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Fundamental solutions and smoothness in Schrödinger problems with applications to quantum fluids

PHD CANDIDATE
Michele D'Amico

ADVISORS
Pierangelo Marcati
Gran Sasso Science Institute
and
Department of Information Engineering,
Computer Science and Mathematics,
University of L'Aquila

Paolo Antonelli
Gran Sasso Science Institute

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Thesis Referees

Prof. Rémi Carles (CNRS and Université Montpellier IMAG)
Prof. Vladimir Gueorguiev (University of Pisa)
Prof. Alessandro Michelangeli (SISSA, Trieste)

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Contents

Introduction

The nonlinear Schrödinger (NLS) equation

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + V(t,x)\psi + f'(|\psi|^2)\psi\,, \tag{1}$$

is ubiquitous in physics, as it arises as an effective description of many importart physical models, like Bose-Einstein condensation [55, 60, 105], superfluidity [32, 33], nonlinear optics [78], wave propagation in nonlinear media [130], deep water waves [129], plasma physics [128]. Moreover, it is also relevant to consider the hydrodynamical formulation associated to wave function dynamics ([86]). It is often computationally less expensive and nevertheless it provides the observable quantities measured in experiments. For the linear Schrödinger equation describing the dynamics of a quantum particle, the hydrodynamical formulation was pointed out since the early days of quantum mechanics by Madelung [89], who attempted to give a description to the quantum system in terms of the dynamics for the probability density and the phase of the wave function. This formulation was later resumed by Landau ([76, 85]) to describe nonlinear phenomena in superfluidity. A strict relation between the Landau two-fluid model and nonlinear Schrödinger equations (Gross-Pitaevskii equations [61, 104]) was already noticed in the steady case by Ginzburg and Pitaevskii in [55].

This analogy was further exploited in many other physical contexts, see for instance superconductivity [33], semiconductor devices [48], dense astrophysical plasmas, laser plasmas [63, 64, 111, 112] and Bose-Einstein condensation at finite temperatures [58, 59, 105].

In his theory Landau distinguished between the superfluid, inviscid, flow and the normal, viscous, flow and used the hydrodynamic formulation to describe the interactions between the two fluids. At finite temperatures, close to the transition temperature, the interactions between the two fluids, mutually exchanging mass and momentum, are non-negligible and they produce some dissipative effects ([76]). Unfortunately Landau's two-fluid model description of superfluidity almost completely lacks of a rigorous mathematical treatment. Amongst the few analytical results we cite [5], where a simplified model was proposed; there the authors show the existence of global in time finite energy weak solutions for a toy model. The strategy, following also the theory developed in [6, 8], consists in studying the underlying wave function dynamics for the superfluid part and in constructing a sequence of approximate solutions. However the result is partial, since a finer analysis of the main linear propagator for the superfluid part is missing and the results available in the literature ([2, 17, 39]) were not suitable to develop a complete theory even for the toy model. This is indeed the main motivation for the study developed in the first part of this thesis, which concerns the construction of the fundamental solution for a linear Schrödinger equation with a time dependent potential and the study of its dispersive properties.

More precisely, in the model introduced in [5], the normal fluid is not influenced by the dynamics of the superfluid, which on the other hand interacts with the former through a collision term. The normal fluid, being a classical one, evolves according to the compressible Navier-Stokes equation and, once solved by using the results available in literature ([28, 31, 88]), can be used as a given term in the equations for the superfluid. The idea now is to study the latter one by means of the associated wave function dynamics. However, for this model, the wave function satisfies a nonlinear Schrödinger equation like (1) with a potential V such that $V = V_p + V_\infty$, where $V_p \in L^2_t W^{1,6}_x$, $V_\infty \in C^\infty(\mathbb{R}^n)$ for $a.e.$ $t \in \mathbb{R}_+$ and for any $|\alpha| \geq 1$, $\partial^\alpha_x V_\infty \in L^2_t L^\infty_x$. Chapter 1 is devoted to study the fundamental solution for the free Schrödinger equation with a scalar potential enjoying the same properties as V_∞. This will be a first step towards establishing a more satisfactory existence theory for the model introduced in [5]. More details will be given in Appendix B.
When dealing with such potentials, which cannot be considered as Kato perturbations of the free Laplacian, a suitable strategy, at a formal level, consists in using the classical methods of Feynman path integrals. In the literature there are different approaches for a rigorous mathematical treatment of Feynman integrals, for instance in [2] it is proposed an approach based on an infinite dimensional generalization of Fresnel integral transformations. Our approach instead follows directly from the methods developed by Fujiwara ([36]) and his school by using the so called time slicing approximation technique. The strategy consists in constructing the propagator by means of an oscillatory integral operator, where the phase is given by the classical action, computed along classical trajectories. In particular let $\Delta: \ s = t_0 < t_1 < \cdots < t_L = t$ be an arbitrary subdivision of the time interval $[s, t]$ and $x_j \in \mathbb{R}^n$, $j = 0, 1, \ldots, L-1$. Denote by $\gamma_\Delta = \gamma_\Delta(x_0, x_1, \ldots, x_L, x)$ the piecewise classical path joining (t_j, x_j), then

$$C(\Delta)\varphi = \prod_{j=1}^{L} \frac{1}{(2\pi i(t_j - t_{j-1}))^{\frac{n}{2}}} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} e^{iS(\gamma_\Delta)} \varphi(x_0) dx_0 dx_1 \cdots dx_{L-1},$$

yields a suitable approximation for the fundamental solution. Fujiwara in [36] proved that $C(\Delta)$ converges to the propagator in the topology of the operator norm on $L^2(\mathbb{R}^n)$, in the case when $V \in L^\infty_t L^\infty_{loc,x}$ and $\partial^\alpha_x V \in L^\infty_t L^\infty_x$, for $|\alpha| \geq 2$. Motivated by the toy model for superfluidity studied in [5], we study in Chapter 1 the construction of the fundamental solution for a Schrödinger equation when the potential satisfies $V \in L^2_t L^\infty_{loc,x}$ and $\partial^\alpha_x V \in L^2_t L^\infty_x$. We first analyze the classical orbits, which in our case turn out to be non-Lipschitz in time because of our assumptions on the potential. Therefore we are forced to study the problem in its integral formulation. As a consequence of this, we obtain orbits whose regularity is only $\mathscr{C}^{1/2}$. However, this integral formulation is fully compatible with the Fujiwara construction when differentiating with respect to initial data. Despite the Lagrangian is only square integrable in time, we can still define the action, but differently from Fujiwara we cannot use the Rademacher's theorem and infer the almost everywhere differentiability of the action in the product space $\mathbb{R}_t \times \mathbb{R}_s \times \mathbb{R}^d_x \times \mathbb{R}^d_y$. The lack of almost everywhere differentiability introduces a major difficulty, because the parametrix, constructed as an oscillatory integral operator whose phase is the action, therefore cannot be differentiated almost everywhere. To overcome this problem we approximate the action by using orbits generated by a smoothed potential which allows to pass to the limit both the action and the paramtetrix. The construction of the fundamental solution then does not present major difficulties. Furthermore this construction yields also a dispersive estimate which holds for small times. By exploiting the result by Keel-Tao in [73] (see also [15, 16]) it is then possible to infer the whole set of Strichartz estimates. We want to stress here that in Chapter 1, we will strictly follow Fujiwara's notation, in order to allow the readers, already familiar with the paper [36], to fully understand the main differences between our framework and Fujiwara's one.

The next natural question which arises is the local smoothing associated to the propagator just constructed. This issue is discussed in Chapter 2. The local smoothing for dispersive equations was first observed by Kato in [71] for the KdV equation. Then it was further developed by many authors, see for example [22], [121], [113]. In particular in [126] Yajima proved the local smoothing property for (a generalization of) the class of propagators constructed by Fujiwara in [36]. By following Yajima's techniques in [126], we focus the attention on a particular choice of the potential, which is representative in the class of potentials we studied in advance. In particular we choose $V(t,x) = a(t)b(x)$, with $a \in L^2_t$ and b quadratic.

In the last Chapter we study the Cauchy problem for a nonlinear Maxwell-Schrödinger system. The motivation for investigating the Maxwell-Schrödinger with the presence of a nonlinear potential can again be found in a class of quantum hydrodynamical systems with a nontrivial pressure tensor. More precisely, in some physical contexts (for example dense astrophysical plasmas, like in white dwarfs) electromagnetic fields play a relevant role, so we need to consider them in our hydrodynamical system. In particular it can be seen that, under suitable physical conditions, the pressure term appearing in the quantum hydrodynamical systems can be approximated by a power law (see Section 3.5); so the introduction of the power-type nonlinearity in the Maxwell-Schrödinger system is fundamental to recover the nontrivial pressure term in the QMHD. This leads to the analysis of a class of quantum magnetohydrodynamic (QMHD) equations, whose wave function dynamics analogue is given by a Maxwell-Schrödinger system with a power-type nonlinearity.
We first prove a local strong well-posedness result in $H^2 \times H^{\frac{3}{2}}$. Our strategy relies

on the construction of the evolution operator associated to the magnetic Laplacian, by using Kato theory; we then perform a fixed point iteration scheme, by means of Duhamel's formula. Unfortunately, due to the presence of the power-type nonlinearity and the lack of intrinsic Strichartz type estimates for the magnetic Laplacian, the local theory can not be extended globally in time, as on the contrary occurs without the nonlinear power-type potential [97, 98]. To overcome these difficulties, we approximate our system by using Yosida type regularization and we obtain the global well-posedness for the approximating system in $H^2 \times H^{\frac{3}{2}}$. Then, by passing to the limit, we obtain a global in time finite energy weak solution. As a byproduct of our existence result, we can then show the existence of weak solutions in the energy space to the QMHD system.

For the convenience of the reader we include two appendices. In Appendix A we collect some results on oscillatory integral operators; in Appendix B we summarize the key points about the connection between nonlinear Schrödinger equations and quantum hydrodynamics systems, with an application to the study of a two-fluid model.

Chapter 1

A construction of the fundamental solution for a Schrödinger equation with time dependent potential

1.1 Feynman Path Integrals

In this section, we recall some ideas related to the Feyman path integrals, contained in [2, 91]. We refer to these books for more details.

In 1948, Feynman introduced the notion of path integrals in [34], where he proposed a new approach to non-relativistic quantum mechanics. He developed an earlier idea suggested by Dirac in [25], regarding the analogue of the concepts of classical Lagrangian and action in quantum mechanics. It is well known that in classical mechanics the action functional, defined as the integral with respect to time of the Lagrangian, plays a crucial role in determining the dynamics of a system. Indeed the classical trajectory of a particle can be obtained as the stationary point of the action functional.
If we consider a particle, with mass m, position x and velocity $\dot{x}$, subject to the action of a potential V, the classical Lagrangian is given by

$$L(x,\dot{x}) := \frac{m}{2}\dot{x}^2 - V(x).$$

Feynman's formulation of quantum theory generalizes the action principle in the following sense: to a quantum system can not be assigned a unique (classical) trajectory; one has to take into account every possible path from one quantum state to another and then to sum over all the possibilities, a "sum over all possible histories of the system". We stress here that Feynman's description reintroduces to quantum mechanics the notion of trajectory, which was meaningless in the traditional formulations of quantum theory, according to Heisenberg's uncertainty principle. These ideas are summarized in the first postulate in [34]:

> *"If an ideal measurement is performed to determine whether a particle has a path lying in a region of space-time, then the probability that the result will be affirmative is the absolute square of a sum of complex contributions, one from each path in the region*".

So, roughly speaking, in order to know the probability amplitude for a quantum particle to reach the position x_f at time t_f, starting from the position x_i at time t_i, one has to consider all the possible paths connecting the points x_i and x_f and integrate over all of them. The question, now, is how to weight each individual path. The answer is contained in the second postulate in [34]:

> *"The paths contribute equally in magnitude, but the phase of their contribution is the classical action (in units of ħ); i.e., the time integral of the Lagrangian taken along the path*".

This means that, if $\gamma(t)$ is a given path, it carries a contibution which is proportional to $e^{i\hbar^{-1}S[\gamma(t)]}$, where $S[\gamma(t)]$ denotes the action along the pathh γ, that is

$$S[\gamma] = \int_{t_i}^{t_f} L(\gamma(t), \dot{\gamma}(t))dt\,, \tag{1.1}$$

with L being the classical Lagrangian along the path $\gamma(t)$.
So, in order to determine the probability that a particle occupies a certain location in space-time, it is necessary to sum all the contributions. In this way path integrals come out and they can be expressed symbolically as

$$\int e^{i\hbar^{-1}S[\gamma]}\mathscr{D}[\gamma]\,, \tag{1.2}$$

where the integration is over the space of paths γ which satisfy the boundary conditions $\gamma(t_i) = x_i$ and $\gamma(t_f) = x_f$. Here $\mathscr{D}[\gamma]$ stands for a Lebesgue-type measure on the space of paths. With this choice, it is clear that the classical path is singled out, when $\hbar \to 0$; indeed when $\hbar$ is regarded as a small parameter, the phase factor $e^{\frac{i}{\hbar}S[\gamma]}$ becomes a rapidily oscillating function. It follows from the stationary phase method, that the main contribution in (1.2) is given by the paths that make the action stationary, that is the classical trajectories.

A physical intuition, behind the Feynman path integrals, can be found in the famous two-slit experiment. Let us consider a source of electrons, placed in a plane A. The electrons, starting from A, pass throuh a screen B, which has two slits and they are detected in a plane C. By moving the detector in C it is possible to measure the intensity of the electrons current in different places in C. According to classical physics, the flux of the particles should be localized in two different places in C, depending on which hole in B they passed through. Actually, by reproducing the experiment, an interference pattern appears on the screen C (see Figure 1.1).

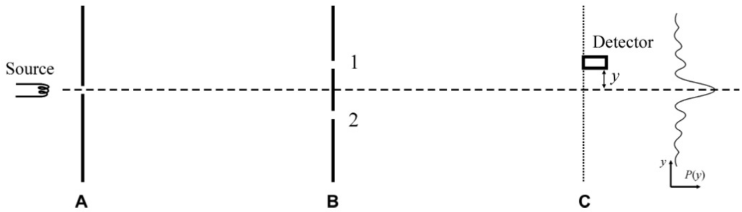

Figure 1.1: Two-slit experiment

This suggests that, if we are interested in the probability of finding an electron in a specific location in C, we have to sum the probability amplitudes of the two possible paths of the particle, through the first or the second slit. One can think to generalize this result by inserting more holes or multiple screens between A and the detector in C. As a result, there are more paths to be taken into account to get the final amplitude of the electrons propagating through the screens. These considerations lead to the concept of the sum over all possible paths and so to *path integrals*, which can be regarded as an infinite-slit experiment.

In order to have a heuristic idea of Feynman path integrals, let us consider the case of a non-relativistic particle with mass m, which moves in $\mathbb{R}^n$. The particle is subject to an external potential $V(x)$, under the assumptions to be a bounded continuous real valued function of $\mathbb{R}^n$ (see [2]). The classical Lagrangian associated to the system is

$$L(x,\dot{x}) = \frac{m}{2}\dot{x}^2 - V(x).$$

As already recalled, the principle of least action tells us that the trajectory followed by the particle to reach x_f at time t_f, starting from x_i at time t_i, is obtained as a stationary point of (1.1) over the space of paths γ that leave the extremal points x_i and x_f fixed. In the quantum mechanical picture, the state of a particle at time t is described by a unitary vector $\psi(x,t)$, called wave function, which solves the Schrödinger equation

$$i\hbar\partial_t\psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi(t,x) + V(t,x)\psi(t,x). \tag{1.3}$$

It is well known that, under suitable assumptions on the potential and on the domain of the operator , H is self-adjoint. In the three dimensional case, for instance, if $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, then $-\Delta + V(x)$ is self-adjoint on $D(-\Delta)$ (see [106]). So, once the inital datum $\psi(x,0) = \varphi(x)$ is specified, the solution of (1.3) is given by

$$\psi(t,x) = e^{-\frac{i}{\hbar}tH}\varphi(x). \tag{1.4}$$

By means of the Lie-Trotter-Kato formula, (1.4) becomes

$$\psi(t,x) = \lim_{k\to\infty}\left(e^{-\frac{i}{\hbar}\frac{t}{k}V}e^{-\frac{i}{\hbar}\frac{t}{k}H_0}\right)^k, \tag{1.5}$$

where $H_0 = -\frac{\hbar^2}{2m}\Delta$. By using the Green function of the quantum free evolution operator, we have

$$e^{-\frac{i}{\hbar}tH_0}\varphi(x) = \left(2\pi i\frac{\hbar}{m}t\right)^{-\frac{n}{2}} \int e^{i\frac{m(x-y)^2}{2\hbar t}}\varphi(y)dy\,.$$

Hence

$$\psi(t,x)=\lim_{k\to\infty}\left(2\pi i\frac{\hbar}{m}\frac{t}{k}\right)^{-k\frac{n}{2}}\int_{\mathbb{R}^{nk}} e^{-i\frac{i}{\hbar}\sum_{j=1}^{k}\left[\frac{m}{2}\frac{(x_j-x_{j-1})^2}{(t/n)^2}-V(x_j)\right]\frac{t}{k}}\varphi(x_0)dx_0\cdots dx_{k-1}. \tag{1.6}$$

Expression (1.6) can be interpreted as the finite dimensional approximation of the path integral (1.2); indeed let γ be a continuous path satisfying $\gamma(t)=x$ and $\gamma(t_j)=x_j$, $j=0,\dots,k$, with $t_j = tj/k$, $x_0,\dots,x_{k-1}$ given points in $\mathbb{R}^n$ and $x_k = x$. Then the exponent in the integrand in (1.6) can be regarded as a Riemann approximation of (1.1) along γ, that is

$$S[\gamma]=\int_0^t\left(\frac{m}{2}\dot\gamma(\tau)^2-V(\gamma(\tau))\right)=\lim_{k\to\infty}\sum_{j=1}^{k}\Big[\frac{m}{2}\frac{(x_j-x_{j-1})^2}{(t/n)^2}-V(x_j)\Big]\frac{t}{k}\,.$$

This means that we are approximating γ with a piecewise line γ_k defined as

$$\gamma_k(\tau):=x_j+\frac{(x_{j+1}-x_j)}{\frac{t}{n}}(\sigma-tj/n),\quad s\in[tj/n,t(j+1)/n],\quad j=0,\dots,k-1\,.$$

In the mathematical literature, we can find several attempts to give a rigorous meaning to Feynman's path integrals. Indeed the interpretetion of formula (1.2), which is just heuristic, leads to mathematical difficulties. The first problem one has to face deals with integration over the space of paths, which is an infinite dimensional space. Then one has to specify the meaning of $\mathscr{D}[\gamma]$ in (1.2); it turns out that a Lebesgue-type measure cannot be defined on an infinite dimensional Hilbert (or Banach) space (see [91]); in 1960 Cameron showed in [14], that is not even possible to construct this measure as a limit of finite dimensional approximations. It follows that the expression $\mathscr{D}[\gamma]$ is mathematically meaningless.
Actually a theory for integration in spaces of continuous functions was already known also before the introduction of path integrals by Feynman; in particular we mention, as a seminal work, [123], where in 1923 Wiener proved the existence of Brownian motion (see also [99, 124, 127]).
One way to give a rigorous mathematical definition of Feynman's path integrals, is that of "analytic continuation", which is based on the expression of the solution of the heat equation

$$\partial_t u(t,x)=\sigma\Delta u(t,x)-V(x)u(t,x)\,, \tag{1.7}$$

(σ being a positive constant) as an integral with respect to the Wiener measure, found by Kac in [68], by proving that the solution of (1.7) admits the following representation

$$u(t,x) = \int e^{-\int_0^t V(\gamma(\tau)+x)d\tau}\varphi(\gamma(0)+x)dW(\gamma), \tag{1.8}$$

where $dW(\gamma)$ is the Wiener measure for the Brownian motion defined on continuous paths $\gamma(\tau)$, $0 \leq \tau \leq t$, with $\gamma(t) = 0$. Expression (1.8) is mathematically rigorous and it is known as *Feynman-Kac formula*. By noting that (1.7) is analoguous to the Schrödinger equation (1.3) if t is replaced by $-it$, one can define the Feynman path integral (1.2) as the analytic continuation of (1.8) to purely imaginary t. This argument can be made rigorous under suitable assumptions on the potential and on the initial datum. This procedure results to be successful with potentials that are a sum of a quadratic part plus a bounded potential with singularities, potentials which exhibit a particular polynomial growth or an exponential growth if they are the Laplace transform of measures. In this direction one can see [11, 14, 50, 100]. Another possibility to rigorously define path integrals consists in regarding Feynman measure as an infinite dimensional distribution, see for instance [92].
An alternative approach, in the mathematical justification of Feynman path integrals, was proposed by Itô in [66] and further developed by Albeverio and Høegh-Krohn in [2]. Itô considered the space of paths γ as a separable Hilbert space with norm $|\gamma|^2 = \int_0^t \dot{\gamma}(\tau)d\tau$. The Feynman path integral is recovered as an infinite dimensional Fresnel integral in the Hilbert space. This procedure works for potentials which are the sum of a quadratic form and the Fourier transform of a complex measure of bounded variation.
Another possible method is the so-called "time slicing approximation" of Feynman path integrals, developed by Fujiwara and Kumano-go (see [36, 38, 40–46]). The starting idea is to construct a subdivision Δ of the time interval $[s,t]$

$$s = t_0 < t_1 < \cdots < t_L = t\,.$$

We denote with $\omega(\Delta) = \sup_{j=1,\dots,L-1} |t_j - t_{j-1}|$. For each j we consider a point x_j. Now the path γ in the expression (1.2), is approximated by a piecewise classical path, that is the solution (which is unique under suitable assumptions on the potential and for sufficiently small $|t-s|$) of the classical equation of motion

$$m\ddot{\gamma}(\tau) = -\nabla V(\tau, \gamma(\tau))\,,$$

with boundary conditions $\gamma(t_j) = x_j$ and $\gamma(t_{j+1}) = x_{j+1}$. So, if we denote with γ_Δ the approximation of γ, we can define the path integral as

$$\int e^{\frac{i}{\hbar}S(\gamma)}f(\gamma) = \lim_{\omega(\Delta)\to 0} \prod_{j=1}^{L} \frac{1}{(2\pi i\hbar(t_j - t_{j-1}))^{\frac{n}{2}}} \int_{\mathbb{R}^{Ln}} e^{\frac{i}{\hbar}S(\gamma_\Delta)}f(\gamma_\Delta)\prod_{j=1}^{L} dx_j\,.$$

In this Chapter we will focus on this last procedure.

1.2 Presentation of the problem

In this chapter we consider a Schrödinger equation for a wave function subject to a time dependent potential

$$\begin{cases} -i\frac{\partial}{\partial t}u(t,x) - \frac{1}{2}\Delta u(t,x) + V(t,x)u(t,x) = 0\,, \\ u(s,x) = \varphi(x)\,, \end{cases} \tag{1.9}$$

with $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ ($\hbar = 1$ here). The purpose of our study is to prove the existence and uniqueness of solutions to (1.9) under some suitable hypotheses on the potential.

Equation (1.9) is of fundamental importance in quantum mechanics and many other contexts, so it is a basic mathematical question to see under which assumptions it is possible to construct its fundamental solution and which integrability/regularity properties it satisfies.

In the mathematical literature there is a variety of methods available to study this problem. In the case when the potential can be considered as a perturbation of the free evolution given by the Laplacian, then it is possible to apply the abstract theory of evolution operators (see for example [69, 106] and references therein). Alternatively, by exploiting the smoothing estimates enjoyed by the free propagator, it is also possible to study the problem by a variation of constants formula, see [95, 125]. Moreover, under some suitable assumptions on the potential, we can also exploit resolvent techniques to even infer some analogue dispersive estimates as for the free Schrödinger operator, see for example [56, 107].

However, there is a number of physically interesting potentials that cannot be treated in a perturbative manner. In some cases we can adopt another strategy in order to construct the fundamental solution to (1.9), which consists in constructing directly the integral operator by means of a semiclassical approximation technique. This latter strategy is inspired by Feynman's formulation of quantum mechanics through path integrals. More specifically the solution to (1.9) is given by an oscillating integral operator whose phase is given by the classical action and the integral is performed over a suitable set of paths, hence on an infinite dimensional functional space. This theory received a lot of attention in the physical literature because of its broad applications and was rigorously studied in the mathematical literature by many authors, using different approaches (see Section 1.1). We mention [2] where the whole theory is developed by using an infinite generalization of Fresnel integrals; there the authors can treat potentials which are polynomials or the Fourier transform of a complex measure with bounded variation. This theory is robust enough so that it is possible to study the method of stationary phase for Feynman path integrals [3]. Other approaches were proposed, see for example [14, 100] where the Feynman path integral is defined by the analytic continuation of the Wiener integral, or [90] where the author uses Poisson processes to define it. For more insights about different approaches we refer to [2, 11, 14, 50, 66, 74, 92, 100] and their references.

In this dissertation we adopt the "time slicing" approach developed by Fujiwara [36] and his school. More precisely the classical action in the phase of the oscillatory integral operator is computed along piecewise classical paths. Then the convergence is proven by exploiting the L^2 theory of oscillating integral transformations. As a result they prove the convergence of the propagator in the norm operator topology for any time dependent potential $V(t,x)$ such that for almost every $t \in \mathbb{R}$, $V(t,x)$ is a real-valued $\mathscr{C}^\infty$ function and $V \in L^\infty_t L^\infty_{x,loc}$, $\partial_x^\alpha V \in L^\infty(\mathbb{R} \times \mathbb{R}^n)$ for any $|\alpha| \geq 2$, i.e. the potential grows at most quadratically in space at infinity. Let us notice that altough the approach in [2] allows to consider arbitrary polynomials, here the main advantage is that the convergence holds in the uniform operator topology. The approach by Fujiwara was further developed, see for example [125], where Yajima considers also magnetic potentials and, by combining the Fujiwara's approach with Kato's pertubration theory, extends the result on a large class of (electro-magnetic) potentials. A slightly different approach is considered in [65, 82, 84], where the action is not computed along classical trajectories, but rather on straight lines, in the spirit of finite difference methods. The time slicing approach was extensively studied in the literature, see also [37, 38, 40, 47, 77, 81, 83, 84]. Moreover, in [101] it was shown that it is possible also to consider a class of non-smooth (in space) potentials.

We are interested in relaxing the assumptions on the time integrability for the potential. More specifically, we want to consider a class of potentials $V(t,x)$ such that

(V-I) $V(t,x)$ is a measurable function of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and for almost every $t \in \mathbb{R}$, $V(t,\cdot) \in \mathscr{C}^\infty(\mathbb{R}^n)$.

(V-II) $V \in L^2_t L^\infty_{\mathrm{loc},x}$.

(V-III) For any $|\alpha| \geq 2$, $\partial_x^\alpha V \in L^2_t L^\infty_x$.

Our interest in this problem stems from the attempt of studying a class of quantum hydrodynamic systems arising in the Landau's two-fluid theory for superfluidity at finite temperatures. In order to develop a self-consistent theory for finite energy weak solutions to quantum hydrodynamic systems it is possible to study the existence of solutions for the underlying wave function dynamics. Then, by means of the Madelung transformations and of a polar factorization technique, it is possible to show the existence of finite energy solutions to the hydrodynamical system [6, 8, 18]. It turns out that, when studying a toy system related to Landau's two-fluid model, we encounter a nonlinear Schrödinger equation whose linear part is given by (1.9). For more details about the relation between (1.9) and the quantum hydrodynamic systems we address the interested reader to [5, 18] and Section B.2.

Besides applications to quantum fluid dynamics, the problem is interesting as in our setup the trajectories are not Lipschitz continuous anymore. Indeed, under our assumptions on the potential, the classical paths are only Hölder continuous with exponent $1/2$. Consequently the action inherits the same rough regularity and we have to face the problem of its differentiability in order to prove that it satisfies the Hamilton-Jacobi equation.

The chapter is organized as follows. In Section 1.3 we study the classical mechanics associated to (1.9) in order to prove that the phase function $S(t,s,x,y)$ is well defined if $|t-s|$ is sufficiently small. In Section 1.4 we study the basic properties of the classical action $S(t,s,x,y)$. In Section 1.5 we analyze the behaviour of the integral transformation E(t,s), by proving that it is an approximate solution of (1.9). In Section 1.6 we prove the convergence of the time slicing approximation procedure to the fundamental solution of (1.9). In the Appendix A we collect some results on the L^2 theory of oscillatory integral transformations.

As already stressed in the Introduction of this dissertation, in the following sections we strictly follow Fujiwara's notation in [36].

1.3 Classical orbits

In this section we analyze the flow generated by the Hamiltonian

$$H(t,x,\xi) = \frac{1}{2}|\xi|^2 + V(t,x),$$

which is described by the Hamilton canonical differential equations

$$\frac{dx}{dt} = \xi, \qquad \frac{d\xi}{dt} = -\nabla_x V(t,x). \tag{1.10}$$

In particular, we will see that there exists a generating function of the flow, that is the classical action $S(t,s,x,y)$, for small $|t-s|$, which will be used as the phase function for the construction of the parametrix in Section 1.5. Let $x(t) = x(t,s,y,\eta)$ and $\xi(t) = \xi(t,s,y,\eta)$ be the solution of (1.10) with initial conditions

$$x(s) = y, \qquad \xi(s) = \eta. \tag{1.11}$$

$(x(t,s,x,y), \xi(t,s,x,y))$ is the classical orbit in the phase space. The Cauchy problem (1.10)-(1.11) is equivalent to the integral equation

$$x(t) = y + \int_s^t \xi(\tau)d\tau \tag{1.12}$$

$$\xi(t) = \eta - \int_s^t \nabla_x V(\tau, x(\tau))d\tau. \tag{1.13}$$

Throughout this chapter we make the assumptions (V-I)-(V-III) on the potential $V(t,x)$ and we conduct the discussion on $(-T,T)$, with $T>0$.

In order to prove the existence of a unique solution for the previous Cauchy problem, set $X=(x,\xi)$, $F(t,X)=(\xi,-\nabla_x V(t,x))$ and $X_s=(y,\eta)$. With this notation, the integral equations (1.12), (1.13) become

$$X(t) = X_s + \int_s^t F(\tau, X(\tau))d\tau, \tag{1.14}$$

For $\rho > 0$ let

$$Z_\rho := \left\{ X \in C([s,s+\alpha]) \text{ s.t. } \sup_{t\in[s,s+\alpha]} |X(t) - X_s| \le \rho \right\}.$$

Now we define the map

$$\Phi(X)(t) = X_s + \int_s^t F(\tau, X(\tau))d\tau\,. \tag{1.15}$$

By the assumptions on the potential it follows that, for almost every t

$$|F(t,X)| \le |X(t)| + C(1+|X(t)|)\|\nabla_x^2 V(t)\|_{L_x^\infty(\mathbb{R}^n)}\,. \tag{1.16}$$

For $X \in Z_\rho$ and $|t-s| \le \alpha$ we have, by using (1.16),

$$\begin{aligned} |\Phi(X)(t) - X_s| &\le \int_s^t |F(\tau, X(\tau)|d\tau \\ &\le (\rho + |X_s|)\alpha + C\sqrt{\alpha}M + C\sqrt{\alpha}(\rho + |X_s|)M\,, \end{aligned}$$

where $M = \|\nabla_x^2 V\|_{L_t^2 L_x^\infty}$. By choosing $\alpha \le 1$ and $\rho = \max(1, |X_s|)$, we get

$$|\Phi(X)(t) - X_s| \le \rho\sqrt{\alpha}(2+3CM)\,.$$

By choosing

$$\alpha \le \frac{1}{(2+3CM)^2}$$

we obtain that Φ maps Z_ρ into itself.
Moreover

$$\begin{aligned} |\Phi(X)(t) - \Phi(\tilde{X})(t)| &\le \int_s^t |F(\tau, X(\tau)) - F(\tau, \tilde{X}(\tau))|d\tau \\ &\le \int_s^t |X(\tau) - \tilde{X}(\tau)|(1 + \|\nabla_x^2 V(\tau)\|_{L_x^\infty(\mathbb{R}^n)})d\tau \\ &\le \sqrt{\alpha}(1+M) \sup_{t\in[s,s+\alpha]} |X(t) - \tilde{X}(t)|\,. \end{aligned}$$

So, by choosing α sufficiently small, in particular

$$\alpha = \min\left\{ \frac{1}{(2+3CM)^2}, \frac{1}{(1+M)^2} \right\},$$

we can perform a standard fixed point iteration scheme, getting the existence of a unique local solution. It is straightforward to show that it can be extended to a maximal solution in (T_{min}, T_{max}). (Actually, in this way, the uniqueness is established just in Z_ρ, but it can be easily extended, by using a standard continuity argument). In order to get a global solution, we assume that T_{max} is finite; it follows that

$$\lim_{t\to T_{max}^-} |X(t)| = +\infty\,. \tag{1.17}$$

By using the fact that $|\nabla_x V(t,x(t))| \lesssim (1+|x(t)|)\|\nabla_x^2 V(t)\|_{L_x^\infty}$ for a.e. t in (1.14), we get

$$|X(t)| \le C_{y,\eta,T_{max},s} + C\int_s^t (1+\|\nabla_x^2 V(\tau)\|_{L_x^\infty})|X(\tau)|d\tau\,, \tag{1.18}$$

where $M = \|\nabla_x^2 V(t)\|_{L_t^2 L_x^\infty}$. By the Gronwall's inequality it follows that

$$|X(t)| \le C_{y,\eta,T_{max},s} e^{C(|t-s|+M\sqrt{|t-s|})}\,, \tag{1.19}$$

which contradicts (1.17).
So we have proved the following

Proposition 1.3.1. *For any $t,s\in\mathbb{R}$, the system* (1.10) *has a unique solution $x(t) = x(t,s,y,\eta)$ and $\xi(t) = \xi(t,s,y,\eta)$. $x(t)$ is of class C^1 in t and $\xi(t)$ is absolutely continuous in t.*

Remark 1.3.2. *Under Fujiwara's assumptions on the potential $V(t,x)$, the solution $X(t)$ is Lipschitz continuous in time; in our case the solution is only $C^{\frac{1}{2}}$-continuous in time, since*

$$|X(t+h)-X(t)| \le \int_t^{t+h} |F(\tau,x(\tau))|d\tau \lesssim \sqrt{h}(1+\|\nabla_x^2 V\|_{L_t^2 L_x^\infty})$$

In order to prove that for sufficiently small $|t-s|$, the phase function $S(t,s,x,y)$ is well defined, we want to write η as a function of (t,s,x,y). For this reason we shall study derivatives of $x(t)$ and $\xi(t)$ with respect to the initial values (y,η). Let u be any of the $2n$—variables y_j, η_j, $j=1,2,\cdots,n$. Differentiating both sides of (1.12) and (1.13) with respect to u, we obtain the equations

$$\frac{\partial x}{\partial u}(t) = \frac{\partial y}{\partial u} + \int_s^t \frac{\partial \xi}{\partial u}(\tau)d\tau, \tag{1.20}$$

$$\frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial u} - \int_s^t \partial_x^2 V(\tau,x(\tau))\frac{\partial x}{\partial u}(\tau)d\tau \tag{1.21}$$

The following Proposition shows that the map $(x(t,s),\xi(t,s))$ is globally Lipschitz, with respect to the space variables, if $|t-s|\le T$.

Proposition 1.3.3. *For every $T>0$ there exists a constant $C(T,M)$ depending on T and $\|\partial_x^2 V\|_{L_t^2 L_x^\infty}$ such that, for $|t-s|\le T$,*

$$\left\|\frac{\partial x}{\partial u}(t)\right\|_{L^\infty(\mathbb{R}_y^n\times\mathbb{R}_\eta^n)} \le C, \quad \left\|\frac{\partial \xi}{\partial u}(t)\right\|_{L^\infty(\mathbb{R}_y^n\times\mathbb{R}_\eta^n)} \le C \tag{1.22}$$

Proof. We have that

$$\left|\frac{\partial x}{\partial u}(t)\right| \le \left|\frac{\partial y}{\partial u}\right| + \int_s^t \left|\frac{\partial \xi}{\partial u}(\tau)\right| d\tau \le \left|\frac{\partial y}{\partial u}\right| + \int_s^t \left\|\frac{\partial \xi}{\partial u}(\tau)\right\|_{L^\infty(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)} d\tau\,.$$

In a similar way

$$\begin{aligned}\left|\frac{\partial \xi}{\partial u}(t)\right| &\le \left|\frac{\partial \eta}{\partial u}\right| + \int_s^t \|\partial_x^2 V\|_{L^\infty(\mathbb{R}^n)}\left\|\frac{\partial x}{\partial u}(\tau)\right\|_{L^\infty(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)} d\tau \\ &\le \left|\frac{\partial \eta}{\partial u}\right| + \int_s^t \left(1+\|\partial_x^2 V(\tau)\|_{L^\infty_x}\right)\left\|\frac{\partial x}{\partial u}(\tau)\right\|_{L^\infty(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)} d\tau\,.\end{aligned}$$

It follows that

$$\begin{aligned}\left|\frac{\partial x}{\partial u}(t)\right| + \left|\frac{\partial \xi}{\partial u}(t)\right| \le \left|\frac{\partial y}{\partial u}\right| + \left|\frac{\partial \eta}{\partial u}\right| \\ + \int_s^t \left(1+\|\partial_x^2 V\|_{L^2_tL^\infty_x}\right)\left(\left\|\frac{\partial \xi}{\partial u}(\tau)\right\|_{L^\infty(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)} + \left\|\frac{\partial x}{\partial u}(\tau)\right\|_{L^\infty(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)}\right)\end{aligned}$$

By setting $M := \|\partial_x^2 V\|_{L^2_tL^\infty_x}$ and by using Gronwall's inequality, we get

$$\left\|\frac{\partial x}{\partial u}(t)\right\|_{L^\infty(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)} + \left\|\frac{\partial \xi}{\partial u}(t)\right\|_{L^\infty(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)} \le C\exp(|t-s| + M\sqrt{|t-s|}) \le C(T,M)$$

□

In particular, we can prove the following result.

Proposition 1.3.4. *For every $T>0$ there exists a constant $C=C(T,M)>0$ such that, for $|t-s|\le T$,*

$$\left\|\frac{\partial \xi}{\partial \eta}(t) - I\right\|_{L^\infty(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)} \le MC|t-s|^{\frac{3}{2}}, \quad \left\|\frac{\partial \xi}{\partial y}(t)\right\|_{L^\infty(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)} \le MC\sqrt{|t-s|}, \tag{1.23}$$

$$\left\|\frac{\partial x}{\partial y} - I\right\|_{L^\infty(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)} \le MC|t-s|^{\frac{3}{2}}, \quad \left\|\frac{\partial x}{\partial \eta}(t)\right\|_{L^\infty(\mathbb{R}^n_y\times\mathbb{R}^n_\eta)} \le C|t-s|, \tag{1.24}$$

where I is the identity matrix.

Proof. From (1.12) and (1.13) we have

$$\frac{\partial x}{\partial u}(t) - \frac{\partial y}{\partial u} = \int_s^t \frac{\partial \xi}{\partial u}(\tau)d\tau \tag{1.25}$$

$$\frac{\partial \xi}{\partial u}(t) - \frac{\partial \eta}{\partial u} + \int_s^t \partial_x^2 V(\tau, x(\tau))\frac{\partial y}{\partial u}d\tau = -\int_s^t\int_s^\tau \partial_x^2 V(\tau, x(\tau))\frac{\partial \xi}{\partial u}(\sigma)d\sigma d\tau\,. \tag{1.26}$$

Let's start with the first inequality in (1.23). By using (1.26) and (1.22) we have

$$\left|\frac{\partial \xi}{\partial \eta}(t)-I\right| \leq \int_s^t \int_s^\tau \|\partial_x^2 V(\tau)\|_{L_x^\infty} \left\|\frac{\partial \xi}{\partial \eta}(\sigma)\right\|_{L^\infty(\mathbb{R}_y^n\times\mathbb{R}_\eta^n)} d\sigma d\tau$$

$$\leq C(T,M)\int_s^t \|\partial_x^2 V(\tau)\|_{L_x^\infty}(\tau-s)d\tau \leq C(T,M)M|t-s|^{\frac{3}{2}},$$

so the desired result follows. The other inequalities can be obtained in the same way, by combining (1.25) and (1.26) with (1.22). □

By using the above propositions, we can easily prove the following estimate.

Proposition 1.3.5. *Let k any integer ≥ 2 and $|t-s| \leq T$. Then there exists a positive constant $C_k = C_k(T,M)$ such that*

$$\sum_{2\leq|\alpha|+|\beta|\leq k} \left\{|t-s|^{-|\beta|-\frac{3}{2}}\left|\left(\frac{\partial}{\partial y}\right)^\alpha\left(\frac{\partial}{\partial \eta}\right)^\beta x(t,s,y,\eta)\right| + |t-s|^{-|\beta|-\frac{1}{2}}\left|\left(\frac{\partial}{\partial y}\right)^\alpha\left(\frac{\partial}{\partial \eta}\right)^\beta \xi(t,s,y,\eta)\right|\right\} \leq C_k, \tag{1.27}$$

for any pair of multi-indices α and β such that $2 \leq |\alpha|+|\beta| \leq k$.

Proof. From (1.12) and (1.13), we have that for $|\alpha|+|\beta| \geq 2$,

$$\partial_y^\alpha \partial_\eta^\beta x(t,s,y,\eta) = \int_s^t \partial_y^\alpha \partial_\eta^\beta \xi(\tau,s,y,\eta)d\tau \tag{1.28}$$

and

$$\partial_y^\alpha \partial_\eta^\beta \xi(t,s,y,\eta) = -\int_s^t \sum_{j=1}^n (\partial_{x_j}\nabla_x V(\tau,x(\tau))\partial_y^\alpha\partial_\eta^\beta x_j(\tau,s,y,\eta)d\tau \tag{1.29}$$

$$-\int_s^t f_{\alpha\beta}(\tau,s,y,\eta)d\tau, \tag{1.30}$$

where $f_{\alpha\beta}(\tau,s,y,\eta)$ is a linear combination of terms of the form

$$(\partial_x^\gamma \nabla_x V)(\tau,x(\tau))\big(\partial_y^{\nu_1}\partial_\eta^{\mu_1}x_{j_1}(\tau)\big)\cdots\big(\partial_y^{\nu_{|\gamma|}}\partial_\eta^{\mu_{|\gamma|}}x_{j_{|\gamma|}}(\tau)\big), \tag{1.31}$$

where $j_1,\ldots,j_{|\gamma|} \in \{1,\ldots,n\}$, $\nu_1+\ldots,\nu_{|\gamma|} = \alpha$, $\mu_1+\cdots+\mu_{|\gamma|} = \beta$, $|\nu_j|+|\mu_j| \geq 1$ for $j = 1,\ldots,|\gamma|$ and $2 \leq |\gamma| \leq |\alpha|+|\beta|$. The proof proceeds by induction on $|\alpha|+|\beta|$. If $|\alpha|+|\beta| = 2$, (1.27) is a simple consequence of Proposition 1.3.4. Let us suppose that (1.27) is satisfied for $|\alpha|+|\beta| \leq m$ and prove it for $|\alpha|+|\beta| = m+1$. To deal with the terms in (1.31) we note that

$$|\partial_y^{\nu_j}\partial_\eta^{\mu_j}x(\tau)| \leq C(T)|\tau-s|^{|\mu_j|}.$$

This holds for $|\nu_j|+|\mu_j|=1$ by using Proposition 1.3.4 and by the inductive hypothesis if $|\nu_j|+|\mu_j|\geq 2$. Since $|\mu_1|+\cdots+\mu_{|\gamma|}=\beta$, it follows that

$$\left|\left(\partial_y^{\nu_1}\partial_\eta^{\mu_1}x_{j_1}(\tau)\right)\cdots\left(\partial_y^{\nu_{|\gamma|}}\partial_\eta^{\mu_{|\gamma|}}x_{j_{|\gamma|}}(\tau)\right)\right|\leq C(T)|\tau-s|^\beta .$$

By putting all together we get

$$|\partial_y^\alpha\partial_\eta^\beta\xi(\tau)|\leq C\int_s^t \|\nabla_x^2 V(\tau)\|_{L_x^\infty}|\partial_y^\alpha\partial_\eta^\beta x(\tau)|d\tau+C|t-s|^{\beta+\frac{1}{2}} .$$

This inequality, together with (1.28) and Gronwall's inequality gives (1.27). □

As suggested by Proposition 1.3.5, we introduce the new variable $\zeta=(t-s)\eta$ and we consider the map

$$(u,\zeta)\mapsto(\tilde{x}(t,s,y,\zeta),\tilde{\xi}(t,s,y,\zeta)):=(x(t,s,y,\zeta/(t-s)),(t-s)\xi(t,s,y,\zeta/(t-s)) .$$

Then, we have

Proposition 1.3.6. *We have, for $j,k=1,\dots,n$, $t\neq s$,*

$$\frac{\partial\tilde{x}_j}{\partial\zeta_k}=\delta_{jk}-(t-s)^{\frac{3}{2}}a_{jk}(t,s,y,\zeta), \tag{1.32}$$

$$\frac{\partial\tilde{\xi}_j}{\partial\zeta_k}=\delta_{jk}-(t-s)^{\frac{3}{2}}b_{jk}(t,s,y,\zeta), \tag{1.33}$$

$$\frac{\partial\tilde{x}_j}{\partial y_k}=\delta_{jk}-(t-s)^{\frac{3}{2}}c_{jk}(t,s,y,\zeta), \tag{1.34}$$

where $a_{jk}(t,s,y,\zeta)$, $b_{jk}(t,s,y,\zeta)$, $c_{jk}(t,s,y,\zeta)$ belong to a bounded set in the function space of Schwartz $\mathscr{B}(\mathbb{R}_y^n\times\mathbb{R}_\zeta^n)$ (see Appendix A for the definition) if $0<|t-s|\leq T$, for every fixed $T>0$.

Proof. Since $t\neq s$, we have that (1.33) defines the functions b_{jk}, so

$$|b_{jk}(t,s,y,\zeta)|\leq C(M,T), \tag{1.35}$$

by using Proposition 1.3.4. Now we need to estimate the derivative of $b_{jk}(t,s,y,\zeta)$:

$$\left(\frac{\partial}{\partial y}\right)^\alpha\left(\frac{\partial}{\partial\eta}\right)^\beta b_{jk}(t,s,y,\zeta)=(t-s)^{-\frac{3}{2}-|\beta|}\left(\frac{\partial}{\partial y}\right)^\alpha\left(\frac{\partial}{\partial\eta}\right)^\beta\frac{\partial\xi_j}{\partial\eta_k}(t,s,y,\zeta/(t-s)) .$$

Proposition 1.3.5 yields the estimate

$$\begin{aligned}\left|\left(\frac{\partial}{\partial y}\right)^\alpha\left(\frac{\partial}{\partial\eta}\right)^\beta b_{jk}(t,s,y,\zeta)\right|&\leq|(t-s)|^{-\frac{3}{2}-|\beta|}\left|\left(\frac{\partial}{\partial y}\right)^\alpha\left(\frac{\partial}{\partial\eta}\right)^\beta\frac{\partial\xi_j}{\partial\eta_k}(t,s,y,\zeta/(t-s))\right|\\&\leq C(M,T) .\end{aligned} \tag{1.36}$$

Thus $\{b_{jk}(t,s,y,\zeta)\}_{t,s}$ belong to a bounded set of $\mathscr{B}(\mathbb{R}^n_y \times \mathbb{R}^n_\zeta)$. We can prove that $\{c_{jk}(t,s,y,\zeta)\}_{t,s}$ is bounded in $\mathscr{B}(\mathbb{R}^n_y \times \mathbb{R}^n_\zeta)$ in just the same manner. Moreover we can easily prove that

$$a_{jk}(t,s,y,\zeta) = (t-s)^{-\frac{5}{2}} \int_s^t (\tau - s)^{\frac{3}{2}} b_{jk}(\tau,s,y,\zeta) d\tau \,. \tag{1.37}$$

It follows form (1.35), (1.36) and (1.37) that $\{a_{jk}(t,s,y,\zeta)\}_{t,s}$ belong to a bounded set in $\mathscr{B}(\mathbb{R}^n_y \times \mathbb{R}^n_\zeta)$, if $|t-s| \le T$. □

Now we can study the invertibility of the map

$$\zeta \mapsto \tilde{x}(t,s,y,\zeta)\,.$$

Proposition 1.3.7. *There exists $\delta > 0$, depending on T, such that for $0 < |t-s| \le \delta$ and $y \in \mathbb{R}^n$, the map*

$$\zeta \mapsto \tilde{x}(t,s,y,\zeta) = x(t,s,y,\zeta/(t-s))$$

is invertible in $\mathbb{R}^n$. Moreover the following equality holds

$$\frac{\partial \zeta_j}{\partial \tilde{x}_k}(t,s,\tilde{x},y) = \delta_{jk} - (t-s)^{\frac{3}{2}} d_{jk}(t,s,\tilde{x},y)\,, \tag{1.38}$$

where the functions $d_{jk}(t,s,\tilde{x},y)$ belong to a bounded subset in $\mathscr{B}(\mathbb{R}^n_{\tilde{x}} \times \mathbb{R}^n_y)$.

Proof. By using Proposition 1.3.6, we get

$$\det \frac{\partial \tilde{x}}{\partial \zeta} = 1 - (t-s)^{\frac{3}{2}} f(t,s,y,\zeta)\,, \tag{1.39}$$

for some function $f(t,s,y,\zeta)$, belonging to a bounded set in $\mathscr{B}(\mathbb{R}^n_y \times \mathbb{R}^n_\zeta)$. Choose δ such that $(t-s)^{\frac{3}{2}} \|f(t,s,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^n_y \times \mathbb{R}^n_\zeta)} \le \frac{1}{2}$ holds for any $(y,\zeta) \in \mathbb{R}^{2n}$ and $0 < |t-s| \le \delta$.
In order to prove (1.38) we can proceed in the same way of Proposition 1.3.6. First of all we have that

$$\frac{\partial \zeta}{\partial \tilde{x}}(t,s,\tilde{x},y) = \left[\frac{\partial \tilde{x}}{\partial \zeta}(t,s,y,\zeta)\right]^{-1} \quad \zeta = \zeta(t,s,\tilde{x},y)\,. \tag{1.40}$$

In order to prove (1.38), it is sufficient to show that

$$\left[\frac{\partial \tilde{x}_j(t,s,y,\zeta)}{\partial \zeta_k}\right]^{-1} = \delta_{jk} - (t-s)^{\frac{3}{2}} d_{jk}(t,s,y,\zeta)\,. \tag{1.41}$$

From (1.39) we have that

$$\left(\det \frac{\partial \tilde{x}}{\partial \zeta}\right)^{-1} = 1 + (t-s)^{\frac{3}{2}} \tilde{a}(t,s,y,\zeta)\,, \tag{1.42}$$

where

$$\tilde{a}(t,s,y,\zeta)=\frac{f(t,s,y,\zeta)}{1-(t-s)^{\frac{3}{2}}f(t,s,y,\zeta)}\,.$$

Clearly $\tilde{a}$ belongs to a bounded subset of $\mathscr{B}(\mathbb{R}^{2n})$. This, together with (1.32), gives (1.38).

□

1.4 Classical action

Throughout this section, we always assume that $0 < t-s \leq \delta$. Thus the function $\eta(t,s,x,y)$ is well defined. Therefore, the curve

$$\tau \mapsto x(\tau)=x(\tau,s,y,\eta(t,s,x,y))$$

is the unique classical orbit starting from y at time s and reaching x at time t.
Now we can define the classical action $S(t,s,x,y)$ as

$$S(t,s,x,y)=\int_s^t L(\tau,x(\tau),\dot{x}(\tau))d\tau\,, \tag{1.43}$$

where

$$L(\tau,x(\tau),\dot{x}(\tau))=\frac{1}{2}|\dot{x}(\tau)|^2-V(\tau,x(\tau))\,. \tag{1.44}$$

It is well known that $S(t,s,x,y)$ satisfies the Hamilton Jacobi equation if the potential $V(t,x)$ is continuous in (t,x). To prove this one has to exploit the total differentiability of the action at every point. Since we do not assume the continuity with respect to time of $V(t,x)$, we cannot expect this. Fujiwara in [36] solves this problem by showing that it is sufficient to have that $S(t,s,x,y)$ is totally differentiable almost everywhere. This property follows from Rademacher's theorem, since under the assumptions on the potential in [36], the classical action is a locally Lipschitz function of (t,x,y), for every fixed s. So, there exists a zero measure set $Z \subset (s-\delta,s+\delta)\times\mathbb{R}^{2n}$ such that $S(t,s,x,y)$ is totally differentiable at every $(t,x,y)\notin Z$; moreover at every $(t,x,y)\notin Z$

$$\frac{\partial}{\partial t}S(t,s,x,y)+\frac{1}{2}\left|\frac{\partial}{\partial x}S(t,s,x,y)\right|^2+V(t,x)=0\,.$$

As a consequence of the Remark (1.3.2), under our hypothesis on the potential, $S(t,s,x,y)$ is not locally Lipschitz continuous in (t,x,y), since with respect to time it is just $C^{\frac{1}{2}}$-continuous. On the other hand, from the definition (1.43), it follows that it is absolutely continuous with respect to time, for every x and y fixed. So it is differentiable almost everywhere in t, but the zero measure set out of which this property holds depends on x and y.

To overcome this difficulty we introduce the following smooth regularization in time of the potential

$$V_\varepsilon(\cdot,x) = V(\cdot,x) * \rho_\varepsilon, \quad 0<\varepsilon\le 1, \tag{1.45}$$

where $\rho_\varepsilon(t) = \varepsilon^{-1}\rho(\varepsilon^{-1}t)$ is a standard mollifier in $\mathbb{R}$. From the properties of mollifiers, it follows that for each $x\in\mathbb{R}^n$, $V_\varepsilon(t,x)\to V(t,x)$, for almost every t as $\varepsilon\to 0$; moreover for each $|\alpha|\ge 2$, $\partial_x^\alpha V_\varepsilon \to \partial_x^\alpha V$ in $L_t^2 L_x^\infty$, when $\varepsilon\to 0$ and hence $\|\partial_x^\alpha V_\varepsilon\|_{L_t^2L_x^\infty}$ is uniformly bounded with respect to ε.
Now we consider the following approximate Hamilton system

$$x_\varepsilon(y) = y + \int_s^t \xi_\varepsilon(\tau)d\tau, \quad \xi_\varepsilon(t) = \eta - \int_s^t \frac{\partial}{\partial x}V_\varepsilon(\tau, x_\varepsilon(\tau))d\tau\,. \tag{1.46}$$

Lemma 1.4.1. *For fixed $s\in\mathbb{R}$, the solutions $x_\varepsilon(t,s,y,\eta)$, $\xi_\varepsilon(t,s,y,\eta)$ are bounded on the compact subsets of $\mathbb{R}\times\mathbb{R}^{2n}$, uniformly with respect to ε.*

Proof. Let $\tilde{T}>0$ such that $|t|\le\tilde{T}$. We put $M_\varepsilon = \|\partial_x^2 V_\varepsilon\|_{L_t^2L_x^\infty}$, which is uniformly bounded with respect to ε. From (1.12) and (1.14) we get

$$\begin{aligned}|x_\varepsilon(t)| &\le |y| + \int_s^t |\xi_\varepsilon(\tau)|d\tau \le |y| + |\eta||t-s| + \int_s^t (t-\tau)(1+|x_\varepsilon(\tau)|)\|\nabla_x^2 V_\varepsilon(\tau)\|_{L_x^\infty} \\ &\le |y| + (\tilde{T}+|s|)|\eta| + (\tilde{T}+|s|)^{\frac{3}{2}}M_\varepsilon + \int_s^t (t-\tau)\|\nabla_x^2 V_\varepsilon(\tau)\|_{L_x^\infty}|x_\varepsilon(\tau)|d\tau\end{aligned}$$

By using Gronwall's inequality we obtain

$$|x_\varepsilon(t)| \le C_{s,y,\eta,M_\varepsilon,\tilde{T}}\,.$$

In the same way we can proceed for $\xi_\varepsilon(t)$ and this concludes the proof. □

Lemma 1.4.2. *For fixed $s\in\mathbb{R}$, we have that $x_\varepsilon(t,s,y,\eta)$ converges to $x(t,s,y,\eta)$ and $\xi_\varepsilon(t,s,y,\eta)$ converges to $\xi(t,s,y,\eta)$, uniformly on the compact subset of $\mathbb{R}\times\mathbb{R}^{2n}$.*

Proof. We want to study the convergence of the approximate orbits. We have

$$\begin{aligned}&|\xi_\varepsilon(t)-\xi(t)| \le \int_s^t \left|\frac{\partial V}{\partial x}(\tau,x(\tau)) - \frac{\partial V_\varepsilon}{\partial x}(\tau,x_\varepsilon(\tau))d\tau\right| \\ &\le \int_s^t \left|\frac{\partial V}{\partial x}(\tau,x(\tau)) - \frac{\partial V}{\partial x}(\tau,x_\varepsilon(\tau))d\tau\right| + \int_s^t \left|\frac{\partial V}{\partial x}(\tau,x_\varepsilon(\tau)) - \frac{\partial V_\varepsilon}{\partial x}(\tau,x_\varepsilon(\tau))d\tau\right| \\ &=: I_1 + I_2\,.\end{aligned}$$

For I_1

$$I_1 = \int_s^t \left|\int_0^1 \frac{\partial}{\partial\theta}\nabla_x V(\tau, \theta x(\tau) + (1-\theta)x_\varepsilon(\tau)d\theta\right|d\tau \le \int_s^t \|\partial_x^2 V\|_{L_x^\infty}|x(\tau)-x_\varepsilon(\tau)|d\tau$$

The second integral I_2 tends to zero, when $\varepsilon \to 0$. Indeed, from Lemma 1.4.1 it follows that for fixed $\tilde{T} > 0$, there exists a ball B in $\mathbb{R}^{2n}$, where $x_\varepsilon(t,s,y,\eta)$ takes values for $|t| \le \tilde{T}$ and $(y,\eta) \in B$. Since

$$I_2 \le \int_s^t \left\| \frac{\partial V}{\partial x}(\tau,\cdot) - \frac{\partial V_\varepsilon}{\partial x}(\tau,\cdot) \right\|_{L_x^\infty(B)} d\tau \le \sqrt{t-s} \left\| \frac{\partial V}{\partial x}(\tau,\cdot) - \frac{\partial V_\varepsilon}{\partial x}(\tau,\cdot) \right\|_{L_t^2 L_x^\infty(B)},$$

we get $I_2 \to 0$, as $\varepsilon \to 0$, by exploiting the properties of the mollification. Moreover

$$|x(t) - x_\varepsilon(t)| \le \int_s^t |\xi(\tau) - \xi_\varepsilon(\tau)| d\tau .$$

By putting all together and by using Gronwall's inequality we get that

$$|\xi(t) - \xi_\varepsilon(t)| + |x(t) - x_\varepsilon(t)| \to 0 .$$

□

Lemma 1.4.3. *For $|t-s| \le T$, we have that*

$$\frac{\partial x_\epsilon}{\partial \eta}(t,s,y,\eta) \to \frac{\partial x}{\partial \eta}(t,s,y,\eta) \tag{1.47}$$

$$\frac{\partial \xi_\epsilon}{\partial \eta}(t,s,y,\eta) \to \frac{\partial \xi}{\partial \eta}(t,s,y,\eta), \tag{1.48}$$

pointwise, as $\epsilon \to 0$.

Proof. We have

$$\left| \frac{\partial x_\epsilon}{\partial \eta}(t) - \frac{\partial x}{\partial \eta}(t) \right| \le \int_s^t \left| \frac{\partial \xi_\epsilon}{\partial \eta}(\tau) - \frac{\partial \xi}{\partial \eta}(\tau) \right| d\tau .$$

Then

$$\left| \frac{\partial \xi_\epsilon}{\partial \eta}(t) - \frac{\partial \xi}{\partial \eta}(t) \right| \le \underbrace{\int_s^t \left| \frac{\partial^2 V_\epsilon}{\partial x^2}(\tau, x_\epsilon(\tau)) - \frac{\partial^2 V}{\partial x^2}(\tau, x(\tau)) \right| \left| \frac{\partial x_\epsilon}{\partial \eta}(\tau) \right| d\tau}_{I_1}$$

$$+ \int_s^t \left| \frac{\partial^2 V}{\partial x^2} \right| \left| \frac{\partial x_\epsilon}{\partial \eta}(\tau) - \frac{\partial x}{\partial \eta}(\tau) \right| d\tau .$$

Now, by using Proposition 1.3.4, we have

$$\begin{aligned} I_1 &\le C(T,M) \int_s^t (\tau - s) \left| \frac{\partial^2 V_\epsilon}{\partial x^2}(\tau, x_\epsilon(\tau)) - \frac{\partial^2 V_\epsilon}{\partial x^2}(\tau, x(\tau)) \right| d\tau \\ &+ \int_s^t (\tau - s) \left| \frac{\partial^2 V_\epsilon}{\partial x^2}(\tau, x(\tau)) - \frac{\partial^2 V}{\partial x^2}(\tau, x(\tau)) \right| d\tau \\ &\le C(T,M) \int_s^t (\tau - s) \|\partial_x^2 V_\epsilon\|_{L_x^\infty} |x_\epsilon(\tau) - x(\tau)| d\tau \\ &+ (t-s)^{\frac{3}{2}} \|\partial_x^2 V_\epsilon(\tau,\cdot) - \partial_x^2 V(\tau,\cdot)\|_{L_t^2 L_x^\infty} \to 0 . \end{aligned}$$

The first integral in the last lines goes to 0 by the dominated convergence theorem, the second one because of the properties of the regularization.
The thesis follows by Gronwall's inequality. □

We can define, for each fixed $\varepsilon > 0$, the approximate classical action $S_\varepsilon(t,s,x,y)$ as

$$S_\varepsilon(t,s,x,y) := \int_s^t L(\tau, x_\varepsilon(\tau), \dot{x}_\varepsilon(\tau))d\tau\,. \tag{1.49}$$

Lemma 1.4.4.

$$S_\varepsilon(t,s,x,y) \to S(t,s,x,y) \;\; \textit{pointwise as} \;\; \varepsilon \to 0\,,$$

if $0 < t-s \le \delta$.

Proof.

$$\begin{aligned} S_\varepsilon(t,s,x,y) - S(t,s,x,y) &= \frac{1}{2}\int_s^t \{|\xi_\varepsilon(\tau,s,y,\eta_\varepsilon(t,s,x,y))|^2 - |\xi(\tau,s,y,\eta(t,s,x,y))|^2\} \\ &+ \int_s^t \{V(\tau,x(\tau)) - V_\varepsilon(\tau,x_\varepsilon(\tau))\}d\tau =: I_1 + I_2 \end{aligned}$$

By using Lemma 1.4.2, the fact that $\eta_\varepsilon(t,s,x,y) \to \eta(t,s,x,y)$ pointwise (which easily follows by contradiction by recalling that $\eta = \zeta/(t-s)$ and by using Proposition 1.3.7 and Lemma 1.4.2) and the dominated convergence theorem, it follows that $I_1 \to 0$ as $\varepsilon \to 0$.
Moreover, we have that $I_2 := I_2^1 + I_2^2$, where

$$\begin{aligned} I_2^1 &= \int_s^t \{V_\varepsilon(\tau,x_\varepsilon(\tau)) - V_\varepsilon(\tau,x(\tau))\}d\tau\,, \\ I_2^2 &= \int_s^t \{V_\varepsilon(\tau,x(\tau)) - V(\tau,x(\tau))\}d\tau\,. \end{aligned}$$

It holds that

$$\begin{aligned} |I_2^1| &\le \int_s^t (1+|x_\varepsilon(\tau)|+|x(\tau)|)\|\nabla_x^2 V_\varepsilon(\tau)\|_{L_x^\infty}|x_\varepsilon(\tau)-x(\tau)|d\tau \\ &\le \|\nabla_x^2 V_\varepsilon\|_{L_t^2 L_x^\infty}\left(\int_s^t (1+|x_\varepsilon(\tau)|+|x(\tau)|)^2|x_\varepsilon(\tau)-x(\tau)|^2 d\tau\right)^{\frac{1}{2}}, \end{aligned}$$

which goes to zero as $\varepsilon \to 0$, by combining the dominated convergence theorem with Lemmas 1.4.1, 1.4.2 and the fact that $\|\nabla_x^2 V_\varepsilon\|_{L_t^2 L_x^\infty}$ is uniformly bounded with respect to ε.

Moreover $I_2^2 \to 0$, as $\varepsilon \to 0$, by using the fact that, if if $|t-s| \leq \delta$, there exists a ball B in $\mathbb{R}^{2n}$ where $x(\tau)$ takes values and that $V_\varepsilon \to V$ in $L^2_t L^\infty_{loc,x}$ as $\varepsilon \to 0$. Indeed

$$
I_2^2 \leq \int_s^t \|V_\varepsilon(\tau,\cdot) - V(\tau,\cdot)\|_{L^\infty_x(B)}\,.
$$

□

Proposition 1.4.5. *Assume that* $0 < t-s \leq \delta$. *Then,* $S(t,s,x,y)$ *is of class* C^∞ *in* (x,y) *if* t *and* s *are fixed. Moreover the following equalities hold:*

$$
\partial_{x_j} S(t,s,x,y) = \xi_j(t,s,y,\eta(t,s,x,y)) \tag{1.50}
$$

$$
\partial_{y_j} S(t,s,x,y) = -\eta_j(t,s,x,y)\,. \tag{1.51}
$$

Proof. Equalities (1.50) and (1.51) follow by direct computations, by using integration by parts in (1.43). Indeed, let

$$
S_0(t,s,y,\eta) = S(t,s,x(t,s,y,\eta),y) = \int_s^t L(\tau,x(\tau),\dot{x}(\tau))d\tau\,, \tag{1.52}
$$

where $x(\tau) = x(\tau,s,y,\eta)$ is the unique solution of Hamilton equations (1.10). Clearly $S(t,s,x,y) = S_0(t,s,y,\eta(t,s,x,y))$. By using the chain rule, we have

$$
\partial_{x_j} S(t,s,x,y) = \partial_{\eta_k} S_0(t,s,y,\eta(t,s,x,y))\partial_{x_j}\eta_k(t,s,x,y)\,. \tag{1.53}
$$

By integrating by parts in (1.52) and by using Einstein convention on repeated indices, we get

$$
\begin{aligned}
\partial_{\eta_k} S_0(t,s,y,\eta) &= \int_s^t \partial_{x_h} L(\tau,x(\tau),\dot{x}(\tau))\partial_{\eta_k} x_h(\tau) + \partial_{\dot{x}_h} L(\tau,x(\tau),\dot{x}(\tau))\partial_{\eta_k}\dot{x}_h(\tau)d\tau \\
&= \int_s^t \partial_{\eta_k} x_h(\tau)[\partial_{x_h} L(\tau,x(\tau),\dot{x}(\tau)) - \partial_\tau \partial_{\dot{x}_h} L(\tau,x(\tau),\dot{x}(\tau))] \\
&\quad + \xi_h(\tau,s,y,\eta)\partial_{\eta_k} x_h(\tau,s,y,\eta)\Big|_{\tau=s}^{\tau=t} \\
&= \xi_h(t,s,y,\eta)\partial_{\eta_k} x_h(t,s,y,\eta)\,,
\end{aligned} \tag{1.54}
$$

where we used Hamilton's equations (1.10) equation in order to write the last equality. Indeed, from (1.44), it follows

$$
\partial_{x_h} L(\tau,x(\tau),\dot{x}(\tau)) - \partial_\tau \partial_{\dot{x}_h} L(\tau,x(\tau),\dot{x}(\tau)) = -\partial_{x_h} V(\tau,x(\tau)) - \dot{\xi}_h(\tau) = 0\,.
$$

Plugging (1.54) in (1.53) we obtain

$$
\partial_{x_j} S(t,s,x,y) = \xi_h(t,s,y,\eta)\partial_{\eta_k} x_h(t,s,y,\eta)\Big|_{\eta=\eta(t,s,x,y)} \partial_{x_j}\eta_k(t,s,x,y)\,. \tag{1.55}
$$

On the other hand, by differentiating with respect to x_j both sides of the identity $x_h = x_h(t,s,y,\eta(t,s,x,y))$ we have

$$\delta_{hj} = \partial_{\eta_k} x_h(t,s,y,\eta(t,s,x,y))\partial_{x_j}\eta_k(t,s,x,y). \tag{1.56}$$

By using (1.56) in (1.55) we get (1.50). We can prove (1.51) in a similar way. □

Proposition 1.4.6. *Let s be fixed. Then for every $x, y \in \mathbb{R}^n$ there exists a zero measure set $Z_{x,y}$, which depends on x and y, such that for each $t \notin Z_{x,y}$, the classical action $S(t,s,x,y)$ satisfies the Hamilton Jacobi partial differential equation*

$$\partial_t S(t,s,x,y) + \frac{1}{2}|\nabla_x S(t,s,x,y)|^2 + V(t,x) = 0. \tag{1.57}$$

Proof. We know, by [36] (the regularized potential V_ε satisfies Fujiwara's assumptions), that for each $\varepsilon > 0$

$$\partial_t S_\varepsilon(t,s,x,y) + \frac{1}{2}|\nabla_x S_\varepsilon(t,s,x,y)|^2 + V_\varepsilon(t,x) = 0,$$

for almost every $t \in \mathbb{R}$ and all $(x,y) \in \mathbb{R}^{2n}$. So it follows that

$$S_\varepsilon(t,s,x,y) + \frac{1}{2}\int_s^t |\nabla_x S_\varepsilon(\tau,s,x,y)|^2 d\tau + \int_s^t V_\varepsilon(\tau,x)d\tau = 0, \tag{1.58}$$

where we used the fact that $S_\varepsilon(s,s,x,y) = 0$. From Proposition 1.4.4, we know that $S_\varepsilon(s,t,x,y) \to S(t,s,x,y)$ pointwise as $\varepsilon \to 0$. By using Proposition 1.4.5, Lemmata 1.4.1, 1.4.2 and the dominated convergence theorem (for x and y in a bounded set), we get that

$$\int_s^t |\nabla_x S_\varepsilon(\tau,s,x,y)|^2 d\tau \to \int_s^t |\nabla_x S(\tau,s,x,y)|^2 d\tau,$$

as $\varepsilon \to 0$. Regarding the last term in (1.58) we have

$$\begin{aligned}\int_s^t |V_\varepsilon(\tau,x) - V(\tau,x)|d\tau &\le \int_s^t \|V_\varepsilon(\tau) - V(\tau)\|_{L^\infty_x(B)} d\tau \\ &\le \sqrt{t-s}\|V_\varepsilon - V\|_{L^2_t L^\infty_x(B)},\end{aligned}$$

where B is a closed ball containing x. Since $V_\varepsilon \to V$ in $L^2_t L^\infty_{x,\mathrm{loc}}$, as $\varepsilon \to 0$, by passing to the limit in (1.58), we get

$$S(t,s,x,y) + \frac{1}{2}\int_s^t |\nabla_x S(\tau,s,x,y)|^2 d\tau + \int_s^t V(\tau,x)d\tau = 0. \tag{1.59}$$

Since $S(t,s,x,y)$ is absolutely continuous in t, differentiating a.e. with respect to t in (1.59), we get (1.57).

We include here an alternative proof of (1.57).
From (1.43), it follows that $S(t,s,x,y)$ is absolutely continuous with respect to t for every fixed $x,y \in \mathbb{R}^n$; hence , there exists a zero measure set $Z_{x,y} \subset \mathbb{R}$, depending on x and y, such that $\partial_t S(t,s,x,y)$ exists for every $t \notin Z_{x,y}$.

By the chain rule, we have

$$\frac{d}{dt}S(t,s,x,y) = \partial_t S_0(t,s,y,\eta(t,s,x,y)) + \partial_{\eta_k} S_0(t,s,y,\eta(t,s,x,y))\partial_t \eta_k(t,s,x,y), \tag{1.60}$$

for every $t \notin Z_{x,y}$. From (1.52) it follow that

$$\partial_t S_0(t,s,y,\eta) = \frac{1}{2}|\xi(t,s,y,\eta)|^2 - V(t,x(t,s,y,\eta)). \tag{1.61}$$

Plugging (1.61) and (1.54) into (1.60) we get

$$\begin{aligned}\partial_t S(t,s,x,y) = {} & \frac{1}{2}|\xi(t,s,y,\eta(t,s,x,y))|^2 - V(t,x(t,s,y,\eta(t,s,x,y))) \\ & + \xi_h(t,s,y,\eta(t,s,x,y))\partial_{\eta_k} x_h(t,s,y,\eta(t,s,x,y))\partial_t \eta_k(t,s,x,y).\end{aligned} \tag{1.62}$$

Now, we differentiate with respect to time the identity $x_h = x_h(t,s,y,\eta(t,s,x,y))$ and we obtain

$$\begin{aligned}0 &= \partial_t x_h(t,s,y,\eta(t,s,x,y)) + \partial_{\eta_k} x_h(t,s,y,\eta(t,s,x,y))\partial_t \eta_k(t,s,x,y) \\ &= \xi_h(t,s,y,\eta(t,s,x,y)) + \partial_{\eta_k} x_h(t,s,y,\eta(t,s,x,y))\partial_t \eta_k(t,s,x,y).\end{aligned} \tag{1.63}$$

By using (1.63) into (1.62) we have

$$\partial_t S(t,s,x,y) = -\frac{1}{2}|\xi(t,s,y,\eta(t,s,x,y))|^2 - V(t,x(t,s,y,\eta(t,s,x,y))), \tag{1.64}$$

which is (1.57) in virtue of (1.50). □

The following proposition states an important property of the action.

Proposition 1.4.7. *Assume that* $0 < t - s \le \delta$. *Then we have*

$$S(t,s,x,y) = \frac{1}{2}\frac{|x-y|^2}{t-s} + \sqrt{t-s}\,\omega(t,s,x,y). \tag{1.65}$$

For any pair of multi-indices α *and* β *with length* $|\alpha| + |\beta| \ge 2$, *there exists a constant* $C_{\alpha\beta M}$ *such that*

$$|\partial_x^\alpha \partial_y^\beta \omega(t,s,x,y)| \le C_{\alpha\beta M}\,, \tag{1.66}$$

where $C_{\alpha\beta}$ *is independent of* (t,s) *and* (x,y).

Proof. Since $t > s$, $\omega(t,s,x,y)$ is defined by (1.65), hence

$$\omega(t,s,x,y) = (t-s)^{-\frac{3}{2}}\left[(t-s)S(t,s,x,y) - \frac{1}{2}|x-y|^2\right].$$

The Proposition 1.4.5 implies that

$$\partial^2_{x_jx_k}\omega(t,s,x,y)=(t-s)^{-\frac{3}{2}}(\partial_{x_k}((t-s)\xi_j(t,s,y,\eta(t,s,x,y)))-\delta_{jk}).$$

Moreover, since $\zeta=(t-s)\eta$,

$$\partial_{x_k}((t-s)\xi_j)=\sum_m \partial_{\zeta_m}((t-s)\xi_j)\partial_{x_k}\zeta_m.$$

We now apply the Propositions 1.3.6 and 1.3.7 and we obtain

$$\begin{aligned}\partial^2_{x_jx_k}\omega(t,s,x,y)&=-b_{jk}(t,s,y,\zeta(t,s,x,y))-d_{jk}(t,s,x,y)\\&\quad+(t-s)^{\frac{3}{2}}\sum_{m=1}^{d} b_{jm}(t,s,y,\zeta(t,s,x,y))d_{mk}(t,s,x,y),\end{aligned}$$

where the functions b_{jk} and d_{jk} are defined in (1.33) and (1.38). This proves that $\partial_{x_jx_k}\omega(t,s,x,y)$ form a bounded set in $\mathscr{B}(\mathbb{R}^n_x\times\mathbb{R}^n_y)$.
Similar discussions for $\partial^2_{x_jy_k}\omega(t,s,x,y)$ and $\partial^2_{y_jx_k}\omega(t,s,x,y)$ prove the proposition.
□

Lemma 1.4.8. *Let* $S_\epsilon(t,s,x,y)=\frac{1}{2}\frac{|x-y|^2}{t-s}+\sqrt{t-s}\,\omega_\epsilon(t,s,x,y)$*. Then we have that*

$$\Delta_x\omega_\epsilon(t,s,x,y)\to\Delta_x\omega(t,s,x,y),$$

pointwise as $\epsilon\to 0$*, if* $0<t-s\le\delta$*.*

Proof. We have that

$$\sqrt{|t-s|}\Delta_x\omega_\epsilon(t,s,x,y)=\Delta_x S_\epsilon(t,s,x,y)-\frac{n}{t-s}.$$

By using Proposition 1.4.5, we get

$$\Delta_x S_\epsilon(t,s,x,y)=\sum_k\frac{\partial\xi_{\epsilon j}}{\partial\eta_k}(t,s,y,\eta_\varepsilon(t,s,x,y))\frac{\partial\eta_{\varepsilon k}}{\partial x_j}(t,s,x,y).$$

So, by using (1.47), (1.48) together with the fact that $\eta_\varepsilon(t,s,x,y)\to\eta(t,s,x,y)$, we get

$$\Delta_x S_\varepsilon(t,s,x,y)\to\Delta_x S(t,s,x,y),\tag{1.67}$$

pointwise as $\varepsilon\to 0$. □

1.5 Parametrices

We assume throughout this section that $0 < t-s < \delta$, where δ is the constant appearing in Proposition 1.3.7. For any $\varphi \in \mathscr{B}(\mathbb{R}^n)$ we define the integral operators $E(t,s)$ as

$$E(t,s)\varphi(x) = \int_{\mathbb{R}^n} e(t,s,x,y)\varphi(y)dy\,, \tag{1.68}$$

where

$$e(t,s,x,y) = \left(\frac{1}{2\pi i(t-s)}\right)^{\frac{n}{2}} e^{iS(t,s,x,y)}\,.$$

In the following proposition we prove that the operator $E(t,s)$ is a bounded linear operator in $L^2(\mathbb{R}^n)$.

Proposition 1.5.1. *There exists a positive constant $\gamma_0 = \gamma_0(\delta, M)$ such that*

$$\|E(t,s)\varphi\|_{L^2(\mathbb{R}^n)} \leq \gamma_0 \|\varphi\|_{L^2(\mathbb{R}^n)}\,,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Proof. Let $\nu = (t-s)^{-1}$ and

$$\phi(t,s,x,y) = (t-s)S(t,s,x,y) = \frac{1}{2}|x-y|^2 + (t-s)^{\frac{3}{2}}\omega(t,s,x,y)\,.$$

Then $E(t,s)$ can be written as an oscillatory integral transformation as follows

$$E(t,s)\varphi(x) = \left(\frac{\nu}{2\pi i}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\nu\phi(t,s,x,y)}\varphi(y)dy\,. \tag{1.69}$$

So $E(t,s) = I(t,s;1)$ (see Appendix A). By Proposition 1.4.5 we have that

$$\frac{\partial^2}{\partial x_j \partial y_k}\phi(t,s,x,y) = -\frac{\partial \xi_k}{\partial x_j}(t,s,x,y)\,.$$

Proposition 1.3.6 implies that there exists a constant ρ such that

$$\left|\det \frac{\partial^2}{\partial x_j \partial y_k}\phi(t,s,x,y)\right| > \rho\,.$$

Since Proposition 1.4.7 holds, we can apply Theorem A.0.5 of the Appendix A to the integral transformation (1.69), getting the thesis. We remark that the constant γ_0 (see (A.1) in the Appendix A) depends only on δ and M in virtue of Proposition 1.4.7. □

Definition 1.5.2. *Let*

$$W = \left\{ f \in L^2(\mathbb{R}^n) \,\middle|\, (1+|x|^2)f \in L^2(\mathbb{R}^n) \text{ and } \left(\frac{\partial}{\partial x}\right)^\alpha f \in L^2(\mathbb{R}^n) \text{ for any } \alpha \text{ with } |\alpha| \leq 2 \right\}.$$

The space W is a Hilbert space equipped with the norm

$$\|f\|_W^2 = \|(1+|\cdot|^2)f\|_{L^2(\mathbb{R}^n)}^2 + \sum_{|\alpha|\le 2} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha f \right\|_{L^2(\mathbb{R}^n)}^2 .$$

We note that $W = H^2 \cap \mathscr{F}(H^2)$, which is related to the harmonic oscillator. Let $H(t)$ be the minimal closed extension, with respect to L^2, of the differential operator $-\frac{1}{2}\Delta + V(t,x)$ restricted to $C_0^\infty(\mathbb{R}^n)$. We shall denote its domain with $D(H(t))$. Under the assumptions on the potential we have that $|V(t,x)| \le C(1+|x|^2)\|\nabla_x^2 V(t)\|_{L_x^\infty}$ for almost every $t \in \mathbb{R}$. So it follows that the Hamiltonian operator $H(t)$ is a bounded linear map from W to $L^2(\mathbb{R}^n)$ for a.e. $t \in \mathbb{R}$.

Proposition 1.5.3. *E(t,s) is a continuous mapping of W into itself.*

Proof. The proof is an immediate consequence of Theorem A.0.6 of the Appendix. □

Proposition 1.5.4. *If $\varphi \in W$, then, $E(t,s)\varphi \in D(H(t))$ for almost every $t \in \mathbb{R}$.*

Proof. As a consequence of Proposition 1.5.3, we have only to prove that $D(H(t)) \supset W$ for a.e.$t \in \mathbb{R}$. Let $\varphi \in W$. Then $\Delta\varphi \in L^2(\mathbb{R}^n)$ and $(1+|\cdot|^2)\varphi \in L^2(\mathbb{R}^n)$. Moreover $|V(t,x)| \le C(1+|x|^2)\|\nabla_x^2 V(t)\|_{L_x^\infty}$ for almost every $t \in \mathbb{R}$. Therefore, we have

$$-\frac{1}{2}\Delta\varphi + V(t,x)\varphi \in L^2(\mathbb{R}^n).$$

Hence the Hamiltonian operator $H(t)$ is a bounded linear map from W to $L^2(\mathbb{R}^n)$ for a.e. $t \in \mathbb{R}$. Now, let $\varphi \in W$. Since $C_o^\infty(\mathbb{R}^n)$ is dense in W, there exists a sequence $\varphi_n \in C_0^\infty(\mathbb{R}^n)$, which converges to φ in W. Morevor, $H(t)\varphi_n$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$; indeed

$$\|H(t)\varphi\|_{L^2(\mathbb{R}^n)} \lesssim \|\varphi_n\|_W .$$

Thus $H(t)\varphi_n$ converges to some $\psi \in L^2(\mathbb{R}^n)$. It follows that $\varphi \in D(H(t))$ for almost every $t \in \mathbb{R}$. □

Now, for each $\varepsilon > 0$ we can define the parametrix

$$E_\varepsilon(t,s)\varphi(x) = \left(\frac{1}{2\pi i(t-s)}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{iS_\varepsilon(t,s,x,y)}\varphi(y)dy , \tag{1.70}$$

where $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and S_ε is defined in (1.49).

Remark 1.5.5. *Proceeding as in Proposition 1.5.1, we can prove the existence of a constant $\tilde{\gamma}_0$, which is independent with respect to ε, such that*

$$\|E_\varepsilon(t,s)\varphi\|_{L^2(\mathbb{R}^n)} \le \tilde{\gamma}_0\|\varphi\|_{L^2(\mathbb{R}^n)} . \tag{1.71}$$

Indeed, as in Proposition 1.5.1, $\tilde{\gamma}_0$ depends only on δ and $\|\nabla_x^2 V_\varepsilon\|_{L_t^2 L_x^\infty}$ is uniformly bounded with respect to ε.

Proposition 1.5.6. *There exists a positive constant C, which is independent of ε, such that*

$$\|E_\varepsilon(t,s)\varphi\|_W \le C\|\varphi\|_W \tag{1.72}$$

Proof. This is an immediate consequence of Theorem A.0.6 of the Appendix. Indeed, by using Propositions 1.3.5, 1.3.6 and 1.4.7, we have that all the assumptions of Theorem A.0.6 are satisfied. The constant depends on $\|\partial_x^2 V_\varepsilon\|_{L_t^2 L_x^\infty}$, which is uniformly bounded with respect to ε. □

Proposition 1.5.7.

$$E_\varepsilon(t,s)\varphi(x) \to E(t,s)\varphi(x),$$

for every $x \in \mathbb{R}^n$, as $\varepsilon \to 0$.

Proof. The result follows by Lemma 1.4.4 and the dominated convergence theorem, because $\varphi \in \mathscr{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. □

The following result shows that $E(t,s)$ converges strongly to the identity operator as $t \to s$.

Proposition 1.5.8. *1. For every $\varphi \in L^2(\mathbb{R}^n)$, we have*

$$\lim_{t\to s} E(t,s)\varphi = \varphi\,,$$

in $L^2(\mathbb{R}^n)$.

2. If we set $E(s,s) = I$, then the correspondence $(s,t) \mapsto E(t,s)\varphi$ gives a strongly continuous function with values in $L^2(\mathbb{R}^n)$.

Proof. First of all we note that we have only to prove (1) for $\varphi \in C_0^\infty(\mathbb{R}^n)$ since Proposition 1.5.1 holds. From Proposition 1.5.7 we know that

$$E_\varepsilon(t,s)\varphi(x) \to E(t,s)\varphi(x),$$

for every $x \in \mathbb{R}^n$, as $\varepsilon \to 0$. By using the Fatou's Lemma and the definition of lim inf, we have that for every $\mu > 0$ there exists ε_0 such that

$$\begin{aligned}\|E(t,s)\varphi - \varphi\|_{L^2(\mathbb{R}^n)} &\le \liminf_{\varepsilon\to 0}\|E_\varepsilon(t,s)\varphi - \varphi\|_{L^2(\mathbb{R}^n)} \\ &\le \|E_{\varepsilon_0}(t,s)\varphi - \varphi\|_{L^2(\mathbb{R}^n)} + \mu\,.\end{aligned}$$

Now, we know that the regularized potential satisfies Fujiwara's assumptions; so it follows (see Proposition 4.3 in [36]) that

$$\lim_{t\to s}\|E_{\varepsilon_0}(t,s)\varphi - \varphi\|_2 = 0$$

and therefore we conclude that

$$\limsup_{t\to s} \|E(t,s)\varphi - \varphi\|_{L^2(\mathbb{R}^n)} \leq \mu\,.$$

By the arbitrariness of μ we get the desired result.
We can prove (2) by using the fact that for every $\varepsilon > 0$ we have

$$\lim_{t\to t',\, s\to s'} \|E_\varepsilon(t,s)\varphi - E_\varepsilon(t',s')\varphi\|_{L^2(\mathbb{R}^n)} = 0\,,$$

from [36]. By using the Fatou Lemma as above we get the desired result. □

At this point we need to show that the $E(t,s)\varphi(x)$ is an approximate solution of the Schrödinger equation (1.9). As already stressed, under our assumptions on the potential $V(t,x)$, for every fixed $x, y \in \mathbb{R}^n$ the time derivative of $S(t,s,x,y)$ exists almost everywhere, but the zero measure set depends on x and y. This prevents us from justifying the exchange of the order of derivation and integration, so we cannot expect to have that

$$\frac{\partial}{\partial t}E(t,s,x,y)\varphi(x) = \int_{\mathbb{R}^n} \frac{\partial}{\partial t}e(t,s,x,y)\varphi(y)dy\,,$$

as in [36]. To overcome this difficulty we use the regularization of the potential introduced in the previous Section. So, for each fixed $\varepsilon > 0$, proceeding as in [36], we have that

$$\Big(-i\partial_t - \frac{1}{2}\Delta + V_\varepsilon(t,x)\Big)E_\varepsilon(t,s)\varphi(x) = G_\varepsilon(t,s)\varphi\,, \tag{1.73}$$

at almost every $t \in (s-\delta, s+\delta)$ for any $\varphi \in C_c^\infty(\mathbb{R}^n)$, with

$$G_\varepsilon(t,s)\varphi(x) = \frac{i\sqrt{t-s}}{(2\pi i(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{iS_\varepsilon(t,s,x,y)}\Delta_x \omega_\varepsilon(t,s,x,y)\varphi(y)dy\,.$$

Proposition 1.5.9. *There exists a positive constant C, which is idependent of ε, such that*

$$\|G_\varepsilon(t,s)\varphi\|_{L^2(\mathbb{R}^n)} \leq C\sqrt{|t-s|}\|\varphi\|_{L^2(\mathbb{R}^n)} \tag{1.74}$$

Proof. This is a simple consequence of Theorem A.0.5 in the Appendix; the constant C is independent on ε since it depends on $\|\nabla_x^2 V_\varepsilon\|_{L_t^2 L_x^\infty}$, which is constant. □

Lemma 1.5.10.

$$G_\varepsilon(t,s)\varphi(x) \to G(t,s)\varphi(x)\,,$$

for every $x \in \mathbb{R}^n$, as $\varepsilon \to 0$. Here $G(t,s)$ is given by

$$G(t,s)\varphi(x) = \frac{i\sqrt{t-s}}{(2\pi i(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{iS(t,s,x,y)}\Delta_x \omega(t,s,x,y)\varphi(y)dy\,. \tag{1.75}$$

Proof. The assertion follows from Lemma 1.4.8 and the dominated convergence theorem. □

We note that Proposition 1.5.9 and Lemma 1.5.10 imply that

$$G_\varepsilon(t,s)\varphi \rightharpoonup G(t,s)\varphi \qquad \text{in } L^2(\mathbb{R}^n) \ \text{(weakly)}. \tag{1.76}$$

By the semicontinuity of the norm, it follows

$$\|G(t,s)\varphi\|_{L^2(\mathbb{R}^n)} \leq C\sqrt{t-s}\|\varphi\|_{L^2(\mathbb{R}^n)}. \tag{1.77}$$

At this point we can explain why $E(t,s)\varphi(x)$ is an approximation of the solution of (1.9).

Proposition 1.5.11. *Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then we have that*

$$\Big(i\partial_t + \frac{1}{2}\Delta - V(t,x)\Big)E(t,s)\varphi(x) = -G(t,s)\varphi\,,$$

in $\mathscr{D}'([s,t]\times\mathbb{R}^n)$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n) \subset W$. For each $\varepsilon > 0$ we have that

$$\Big(i\partial_t + \frac{1}{2}\Delta - V_\varepsilon(t,x)\Big)E_\varepsilon(t,s)\varphi(x) = -G_\varepsilon(t,s)\varphi\,,$$

for almost every $t \in (s-\delta, s+\delta)$. For every $\lambda \in C_0^\infty([s,t]\times\mathbb{R}^n)$ we have

$$\begin{aligned} i\int_{\mathbb{R}^n} E_\varepsilon(t,s)\varphi(x)\lambda(t,x)dx - i\int_{\mathbb{R}^n}\varphi(x)\lambda(s,x)dx - i\int_s^t\int_{\mathbb{R}^n} E_\varepsilon(\tau,s)\varphi(x)\partial_\tau\lambda(\tau,x)dxd\tau \\ = \int_s^t\int_{\mathbb{R}^n} E_\varepsilon(\tau,s)\varphi(x)\Big(-\frac{1}{2}\Delta + V_\varepsilon(\tau,x)\Big)\lambda(\tau,x)dxd\tau \\ - \int_s^t\int_{\mathbb{R}^n} G_\varepsilon(\tau,x)\varphi(x)\lambda(\tau,x)dxd\tau\,. \end{aligned} \tag{1.78}$$

From Propositions 1.5.3 and 1.5.7 we get

$$E_\varepsilon(t,s)\varphi \rightharpoonup E(t,s)\varphi \qquad \text{weakly in } L^2_{t,x}\,.$$

Moreover

$$V_\varepsilon \to V \qquad \text{strongly in } L^2_t L^\infty_{loc,x}\,.$$

From Proposition 1.5.9 and Lemma 1.5.10 it follows

$$G_\varepsilon(t,s)\varphi \rightharpoonup G(t,s)\varphi \qquad \text{in } L^2_{t,x}\,.$$

By passing to the limit, as $\varepsilon \to 0$, in (1.78), we get the thesis. □

1.6 Construction of the fundamental solution

The construction of the fundamental solution is based on the approximate evolution properties of $E(t,s)$, which will be stated in the following propositions.

Remark 1.6.1. *By proceeding as for* (1.73)*, we get that*

$$-i\partial_s E_\varepsilon(t,s)\varphi(x) = E_\varepsilon(t,s)\Big(-\frac{1}{2}\Delta + V_\varepsilon(t,x)\Big)\varphi(x) + \tilde{G}_\varepsilon(t,s)\varphi(x), \tag{1.79}$$

where

$$\tilde{G}_\varepsilon(t,\sigma)\varphi(x) = \left(\frac{1}{2\pi i(t-\sigma)}\right)^{\frac{n}{2}} \frac{\sqrt{t-\sigma}}{2i}\int_{\mathbb{R}^n} \Delta_y \omega_\varepsilon(t,\sigma,x,y) e^{iS_\varepsilon(t,s,x,y)}\varphi(y)dy\,.$$

As for (1.74)*, we can prove*

$$\|\tilde{G}_\varepsilon(t,s)\varphi\|_{L^2(\mathbb{R}^n)} \le C\sqrt{|t-s|}\|\varphi\|_{L^2(\mathbb{R}^n)}, \tag{1.80}$$

with C independent on ε.

Proposition 1.6.2. *There exists a positive constant C_1 such that*

$$\|E(t,s_1)E(s_1,s) - E(t,s)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C_1(|t-s_1|^{\frac{3}{2}} + |s_1-s|^{\frac{3}{3}}). \tag{1.81}$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then, we have

$$(E_\varepsilon(t,s_1)E_\varepsilon(s_1,s) - E_\varepsilon(t,s))\varphi = \int_s^{s_1} \frac{d}{d\sigma}(E_\varepsilon(t,\sigma)E_\varepsilon(\sigma,s)\varphi)d\sigma\,, \tag{1.82}$$

for each $\varepsilon > 0$. Thus, from (1.73) and (1.79) it follows (see [36]) that

$$\left\|\frac{d}{d\sigma}E_\varepsilon(t,\sigma)E_\varepsilon(\sigma,s)\varphi\right\|_{L^2(\mathbb{R}^n)} = \|i\tilde{G}_\varepsilon(t,\sigma)E_\varepsilon(\sigma,s)\varphi - iE_\varepsilon(t,\sigma)G_\varepsilon(\sigma,s)\varphi\|_{L^2(\mathbb{R}^n)}\,.$$

From (1.71), (1.74) and (1.80) we get that

$$\left\|\frac{d}{d\sigma}E_\varepsilon(t,\sigma)E_\varepsilon(\sigma,s)\varphi\right\|_{L^2(\mathbb{R}^n)} \le C(\sqrt{|t-\sigma|} + \sqrt{|\sigma-s|})\|\varphi\|_{L^2(\mathbb{R}^n)}\,.$$

The last inequality together with (1.82), gives

$$\|E_\varepsilon(t,s_1)E_\varepsilon(s_1,s) - E_\varepsilon(t,s)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C_1(|t-s_1|^{\frac{3}{2}} + |s_1-s|^{\frac{3}{3}}), \tag{1.83}$$

where C_1 does not depend on ε. By using the Fatou's lemma and the definition of the liminf as in Proposition 1.5.8, we get the thesis. $\square$

Proposition 1.6.3. *Let $0 < t-s \le \delta$. There exists a positive constant C_2 such that for any $\varphi \in L^2(\mathbb{R}^n)$,*

$$\|E(t,s)\varphi\|_{L^2(\mathbb{R}^n)} \le e^{C_2(t-s)^{\frac{3}{2}}}\|\varphi\|_{L^2(\mathbb{R}^n)} \tag{1.84}$$

Proof. The proof proceeds as in Proposition 1.6.2. □

The estimate (1.84) tells us that we can take γ_0 in Proposition (1.5.1) as $\gamma_0 = e^{C_2(t-s)^{\frac{3}{2}}}$. Now we are ready to construct the propagator for the equation (1.9), by following Feyman's construction of the approximating sequence of the fundamental solution. Let s,t be arbitrary real numbers such that $-T < s < t < T$. Let us consider a subdivision Δ of the interval $[s,t]$ such that

$$\Delta : \qquad t_0 = s < t_1 < t_2 < \ldots < t_{l-1} < t_l = t\,,$$

with

$$\omega(\Delta) = \max_{1\leq j\leq l} |t_j - t_{j-1}|$$

the maximal size of the subintervals. We introduce the iterated integral operator $E(\Delta|t,s)$ as follows

Definition 1.6.4. *Let Δ be any subdivision of the interval $[s,t]$ as above.*

$$E(\Delta|t,s) = E(t,t_{l-1})E(t_{l-1},t_{l-2})\cdots E(t_2,t_1)E(t_1,s)\,. \tag{1.85}$$

Then we have

$$E(\Delta|t,s)\varphi(x) = \int_{\mathbb{R}^n} I(\Delta|t,s,x,y)\varphi(y)dy\,,$$

where

$$I(\Delta|t,s,x,y) = \prod_{j=1}^{L}\left(\frac{1}{2\pi i(t_j - t_{j-1})}\right)^{\frac{n}{2}}$$
$$\times \int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}\prod_{j=1}^{L}\exp\left\{\sum_{j=1}^{L} S(t_j,t_{j-1},x_j,x_{j-1})\right\}\prod_{j=1}^{L} dx_j\,,$$

with $x_0 = y$ and $x_L = x$. We want to prove that

$$\lim_{\omega(\Delta)\to 0} I(\Delta|t,s,x,y) \tag{1.86}$$

exists and equals the kernel function of the fundamental solution for the Schrödinger equation (1.9).
To construct the propagator for (1.9), we shall use the following theorem, which deals with the existence of the limit (1.86).

Theorem 1.6.5. *Let $\{F(t,s)|(t,s)\in[-T,T]\times[-T,T]\}$ be a family of linear operators acting on $L^2(\mathbb{R}^n)$ with the following properties:*

- *$F(t,s)$ is a bounded operator on $L^2(\mathbb{R}^n)$ and there exists a constant $C_1 > 0$ and $\gamma_1 > 0$ such that*

$$\|F(t,s)\varphi\|_{L^2(\mathbb{R}^n)} \leq e^{C_1|t-s|^{\gamma_1}}\|\varphi\|_{L^2(\mathbb{R}^n)}\,. \tag{1.87}$$

- *There exist $\alpha > 1$ and C_2 such that for any $\varphi \in L^2(\mathbb{R}^n)$,*

$$\|(F(t,s_1)F(s_1,s) - F(t,s))\varphi\|_{L^2(\mathbb{R}^n)} \leq C_2(|t-s_1|^\alpha + |s_1 - s|^\alpha)|\varphi\|_{L^2(\mathbb{R}^n)}. \tag{1.88}$$

- *For $\varphi \in \mathscr{S}(\mathbb{R}^n)$, $F(t,s)\varphi$ is a $L^2(\mathbb{R}^n)$-valued strongly continuous function in $(t,s) \in \mathbb{R}^2$ and it satisfies*

$$\begin{cases} F(s,s)\varphi = \varphi \quad \textit{for any } s \in \mathbb{R}, \\ \lim_{t\to s} \|F(t,s)\varphi - \varphi\|_{L^2(\mathbb{R}^n)} = 0. \end{cases}$$

Let Δ be a subdivision of the interval $[s,t]$. We put

$$F(\Delta|t,s) = F(t,t_{l-1})F(t_{l-1},t_{l-2})\cdots F(t_1,s).$$

Then there exists a bounded linear operator $U(t,s)$ in $L^2(\mathbb{R}^n)$ such that

$$\lim_{\omega(\Delta)\to 0} \|U(t,s) - F(\Delta|t,s)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = 0, \tag{1.89}$$

in the norm of bounded operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. More precisely, there exists a constant γ such that

$$\|U(t,s) - F(\Delta|t,s)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq \gamma|t-s|\omega(\Delta)^{\alpha-1}. \tag{1.90}$$

For the proof, we prepare the following lemmas:

Lemma 1.6.6. *Assume $|t-s| \leq \delta$. Let $\Delta = \{t_j\}$, with $t_j = s + jL^{-1}(t-s)$, for $j = 0,1,\ldots,L$ and $\delta(\Delta) = L^{-1}|t-s|$. We have*

$$\|(F(t,s) - F(\Delta|t,s))\varphi\|_{L^2(\mathbb{R}^n)} \lesssim |t-s|^\alpha e^{C_1|t-s|}\|\varphi\|_{L^2(\mathbb{R}^n)},$$

where $F(\Delta|t,s)\varphi = F(t_L,t_{L-1})\cdots F(t_1,t_0)\varphi$.

Proof. The proof is just an adaptation of the same result contained in Lemma 5.7 in [36] (where $\alpha = 2$). Let us assume to have a finite sequence $\Delta^{(0)}, \Delta^{(1)}, \ldots, \Delta^{k+1}$ of subdivisions of the interval $[s,t]$ such that

(a) $k \leq \log_2 L + 2$.

(b) $\Delta^{(m)}$ is a refinement of $\Delta^{(m-1)}$.

(c) $\Delta^{(0)}$ is the trivial subdivision $[s,t]$ and $\Delta^{(k+1)} = \Delta$.

(d) The following estimates hold for any $m = 1,2,\ldots,k+1$:

$$\|F(\Delta^{(m)}|t,s) - F(\Delta^{(m-1)}|t,s)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C|t-s|^\alpha(2^{1-m} + 4L^{-1})e^{C_1|t-s|}. \tag{1.91}$$

Then, it follows

$$\begin{aligned}\|F(\Delta|t,s)-F(t,s)\|_{L^2(\mathbb{R}^n)} &\leq \sum_{m=1}^{k+1} \|F(\Delta^{(m)}|t,s)-F(\Delta^{(m-1)}|t,s)\|_{L^2(\mathbb{R}^n)} \\ &\leq C|t-s|^{\alpha}(2+4(\log_2 L+3)L^{-1})\exp(C_1|t-s|)\,.\end{aligned}$$

The desired estimate follows by noting that $\sup L^{-1}\log_2 L < \infty$. So to conclude the proof we need to construct the sequence of subdivisions $\Delta^{(m)}$. Let us call $J_1 = [s,t_1],\ldots;J_L = [t_{L-1},t]$ elementary intervals and let us the note with I the whole interval $[s,t]$. We put $\tau_0^1 = s$, $\tau_1^1 = t_{[L/2]}$ and $\tau_2^1 = t$, where $[\cdot]$ is the integer part function. We divide I into two intervals

$$\Delta^{(1)}; \qquad I_1^1 = [s,\tau_1^1], \qquad I_2^1 = [\tau_1^1,t]\,,$$

which we call intervals of the first generation. The intervals of the second generation are

$$\Delta^{(2)}; \quad I_1^2 = [\tau_0^2,\tau_1^2], \quad I_2^2 = [\tau_1^2,\tau_2^2], \quad I_3^2 = [\tau_2^2,\tau_3^2], \quad I_4^2 = [\tau_3^2,\tau_4^2]\,,$$

where

$$\begin{aligned}&\tau_0^2 = s, && \tau_1^2 = \max\{t_j|t_j \leq 2^{-1}(\tau_0^1+\tau_1^1)\}\,,\\ &\tau_2^2 = \tau_1^1, && \tau_3^2 = \max\{t_j|t_j \leq 2^{-1}(\tau_1^1+\tau_2^1)\}\,,\\ &\tau_4^2 = \tau_2^1\,.\end{aligned}$$

Iterating this process m—times, we obtain intervals $I_1^m, I_2^m, \ldots, I_{2^m}^m$ of m-th generation, where $I_j^m = [\tau_{j-1}^m, \tau_j^m]$ with $\tau_{2j}^m = \tau_j^{m-1}$ and

$$\tau_{2j+1}^m = \max\{t_j|t_j \leq 2^{-1}(\tau_j^{m-1}+\tau_{j+1}^{m-1})\}\,.$$

Clearly

$$I_j^{m-1} = I_{2j-1}^m \cup I_{2j}^m\,.$$

By definition, we have

$$\left|I_{2j-1}^m\right| \leq 2^{-1}\left|I_j^{m-1}\right|, \qquad \left|I_{2j}^m\right| \leq 2^{-1}\left|I_j^{m-1}\right| + L^{-1}\left|t-s\right|.$$

So it follows that

$$\left|I_j^m\right| \leq |t-s|(2^{-m}+L^{-1}(1+2^{-1}+\cdots+2^{1-m})) < |t-s|(2^{-m}+2L^{-1})\,.$$

If $k \geq \log_2 L$, then

$$\left|I_j^k\right| < 3L^{-1}|t-s|\,.$$

From the last inequality, it follows that I_j^k is a union of at most two elementary intervals. Therefore every interval I_j^{k+1} of $(k+1)$th generation is a single point or it coincides with one of elementary intervals, which means that $\Delta^{(k+1)} = \Delta$.

The sequence of subdivisions we have constructed satisfies the conditions (a), (b) and (c). We need to prove (1.91). Let I_j^{m-1} any interval of $(m-1)$–th generation; in $\Delta^{(m)}$ it is divided into

$$I_{2j-1}^m = [\tau_{2j-2}^m, \tau_{2j-1}^m], \qquad \text{and} \qquad I_{2j}^m = [\tau_{2j-1}^m, \tau_{2j}^m].$$

By using (1.88), we get

$$\begin{aligned}\|F(\tau_j^{m-1}, \tau_{j-1}^{m-1}) - F(\tau_{2j}^m, \tau_{2j-1}^m)F(\tau_{2j-1}^m, \tau_{2j-2}^m)\|_{L^2(\mathbb{R}^n)} \\ \leq C_2\big(\big|I_{2j-1}^m\big|^\alpha + \big|I_{2j}^m\big|^\alpha\big).\end{aligned}$$

This, together with (1.87), gives

$$\|F(\Delta^{(m-1)}|t,s) - F(\Delta^{(m)}|t,s)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}$$

$$\begin{aligned}\leq \sum_{j=0}^{2m-1} C_2\big(\big|I_{2j-1}^m\big|^\alpha + \big|I_{2j}^m\big|^\alpha\big)\exp\big(C_1\big(\big|I_{2^{m-1}}^{m-1}\big|^\alpha + \cdots + \big|I_{j+1}^{m-1}\big|^\alpha + \big|I_{2j+1}^m\big|^\alpha \\ + \big|I_{2j-1}^m\big|^\alpha + \cdots + \big|I_1^m\big|^\alpha\big)\big) \leq C_2|t-s|^\alpha(2^{1-m} + 4L^{-1})\exp(C_1|t-s|).\end{aligned}$$

This proves (1.91) (see Lemma 5.7 of [36]). □

Lemma 1.6.7. *Let two subdivisions of $[s,t]$ be given by*

$$\begin{aligned}\Delta_1 &: s = t_0 < t_1 < \cdots < t_{L-1} < t_L = t,, \\ \Delta_2 &: s = s_o < s_1 < \cdots < s_{M-1} < s_M = t.\end{aligned}$$

Assuming that $\delta(\Delta_1) < \delta$ and $\delta(\Delta_2) < \delta$, we get

$$\|(F(\Delta_1|y,s) - F(\Delta_2|t,s))\varphi\|_{L^2(\mathbb{R}^n)} \lesssim |t-s|(\delta(\Delta_1)^{\alpha-1} + \delta(\Delta_2)^{\alpha-1})e^{C_1|t-s|}\|\varphi\|_{L^2(\mathbb{R}^n)}. \tag{1.92}$$

Proof. The proof proceeds verbatim as in Lemma 5.8 in [36]. We begin with proving (1.92) under the assumption that all the ratios $(t-s)^{-1}(t_j - t_{j-1})$ and $(t-s)^{-1}(s_k - s_{k-1})$ are rational. Under this conditions, there exists a common refinement Δ_3 of the subdivisions Δ_1 and Δ_2 with intervals of the same length. Let us denote

$$\Delta_3; \qquad s = \tau_0 < \tau_1 < \cdots < \tau_k = t,$$

and $\delta(\Delta_3) = |\tau_j - \tau_{j-1}| = K^{-1}|t-s|$. Any interval $[t_{j-1}, t_j]$ of Δ_1 is divided into $[\tau_k, \tau_{k+1}], \ldots, [\tau_{k+m-1}, \tau_{k+m}]$ in Δ_3. By applying Lemma 1.6.6 to $[t_{j-1}, t_j]$, we get

$$\|F(t_j, t_{j-1}) - F(\tau_{k+m}, \tau_{k+m-1})\cdots F(\tau_{k+1}, \tau_k)\|_{L^2(\mathbb{R}^n)} \lesssim |t_j - t_{j-1}|^\alpha e^{C_1|t-t_{j-1}|}. \tag{1.93}$$

Using this and (1.87)

$$\begin{aligned}
&\|F(\Delta_1|t,s)-F(\Delta_3|t,s)\|_{L^2(\mathbb{R}^n)}\\
&\quad\lesssim \sum_{j=0}^{L-1}|t_j-t_{j-1}|^{\alpha}\exp(C_1|t_j-t_{j-1}|+|t-t_{L-1}|^{\gamma_1}+\dots+|t_{j+1}-t_j|^{\gamma_1}\\
&\quad+|\tau_k-\tau_{k-1}|^{\gamma_1}+\dots+|\tau_1-s|^{\gamma_1})\\
&\quad\lesssim |t-s|\delta(\Delta_1)^{\alpha-1}\exp(C_1|t-s|)\,.
\end{aligned} \tag{1.94}$$

In the same way we can prove a similar estimate for $\|E(\Delta_2|t,s)-E(\Delta_3|t,s)\|_{L^2(\mathbb{R}^n)}$,hence we have proved (1.92) when the ratios $(t-s)^{-1}(t_j-t_{j-1})$ and $(t-s)^{-1}(s_k-s_{k-1})$ are rational. In the case of general sundivisions Δ_1 and Δ_2 we can prove (1.92), by exploiting the density of the rational numbers in $\mathbb{R}$ (see also Lemma 5.8 of [36]). □

Proof of Theorem 1.6.5. The proof corresponds to the proof of Theorem 1 in [36], so is omitted. □

Now, we put $F(t,s)=E(t,s)$. All the assumptions of Theorem 1.6.5 are satisfied: they follow by using Proposition 1.6.3 with $\gamma_1=\frac{3}{2}$, (1.83) with $\alpha=\frac{3}{2}$ and Proposition 1.5.8. Thus, we have constructed a family $\{U(t,s)|t,s\in[-T,T]\}$ of operators such that

$$\lim_{\omega(\Delta)\to 0}\|U(t,s)-E(\Delta|t,s)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}=0\,.$$

The following proposition shows that $U(t,s)$ has the evolution property.

Proposition 1.6.8. *For any $s,t,r\in\mathbb{R}$*

$$U(t,r)U(r,s)=U(t,s)\,.$$

Proof. Let $s<r<t$; take the subdivision Δ such that $\Delta=\Delta_l\cup\Delta_r$, where

$$\begin{aligned}
\Delta_l &: s=t_0<t_1<\ldots<t_l=r<t_{l+1}\,,\\
\Delta_r &: r=t_l<t_{l+1}<\ldots<t_M=t\,.
\end{aligned}$$

Then, by noting that $E(\Delta_r|t,s)E(\Delta_l|r,s)=E(\Delta|t,s)$, we get

$$\begin{aligned}
\|U(t,r)U(r,s)\varphi-U(t,s)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} &\le \|(U(t,r)-E(\Delta_r|t,r))U(r,s)\varphi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}\\
&+\|E(\Delta_r|t,r)(U(r,s)-E(\Delta_l|r,s))\varphi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}\\
&+\|(E(\Delta_r|t,r)E(\Delta_l|r,s)-U(t,s))\varphi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}\to 0 \text{ as } \omega(\Delta)\to 0\,.
\end{aligned}$$

□

Now we prove that $U(t,s)$ is actually the fundamental solution of (1.9).

Theorem 1.6.9. *For any t,s the operator $U(t,s)$ maps the space W into itself.*

Proof. Let us consider an arbitrary subdivision Δ of the interval $[s,t]$:

$$\Delta: \qquad s = t_0 < t_1 < \cdots < t_L = t\,.$$

Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. By applying repeatedly Theorem A.0.6 of the Appendix, we get the estimate

$$\|E(\Delta|t,s)\varphi\|_W \le C(T)\|\varphi\|_W\,.$$

It follows that the set $\Sigma = \{E(\Delta|t,s)\varphi\}$ is bounded in W when $\omega(\Delta) \to 0$. Consequently, Σ has at least one accumulation point in the weak topology of W. On the other hand, since W is continuously embedded in $L^2(\mathbb{R}^n)$, every accumulation point of Σ in W must coincide with $U(t,s)\varphi$. Hence, $U(t,s)\varphi \in W$. □

Theorem 1.6.10. *For any $\varphi \in W$, we have*

$$i\frac{\partial}{\partial t}U(t,s)\varphi = -\frac{1}{2}\Delta U(t,s)\varphi + V(t,x)U(t,s)\varphi\,,$$

in $\mathscr{D}'([s,t]\times\mathbb{R}^n)$.

Proof. We consider a subdivision Δ of the interval $[s,t]$

$$s = t_0 < t_1 < \cdots < t_L = t\,,$$

with $\omega(\Delta)$ sufficiently small. For any $\sigma \in [s,t]$, there exists j such that $\sigma \in [t_j, t_{j+1}]$ and we put

$$F(\sigma) = E(\sigma,t_j)E(t_j,t_{j-1})\cdots E(t_1,s)\,.$$

From Theorem 1.6.5 we know that, for every $\psi \in C_0^\infty(\mathbb{R}^n)$,

$$(F(\sigma)\varphi,\psi) \to (U(\sigma,s)\varphi,\psi)\,, \tag{1.95}$$

when $\omega(\Delta) \to 0$. Moreover, by using Proposition 1.5.11, we have

$$\left(i\partial_t + \frac{1}{2}\Delta - V(t,x)\right)F(\sigma)\varphi = -G(\sigma,t_j)F(t_j)\varphi\,, \tag{1.96}$$

in $\mathscr{D}'([t_j,t_{j+1}]\times\mathbb{R}^n)$. Now, we consider $\lambda \in C_0^\infty([s,t]\times\mathbb{R}^n)$, and we put $f(t,x) = (i\partial_t + \frac{1}{2}\Delta - V(t,x))\lambda(t,x)$. We have

$$\begin{aligned} i\int_{\mathbb{R}^n} F(t)\varphi(x)\lambda(t,x)dx - i\int_{\mathbb{R}^n} F(s)\varphi(x)\lambda(s,x)dx + \int_s^t\int_{\mathbb{R}^n} F(\sigma)\varphi(x)f(\sigma,x)dxd\sigma \\ = i\int_{\mathbb{R}^n} F(t)\varphi(x)\lambda(t,x)dx - i\int_{\mathbb{R}^n} F(s)\varphi(x)\lambda(s,x)dx \\ + \sum_{j=0}^{L-1}\int_{t_j}^{t_{j+1}}\int_{\mathbb{R}^n} F(\sigma)\varphi(x)f(\sigma,x)dxd\sigma \end{aligned} \tag{1.97}$$

By Proposition 1.5.11, we know that

$$\int_{t_j}^{t_{j+1}}\int_{\mathbb{R}^n} F(\sigma)\varphi(x)f(\sigma,x)dxd\sigma = -\int_{t_j}^{t_{j+1}}\int_{\mathbb{R}^n} G(\sigma,t_j)\varphi(x)\lambda(\sigma,x)dxd\sigma + i\int_{\mathbb{R}^n} F(t_j)\varphi(x)\lambda(t_j,x)dx - i\int_{\mathbb{R}^n} F(t_{j+1})\varphi(x)\lambda(t_{j+1},x)dx\,. \tag{1.98}$$

By plugging (1.98) into (1.97), we get

$$i\int_{\mathbb{R}^n} F(t)\varphi(x)\lambda(t,x)dx - i\int_{\mathbb{R}^n} F(s)\varphi(x)\lambda(s,x)dx + \int_s^t\int_{\mathbb{R}^n} F(\sigma)\varphi(x)f(\sigma,x)dxd\sigma = \sum_{j=0}^{L-1}\int_{t_j}^{t_{j+1}}\int_{\mathbb{R}^n} G(\sigma,t_j)\varphi(x)\lambda(\sigma,x)dxd\sigma\,. \tag{1.99}$$

By using (1.77)

$$\left|\sum_{j=0}^{L-1}\int_{t_j}^{t_{j+1}}\int_{\mathbb{R}^n} G(\sigma,t_j)\varphi(x)\lambda(\sigma,x)dxd\sigma\right| \leq \sum_{j=0}^{L-1}\int_{t_j}^{t_{j+1}} \|G(\sigma,t_j)\varphi\|_{L^2(\mathbb{R}^n)}\|\lambda(t)\|_{L^2(\mathbb{R}^n)}d\sigma \lesssim L\omega(\Delta)^{\frac{3}{2}}\|\varphi\|_{L^2(\mathbb{R}^n)}\,. \tag{1.100}$$

Then, by passing to the limit $\omega(\Delta)\to 0$ in (1.99) and by using (1.95) , we get the thesis. □

Remark 1.6.11. *If $\varphi \in W$, it follows that $H(t)U(t,s)\varphi \in L^2(\mathbb{R}^n)$ for a.e. t. By using Theorem 1.6.10, we have that*

$$i\frac{\partial}{\partial t}U(t,s)\varphi = H(t)U(t,s)\varphi\,,$$

in $L^2(\mathbb{R}^n)$ for almost every t.

Chapter 2

Local Smoothing

2.1 Introduction

The goal of this chapter is the study of the local smoothing property for the Schrödinger equation (1.9), under the assumptions (V-I)-(V-III) on the potential $V(t,x)$. The local smoothing estimate refers to the property of solutions to a linear homogeneous dispersive equation to gain some regularity with respect to the initial data, on average in time and locally in space. The local smoothing property represents a useful tool in the analysis of nonlinear problems, for instance in the study of decaying properties of solutions ([67]) and in global existence issues, expecially in the case of the Schrödinger equations with derivatives in the nonlinearity ([75]).

It is well known that the solution to the free Schrödinger equation satisfies the following smoothing estimate

$$\|\langle x\rangle^{-s}(-\Delta)^{\frac{1}{2}}e^{it\Delta}u\|_{L_t^2L_x^2} \leq C\|u\|_{L^2(\mathbb{R}^n)} \qquad s>\frac{1}{2}.$$

This kind of inequality was proved by Ben-Artzi and Klainerman in [13] for $n\geq 3$ and by Chihara in [20] for $n=2$.
Actually the first result concerning global smoothing effect for unitary operators goes back to Kato in [72]. In this work the author introduced the notion of H-smooth operators. Let $L:\mathscr{H}\mapsto\tilde{\mathscr{H}}$ ($\mathscr{H}$ and $\tilde{\mathscr{H}}$ Hilbert spaces) be a densely defined closed operator; we say that L is H-smooth if

$$\sup_{Im\zeta\neq 0}|((H-\zeta)^{-1}L^*\tilde{u},L^*\tilde{u})|\leq C\|\tilde{u}\|^2_{\tilde{\mathscr{H}}}, \quad \tilde{u}\in D(L^*)\subset\tilde{\mathscr{H}},$$

where H is a selfadjoint operator in $\mathscr{H}$ such that the resolvent $(H-\zeta)^{-1}$ is defined (so at least for $Im\zeta\neq 0$). The relationship between H-smooth operators and smoothing effects is the following: an operator L is H-smooth if and only if

$$\int_{-\infty}^{\infty}\|Le^{-itH}u\|^2_{\tilde{\mathscr{H}}}dt\leq C\|u\|_{\mathscr{H}},$$

and in particular, $e^{-itH}u \in D(L)$ for almost every $t \in \mathbb{R}$.
Later on, Strichartz, studying properties of Fourier transform in [115], proved that

$$\left(\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n} |e^{it\Delta}u|^{2(n+2)/n} dx\,dt\right)^{n/2(n+2)} \leq C\|u\|_{L^2(\mathbb{R}^n)},$$

which tells us that, if $u \in L^2(\mathbb{R}^n)$ then $e^{it\Delta}u \in L^{\frac{2(n+2)}{n}}(\mathbb{R}^n)$ for almost every t. This means that the free Schrödinger propagator improves L^p-smoothness.
In [54], the authors improved this result by proving that

$$\left(\int\left\{\int |e^{it\Delta}u|^p dx\right\}^{\frac{\theta}{p}}\right)^{\frac{1}{\theta}} \leq C\|u\|_{L^2(\mathbb{R}^n)},$$

for $0 \leq 2/\theta = n(1/2 - 1/p) < 1$, which means that for every $u \in L^2(\mathbb{R}^n)$, $e^{it\Delta}u \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $2 \leq p < 2n/(n-2)$ for a.e $t \in \mathbb{R}$.
The local smoothing effect was first established by Kato in [71]; he observed that the solutions of the Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad t, x \in \mathbb{R}, \\ u(0) = u_0. \end{cases}$$

with $u(0) = u_0$, have an improvement of the differentiability property. In particular

$$\int_{-T}^{T}\int_{R}^{R} |\partial_x u(x,t)|^2 dx\,dt \leq C(T,R)\|u_0\|^2_{L^2(\mathbb{R}^n)}, \tag{2.1}$$

which means that the solution is one derivative smoother than the inital datum. In [80], Kruzhkov and Faminskii obtained a similar result independently.
Subsequently, but simultaneously, Constantin and Saut ([22]), Sjölin ([113]) and Vega ([121]) found that estimates of the type in (2.1) are possessed by the unitary groups generated by dispersive equations. This is a consequence of the dispersive nature of the linear part of the equation. Most of the physically relevant dispersive equations and systems (K-dV, Benjamin-Ono, Boussinesq, Schrödinger) show the local smoothing property.

If we consider an operator $P(D)$ with real symbol $P(\xi)$, such that $P(\xi) \sim |\xi|^\alpha$ at infinity, for a real positive α, we get

$$\left(\int_{-T}^{T}\int_{|x|\leq R} |(-\Delta)^{(\alpha-1)/4} e^{-itP(D)}u_0(x)|^2 dx\,dt\right)^{\frac{1}{2}} \leq C(T,R)\|u_0\|_{L^2(\mathbb{R}^n)}. \tag{2.2}$$

This last inequality tells us that if $u_0 \in L^2(\mathbb{R}^n)$, the solutions $e^{itP(D)}u_0 \in H^{(\alpha-1)/2}_{loc}(\mathbb{R}^n)$ for almost every t.
In [22], the authors extended Kato's result (2.1) to general linear dispersive equations. On the other hand, in [113] and [121], the authors got inequality (2.2) with

$\alpha = 2$ implicitly, while studying the problem of finding the value of s such that the following limit exists in $H^s(\mathbb{R}^n)$:

$$\lim_{t\downarrow 0} e^{it\Delta}u_0(x) = u_0(x), \text{ for almost every } x \in \mathbb{R}^n .$$

While Kato's original proof of the smoothing effects (2.1) relies on energy estimates, the generalization of the result to general linear dispersive equations is based on a Fourier transform argument.
In particular, for the free Schrödinger propagator, it is showed that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |\phi(t,x)(1-\Delta)^{\frac{1}{4}} e^{it\Delta}u(x)|^2 dx dt \le C\|u\|^2_{L^2(\mathbb{R}^n)}, \quad u \in L^2(\mathbb{R}^n), \tag{2.3}$$

where $\phi \in C_0^\infty(\mathbb{R}^{n+1})$. For Schrödinger equations, the smoothing property has been extended to the case with a potential. There is a wide literature in this direction, when the potential V has enough regularity and decays fast at infinity; we refer to [30, 107, 108] and references therein. Yajima, in [126], extended (2.3) to the problem

$$-i\partial_t u(t,x) - (\nabla - iA(t,x))^2 u + V(t,x)u(t,x) = 0,$$

where $V(t,x)$ is a smooth potential growing subquadratically at infinity and the magnetic potential $A(t,x)$ has a sublinear growth and satisfies suitable regularity assumptions. This result was generalized by Doi in [26, 27] to equations of the form

$$i\partial_t u(t,x) - \sum(-i\partial_{x_j} - iA_j(t,x))g^{jk}(x)(-i\partial_{x_k} - iA_k(t,x))u(t,x) + V(t,x)u(t,x) = 0,$$

where the the metric $g^{jk}(x)$ satsfies suitable assumptions.
In this Chapter we investigate the possibility to extend (2.3) to the propagator of Schrödinger equations (1.9) with a time dependent potential, which may increase subquadratically at infinity but with rough integrability in time. We know, from Chapter 1, that under the assumptions (V-I)-(V-III), the equation (1.9) generates a unique propagator $U(t,s)$ in $L^2(\mathbb{R}^n)$, which gives the unique solution of (1.9).
The analysis led in Chapter 1 is focused on potentials, which are not locally bounded in time as in [36], so they may exhibit local singularities (in time), which however are L^2-integrable. As a preliminary step towards a further investigation of the local smoothing property for equation (1.9), with potentials studied in Chapter 1, we examine the case of a potential with a local singularity in time. More specifically, we deal with the following

$$V(t,x) = \frac{1}{2}t^{-\frac{1}{4}}|x|^2 . \tag{2.4}$$

We stress here that the potential (2.4) does not fall in the class considered in [126]; indeed, with this choice, we are relaxing the assumption on the integrability with respect to time by considering a potential $V(t,x)$ which belongs to $L^2_t L^\infty_{loc,x}$ and not to $L^\infty_t L^\infty_{loc,x}$.
The result we would like to prove, by proceeding as in [126], is the following

Theorem 2.1.1. *Let $V(t,x)$ be defined by* (2.4). *Let $T > 0$ be sufficiently small, $\mu > \frac{1}{2}$ and $\rho \geq 0$. Then there exists a constant $C_{\rho\mu} > 0$ such that for $s \in \mathbb{R}$*

$$\int_{s-T}^{s+T} \|\langle x\rangle^{-\mu-\rho}\langle D\rangle^{\rho} U(t,s)f\|^2_{L^2(\mathbb{R}^n)} \leq C_{\rho\mu}\|\langle D\rangle^{\rho-\frac{1}{2}} f\|_{L^2(\mathbb{R}^n)} \quad f \in \mathscr{S}(\mathbb{R}^n), \tag{2.5}$$

where $\langle x\rangle = \sqrt{1+|x|^2}$, $\langle D\rangle = (1-\Delta)^{\frac{1}{2}}$ and $U(t,s)$ is the propagator associated to the equation (1.9)*, with $V(t,x)$ defined by* (2.4).

2.2 Representation formula for the propagator

In [126], the proof of Theorem 2.1.1 is strongly based on the representation formula of the propagator $U(t,s)$ of (1.9) as an oscillatory integral operator, which is derived in this Section.

Let $T > 0$. We say that a function $a(t,s,x,y)$ belongs to the amplitude class *Amp*, $a(t,s,x,y) \in Amp$, when for any α and β,

$$|\partial_x^{\alpha}\partial_y^{\beta} a(t,s,x,y)| \leq C_{\alpha\beta}, \quad |t-s| < T, \quad x,y \in \mathbb{R}^n.$$

We denote with $I(t,s,a)$ the oscillatory integral operator with the phase $S(t,s,x,y)$ and amplitude $a(t,s,x,y)$:

$$I(t,s,a)f(x) = \frac{1}{(2\pi i(t-s))^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{iS(t,s,x,y)}a(t,s,x,y)f(y)dy.$$

We want to show that the propagator $U(t,s)$ for (1.9) admits the integral representation $U(t,s) = I(t,s;k)$, where $k \in Amp$. We denote with $\|\cdot\|_m$ the norm defined by

$$\|f\|_m = \sum_{|\alpha|+|\beta|\leq m}\sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n}\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial y}\right)^{\beta} f(x,y)\right|,$$

for $m = 0,1,2,\ldots$ The following Lemma contains some useful properties of oscillatory integral operators.

Lemma 2.2.1. *There exists a positive number T such that, for $|t-s| \leq T$, the following statements hold true.*

(i) There exists a constant $\gamma_0(T)$ such that, for $|t-s| \leq T$

$$\|I(t,s;a)\varphi\|_{L^2(\mathbb{R}^n)} \leq \gamma_0(T)\|a\|_{2n+1}|\varphi\|_{L^2(\mathbb{R}^n)},.$$

(ii) For $a, b \in Amp$, there exists $c \in Amp$ such that

$$I(t,r;a)I(r,s;b) = I(t,s;c).$$

(iii) For subdivisions $\Omega : s = s_0 < s_1 < \cdots < s_L = t$ and $a_1, a_2, \ldots, a_L \in Amp$, there exists $a \in Amp$ such that $I(t, s_{L-1}; a_L) \cdot I(s_{L-1}, s_{L-2}; a_{L-1}) \cdots I(s_1, s; a_1) = I(t, s; a)$. For $m = 0, 1, \ldots$,

$$\|a(t, s_{L-1}, \ldots, s_1, s)\|_m \leq \kappa(m)^L \prod_{j=1}^{L} \|a_j\|_{R(m)},$$

where $\kappa(m)$ and $R(m)$ are some positive constants, independent of L, the subdivision Ω and functions $a_j \in Amp$ (here $a(t, s_{L-1}, \ldots, s_1, s)$ is just a notation to denote the amplitude function a, as in the equation A.7 in [38]).

Proof. See Theorem *A*.2 in [38]. □

In the remaining part of this section, we assume that $|t-s|$ is sufficiently small, in order to exploit the results of Chapter 1.
We know from the previous Chapter, that the integral operator

$$E(t,s)\varphi(x) = \frac{1}{(2\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{iS(t,s,x,y)} \varphi(y) dy\,,$$

is an approximate solution of the Schrödinger equation (1.9), in the sense that

$$i\partial_t E(t,s)\varphi = \left(\frac{1}{2}\Delta + V\right) E(t,s)\varphi - G(t,s)\varphi\,,$$

in $L^2(\mathbb{R}^n)$ for almost every t, where

$$G(t,s)\varphi(x) = \frac{i\sqrt{t-s}}{(2\pi i(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{iS(t,s,x,y)} \Delta_x \omega(t,s,x,y)\varphi(y) dy\,. \tag{2.6}$$

Since $U(t,s)$ is the fundamental solution of (1.9), we can use Duhamel's formula to write

$$E(t,s)\varphi = U(t,s)\varphi + i \int_s^t U(t,\sigma) G(\sigma,s)\varphi d\sigma\,. \tag{2.7}$$

So, we are looking for an operator-valued function $F(t,s)$, which allows us to write the fundamental solution as

$$U(t,s) = E(t,s) + E(t,s)\#F(t,s)\,, \tag{2.8}$$

where

$$E\#F(t,s) = \int_s^t E(t,\sigma)F(\sigma,s)d\sigma\,.$$

At this point we consider (2.7) as an operator equation for $U(t,s)$ and we solve it by successive approximation. At least formally we have that

$$\begin{aligned} U(t,s) = E(t,s) - i(E\#G)(t,s) + (-i)^2(E\#G\#G)(t,s) \\ + (-i)^3(E\#G\#G)(t,s) + \cdots \end{aligned} \tag{2.9}$$

We have to study the convergence of the right hand side of (2.9) in the uniform operator topology. It follows from (2.6) that

$$G(t,s)=(2i)^{-1}I(t,s;c(t,s)),$$

with

$$c(t,s)=\sqrt{t-s}\Delta_x\omega(t,s,x,y).$$

Moreover, the estimate (1.77) of Chapter 1, gives

$$\|G(t,s)\varphi\|_{L^2(\mathbb{R}^n)}\leq\gamma_1\sqrt{t-s}\|\varphi\|_{L^2(\mathbb{R}^n)}. \tag{2.10}$$

Proposition 2.2.2. *We can estimate the norm of the product of k—factors as*

$$\|G\#\cdots\#G(t,s)\varphi\|_{L^2(\mathbb{R}^n)}\leq\frac{\gamma_1^k\Gamma(\frac{3}{2})^k}{\Gamma(\frac{3}{2}k)}|t-s|^{\frac{3}{2}k-1}\|\varphi\|_{L^2(\mathbb{R}^n)} \tag{2.11}$$

Proof. The proof proceeds by induction. We begin with the case $k=2$; we have

$$G\#G(t,s)=\int_s^t G(t,\sigma)G(\sigma,s)d\sigma.$$

By using (2.10), we get

$$\begin{aligned}\|G\#G(t,s)\|&\leq\gamma_1^2\int_s^t\sqrt{t-\sigma}\sqrt{\sigma-s}d\sigma=\gamma_1^2(t-s)^2\int_0^1(1-x)^{\frac{1}{2}}x^{\frac{1}{2}}dx\\&=\gamma_1^2(t-s)^2\frac{\Gamma(\frac{3}{2})^2}{\Gamma(3)},\end{aligned}$$

where we used in the integral the substitution $\sigma=(t-s)x+s$ and the fact ([57]) that

$$\int_0^1(1-x)^{\alpha-1}x^{\beta-1}dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Now we assume that (2.11) holds for k factors. For $k+1$ factors we have

$$G\#\cdots\#G(t,s)=\int_s^t d\sigma_kG(t,\sigma_k)\int_s^{\sigma_k}G(\sigma_k,\sigma_{k-1})\cdots\int_s^{\sigma_1}d\sigma_1G(\sigma_1,s).$$

By using the inductive hypothesis we get

$$\begin{aligned}\|G\#\cdots\#G(t,s)\|&\leq\frac{\gamma_1^{k+1}\Gamma(\frac{3}{2})^k}{\Gamma(\frac{3}{2}k)}\int_s^t\sqrt{t-\sigma_k}(\sigma_k-s)^{\frac{3}{2}k-1}d\sigma_k\\&\leq\gamma_1^{k+1}|t-s|^{\frac{3}{2}(k+1)-1}\frac{\Gamma(\frac{3}{2})^k}{\Gamma(\frac{3}{2}k)}\int_0^1(1-x)^{\frac{1}{2}}x^{\frac{3}{2}k-1}dx\\&=\gamma_1^{k+1}|t-s|^{\frac{3}{2}(k+1)-1}\frac{\Gamma(\frac{3}{2})^k}{\Gamma(\frac{3}{2}k)}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2}k)}{\Gamma(\frac{3}{2}(k+1))}\\&=\gamma_1^{k+1}|t-s|^{\frac{3}{2}(k+1)-1}\frac{\Gamma(\frac{3}{2})^{k+1}}{\Gamma(\frac{3}{2}(k+1))}.\end{aligned}$$

□

Proposition 2.2.3. *There exists an amplitude function $c_k(t,s,x,y) \in Amp$ such that*

$$G\# \cdots \# G(t,s) = (2i)^{-k} I(t,s;c_k(t,s)),$$

for $k \geq 2$. Moreover for any integer $m > 0$ there exists a constant $\gamma_2 = \gamma_2(m,n,T)$ such that

$$\|c_k(t,s)\|_m \leq \frac{\gamma_2^k}{\Gamma(\frac{3}{2}k)} |t-s|^{\frac{3}{2}k-1}. \tag{2.12}$$

Proof. Let's consider an arbitrary subdivision of $[s,t]$

$$\Delta : s = \sigma_0 < \sigma_1 < \cdots < \sigma_k = t.$$

By applying Lemma 2.2.1 repeatedly, we have that

$$G(t,\sigma_{k-1})G(\sigma_{k-1},\sigma_{k-2}) \cdots G(\sigma_1,s) = (2i)^{-k} I(t,s;b_k(t,\sigma_{k-1},\ldots,\sigma_1,s)),$$

with $b_k(t,\sigma_{k-1},\ldots,\sigma_1,s)) \in Amp$. It follows that

$$G\# \cdots \# G(t,s) = (2i)^{-k} I(t,s;c_k(t,s)),$$

by putting

$$c_k(t,s,x,y) = \int_s^t d\sigma_{k-1} \int_s^{\sigma_{k-2}} \cdots \int_s^{\sigma_1} b_k(t,\sigma_{k-1},\ldots,\sigma_1,s).$$

In order to prove (2.12), we first recall that, by the properties of the function $\omega(t,s,x,y)$, we have

$$\|c(t,s)\|_m \leq A\sqrt{t-s}, \tag{2.13}$$

with $A = A(m,T)$. Then it follows, from (2.13) and Lemma 2.2.1, that

$$\begin{aligned} \|b_k(t,\sigma_{k-1},\ldots,\sigma_1,s)\|_m &\leq \kappa(m)^k \prod_{j=1}^{k} \|c(\sigma_j,\sigma_{j-1})\|_{R(m)} \\ &\leq \kappa(m)^k A_1^k \prod_{j=1}^{k} \sqrt{\sigma_j - \sigma_{j-1}}, \end{aligned} \tag{2.14}$$

with $A_1 = A_1(R(m),T)$. Clearly, by the expression for c_k, it follows that

$$\|c_k(t,s)\|_m \leq \frac{\kappa(m)^k A_1^k \Gamma(\frac{3}{2})^k}{\Gamma(\frac{3}{2}k)} |t-s|^{\frac{3}{2}k-1},$$

which is (2.12) with $\gamma_2 = \kappa(m) A_1 \Gamma(\frac{3}{2})$. □

Proposition 2.2.4. *There exists an amplitude function $k(t,s,x,y) \in Amp$ such that*

$$U(t,s) = I(t,s;k(t,s)).$$

Proof. By using (2.12), it follows that

$$\sum_{k=2}^{\infty} \|(2i)^{-k} c_k(t,s)\|_m \leq \sum_{k=2}^{\infty} \frac{2^{-k}\gamma_2^k}{\Gamma(\frac{3}{2}k)} |t-s|^{\frac{3}{2}k-1} < \infty$$

Indeed, by using the ratio test, with $s_k = \frac{2^{-k}\gamma_2^k}{\Gamma(\frac{3}{2}k)}$, we get

$$\frac{s_{k+1}}{s_k} = \kappa(m) A_1 \Gamma(\frac{3}{2}) \frac{\Gamma(\frac{3}{2}k)}{\Gamma(\frac{3}{2}k+\frac{3}{2})}.$$

By Stirling's formula it follows that

$$\frac{\Gamma(\frac{3}{2}k)}{\Gamma(\frac{3}{2}k+\frac{3}{2})} \sim e^{\frac{3}{2}} \frac{\left(\frac{3}{2}k\right)^{\frac{3}{2}k-\frac{1}{2}}}{\left(\frac{3}{2}k+\frac{3}{2}\right)^{\frac{3}{2}k+1}} \to 0.$$

This proves that the series $\sum_k (2i)^{-k} c_k(t,s,x,y)$ converges in the space $\mathscr{B}(\mathbb{R}^n \times \mathbb{R}^n)$. We denote with $c_\infty(t,s,x,y)$ its limit. It is obvious that

$$\sum_{j=1}^{k} G\# \cdots \# G(t,s) = I\left(t,s; \sum_{j_1}^{k} (2i)^{-j} c_j(t,s)\right). \tag{2.15}$$

We want to pass to the limit $k \to \infty$ in (2.15). The left hand side converges to $F(t,s)$ in the uniform operator topology by Proposition 2.2.2. By using Lemma 2.2.1 and the convergence of the series $\sum_k (2i)^{-k} c_k(t,s,x,y)$ in $\mathscr{B}(\mathbb{R}^n \times \mathbb{R}^n)$, we get that the right hand side of (2.15) converges to $I(t,s;c_\infty(t,s))$ in the uniform operator topology. It follows that

$$F(t,s) = I(t,s;c_\infty(t,s)).$$

We know that $E(t,s) = I(t,s;1)$; so by Lemma 2.2.1 there exists an amplitude function $p(t,\sigma,s)$ such that

$$I(t,s;p(t,\sigma,s)) = I(t,\sigma;1) I(\sigma,s;c_\infty(\sigma,s)).$$

Then

$$E\#F(t,s) = I(t,s;r(t,s)),$$

with

$$r(t,s,x,y) = \int_s^t p(t,\sigma,s,x,y) d\sigma.$$

Thus it follows that

$$U(t,s) = I(t,s;k(t,s)),$$

with $k(t,s) = 1 + r(t,s)$. □

2.3 Local smoothing estimate

As already stressed in the previous Section, we start by constructing the evolution operator associated to (1.9) with the potential (2.4). In order to do this we exploit the results contained in Chapter 1.

Let $H(\tau,x,\xi)=\frac{1}{2}|\xi|^2+\frac{1}{2}\tau^{-\frac{1}{4}}|x|^2$ be the Hamiltonian associated with (1.9). The starting point for the construction of the propagator for (1.9) is to find the solution of the Hamilton's equation

$$\begin{cases} \ddot{x}(\tau)+\tau^{-\frac{1}{4}}x(\tau)=0\,, \\ x(s)=y\,, \\ x(t)=x\,, \end{cases} \tag{2.16}$$

with $x,y\in\mathbb{R}^n$. Equation (2.16) is a generalized form of Bessel's ordinary differential equations (also reminiscent of Airy's equation). In order to see this, we firstly introduce the change of variables $x=\sqrt{\tau}w$; in this way the equation (2.16) becomes

$$\tau^2\ddot{w}(\tau)+\tau\dot{w}(\tau)+\left(\tau^{\frac{7}{4}}-\frac{1}{4}\right)w(\tau)=0\,. \tag{2.17}$$

By putting $\tau^{\frac{7}{8}}=\frac{7}{8}u$ and by rewriting the equation (2.17), we end up with the following

$$u^2w''(u)+uw'(u)+\left(u^2-\frac{16}{49}\right)w(u)=0\,, \tag{2.18}$$

which is the Bessel's ordinary differential equation. It is well known ([1]) that the solution of (2.18) is given by

$$w(u)=c_1J_{4/7}(u)+c_2J_{-4/7}(u)\,, \tag{2.19}$$

where

$$J_\nu(u)=\left(\frac{u}{2}\right)^\nu\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{u}{2}\right)^{2n} \tag{2.20}$$

are the so-called Bessel functions of the first type. Finally the solution of (2.16) reads as

$$x(\tau)=c_1\sqrt{\tau}J_{\frac{4}{7}}\left(\frac{8}{7}\tau^{\frac{7}{8}}\right)+c_2\sqrt{\tau}J_{-\frac{4}{7}}\left(\frac{8}{7}\tau^{\frac{7}{8}}\right). \tag{2.21}$$

In order to determine the constants c_1 and c_2, we impose the boundary conditions $x(s)=y$ and $x(t)=x$ to obtain

$$c_1(t,s,x,y)=\frac{x}{\sqrt{t}a(t,s)}J_{-\frac{4}{7}}\left(\frac{8}{7}s^{\frac{7}{8}}\right)-\frac{y}{\sqrt{s}a(t,s)}J_{-\frac{4}{7}}\left(\frac{8}{7}t^{\frac{7}{8}}\right), \tag{2.22}$$

$$c_2(t,s,x,y)=-\frac{x}{\sqrt{t}a(t,s)}J_{\frac{4}{7}}\left(\frac{8}{7}s^{\frac{7}{8}}\right)+\frac{y}{\sqrt{s}a(t,s)}J_{\frac{4}{7}}\left(\frac{8}{7}t^{\frac{7}{8}}\right), \tag{2.23}$$

with

$$a(t,s)=J_{-\frac{4}{7}}\left(\frac{8}{7}s^{\frac{7}{8}}\right)J_{\frac{4}{7}}\left(\frac{8}{7}t^{\frac{7}{8}}\right)-J_{-\frac{4}{7}}\left(\frac{8}{7}t^{\frac{7}{8}}\right)J_{\frac{4}{7}}\left(\frac{8}{7}s^{\frac{7}{8}}\right). \tag{2.24}$$

Moreover we have

$$\dot{x}(\tau) = c_1 \tau^{\frac{3}{8}} J_{-\frac{3}{7}}\left(\frac{8}{7}\tau^{\frac{7}{8}}\right) - c_2 \tau^{\frac{3}{8}} J_{\frac{3}{7}}\left(\frac{8}{7}\tau^{\frac{7}{8}}\right). \tag{2.25}$$

In order to derive (2.25), we used the following properties of Bessel's functions

$$\frac{d}{du} J_\nu(u) = \frac{1}{2}(J_{\nu-1}(u) - J_{\nu+1}(u)) \tag{2.26}$$

and

$$u J_{\nu+1}(u) = 2\nu J_\nu(u) - u J_{\nu-1}(u). \tag{2.27}$$

We denote with $S(t,s,x,y)$, for $0 < t-s \leq T$ and $x, y \in \mathbb{R}^n$, the classical action integral along the path $x(\tau)$:

$$S(t,s,x,y) = \int_s^t L(\tau, x(\tau), \dot{x}(\tau)) d\tau, \tag{2.28}$$

where $L(\tau, x(\tau), \dot{x}(\tau)) = \frac{1}{2}|\dot{x}(\tau)|^2 - \frac{1}{2}\tau^{-\frac{1}{4}}|x(\tau)|^2$ is the Lagrangian corresponding to the Hamiltonian H. By integrating by parts in (2.28), we get that

$$S(t,s,x,y) = \frac{1}{2}(x\dot{x}(t) - y\dot{x}(s)),$$

that is

$$\begin{aligned} S(t,s,x,y) = \frac{1}{2}\Big\{ & c_1(t,s,x,y)\left(t^{\frac{3}{8}} x J_{-\frac{3}{7}}\left(\frac{8}{7}t^{\frac{7}{8}}\right) - s^{\frac{3}{8}} y J_{-\frac{3}{7}}\left(\frac{8}{7}s^{\frac{7}{8}}\right)\right) \\ & - c_2(t,s,x,y)\left(t^{\frac{3}{8}} x J_{\frac{3}{7}}\left(\frac{8}{7}t^{\frac{7}{8}}\right) - s^{\frac{3}{8}} y J_{\frac{3}{7}}\left(\frac{8}{7}s^{\frac{7}{8}}\right)\right)\Big\}. \end{aligned} \tag{2.29}$$

Once we have defined the classical action, we can construct the propagator $U(t,s)$, by using $S(t,s,x,y)$ as the phase function for the oscillatory integral operator. Indeed, it follows from Section 2.2, that there exists a function $e(t,s,x,y)$ such that for any α and β

$$|\partial_x^\alpha \partial_y^\beta e(t,s,x,y)| \leq C_{\alpha\beta},$$

for $0 < t-s < T$ and $x, y \in \mathbb{R}^n$ and the propagator $U(t,s)$ admits the following representation formula

$$U(t,s)f(x) = \frac{1}{(2\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{iS(t,s,x,y)} e(t,s,x,y) f(y) dy. \tag{2.30}$$

Now we can start with the proof of Theorem 2.1.1. Without loss of generality we may assume $\frac{1}{2} < \mu \leq 1$. We have that

$$\begin{aligned} \|\langle x\rangle^{-\mu} U(t,s)f\|^2_{L^2(\mathbb{R}^n)} = & \frac{1}{(2\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i(S(t,s,x,y)-S(t,s,x,z))} \\ & \langle x\rangle^{-2\mu} e(t,s,x,y)\overline{e(t,s,x,z)} f(y)\overline{f(z)} dy dz dx. \end{aligned} \tag{2.31}$$

By straightforward computations it follows that

$$
\begin{aligned}
&S(t,s,x,y)-S(t,s,x,z)\\
&=(z-y)\Bigg\{\frac{x}{2}\Bigg[\frac{t^{\frac{3}{8}}}{\sqrt{s}a(t,s)}\Big(J_{-\frac{3}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)J_{-\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)+J_{\frac{3}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)J_{\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)\Big)\\
&+\frac{s^{\frac{3}{8}}}{\sqrt{t}a(t,s)}\Big(J_{-\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{-\frac{4}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)+J_{\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{\frac{4}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)\Big)\Bigg]\\
&-\frac{s^{\frac{3}{8}}}{\sqrt{s}a(t,s)}\frac{y+z}{2}\Big(J_{-\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{-\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)+J_{\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)\Big)\Bigg\}
\end{aligned}
\tag{2.32}
$$

Now we make the change of variables $x=\xi$, with

$$
\begin{aligned}
\xi=&\frac{x}{2}\Bigg[\frac{t^{\frac{3}{8}}}{\sqrt{s}a(t,s)}\Big(J_{-\frac{3}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)J_{-\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)+J_{\frac{3}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)J_{\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)\Big)\\
&+\frac{s^{\frac{3}{8}}}{\sqrt{t}a(t,s)}\Big(J_{-\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{-\frac{4}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)+J_{\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{\frac{4}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)\Big)\Bigg]\\
&-\frac{s^{\frac{3}{8}}}{\sqrt{s}a(t,s)}\frac{y+z}{2}\Big(J_{-\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{-\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)+J_{\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)\Big)
\end{aligned}
\tag{2.33}
$$

and the right handside of (2.31) becomes

$$
\begin{aligned}
\frac{A(t,s)}{(2\pi(t-s))^n}\int_{\mathbb{R}^n}&e^{i(z-y)\xi}\langle G(t,s,y,z,\xi)\rangle^{-2\mu}e(t,s,x,y)\overline{e(t,s,x,z)}\\
&f(y)\overline{f(z)}dydzd\xi\,,
\end{aligned}
\tag{2.34}
$$

where

$$
\begin{aligned}
G(t,s,y,z,\xi)=&\,2\frac{\sqrt{ts}a(t,s)}{h(t,s)}\xi\\
&+\frac{\sqrt{ts}^{\frac{3}{8}}\Big(J_{-\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{-\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)+J_{\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)\Big)}{h(t,s)}\Big(\frac{y+z}{2}\Big),
\end{aligned}
\tag{2.35}
$$

with

$$
\begin{aligned}
h(t,s)=&\,t^{\frac{7}{8}}\Big(J_{-\frac{3}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)J_{-\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)+J_{\frac{3}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)J_{\frac{4}{7}}\Big(\frac{8}{7}t^{\frac{7}{8}}\Big)\Big)\\
&+s^{\frac{7}{8}}\Big(J_{-\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{-\frac{4}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)+J_{\frac{3}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)J_{\frac{4}{7}}\Big(\frac{8}{7}s^{\frac{7}{8}}\Big)\Big)
\end{aligned}
\tag{2.36}
$$

and

$$
A(t,s)=\left|2\frac{\sqrt{ts}a(t,s)}{h(t,s)}\right|^n. \tag{2.37}
$$

By using Taylor expansion, exploiting the fact that $(t-s)$ is sufficiently small, and (2.27) we get

$$\sqrt{ts}a(t,s)=(t-s)s^{\frac{7}{8}}\left(J_{-\frac{4}{7}}\left(\frac{8}{7}s^{\frac{7}{8}}\right)J_{-\frac{3}{7}}\left(\frac{8}{7}s^{\frac{7}{8}}\right)+J_{\frac{4}{7}}\left(\frac{8}{7}s^{\frac{7}{8}}\right)J_{\frac{3}{7}}\left(\frac{8}{7}s^{\frac{7}{8}}\right)\right)+o((t-s)^2)\,. \tag{2.38}$$

Then, for $t-s$ small,

$$G(t,s)\sim(t-s)\xi+\frac{y+z}{2}\,, \tag{2.39}$$

and

$$A(t,s)\sim(t-s)^n\,. \tag{2.40}$$

So, by using (2.39) and (2.40), we obtain that, for $(t-s)$ small, the right handside of (2.31) behaves like

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{i(z-y)\xi}\langle(t-s)\xi+\frac{y+z}{2}\rangle^{-2\mu}e(t,s,x,y)\overline{e(t,s,x,z)}f(y)\overline{f(z)}dydxd\xi\,. \tag{2.41}$$

In order to complete the proof of Theorem 2.1.1, we need to estimate

$$\int_{s-T}^{s+T}\|\langle x\rangle^{-\mu}U(t,s)f\|^2_{L^2(\mathbb{R}^n)}dt\,.$$

First of all note that

$$\begin{aligned}\int_{s-T}^{s+T}\langle(t-s)\xi+\frac{y+z}{2}\rangle^{-2\mu}dt&\le\int_{-\infty}^{+\infty}\langle(t-s)\xi+\frac{y+z}{2}\rangle^{-2\mu}dt\\&\lesssim\langle\xi\rangle^{-1}\,.\end{aligned} \tag{2.42}$$

This follows from the change of variables $t'=|t-s||\xi|-\frac{y+z}{2}$ and the fact that $\mu>\frac{1}{2}$. By (2.42) and the boundedness of the function $e(t,s,x,y)$, we have that

$$F(s,y,\xi,z)=\int_{s-T}^{s+T}\langle(t-s)\xi+\frac{y+z}{2}\rangle^{-2\mu}e(t,s,x,y)\overline{e(t,s,x,z)}dt \tag{2.43}$$

is such that $\{\langle\xi\rangle F(s,\cdot,\cdot,\cdot)\,:\,s\in\mathbb{R}\}$ is bounded in $\mathscr{B}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$. Finally, by applying the Calderon-Vaillancourt theorem ([119]), we get

$$\begin{aligned}&\int_{s-T}^{s+T}\|\langle x\rangle^{-\mu}U(t,s)f\|^2_{L^2(\mathbb{R}^n)}dt\\&\quad=(2\pi)^{-n}\int e^{i(z-y)\cdot\xi}F(t,s,y,\xi,z)f(y)\overline{f(z)}dydzd\xi\le C\|\langle D\rangle^{-\frac{1}{2}}f\|^2_{L^2(\mathbb{R}^n)}\,.\end{aligned} \tag{2.44}$$

This concludes the proof of Theorem 2.1.1.

Chapter 3

Maxwell-Schrödinger system

3.1 Presentation of the problem

In this chapter we investigate the existence of local and global in time solutions to the 3–D nonlinear Maxwell-Schrödinger system. Let us consider a non-relativistic quantum particle, described by a wave function u, interacting with the self-generated (classical) electro-magnetic field and subject to a self-consistent power-like interaction potential, of the form $|u|^{2(\gamma-1)}u$. The Maxwell's equations for the electromagnetic fields read as

$$\begin{cases} \operatorname{div} E = \rho\,, \\ \operatorname{div} B = 0\,, \\ \partial_t B = -\nabla \wedge E\,, \\ \partial_t E = \nabla \wedge B - J\,, \end{cases} \tag{3.1}$$

where E and B stand for the electric and magnetic fields respectively, ρ and J are the total electric and current charge densities, respectively (all the physical constants are normalized to one). From the equation $\operatorname{div} B = 0$ we may infer that B is the curl of a vector field A, that is $B = \nabla \wedge A$. By plugging this expression for B into the Faraday's law we obtain

$$\nabla \wedge (E + \partial_t A) = 0\,, \tag{3.2}$$

which implies that $E + \partial_t A$ must be the gradient of a scalar field φ, that is $E = -\partial_t A - \nabla\varphi$. Putting all togehter we get

$$\begin{cases} B = \nabla \wedge A, \quad E = -\partial_t A - \nabla\varphi\,, \\ -\partial_t \operatorname{div} A - \Delta\varphi = \rho\,, \\ \partial_{tt} A - \Delta A + \nabla(\partial_t\varphi + \operatorname{div} A) = J\,. \end{cases} \tag{3.3}$$

It is well known that the choice of the potential vector field (φ, A) is unique up to the gauge transformation

$$A' = A + \nabla\eta\,, \quad \varphi' = \varphi - \partial_t\eta\,,$$

where η is any scalar field. We will adopt the *Coulomb gauge*, namely $\mathrm{div}A = 0$. Hence, the system (3.3) becomes

$$\begin{cases} B = \nabla \wedge A, \quad E = -\partial_t A - \nabla \varphi\,, \\ \partial_{tt} A - \Delta A = \mathbb{P}J\,, \\ -\Delta \varphi = \rho\,, \end{cases} \tag{3.4}$$

where $\mathbb{P} = \mathbb{I} - \nabla\,\mathrm{div}\,\Delta^{-1}$ is the Leray-Helmholtz projection operator onto divergence free vector fields. We recall here that if $f = \nabla g + k$, where $\mathrm{div}\,k = 0$, then $\mathbb{P}f = k$. It follows that the term $\nabla \partial_t \varphi$ in the third equation of (3.3) is dropped by operating $\mathbb{P}$.

In this chapter we investigate the existence of local and gloabl solutions to the following Maxwell-Schrödinger system

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta_A u + \varphi u + |u|^{2(\gamma-1)} u \\ \Box A = \mathbb{P}J(u,A) \end{cases} \tag{3.5}$$

with the initial data

$$u(0) = u_0,\ A(0) = A_0,\ \partial_t A(0) = A_1.$$

Here all the physical constants are normalized to 1, $\Delta_A = (\nabla - iA)^2$ denotes the magnetic Laplacian, $\varphi = \varphi(\rho) = (-\Delta)^{-1}\rho$, with $\rho := |u|^2$, represents the Hartree-type electrostatic potential, while the power nonlinearity describes the self-consistent interaction potential. $J(u,A) = \mathrm{Im}(\bar{u}(\nabla - iA)u)$ is the electric current density.

As already said, the Maxwell-Schrödinger system

$$\begin{cases} i\partial_t u = -\dfrac{1}{2}\Delta_A u + \varphi u \\ -\Delta\varphi - \partial_t\,\mathrm{div}A = \rho \\ \Box A + \nabla(\partial_t \varphi + \mathrm{div}A) = J, \end{cases} \tag{3.6}$$

is used in the literature to describe the dynamics of a charged non-relativistic quantum particle, subject to its self-generated (classical) electro-magnetic field, see for instance [35, 109]. In particular the Maxwell-Schrödinger system (3.6) can be seen as a classical approximation to the quantum field equations for an electro-dynamical non-relativistic many body system. It is well known to be invariant under the gauge transformation

$$(u, A, \varphi) \mapsto (u', A', \varphi') = (e^{i\lambda}u, A + \nabla\lambda, \varphi - \partial_t \lambda), \tag{3.7}$$

therefore for our convenience we can decide to work in the Coulomb gauge, namely by assuming $\mathrm{div}A = 0$. Consequently under this gauge the system (3.6) takes the form

$$\begin{cases} i\partial_t u = -\dfrac{1}{2}\Delta_A u + ((4\pi|x|)^{-1} * |u|^2)u \\ \Box A = \mathbb{P}J(u,A). \end{cases}$$

It is straightforward to verify that also power-type nonlinearities of the previous form are gauge invariant.
The Maxwell-Schrödinger system (3.6) has been widely studied in the mathematical literature in the various choice of gauges. For instance, among the first mathematical treatments, we mention [96, 120], where the authors studied the local and global well-posedness in high regularity spaces by means of the Lorentz gauge. The global existence of finite energy weak solutions has been investigated in [62], by using the method of vanishing viscosity. However the uniqueness and the global well-posedness of the finite energy weak solutions is not easily achievable with this approach. In [97, 98], by using the semigroup associated to the magnetic Laplacian following Kato's theory [69, 70] and hence by means of a fixed point argument, the authors obtained global well-posedness with higher order Sobolev regularity.

More recently a global well-posedness result in the energy space has been proven in [12] by using the analysis of a short time wave packet parametrix for the magnetic Schrödinger equation and the related linear, bilinear, and trilinear estimates. Therefore strong H^1 solutions to (3.6) are obtained as the unique strong limit of H^2 solutions. Moreover, in the same paper, the authors obtained a continuous dependence on initial data in the energy space. The asymptotic behavior and the long-range scattering of solutions to (3.6) has been studied for instance in [52, 53, 110] (see also the references therein). The global well-posedness in the space of energy for the 2D Maxwell-Schrödinger system in Lorentz gauge has been investigated by [122] .
We focus on the Cauchy problem for the Maxwell-Schrödinger system with a power-type nonlinearity; our interest in this problem is motivated by the possibility to develop a general theory for quantum fluids in presence of self-induced electromagnetic interacting fields. The related Quantum Magneto-Hydrodynamic (QMHD) systems, with a nontrivial pressure tensor, arise in the description of quantum plasmas, for example in astrophysics, where magnetic fields and quantum effects are non negligible, see [63, 64, 111, 112] and the references therein. The hydrodynamic equations describing a bipolar gas of ions and electrons can be recovered from the Maxwell-Schrödinger system (3.5) by applying the Madelung transforms as done in [6], where the authors studied a general class of quantum fluids in the non-magnetic case. We refer to the Section 3.5 for a more detailed discussion concerning the connection between QMHD and the Maxwell-Schrödinger system (3.5).
We state in the sequel the two main results of this chapter. The first one regards the local well-posedness theory for (3.5) in $H^2(\mathbb{R}^3)\times H^{\frac{3}{2}}(\mathbb{R}^3)$. More precisely let us denote by

$$X := \left\{(u_0,A_0,A_1)\in H^2(\mathbb{R}^3)\times H^{3/2}(\mathbb{R}^3)\times H^{1/2}(\mathbb{R}^3)\text{ s.t. } \operatorname{div}A_0=\operatorname{div}A_1=0\right\}. \tag{3.8}$$

Theorem 3.1.1 (Local wellposedness). *Let $\gamma > \frac{3}{2}$. For all $(u_0,A_0,A_1)\in X$ there exist a (maximal) time $0 < T_{max} \le \infty$, depending on $\|(u_0,A_0,A_1)\|_X$, and a unique (maximal) solution (u,A) to (3.5) such that*

- $u \in C([0,T_{max});H^2(\mathbb{R}^3))$.
- $A \in C([0,T_{max});H^{\frac{3}{2}}(\mathbb{R}^3)) \cap C^1([0,T_{max});H^{\frac{1}{2}}(\mathbb{R}^3))$, $\mathrm{div}A = 0$.
- *The solution depends continuously on the initial data in the following sense:if we consider a sequence of initial data $\{(u_{0,n},A_{0,n},A_{1,n})\}$ converging to $\{(u_0,A_0,A_1)\}$ in X as $n \to \infty$, then the corresponding sequence of solutions $\{(u_n,A_n,\partial_t A_n)\}$ with initial data $\{(u_{0,n},A_{0,n},A_{1,n})\}$ converges to $\{(u,A,\partial_t A)\}$ in $C(I;X)$ for any compact interval $I \subset [0,T_{max})$.*

The following blowup alternative holds: either $T_{max} = \infty$ or $T_{max} < \infty$ and

$$\lim_{t \to T_{max}^-} (\|u(t)\|_{H^2} + \|A(t)\|_{H^{3/2}} + \|\partial_t A(t)\|_{H^{1/2}}) = \infty.$$

Our proof plays on the construction of the evolution operator associated to the magnetic Laplacian, based on Kato's approach [69, 70], then we perform a fixed point argument to approximate the solutions to the Maxwell-Schrödinger system by the classical Picard iteration. Differently from [97], in our case the solutions obtained by this method cannot be extended globally in time, indeed the power-type nonlinearity does not lead to a Gronwall type inequality capable to bound the higher order norms of the solution at any time, see also [103] for a similar problem. To circumvent this difficulty we regularize the system (3.5) by making use of the so-called Yosida approximations of the identity; hence we are able to get the global well-posedness for the approximating system in $H^2(\mathbb{R}^3) \times H^{3/2}(\mathbb{R}^3)$. Moreover, by using the uniform bounds provided by the higher order energy, defined by the norm of X, we prove the existence of a finite energy weak solution to (3.5), in the sense defined in [62]. This is established by the following theorem.

Theorem 3.1.2 (Global Weak Solutions)**.** *Let $1 < \gamma < 3$, $(u_0,A_0,A_1) \in X$, then there exists, globally in time, a finite energy weak solution (u,A) to* (3.5)*, such that $u \in L^\infty(\mathbb{R}_+;H^1(\mathbb{R}^3))$, $A \in L^\infty(\mathbb{R}_+;H^1(\mathbb{R}^3)) \cap W^{1,\infty}(\mathbb{R}_+;L^2(\mathbb{R}^3))$.*

Remark 3.1.3. *The same results can be obtained in a straightforward way, by using the previous results on the Coulomb gauge, in any other admissible gauge.*

Remark 3.1.4. *It is possible to include a Hartree (nonlocal) nonlinear potential of the form $(|\cdot|^{-\alpha} * |u|^2)u$, with $0 < \alpha < 3$. It can be dealt in the same fashion as for power nonlinearities.*

With those results at hand, we want to develop a suitable theory in the energy space for the QMHD system (3.61). The major obstacle in this direction, which is also the major difference with respect to the usual QHD theory, regards the possibility to give sense to the nonlinear term related to the Lorentz force. For sake of simplicity let us consider the case without the nonlinear potential. Let us recall the definition of the macroscopic hydrodynamic variables via the so-called Madelung transformations, namely

$$\rho := |u|^2 \qquad J := \mathrm{Re}(\bar{u}(-i\nabla - A)u)$$

From the Maxwell equations we have

$$E = -\partial_t A - \nabla\varphi \qquad B = \nabla \wedge A \qquad F_L := \rho E + J \wedge B,$$

where E, B, F_L, φ denote the Electric field, the Magnetic field, the Lorentz force and the (scalar) electrostatic potential, respectively. The fields equations are supplemented by the involution of the magnetic field and in the Coulomb gauge by the Poisson equation (here all the physical constants are normalized to one), namely

$$\operatorname{div} B = 0, \qquad \operatorname{div} E = -\Delta\phi = \rho$$

The usual energy estimates on the Maxwell-Schrödinger system (3.5), as we will see in the Section 3.5, lead to $\frac{J}{\sqrt{\rho}} \in L_t^\infty L_x^2$, $\nabla\sqrt{\rho} \in L_t^\infty L_x^2$, $J \in L_t^\infty L_x^{3/2}$, $B \in L_t^\infty L_x^2$, $\nabla \wedge J \in L_t^\infty L_x^1 \cap L_t^\infty W_x^{-1,3/2}$. Unfortunately these bounds are not sufficient to apply the compensated compactness of Tartar [93, 94, 118] and in particular the argument in the Lecture 40 of [117], indeed $J \notin L_x^2$ and $B \notin L_x^3$ (the boundedness in at least one of these norms would be sufficient). Therefore the analysis of the Lorentz force for finite energy solutions needs still to be better understood. In [9], the authors investigate the weak stability of the Lorentz force by a detailed frequency analysis, in the case of incompressible dynamics (where $J \in L^2$). In [12] the authors obtain a global well-posedness result, in the sense that finite energy strong solutions are the unique limit of H^2 regular solutions, but however these solutions do not allow to treat the Lorentz force term. The results of [97, 98], obtained without the nonlinear potential, include global well-posedness in higher order Sobolev spaces which, combined with the methods of [6, 8], allows instead to analyze the pressureless QMHD case.
The additional difficulty introduced by the power nonlinearity in the Maxwell-Schrödinger system (3.5) in 3–D, namely a nonlinear pressure term in the QMHD system, cannot be easily managed. Usually the proof of higher order well-posedness for the NLS, combines higher order energy estimates with the use of sharp Strichartz estimates. However, to our knowledge, there are not intrinsic Strichartz estimates for (3.5); actually there are many Strichartz estimates available in the literature for the Schrödinger equations with a prescribed magnetic potential, but our solution to (3.5) does not fall in that class. On the other hand, a brute force higher order energy estimate would end up in a superlinear Gronwall inequality and hence into an upper bound which blows up in finite time.
Our theory deals with the presence of a hydrodynamic pressure and it will provide the existence of local in time finite energy weak solutions for the QMHD.

This chapter is organized as follows. In Section 3.2 we collect some estimates which will be used afterwards and we study the evolution operator associated to the linear magnetic Schrödinger equation. In Section 3.3 we prove Theorem 3.1.1. In Section 3.4 we introduce an approximating system to (3.5) for which we show global existence of solutions and then we pass to the limit, proving Theorem 3.1.2. Finally, in Section 3.5 we discuss about the application of our main results to the existence theory for the QMHD system.

3.2 Notation and Preliminaries

In this Section we introduce the notation and we review some preliminary results we are going to use throughout the chapter.
Let A, B be two quantities, we say $A \lesssim B$ if $A \leq CB$ for some constant $C > 0$. We denote by $L^p(\mathbb{R}^3)$ the usual Lebesgue spaces, $H^{s,p}(\mathbb{R}^3)$ are the Sobolev spaces defined throught the norms $\|f\|_{H^{s,p}} := \|(1-\Delta)^{s/2} f\|_{L^p}$. For a given reflexive Banach space $\mathscr{X}$ we let $C([0,T];\mathscr{X})$ (resp. $C^1([0,T];\mathscr{X})$)) denote the space of continuous (resp. differentiable) maps $[0,T] \to \mathscr{X}$. Analogously, $L^p(0,T;\mathscr{X})$ is the space of functions whose Bochner integral $\|f\|_{L^p(0,T;\mathscr{X})} := \left(\int_0^T \|f(t)\|_{\mathscr{X}}\, dt\right)^{1/p}$ is finite.

Lemma 3.2.1 (Generalized Kato-Ponce inequality). *Suppose* $1 < p < \infty$, $s \geq 0$, $\alpha \geq 0$, $\beta \geq 0$ *and* $\frac{1}{p} = \frac{1}{p_i} + \frac{1}{q_i}$ *with* $i = 1, 2$, $1 < q_1 \leq \infty$, $1 < p_2 \leq \infty$. *Setting* $\Lambda^s = (I - \Delta)^{\frac{s}{2}}$ *we have*

$$\|\Lambda^s(f_1 f_2)\|_{L^p(\mathbb{R}^3)} \lesssim \|\Lambda^{s+\alpha}(f_1)\|_{L^{p_1}(\mathbb{R}^3)} \|\Lambda^{-\alpha}(f_2)\|_{L^{q_1}(\mathbb{R}^3)} + \|\Lambda^{-\beta}(f_1)\|_{L^{p_2}(\mathbb{R}^3)} \|\Lambda^{s+\beta}(f_2)\|_{L^{q_2}(\mathbb{R}^3)}$$

Proof. Those estimates are generalization of Kato-Ponce commutator estimates, for a proof of this Lemma see for example Theorem 1.4 in [79]. □

Lemma 3.2.2. *Let* p, q *be such that* $1 \leq q < \frac{3}{2} < p \leq \infty$, *then*

$$\|(-\Delta)^{-1} f\|_{L^\infty} \lesssim \|f\|_{L^p}^{\theta} \|f\|_{L^q}^{1-\theta}, \tag{3.9}$$

where $\theta \in (0,1)$ *is given by* $\theta = \frac{p'(3-2q)}{3(q-p'(q-1))}$. *Furthermore, the following estimates hold*

$$\|(-\Delta)^{-1}(f_1 f_2) f_3\|_{L^2(\mathbb{R}^3)} \lesssim \|f_1\|_{L^2(\mathbb{R}^3)} \|f_2\|_{L^3(\mathbb{R}^3)} \|f_3\|_{L^3(\mathbb{R}^3)} \tag{3.10}$$

$$\|(-\Delta)^{-1} |f|^2\|_{L^\infty(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)}^2 + \|f\|_{L^\infty(\mathbb{R}^3)}^2 \tag{3.11}$$

Proof. Let $R > 0$, then we have

$$4\pi((-\Delta)^{-1} f)(x) = \int \frac{1}{|y|} f(x-y)\, dx = \int_{|y|<R} \frac{1}{|y|} f(x-y)\, dx + \int_{|y|\geq R} \frac{1}{|y|} f(x-y)\, dx,$$

then by Hölder's inequality we have

$$\|(-\Delta)^{-1} f\|_{L^\infty} \lesssim \left(\int_{|y|<R} |y|^{-p'}\, dy\right)^{1/p'} \|f\|_{L^p} + \left(\int_{|y|\geq R} |y|^{-q'}\, dy\right)^{1/q'} \|f\|_{L^q}.$$

The two integrals on the right hand side are finite by the assumptions on p, q. By optimizing the above inequality in R we then get (3.9). To prove (3.10) we apply Hölder and Hardy-Littlewood-Sobolev inequality to get

$$\|(-\Delta)^{-1}(f_1 f_2) f_3\|_{L^2} \leq \|(-\Delta)^{-1}(f_1 f_2)\|_{L^6} \|f_3\|_{L^3} \lesssim \|f_1 f_2\|_{L^{6/5}} \|f_3\|_{L^3}.$$

Using again Hölder inequality for $\|f_1 f_2\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}$ we get (3.10).
Inequality (3.11) follows from (3.9) by choosing $p = \infty, q = 1$ and by applying Young's inequality. □

Next Lemma will be useful to estimate the Hartree term in the fixed point argument in Section 3.3.

Lemma 3.2.3. *Let $u \in H^2(\mathbb{R}^3)$, then*

$$\|(-\Delta)^{-1}(|u|^2)u\|_{H^2} \lesssim \|u\|_{H^{3/4}}^2 \|u\|_{H^2}. \tag{3.12}$$

Proof. We have

$$\begin{aligned}\|(-\Delta)^{-1}(|u|^2)u\|_{H^2} \lesssim & \|(-\Delta)^{-1}(1-\Delta)(|u|^2)u\|_{L^2} + \|(-\Delta)^{-1}(|u|^2)(1-\Delta)u\|_{L^2} \\ \lesssim & \|(-\Delta)^{-1}(1-\Delta)|u|^2\|_{L^6}\|u\|_{L^3} + \|(-\Delta)^{-1}|u|^2\|_{L^\infty}\|(1-\Delta)u\|_{L^2}.\end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality we have

$$\|(-\Delta)^{-1}(1-\Delta)|u|^2\|_{L^6} \lesssim \|(1-\Delta)|u|^2\|_{L^{6/5}} \lesssim \|u\|_{L^3}\|(1-\Delta)u\|_{L^2},$$

where the last inequality follows from Lemma 3.2.1. On the other hand, by using (3.9), with p, q sufficiently close to $\frac{3}{2}$, and Sobolev embedding we see that

$$\|(-\Delta)^{-1}|u|^2\|_{L^\infty} \lesssim \|u\|_{H^{\frac{1}{2}+\varepsilon}}^2.$$

Consequently,

$$\|(-\Delta)^{-1}(|u|^2)u\|_{H^2} \lesssim \|u\|_{H^{\frac{1}{2}+\varepsilon}}^2 \|u\|_{H^2}.$$

□

Lemma 3.2.4. *Let $A \in H^1(\mathbb{R}^3)$ and $u \in H^2(\mathbb{R}^3)$. Then the following estimates hold:*

$$\|(\nabla - iA)u\|_{H^1(\mathbb{R}^3)} \lesssim (1 + \|A\|_{H^1(\mathbb{R}^3)})\|u\|_{H^2(\mathbb{R}^3)}, \tag{3.13}$$

$$\|\mathbb{P}J(u,A)\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \lesssim \|u\|_{H^1(\mathbb{R}^3)}\|u\|_{H^2(\mathbb{R}^3)} + \|A\|_{H^1(\mathbb{R}^3)}\|u\|_{H^2(\mathbb{R}^3)}^2, \tag{3.14}$$

$$\|\Delta_A u\|_{L^2} \lesssim \|u\|_{H^2} + \|A\|_{H^1}^4 \|u\|_{L^2}, \tag{3.15}$$

$$\|u\|_{H^2} \lesssim \|\Delta_A u\|_{L^2} + \|A\|_{H^1}^4 \|u\|_{L^2}, \tag{3.16}$$

$$\|(\nabla + iA)u\|_{L^6} \lesssim \|u\|_{H^2} + \|A\|_{H^1}^4 \|u\|_{L^2}. \tag{3.17}$$

Proof. We begin with the proof of (3.13). By using Lemma 3.2.1 we have

$$\begin{aligned}\|(\nabla - iA)u\|_{H^1(\mathbb{R}^3)} &\leq \|\nabla u\|_{H^1(\mathbb{R}^3)} + \|Au\|_{H^1(\mathbb{R}^3)} \\ &\lesssim \|u\|_{H^2(\mathbb{R}^3)} + \|A\|_{H^1(\mathbb{R}^3)}\|u\|_{L^\infty(\mathbb{R}^3)} + \|A\|_{L^6(\mathbb{R}^3)}\|u\|_{W^{1,3}(\mathbb{R}^3)} \\ &\lesssim \|u\|_{H^2(\mathbb{R}^3)} + \|A\|_{H^1(\mathbb{R}^3)}\|u\|_{H^2(\mathbb{R}^3)} + \|A\|_{H^1(\mathbb{R}^3)}\|u\|_{H^{\frac{3}{2}}(\mathbb{R}^3)},\end{aligned}$$

where in the last inequality we used the Sobolev embedding theorem. Thus (3.13) is proved. We now consider (3.14); by Lemma 3.2.1,

$$\begin{aligned}\|\overline{u}\nabla u\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} &\lesssim \|\overline{u}\|_{W^{\frac{1}{2},3}(\mathbb{R}^3)}\|\nabla u\|_{L^6(\mathbb{R}^3)} + \|\overline{u}\|_{L^6(\mathbb{R}^3)}\|\nabla u\|_{W^{\frac{1}{2},3}(\mathbb{R}^3)}\\ &\lesssim \|\overline{u}\|_{W^{\frac{1}{2},3}(\mathbb{R}^3)}\|\nabla u\|_{H^1(\mathbb{R}^3)} + \|\overline{u}\|_{H^1(\mathbb{R}^3)}\|\nabla u\|_{W^{\frac{1}{2},3}(\mathbb{R}^3)}\\ &\lesssim \|u\|_{H^1(\mathbb{R}^3)}\|u\|_{H^2(\mathbb{R}^3)}\end{aligned}$$

and

$$\begin{aligned}\|A|u|^2\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} &\lesssim \|A\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}\|u\|^2_{L^\infty(\mathbb{R}^3)} + \|A\|_{L^6}\||u|^2\|_{W^{\frac{1}{2},3}(\mathbb{R}^3)}\\ &\lesssim \|A\|_{H^1(\mathbb{R}^3)}\|u\|^2_{H^2(\mathbb{R}^3)}.\end{aligned}$$

By adding the two estimates above we then obtain

$$\|\mathbb{P}J\|_{H^{1/2}} \lesssim \|J\|_{H^{1/2}} \lesssim \|u\|_{H^1}\|u\|_{H^2} + \|A\|_{H^1}\|u\|^2_{H^2}.$$

For (3.15) we have

$$\begin{aligned}\|\Delta_A u\|_{L^2} \lesssim&\|u\|_{H^2} + \|A\cdot\nabla u\|_{L^2} + \||A|^2u\|_{L^2}\\ \lesssim&\|u\|_{H^2} + \|A\|_{L^6}\|\nabla u\|_{L^3} + \|A\|^2_{L^6}\|u\|_{L^6}\\ \lesssim&\|u\|_{H^2} + \|A\|_{H^1}\|u\|_{H^{3/2}} + \|A\|^2_{H^1}\|u\|_{H^1}\\ \lesssim&\|u\|_{H^2} + \|A\|_{H^1}\|u\|^{1/4}_{L^2}\|u\|^{3/4}_{H^2} + \|A\|^2_{H^1}\|u\|^{1/2}_{L^2}\|u\|^{1/2}_{H^2}.\end{aligned}$$

By using Young's inequality we obtain (3.15). Estimate (3.16) is proved in an analogous way. Finally, for (3.17) we have

$$\begin{aligned}\|(\nabla+iA)u\|_{L^6} \lesssim&\|\nabla(\nabla+iA)u\|_{L^2}\\ \lesssim&\|\Delta_A u\|_{L^2} + \|A(\nabla+iA)u\|_{L^2}\\ \lesssim&\|\Delta_A u\|_{L^2} + \|A\|_{H^1}\|u\|_{H^{3/2}} + \|A\|^2_{H^1}\|u\|_{H^1}\end{aligned}$$

and proceed as for the previous estimates. □

Let us now state the Strichartz estimates for the wave equation we are going to use later. For a proof see for example [51, 116] and references therein.

Lemma 3.2.5 (Strichartz estimates for the wave equation)**.** *Let I be a time interval, and let $B : I\times\mathbb{R}^3 \to \mathbb{C}$ be a solution to the wave equation $\Box B = F$ with initial data $B(0) = B_0$, $\partial_t B(0) = B_1$, such that $B_0 \in \dot{H}^s(\mathbb{R}^3)$, $B_1 \in \dot{H}^{s-1}(\mathbb{R}^3)$ and $F \in L_t^{\tilde{q}'}L_x^{\tilde{r}'}(I\times\mathbb{R}^3)$, whenever $s \geq 0$, $2 \leq q, \tilde{q} \leq \infty$ and $2 \leq r, \tilde{r} < \infty$ obey the scaling condition*

$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - s = \frac{1}{\tilde{q}'} + \frac{3}{\tilde{r}'} - 2$$

and the wave admissibility condition

$$\frac{1}{q} + \frac{1}{r}, \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} \leq \frac{1}{2}.$$

Then the following estimate holds

$$\begin{aligned}\|B\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} + \|B\|_{C_t\dot{H}_x^s(I\times\mathbb{R}^3)} + \|\partial_t B\|_{C_t\dot{H}_x^{s-1}(I\times\mathbb{R}^3)}\\ \lesssim \|B_0\|_{\dot{H}^s(\mathbb{R}^3)} + \|B_1\|_{\dot{H}^{s-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^3)}.\end{aligned}$$

As a consequence we also obtain the following energy estimate.

Lemma 3.2.6. *Let $s\in\mathbb{R}$, $B_0\in H^s(\mathbb{R}^3)$, $B_1\in H^{s-1}(\mathbb{R}^3)$ and $F\in L^1([0,T];H^{s-1}(\mathbb{R}^3))$, $T>0$, then*

$$B\in C([0,T];H^s(\mathbb{R}^3))\cap H^{s-1}([0,T];H^{s-1}(\mathbb{R}^3))$$

defined as in previous Lemma satisfies

$$\begin{aligned}\|B\|_{C_tH_x^s([0,T]\times\mathbb{R}^3)} + \|\partial_t B\|_{C_tH^{s-1}([0,T]\times\mathbb{R}^3)}\\ \lesssim (1+T)(\|B_0\|_{H^s(\mathbb{R}^3)} + \|B_1\|_{H^{s-1}(\mathbb{R}^3)} + \|F\|_{L_t^1H_x^{s-1}([0,T]\times\mathbb{R}^3)}).\end{aligned} \tag{3.18}$$

We conclude this Section by recalling some results concerning the Schrödinger propagator associated to the magnetic Laplacian Δ_A. More precisely, let A be a given, time dependent, divergence-free vector field, we then consider the following initial value problem

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta_A u\\ u(s) = f,\end{cases} \tag{3.19}$$

and we study the properties of its solution.

Proposition 3.2.7. *Let $0<T<\infty$ and let us assume that $A\in C([0,T];H^1(\mathbb{R}^3))$, $\partial_t A\in L^1([0,T];L^3(\mathbb{R}^3))$. Then there exists a unique*

$$u\in C([0,T];H^2(\mathbb{R}^3))\cap C^1([0,T];L^2(\mathbb{R}^3)),$$

which solves (3.19). *Moreover, it holds*

$$\|u\|_{L^\infty(0,T;H^2(\mathbb{R}^3))} \lesssim \|f\|_{H^2}\left(1+\|A\|^4_{L_t^\infty H_x^1}\right)e^{C\|\partial_t A\|_{L_t^1L_x^3}}. \tag{3.20}$$

Proof. The existence and uniquenss for u is already known for a sufficiently smooth magnetic field A, see [69, 70]. Therefore we only need to prove (3.20) as an a priori estimate: the general result stated in this Proposition will then follow by a standard density argument (for more details see Lemma 2.1 in [98]). Let us then consider the norm $\|\Delta_A u\|_{L^2}$; by differentiating it in time we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|\Delta_A u\|^2_{L^2}\right) = \int \mathrm{Re}\left\{\overline{\Delta_A u}\partial_t\Delta_A u\right\}dx = \int \mathrm{Re}\left\{\overline{\Delta_A u}\left(\frac{i}{2}\Delta_A^2 u + [\partial_t,\Delta_A]u\right)\right\}dx\\ \le \|\Delta_A u\|_{L^2}\|\partial_t A\cdot(\nabla+iA)u\|_{L^2}\\ \le \|\Delta_A u\|_{L^2}\|\partial_t A\|_{L^3}\|(\nabla+iA)u\|_{L^6}.\end{aligned}$$

Now by (3.17) we have

$$\|(\nabla+iA)u\|_{L^6} \le \|u\|_{H^2} + \|A\|^4_{H^1}\|u\|_{L^2}.$$

Consequently,

$$\frac{d}{dt}\left(\|\Delta_A u\|_{L^2}\right) \leq \|\partial_t A\|_{L^3}\left(\|u\|_{H^2} + \|A\|_{H^1}^4\|u\|_{L^2}\right).$$

Let $L = 1+\|A\|^4_{L_t^\infty H_x^1([0,T]\times\mathbb{R}^3)}$, then by the conservation of the total mass, $\|u(t)\|_{L^2} = \|f\|_{L^2}$, we have

$$\frac{d}{dt}\left(\|\Delta_A u\|_{L^2} + L\|u\|_{L^2}\right) \lesssim \|\partial_t A\|_{L^3}\left(\|u\|_{H^2} + L\|u\|_{L^2}\right).$$

By using (3.16) we then get

$$\frac{d}{dt}\left(\|\Delta_A u\|_{L^2} + L\|u\|_{L^2}\right) \lesssim \|\partial_t A\|_{L^3}\left(\|\Delta_A u\|_{L^2} + L\|u\|_{L^2}\right),$$

and by Gronwall

$$\|\Delta_A u\|_{L^2} + L\|u\|_{L^2} \lesssim \left(L\|f\|_{L^2} + \|\Delta_{A_0} f\|_{L^2}\right) e^{C\|\partial_t A\|_{L_t^1 L_x^3}}.$$

Again by using (3.16) we also get

$$\|u(t)\|_{H^2} \lesssim \|f\|_{H^2}(1+\|A\|^4_{L_t^\infty H_x^1})e^{C\|\partial_t A\|_{L_t^1 L_x^3}},$$

so that (3.20) is satisfied and the Proposition is proved. □

From Proposition 3.2.7 we can then define the propagator $U_A(t,\tau)$ associated to (3.19), i.e. $U_A(t,\tau)f = u(t)$, where u is the solution in Proposition 3.2.7, and U_A satisfies the following properties:

- $U_A(t,\tau)H^2 \subset H^2$, for any $t \in [0,T]$;
- $U_A(t,t) = \mathbb{I}$;
- $U_A(t_1,t_2)U_A(t_2,t_3) = U_A(t_1,t_3)$, for any $t_1, t_2, t_3 \in [0,T]$.

Moreover, by (3.20) we have

$$\mathcal{K}_2 := \sup_{t,\tau\in[0,T]} \|U_A(t,\tau)\|_{H^2\to H^2} \lesssim \left(1+\|A\|^4_{L_t^\infty H^1}\right) e^{C\|\partial_t A\|_{L_t^1 L_x^3}}.$$

From the unitarity of $U_A(t,\tau)$ in L^2, $\|U_A(t,\tau)f\|_{L^2} = \|f\|_{L^2}$, and by interpolation, we can then infer

$$\sup_{t,\tau\in[0,T]} \|U_A(t,\tau)f\|_{H^s\to H^s} < \infty, \quad \forall\, s\in[0,2].$$

Proposition 3.2.8. *Let $A \in L^\infty([0,T];H^1(\mathbb{R}^3)) \cap W^{1,1}([0,T];L^3)$, $f \in L^1([0,T];H^{-2}(\mathbb{R}^3))$ and let $v \in C([0,T];L^2(\mathbb{R}^3)) \cap W^{1,1}([0,T];H^{-2}(\mathbb{R}^3))$ be solution to*

$$i\partial_t v = -\frac{1}{2}\Delta_A v + f\,.$$

Then for every $t_0 \in [0,T]$,

$$v(t) = U_A(t,t_0) - i\int_{t_0}^t U_A(t,s)f(s)ds\,.$$

Proof. See [97] for a proof of Proposition 3.2.8. □

3.3 Local well-posedness

In this Section we are going to prove the local well-posedness result stated in Theorem 3.1.1 by using a fixed point argument. We split the proof into two parts: in Proposition 3.3.1 we are going to show the existence and uniqueness of a local solution by means of a fixed point argument, then Proposition 3.3.8 will be about the continuous dependence of the solution on the initial data.

Proposition 3.3.1. *Let $\gamma > \frac{3}{2}$. For all $(u_0,A_0,A_1) \in X$ there exists $T_{max} > 0$ and a unique maximal solution (u,A) to (3.5) such that $u \in C([0,T_{max});H^2(\mathbb{R}^3))$, $A \in C([0,T_{max});H^{\frac{3}{2}}(\mathbb{R}^3) \cap C^1([0,T_{max});H^{\frac{1}{2}}(\mathbb{R}^3))$, $\operatorname{div} A = 0$. Moreover the following blowup alternative holds true: if $T_{max} < \infty$, then*

$$\lim_{t\to T_{max}^-} (\|u(t)\|_{H^2} + \|A(t)\|_{H^{3/2}} + \|\partial_t A(t)\|_{H^{1/2}}) = \infty.$$

Proof. First of all, let us define the space

$$\begin{aligned} X_T := \{(u,A) \text{ s.t. } & u \in C([0,T];H^2(\mathbb{R}^3)), A \in C([0,T];H^{\frac{3}{2}}(\mathbb{R}^3)) \cap C^1([0,T];H^{\frac{1}{2}}(\mathbb{R}^3)), \\ & \operatorname{div} A = 0, \|u\|_{L_t^\infty H_x^2(\mathbb{R}^3)} \le R_1,\ \|A\|_{L_t^\infty H^{\frac{3}{2}}(\mathbb{R}^3)} + \|\partial_t A\|_{L_t^\infty H_x^{\frac{1}{2}}(\mathbb{R}^3)} \le R_2\}\,, \end{aligned} \tag{3.21}$$

where $R_1, R_2, T > 0$ will be chosen later. It is straightforward to see that X_T, endowed with the distance

$$d((u_1,A_1),(u_2,A_2)) = \max\{\|u_1-u_2\|_{L_t^\infty L_x^2(\mathbb{R}^3)}, \|A_1-A_2\|_{L_t^4 L_x^4(\mathbb{R}^3)}\}\,, \tag{3.22}$$

is a complete metric space. We also define

$$\|(u,A)\|_{X_T} := \|u\|_{L^\infty(0,T;H^2(\mathbb{R}^3))} + \|A\|_{L^\infty(0,T;H^{3/2}(\mathbb{R}^3))} + \|\partial_t A\|_{L^\infty(0,T;H^{1/2}(\mathbb{R}^3))}. \tag{3.23}$$

Let $(u_0,A_0,A_1) \in X$, where X is defined in (3.8); we define the map Φ on X_T, $(v,B) = \Phi(u,A)$, $(u,A) \in X_T$, where

$$v(t) = \Phi(u)(t) = U_A(t,0)u_0 - i\int_0^t U_A(t,s)(\varphi u + |u|^{2(\gamma-1)}u)(s)ds \tag{3.24}$$

and

$$B(t)=\Phi(A)(t)=\cos(t\sqrt{-\Delta})A_0+\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}A_1+\int_0^t\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\mathbb{P}J(u,A)(s)ds$$

Let us first show that Φ maps X_T into itself. By (3.20) we have that for any $s\in[0,T]$,

$$\|U_A(t,s)f\|_{H^2}\lesssim\|f\|_{H^2}\left(1+\|A\|^4_{L^\infty_tH^1_x}\right)e^{C\|\partial_tA\|_{L^1_tL^3_x}}$$

and since $\|\partial_tA\|_{L^1_tL^3_x}\lesssim T\|\partial_tA\|_{L^\infty_tH^{1/2}_x}$, we have

$$\|U_A(t,s)f\|_{H^2}\le C(1+R_2^4)e^{CTR_2}\|f\|_{H^2}.$$

Let us consider the nonlinear terms in (3.24). Since $\gamma>\frac{3}{2}$ the function $z\mapsto|z|^{2(\gamma-1)}z$ is $C^2(\mathbb{C};\mathbb{C})$, then by the Sobolev embedding $H^2\hookrightarrow L^\infty$ and by Lemma 3.2.1 we have

$$\||u|^{2(\gamma-1)}u\|_{L^\infty_tH^2_x}\lesssim\|u\|^{2\gamma-1}_{L^\infty_tH^2_x}\lesssim R_1^{2\gamma-1}.$$

Furthermore, from (3.12) we have

$$\|\varphi u\|_{L^\infty_tH^2_x}\lesssim\|u\|^2_{L^\infty_tH^{3/4}_x}\|u\|_{L^\infty_tH^2_x}\lesssim R_1^3,$$

so that by putting everything together, we obtain

$$\|v\|_{L^\infty_tH^2_x}\le C_1(1+R_2^4)\exp(CTR_2)\Big(\|u_0\|_{H^2}+TR_1^{2\gamma-1}+TR_1^3\Big).$$

On the other hand, by using the Strichartz estimates for the wave equation stated in Lemma 3.2.5 we have

$$\|B\|_{L^\infty_tH^{3/2}_x}+\|\partial_tB\|_{L^\infty_tH^{1/2}_x}\lesssim(1+T)\Big(\|A_0\|_{H^{3/2}}+\|A_1\|_{H^{1/2}}+\|\mathbb{P}J\|_{L^1_tH^{1/2}_x}\Big).$$

By (3.14) we have

$$\|\mathbb{P}J\|_{L^\infty_tH^{1/2}_x}\lesssim R_1^2(1+R_1),$$

so that

$$\|B\|_{L^\infty_tH^{3/2}_x}+\|\partial_tB\|_{L^\infty_tH^{1/2}_x}\le C_2(1+T)\left(\|A_0\|_{H^{3/2}}+\|A_1\|_{H^{1/2}}+TR_1^2(1+R_1)\right).$$

Let us now choose R_1,R_2,T; without loss of generality we can assume that $T<1$. Let

$$R_2:=4C_2\|A_0\|_{H^{3/2}}+\|A_1\|_{H^{1/2}}$$
$$R_1:=2C_1(1+R_2^4)e^{CR_2}\|u_0\|_{H^2}$$

Then

$$\|v\|_{L^\infty_tH^2_x(\mathbb{R}^3)}\le\frac{R_1}{2}+C_1(1+R_2^4)Te^{R_2}R_1(R_1^{2(\gamma-1)}+R_1^2)$$
$$\|B\|_{L^\infty_tH^{\frac{3}{2}}_x(\mathbb{R}^3)}+\|\partial_tB\|_{L^\infty_tH^{\frac{1}{2}}_x(\mathbb{R}^3)}\le\frac{R_2}{2}+2C_2R_1^2(1+R_2)T$$

Now by choosing T such that

$$\max\left\{C_1(1+R_2^4)e^{CR_2}(R_1^{2(\gamma-1)}+R_1^2)T, \frac{2C_2R_1^2(1+R_2)}{R_2}T\right\} < \frac{1}{2},$$

we get that Φ maps X_T into itself.

We now prove that, possibly choosing a smaller value for $T>0$, the map Φ is indeed a contraction on X_T. Let us define

$$(v,B)=\Phi(u,A)$$
$$(v',B')=\Phi(u',A').$$

In view of Lemma 3.2.8, we have that

$$i\partial_t v = -\Delta_A v + \varphi(u)u + |u|^{2(\gamma-1)}u, \qquad v(0)=u_0\,, \tag{3.25}$$

$$i\partial_t v' = -\Delta_{A'} v' + \varphi(u')u' + |u'|^{2(\gamma-1)}u', \qquad v'(0)=u_0\,, \tag{3.26}$$

By writing the difference of the equations (3.25) and (3.26) for v and v' we get

$$i\partial_t(v-v') = -\Delta_A(v-v') + F, \qquad (v-v')(0)=0\,, \tag{3.27}$$

where F is given by

$$\begin{aligned} F &= 2i(A-A')\cdot\nabla v' + \frac{1}{2}(|A|^2-|A'|^2)v' + (\varphi(|u|^2)-\varphi(|u'|^2))u' \\ &\quad + \varphi(|u|^2)(u-u') + |u|^{2(\gamma-1)}u - |u'|^{2(\gamma-1)}u' =: \sum_{j=1}^{5} F_j. \end{aligned} \tag{3.28}$$

Again, by using Lemma 3.2.8, it follows that

$$(v-v')(t) = -i\int_0^t U_A(t,s)F(s)\,ds,$$

Hence we have

$$\|v-v'\|_{L_t^\infty L_x^2} \lesssim \sum_j \|F_j\|_{L_t^1 L_x^2}. \tag{3.29}$$

We now estimate term by term; by using Hölder's inequality, Sobolev embedding and the fact that v, $v'\in X_T$, we have

$$\begin{aligned} \|F_1\|_{L_t^1L_x^2} &= 2\int_0^T \|(A-A')\nabla v'\|_{L_x^2} \lesssim \int_0^T \|A-A'\|_{L_x^4}\|\nabla v'\|_{L_x^4} \\ &\lesssim \int_0^T \|A-A'\|_{L_x^4}\|\nabla v'\|_{H_x^1} \lesssim \|\nabla v'\|_{L_t^{\frac{4}{3}}H^1}\|A-A'\|_{L_{t,x}^4} \\ &\lesssim T^{3/4}\|\nabla v'\|_{L_t^\infty H_x^1}\|A-A'\|_{L_{t,x}^4}, \end{aligned}$$

and analogously

$$\|F_2\|_{L_t^1 L_x^2} \lesssim T^{3/4} \|v'\|_{L_{t,x}^\infty} \left(\|A\|_{L_t^\infty H_x^1} + \|A'\|_{L_t^\infty H_x^1} \right) \|A - A'\|_{L_{t,x}^4}$$

By using (3.10), the third term is estimated by

$$\|F_3\|_{L_t^1 L_x^2} \lesssim T \left(\|u\|_{L_t^\infty H_x^1} + \|u'\|_{L_t^\infty H_x^1} \right) \|u'\|_{L_t^\infty H_x^1} \|u - u'\|_{L_t^\infty L_x^2}.$$

For the term F_4 we use (3.11) and Sobolev embedding to get

$$\|F_4\|_{L_t^1 L_x^2} \lesssim T \|u\|_{L_t^\infty H_x^2}^2 \|u - u'\|_{L_t^\infty L_x^2}.$$

The last term is estimated by

$$\|F_5\|_{L^2(\mathbb{R}^3)} \lesssim (\|u\|_{L^\infty}^{2(\gamma-1)} + \|u'\|_{L^\infty}^{2(\gamma-1)}) \|u - u'\|_{L^2(\mathbb{R}^3)} \lesssim R_1^{2(\gamma-1)} \|u - u'\|_{L^2(\mathbb{R}^3)},$$

where we used the following inequality

$$||u|^{2(\gamma-1)} u - |u'|^{2(\gamma-2)} u_2| \lesssim (|u_1|^{2(\gamma-1)} + |u'|^{2(\gamma-1)}) |u - u'|$$

By putting everything together in (3.29), and by using Hölder's inequality in time, we obtain

$$\|v - v'\|_{L_t^\infty L_x^2} \lesssim (T^{3/4} + T) C(R_1, R_2) d((u, A), (u', A')). \tag{3.30}$$

Analogously, for B, B' we write

$$(B - B')(t) = \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} G(s)\, ds, \tag{3.31}$$

where $G = \sum_{j=1}^3 G_j$ is given by:

$$G = \mathbb{P}\, \mathrm{Im}\{ (\overline{u - u'})(\nabla - iA) u - iu\overline{u'}(A - A') - (u - u')(\nabla + iA')\overline{u'} \}.$$

Here we have used the fact that $\mathbb{P}(\overline{u'} \nabla (u - u')) = -\mathbb{P}((u - u') \nabla \overline{u'})$. Using the Strichartz estimates in Lemma 3.2.5 with $q = r = \tilde{q} = \tilde{r} = 4$, we get

$$\|B - B'\|_{L_{t,x}^4} \lesssim \|G\|_{L_{t,x}^{\frac{4}{3}}} \tag{3.32}$$

We estimate the three terms in G. The terms G_1 and G_3 are treated similarly, by Sobolev embedding and by using (3.13) we have

$$\|G_1\|_{L_{t,x}^{4/3}} + \|G_3\|_{L_{t,x}^{4/3}} \lesssim$$
$$T^{3/4} \left(1 + \|A\|_{L_t^\infty H_x^1} + \|A'\|_{L_t^\infty H_x^1} \right) \left(\|u\|_{L_t^\infty H_x^2} + \|u'\|_{L_t^\infty H_x^2} \right) \|u - u'\|_{L_t^\infty L_x^2}.$$

By using Hölder's inequality, G_2 is bounded by

$$\|G_2\|_{L_{t,x}^{4/3}} \lesssim T^{1/2} \|u\|_{L_t^\infty H_x^1} \|u'\|_{L_t^\infty H_x^1} \|A - A'\|_{L_{t,x}^4}.$$

Resuming, by estimating the terms in (3.32) we obtain

$$\|B - B'\|_{L^4_{t,x}} \lesssim (T^{1/2} + T^{3/4}) C(R_1, R_2) d((u,A),(u',A')). \tag{3.33}$$

By summing up (3.30) and (3.33), we finally get

$$d((v,B),(v',B')) \leq (T^{1/2} + T) C(R_1, R_2) d((u,A),(u',A')).$$

Thus, if $T > 0$ is chosen sufficiently small, then Φ is a contraction. This proves that for any initial data $(u_0, A_0, A_1) \in X$, there exists a unique local solution (u,A) to (3.5) in X_T such that

$$u \in C([0,T]; H^2(\mathbb{R}^3)), \quad A \in C([0,T]; H^{3/2}(\mathbb{R}^3)) \cap C^1([0,T]; H^{1/2}(\mathbb{R}^3)).$$

By a standard argument it is straightforward to show that it may be extended to a maximal solution (u,A), with $u \in C([0,T_{max}); H^2(\mathbb{R}^3))$, $A \in C([0,T_{max}); H^{3/2}(\mathbb{R}^3)) \cap C^1([0,T_{max}); H^{1/2}(\mathbb{R}^3))$ and that the blow-up alternative holds true, namely if $T_{max} < \infty$ then we have

$$\lim_{t \to T_{max}^-} (\|u(t)\|_{H^2} + \|A(t)\|_{H^{3/2}} + \|\partial_t A(t)\|_{H^{1/2}}) = \infty.$$

□

Proposition 3.3.8 states the continuous dependence of solution on the initial data. Its proof goes through a series of technical lemmas and it follows this strategy: first we prove the continuous dependence for more regular solutions, then by an approximation argument we prove the general result for solutions $(u,A) \in X$. This will finish the proof of Theorem 3.1.1. In the remaining part of the Section we state the Proposition and the Lemmas needed to prove the continuous dependence for regular solutions. Then we show how to extend it to arbitrary solutions $(u,A) \in X$.

We consider two different solutions (u,A), (u',A') emanated from two sets of initial data $(u_0, A_0, A_1), (u'_0, A'_0, A'_1) \in X$ such that

$$\|(u_0, A_0, A_1)\|_X, \|(u'_0, A'_0, A'_1)\|_X \leq L.$$

Then there exists a positive time, say T, such that both soutions (u,A), (u',A') exist in $[0,T]$. Moreover we assume that there exists a constant $R > 0$ such that in $[0,T]$

$$\|(u, A, \partial_t A)\|_{X_T}, \ \|(u', A', \partial_t A')\|_{X_T} \leq R. \tag{3.34}$$

We are also going to exploit the uniform bounds given by the total energy of system (3.5),

$$E(t) = \int \frac{1}{2}|(\nabla - iA)u|^2 + \frac{1}{2}|\partial_t A|^2 + \frac{1}{2}|\nabla A|^2 + \frac{1}{2}|\nabla \varphi|^2 + \frac{1}{\gamma}|u|^{2\gamma}\, dx. \tag{3.35}$$

It is straightforward to see that it is conserved along the flow of solutions to (3.5) (see [97]); thus if (u,A), resp. (u',A'), is the solutions emanated from (u_0,A_0,A_1), resp. (u'_0,A'_0,A'_1), then we may consider $E>0$ such that

$$E(t), E'(t) \leq E,$$

where $E(t)$, resp. $E'(t)$, is the total energy associated to (u,A), resp. (u',A').
We remark here that in the following lemmas, the discussion will be led for a sufficiently small $T=T(R)$; indeed, if it is not the case, one can divide the interval $[0,T]$ into small subintervals and repeatedly use the estimates obtained for short intervals.

Lemma 3.3.2. *Let* $(u,A),(u',A')$ *be solutions to* (3.5) *defined as above, then we have*

$$\begin{aligned}\|u-u'\|_{L_t^\infty H_x^2} \lesssim & \|\partial_t(u-u')(0)\|_{L^2} + T\|\partial_t u'\|_{L_t^\infty H_x^2}\left(\|A-A'\|_{L_t^\infty H_x^{1/2}} + \|u-u'\|_{L_t^\infty L_x^2}\right) \\ & + T\|(u,A)-(u',A')\|_{X_T},\end{aligned}$$

where the constant depends only on R,E *defined as above.*

We split the proof of Lemma 3.3.2 into two steps; see the two Lemmas 3.3.3 and 3.3.4 below.

Lemma 3.3.3. *We have*

$$\begin{aligned}\|u-u'\|_{L_t^\infty H_x^2(\mathbb{R}^3)} \lesssim_{R,E} & \|\partial_t(u-u')\|_{L_t^\infty L_x^2(\mathbb{R}^3)} + \|u-u'\|_{L_t^\infty L_x^2(\mathbb{R}^3)} \\ & + \|A-A'\|_{L_t^\infty H_x^{\frac{1}{2}}(\mathbb{R}^3)}.\end{aligned} \tag{3.36}$$

Proof. Let us consider the equation for the difference $u-u'$; by using (3.27) we have

$$i\partial_t(u-u') = -\Delta(u-u') + 2iA\cdot\nabla(u-u') + |A|^2(u-u') + F \tag{3.37}$$

where

$$\begin{aligned}F = & 2i(A-A')\cdot\nabla u' + (|A|^2-|A'|^2)u' + (\varphi(|u|^2)-\varphi(|u'|^2))u' \\ & + \varphi(|u|^2)(u-u') + |u|^{2(\gamma-1)}u - |u'|^{2(\gamma-1)}u'.\end{aligned}$$

This implies

$$\begin{aligned}\|\Delta(u-u')\|_{L^2(\mathbb{R}^3)} \leq & \|\partial_t(u-u')\|_{L^2(\mathbb{R}^3)} + \|A\cdot\nabla(u-u')\|_{L^2(\mathbb{R}^3)} \\ & + \||A|^2(u-u')\|_{L^2(\mathbb{R}^3)} + \|F\|_{L^2(\mathbb{R}^3)}\end{aligned}$$

From Hölder's inequality and Sobolev embedding theorem we have

$$\begin{aligned}\|A\cdot\nabla(u-u')\|_{L^2(\mathbb{R}^3)} \leq & \|A\|_{L^6(\mathbb{R}^3)}\|\nabla(u-u')\|_{L^3(\mathbb{R}^3)} \lesssim \|\nabla A\|_{L^2(\mathbb{R}^3)}\|u-u'\|_{H^{\frac{3}{2}}} \\ \lesssim_E & \|u-u'\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \lesssim_E \|u-u'\|^{\frac{1}{4}}_{L^2(\mathbb{R}^3)}\|u-u'\|^{\frac{3}{4}}_{H^2(\mathbb{R}^3)} \\ \lesssim_E & C(\varepsilon)\|u-u'\|_{L^2(\mathbb{R}^3)} + \varepsilon\|u-u'\|_{H^2(\mathbb{R}^3)},\end{aligned}$$

where we do not consider the explicit dependence on the constants on R and E. Similarly we have

$$\||A|^2(u-u')\|_{L^2(\mathbb{R}^3)} \lesssim \|A\|_{L^6}^2 \|u-u'\|_{H^1(\mathbb{R}^3)} \lesssim_E C(\varepsilon)\|u-u'\|_{L^2(\mathbb{R}^3)} + \varepsilon\|u-u'\|_{H^2(\mathbb{R}^3)}$$

We can deal with F as already done previously, getting

$$\|F\|_{L^2(\mathbb{R}^3)} \lesssim_{R,E} \|A-A'\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} + \|u-u'\|_{L^2(\mathbb{R}^3)}$$

Finally, putting all together the previous inequalities, we have

$$\begin{aligned}\|u-u'\|_{L_t^\infty H_x^2(\mathbb{R}^3)} &\leq \|u-u'\|_{L_t^\infty L_x^2(\mathbb{R}^3)} + \|\Delta(u-u')\|_{L_t^\infty L_x^2(\mathbb{R}^3)} \\ &\lesssim_{R,E} C(\varepsilon)\|u-u'\|_{L_t^\infty L_x^2(\mathbb{R}^3)} + \|\partial_t(u-u')\|_{L_t^\infty L_x^2(\mathbb{R}^3)} \\ &\quad + \|A-A'\|_{L_t^\infty H_x^{\frac{1}{2}}(\mathbb{R}^3)} + \varepsilon\|u-u'\|_{L_t^\infty H_x^2(\mathbb{R}^3)}.\end{aligned}$$

Now, by choosing ε sufficently small, we get (3.36). □

We note that, as a consequence of the previous computations and equation (3.37), we can prove that

$$\|\partial_t(u-u')\|_{L_t^\infty L_x^2(\mathbb{R}^3)} \lesssim_{R,E} \|u-u'\|_{L_t^\infty H_x^2(\mathbb{R}^3)} + \|A-A'\|_{L_t^\infty H_x^{\frac{1}{2}}(\mathbb{R}^3)} \tag{3.38}$$

In order to estimate the term $\|\partial_t(u-u')\|_{L_t^\infty L_x^2(\mathbb{R}^3)}$ we use next lemma.

Lemma 3.3.4. *The following inequality holds:*

$$\begin{aligned}\|\partial_t u - \partial_t u'\|_{L_t^\infty L_x^2(\mathbb{R}^3)} &\lesssim_{R,E} \|\partial_t(u-u')(0)\|_{L^2(\mathbb{R}^3)} \\ &+ T\|\partial_t u'\|_{L_t^\infty H_x^2(\mathbb{R}^3)}\Big(\|A-A'\|_{L_t^\infty H_x^{\frac{1}{2}}(\mathbb{R}^3)} + \|u-u'\|_{L_t^\infty L_x^2(\mathbb{R}^3)}\Big) \\ &+ T\|(u-u', A-A', \partial_t A - \partial_t A')\|_{X_T}\end{aligned} \tag{3.39}$$

Proof. We start by differentiating in time the equation

$$i\partial_t u = -\Delta_A u + \varphi(u)u + |u|^{2(\gamma-1)}u.$$

We then get

$$i\partial_t^2 u = -\Delta_A \partial_t u + \varphi(u)\partial_t u + (2i\partial_t A(\nabla - iA) + \partial_t\varphi)u + \partial_t(|u|^{2(\gamma-1)}u)$$

Writing the corresponding equation for $\partial_t^2 u'$ and taking the difference with the previous one we get

$$i\partial_t^2(u-u') = -\Delta_A(\partial_t u - \partial_t u') + F, \tag{3.40}$$

where F is given by

$$\begin{aligned}F = &\left[2i(A-A')\left(\nabla - \frac{i}{2}(A+A')\right) + (\varphi - \varphi')\right]\partial_t u' + \varphi(\partial_t u - \partial_t u') \\ &+ (2i\partial_t A(\nabla - iA) + \partial_t\varphi)(u-u') + \partial_t(|u|^{2(\gamma-1)}u - |u'|^{2(\gamma-1)}u') \\ &+ (2i\partial_t(A-A')(\nabla - iA) - 2i(A-A')\partial_t A' + \partial_t(\varphi-\varphi'))u'.\end{aligned} \tag{3.41}$$

Using the unitarity in $L^2(\mathbb{R}^3)$ of $U_A(t,s)$ we get

$$\|\partial_t(u-u')(t)\|_{L^2(\mathbb{R}^3)} \le \|\partial_t(u-u')(0)\|_{L^2} + \int_0^t \|F(s)\|_{L^2(\mathbb{R}^3)} ds\,. \tag{3.42}$$

We estimate the inhomogenous term F, we have

$$\begin{aligned} &\left\| \Big[2i(A-A')(\nabla - \frac{i}{2}(A+A')) + (\varphi - \varphi') \Big] \partial_t u' \right\|_{L^2(\mathbb{R}^3)} \\ &\qquad \lesssim_{R,E} \left(\|u-u'\|_{L^2(\mathbb{R}^3)} + \|A-A'\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \right) \|\partial_t u'\|_{H^2(\mathbb{R}^3)} \end{aligned}$$

This inequality follows from

$$\begin{aligned} &\left\| (A-A')\Big(\nabla - \frac{i}{2}(A+A') \Big) \partial_t u' \right\|_{L^2(\mathbb{R}^3)} \\ &\le \|A-A'\|_{L^3(\mathbb{R}^3)} \left\| \Big(\nabla - \frac{i}{2}(A+A') \Big) \partial_t u' \right\|_{L^6(\mathbb{R}^3)} \\ &\lesssim \|A-A'\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \Big\{ \|\nabla \partial_t u'\|_{H^1(\mathbb{R}^3)} + \|A+A'\|_{L^6(\mathbb{R}^3)} \|\partial_t u'\|_{L^\infty(\mathbb{R}^3)} \Big\} \\ &\lesssim \|A-A'\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \|\partial_t u'\|_{H^2(\mathbb{R}^3)} \Big(1 + \|\nabla A\|_{L^2(\mathbb{R}^3)} + \|\nabla A'\|_{L^2(\mathbb{R}^3)} \Big) \\ &\lesssim_E \|A-A'\|_{H^{\frac{1}{2}}(\mathbb{R}^3)} \|\partial_t u'\|_{H^2(\mathbb{R}^3)} \end{aligned}$$

and

$$\begin{aligned} &\|(\varphi - \varphi')\partial_t u'\|_{L^2(\mathbb{R}^3)} \le \|\Delta^{-1}((u-u')\overline{u} + \overline{(u-u')}u')\partial_t u'\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|u-u'\|_{L^2(\mathbb{R}^3)} \|\overline{u}\|_{L^3(\mathbb{R}^3)} \|\partial_t u'\|_{L^3(\mathbb{R}^3)} + \|\overline{(u-u')}\|_{L^2(\mathbb{R}^3)} \|u'\|_{L^3(\mathbb{R}^3)} \|\partial_t u'\|_{L^3(\mathbb{R}^3)} \\ &\lesssim_{M,E} \|u-u'\|_{L^2(\mathbb{R}^3)} \|\partial_t u'\|_{H^2(\mathbb{R}^3)} \end{aligned}$$

where $M = \|u\|^2_{L^2(\mathbb{R}^3)}$ and we used Hölder inequality, the Sobolev embeddings $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, $H^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ and (3.10).
Furthermore, from (3.11) we may infer

$$\|\varphi(\partial_t u - \partial_t u')\|_{L^2(\mathbb{R}^3)} \lesssim_R \|\partial_t u - \partial_t u'\|_{L^2(\mathbb{R}^3)}\,.$$

Again,

$$\begin{aligned} \|2i\partial_t A(\nabla - iA)(u-u')\|_{L^2(\mathbb{R}^3)} &\lesssim \|\partial_t A\|_{L^3(\mathbb{R}^3)} \|(\nabla - iA)(u-u')\|_{L^6} \\ &\lesssim_{R,E} \|u-u'\|_{H^2(\mathbb{R}^3)} \end{aligned}$$

and, by using (3.13) and (3.10),

$$\begin{aligned} \|\partial_t \varphi(u-u')\|_{L^2(\mathbb{R}^3)} &\lesssim \|(\Delta^{-1}(2\,\mathrm{Re}(\overline{u}\partial_t u)))(u-u')\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|\partial_t u\|_{L^2(\mathbb{R}^3)} \|\overline{u}\|_{L^3(\mathbb{R}^3)} \|u-u'\|_{H^2(\mathbb{R}^3)} \\ &\lesssim_R \|u-u'\|_{H^2(\mathbb{R}^3)}\,. \end{aligned}$$

Observe that one has

$$\partial_t(|u|^{2(\gamma-1)}u) = \gamma|u|^{2(\gamma-1)}\partial_t u + (\gamma-1)|u|^{2(\gamma-2)}u^2\partial_t\overline{u},$$

therefore it follows

$$\begin{aligned}\partial_t(|u|^{2(\gamma-1)}u - |u'|^{2(\gamma-1)}u') = &\gamma\partial_t u(|u|^{2(\gamma-1)} - |u'|^{2(\gamma-1)}) + \gamma|u'|^{2(\gamma-1)}\partial_t(u-u')\\ &+ (\gamma-1)\partial_t\overline{u}(|u|^{2(\gamma-2)}u^2 - |u'|^{2(\gamma-2)}u'^2)\\ &+ (\gamma-1)|u'|^{2(\gamma-2)}u^2\partial_t(\overline{u-u'})\end{aligned}$$

We then have

$$\|\partial_t(|u|^2u - |u'|^2u')\|_{L^2(\mathbb{R}^3)} \lesssim_R \|\partial_t(u-u')\|_{L^2(\mathbb{R}^3)} + \|u-u'\|_{H^2(\mathbb{R}^3)},$$

where we used the following two inequalities

$$\begin{aligned}\left||z|^{2(\gamma-1)} - |z'|^{2(\gamma-1)}\right| &\lesssim \left(|z|^{2\gamma-3} + |z'|^{2\gamma-3}\right)|z-z'|\\ \left||z|^{2(\gamma-2)}z^2 - |z'|^{2(\gamma-2)}z'^2\right| &\lesssim \left(|z|^{2\gamma-3} + |z'|^{2\gamma-3}\right)|z-z'|,\end{aligned}$$

which follow from the assumption $\gamma > \frac{3}{2}$, that ensures to deal with a locally Lipschitz nonlinearity. For the last term, with similar computations, we have

$$\|\partial_t(A-A')(\nabla - iA)u'\|_{L^2(\mathbb{R}^3)} \lesssim_{R,E} \|\partial_t(A-A')\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}$$

$$\begin{aligned}\|\partial_t A'(A-A')u'\|_{L^2(\mathbb{R}^3)} &\lesssim \|\partial_t A'\|_{L^3(\mathbb{R}^3)}\|A-A'\|_{L^6(\mathbb{R}^3)}\|u'\|_{L^\infty(\mathbb{R}^3)}\\ &\lesssim_R \|A-A'\|_{H^{\frac{3}{2}}(\mathbb{R}^3)}\end{aligned}$$

$$\|(\partial_t\varphi - \partial_t\varphi')u'\|_{L^2(\mathbb{R}^3)} \lesssim_R \|\partial_t u - \partial_t u'\|_{L^2(\mathbb{R}^3)} + \|u-u'\|_{H^2(\mathbb{R}^3)}.$$

By putting everything together, we obtain

$$\begin{aligned}\|\partial_t u - \partial_t u'\|_{L^\infty_t L^2_x(\mathbb{R}^3)} \lesssim_{R,E} &\|\partial_t(u-u')(0)\|_{L^2(\mathbb{R}^3)} + T\|\partial_t u - \partial_t u'\|_{L^\infty_t L^2_x(\mathbb{R}^3)}\\ &+ T\|\partial_t u'\|_{L^\infty_t H^2_x(\mathbb{R}^3)}\big(\|A-A'\|_{L^\infty_t H^{\frac{1}{2}}_x(\mathbb{R}^3)} + \|u-u'\|_{L^\infty_t L^2_x(\mathbb{R}^3)}\big)\\ &+ T\big(\|u-u'\|_{L^\infty_t H^2_x(\mathbb{R}^3)} + \|A-A'\|_{L^\infty_t H^{\frac{3}{2}}_x(\mathbb{R}^3)} + \|\partial_t(A-A')\|_{L^\infty_t H^{\frac{1}{2}}_x(\mathbb{R}^3)}\big),\end{aligned}$$

which gives (3.39), by using (3.38) for the term $\|\partial_t(u-u')\|_{L^\infty_t L^2_x(\mathbb{R}^3)}$ in the righthand side of the previous inequality. □

Proof of Lemma 3.3.2. The proof clearly follows by putting together Lemmas 3.3.3 and 3.3.4. □

Lemma 3.3.5. *Let $(u,A),(u',A')$ be solutions to (3.5) defined as above, then we have*

$$\|u-u'\|_{L^\infty_t L^2_x} + \|A-A'\|_{L^\infty_t H^{1/2}_x} \lesssim \|(u_0,A_0,A_1) - (u_0',A_0',A_1')\|_{L^2\times H^{1/2}\times H^{-1/2}}, \quad (3.43)$$

where the constant depends only on R,E defined as above.

Proof. Writing the difference equation for A and A' we get

$$\Box(A-A') = G\,,$$

with

$$G = \mathbb{P}\,\mathrm{Im}\{(\overline{u-u'})(\nabla - iA)u - iu\overline{u'}(A-A') - (u-u')(\nabla + iA')\overline{u'}\}$$

where we used the fact that $\mathbb{P}(\overline{u'}\nabla(u-u')) = -\mathbb{P}((u-u')\nabla\overline{u'})$. By applying the energy estimate (3.18) we get

$$\begin{aligned}\|A-A'\|_{L_t^\infty H_x^{\frac12}(\mathbb{R}^3)} &\lesssim (1+T)\|(A_0-A_0', A_1-A_1')\|_{H^{\frac12}(\mathbb{R}^3)\times H^{-\frac12}(\mathbb{R}^3)}\\ &+(1+T)\|G\|_{L_t^1 H_x^{-\frac12}(\mathbb{R}^3)}\end{aligned}$$

Using the embedding $L^{\frac32}(\mathbb{R}^3) \hookrightarrow H^{-\frac12}(\mathbb{R}^3)$ we have

$$\begin{aligned}\|(u-u')(\nabla - iA)u\|_{L^{\frac32}(\mathbb{R}^3)} &\le \|u-u'\|_{L^2(\mathbb{R}^3)}\|(\nabla - iA)u\|_{L^6(\mathbb{R}^3)}\\ &\lesssim \|u-u'\|_{L^2(\mathbb{R}^3)}\big\{\|\nabla u\|_{H^1(\mathbb{R}^3)} + \|Au\|_{L^6(\mathbb{R}^3)}\big\}\\ &\lesssim \|u-u'\|_{L^2(\mathbb{R}^3)}\|u\|_{H^2(\mathbb{R}^3)}\big(1+\|\nabla A\|_{L^2(\mathbb{R}^3)}\big)\\ &\lesssim_{R,E} \|u-u'\|_{L^2(\mathbb{R}^3)}\,.\end{aligned}$$

Analogously

$$\|(u-u')(\nabla + iA)\overline{u'}\|_{L^{\frac32}(\mathbb{R}^3)} \lesssim_{R,E} \|u-u'\|_{L^2(\mathbb{R}^3)}$$

and

$$\|u\overline{u'}(A-A')\|_{L^{\frac32}(\mathbb{R}^3)} \lesssim_R \|A-A'\|_{H^{\frac12}(\mathbb{R}^3)}$$

Hence we get

$$\begin{aligned}\|A-A'\|_{L_t^\infty H^{\frac12}(\mathbb{R}^3)} &\lesssim_{R,E} (1+T)\|(A_0-A_0', A_1-A_1')\|_{H^{\frac12}(\mathbb{R}^3)\times H^{-\frac12}(\mathbb{R}^3)}\\ &+T(1+T)\{\|A-A'\|_{L_t^\infty H^{\frac12}(\mathbb{R}^3)} + \|u-u'\|_{L_t^\infty L^2(\mathbb{R}^3)}\}\,.\end{aligned}$$

In a similar way, using the difference of the equations for u and u' we get

$$\begin{aligned}\|u-u'\|_{L_t^\infty L^2(\mathbb{R}^3)} &\lesssim_{R,E} \|u_0-u_0'\|_{L^2(\mathbb{R}^3)}\\ &+T\big\{\|A-A'\|_{L_t^\infty H^{\frac12}(\mathbb{R}^3)} + \|u-u'\|_{L_t^\infty L^2(\mathbb{R}^3)}\big\}\end{aligned}$$

Putting all togheter

$$\begin{aligned}&\|A-A'\|_{L_t^\infty H^{\frac12}(\mathbb{R}^3)} + \|u-u'\|_{L_t^\infty L^2(\mathbb{R}^3)}\\ &\qquad\lesssim_{R,E} (1+T)\|(u_0-u_0', A_0-A_0', A_1-A_1')\|_{L^2(\mathbb{R}^3)\times H^{\frac12}(\mathbb{R}^3)\times H^{-\frac12}(\mathbb{R}^3)}\\ &\qquad+T(1+T)\{\|A-A'\|_{L_t^\infty H^{\frac12}(\mathbb{R}^3)} + \|u-u'\|_{L_t^\infty L^2(\mathbb{R}^3)}\}\,.\end{aligned}$$

So, taking T sufficiently small, we get (3.43). □

Lemma 3.3.6. *We have*

$$\|\partial_t u\|_{L_t^\infty H_x^2} \lesssim \|u_0\|_{H^4} + \|A_0\|_{H^{5/2}} + \|A_1\|_{H^{3/2}},$$

where the constant depends only on T, R, E.

In order to prove Lemma 3.3.6, we will use the following.

Lemma 3.3.7. *The following estimate holds:*

$$\|\partial_t u\|_{L_t^\infty H^2(\mathbb{R}^3)} \le \|\partial_t^2 u\|_{L^2(\mathbb{R}^3)} + C(E,R) \tag{3.44}$$

Proof. From the equation

$$\begin{aligned} i\partial_{tt}^2 u &= -\Delta\partial_t u + 2iA\cdot\nabla\partial_t u + |A|^2\partial_t u + 2i\partial_t A\cdot\nabla u + 2A\cdot\partial_t A u \\ &\quad + \partial_t\varphi u + \varphi\partial_t u + \partial_t(|u|^{2(\gamma-1)}u) \end{aligned} \tag{3.45}$$

we can estimate $\|\partial_t u\|_{H^2(\mathbb{R}^3)}$. Indeed

$$\|\partial_t u\|_{H^2(\mathbb{R}^3)} \le \|\partial_t u\|_{L^2(\mathbb{R}^3)} + \|\Delta\partial_t u\|_{L^2(\mathbb{R}^3)} \le C(R) + \|\Delta\partial_t u\|_{L^2(\mathbb{R}^3)}$$

So we have

$$\begin{aligned} \|\Delta\partial_t u\|_{L^2(\mathbb{R}^3)} \le \|\partial_{tt}^2 u\|_{L^2(\mathbb{R}^3)} &+ \|A\cdot\nabla\partial_t u\|_{L^2(\mathbb{R}^3)} + \||A|^2\partial_t u\|_{L^2(\mathbb{R}^3)} \\ &+ \|\partial_t A\cdot\nabla u\|_{L^2(\mathbb{R}^3)} + \|A\cdot\partial_t A u\|_{L^2(\mathbb{R}^3)} \\ &+ \|\partial_t\varphi u + \varphi\partial_t u\|_{L^2(\mathbb{R}^3)} + \|\partial_t(|u|^{2(\gamma-1)}u)\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

We begin with the estimate of the right-hand side of the previous inequality.

$$\begin{aligned} &\|A\cdot\nabla\partial_t u\|_{L^2(\mathbb{R}^3)} \lesssim \|A\|_{L^6(\mathbb{R}^3)}\|\nabla\partial_t u\|_{L^3(\mathbb{R}^3)} \lesssim \|\nabla A\|_{L^2(\mathbb{R}^3)}\|\partial_t u\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \\ &\lesssim \sqrt{E}\|\partial_t u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}}\|\partial_t u\|_{H^2(\mathbb{R}^3)}^{\frac{3}{4}} \lesssim \sqrt{E}(C(\varepsilon)\|\partial_t u\|_{L^2(\mathbb{R}^3)} + \varepsilon\|\partial_t u\|_{H^2(\mathbb{R}^3)}) \\ &\lesssim C(E,R) + C(E)\varepsilon\|\partial_t u\|_{H^2(\mathbb{R}^3)}, \end{aligned}$$

where we used the fact that $\|\partial_t u\|_{L_t^\infty L^2} \le C(R)$, which follows by using the equation for u together with the assumption (3.34). In the same way

$$\begin{aligned} \||A|^2\partial_t u\|_{L^2(\mathbb{R}^3)} &\lesssim \|A\|_{L^6(\mathbb{R}^3)}^2\|\partial_t u\|_{L^6(\mathbb{R}^3)} \lesssim \|\nabla A\|_{L^2(\mathbb{R}^3)}^2\|\partial_t u\|_{H^1(\mathbb{R}^3)} \\ &\lesssim (1 + \varepsilon\|\partial_t u\|_{H^2(\mathbb{R}^3)}) \end{aligned}$$

The other terms are all bounded by $C(R)$; for instance

$$\|\partial_t A\nabla u\|_{L^2(\mathbb{R}^3)} \lesssim \|\partial_t A\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}\|u\|_{H^2(\mathbb{R}^3)} \le C(R)$$

or

$$\begin{aligned} \|\partial_t(|u|^{2(\gamma-1)}u)\|_{L^2(\mathbb{R}^3)} &\lesssim \||u|^{2(\gamma-1)}\partial_t u\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \|u\|_{L^\infty}^{2(\gamma-1)}\|\partial_t u\|_{L^2(\mathbb{R}^3)} \le C(R) \end{aligned}$$

We can deal with the remaining terms analogously. Finally we get

$$\|\partial_t u\|_{H^2(\mathbb{R}^3)} \lesssim \|\partial_{tt}^2 u\|_{L^2(\mathbb{R}^3)} + C(E,R) + C(R)\varepsilon\|\partial_t u\|_{H^2(\mathbb{R}^3)}$$

which gives (3.44) for sufficiently small ε. □

Proof of Lemma 3.3.6. To complete the estimates we have to deal with $\|\partial_t^2 u\|_{L_t^\infty L^2(\mathbb{R}^3)}$. We write the equation for the time derivative $\partial_t^2 u$

$$i\partial_t^3 u = -\Delta \partial_t^2 u + 2iA\cdot\nabla\partial_t^2 u + |A|^2\partial_t^2 u + G$$

where

$$\begin{aligned} G &= 4i\partial_t A\cdot\nabla\partial_t u + 4A\cdot\partial_t A\partial_t u + 2i\partial_t^2 A(\nabla u - iAu) + 2(\partial_t A)^2 u \\ &\quad + 2\partial_t\varphi\partial_t u + \partial_t^2\varphi u + \partial_t^2 u\varphi + \partial_t^2(|u|^{2(\gamma-1)}u) \end{aligned}$$

Using Duhamel's representation in Proposition (3.2.8) we have

$$\|\partial_t^2 u\|_{L_t^\infty L^2(\mathbb{R}^3)} \lesssim \|\partial_{tt}u(0)\|_{L^2(\mathbb{R}^3)} + T\|G\|_{L_t^\infty L^2(\mathbb{R}^3)}$$

Proceeding as before we finally get

$$\begin{aligned} \|\partial_t^2 u\|_{L^2(\mathbb{R}^3)} &\lesssim \|\partial_t^2 u(0)\|_{L^2(\mathbb{R}^3)} \\ &\quad + TC(R,E)\Big\{\|\partial_t u\|_{L_t^\infty H^2(\mathbb{R}^3)} + \|A\|_{L_t^\infty H^{\frac{5}{2}}(\mathbb{R}^3)} + \|\partial_t A\|_{L_t^\infty H^{\frac{3}{2}}(\mathbb{R}^3)}\Big\} \end{aligned}$$

We estimate the right-hand side of the previous inequality. From (3.18), we have

$$\begin{aligned} \|A\|_{L_t^\infty H^{\frac{5}{2}}(\mathbb{R}^3)} + \|\partial_t A\|_{L_t^\infty H^{\frac{3}{2}}(\mathbb{R}^3)} &\lesssim (1+T)\|(A_0,A_1)\|_{H^{\frac{5}{2}}(\mathbb{R}^3)\times H^{\frac{3}{2}}(\mathbb{R}^3)} \\ &\quad + T(1+T)\|J\|_{L_t^\infty H^{\frac{3}{2}}(\mathbb{R}^3)} \end{aligned}$$

For the term with J, proceeding as in (3.14), we have, by using Lemma 3.2.1, that

$$\begin{aligned} \|\bar{u}\nabla u\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} &\lesssim \|\bar{u}\|_{L^\infty(\mathbb{R}^3)}\|\nabla u\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} + \|\bar{u}\|_{H^{\frac{3}{2}}(\mathbb{R}^3)}\|\nabla u\|_{L^\infty(\mathbb{R}^3)} \\ &\lesssim C(R)\|u\|_{H^4(\mathbb{R}^3)} \end{aligned}$$

and

$$\|A|u|^2\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \lesssim C(R,E).$$

So

$$\begin{aligned} \|A\|_{L_t^\infty H^{\frac{5}{2}}(\mathbb{R}^3)} + \|\partial_t A\|_{L_t^\infty H^{\frac{3}{2}}(\mathbb{R}^3)} &\lesssim (1+T)\|(A_0,A_1)\|_{H^{\frac{5}{2}}(\mathbb{R}^3)\times H^{\frac{3}{2}}(\mathbb{R}^3)} \\ &\quad + T(1+T)\|u\|_{L^\infty H^4(\mathbb{R}^3)}. \end{aligned}$$

Moreover since $\|u\|_{H^4(\mathbb{R}^3)} \lesssim \|u\|_{L^2(\mathbb{R}^3)} + \|\Delta u\|_{H^2(\mathbb{R}^3)}$ and by the equation for u it follows

$$\|\Delta u\|_{H^2(\mathbb{R}^3)} \lesssim \|\partial_t u\|_{H^2(\mathbb{R}^3)} + \|A\cdot\nabla u + |A|^2 u\|_{H^2} + \|\varphi u\|_{H^2} + \||u|^{2(\gamma-1)}u\|_{H^2(\mathbb{R}^3)},$$

then by estimating the right-hand side as before, we obtain

$$\|u\|_{L_t^\infty H^4(\mathbb{R}^3)} \lesssim C(E,R)\Big(\|\partial_t u\|_{L_t^\infty H^2(\mathbb{R}^3)} + \|A\|_{L_t^\infty H^{\frac{5}{2}}(\mathbb{R}^3)}\Big).$$

Putting all together

$$\|\partial_t u\|_{L_t^\infty H^2(\mathbb{R}^3)} \lesssim \|\partial_{tt} u(0)\|_{L^2(\mathbb{R}^3)} + \|(A_0,A_1)\|_{H^{\frac{5}{2}}(\mathbb{R}^3)\times H^{\frac{3}{2}}(\mathbb{R}^3)},$$

moreover one has

$$\|\partial_{tt} u(0)\|_{L^2(\mathbb{R}^3)} \lesssim \|u_0\|_{H^4(\mathbb{R}^3)} + C(E,R)\|A_0\|_{H^{\frac{5}{2}}(\mathbb{R}^3)}.$$

Indeed, from the equation (3.45), evaluated at time $t=0$, it follows that

$$\|\partial_{tt} u(0)\|_{L^2} \lesssim_{R,E} \|\partial_t u(0)\|_{H^2} + C(E,R).$$

On the other hand, by doing the same with the equation for $\partial_t u$, we get

$$\|\partial_t u(0)\|_{H^2} \lesssim \|u_0\|_{H^4} + C(E,R)\|A_0\|_{H^{\frac{5}{2}}}.$$

Then we obtain

$$\|\partial_t u\|_{L_t^\infty H^2(\mathbb{R}^3)} \lesssim \|u_0\|_{H^4(\mathbb{R}^3)} + \|(A_0,A_1)\|_{H^{\frac{5}{2}}(\mathbb{R}^3)\times H^{\frac{3}{2}}(\mathbb{R}^3)}. \tag{3.46}$$

□

By combining the above Lemmas, it is possible to show the continuous dependence for solutions whose initial data are $(u_0,A_0,A_1) \in H^4 \times H^{5/2} \times H^{3/2}$. Indeed Lemmas 3.3.2, 3.3.5 and 3.3.6 imply the following estimate

$$\begin{aligned}\|(u,A)-(u',A')\|_{X_T} &\lesssim \|(u_0,A_0,A_1)-(u_0',A_0',A_1')\|_X \\ &+ T\left(\|u_0'\|_{H^4} + \|A_0'\|_{H^{5/2}} + \|A_1'\|_{H^{3/2}}\right)\|(u_0,A_0,A_1)-(u_0',A_0',A_1')\|_{L^2\times H^{1/2}\times H^{-1/2}} \\ &+ T\|(u,A)-(u',A')\|_{X_T}.\end{aligned}$$

A straightforward bootstrap argument (by choosing T sufficiently small) yields to

$$\begin{aligned}\|(u,A)-(u',A')\|_{X_T} &\lesssim \|(u_0,A_0,A_1)-(u_0',A_0',A_1')\|_X \\ &+ T\left(\|u_0'\|_{H^4} + \|A_0'\|_{H^{5/2}} + \|A_1'\|_{H^{3/2}}\right)\|(u_0,A_0,A_1)-(u_0',A_0',A_1')\|_{L^2\times H^{1/2}\times H^{-1/2}}.\end{aligned} \tag{3.47}$$

Proposition 3.3.8. *[Continuous dependence on the initial data]*
Let $T>0$. The mapping $(u_0,A_0,A_1) \mapsto (u,A,\partial_t A)$, , where (u,A) is the solution to (3.5)*, is continuous as a mapping from X to $C([0,T];X)$.*

Proof. In the following we take $T>0$ sufficiently small. Iterating the process below, the result can be extended to any compact subset of $[0,T_{max})$. Let us consider initial data $(u_0,A_0,A_1) \in X$. Let now $\{(u_{0,n},A_{0,n},A_{1,n})\} \subset X$ be a sequence converging to $(u_0,A_0,A_1) \in X$. Then $\{(u_{0,n},A_{0,n},A_{1,n})\}$ is bounded in X and, accordingly, the assumption (3.34) is satisfied.
Let us define a mollifier $\eta^\delta(x) = \delta^{-3}\eta(x/\delta)$, $\delta>0$, where $\eta \in C_c^\infty(\mathbb{R}^3)$ is a smooth,

radial function with $\int \eta = 1$. We define $u_0^\delta = \eta^\delta * u_0$, $A_0^{\delta^2} = \eta^{\delta^2} * A_0$, $A_1^{\delta^2} = \eta^{\delta^2} * A_1$. It is straightforward to check that this definition implies

$$\|u_0^\delta\|_{H^4} + \|A_0^{\delta^2}\|_{H^{5/2}} + \|A_1^{\delta^2}\|_{H^{3/2}} \lesssim \delta^{-2} (\|u_0\|_{H^2} + \|A_0\|_{H^{3/2}} + \|A_1\|_{H^{1/2}})$$

and that

$$\|u_0 - u_0^\delta\|_{L^2} + \|A_0 - A_0^{\delta^2}\|_{H^{1/2}} + \|A_1 - A_1^{\delta^2}\|_{H^{-1/2}} = 0(\delta^2).$$

By using (3.47) above we then infer

$$\begin{aligned} &\|(u,A) - (u^\delta, A^{\delta^2})\|_{X_T} \lesssim \|(u_0,A_0,A_1) - (u_0^\delta, A_0^{\delta^2}, A_1^{\delta^2})\|_X \\ &\quad + T\left(\|u_0^\delta\|_{H^4} + \|A_0^{\delta^2}\|_{H^{5/2}} + \|A_1^{\delta^2}\|_{H^{3/2}}\right) \|(u_0,A_0,A_1) - (u_0^\delta, A_0^{\delta^2}, A_1^{\delta^2})\|_{L^2 \times H^{1/2} \times H^{-1/2}} \\ &\lesssim \|(u_0,A_0,A_1) - (u_0^\delta, A_0^{\delta^2}, A_1^{\delta^2})\|_X + TO(\delta^2) o(\delta^{-2}). \end{aligned} \tag{3.48}$$

Consequently we have that (u^δ, A^{δ^2}) converges to (u,A) in X_T as $\delta \to 0$. We want to prove that the solutions (u_n, A_n) emanated from $(u_{0,n}, A_{0,n}, A_{1,n})$ converge to (u,A) in X_T. To do this, we regularize the initial data by considering $(u_{0,n}^\delta, A_{0,n}^{\delta^2}, A_{1,n}^{\delta^2})$. From (3.48) we know that $\{(u_n^\delta, A_n^{\delta^2})\}$ converges to $(u_, A_n)$ in X_T, as $\delta \to 0$, where (u_n, A_n) is the solution to (3.5) with initial data $(u_{0,n}, A_{0,n}, A_{1,n})$. On the other hand, $\{(u_{0,n}^\delta, A_{0,n}^{\delta^2}, A_{1,n}^{\delta^2})\}$ generate regular solutions, so that by (3.47) we have that $\{(u_n^\delta, A_n^{\delta^2})\}$ converges to (u^δ, A^{δ^2}) in X_T, for $n \to \infty$. The triangular inequality then yields the convergence of (u_n, A_n) to (u,A) in X_T. □

3.4 Global existence

In the previous Section we proved the local well-posedness of (3.5) in $H^2 \times H^{3/2}$. However, the presence of the power-type nonlinearity in (3.5) prevents from obtaining a global bound for $\|(u(t), A(t), \partial_t A(t))\|_X$. This is different, for example, from what can be proven in [97]. Indeed, while in the case of Hartree nonlinearity it is possible to use (3.12) which is linear in the higher order norm, in the case of the power-type nonlinearity one has

$$\||u|^{2(\gamma-1)} u\|_{H^2(\mathbb{R}^3)} \lesssim \|u\|_{L^\infty(\mathbb{R}^3)}^{2(\gamma-1)} \|u\|_{H^2(\mathbb{R}^3)},$$

which requires to bound u in $H^s(\mathbb{R}^3)$, with $s > \frac{3}{2}$. Therefore it follows that the related Gronwall type inequality becomes superlinear in the higher order norm, hence it blows up in finite time.

The major obstacle in this direction, as already stressed in the first section of this chapter, is represented by the lack of intrinsic Strichartz estimates for the Maxwell-Schrödinger system (3.5); indeed, this kind of estimates are a powerful tool in the analysis of the global existence of solutions in the field of nonlinear Schrödinger equations, as well as wave equations, as we recall briefly here.

We say that (q,r) are admissible Schrödinger exponents in $\mathbb{R}^n$, if $2 \le q \le \infty$, $2 \le r \le \frac{2n}{n-2}$ ($2 \le r \le \infty$ if $n=1$, $2 \le r < \infty$ if $n=2$) and

$$\frac{1}{q} = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{r}\right).$$

Now we can define the Strichartz norms in a space-time slab $I \times \mathbb{R}^3$ as

$$\|u\|_{S^0(I\times\mathbb{R}^3)} := \sup_{(q,r)\,\text{admissible}} \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)},$$

and for any $k \ge 1$

$$\|u\|_{\dot{S}^k} = \|\nabla^k u\|_{S^0}.$$

If we consider a Schwartz solution u to the non homogeneous Schrödinger $iu_t + \Delta u = F(u)$, then

$$\|u\|_{\dot{S}^k(I\times\mathbb{R}^3)} \lesssim \|u(t_0)\|_{\dot{H}^k(\mathbb{R}^3)} + C\|\nabla^k F\|_{L_t^{q'} L_x^{r'}(I\times\mathbb{R}^3)} \tag{3.49}$$

Let us suppose to have a quintic nonlineraity $F(u)$ (the energy-critical case in $\mathbb{R}^3$). Let $k = 0,1,2$ and u a finite energy solution such that $\|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \le R$, then, if $t_0 \in I$ and $u(t_0) \in \dot{H}^k$, we have

$$\|u\|_{\dot{S}^k(I\times\mathbb{R}^3)} \le C(R,E)\|u(t_0)\|_{\dot{H}^k}. \tag{3.50}$$

This follows by combining the following inequality

$$\|\nabla^k(F(u))\|_{L_t^1 L_x^2} \lesssim \|u\|_{L^{10}_{x,t}} \|u\|_{\dot{S}^k} \|u\|^3_{\dot{S}^1},$$

where the norm $\|u\|_{\dot{S}^1}$ is controlled by the energy bounds, with the Strichartz estimate(3.49)(see [21]). This leads to the global existence of solutions. To our knowledge, there are not intrinsic Strichartz estimates for (3.5); actually there are many Strichartz estimates available in the literature for the Schrödinger equations with a prescribed magnetic potential (see for instance [23, 24, 114] and references therein), but our solution to (3.5) does not fall in that class.

Our strategy to investigate global in time existence is based on the regularization of the nonlinear terms, provided by the classical Yosida approximations of the identity. We then consider the following approximating system

$$\begin{cases} iu_t^\varepsilon = -\dfrac{1}{2}\Delta_{\tilde{A}^\varepsilon} u^\varepsilon + \varphi^\varepsilon u^\varepsilon + N^\varepsilon(u^\varepsilon) \\ \Box A^\varepsilon = \mathscr{J}^\varepsilon \mathbb{P} J^\varepsilon \\ u^\varepsilon(0) = u_0,\ A^\varepsilon(0) = A_0,\ \partial_t A^\varepsilon(0) = A_1, \end{cases} \tag{3.51}$$

where $\mathscr{J}^\varepsilon = (I - \varepsilon\Delta)^{-1}$, $\tilde{A}^\varepsilon = \mathscr{J}^\varepsilon A^\varepsilon$, $N^\varepsilon(u^\varepsilon) = \mathscr{J}^\varepsilon\left(|\mathscr{J}^\varepsilon u^\varepsilon|^{2(\gamma-1)}\mathscr{J}^\varepsilon u^\varepsilon\right)$, $J^\varepsilon = J(u^\varepsilon, A^\varepsilon)$, $\varphi^\varepsilon = \varphi(|u^\varepsilon|^2)$ and we denote $\nabla_{\tilde{A}^\varepsilon} = \nabla - i\tilde{A}^\varepsilon$. The total energy of this approximating system is given by

$$E = \int_{\mathbb{R}^3} \left\{|\nabla_{\tilde{A}^\varepsilon} u^\varepsilon|^2 + \frac{1}{2}|\nabla\varphi^\varepsilon|^2 + \frac{1}{2}|\nabla A^\varepsilon|^2 + \frac{1}{2}|\partial_t A^\varepsilon|^2 + \frac{1}{\gamma}|\mathscr{J}^\varepsilon u^\varepsilon|^{2\gamma}\right\} dx \tag{3.52}$$

which is conserved along the flow of solutions. A local well-posedness result, analogous to Theorem 3.1.1, can be proved for the system (3.51) in a straightforward way. We remark that, from now, several constants will depend on ε and they are unbounded if $\varepsilon \to 0$.

Proposition 3.4.1. *For all* $(u_0, A_0, A_1) \in X$, *there exists* $T^{\varepsilon}_{max} > 0$ *and a unique maximal solution* $(u^{\varepsilon}, A^{\varepsilon})$ *to* (3.51) *such that*

- $u^{\varepsilon} \in C([0, T^{\varepsilon}_{max}); H^2(\mathbb{R}^3))$,
- $A^{\varepsilon} \in C([0, T^{\varepsilon}_{max}); H^{3/2}(\mathbb{R}^3)) \cap C^1([0, T^{\varepsilon}_{max}); H^{1/2}(\mathbb{R}^3))$

and the usual blow-up alternative holds true (see Theorem 3.1.1). Moreover, the solution depends continuously on the initial data.

Proof. We only remark here that the local well-posedness result for system (3.51) holds for any $\gamma \in (1, \infty)$, while in Theorem 3.1.1 we restrict the range to $\gamma \in (\frac{3}{2}, \infty)$. Indeed, because of the Yosida regularisation, we have

$$\|N^{\varepsilon}(u^{\varepsilon})\|_{H^2} \lesssim \| |\mathcal{J}^{\varepsilon} u^{\varepsilon}|^{2(\gamma-1)} \mathcal{J}^{\varepsilon} u^{\varepsilon} \|_{L^2} \lesssim \|u^{\varepsilon}\|^{2\gamma-1}_{H^2}.$$

□

From now on, several constants will depend on ε, and are unbounded as epsilon goes to zero.
The regularisation of the nonlinear terms yields indeed the global existence of solutions.

Proposition 3.4.2. *The solution obtained in Proposition (3.4.1) exists globally in time, namely* $\|(u^{\varepsilon}(t), A^{\varepsilon}(t), \partial_t A^{\varepsilon}(t))\|_X$ *belongs to* L^{∞}_{loc} *in time.*

The proof of Proposition 3.4.2 is based on the following

Lemma 3.4.3. *Let* $\varepsilon > 0$, *then for every* $t \in \mathbb{R}$,

$$\|u^{\varepsilon}(t)\|_{H^2} \leq C_1(\|u_0\|_{L^2}, E) e^{C_2 t \|\partial_t A^{\varepsilon}\|_{L^{\infty}_t H^{1/2}_x}}. \tag{3.53}$$

Proof. By (3.16) we have

$$\begin{aligned} \|u^{\varepsilon}\|_{H^2} \lesssim & \|\Delta_{\tilde{A}^{\varepsilon}} u^{\varepsilon}\|_{L^2} + \|A^{\varepsilon}\|^4_{H^1} \|u^{\varepsilon}\|_{L^2} \\ \leq & C(\|u_0\|_{L^2}, E) \|\Delta_{\tilde{A}^{\varepsilon}} u^{\varepsilon}\|_{L^2}, \end{aligned}$$

therefore it is convenient to estimate the norm $\|\Delta_{\tilde{A}^{\varepsilon}} u^{\varepsilon}\|_{L^2}$ instead of $\|u\|_{H^2(\mathbb{R}^3)}$. By a standard energy method (see the proof of Proposition 3.2.7) it follows that

$$\frac{d}{dt}\left(\|(\Delta_{\tilde{A}^{\varepsilon}} u^{\varepsilon})(t)\|_{L^2}\right) \leq \|\Delta_{\tilde{A}^{\varepsilon}}(\varphi^{\varepsilon} u^{\varepsilon})\|_{L^2} + \|\Delta_{\tilde{A}^{\varepsilon}} N^{\varepsilon}(u^{\varepsilon})\|_{L^2} + \|[\partial_t, \Delta_{\tilde{A}^{\varepsilon}}] u^{\varepsilon}\|_{L^2}.$$

The first term can be estimated by using (3.15) and (3.12),

$$\begin{aligned}\|\Delta_{\tilde{A}^\varepsilon}(\varphi^\varepsilon u^\varepsilon)\|_{L^2} \lesssim & \|\varphi^\varepsilon u^\varepsilon\|_{H^2} + \|A^\varepsilon\|_{H^1}^4 \|\varphi^\varepsilon u^\varepsilon\|_{L^2} \\ \lesssim & \|u^\varepsilon\|_{H^{3/4}}^2 \left(\|\Delta_{\tilde{A}^\varepsilon} u^\varepsilon\|_{L^2} + \|A^\varepsilon\|_{H^1}^4 \|u^\varepsilon\|_{L^2}\right) \\ \leq & C(\|u_0\|_{L^2}, E)\|\Delta_{\tilde{A}^\varepsilon} u^\varepsilon\|_{L^2}.\end{aligned}$$

The nonlinear term $N^\varepsilon(u^\varepsilon)$ can be controlled by exploiting the regularization given by $\mathscr{J}^\varepsilon$

$$\begin{aligned}\|\Delta_{\tilde{A}^\varepsilon} N^\varepsilon(u^\varepsilon)\|_{L^2} \lesssim & \|N^\varepsilon(u^\varepsilon)\|_{H^2} + \|A^\varepsilon\|_{H^1}^4 \|N^\varepsilon(u^\varepsilon)\|_{L^2} \\ \lesssim & \||\mathscr{J}^\varepsilon u^\varepsilon|^{2(\gamma-1)} \mathscr{J}^\varepsilon u^\varepsilon\|_{L^2} + \|A^\varepsilon\|_{H^1}^4 \|N^\varepsilon(u^\varepsilon)\|_{L^2} \\ \lesssim & \||\mathscr{J}^\varepsilon u^\varepsilon|^{2(\gamma-1)}\|_{L^2} \|\mathscr{J}^\varepsilon u^\varepsilon\|_{L^\infty} + \|A^\varepsilon\|_{H^1}^4 \|N^\varepsilon(u^\varepsilon)\|_{L^2} \\ \lesssim & \|\mathscr{J}^\varepsilon u^\varepsilon\|_{L^{4(\gamma-1)}}^{2(\gamma-1)} \|u^\varepsilon\|_{H^2} + \|A^\varepsilon\|_{H^1}^4 \|u^\varepsilon\|_{L^2} \\ \lesssim & \|\mathscr{J}^\varepsilon u^\varepsilon\|_{H^2}^{2(\gamma-1)} \|u^\varepsilon\|_{H^2} + \|A^\varepsilon\|_{H^1}^4 \|u^\varepsilon\|_{L^2} \\ \lesssim & \|u^\varepsilon\|_{H^2} + \|A^\varepsilon\|_{H^1}^4 \|u^\varepsilon\|_{L^2},\end{aligned}$$

where we used the Sobolev embeddings $H^2(\mathbb{R}^3) \hookrightarrow L^{4(\gamma-1)}(\mathbb{R}^3)$, $H^4(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ and the conservation of the mass. The commutator $[\partial_t, \Delta_{\tilde{A}^\varepsilon}]u^\varepsilon = 2\partial_t \tilde{A}^\varepsilon(\nabla + i\tilde{A}^\varepsilon)u^\varepsilon$ can be estimated by using the Hölder's inequality and the Sobolev embedding

$$\begin{aligned}\|\partial_t \tilde{A}^\varepsilon \cdot (\nabla + i\tilde{A}^\varepsilon)u^\varepsilon\|_{L^2} \leq & \|\partial_t \tilde{A}^\varepsilon\|_{L^3} \|(\nabla + i\tilde{A}^\varepsilon)u^\varepsilon\|_{L^6} \\ \lesssim & \|\partial_t A^\varepsilon\|_{H^{1/2}} \left(\|\Delta_{\tilde{A}^\varepsilon} u^\varepsilon\|_{L^2} + \|A^\varepsilon\|_{H^1}^4 \|u^\varepsilon\|_{L^2}\right).\end{aligned}$$

By summing up the previous three terms

$$\frac{d}{dt}\left(\|(\Delta_{\tilde{A}^\varepsilon} u^\varepsilon)(t)\|_{L^2}\right) \leq C(\|u_0\|_{L^2}, E)\|\partial_t A^\varepsilon\|_{H^{1/2}} \|\Delta_{\tilde{A}^\varepsilon} u^\varepsilon\|_{L^2},$$

hence (3.53). □

Proof of Proposition 3.4.2. In order to get a bound on the H^2 norm of the approximating solution u^ϵ, by Lemma 3.4.3 it is sufficient to control $\|\partial_t A^\varepsilon\|_{L_t^\infty H_x^{1/2}}$. Using the energy estimate for the wave equation

$$\|A^\varepsilon\|_{L_t^\infty H_x^{3/2}} + \|\partial_t A^\varepsilon\|_{L_t^\infty H_x^{1/2}} \lesssim C(T)\Big(\|A_0\|_{H^{3/2}} + \|A_1\|_{H^{1/2}} + \|\mathscr{J}^\varepsilon \mathbb{P} J^\varepsilon\|_{L_t^\infty H_x^{1/2}}\Big),$$

and, by exploiting the Yosida regularization, we get

$$\|\mathscr{J}^\varepsilon \mathbb{P} J^\varepsilon\|_{L_t^\infty H_x^{1/2}} \lesssim \|\mathbb{P} J^\varepsilon\|_{L_t^\infty H_x^{-1/2}} \lesssim \|J^\varepsilon\|_{L_t^\infty L_x^{3/2}} \leq C(E).$$

It follows that $\|A^\varepsilon(t)\|_{H^{3/2}} + \|\partial_t A^\varepsilon(t)\|_{H^{1/2}}$ is uniformly bounded on compact time intervals and consequently by (3.53) also $\|u^\varepsilon(t)\|_{H^2}$ is finite. Hence, by the blow-up alternative, the solution $(u^\varepsilon, A^\varepsilon)$ to (3.51) exists globally in time. □

Now we conclude the proof of Theorem 3.1.2 by showing that $(u^\varepsilon, A^\varepsilon)$ converges to a solution to (3.5), as $\varepsilon \to 0$.
The conservation of mass and energy yields the following a priori bounds

$$\begin{aligned} &\|u^\varepsilon\|_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^3)} \le C, \\ &\|A^\varepsilon\|_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^3)} \le C, \quad \|\partial_t A^\varepsilon\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^3)} \le C, \end{aligned} \tag{3.54}$$

which imply that, up to subsequences, there exist $u \in L_t^\infty H_x^1$, $A \in L_t^\infty H_x^1 \cap W_t^{1,\infty} L_x^2$, such that

$$u^\varepsilon \overset{*}{\rightharpoonup} u \quad \text{in} \quad L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^3) \tag{3.55}$$

$$A^\varepsilon \overset{*}{\rightharpoonup} A \quad \text{in} \quad L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^3) \tag{3.56}$$

$$\partial_t A^\varepsilon \overset{*}{\rightharpoonup} \partial_t A \quad \text{in} \quad L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^3) \tag{3.57}$$

Remark 3.4.4. *We want to stress here that, actually, the conservation of mass and energy gives a uniform bound on* $\|u\|_{H_A^1} := \|(\nabla - iA)u\|_{L^2(\mathbb{R}^3)}$*, from which the first inequality in* (3.54) *easily follows. Indeed*

$$\begin{aligned} \|u\|_{H^1} &\le \|u\|_{H_A^1} + \|Au\|_{L^2} \le E + \|A\|_{L^6}\|u\|_{L^3} \\ &\le E + \|A\|_{H^1}\|u\|_{L^3} \lesssim_E 1 + \|u\|_{L^3} \\ &\lesssim_E 1 + \|u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}} \lesssim_{M,E} 1, \end{aligned}$$

where we used the Young inequality and we put $M = \|u\|_{L^2}$.

Proposition 3.4.5. *The weak limit* (u, A) *in (3.55), (3.56) is a finite energy weak solution to the Cauchy problem (3.5), with initial datum* (u_0, A_0, A_1).

Proof. Let us consider u^ε, by using equation (3.51) and the a priori bounds given by the energy we have $\{\partial_t u^\varepsilon\}$ is uniformly bounded in $L^\infty(\mathbb{R}; H^{-1}(\mathbb{R}^3))$. Indeed, for the right handside of (3.51), we have

$$\begin{aligned} \|A^\varepsilon \cdot \nabla u^\varepsilon\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} &\le C\|u^\varepsilon\|_{H^1(\mathbb{R}^3)}\|\nabla \tilde{A}^\varepsilon\|_{L^2(\mathbb{R}^3)} \le C \\ \||A^\varepsilon|^2 u^\varepsilon\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} &\le C\|\nabla A^\varepsilon\|_{L(\mathbb{R}^3)}^2\|u^\varepsilon\|_{H^1(\mathbb{R}^3)} \le C \\ \|\varphi(u^\varepsilon)u^\varepsilon\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} &\le C\|u^\varepsilon\|_{H^1(\mathbb{R}^3)}^3 \le C. \end{aligned}$$

Moreover

$$\begin{aligned} \|N^\varepsilon(u^\varepsilon)\|_{L^{\frac{2\gamma}{2\gamma-1}}} &\le \||\mathcal{J}^\varepsilon u^\varepsilon|^{2(\gamma-1)}\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3)}\|\mathcal{J}^\varepsilon u^\varepsilon\|_{L^{2\gamma}(\mathbb{R}^3)} \\ &\le \|\mathcal{J}^\varepsilon u^\varepsilon\|_{L^{2\gamma}(\mathbb{R}^3)}^{2\gamma-1} \le C, \end{aligned}$$

with $\frac{2\gamma}{2\gamma-1} \in (\frac{6}{5}, 2]$. Hence, by using the Aubin-Lions lemma and from the assumption $1 < \gamma < 3$ we may infer

$$u^\varepsilon \to u \quad \text{in} \quad L_{loc}^4(\mathbb{R}\times\mathbb{R}^3) \cap L_{loc}^{2\gamma}(\mathbb{R}\times\mathbb{R}^3). \tag{3.58}$$

This also implies that $|u^\varepsilon|^2 \rightharpoonup |u|^2$ in $L^2_t L^{6/5}_x$, and consequently, since Δ^{-1} is bounded from $L^{\frac{6}{5}}$ to L^6, we obtain

$$(-\Delta)^{-1}(|u^\varepsilon|^2) \rightharpoonup (-\Delta)^{-1}(|u|^2), \quad \text{in } L^2_t L^6_x. \tag{3.59}$$

Analogously for A^ε, the a priori bounds yield

$$A^\varepsilon \to A \;\text{ in }\; L^4_{loc}(\mathbb{R}\times\mathbb{R}^3)\,. \tag{3.60}$$

We are now able to show the convergence for the nonlinear terms $K^\varepsilon(u^\varepsilon, A^\varepsilon)$, $N^\varepsilon(u^\varepsilon)$, $\mathscr{J}^\varepsilon \mathbb{P} J^\varepsilon$, where

$$K^\varepsilon(u^\varepsilon, \tilde{A}^\varepsilon) = i\tilde{A}^\varepsilon\cdot\nabla u^\varepsilon + \frac{1}{2}|\tilde{A}^\varepsilon|^2 u^\varepsilon + \varphi(u^\varepsilon)u^\varepsilon\,,$$

Indeed, by using the convergences (3.55)-(3.60) we may conclude

$$\begin{aligned} K^\varepsilon(u^\varepsilon, \tilde{A}^\varepsilon) &\rightharpoonup K(u,A) \;\text{ in }\; L^{\frac{4}{3}}_{loc}(\mathbb{R}\times\mathbb{R}^3)\,,\\ \mathbb{P}J(u^\varepsilon, \tilde{A}^\varepsilon) &\rightharpoonup \mathbb{P}J(u,A) \;\text{ in }\; L^{\frac{4}{3}}_{loc}(\mathbb{R}\times\mathbb{R}^3)\,,\\ N^\varepsilon(u^\varepsilon) &\rightharpoonup N(u), \quad \text{in } L^{\frac{2\gamma}{2\gamma-1}}_{loc}(\mathbb{R}\times\mathbb{R}^3). \end{aligned}$$

It remains to see that the initial condition is satisfied. We have that $\partial_t A \in L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^3)$ and $\partial_t^2 A, \partial_t u \in L^\infty_t H^{-1}_x(\mathbb{R}\times\mathbb{R}^3)$, and consequently $(u, A, \partial_t A) \in C(\mathbb{R}; H^{-1}\times L^2\times H^{-1})$. Moreover, the energy bounds imply $(u, A, \partial_t A) \in L^\infty(\mathbb{R}; H^1\times H^1\times L^2)$ and hence we may also infer the weak continuity $(u, A, \partial_t A) \in C_w(\mathbb{R}; H^1\times H^1\times L^2)$.

Since $A^\varepsilon \in L^2(0,T; H^1(\mathbb{R}^3))$ and $\partial_t A^\varepsilon \in L^2(0,T; L^2(\mathbb{R}^3))$, integrating by parts we have

$$\int_0^T \langle A^\varepsilon(t)\partial_t f(t) + \partial_t A^\varepsilon(t) f(t), \varphi\rangle_{H^1, H^{-1}}\, ds = -\langle A_0, \varphi\rangle$$

for every $\varphi \in L^2(\mathbb{R}^3)$ and all $f \in C^\infty(\mathbb{R})$ with $f(0) = 1$ and $f(T) = 0$. As $\varepsilon \to 0$ we obtain

$$\int_0^T \{A(t)\partial_t f(t) + \partial_t A(t) h(t)\}\, dt = -A_0$$

in $L^2(\mathbb{R}^3)$, which implies $A|_{t=0} = A_0$. Now we have

$$\int_0^T \left\langle \partial_t A^\varepsilon \partial_t f(t) + \{\Delta A^\varepsilon - \mathbb{P}J(u^\varepsilon, \tilde{A}^\varepsilon)\} f(t), \eta \right\rangle = \langle A_1, \eta\rangle\;,$$

and as $\varepsilon \to 0$, we find

$$\int_0^T \{\partial_t A(t)\partial_t f(t) + \partial_t^2 A(t) f(t)\} = A_1$$

in $H^{-1}(\mathbb{R}^3)$, which gives us $\partial_t A|_{t=0} = A_1$. Applying the same argument to u^ε we deduce that $u|_{t=0} = u_0$. □

3.5 Quantum Magnetohydrodynamics

Our last Section is devoted to point out the relation between the nonlinear Maxwell-Schrödinger system (3.5) and quantum magnetohydrodynamic (QMHD) models. Such hydrodynamic systems have been introduced in the physics literature, motivated by various applications to semiconductor devices, dense astrophysical plasmas (e.g. in white dwarfs), or laser plasmas [63, 64, 111, 112].

The quantum magnetohydrodynamic equations consist of the continuity equation and the electron and ion momentum equations (here i =ions and e =electrons)

$$\frac{\partial n_{e,i}}{\partial t} + \nabla \cdot (n_{e,i}\mathbf{u}_{e,i}) = 0,$$

$$n_e m_e \left(\frac{\partial}{\partial t} + \mathbf{u}_e \cdot \nabla \right) \mathbf{u}_e = -n_e e \left(\mathbf{E} + \frac{1}{c}\mathbf{u}_e \times \mathbf{B} \right) - \nabla P_e + n_e \mathbf{F}_{Qe},$$

$$n_i m_i \left(\frac{\partial}{\partial t} + \mathbf{u}_i \cdot \nabla \right) \mathbf{u}_i = Z_i e n_i \left(\mathbf{E} + \frac{1}{c}\mathbf{u}_i \times \mathbf{B} \right),$$

the Faraday law

$$c\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

and the Maxwell equation including the magnetization spin current

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\left(\mathbf{J}_p + \mathbf{J}_m \right) + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

where n_j and m_j are the number density and the mass of the particle species j respectively, u_j is the particle fluid velocity, Z_i is the ion charge state, $\mathbf{J}_p = -n_e e \mathbf{u}_e + Z_i n_i e \mathbf{u}_i$ is the plasma current density and $\mathbf{J}_m = \nabla \times \mathbf{M}$ is the electron magnetization spin current density, with $\mathbf{M} = (n_e \mu_B^2 / k_B T_{Fe})\mathbf{B}$, where $\mu_B = e\hbar/2m_e c$ is the Bohr magneton (magnetic moment of an electron caused by either its orbital or spin angular momentum). The term $\mathbf{F}_{Qe}$ is the sum of the quantum Bohm potential and intrinsic angular momentum spin forces

$$\mathbf{F}_{Qe} = \nabla \left(\frac{\nabla^2 \sqrt{n_e}}{\sqrt{n_e}} \right) - \frac{\mu_B^2}{k_B T_{Fe}} \nabla B,$$

where $B = |\mathbf{B}|$. The pressure for degenerate (close to zero temperature) electrons is given by ([29])

$$P_e = \frac{4eB(2m_e)^{1/2} E_F^{3/2}}{3(2\pi)^2 \hbar^2 c} \left[1 + 2\sum_{n_L=1}^{n_{max}} \left(1 - \frac{n_L \hbar \omega_B}{E_F} \right)^{3/2} \right],$$

where p_F and E_F are the Fermi momentum and the Fermi energy respectively. If $\frac{p_F^2}{2m} < \hbar\omega_B$ (strong Landau quantization condition) one has the following density scaling laws

- If $B=0$ then $p_F \approx n_e^{1/3}, E_F \approx n_e^{2/3}, P_e \approx n_e^{5/3}$
- If $B>0$ then $p_F \approx n_e, E_F \approx n_e^2, P_e \approx n_e^3$

As we can see, under certain conditions, the pressure term can be approximated by a power law and this motivates the introduction of the nonlinear power-like potentials in (3.5).
The above equations are written for a two species charged particle system (bipolar quantum fluid model). As a simplification, we focus the attention on a one-species charged quantum plasma, with self-generated electromagnetic fields, whose dynamics is described by

$$\begin{cases} \partial_t \rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}\left(\dfrac{J \otimes J}{\rho}\right) + \nabla P(\rho) = \rho E + J \wedge B + \dfrac{1}{2}\rho \nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases} \tag{3.61}$$

where ρ denotes the charge density and J the current density of the quantum fluid. Here all the constants are normalized to one. The pressure term $P(\rho)$ is assumed to be isentropic of the form $P(\rho) = \frac{\gamma-1}{\gamma}\rho^\gamma$, $1<\gamma<3$. The last term in the equation for the current density can be written in different ways

$$\frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \frac{1}{4}\nabla\Delta\rho - \operatorname{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}) = \frac{1}{4}\operatorname{div}(\rho\nabla^2\log\rho). \tag{3.62}$$

and it can be seen as a self-consistent quantum potential (the so called Bohm potential) or as a quantum correction to the stress tensor. Mathematically speaking, this is a third order nonlinear dispersive term. The hydrodynamical system above is complemented by the Maxwell equations for the electromagnetic fields E and B

$$\begin{cases} \operatorname{div} E = \rho, \quad \nabla \wedge E = -\partial_t B \\ \operatorname{div} B = 0, \quad \nabla \wedge B = J + \partial_t E. \end{cases} \tag{3.63}$$

In recent years a global existence theory of finite energy weak solutions for a class a quantum hydrodynamic systems has been established in [6–8]. By means of a polar factorization techinque it is possible to define the hydrodynamic quantities by considering the Madelung transform of a wave function solution to a nonlinear Schrödinger equation. In this way the definition of the velocity field in the nodal regions is no longer needed. We also mention in the H^2 case the construction given in [49]. Furthermore it could be interesting to consider also confining potentials as in [4], generated by external magnetic fields. The aim of this Section is to show the existence of a finite energy weak solution to (3.61)-(3.63) by taking advantage of our results on the system (3.5).

Definition 3.5.1. *We define the total mass and energy for the system* (3.61)-(3.63) *as*

$$M(t) := \int_{\mathbb{R}^3} \rho(t,x)\,dx, \tag{3.64}$$

$$E(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}|\Lambda|^2 + f(\rho) + \frac{1}{2}|\partial_t A|^2 + \frac{1}{2}|\nabla A|^2 + \frac{1}{2}|\nabla\phi|^2\,dx\,, \quad (3.65)$$

where $f(\rho) = \frac{1}{\gamma}\rho^\gamma$.

Definition 3.5.2. *Let* $\rho_0, J_0, E_0, B_0 \in L^1_{loc}(\mathbb{R}^3)$ *such that* $M(0)$ *and* $E(0)$ *are finite. A finite energy weak solution to system* (3.61)-(3.63) *in the space-time slab* $[0,T)\times\mathbb{R}^3$ *is given by a quadruple* $(\sqrt{\rho},\Lambda,\phi,A)$ *such that*

1. $\sqrt{\rho}\in L^\infty([0,T);H^1(\mathbb{R}^3))$, $\Lambda\in L^\infty([0,T);L^2(\mathbb{R}^3))$, $\phi\in L^\infty([0,T);H^1(\mathbb{R}^3))$, $A\in L^\infty([0,T);H^1(\mathbb{R}^3))\cap W^{1,\infty}([0,T);L^2(\mathbb{R}^3))$;

2. $\rho := (\sqrt{\rho})^2$, $J := \sqrt{\rho}\Lambda$, $E := -\partial_t A - \nabla\varphi$, $B := \nabla\wedge A$;

3. $J\in L^2([0,T);L^2_{loc}(\mathbb{R}^3))$;

4. $\forall\ \eta\in C_c^\infty([0,T)\times\mathbb{R}^3)$,
$$\int_0^T\int_{\mathbb{R}^3} \rho\partial_t\eta + J\cdot\nabla\eta\,dxdt + \int_{\mathbb{R}^3}\rho_0(x)\eta(0,x)\,dx = 0;$$

5. $\forall\zeta\in C_c^\infty([0,T)\times\mathbb{R}^3;\mathbb{R}^3)$,
$$\int_0^T\int_{\mathbb{R}^3} J\cdot\partial_t\zeta + \Lambda\otimes\Lambda : \nabla\zeta + P(\rho)\,\mathrm{div}\,\zeta + \rho E\cdot\zeta + (J\wedge B)\cdot\zeta$$
$$+\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} : \nabla\zeta + \frac{1}{4}\rho\Delta\,\mathrm{div}\,\zeta\,dxdt + \int_{\mathbb{R}^3} J_0(x)\cdot\zeta(0,X)\,dx = 0;$$

6. E, B *satisfy* (3.63) *in* $[0,T)\times\mathbb{R}^3$ *in the sense of distributions;*

7. *(finite energy) The total mass and energy defined by* (3.64) *and* (3.65) *respectively, are finite for every* $t\in[0,T)$.

Proposition 3.5.3. *Let* (ρ_0, J_0, B_0, E_0) *be such that* $\rho_0 := |u_0|^2$, $J_0 := \mathrm{Re}(\bar{u}_0(-i\nabla - A_0)u_0)$, $B_0 := \nabla\wedge A_0$, $E_0 := -A_1 - \nabla\phi_0$, $\phi_0 := (-\Delta)^{-1}|u_0|^2$ *for some* $(u_0, A_0, A_1)\in X$, *then there exists* $T_{max} > 0$ *such that* $(\sqrt{\rho},\Lambda,\phi,A)$ *is a finite energy weak solution to* (3.61)-(3.63) *with initial data* (ρ_0, J_0, B_0, E_0) *in the space-time slab* $[0,T_{max})\times\mathbb{R}^3$. *Moreover, the total mass and energy are conserved for all* $t\in[0,T_{max})$.

To prove this Proposition we are going to use a polar factorization argument, in analogy with the electrostatic case treated in [6, 8] (see Appendix B).
Given any complex valued fuction $u\in H^1(\mathbb{R}^3)$, we may define the set of its polar factors as

$$P(u) := \{\varphi\in L^\infty(\mathbb{R}^3)\ :\ \|\varphi\|_{L^\infty}\le 1, u = \sqrt{\rho}\varphi\ \text{ a.e. in } \mathbb{R}^3\},$$

where $\sqrt{\rho} := |u|$. Thus, for any $\varphi \in P(u)$, we have $|\varphi| = 1$ $\sqrt{\rho}\,dx$ a.e. in $\mathbb{R}^3$ and φ is uniquely defined $\sqrt{\rho}\,dx$ a.e. in $\mathbb{R}^3$. Clearly the polar factor is not uniquely defined in the nodal regions, i.e. in the set $\{\rho = 0\}$.
In the following Lemma we exploit the polar factorization of a given wave function ψ in order to define the hydrodynamical quantities associated to ψ. This approach overcomes the WKB ansatz in the finite energy framework and allows to define the hydrodynamical quantities almost everywhere in the space, without passing through the construction of the velocity field, which is not uniquely defined in the nodal region. Furthermore, we show how this definition which uses the polar factorization is stable in $H^1(\mathbb{R}^3)$.

Lemma 3.5.4. *Let $u \in H^1(\mathbb{R}^3)$, $A \in L^3(\mathbb{R}^3)$, and let $\sqrt{\rho} := |u|$, $\varphi \in P(u)$. Let us define $\Lambda := \mathrm{Re}(\bar{\varphi}(-i\nabla - A)u) \in L^2(\mathbb{R}^3)$, then we have*

- $\sqrt{\rho} \in H^1(\mathbb{R}^3)$ *and* $\nabla\sqrt{\rho} = \mathrm{Re}(\bar{\varphi}\nabla u)$*;*
- *the following identity holds a.e. in* $\mathbb{R}^3$*,*

$$\mathrm{Re}\{\overline{(-i\nabla - A)u} \otimes (-i\nabla - A)u\} = \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} + \Lambda \otimes \Lambda. \tag{3.66}$$

Moreover, let $\{u_n\} \subset H^1(\mathbb{R}^3)$, $\{A_n\} \subset L^3(\mathbb{R}^3)$ be such that u_n converges strongly to u in H^1 and A_n converges strongly to A in L^3, then we have

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \quad \Lambda_n \to \Lambda, \quad \text{in } L^2(\mathbb{R}^3),$$

where $\sqrt{\rho_n} := |u_n|$, $\Lambda_n := \mathrm{Re}(\bar{\varphi}_n(-i\nabla - A_n)u_n)$.

Proof. Let $u \in H^1(\mathbb{R}^3)$ and let us consider a sequence of smooth functions converging to u, $\{u_n\} \subset C_c^\infty(\mathbb{R}^3)$, $u_n \to u$ in $H^1(\mathbb{R}^3)$. For each u_n we may define

$$\varphi_n(x) := \begin{cases} \dfrac{u_n(x)}{|u_n(x)|} & \text{if } u_n(x) \neq 0 \\ 0 & \text{if } u_n(x) = 0. \end{cases}$$

The φ_n's are clearly polar factors for the wave functions u_n. Since $\|\varphi_n\|_{L^\infty} \leq 1$, then (up to subsequences) there exists $\varphi \in L^\infty(\mathbb{R}^3)$ such that

$$\varphi_n \overset{*}{\rightharpoonup} \varphi, \quad L^\infty(\mathbb{R}^d). \tag{3.67}$$

It is easy to check that φ is indeed a polar factor for u. Since $\{u_n\} \subset C_c^\infty(\mathbb{R}^3)$, we have

$$\nabla\sqrt{\rho_n} = \mathrm{Re}(\bar{\varphi}_n\nabla u_n), \quad \text{a.e. in } \mathbb{R}^3.$$

It follows from the convergence above

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \quad L^2(\mathbb{R}^3)$$
$$\mathrm{Re}(\bar{\varphi}_n\nabla u_n) \rightharpoonup \mathrm{Re}(\bar{\varphi}\nabla u), \quad L^2(\mathbb{R}^3),$$

thus $\nabla\sqrt{\rho} = \mathrm{Re}(\bar{\varphi}\nabla u)$ in $L^2(\mathbb{R}^3)$ and consequently the equality holds a.e. in $\mathbb{R}^3$. It should be noted that here we have $\nabla\sqrt{\rho} = \mathrm{Re}(\bar{\varphi}\nabla u)$, where φ is the weak$-*$ limit in (3.67). However the identity above for $\nabla\sqrt{\rho}$ does not depend on the choice of φ. Indeed, by Theorem 6.19 in [87] we have $\nabla u = 0$ for almost every $x \in u^{-1}(\{0\})$ and, on the other hand, φ is uniquely determined on $\{x \in \mathbb{R}^3 : |u(x)| > 0\}$ almost everywhere. Consequently, for any $\varphi_1, \varphi_2 \in P(u)$, we have $\mathrm{Re}(\bar{\varphi}_1\nabla u) = \mathrm{Re}(\bar{\varphi}_2\nabla u) = \nabla\sqrt{\rho}$. The same argument applies for $\Lambda := \mathrm{Re}(\bar{\varphi}(-\nabla - A)u)$. Let us now prove the identity (B.8). Recall that we have $|\varphi| = 1$ $\sqrt{\rho}\,dx$ a.e. in $\mathbb{R}^3$, hence again by invoking Theorem 6.19 in [87] we have

$$\begin{aligned}
\mathrm{Re}\{\overline{(-i\nabla - A)u} \otimes (-i\nabla - A)u\} =& \mathrm{Re}\left\{\left(\varphi\overline{(-i\nabla - A)u}\right) \otimes (\bar{\varphi}(-i\nabla - A)u)\right\} \\
=& \mathrm{Re}\{\varphi\overline{(-i\nabla - A)u}\} \otimes \mathrm{Re}\{\bar{\varphi}(-i\nabla - A)u\} \\
&- \mathrm{Im}\{\varphi\overline{(-i\nabla - A)u}\} \otimes \mathrm{Im}\{\bar{\varphi}(-i\nabla - A)u\} \\
=& \Lambda \otimes \Lambda + \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho},
\end{aligned}$$

a.e. in $\mathbb{R}^3$. Furthermore, by taking the trace on both sides of the above equality we furthermore obtain

$$|(-i\nabla - A)u|^2 = |\nabla\sqrt{\rho}|^2 + |\Lambda|^2. \tag{3.68}$$

For the second part of the Lemma, let us consider a sequence $\{u_n\} \subset H^1$ strongly converging to $u \in H^1$ and vector fields $\{A_n\} \subset L^3$ strongly converging to $A \in L^3$. As before it is straightforward to show that

$$\begin{aligned}
\mathrm{Re}(\bar{\varphi}_n\nabla u_n) &\rightharpoonup \mathrm{Re}(\bar{\varphi}\nabla u), \quad L^2 \\
\mathrm{Re}(\bar{\varphi}_n(-i\nabla - A_n)u_n) &\rightharpoonup \mathrm{Re}(\bar{\varphi}(-i\nabla - A)u), \quad L^2.
\end{aligned}$$

Moreover, from (3.68), the strong convergence of u_n and the weak convergence for $\nabla\sqrt{\rho_n}, \Lambda_n$, we obtain

$$\begin{aligned}
\|(-i\nabla - A)u\|_{L^2}^2 =& \|\nabla\sqrt{\rho}\|_{L^2}^2 + \|\Lambda\|_{L^2}^2 \leq \liminf_{n\to\infty}\left(\|\nabla\sqrt{\rho_n}\|_{L^2}^2 + \|\Lambda_n\|_{L^2}^2\right) \\
=& \lim_{n\to\infty}\|(-i\nabla - A_n)u_n\|_{L^2}^2 = \|(-i\nabla - A)u\|_{L^2}^2.
\end{aligned}$$

Hence, we obtain $\|\nabla\sqrt{\rho_n}\|_{L^2} \to \|\nabla\sqrt{\rho}\|_{L^2}$ and $\|\Lambda_n\|_{L^2} \to \|\Lambda\|_{L^2}$. Consequently, from the weak convergence in L^2 and the convergence of the L^2 norms we may infer the strong convergence

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \quad \Lambda_n \to \Lambda, \quad \text{in } L^2(\mathbb{R}^3).$$

□

In view of Lemma 3.5.4 we can now prove Proposition 3.5.3. Let $(u_0, A_0, A_1) \in X$ be given, then by our main Theorem 3.1.1 there exists a unique solution (u, A) to (3.5) in $[0, T_{max}) \times \mathbb{R}^3$ such that $u \in C([0, T_{max}); H^2(\mathbb{R}^3)), A \in C([0, T_{max}); H^{3/2}(\mathbb{R}^3)) \cap C^1([0, T_{max}); H^{1/2}(\mathbb{R}^3))$. Let us now define $\sqrt{\rho} := |u|$, $\Lambda := \mathrm{Re}(\bar{\varphi}(-i\nabla + A)u)$, where

φ is a polar factor for u, and let $\phi := (-\Delta)^{-1}\rho$. By differentiating ρ with respect to time we have

$$\begin{aligned}\partial_t\rho =& 2\,\mathrm{Re}\left\{\bar u\left(-\frac{i}{2}(-i\nabla-A)^2u - i\phi u - i|u|^{2(\gamma-1)}u\right)\right\}\\ =& \mathrm{Im}\left\{\bar u(-i\nabla-A)^2u\right\}\\ =& \mathrm{Im}\left\{-i\,\mathrm{div}\left(\bar u(-i\nabla-A)u + \overline{(-i\nabla-A)u}\cdot(-i\nabla-A)u\right)\right\}\\ =& -\mathrm{div}\left(\mathrm{Re}(\bar u(-i\nabla-A)u)\right).\end{aligned}$$

Hence by defining $J = \mathrm{Re}\,(\bar u(-i\nabla-A)u) = \sqrt{\rho}\Lambda$ we obtain the continuity equation for ρ

$$\partial_t\rho + \mathrm{div}\,J = 0.$$

Now let us differentiate J with respect to time,

$$\begin{aligned}\partial_t J =& \mathrm{Re}\left\{\left(\frac{i}{2}\overline{(-i\nabla-A)^2u} + i\phi\bar u + i|u|^{2(\gamma-1)}\bar u\right)(-i\nabla-A)u\right\}\\ &+ \mathrm{Re}\left\{\bar u(-i\nabla-A)\left(-\frac{i}{2}(-i\nabla-A)^2u - i\phi u - i|u|^{2(\gamma-1)}u\right)\right\} - \rho\,\partial_t A\\ =& \frac{1}{2}\,\mathrm{Im}\left\{\bar u(-i\nabla-A)\left((-i\nabla-A)^2u\right) - \overline{(-i\nabla-A)^2u}(-i\nabla-A)u\right\}\\ &+ \mathrm{Re}\left\{\bar u(\phi + |u|^{2(\gamma-1)})\nabla u - \bar u\nabla\left(\phi u + |u|^{2(\gamma-1)}u\right)\right\} - \rho\,\partial_t A.\end{aligned}$$

Now the last line equals $\rho\nabla\phi - \rho\nabla\rho^{\gamma-1} - \rho\,\partial_t A = \rho(-\partial_t A - \nabla\phi) + \nabla P(\rho)$, where $P(\rho) = \frac{\gamma-1}{\gamma}\rho^\gamma$. After some tedious but rather straightforward calculations we may see that

$$\begin{aligned}&\frac{1}{2}\,\mathrm{Im}\left\{\bar u(-i\nabla-A)\left((-i\nabla-A)^2u\right) - \overline{(-i\nabla-A)^2u}(-i\nabla-A)u\right\}\\ &\qquad = \frac{1}{4}\nabla\Delta\rho - \mathrm{div}\left(\mathrm{Re}\left\{\overline{(-i\nabla-A)u}\otimes(-i\nabla-A)u\right\}\right) + J\wedge(\nabla\wedge A).\end{aligned}$$

By putting everything together we then obtain

$$\begin{aligned}&\partial_t J + \mathrm{div}\left(\mathrm{Re}\{\overline{(-i\nabla-A)u}\otimes(-i\nabla-A)u\}\right) + \nabla P(\rho) =\\ &\qquad\qquad \rho(-\partial_t A - \nabla\phi) + J\wedge(\nabla\wedge A) + \frac{1}{4}\nabla\Delta\rho.\end{aligned}$$

We now use the polar factorization Lemma to infer that

$$\mathrm{Re}\{\overline{(-i\nabla-A)u}\otimes(-i\nabla-A)u\} = \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} + \Lambda\otimes\Lambda$$

and consequently we get

$$\partial_t J + \mathrm{div}(\Lambda\otimes\Lambda) + \nabla P(\rho) = \rho E + J\wedge B + \frac{1}{4}\nabla\Delta\rho - \mathrm{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}).$$

By recalling identity (3.62) we see that this is the equation for the current density in the QMHD system (3.61). The above calculations are rigorous only when (u, A) are sufficiently regular, however for solutions to (3.5) considered in Theorem 3.1.1 they can be rigorously justified in the weak sense, namely in the sense of Definition 3.5.2 by regularizing the initial data and by exploiting the continuous dependence showed in Proposition 3.3.8 and the H^1-stability of the polar factorization stated in Lemma 3.5.4.
It only remains to prove that E, B satisfy the Maxwell equations, but this comes in a straightforward way from the wave equation in (3.5) and the definitions $E = -\partial_t A - \nabla\phi$, $B = \nabla \wedge A$.
Finally we remark that for solutions (u, A) to (3.5) considered in Theorem 3.1.1 the total energy (3.35) is conserved. Again by using Lemma 3.5.4 we see that the energy in (3.35) equals the one defined in (3.65) this equals the energy defined in (3.65). This concludes the proof of Proposition 3.5.3.

Appendix A

Oscillatory integral operators

We collect here some results about oscillatory integral transformations which are used in the Chapter 1.
Let consider integral transformations of the following type:

$$I(t,s;a)f(x) = \nu^{\frac{n}{2}} \int_{\mathbb{R}^n} a(x,y)e^{i\nu\phi(x,y)} f(y)dy\,,$$

where $\nu \geq 1$ is a parameter. We denote with $\mathscr{B}(\mathbb{R}^n)$ the Schwartz space of bounded C^∞-functions with bounded derivatives:

$$\mathscr{B}(\mathbb{R}^n) = \left\{ f : \|f\|_m = \sup_{x\in\mathbb{R}^n} \sum_{|\alpha|\leq m} |\partial_x^\alpha f(x)| < \infty,\, m = 0,1,\dots \right\},$$

which is $W^{\infty,\infty}(\mathbb{R}^n)$ as a Sobolev space. We assume that the phase function $\phi(x,y)$ and the amplitude function $a(x,y)$ satisfy the following assumptions:

(A-I) $\phi(x,y)$ is a real valued C^∞ function of $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

(A-II) There exists a positive constant δ_0 such that

$$\left| \det \frac{\partial^2}{\partial x \partial y} \phi(x,y) \right| \geq \delta_0\,.$$

(A-III) For every multi-indices α, β, with $|\alpha|+|\beta| \geq 2$, there exists a positive constant $C_{\alpha\beta}$ such that

$$\left| \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial y}\right)^\beta \phi(x,y) \right| \leq C_{\alpha\beta}\,.$$

(A-IV) $a(x,y) \in \mathscr{B}(\mathbb{R}^n \times \mathbb{R}^n)$.

For any integer $l \geq 0$ we define

$$\delta(2,l+2) = \max_{d(\phi)} \sum_{|\alpha|+|\beta|\leq l} \sup_{(x,y)} \left| \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial y}\right)^\beta d(\phi)(x,y) \right|,$$

where $d(\phi)(x,y)$ denotes each of the entries of the matrix $D(\phi)(x,y) = \frac{\partial^2}{\partial x \partial y}\phi(x,y)$. We write as an abbreviation $\delta_2 = \delta(2,2)$ and we set $\tilde{\delta} = C(n)\delta_0\delta_2^{1-n}$.

Theorem A.0.5. *Assume that (A-I), (A-II), (A-III) and (A-IV) hold. Then there exists a positive constant K such that the estimate*

$$\|I(t,s;a)f\|_{L^2(\mathbb{R}^n)} \leq K\|f\|_{L^2(\mathbb{R}^n)},$$

for any $f \in C_0^\infty(\mathbb{R}^n)$. In particular we can take

$$K = C(n)(1+\tilde{\delta}^{\frac{n}{2}})(1+\delta(2,8n+4))^{3n+2}\|a\|_{2n+1}. \tag{A.1}$$

Proof. See [10]. □

Theorem A.0.6. *Assume that $a \in \mathscr{B}(\mathbb{R}^n \times \mathbb{R}^n)$. Then, there exist positive constants C_1 and C_2 such that for any $f \in C_0^\infty(\mathbb{R}^n)$*

$$\|x^\alpha I(t,s;a)f\|_{L^2(\mathbb{R}^n)} \leq C_1\|a\|_m\|f\|_W,$$

$$\left\|\left(\frac{\partial}{\partial x}\right)^\alpha I(t,s;a)f\right\|_{L^2(\mathbb{R}^n)} \leq C_2\|a\|_m\|f\|_W,$$

with some positive integer m if $|\alpha| \leq 2$.

Proof. See [38]. □

Appendix B

QHD and Nonlinear Schrödinger equations

B.1 From NLS to QHD

In this section we recall some results regarding the correspondence between the Quantum Hydrodynamic system

$$
\begin{cases}
\partial_t \rho + \operatorname{div} J = 0\,, \\
\partial_t J + \operatorname{div}\left(\frac{J \otimes J}{\rho}\right) + \nabla P(\rho) + \rho \nabla V = \frac{1}{2}\rho \nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \\
-\Delta V = \rho\,,
\end{cases}
\tag{B.1}
$$

and the Schrödinger-Poisson system

$$
\begin{cases}
i\partial_t \psi = -\frac{1}{2}\Delta\psi + |\psi|^{p-1}\psi + V\psi\,, \\
-\Delta V = |\psi|^2\,,
\end{cases}
\tag{B.2}
$$

where $P(\rho) = \frac{p-1}{p+1}\rho^{(p+1)/2}$ and $1 \leq p < 5$ and $x \in \mathbb{R}^3$. One way to prove this equivalence is the WKB ansatz, which consists in expressing the wave function ψ as the product of its amplitude $\sqrt{\rho}$ and its phase S, that is $\psi = \sqrt{\rho}e^{iS}$. This procedure tells us that if we have a solution ψ of (B.2), then the pair (ρ, J), with $J := \rho\nabla S$ is a solution of (B.1). This kind of approach fails in the nodal region, that is $\{\rho = 0\}$, since the phase is not well-defined there. Actually, it is possible to overcome this difficulty by means of a polar factorization technique , which does not require the definition of the velocity field in the vacuum regions (see [6, 8, 18] for more details).

It is well known (see for instance [19]) that the system (B.2) is globally well posed for initial data in $H^1(\mathbb{R}^3)$ and the solution is such that $\psi \in C(\mathbb{R}^3; H^1(\mathbb{R}^3))$. For this reason, for each time $t \in [0, T)$, we can define the following quantities

$$
\rho(t) = |\psi|^2\,, \qquad J(t) = \operatorname{Im}(\overline{\psi(t)}\nabla\psi(t))\,.
\tag{B.3}
$$

The couple (ρ, J) is the candidate solution for (B.1). Let us compute the balance laws for the quantities ρ and J defined in (B.3). Formally we get the following identities

$$\partial_t \rho + \operatorname{div} J = 0\,, \tag{B.4}$$

$$\partial_t J = \frac{1}{4}\nabla\Delta\rho - \operatorname{div}(\operatorname{Re}(\nabla\overline{\psi}\otimes\nabla\psi)) - \frac{p-1}{p+1}\nabla(|\psi|^{p+1}) - \rho\nabla V\,. \tag{B.5}$$

Now we want to rewrite the quadratic term inside the divergence in (B.5), in terms of the hydrodynamical quantities ρ and J. Formally, by multiplying and dividing by $|\psi|^2$, we have

$$\begin{aligned}
\operatorname{Re}(\nabla\overline{\psi}\otimes\nabla\psi) &= \operatorname{Re}\left(\frac{(\psi\nabla\overline{\psi})\otimes(\overline{\psi}\nabla\psi)}{|\psi|^2}\right) \\
&= \frac{1}{\rho}(\operatorname{Re}(\overline{\psi}\nabla\psi)\otimes\operatorname{Re}(\overline{\psi}\nabla\psi) + \operatorname{Im}(\overline{\psi}\nabla\psi)\otimes\operatorname{Im}(\overline{\psi}\nabla\psi)) \\
&= \frac{1}{\rho}\left[\frac{1}{2}\nabla\rho\otimes\frac{1}{2}\nabla\rho + J\otimes J\right] \\
&= \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} + \frac{J\otimes J}{\rho}\,.
\end{aligned} \tag{B.6}$$

By putting (B.6) into (B.5), and by noting that

$$\frac{1}{2}\rho\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \frac{1}{4}\Delta\nabla\rho - \operatorname{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho})\,, \tag{B.7}$$

we get the formal equivalence between (B.1) and (B.2). As already stressed, these computations are just formal, since in order to obtain (B.6), we have divided by $|\psi|^2$. Again, as in the case of the WKB ansatz mentioned before, we have to face the problem of the vacuum, that is the region $\{\psi = 0\}$.

To circumvent this difficulty, we can use a polar factorisation technique, which allows us to decompose the wave function ψ in terms of its amplitude $|\psi|$ and its phase ϕ. Let ψ be a wave function in $L^2(\mathbb{R}^3)$. We define the set

$$P(\psi) := \{\varphi \in L^\infty(\mathbb{R}^3)\ :\ \|\varphi\|_{L^\infty} \le 1, \psi = \sqrt{\rho}\varphi \ \text{ a.e. in } \mathbb{R}^3\},$$

where $\sqrt{\rho} := |\psi|$.
The next lemma explains how to connect the bilinear term $\operatorname{Re}\operatorname{div}(\nabla\overline{\psi}\otimes\nabla\psi)$ to the hydrodynamical quantities.

Lemma B.1.1. *Let $\psi \in H^1(\mathbb{R}^3)$, $\sqrt{\rho} := |\psi|$ and $\varphi \in P(\psi)$. Let us define $\Lambda := \operatorname{Im}(\bar{\varphi}\nabla\psi) \in L^2(\mathbb{R}^3)$, then we have*

- $\sqrt{\rho} \in H^1(\mathbb{R}^3)$ *and* $\nabla\sqrt{\rho} = \operatorname{Re}(\bar{\varphi}\nabla\psi)$;
- *the following identity holds a.e. in $\mathbb{R}^3$,*

$$\operatorname{Re}\{\nabla\overline{\psi}\otimes\nabla\psi\} = \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} + \Lambda\otimes\Lambda. \tag{B.8}$$

Moreover, let $\{\psi_n\} \subset H^1(\mathbb{R}^3)$ *such that* ψ_n *converges strongly to* ψ *in* H^1*; then we have*

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \quad \Lambda_n \to \Lambda, \quad \text{in } L^2(\mathbb{R}^3),$$

where $\sqrt{\rho_n} := |\psi_n|$, $\Lambda_n := \mathrm{Re}(\overline{\varphi}_n \nabla\psi_n)$.

Proof. See [6]. □

Thanks to Lemma B.1.1, the identity

$$\mathrm{Re}(\nabla\overline{\psi} \otimes \nabla\psi) = \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} + \Lambda \otimes \Lambda \tag{B.9}$$

is justified; clearly $J = \sqrt{\rho}\Lambda$. In this way we recover a solution for the system (B.1), by sarting from a solution to the equation (B.2). Actually the above discussion is rigorous only for smooth enough solutions ψ of (B.2). In order to deal with $\psi \in H^1(\mathbb{R}^3)$, one can use a density argument, exploiting the persistence of regularity for the solutions of the nonlinear Schrödinger equations and the stability of the polar factorisation in $H^1(\mathbb{R}^3)$, contained in the last statement of Lemma B.1.1.

B.2 Two-fluid model

Our interest on (1.9) is motivated by the attempt to study a class of two-fluid hydrodynamic systems. This kind of models arise in several physical phenomena, such as superfluidity [76] or Bose-Einstein condensation at finite temperatures [58, 59]. A typical example in this direction is given by liquid Helium; indeed when it is cooled to the temperature of $T_\lambda = 2.172\,K$ it exhibits a phase transition: above the critical temperature T_λ, Helium behaves like a viscous Newtonian fluid; this is what, usually, is called normal fluid. Below T_λ, instead, quantum effects become relevant and so the liquid Helium is described by a frictionless quantum fluid, usually called superfluid. The same kind of behaviour is shown by dilute Bose condensed gases at a temperature lower than the critical condensation temperature, but sufficiently far from absolute zero, in order to have distinguished condensate and non-condensate part. The class of two-fluid model we are interested in reads as

$$\begin{cases} \partial_t \rho_s + \mathrm{div}\, J_s = 0 \\ \partial_t(J_s) + \mathrm{div}\left(\frac{J_s \otimes J_s}{\rho_s}\right) + \nabla P_s(\rho_s) = \frac{1}{2}\rho_s \nabla\left(\frac{\Delta\sqrt{\rho_s}}{\sqrt{\rho_s}}\right) - (J_s - \mathbb{Q}v_n) \\ \partial_t \rho_n + \mathrm{div}(\rho_n v_n) = 0 \\ \partial_t(\rho_n v_n) + \mathrm{div}(\rho_n v_n \otimes v_n) + \nabla P_n(\rho_n) = \eta\Delta v_n + \frac{\eta}{3}\nabla\,\mathrm{div}\, v_n, \end{cases} \tag{B.10}$$

where ρ_s, J_s denote the superfluid mass and current density, respectively, and ρ_n, v_n the mass density and the velocity field for the normal fluid. P_s and P_n are self-consistent pressure terms, η is the viscosity in the equation for normal fluid and $\mathbb{Q} = -(-\Delta)^{-1}\nabla\,\mathrm{div}$. It is clear from the system (B.10), that the dynamics of the normal fluid is not affected by the superfluid; on the other hand the latter interacts with the former through the collision term $J_1 - \rho_1\mathbb{Q}v_2$. The dynamics of the normal fluid, which is a classical one, is described by the Navier-Stokes equations for compressible fluids,

$$\begin{cases} \partial_t \rho_n + \operatorname{div}(\rho_n v_n) = 0\,, \\ \partial_t(\rho_n v_n) + \operatorname{div}(\rho_n v_n \otimes v_n) + \nabla P_n(\rho_n) = \eta \Delta v_n + \frac{\eta}{3} \nabla \operatorname{div} v_n\,, \\ \rho_n(0) = \rho_{n,0}\,, \quad (\rho_n v_n)(0) = J_{2,0}\,, \end{cases} \tag{B.11}$$

where the viscosity coefficient η is constant. The compressible Navier-Stokes system (B.11) can be solved by using the results available in the mathematical literature (see for instance [28, 31, 88]).
At this point it can be seen as a given term in the equation for the superfluid, which evolves according to the following Cauchy problem

$$\begin{cases} \partial_t \rho_s + \operatorname{div} J_s = 0 \\ \partial_t(J_s) + \operatorname{div}\left(\frac{J_s \otimes J_s}{\rho_s}\right) + \nabla P_s(\rho_s) - \mathbb{Q} v_n = \frac{\hbar^2}{2} \rho_s \nabla\left(\frac{\Delta\sqrt{\rho_s}}{\sqrt{\rho_s}}\right) - J_s\,, \\ \rho_s(0) = \rho_{s,0}\,, \quad J_s(0) = J_{s,0}\,. \end{cases} \tag{B.12}$$

As a first step to solve the system (B.12), we consider the case without collision, that is we neglect the term J_s in the right handside of the second equation in (B.12). In the light of the results in Section B.1, we get that the system (B.12) is equivalent to the following Schrödinger equation

$$\begin{cases} i\partial_t \psi = -\frac{1}{2}\Delta\psi + \tilde{V}\psi + f(|\psi|^2)\psi\,, \\ \psi(0) = 0\,, \end{cases} \tag{B.13}$$

where $\tilde{V} = (-\Delta)^{-1} \operatorname{div} v_n$ and $\hbar = 1$. Hence the information we have on $\tilde{V}$ is that $\nabla \tilde{V} = \mathbb{Q} v_n \in L^2_t L^6_x(\mathbb{R}_+ \times \mathbb{R}^n)$. By using Theorem 2.2 in [102], it can be proved that $\tilde{V} = V + V_\infty$, where $V \in L^2_t W^{1,6}_x$, $V_\infty \in C^\infty(\mathbb{R}^n)$ for $a.e.\ t \in \mathbb{R}_+$ and for any $|\alpha| \geq 1$, $\partial^\alpha_x V_\infty \in L^2_t L^\infty_x$. Thus we need to study the properties of the propagator associated with the following Cauchy problem

$$\begin{cases} i\partial_t \psi = -\frac{1}{2}\Delta\psi + V_\infty\psi\,, \\ \psi(0) = 0\,, \end{cases} \tag{B.14}$$

where V_∞ has the properties listed above. This is just (1.9). Once we have a solution of (B.14), we can construct a finite energy weak solution of (B.12) without the collision term, by means of a polar factorisation technique (see Section B.1). Then, in order to solve the general case, a fractional step method can be used. This allows to get an approximate solution for (B.12), by splitting the problems into two separate steps. First of all, one solves the QHD problem without collisions; then the collisional problem, without QHD, is solved. At this point one starts again with the non-collisional QHD problem. We refer the reader to [5, 6, 8] for further details.

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