(0,\cdot)\|_{L^2} < \eta \qquad \text{for any } n, m > N_1. \tag{4.2.33}$$

We deal now with $(II), (III)$: by Young's convolution inequality, the strong convergence of the approximating vorticity ω^n in $C((0,T);L^p(\mathbb{R}^2))$ and the bound $\{\omega^n\}_n \subset L^\infty((0,T);L^1 \cap L^p(\mathbb{R}^2))$ we conclude that there exists N_2 such that

$$\|(aK) * (\omega^n(t,\cdot) - \omega^m(t,\cdot))\|_{L^2} + \|(aK) * (\omega^n(0,\cdot) - \omega^m(0,\cdot))\|_{L^2} < C\eta, \tag{4.2.34}$$

for any $n, m > N_2$. Again arguing as in the proof of Lemma 4.2.3, we have the following estimate for (IV)

$$\begin{aligned} \int_0^t \|\nabla\nabla^\perp[(1-a)K] \star (v^n(s,\cdot)\otimes v^n(s,\cdot) - v^m(s,\cdot)\otimes v^m(s,\cdot)\|_{L^2}\mathrm{d}s \\ \le C\|v_0\|_{L^2}\int_0^t \|v^n(s,\cdot) - v^m(s,\cdot)\|_{L^2}\mathrm{d}s. \end{aligned} \tag{4.2.35}$$

Finally, we deal with (V): by Young's inequality we have that

$$\|\left(\nabla[(1-a)K]\right)\star(F_n(s,\cdot)-F_m(s,\cdot)\|_{L^2}\leq\|\nabla[(1-a)K]\|_{L^2}\|F_n(s,\cdot)-F_m(s,\cdot)\|_{L^1}.$$

Since ∇K is in $L^2(B_1^c)$, a straightforward computation shows that $\nabla[(1-a)K]$ is bounded in L^2. On the other hand, F_n goes to 0 in $L^\infty((0,T);L^1(\mathbb{R}^2))$ so there exists N_3 such that for all $n,m>N_3$ we have that

$$\|\left(\nabla[(1-a)K]\right)\star(F_n(s,\cdot)-F_m(s,\cdot)\|_{L^2}\leq C\eta. \tag{4.2.36}$$

Then, putting together (4.2.33),(4.2.34),(4.2.35) and (4.2.36) we obtain that for all $n,m>N:=\max\{N_1,N_2,N_3\}$

$$\|v^n(t,\cdot)-v^m(t,\cdot)\|_{L^2}\leq C\left(\eta+\int_0^t\|v^n(s,\cdot)-v^m(s,\cdot)\|_{L^2}\mathrm{d}s\right), \tag{4.2.37}$$

and by Gronwall's lemma

$$\|v^n(t,\cdot)-v^m(t,\cdot)\|_{L^2}\leq C(T)\eta. \tag{4.2.38}$$

Taking the supremum in time in (4.2.38) we have the result. □

We are now in position of proving the conservation of the kinetic energy for weak solutions constructed via the vortex-blob approximation.

Theorem 4.2.9. *Let v be a VB-solution and assume that the initial vorticity $\omega_0\in L^p_c(\mathbb{R}^2)$, with $1<p<3/2$, satisfies (4.2.1). Then v is a conservative weak solution. Moreover, if $6/5\leq p<3/2$ the following local energy balance holds*

$$\partial_t\left(\frac{|v|^2}{2}\right)+\operatorname{div}\left(v\left(\frac{|v|^2}{2}+p\right)\right)=0\qquad in\ \mathcal{D}'((0,T)\times\mathbb{R}^2). \tag{4.2.39}$$

Proof. We divide the proof in two steps.

Step 1 *Local balance of the energy.*

Let v^ε constructed by the vortex-blob method as in the definition of VB-solutions. We have that

$$v^\varepsilon\to v\qquad\text{in }L^\infty((0,T);L^q(\mathbb{R}^2)),\qquad\text{for every }2\leq q\leq p^*. \tag{4.2.40}$$

For $q=2$ the convergence (4.2.40) is a consequence of Lemma 4.2.8, while for $2<q\leq p^*$ it follows from Sobolev inequality and the strong convergence of the vorticity. Indeed, by the Calderòn-Zygmund theorem we have that

$$\sup_{t\in[0,T]}\|v^\varepsilon(t,\cdot)-v(t,\cdot)\|_{L^{p^*}}\leq C\sup_{t\in[0,T]}\|\omega^\varepsilon(t,\cdot)-\omega(t,\cdot)\|_{L^p},$$

and by interpolating the spaces L^2 and L^{p^*} the convergence in (4.2.40) holds.
The pressure p^ε solves the following equation

$$-\Delta p^\varepsilon = \operatorname{div}\operatorname{div}(v^\varepsilon \otimes v^\varepsilon),$$

and by elliptic regularity we have that $p^\varepsilon \in L^\infty((0,T); L^q(\mathbb{R}^2))$, where $1 \leq q \leq p^*/2$, with uniform bounds. Therefore there exists a scalar function $p \in L^\infty((0,T); L^1 \cap L^{\frac{p^*}{2}}(\mathbb{R}^2))$ such that

$$p^\varepsilon \overset{*}{\rightharpoonup} p \qquad \text{in } L^\infty((0,T); L^q(\mathbb{R}^2)), \quad \text{for all } 1 < q \leq \frac{p^*}{2}. \tag{4.2.41}$$

Let $\phi \in C_c^\infty((0,T) \times \mathbb{R}^2)$ be a test function. Multiplying the equation (3.4.9) by $v^\varepsilon \phi$ and integrating in space and time we get

$$\int_0^t \int_{\mathbb{R}^2} \frac{|v^\varepsilon|^2}{2} \partial_s \phi \mathrm{d}x \mathrm{d}s + \int_0^t \int_{\mathbb{R}^2} v^\varepsilon \left(\frac{|v^\varepsilon|^2}{2} + p^\varepsilon \right) \nabla \phi \mathrm{d}x \mathrm{d}s \tag{4.2.42}$$

$$= - \int_0^t \int_{\mathbb{R}^2} (K * E_\varepsilon) v^\varepsilon \phi \mathrm{d}x \mathrm{d}s. \tag{4.2.43}$$

We start by considering the error term in (4.2.43): we have that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^2} (K * E_\varepsilon) \cdot v^\varepsilon \phi \mathrm{d}x \mathrm{d}s = - \int_0^t \int_{\mathbb{R}^2} E_\varepsilon (K \star (v^\varepsilon \phi)) \mathrm{d}x \mathrm{d}s \\
&= - \int_0^t \int_{\mathbb{R}^2} (\operatorname{div} F_\varepsilon)(K \star (v^\varepsilon \phi)) \mathrm{d}x \mathrm{d}s = \int_0^t \int_{\mathbb{R}^2} F_\varepsilon \cdot \nabla (K \star (v^\varepsilon \phi)) \mathrm{d}x \mathrm{d}s.
\end{aligned}$$

Then, by Hölder inequality and Calderòn-Zygmund theorem we have that

$$\begin{aligned}
\left| \int_0^t \int_{\mathbb{R}^2} (K * E_\varepsilon) v^\varepsilon \phi \mathrm{d}x \mathrm{d}s \right| &\leq T \sup_{t \in [0,T]} (\|F_\varepsilon(t,\cdot)\|_{L^2} \|\nabla K \star (v^\varepsilon \phi)(t,\cdot)\|_{L^2}) \\
&\leq CT \sup_{t \in [0,T]} (\|F_\varepsilon(t,\cdot)\|_{L^2} \|v^\varepsilon(t,\cdot)\|_{L^2}) \\
&\leq C \delta^{-\frac{7}{3}} \varepsilon^{\frac{1}{3}},
\end{aligned}$$

which goes to 0 as $\varepsilon \to 0$ choosing δ, h in the construction of the approximation as in Lemma 3.4.3. We consider now (4.2.42). By the convergence in (4.2.40) we have that

$$\int_0^t \int_{\mathbb{R}^2} \frac{|v^\varepsilon|^2}{2} \partial_s \phi \mathrm{d}x \to \int_0^t \int_{\mathbb{R}^2} \frac{|v|^2}{2} \partial_s \phi \mathrm{d}x, \quad \text{as } \varepsilon \to 0.$$

We deal now with the second term in (4.2.42). It is here that the restriction to $p \geq \frac{6}{5}$ comes into play: in this range the Sobolev exponent $p^* \geq 3$. Then, the convergences in (4.2.40) and (4.2.41) imply that

$$\int_0^t \int_{\mathbb{R}^2} p^\varepsilon v^\varepsilon \cdot \nabla \phi \mathrm{d}x \mathrm{d}s \to \int_0^t \int_{\mathbb{R}^2} p v \cdot \nabla \phi \mathrm{d}x \mathrm{d}s, \quad \text{as } \varepsilon \to 0,$$

and

$$\int_0^t \int_{\mathbb{R}^2} v^\varepsilon \frac{|v^\varepsilon|^2}{2} \nabla\phi \mathrm{d}x\mathrm{d}s \to \int_0^t \int_{\mathbb{R}^2} v \frac{|v|^2}{2} \nabla\phi \mathrm{d}x\mathrm{d}s, \qquad \text{as } \varepsilon \to 0,$$

and this concludes the proof of (4.2.39).

Step 2 *Conservation of the kinetic energy.*

We prove now that v is a conservative weak solution for any $p > 1$. Multiplying (3.4.9) by v^ε and integrating in space and time we have that

$$\int_{\mathbb{R}^2} |v^\varepsilon|^2(t,x)\mathrm{d}x = \int_{\mathbb{R}^2} |v^\varepsilon|^2(0,x)\mathrm{d}x - \int_0^t \int_{\mathbb{R}^2} (\nabla K \star F_\varepsilon) \cdot v^\varepsilon \mathrm{d}x. \tag{4.2.44}$$

For the second term on the right hand side, by Lemma 3.4.3 we have that

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^2} (\nabla K \star F_\varepsilon) \cdot v^\varepsilon \mathrm{d}x \right| &\leq \|\nabla K \star F_\varepsilon(s,\cdot)\|_{L^2} \|v^\varepsilon(s,\cdot)\|_{L^2} \\ &\leq \|F_\varepsilon(s,\cdot)\|_{L^2} \|v^\varepsilon(s,\cdot)\|_{L^2} \\ &\leq C\delta^{-\frac{7}{3}} \varepsilon^{\frac{1}{3}}, \end{aligned}$$

which goes to 0 as $\varepsilon \to 0$. Then, by the convergence (4.2.26) and letting $\varepsilon \to 0$ in (4.2.44) we have that

$$\int_{\mathbb{R}^2} |v|^2(t,x)\mathrm{d}x = \int_{\mathbb{R}^2} |v_0|^2(x)\mathrm{d}x,$$

which gives the result. □

Note that the local balance of the energy (4.2.39) actually implies the conservation of the L^2 norm of v for $p \geq 6/5$ by choosing properly the test functions. For example, we can choose the test function to be $\phi_R(x) = \phi(x/R)$; letting $\varepsilon \to 0$ and then $R \to \infty$ we obtain the result. Furthermore, the local balance holds true also for (ES).

Appendix

This Appendix is mostly devoted to recalling some results of harmonic analysis which are frequently used troughout this work.

A.1 Equi-integrability

We recall the definition of equi-integrability for a family of functions in L^1 and three related results:

Definition A.1.1. *(Equi-integrability). A bounded family $\{\varphi_i\}_{i\in I} \subset L^1(\mathbb{R}^d)$ is* equi-integrable *if the following two conditions hold:*

(i) $\forall \varepsilon > 0$ there exists a borel set $A \subset \mathbb{R}^d$ with finite measure such that $\displaystyle\int_{\mathbb{R}^d\setminus A} |\varphi_i| \mathrm{d}x \le \varepsilon$ for every $i \in I$;

(ii) $\forall \varepsilon > 0$ there exists $\delta > 0$ such that, for every borel set $E \subset \mathbb{R}^d$ with $\mathscr{L}^d(E) \le \delta$ there holds $\displaystyle\int_E |\varphi_i| \mathrm{d}x \le \varepsilon$ for every $i \in I$.

Theorem A.1.1. ***(Dunford-Pettis)*** *A family $\{\varphi_i\}_{i\in I} \subset L^1(\mathbb{R}^d)$ is equi-integrable if and only if it is relatively compact with respect the weak topology $\sigma(L^1, L^\infty)$.*

Theorem A.1.2. ***(de la Vallée-Poussin)*** *A family $\{\varphi_i\}_{i\in I} \subset L^1(\mathbb{R}^d)$ verifies (ii) in Definition A.1.1 if and only if there exists a non negative, increasing convex function G such that*

$$\lim_{t\to\infty} \frac{G(t)}{t} = +\infty, \quad \sup_{i\in I} \int_{\mathbb{R}^d} G(|\varphi_i|) \mathrm{d}x < +\infty.$$

We finish this subsection with the following lemma

Lemma A.1.3. *Consider a family $\{\varphi_i\}_{i\in I} \subset L^1(\mathbb{R}^d)$ which is bounded in $L^1(\mathbb{R}^d)$. Then this family is equi-integrable if and only if for every $\varepsilon > 0$, there exist a constant C_ε and a Borel set $A_\varepsilon \subset \mathbb{R}^d$ with finite measure such that for every $i \in I$ we can write*

$$\varphi_i = \varphi_i^1 + \varphi_i^2,$$

such that forall $i \in I$

$$\|\varphi_i^1\|_{L^1(\mathbb{R}^d)} \le \varepsilon, \qquad \operatorname{supp}\, \varphi_i^2 \subseteq A_\varepsilon, \qquad \|\varphi_i^2\|_{L^2(\mathbb{R}^d)} \le C_\varepsilon.$$

A.2 Weak Lebesgue spaces

We recall here the definition of the weak Lebesgue space $M^p(\Omega)$, which are also known in the literature as Lorents spaces, Marcinkievicz spaces, and alternatively denoted by $L^{p,\infty}(\Omega)$ or $L^p_w(\Omega)$.

Definition A.2.1. *Let u be a measurable function on an open set $\Omega \subset \mathbb{R}^d$. For any $1 \leq p < \infty$ we define*

$$|||u|||^p_{M^p(\Omega)} = \sup_{\lambda>0} \Big\{ \lambda^p \mathscr{L}^d \left(\{x \in \Omega : |u(x)| > \lambda\} \right) \Big\}, \tag{A.2.1}$$

and we define the weak Lebesgue space $M^p(\Omega)$ as the set of the functions $u : \Omega \to \mathbb{R}$ with $|||u|||_{M^p(\Omega)} < \infty$. By convention, for $p = \infty$ we set $M^\infty(\Omega) = L^\infty(\Omega)$.

In the above definition we use the notation $|||\cdot|||_{M^p(\Omega)}$ with three vertical bars differently from the usual one for the norms, since $|||\cdot|||_{M^p(\Omega)}$ is not a norm because is not subadditive. Moreover, since for every $\lambda > 0$

$$\lambda^p \mathscr{L}^d \left(\{x \in \Omega : |u(x)| > \lambda\} \right) = \int_{|u|>\lambda} \lambda^p \mathrm{d}x \leq \int_{|u|>\lambda} |u(x)|^p \mathrm{d}x \leq \|u\|^p_{L^p(\Omega)},$$

we have the inclusion $L^p(\Omega) \subset M^p(\Omega)$ and in particular $|||u|||_{M^p(\Omega)} \leq \|u\|_{L^p(\Omega)}$. This inclusion is however strict, for example the function $\dfrac{1}{x}$ defined on $(0, 1)$ is in M^1 but not in L^1. In the following lemma we show that we can interpolate the spaces M^1 and M^p, with $p > 1$, obtaining a bound on the L^1 norm.

Lemma A.2.1. *Let $u : \Omega \to [0, \infty)$ be a non-negative measurable function, where $\Omega \subset \mathbb{R}^d$ has finite measure. Then for every $1 < p < \infty$ we have the interpolation estimate*

$$\|u\|_{L^1(\Omega)} \leq \frac{p}{p-1} |||u|||_{M^1(\Omega)} \left[1 + \log \left(\frac{|||u|||_{M^p(\Omega)}}{|||u|||_{M^1(\Omega)}} \mathscr{L}^d(\Omega)^{1-\frac{1}{p}} \right) \right], \tag{A.2.2}$$

while for $p = \infty$ we have

$$\|u\|_{L^1(\Omega)} \leq |||u|||_{M^1(\Omega)} \left[1 + \log \left(\frac{\|u\|_{L^\infty(\Omega)}}{|||u|||_{M^1(\Omega)}} \mathscr{L}^d(\Omega) \right) \right]. \tag{A.2.3}$$

A.3 Maximal function

We recall here the definition of maximal function and we give some relevant estimates

Definition A.3.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, we define Mf the maximal function of f as*

$$Mf(x) = \sup_{r>0} \frac{1}{\mathscr{L}^d(B_r)} \int_{B_r(x)} |f(y)| \mathrm{d}y \qquad \text{for every } x \in \mathbb{R}^d.$$

Similarly we define the local maximal function as

$$M_\lambda f(x) = \sup_{0<r<\lambda} \frac{1}{\mathscr{L}^d(B_r)} \int_{B_r(x)} |f(y)| \mathrm{d}y.$$

Both Mf and $M_\lambda f$ are finite for a.e. $x \in \mathbb{R}^d$ if $f \in L^1(\mathbb{R}^d)$. The following estimates hold true for Mf and $M_\lambda f$.

Lemma A.3.1. *For every $1 < p \leq \infty$ we have the strong estimate*

$$\|Mf\|_{L^p} \leq C_{d,p}\|f\|_{L^p},$$

while the weak estimate for $p = 1$

$$|||Mf|||_{M^1} \leq C_d\|f\|_{L^1}.$$

Lemma A.3.2. *Let $\lambda > 0$, $p > 1$, and $r > 0$. Then for the local maximal function we have the strong estimate*

$$\int_{B_r} |M_\lambda f(x)|^p \mathrm{d}x \leq C_{d,p} \int_{B_{r+\lambda}} |f(x)|^p \mathrm{d}x,$$

and for all $\alpha > 0$ and $p = 1$ the weak estimate

$$\mathscr{L}^d\left(\{x \in B_r : Mf(x) > \alpha\}\right) \leq \frac{C_d}{\alpha} \int_{B_{r+\lambda}} |f(x)| \mathrm{d}x.$$

Definition A.3.2. *Given a family of smooth functions $\{\rho^\nu\}_\nu \in L^\infty_c(\mathbb{R}^d)$, for every function u in $L^1_{\text{loc}}(\mathbb{R}^d)$ we define the $\{\rho^\nu\}$-maximal function of u as*

$$M_{\{\rho^\nu\}}(u)(x) = \sup_\nu \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^d} \rho^\nu_\varepsilon(x-y)u(y)\mathrm{d}y \right| = \sup_\nu \sup_{\varepsilon>0} \left| \left(\rho^\nu_\varepsilon * u\right)(x) \right| \quad \text{for every } x \in \mathbb{R}^d, \tag{A.3.1}$$

where we use the notation $\rho^\nu_\varepsilon(x) = \dfrac{1}{\varepsilon^d}\rho^\nu\left(\dfrac{x}{\varepsilon}\right)$.

A.4 Singular integral operators

We now present different classes of singular kernels and we describe some properties of the relative singular integral operators.

Definition A.4.1. *(Singular kernel) We say that K is a singular kernel on $\mathbb{R}^d$ if*

(i) $K \in \mathcal{S}'(\mathbb{R}^d)$ *and* $\hat{K} \in L^\infty(\mathbb{R}^d)$*;*

(ii) $K|_{\mathbb{R}^d\setminus\{0\}} \in L^1_{\text{loc}}(\mathbb{R}^d)$ *and there exists a constant $A \geq 0$ such that*

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\mathrm{d}x \leq A \qquad \text{for every } y \in \mathbb{R}^d.$$

Theorem A.4.1. *(Calderòn-Zygmund) Let K be a singular kernel and define*

$$Su = K * u \qquad \text{for every } u \in L^2(\mathbb{R}^d),$$

in the sense of multiplication in the Fourier variable. Then for every $1 < p < \infty$ we have the strong estimate

$$\|Su\|_{L^p(\mathbb{R}^d)} \le C(d,p)\left(A + \|\hat{K}\|_{L^\infty}\right)\|u\|_{L^p(\mathbb{R}^d)}, \qquad \text{for every } u \in L^p \cap L^2(\mathbb{R}^d), \tag{A.4.1}$$

and for $p = 1$ we have the weak estimate

$$|||Su|||_{M^1(\mathbb{R}^d)} \le C(d)\left(A + \|\hat{K}\|_{L^\infty}\right)\|u\|_{L^1(\mathbb{R}^d)}, \qquad \text{for every } u \in L^1 \cap L^2(\mathbb{R}^d). \tag{A.4.2}$$

Corollary A.4.2. *The operator S can be extended to the whole $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$, with values in $L^p(\mathbb{R}^d)$, and the estimate* (A.4.1) *holds for every $u \in L^p(\mathbb{R}^d)$. Moreover, he operator S can be extended to the whole $L^1(\mathbb{R}^d)$ with values in $M^1(\mathbb{R}^d)$, and the estimate* (A.4.2) *holds for every $u \in L^1(\mathbb{R}^d)$.*

The following characterization of singular kernels holds true.

Proposition A.4.3. *Consider a function $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ satisfying the following conditions:*

(i) there exists a constant $A \ge 0$ such that

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\mathrm{d}x \le A \qquad \text{for every } y \in \mathbb{R}^d;$$

(ii) there exists a constant A_0 such that

$$\int_{|x|\le R} |x||K(x)|\mathrm{d}x < A_0 R \qquad \text{for every } R > 0;$$

(iii) there exists a constant A_2 such that

$$\left|\int_{R_1<|x|<R_2} K(x)\mathrm{d}x\right| < A_2 \qquad \text{for every } 0 < R_1 < R_2 < \infty.$$

Then K can be extended to a tempered distribution on $\mathbb{R}^d$ which is a singular integral, unique up to a constant times a Dirac masses at the origin. Conversely, every singular kernel on $\mathbb{R}^d \setminus \{0\}$ has a restriction which satisfies (i),(ii),(iii).

Definition A.4.2. *We say that K is a singular integral of fundamental type if the following properties hold:*

(i) $K|_{\mathbb{R}^d\setminus\{0\}} \in C^1(\mathbb{R}^d \setminus \{0\})$;

(ii) there exists a constant C_0 such that

$$|K(x)| \le \frac{C_0}{|x|^d} \qquad \text{for every } x \ne 0;$$

(iii) there exists a constant C_1 such that

$$|\nabla K(x)| \leq \frac{C_0}{|x|^{d+1}} \qquad \text{for every } x \neq 0;$$

(iv) there exists a constant A_2 such that

$$\left| \int_{R_1<|x|<R_2} K(x)\mathrm{d}x \right| < A_2 \qquad \text{for every } 0 < R_1 < R_2 < \infty.$$

Note that the conditions in the previous definition imply the ones of Proposition A.4.3. We finish this section giving a useful cancellation property for operators which are composition of maximal functions and singular integral operator; the result is contained in the following theorem.

Theorem A.4.4. *Let K a singular kernel of fundamental type as in Definition A.4.2 and set $Su = K * u$ for every $u \in L^2(\mathbb{R}^d)$. Let $\{\rho^\nu\}_\nu \in L^\infty_c(\mathbb{R}^d)$ be a family of kernels such that*

$$\operatorname{supp}\ \rho^\nu \subset B_1 \qquad \text{and} \qquad \|\rho^\nu\|_{L^1(\mathbb{R}^d)} \leq Q_1 \qquad \text{for every } \nu.$$

Assume that for every $\varepsilon > 0$ and every ν there holds that $\big(\varepsilon^d K(\varepsilon\cdot)\big) * \rho^\nu \in C^1_b(\mathbb{R}^d)$ with the norm estimate

$$\| \Big(\varepsilon^d K(\varepsilon\cdot)\Big) * \rho^\nu \|_{C(\mathbb{R}^d)} \leq Q_2 \qquad \text{for every } \varepsilon > 0 \text{ and every } \nu.$$

Then the following estimates hold

(i) there exists a constant $C(d)$ such that for every $u \in L^1 \cap L^2(\mathbb{R}^d)$

$$|||M_{\rho^\nu}(Su)|||_{M^1(\mathbb{R}^d)} \leq C(d) \Big(Q_1 + Q_2 \Big(C_0 + C_1 + \|\hat{K}\|_{L^\infty}\Big)\Big) \|u\|_{L^1(\mathbb{R}^d)};$$

(ii) if in addition $\{\rho^\nu\}_\nu \in C^\infty_c(\mathbb{R}^d)$, the previous estimate holds true for all finite measures u on $\mathbb{R}^d$, with the same constant $C(d)$ and with Su defined as a distribution;

(iii) If $Q_3 = \sup_\nu \|\rho^\nu\|_{L^\infty(\mathbb{R}^d)}$ is finite, then there exists a constant $C(d)$ such that

$$\|M_{\rho^\nu}(Su)\|_{L^2(\mathbb{R}^d)} \leq C(d) Q_3 \|\hat{K}\|_{L^\infty} \|u\|_{L^2(\mathbb{R}^d)} \qquad \text{for every } u \in L^2(\mathbb{R}^d).$$

A.5 Estimates on the different quotients

We give here some estimates on different quotients which are used in Section 1.3.

Lemma A.5.1. *Let $f \in BV(\mathbb{R}^d)$ then there exists a negligible set $\mathcal{N} \subset \mathbb{R}^d$ such that*

$$|f(x) - f(y)| \leq C(d)|x-y| \left(MDf(x) + MDf(y)\right), \tag{A.5.1}$$

for every $x, y \in \mathbb{R}^d \setminus \mathcal{N}$, where Du is the distributional derivative of u, which is a measure.

The following local variant of the previous lemma holds true

Lemma A.5.2. *Let* $f \in BV_{\mathrm{loc}}(\mathbb{R}^d)$ *then there exists a negligible set* $N \subset \mathbb{R}^d$ *such that*

$$|f(x) - f(y)| \leq C(d)|x - y| \left(M_\lambda Df(x) + M_\lambda Df(y)\right), \tag{A.5.2}$$

whever $x, y \in \mathbb{R}^d \setminus N$ *and* $|x - y| < \lambda$.

Lemma A.5.3. *Let* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ *which satisfies the regularity assumption (R2c). Then there exists a non-negative function* $U \in M^1(\mathbb{R}^d)$ *and a* $\mathscr{L}^d$*-negiglible set* $\mathcal{N} \subset \mathbb{R}^d$ *such that*

$$|f(x) - f(y)| \leq |x - y| \left(U(x) + U(y)\right) \qquad \textit{for every } x, y \in \mathbb{R}^d \setminus \mathcal{N}.$$

Moreover, we can take U *explicitly as*

$$U = \sum_{j=1}^{d} \sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j}, \xi \in \mathbb{S}^{d-1}\}} \left(S_{jk} g_{jk}\right),$$

where the kernels $\Upsilon^{\xi,j} \in C_c^\infty(\mathbb{R}^d)$ *are explicitly defined for* $\xi \in \mathbb{S}^{d-1}$ *and* $j = 1, ..., d$ *by*

$$\Upsilon^{\xi,j}(w) = h\left(\frac{\xi}{2} - w\right) w_j,$$

and the kernel $h \in C_c^\infty(\mathbb{R}^d)$ *is chosen such that*

$$\int_{\mathbb{R}^d} h(y) \mathrm{d}y = 1, \qquad \mathrm{supp}\ \ h \subset B_{1/2}.$$

BIBLIOGRAPHY

[1] M. Ainzenmann: *On vector fields as generators of flows: a counterexample to Nelson's conjecture.* Ann. Math., **107**, (1978), 287-296.

[2] G. Alberti, S. Bianchini, G. Crippa: *A uniqueness result for the continuity equation in two dimensions.* J. Eur. Math. Soc. (JEMS), **16**, (2014), 201-234.

[3] G. Alberti, S. Bianchini, G. Crippa: *Structure of level sets and Sard-type properties of Lipschitz maps.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 12(**4**), (2013), 863-902

[4] G. Alberti, G. Crippa, A. L. Mazzucato: *Exponential self-similar mixing by incompressible flows.* J. Amer. Math. Soc., **32**, (2019), no. 2

[5] D. M. Ambrose, J. P. Kelliher, M. C. Lopes Filho, H. J. Nussenzveig Lopes: *Serfati solutions to the 2D Euler equations on exterior domain.* J. Differential Equations, **259**, (2015), 4509-4560.

[6] L. Ambrosio: *Transport equation and Cauchy problem for BV vector fields.* Inventiones Mathematicae, **158**, (2004), 227-260.

[7] S. Attanasio, F. Flandoli: *Zero noise solutions of linear transport equations without uniqueness: an example.* C. R. Math. Acad. Sci. Paris, **347**, (2009), no. 13-14, 753-756.

[8] C. Bardos, E. S. Titi, E. Wiedemann: *The vanishing viscosity as a selection principle for the Euler equations: the case of 3D shear flow.* C. R. Math. Acad. Sci. Paris, **350**, (2012), no. 15-16, 757-760.

[9] S. Bianchini, P. Bonicatto: *A uniqueness result for the decomposition of vector fields in* $\mathbb{R}^d$. `http://cvgmt.sns.it/paper/3619/`

[10] J. T. Beale: *The approximation of weak solutions to the 2-D Euler equations by vortex elements.* Multidymensional Hyperbolic Problems and Computations, Edited by J. Glimm and A. Majda, IMA Vol. Math. Appl. **29**, (1991), 23-37.

[11] A. Bohun, F. Bouchut, G. Crippa: *Lagrangian flows for vector fields with anisotropic regularity.* Ann. Inst. H. Poincaré Anal. Non Linéaire, **33**, 6 (2016), 1409-1429.

[12] A. Bohun, F. Bouchut, G. Crippa: *Lagrangian solutions to the 2D Euler system with L^1 vorticity and infinite energy.* Nonlinear Analysis: Theory, Methods & Applications, **132**, (2016), 160-172.

[13] F. Bouchut, G. Crippa: *Lagrangian flows for vector fileds with gradient given by a singular integral.* J. Hyperbolic Diff. Equ., **10**, (2013), 235-282.

[14] A. Bressan: *An ill posed Cauchy problem for a hyperbolic system in two space dimensions.* Rend. Sem. Mat. Univ. Padova, **110**, (2003), 103-117.

[15] T. Buckmaster, V. Vicol: *Nonuniqueness of weak solutions to the Navier-Stokes equation.* Ann. Math., **189**, (2019), 101-144.

[16] A. Cheskidov, M. C. Lopes Filho, H. J. Nussenzveig Lopes, R. Shvydkoy: *Energy conservation in Two-dimensional incompressible ideal fluids.* Comm. Math. Phys., **348**, (2016), 129-143.

[17] G. Ciampa, G. Crippa, S. Spirito: *On smooth approximations of rough vector fields and the selection of flows.* `https://arxiv.org/abs/1902.00710`

[18] G. Ciampa, G. Crippa, S. Spirito: *Smooth approximation is not a selection principle for the transport equation with rough vector field.* `https://arxiv.org/abs/1902.08084`

[19] G. Ciampa, G. Crippa, S. Spirito: *Weak solutions obtained by the vortex method for the 2D Euler equations are Lagrangian and conserve the energy.* `https://arxiv.org/abs/1905.09720`

[20] F. Colombini, T. Luo, J. Rauch: *Uniqueness and nonuniqueness for nonsmooth divergence-free transport.* Séminaire: Équations aux Dérivées Partielles, Exp. No. XXII, École Polytech., Palaiseau, 2003.

[21] G. Crippa, C. De Lellis: *Estimates and regularity results for the DiPerna-Lions flow.* J. Reine Angew. Math., **616**, (2008), 15-46.

[22] G. Crippa, N. Gusev, S. Spirito, E. Wiedemann: *Non-uniqueness and prescribed energy for the continuity equation.* Comm. Math. Sci., **13** (2015), no. 7, 1937-1947.

[23] G. Crippa, C. Nobili, C. Seis, S. Spirito: *Eulerian and Lagrangian solutions to the continuity and Euler equations with L^1 vorticity.* SIAM J. Math. Anal. **49** (2017), no. 5, 3973-3998.

[24] G. Crippa, S. Spirito: *Lagrangian solutions of the 2D Euler equations.* Submitted paper.

[25] G. Crippa, S. Spirito: *Renormalized solutions of the 2D Euler equations.* Comm. Math. Phys., **339**, (2015), 191-198.

[26] C. De Lellis, L. Székelyhidi: *On Admissibility Criteria for Weak Solutions of the Euler Equations*. Arch. Ration. Mech. Anal., **195** (2010), 225-260.

[27] J.-M. Delort: *Existence de nappes de tourbillon en dimension deux.* J. Amer. Math. Soc., **4**, (1991), 553-586.

[28] N. Depauw: *Non unicité des solutions bornés pour un champ de vecteurs BV en dehors d'un hyperplan.* C.R. Math. Sci. Acad. Paris, **337**, (2003), 249-252.

[29] R. J. DiPerna, P.-L. Lions: *Ordinary differential equations, transport theory and Sobolev spaces.* Inventendiones Mathematicae, **98**, (1989), 511-547.

[30] R. J. DiPerna, A. J. Majda: *Concentrations in regularizations for 2-D incompressible flow.* Comm. Pure Appl. Math., **40**, (1987), 301-345.

[31] L. Lichtenstein: *Über einige Existenzproblem der Hydrodynamik homogener, unzusammendrückbarer, reibungsloser Flüssigkeiten und die Helmholtzschen Wirbelsätze.*, M. Z., **23**, (1925), 89-154.

[32] J.-G. Liu, Z. Xin: *Convergence of vortex methods for weak solutions to the 2-D Euler equations with vortex sheets data.* Comm. Pure Appl. Math., **48**, (1995), 611-628.

[33] M. C. Lopes Filho, A. L. Mazzucato and H. J. Nussenzveig Lopes: *Weak solutions, renormalized solutions and enstrophy defects in 2D turbulence.* Arch. Ration. Mech. Anal., **179**, (2006), no. 3, 353-387.

[34] A. J. Majda: *Remarks on Weak Solutions for Vortex Sheets with a Distinguished Sign.* Indiana Univ. Math. J., **42**, (1993), 921-939.

[35] A. J. Majda, A. L. Bertozzi: *Vorticity and incompressible flow*, vol. 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[36] J.-G. Liu, Z. Xin: *Convergence of vortex methods for weak solutions to the 2-D Euler equations with vortex sheets data.* Comm. Pure Appl. Math., **48**, (1995), 611-628.

[37] S. Modena, G. Sattig: *Convex integration solutions to the transport equation with full dimensional concentration.* https://arxiv.org/abs/1902.08521

[38] S. Modena, L. Székelyhidi Jr: *Non-uniqueness for the transport equation with Sobolev vector fields.* Accepted paper on Annals of PDE (2018).

[39] S. Modena, L. Székelyhidi Jr: *Non-renormalized solutions to the continuity equation.* https://arxiv.org/abs/1806.09145

[40] Q. H. Nguyen: *Quantitative estimates for regular Lagrangian flows with BV vector fields.* http://cvgmt.sns.it/media/doc/paper/3848

[41] P. Serfati: *Solutions C^∞ en temps, n-log Lipschitz bornées en espace et équation d'Euler.* C. R. Acad. Sci. Paris Sér. I Math., **320**, (1995), 555-558.

[42] E. M. Stein: *Singular integrals and differentiability properties of functions.* Princeton University Press, 1970.

[43] I. Vecchi, S. J. Wu: *On L^1-vorticity for 2-D incompressible flow.* Manuscripta Math. 78, **4**, (1993), 403-412.

[44] W. Wolibner: *Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogene, incompressible, pendant un temps infiniment long*, Math. Z., **37**, (1933), 698-726.

[45] V. I. Yudovič: *Non-stationary flows of an ideal incompressible fluid.* Ž. Vyčisl. Mat. i Mat. Fiz. 3, (1963), 1032-1066.