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Contents

Introduction

The aim of this thesis is to present some results about controllability of evolution equations by means of a bilinear control.
In the field of control theory of dynamical systems a huge amount of works is devoted to the study of models in which the control enters as an additive term (boundary o locally distributed control), see, for instance, the books [49] and [50] by J.L. Lions. Such controls can, for example, be a source of heat/mass–transfer or a piezoelectric actuator placed on a beam. This kinds of controls describe the effect of external sources of forces on the considered phenomenon. On the other hand, such control systems are not suitable to describe processes that change their physical characteristics in presence of control. This issue is quite common in many biomedical, chemical and nuclear chain reactions as well as in new technologies, like, for instance, the so–called *smart materials*.
For example, in a *nuclear chain reaction* the number of particles involved increases by the interaction with the surrounding medium. In particular, the process of nuclear fission is obtained by the collision of neutrons with active uranium nuclei that leads to the growth of the amount of particles involved in the reaction. These new neutrons start interacting with active nuclei and so the number of such particles keeps increasing. A simplified model of this phenomenon can be represented by the following equation

$$u_t = a^2\Delta u + v(t,x)u, \tag{0.0.1}$$

where $u(t,x) \geq 0$ is the neutron density and the coefficient v is strictly positive since the chain reaction is equivalent to a source of neutrons that is proportional to their concentration.
In order to control an a priori endless chain reaction, the so–called "control roads" are employed. These devices are indeed able to absorb neutrons. The action of the control roads can be associated to the change of sign of the coefficient v in (0.0.1).
It is important to stress that describing a nuclear fission by the action of additive controls would yield to the following equation

$$u_t = a^2\Delta u + v(t,x). \tag{0.0.2}$$

However, the additive locally distributed control $v(t,x)$ in (0.0.2) would represent the possibility to add or withdraw out at will a certain number of neutrons to the reaction, that is obviously not realistic.
Another example of a model that involves a bilinear control is the SMA–composite beam containing NiTi fibers that are able to change the response when heated by an electric current. The equation that describes this phenomenon is

$$u_{tt} + u_{xxxx} + v(t)u = 0, \tag{0.0.3}$$

with $v(t)$ that represents the axial load.

When the multiplicative control depends both on time and space variables, many results of approximate and exact controllability have been obtained for different types of initial/target conditions. For instance, Khapalov in [45] proved a result of non-negative approximate controllability of the $1D$ semilinear parabolic equation. In [46], the same author proved approximate and exact null controllability for a bilinear parabolic system with the reaction term satisfying Newton's law. Paper [39] is devoted to the study of global approximate multiplicative controllability for nonlinear degenerate parabolic problems. In [18] and [19], results of approximate controllability of a one dimensional reaction-diffusion equation via multiplicative control and with sign changing data are proved. Moreover, in [62] the authors presented a result of exact controllability of parabolic equations to special positive target states for large time.

The controllability of equations in which the control is a scalar function depending only on time, however, is a more delicate issue. A structural obstruction to obtain such property for systems of the form

$$\begin{cases} u'(t)+Au(t)+p(t)Bu(t)=0, \quad t>0 \\ u(0)=u_0. \end{cases} \tag{0.0.4}$$

has been presented by Ball, Marsden and Slemrod in [6]. Let X be a infinite dimensional Banach space, let A be the generator of a C^0-semigroup of bounded linear operators on X and let $B: X \to X$ be a bonded linear operator. Then the main result of [6] establishes that system (0.0.4) is not controllable. Indeed, if $u(t;p,u_0)$ denotes the unique solution of (0.0.4), then the attainable set from u_0 defined by

$$S(u_0)=\{u(t;p,u_0);\ t\geq 0,\ p\in L^r_{loc}([0,+\infty),\mathbb{R}),\ r>1\}$$

has a dense complement.

On the other hand, when B is unbounded, the possibility of proving a positive controllability result remains open. This idea of exploiting the unboundness of the operator B was developed by Beauchard and Laurent in [11] for the Schrödinger equation

$$\begin{cases} iu_t+u_{xx}+p(t)\mu(x)u=0, \quad (t,x)\in(0,T)\times(0,1) \\ u(t,0)=u(t,1)=0. \end{cases}$$

For such an equation the authors proved the local exact controllability along the ground state solution (namely, the solution of the free dynamics with initial condition equal to the first eigenfunction of the second order operator) in a stronger topology than the natural one of $X=H^2\cap H^1_0(0,1)$ for which the multiplication operator $Bu=\mu(x)u$ is unbounded. In other terms, the above result could be regarded as a description of the attainable set from an initial submanifold of the original Banach space.

Following the same strategy, Beauchard in [7] studied the wave equation

$$\begin{cases} u_{tt}-u_{xx}-p(t)\mu(x)u=0, \quad (t,x)\in(0,T)\times(0,1) \\ u_x(t,0)=u(t,1)=0 \end{cases}$$

showing that for $T>2$ the system is locally controllable in a stronger topology than the natural one for this problem and for which the operator $Bu=\mu(x)u$ is unbounded.

In both papers [7] and [11] a key point of the analysis is the application of the inverse mapping theorem which is made possible by the controllability of the linearized problem. This is the reason why, for parabolic problems, the above strategy meets an obstruction: the spaces for which one can prove controllability of the linearized equation are not well–adapted to the use of the inverse map technique.

Therefore, for bilinear control problems of parabolic type, a natural question that arises is whether it is possible to steer the solution exactly to a fixed trajectory $\bar{u}(t;\bar{p},\bar{u}_0)$ in finite time $T>0$. Since in this case the target is represented by the evaluation of $\bar{u}$ at time T, and not by a neighborhood of it (as it happens for the classical exact controllability property), the aforementioned work by Ball, Marsden and Slemrod does not represent a hindrance to such kind of result.

The controllability to a target trajectory is not a new property. It has been studied, for instance, in the work [38] by Fernández-Cara, Guerrero, Imanuvilov and Puel for the Navier-Stokes equations, by means of additive controls. The authors obtained controllability to free trajectories by a Carleman estimate and an inverse mapping argument. Such a strategy seems hard to adapt to problems like (0.0.4). In [33] Duprez and Lissy showed the controllability of the Fokker-Plank equation to a target trajectory with a multiplicative control depending on both space and time. However, to our best knowledge, for bilinear control systems of the form (0.0.4) this property has not yet been explored.

Therefore, our first interest regarded the possibility of steering the solution of (0.0.4), with a bilinear control, to a specific uncontrolled trajectory of the equation, namely the ground state solution.

To be more precise, let X be a separable Hilbert space, $A: D(A)\subset X\to X$ be a self-adjoint accretive operator with compact resolvent (see Chapter 1 for more on notation and assumptions) and let $\{\lambda_k\}_{k\in\mathbb{N}^*}$ be the eigenvalues of A, ($\lambda_k\leq\lambda_{k+1}$, $\forall k\in\mathbb{N}^*$), with associated eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}^*}$. Since it is customary to call φ_1 the *ground state* of A, we refer to the function $\psi_1(t)=e^{-\lambda_1 t}\varphi_1$ as the *ground state solution*.

Our main result of Chapter 2 (Theorem 2.1.4) ensures that, if $\{\lambda_k\}_{k\in\mathbb{N}^*}$ satisfy a suitable gap condition (see condition (2.1.3)) and B spreads the ground state in all directions (see condition (2.1.4)), then system (0.0.4) is locally stabilizable to ψ_1 at superexponential rate, that is, one can find a control $p\in L^2_{loc}(0,\infty)$ such that the corresponding solution $u(\cdot)$ of (0.0.4) satisfies

$$\log\|u(t)-\psi_1(t)\|\leq C-e^{\omega t},\qquad \forall t>0, \tag{0.0.5}$$

for suitable constants $C,\omega>0$. This property can be seen as a weak version of the exact controllability to the ground state solution.

An important point to underline is that our approach — based on the moment method for the linearized system — is fully constructive. First, we use the gap condition (2.1.3) to build a biorthogonal family $\{\sigma_k(t)\}_{k\in\mathbb{N}^*}$ to the exponentials $e^{\lambda_k t}$. Then, we apply such a family to construct a control $p(\cdot)$ that steers the linearized system of (0.0.4) exactly to the ground state solution in finite time. Finally, we repeatedly apply such exact controls for the linearized system in order to build a control $p(\cdot)$ for (0.0.4) which achieves (0.0.5).

We point out that our method applies to both cases $\lambda_1=0$ and $\lambda_1>0$, giving an even faster decay rate in the latter case.

In Chapter 3 we address the more delicate issue of the exact controllability to the ground state solution. Under stronger assumptions, we prove that for any $T>0$ such property is enjoyed by the abstract control system (0.0.4). Taking advantage from the stability estimates proved in Chapter 2, we repeat the linearization argument for a suitable sequence of time intervals of decreasing length T_j, with $\sum_{j=1}^{\infty}T_j=T$, in order to construct a control $p\in L^2(0,T)$ such that $u(T;p,u_0)=\psi_1(T)$.

Furthermore, from the local exact controllability property (Theorem 3.1.1) we were able to infer two different kinds of global results (Theorems 3.1.2 and 3.1.3).
We show in Chapter 2 that the superexponential stabilizability result can be applied to several classes of parabolic equations as, for instance, the heat equation with a controlled source term

$$u_t - u_{xx} + p(t)\mu(x)u = 0,$$

with different kinds of boundary conditions, as well as to a variable coefficients equation of the form

$$u_t - ((1+x)^2 u_x)_x + p(t)\mu(x)u = 0,$$

and also to degenerate parabolic equations as

$$u_t - (x^\alpha u_x)_x + p(t)\mu(x)u = 0. \tag{0.0.6}$$

Moreover, in Chapter 3, we observe that all the examples presented in Chapter 2 fulfill also the stronger hypotheses needed to prove the exact controllability to the ground state solution. While checking the validity of the assumptions on A and B of Theorems 2.1.4 and 3.1.1 turned out to be straightforward for almost all the aforementioned examples, it required a more complex and careful analysis for the degenerate equation (0.0.6). However, we were interested in treating this kind of operators since degenerate differential equations describe valuable phenomena in many fields such as in physics, climate dynamics, biology and economics (see, e.g., [40, 34, 22]). Furthermore, the problem of controlling such equations is, by now, a fairly well–developed subject (see, for instance, [21, 22, 24, 23]). Nevertheless, we noticed that very few results are available in the case of degenerate hyperbolic equations. To our best knowledge, a class of degenerate wave equations has been studied from the point of view of control theory in [2], by means of boundary controls, using HUM and multiplier methods, and in [59, 60, 61], where locally distributed controls are considered.
This has been the inspiring reason to study a bilinear control problem for the following equation

$$\begin{cases} w_{tt} - (x^\alpha w_x)_x = p(t)\mu(x)w, & x \in (0,1), t \in (0,T), \\ (x^\alpha w_x)(x=0) = 0, & t \in (0,T), \\ w_x(x=1) = 0, & t \in (0,T), \\ w(x,0) = w_0(x), & x \in (0,1), \\ w_t(x,0) = w_1(x), & x \in (0,1), \end{cases} \tag{0.0.7}$$

where $\alpha \in [0,2)$ is the degereracy parameter ($\alpha = 0$ for the classical wave equation and $\alpha \in (0,2)$ in the degenerate case), $p \in L^2(0,T)$ is a bilinear control, and μ is an admissible potential. The goal of Chapter 4 is to extend to the degenerate case $\alpha \in (0,2)$ the result [7] by Beauchard for the controllability of the classical wave equation.
A further direction of research arose from the analysis of the potentials μ that are suitable for each bilinear control problem. In particular, a necessary condition that the functions μ have to fulfill in [7, 11, 13, 29, 20, 25, 52, 53, 54, 55] and in the examples of [3, 4] is that

$$\langle \mu\varphi_1, \varphi_k \rangle \neq 0, \quad \forall\, k \in \mathbb{N}^*. \tag{0.0.8}$$

Namely, the Fourier coefficients of the multiplication operator by the potential μ, applied to the ground state, must not vanish. This requirement usually appears when facing a moment problem. Even though such condition is satisfied generically, it is not so easy to exhibit a large explicit class of real valued potential μ satisfying it and only few examples of suitable potentials are available in the existing literature. Moreover, these examples are based on the knowledge of the explicit form of the eigenvalues and eigenfunctions. However, the eigenvalues cannot longer be explicitly represented when changing, for instance, the boundary conditions from Dirichlet-Dirichlet to Dirichlet-Robin. Therefore, natural questions that raise in this context are: is it possible to exhibit large classes of functions μ satisfying (0.0.8)? Can we build a general constructive algorithm to define such functions μ? Is it possible to extend Beauchard and Laurent controllability results for the Schrödinger equation and Alabau-Boussouira, Cannarsa and Urbani superexponential stabilization [4] and controllability [3] results for parabolic equations, and further existing results for other equations to more general boundary conditions?

The aim of Chapter 5 is to give positive answers to these questions.

This thesis is organized as follows. In Chapter 1 we introduce the notation and recall some classical results that will be used throughout the work. Chapter 2 is devoted to present our result of rapid stabilization for abstract parabolic equations by mean of bilinear control. We also give applications to several examples of parabolic problems. In Chapter 3 we exhibit and prove our result of exact controllability to the ground state solution for the same class of problems considered in the previous chapter. Bilinear control for hyperbolic degenerate equations is studied in Chapter 4. Finally, in Chapter 5 we present an algorithm to build polynomials of any degree that satisfy (0.0.8). Furthermore, we extend [4, 3, 11] to mixed boundary conditions of Dirichlet-Robin type.

The Appendix A is dedicated to the investigation of spectral properties of degenerate operator.

CHAPTER 1

Preliminaries

Let $(X, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space. We denote by $|| \cdot ||$ the associated norm on X.
Let $A : D(A) \subset X \to X$ be a densely defined linear operator with the following properties:

$$\begin{array}{ll} (a) & A \text{ is self-adjoint,} \\ (b) & A \text{ is accretive: } \langle Ax, x\rangle \geq 0,\ \forall x \in D(A), \\ (c) & \exists\, \lambda > 0 \text{ such that } (\lambda I + A)^{-1} : X \to X \text{ is compact.} \end{array} \tag{1.0.1}$$

We recall that under the above assumptions A is a closed operator and $D(A)$ is itself a Hilbert space with the scalar product

$$(x|y)_{D(A)} = \langle x, y\rangle + \langle Ax, Ay\rangle, \qquad \forall\, x, y \in D(A).$$

Moreover, $-A$ is the infinitesimal generator of a strongly continuous semigroup of contractions on X which will be denoted by e^{-tA}. Furthermore, e^{-tA} is analytic.
In view of the above assumptions, there exists an orthonormal basis $\{\varphi_k\}_{k\in\mathbb{N}^*}$ in X of eigenfunctions of A, that is, $\varphi_k \in D(A)$ and $A\varphi_k = \lambda_k\varphi_k\ \forall\, k \in \mathbb{N}^*$, where $\{\lambda_k\}_{k\in\mathbb{N}^*} \subset \mathbb{R}$ denote the corresponding eigenvalues. We recall that $\lambda_k \geq 0$, $\forall\, k \in \mathbb{N}^*$ and we suppose — without loss of generality — that $\{\lambda_k\}_{k\in\mathbb{N}^*}$ is ordered so that $0 \leq \lambda_k \leq \lambda_{k+1} \to \infty$ as $k \to \infty$. The associated semigroup has the following representation

$$e^{-tA}\varphi = \sum_{k=1}^{\infty}\langle \varphi, \varphi_k\rangle e^{-\lambda_k t}\varphi_k, \quad \forall\, \varphi \in X. \tag{1.0.2}$$

For any $s \geq 0$, we denote by $A^s : D(A^s) \subset X \to X$ the fractional power of A (see [57]). Under our assumptions, such a linear operator is characterized as follows

$$\begin{aligned} &D(A^s) = \left\{x \in X \;\middle|\; \textstyle\sum_{k\in\mathbb{N}^*} \lambda_k^{2s}|\langle x, \varphi_k\rangle|^2 < \infty \right\} \\ &A^s x = \textstyle\sum_{k\in\mathbb{N}^*} \lambda_k^s\langle x, \varphi_k\rangle\varphi_k, \qquad \forall\, x \in D(A^s). \end{aligned} \tag{1.0.3}$$

Let $T > 0$ and consider the problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases} \tag{1.0.4}$$

where $u_0 \in X$ and $f \in L^2(0, T; X)$. We now recall two definitions of solution of problem (1.0.4):

- the function $u \in C([0, T], X)$ defined by

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(s)ds$$

is called the *mild solution* of (1.0.4),

- u is a *strong solution* of (1.0.4) in $L^2(0,T;X)$ if there exists a sequence $\{u_k\} \subseteq H^1(0,T;X) \cap L^2(0,T;D(A))$ such that

$$u_k \to u, \text{ and } u_k' - Au_k \to f \text{ in } L^2(0,T;X),$$

$$u_k(0) \to u_0 \text{ in } X, \text{ as } k \to \infty.$$

The well-posedness of the Cauchy problem (1.0.4) is a classical result (see, for instance, [15]).

Theorem 1.0.1. *Let $u_0 \in X$ and $f \in L^2(0,T;X)$. Under hypothesis* (1.0.1)*, problem* (1.0.4) *has a unique strong solution in $L^2(0,T;X)$. Moreover u belongs to $C([0,T];X)$ and is given by the formula*

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)ds. \tag{1.0.5}$$

Furthermore, there exists a constant $C_0(T) > 0$ such that

$$\sup_{t\in[0,T]} ||u(t)|| \leq C_0(T)\big(||u_0|| + ||f||_{L^2(0,T;X)}\big) \tag{1.0.6}$$

and $C_0(T)$ is non decreasing with respect to T.

Given $T > 0$, we consider the bilinear control problem

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0,T] \\ u(0) = u_0 \end{cases} \tag{1.0.7}$$

where u is the state variable, $B : X \to X$ is a bounded linear operator and $p \in L^2(0,T)$ is the control function.
Given an initial condition $u_0 \in X$ and a control $p \in L^2(0,T)$, we denote by $u(\cdot; u_0, p) : [0,T] \to X$ the corresponding solution of (1.0.7) and we call it a *trajectory* of problem (1.0.7).

Definition 1.0.2. *Let $T > 0$. Given an initial condition $\bar{u}_0 \in X$ and a control $\bar{p} \in L^2(0,T)$ we say that the control system* (1.0.7) *is* locally controllable along $\bar{u}(t;\bar{u}_0,\bar{p})$ *in time T if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $(u_0, u_f) \in X \times X$ with $||u_0 - \bar{u}_0|| \leq \delta$ and $||u_f - \bar{u}(T;\bar{u}_0,\bar{p})|| \leq \delta$, there exists a control $p \in L^2(0,T)$ such that*

$$u(T;u_0,p) = u_f,$$

$$||p - \bar{p}||_{L^2(0,T)} \leq \varepsilon.$$

We recall that, in general, the exact controllability problem along a given trajectory for system (1.0.7) has a negative answer as shown by Ball, Marsden and Slemrod in [6].

Theorem 1.0.3 (Ball, Marsden, Slemrod 1982)**.** *Let X be an infinite dimensional Banach space. Let $-A$ generate a C^0-semigroup of bounded linear operators on X and let $B : X \to X$ be a bounded linear operator. Let $u_0 \in X$ be fixed, and let $u(t;u_0,p)$ be the trajectory of (1.0.7) corresponding to $p \in L^1_{loc}([0,+\infty),\mathbb{R})$. Then, the attainable set from u_0 defined by*

$$S(u_0) = \{u(t;u_0,p);\ t \geq 0,\ p \in L^r_{loc}([0,+\infty),\mathbb{R}),\ r > 1\},$$

is contained in a countable union of compact subsets of X and, in particular, has a dense complement.

A different notion of controllabilty is the *exact controllability to a given trajectory*.

Definition 1.0.4. *Let $T > 0$. Given an initial condition $\bar{u}_0 \in X$ and a control $\bar{p} \in L^2(0,T)$, we say that the control system* (1.0.7) *is* locally exactly controllable to $\bar{u}(t;\bar{u}_0,\bar{p})$ *in time T if there exists $\delta > 0$ such that, for every $u_0 \in X$ with $||u_0 - \bar{u}_0|| \leq \delta$, there exists a control $p \in L^2(0,T)$ such that*

$$u(T;u_0,p) = \bar{u}(T;\bar{u}_0,\bar{p}).$$

Since in this definition of controllability the target set reduces to a point, Theorem 1.0.3 does not represent an obstruction when proving such property for control systems like (1.0.4).

CHAPTER 2

Superexponential stabilizability to trajectories

This chapter is devoted to the study of a weaker notion of controllability to a given trajectory and it is based on [4, 25].

We prove rapid stabilizability to the ground state solution for a class of abstract parabolic equations of the form

$$u'(t) + Au(t) + p(t)Bu(t) = 0, \qquad t \geq 0$$

where the operator A satisfies hypothesis (1.0.1), B is a linear bounded operator and $p(\cdot)$ is the control function. The proof is based on a linearization argument. We prove that the linearized system is exactly controllable and we apply the moment method to build a control $p(\cdot)$ that steers the solution to the ground state in finite time. Finally, we use such a control to bring the solution of the nonlinear equation arbitrarily close to the ground state solution with doubly exponential rate of convergence.

The aforementioned stabilizability result can be used to study several classes of parabolic problems, for which checking the validity of the assumptions on A and B is quite often straightforward. We show the stabilizability property for the heat equation with a controlled source term of the form

$$u_t - u_{xx} + p(t)\mu(x)u = 0$$

with Dirichlet or Neumann boundary conditions, as well as for operators with variable coefficients

$$u_t - ((1+x)^2 u_x)_x + p(t)\mu(x)u = 0,$$

for $3D$ problems with radial data symmetry such as

$$u_t - \Delta u + p(t)\mu(|x|)u = 0,$$

and finally for degenerate parabolic equations of the form

$$u_t(t,x) + (x^\alpha u_x(t,x))_x + p(t)x^{2-\alpha}u(t,x) = 0.$$

2.1 Main result

We are interested in studying the stabilizability of system

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t > 0 \\ u(0) = u_0 \end{cases} \tag{2.1.1}$$

with $p \in L^2_{loc}([0,+\infty))$ to a fixed trajectory. Let X be a Hilbert space equipped with the scalar product $\langle \cdot,\cdot \rangle$. We denote by $\|\cdot\| = \sqrt{\langle \cdot,\cdot \rangle}$ the associated norm and by $B_R(\varphi)$ the open ball of radius $R > 0$, centered in $\varphi \in X$. Given an initial condition $u_0 \in X$ and a control $p \in L^2_{loc}([0,+\infty))$, we denote by $u(\cdot;u_0,p) : [0,+\infty) \to X$ the corresponding solution of (2.1.1).

Definition 2.1.1. *Given an initial condition $\bar{u}_0 \in X$ and a control $\bar{p} \in L^2_{loc}([0,+\infty))$, we say that the control system* (2.1.1) *is* locally stabilizable to $\bar{u}(\cdot;\bar{u}_0,\bar{p})$ *if there exists $\delta > 0$ such that, for every $u_0 \in B_\delta(\bar{u}_0)$, there exists a control $p \in L^2_{loc}([0,+\infty))$ such that*

$$\lim_{t\to+\infty} \|u(t;u_0,p) - \bar{u}(t;\bar{u}_0,\bar{p})\| = 0.$$

Definition 2.1.2. *Given an initial condition $\bar{u}_0 \in X$ and a control $\bar{p} \in L^2_{loc}([0,+\infty))$, we say that the control system* (2.1.1) *is* locally exponentially stabilizable to $\bar{u}(\cdot;\bar{u}_0,\bar{p})$ *if for any $\rho > 0$, there exists $R(\rho) > 0$ such that, for every $u_0 \in B_{R(\rho)}(\bar{u}_0)$, there exists a control $p \in L^2_{loc}([0,+\infty))$ and a constant $M > 0$ such that*

$$\|u(t;u_0,p) - \bar{u}(t;\bar{u}_0,\bar{p})\| \le Me^{-\rho t}, \qquad \forall t > 0.$$

Definition 2.1.3. *Given an initial condition $\bar{u}_0 \in X$ and a control $\bar{p} \in L^2_{loc}([0,+\infty))$, we say that the control system* (2.1.1) *is* locally superexponentially stabilizable to $\bar{u}(\cdot;\bar{u}_0,\bar{p})$ *if for any $\rho > 0$ there exists $R(\rho) > 0$ such that, for every $u_0 \in B_{R(\rho)}(\bar{u}_0)$, there exists a control $p \in L^2_{loc}([0,+\infty))$ such that*

$$\|u(t;u_0,p) - \bar{u}(t;\bar{u}_0,\bar{p})\| \le Me^{-\rho e^{\omega t}}, \qquad \forall t > 0,$$

where $M, \omega > 0$ are suitable constants depending only on A and B.

For any $j \in \mathbb{N}^*$ we set $\psi_j(t) = e^{-\lambda_j t}\varphi_j$ and we call ψ_1 the *ground state solution*. Observe that ψ_j solves (2.1.1) with $p = 0$ and $u_0 = \varphi_j$. We shall study the superexponential stabilizability of (2.1.1) to the trajectory ψ_1.
We observe that if there exists $\nu > 0$ such that $\langle Ax, x\rangle \ge \nu\|x\|^2$, for all $x \in D(A)$, then the semigroup generated by $-A$ satisfies

$$\|e^{-tA}\| \le e^{-\nu t}, \qquad \forall t > 0.$$

If we consider any initial condition $u_0 \in X$, then the evolution of the free dynamics with initial condition u_0 can be represented by the action of the semigroup, $u(t) = e^{-tA}u_0$. Therefore, one can prove easily that, when A is strictly accretive, choosing the control $p = 0$, system (2.1.1) is locally exponentially stabilizable to the trajectory ψ_1. Indeed,

$$\|u(t) - \psi_1(t)\| = \|e^{-tA}u_0 - e^{-tA}\varphi_1\| \le e^{-\nu t}\|u_0 - \varphi_1\| \tag{2.1.2}$$

and this quantity tends to 0 as t goes to $+\infty$.
On the contrary, in the general case of an accretive operator A, we do not have a straightforward choice of p to deduce any stabilizability property of system (2.1.1) to the ground state ψ_1.
The novelty of our work is the construction of a control function p that brings $u(t)$ arbitrary close to $\psi_1(t)$ in a very short time. Namely, we prove that (2.1.1) is locally superexponentially stabilizable to the ground state solution. This can be seen as a weak version of the exact controllability to trajectories.

Let $B : X \to X$ be a bounded linear operator. From now on we denote by C_B the norm of B

$$C_B = \sup_{\varphi \in X,\, \|\varphi\|=1} \|B\varphi\|$$

and, without loss of generality, we suppose $C_B \geq 1$.
We can now state our main result.

Theorem 2.1.4. *Let $A : D(A) \subset X \to X$ be a densely defined linear operator satisfying hypothesis* (1.0.1) *and suppose that there exists a constant $\alpha > 0$ such that the eigenvalues of A fulfill the gap condition*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \alpha, \quad \forall k \in \mathbb{N}^*. \tag{2.1.3}$$

Let $B : X \to X$ be a bounded linear operator and let $\tau > 0$ be such that

$$\begin{aligned} &\langle B\varphi_1, \varphi_k \rangle \neq 0, \qquad \forall k \in \mathbb{N}^*, \\ &\sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2} < +\infty. \end{aligned} \tag{2.1.4}$$

Then, system (2.1.1) *is superexponentially stabilizable to ψ_1.*
Moreover, for every $\rho > 0$ there exists $R_\rho > 0$ such that any $u_0 \in B_{R_\rho}(\varphi_1)$ admits a control $p \in L^2_{loc}([0, +\infty))$ such that the corresponding solution $u(\cdot; u_0, p)$ of (2.1.1) *satisfies*

$$\|u(t) - \psi_1(t)\| \leq M e^{-(\rho e^{\omega t} + \lambda_1 t)}, \qquad \forall t \geq 0, \tag{2.1.5}$$

where M and ω are positive constants depending only on A and B.

To prove Theorem 2.1.4 we first start assuming that the first eigenvalue of A is zero, $\lambda_1 = 0$, and we prove the local superexponential stabilizability of (2.1.1) to the trajectory φ_1. Then, we will recover the general case from this one.
The proof of Theorem 2.1.4 will be built through a series of propositions. The first result is the well-posedness of the problem

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) + f(t) = 0, & t \in [0, T] \\ u(0) = u_0. \end{cases} \tag{2.1.6}$$

We introduce the following notation:

$$\|f\|_{2,0} := \|f\|_{L^2(0,T;X)}, \qquad \forall f \in L^2(0, T; X)$$

$$\|f\|_{\infty,0} := \|f\|_{C([0,T];X)} = \sup_{t \in [0,T]} \|f(t)\|, \qquad \forall f \in C([0, T]; X).$$

Proposition 2.1.5. *Let $T > 0$. If $u_0 \in X$, $p \in L^2(0, T)$ and $f \in L^2(0, T; X)$, then there exists a unique mild solution of* (2.1.6)*, i.e. a function $u \in C([0, T]; X)$ such that the following equality holds in X for every $t \in [0, T]$,*

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}[p(s)Bu(s) + f(s)]ds. \tag{2.1.7}$$

Moreover, there exists a constant $C_1(T) > 0$ such that

$$\|u\|_{\infty,0} \leq C_1(T)(\|u_0\| + \|f\|_{2,0}). \tag{2.1.8}$$

Hereafter, we denote by C a generic positive constant which may differ from line to line even if the symbol remains the same. Constants which play a specific role will be distinguished by an index i.e., C_0, C_B, ...
The proof of the existence of the mild solution of (2.1.6) is given in [6]. For what concerns the bound for the solution u of (2.1.6), it turns out that if $C_0(T)C_B||p||_{L^2(0,T)} \leq 1/2$, then we have inequality (2.1.8) with $C_1 = C_2$ defined by

$$C_2 := 2C_0(T). \tag{2.1.9}$$

Otherwise, to obtain (2.1.8), we proceed subdividing the interval $[0,T]$ into smaller subintervals for which $C_0(T)C_B||p||_{L^2} \leq 1/2$ in all of them, and in this case the constant C_1 of inequality (2.1.8) is defined by

$$C_1 = (1+N)(2C_0(T/N))^N, \tag{2.1.10}$$

where N is the number of subintervals.
Consider the system

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, \quad t \in [0,T] \\ u(0) = u_0, \end{cases} \tag{2.1.11}$$

and the trajectory φ_1 that is a solution of (2.1.11) when $p = 0$, $u_0 = \varphi_1$ and $\lambda_1 = 0$. Set $v := u - \varphi_1$, we observe that v is the solution of the following Cauchy problem

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \quad t \in [0,T] \\ v(0) = v_0 = u_0 - \varphi_1. \end{cases} \tag{2.1.12}$$

Remark 2.1.6. *Applying Theorem 1.0.1, we find that $v \in C([0,T];X)$ is a mild solution of* (2.1.12)*, that is*

$$v(t) = e^{-tA}v_0 - \int_0^t p(s)e^{-(t-s)A}B(v(s) + \varphi_1)ds = V_0(t) + V_1(t), \tag{2.1.13}$$

where

$$V_0(t) := e^{-tA}v_0,$$

$$V_1(t) := -\int_0^t p(s)e^{-(t-s)A}B(v(s) + \varphi_1)ds.$$

Since $p(\cdot)B(v(\cdot) + \varphi_1) \in L^2(0,T;X)$, we have that $V_1 \in H^1(0,T;X) \cap L^2(0,T;D(A))$, while $V_0 \in C^1((0,T];X) \cap C((0,T];D(A))$. Therefore, for every $\varepsilon \in (0,T)$, $v \in H^1(\varepsilon,T;X)$ and for almost every $t \in [\varepsilon,T]$ the following equality holds

$$v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0. \tag{2.1.14}$$

Showing the stabilizability of the solution u of (2.1.11) to the trajectory φ_1 is equivalent to proving the stabilizability to 0 of system (2.1.12): we have to prove that there exists $\delta > 0$ such that, for every initial condition v_0 that satisfies $||v_0|| \leq \delta$, there exists a trajectory-control pair (v,p) such that $\lim_{t\to+\infty} ||v(t)|| = 0$.
For this purpose, we consider the following linearized system

$$\begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\varphi_1 = 0, \quad t \in [0,T] \\ \bar{v}(0) = v_0. \end{cases} \tag{2.1.15}$$

For this linear system we are able to prove the following null controllability result.

Proposition 2.1.7. *Let $T > \tau$ and let A and B be such that* (1.0.1)*,* (2.1.3)*,* (2.1.4) *hold and furthermore we assume $\lambda_1 = 0$. Let $v_0 \in X$. Then, there exists a control $p \in L^2(0,T)$ such that $\bar{v}(T) = 0$.*
Moreover, there exists a constant $C_\alpha(T) > 0$ such that

$$||p||_{L^2(0,T)} \leq C_\alpha(T)\Lambda_T ||v_0|| \tag{2.1.16}$$

where Λ_T is defined in (2.1.18) *and $\alpha > 0$ is the constant in* (2.1.3)*.*

Let us recall the notion of *biorthogonal family* and a result we will use to show the null controllability of the linearized system (2.1.15).

Definition 2.1.8. *Let $\{\zeta_j\}$ and $\{\sigma_k\}$ be two sequences in a Hilbert space H. We say that the two families are* biorthogonal *or that $\{\zeta_j\}$ (resp.$\{\sigma_k\}$) is* biorthogonal *to $\{\sigma_k\}$ (resp. $\{\zeta_j\}$) if*

$$\langle \zeta_j, \sigma_k \rangle_H = \delta_{j,k}, \quad \forall j,k \geq 0$$

where $\delta_{j,k}$ is the Kronecker delta.

The notion of biorthogonal family was used by Fattorini and Russell in [37], where they introduced the moment method. Such a technique was developed later by several authors. We recall below the result proved in [24].

Theorem 2.1.9. *Let $\{\omega_k\}_{k\in\mathbb{N}}$ be an increasing sequence of nonnegative real numbers. Assume that there exists a constant $\alpha > 0$ such that*

$$\forall k \in \mathbb{N}, \quad \sqrt{\omega_{k+1}} - \sqrt{\omega_k} \geq \alpha.$$

Then, there exists a family $\{\sigma_j\}_{j\geq 0}$ which is biorthogonal to the family $\{e^{\omega_k t}\}_{k\geq 0}$ in $L^2(0,T)$, that is,

$$\forall k,j \in \mathbb{N}, \quad \int_0^T \sigma_j(t) e^{\omega_k t} dt = \delta_{jk}.$$

Furthermore, there exist two constants $C_\alpha, C_\alpha(T) > 0$ such that

$$||\sigma_j||^2_{L^2(0,T)} \leq C^2_\alpha(T) e^{-2\omega_j T} e^{C_\alpha \sqrt{\omega_j}/\alpha}, \quad \forall j \in \mathbb{N}. \tag{2.1.17}$$

Remark 2.1.10. *For all $T \in \mathbb{R}$ we define the quantity*

$$\Lambda_T := \left(\sum_{k\in\mathbb{N}^*} \frac{e^{-2\lambda_k T} e^{C_\alpha \sqrt{\lambda_k}/\alpha}}{|\langle B\varphi_1, \varphi_k \rangle|^2} \right)^{1/2} \tag{2.1.18}$$

and we observe that if there exists $\tau > 0$ such that (2.1.4) *holds then, for every $T > \tau$, $\Lambda_T < +\infty$. Furthermore, if $\lambda_1 > 0$ then $\Lambda_T \to 0$ as $T \to +\infty$.*

Thanks to Theorem 2.1.9 and Remark 2.1.10 we are able to prove Proposition 2.1.7:

Proof (of Proposition 2.1.7). For any $v_0 \in X$ and $p \in L^2(0,T)$, it follows from Proposition 2.1.5 that there exists a unique mild solution $\bar{v} \in C^0([0,T],X)$ of (2.1.15) that can be represented by the formula

$$\bar{v}(t) = e^{-tA} v_0 - \int_0^t e^{-(t-s)A} p(s) B\varphi_1 ds. \tag{2.1.19}$$

We want to find $p \in L^2(0,T)$ such that $\bar{v}(T)=0$, thus the following equality must hold

$$\sum_{k\in\mathbb{N}^*}\langle v_0,\varphi_k\rangle e^{-\lambda_k T}\varphi_k = \int_0^T p(s)\sum_{k\in\mathbb{N}^*}\langle B\varphi_1,\varphi_k\rangle e^{-\lambda_k(T-s)}\varphi_k ds. \tag{2.1.20}$$

Since $\{\varphi_k\}_{k\in\mathbb{N}^*}$ is an orthonormal basis of the space X, the equality must hold in every direction and it follows that

$$\langle v_0,\varphi_k\rangle = \int_0^T e^{\lambda_k s}p(s)\langle B\varphi_1,\varphi_k\rangle ds \tag{2.1.21}$$

for every $k\in\mathbb{N}^*$. Therefore, proving null controllability of the linearized system reduces to finding a function $p\in L^2(0,T)$ that satisfies

$$\int_0^T e^{\lambda_k s}p(s)ds = \frac{\langle v_0,\varphi_k\rangle}{\langle B\varphi_1,\varphi_k\rangle} \tag{2.1.22}$$

for all $k\in\mathbb{N}^*$. Thanks to assumption (2.1.3), there exists $\alpha>0$ such that the gap condition $\sqrt{\lambda_{k+1}}-\sqrt{\lambda_k}\geq\alpha$ holds for all $k\in\mathbb{N}^*$. Then, Theorem 2.1.9 ensures the existence of a family $\{\sigma_k\}_{k\in\mathbb{N}^*}$ that is biorthogonal to $\{e^{\lambda_k s}\}_{k\in\mathbb{N}^*}$. Taking $p(s)=\sum_{k\in\mathbb{N}^*}c_k\sigma_k(s)$ one finds that the coefficients c_k are given by $c_k=\frac{\langle v_0,\varphi_k\rangle}{\langle B\varphi_1,\varphi_k\rangle}$, $\forall k\in\mathbb{N}^*$. Thus, in order to show that

$$p(s) := \sum_{k\in\mathbb{N}^*}\frac{\langle v_0,\varphi_k\rangle}{\langle B\varphi_1,\varphi_k\rangle}\sigma_k(s) \tag{2.1.23}$$

is a solution of (2.1.22), it suffices to prove that the series is convergent in $L^2(0,T)$. Indeed,

$$||p||_{L^2(0,T)} \leq \sum_{k\in\mathbb{N}^*}\left|\frac{\langle v_0,\varphi_k\rangle}{\langle B\varphi_1,\varphi_k\rangle}\right| ||\sigma_k||_{L^2(0,T)} \leq ||v_0||\left(\sum_{k\in\mathbb{N}^*}\frac{||\sigma_k||^2_{L^2(0,T)}}{|\langle B\varphi_1,\varphi_k\rangle|^2}\right)^{1/2}$$

and we appeal to estimate (2.1.17) for $\{\sigma_k\}_{k\in\mathbb{N}^*}$, with $\omega_k=\lambda_k$ for all $k\in\mathbb{N}^*$, to obtain that

$$\left(\sum_{k\in\mathbb{N}^*}\frac{||\sigma_k||^2_{L^2(0,T)}}{|\langle B\varphi_1,\varphi_k\rangle|^2}\right)^{1/2} \leq \left(C^2_\alpha(T)\sum_{k\in\mathbb{N}^*}\frac{e^{-2\lambda_k T}e^{C_\alpha\sqrt{\lambda_k}/\alpha}}{|\langle B\varphi_1,\varphi_k\rangle|^2})\right)^{1/2} = C_\alpha(T)\Lambda_T$$

that is finite thanks to hypothesis (2.1.4) and Remark 2.1.10. Thus, the following bound for the L^2-norm of p holds true:

$$||p||_{L^2(0,T)} \leq C_\alpha(T)\Lambda_T||v_0||.$$

□

In Proposition 2.1.7 we have found a control p that steers the solution of the linearized system to 0 in time T. We use such a control in the nonlinear system (2.1.12) to obtain a uniform estimate for the solution $v(t)$.

Proposition 2.1.11. *Let A and B satisfying hypotheses* (1.0.1)*,* (2.1.3)*,* (2.1.4) *and furthermore we assume $\lambda_1=0$. Let $p\in L^2(0,T)$ be defined by the following formula*

$$p(t) = \sum_{k\in\mathbb{N}^*}\frac{\langle v_0,\varphi_k\rangle}{\langle B\varphi_1,\varphi_k\rangle}\sigma_k(t) \tag{2.1.24}$$

where $\{\sigma_k\}_{k\in\mathbb{N}^*}$ is the biorthogonal family to $\{e^{\lambda_k t}\}_{k\in\mathbb{N}^*}$ given by Theorem 2.1.9.

Then, the solution v of (2.1.12) *satisfies*

$$\sup_{t\in[0,T]} ||v(t)||^2 \le e^{C_3(T)\Lambda_T||v_0||+C_BT}(1+C_4(T)\Lambda_T^2)||v_0||^2 \tag{2.1.25}$$

where $C_B \ge 1$ is the norm of the operator B, $C_3(T) := 2\sqrt{T}C_BC_\alpha(T)$, and $C_4(T) := C_BC_\alpha^2(T)$.

Proof. We consider the equation in (2.1.12). Thanks to Remark 2.1.6, since (2.1.14) is satisfied for almost every $t \in [\varepsilon, T]$, we are allowed to take the scalar product with v:

$$\langle v'(t), v(t)\rangle + \langle Av(t), v(t)\rangle + p(t)\langle Bv(t) + B\varphi_1, v(t)\rangle = 0. \tag{2.1.26}$$

Thus, using that B is bounded, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||v(t)||^2 + \langle Av(t), v(t)\rangle &\le C_B\left(|p(t)|||v(t)||^2 + |p(t)|||\varphi_1||||v(t)||\right)\\ &\le C_B\left(|p(t)|||v(t)||^2 + \frac{1}{2}|p(t)|^2 + \frac{1}{2}||v(t)||^2\right)\end{aligned} \tag{2.1.27}$$

and therefore, since A is accretive, we have that

$$\frac{1}{2}\frac{d}{dt}||v(t)||^2 \le C_B\left(|p(t)| + \frac{1}{2}\right)||v(t)||^2 + \frac{1}{2}C_B|p(t)|^2.$$

We integrate the last inequality from ε to t:

$$\int_\varepsilon^t \frac{d}{ds}||v(s)||^2 ds \le 2C_B\int_\varepsilon^t \left(|p(s)| + \frac{1}{2}\right)||v(s)||^2 ds + C_B\int_0^T |p(s)|^2 ds$$

and by Gronwall's inequality, we obtain

$$||v(t)||^2 \le \left(||v(\varepsilon)||^2 + C_B\int_0^T |p(s)|^2 ds\right)e^{2C_B\int_\varepsilon^t(|p(s)|+1/2)}ds$$

and taking the limit $\varepsilon \to 0$ we find that

$$||v(t)||^2 \le \left(||v_0||^2 + C_B\int_0^T |p(s)|^2 ds\right)e^{2C_B\int_0^t(|p(s)|+1/2)}ds.$$

Thus, taking the supremum over the interval $[0, T]$, the last inequality becomes

$$\sup_{t\in[0,T]} ||v(t)||^2 \le e^{C_B\left(2\sqrt{T}||p||_{L^2(0,T)}+T\right)}\left(||v_0||^2 + C_B||p||^2_{L^2(0,T)}\right) \tag{2.1.28}$$

and finally, recalling the estimate (2.1.16) for the L^2-norm of p from Proposition 2.1.7, we get

$$\sup_{t\in[0,T]} ||v(t)||^2 \le e^{C_B\left(2\sqrt{T}C_\alpha(T)\Lambda_T||v_0||+T\right)}\left(1 + C_BC_\alpha^2(T)\Lambda_T^2\right)||v_0||^2. \tag{2.1.29}$$

□

We want now to measure the distance at time T of the solutions of the nonlinear system and the linearized one when using the same control function p built by solving of the moment problem in Proposition 2.1.7.

Therefore, we introduce the function $w(t) := v(t) - \bar{v}(t)$ that satisfies the following Cauchy problem

$$\begin{cases} w'(t) + Aw(t) + p(t)Bv(t) = 0, & t \in [0, T]\\ w(0) = 0.\end{cases} \tag{2.1.30}$$

We define the constant $K_T^2 := C_BC_4(T)\Lambda_T^2 e^{C_3(T)+(C_B+1)T}(1 + C_4(T)\Lambda_T^2)$.

Proposition 2.1.12. *Let A and B satisfy hypotheses* (1.0.1)*,* (2.1.3)*,* (2.1.4)*, and furthermore we assume $\lambda_1 = 0$. Let $T > \tau$, p be defined by* (2.1.24)*, and let $v_0 \in X$ be such that*

$$K_T \|v_0\| \le 1. \tag{2.1.31}$$

Then, it holds that

$$\|w(T)\| \le K_T \|v_0\|^2. \tag{2.1.32}$$

Proof. Observe that $w \in C([0,T];X)$ is the mild solution of (2.1.30). Moreover $w \in H^1(0,T;X) \cap L^2(0,T;D(A))$ and thus w satisfies the equality

$$w'(t) + Aw(t) + p(t)Bv(t) = 0 \tag{2.1.33}$$

for almost every $t \in [0,T]$.
We multiply equation (2.1.33) by $w(t)$ and we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w(t)\|^2 &\le |p(t)|\|Bv(t)\|\|w(t)\| \\ &\le \frac{1}{2}\|w(t)\|^2 + C_B^2 \frac{1}{2}|p(t)|^2\|v(t)\|^2. \end{aligned} \tag{2.1.34}$$

Therefore, applying Gronwall's inequality, taking the supremum over $[0,T]$ and using (2.1.25) and (2.1.16), we get

$$\begin{aligned} \sup_{t\in[0,T]} \|w(t)\|^2 &\le C_B^2 e^T \|p\|^2_{L^2(0,T)} \sup_{t\in[0,T]} \|v(t)\|^2 \\ &\le C_B^2 e^{C_3(T)\Lambda_T\|v_0\| + C_B T + T}(1 + C_4(T)\Lambda_T^2)\|v_0\|^2 \|p\|^2_{L^2(0,T)} \\ &\le C_B^2 C_\alpha^2(T)\Lambda_T^2 e^{C_3(T)\Lambda_T\|v_0\| + (C_B+1)T}(1 + C_4(T)\Lambda_T^2)\|v_0\|^4. \end{aligned} \tag{2.1.35}$$

We can suppose, without loss of generality, that $C_\alpha(T) \ge 1$. Thus, from (2.1.31), we obtain that $\Lambda_T\|v_0\| \le 1$. Therefore,

$$\sup_{t\in[0,T]} \|w(t)\|^2 \le K_T^2 \|v_0\|^4,$$

that implies

$$\|w(T)\| \le K_T \|v_0\|^2. \tag{2.1.36}$$

□

Recalling that $\bar{v}(T) = 0$, we deduce from (2.1.32) that

$$\|v(T)\| \le K_T \|v_0\|^2, \tag{2.1.37}$$

and, moreover,

$$K_T \|v(T)\| \le (K_T\|v_0\|)^2 \le 1. \tag{2.1.38}$$

We observe that we can apply Proposition 2.1.12 to problem (2.1.12) defined in the interval $[T, 2T]$. Indeed, $v_T := v(T)$ that was computed by solving (2.1.12), is the initial condition of the problem

$$\begin{cases} v_t(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, & t \in [T, 2T] \\ v(T) = v_T. \end{cases} \tag{2.1.39}$$

We shift this problem to the interval $[0,T]$ by introducing the variable $s := t-T$ in the above system. If we set $\tilde{v}(s) := v(s+T)$ and $\tilde{p} := p(s+T)$, then $\tilde{v}$ solves

$$\begin{cases} \tilde{v}_t(s) + A\tilde{v}(s) + \tilde{p}(s)B\tilde{v}(s) + \tilde{p}(s)B\varphi_1 = 0, & s \in [0,T] \\ \tilde{v}(0) = v_T. \end{cases} \tag{2.1.40}$$

Here the control $\tilde{p}$ is given by Proposition 2.1.11, with initial condition v_T, that is:

$$\tilde{p}(s) = \sum_{k\in\mathbb{N}^*} \frac{\langle v_T, \varphi_k\rangle}{\langle B\varphi_1, \varphi_k\rangle}\sigma_k(s) \tag{2.1.41}$$

where $\{\sigma_k(s)\}_{k\in\mathbb{N}^*}$ is the biorthogonal family to $\{e^{\lambda_k s}\}_{k\in\mathbb{N}^*}$ in $[0,T]$. Thus, it is possible to bound the L^2-norm of $\tilde{p}$ by

$$||\tilde{p}||_{L^2(0,T)} \leq C_\alpha(T)\Lambda_T||v_T|| \tag{2.1.42}$$

thanks to the estimate for $\{\sigma_k(s)\}_{k\in\mathbb{N}^*}$ given in Theorem 2.1.9. Therefore, for the control p of the linearized system associated to (2.1.39), it holds that

$$||p||_{L^2(T,2T)} = ||\tilde{p}||_{L^2(0,T)} \leq C_\alpha(T)\Lambda_T||v_T||.$$

Finally, thanks to (2.1.38), the hypotheses of Proposition 2.1.12 for problem (2.1.39) are satisfied and we obtain that $||v(2T)|| \leq K_T||v(T)||^2$. Furthermore,

$$K_T||v(2T)|| \leq (K_T||v_0||)^2 \leq 1, \tag{2.1.43}$$

and we can repeat this argument for the next intervals $[2T,3T],[3T,4T],\dots,[(n-1)T,nT],\dots$. Therefore, we deduce that

$$K_T||v(nT)|| \leq 1, \qquad \forall n \in \mathbb{N}^*. \tag{2.1.44}$$

Now, we want to obtain an estimate as (2.1.37) for the solution v of problem (2.1.12) defined in time intervals of the form $[nT,(n+1)T]$, with $n \geq 1$.

Proposition 2.1.13. *Let A and B satisfy hypotheses* (1.0.1)*,* (2.1.3)*,* (2.1.4) *and furthermore we assume $\lambda_1 = 0$. Let $v_0 \in X$ be such that*

$$K_T||v_0|| \leq 1. \tag{2.1.45}$$

Then, the following iterated estimate holds:

$$||v(nT)|| \leq \frac{1}{K_T}(K_T||v_0||)^{2^n}, \qquad \forall n \geq 0. \tag{2.1.46}$$

Proof. We proceed by induction on n. For $n=1$, the formula has been proved in Proposition 2.1.12. We suppose that (2.1.46) holds and we prove the estimate for $v((n+1)T)$: iterating the construction of the solution v of (2.1.12) in consecutive time intervals of the form $[kT,(k+1)T]$ until $k+1=n$, we come to the following problem

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, & t \in [nT,(n+1)T], \\ v(nT) = v_{nT}. \end{cases} \tag{2.1.47}$$

where v_{nT} is the value assumed at time nT by the solution of the same problem solved in the interval $[(n-1)T,nT]$ with initial data $v_{(n-1)T}$. We shift this problem in the time interval $[0,T]$

by introducing the variable $s := t - nT$ and the functions $\tilde{v}(s) = v(s + nT)$, $\tilde{p}(s) = p(s + nT)$. Then, $\tilde{v}$ is the solution of the following Cauchy problem

$$\begin{cases} \tilde{v}_t(s) + A\tilde{v}(s) + \tilde{p}(s)B\tilde{v}(s) + \tilde{p}(s)B\varphi_1 = 0, & s \in [0, T] \\ \tilde{v}(0) = v_{nT}. \end{cases} \tag{2.1.48}$$

The control function $\tilde{p}$ is defined in $[0, T]$ by solving the null controllability problem for the associated linearized system and its L^2-norm can be bound by

$$||\tilde{p}||_{L^2(0,T)} \leq C_\alpha(T)\Lambda_T ||v_{nT}||.$$

Therefore, coming back to the original time interval $[nT, (n+1)T]$ we find that

$$||p||_{L^2(nT,(n+1)T)} = ||\tilde{p}||_{L^2(0,T)} \leq C_\alpha(T)\Lambda_T ||v_{nT}||. \tag{2.1.49}$$

Moreover, since it holds that

$$K_T ||v_{nT}|| \leq 1 \tag{2.1.50}$$

we can use Proposition 2.1.12 for problem (2.1.47), obtaining

$$||v((n+1)T)|| \leq K_T ||v(nT)||^2 \leq K_T \left(\frac{1}{K_T} (K_T ||v_0||)^{2^n} \right)^2 = \frac{1}{K_T} (K_T ||v_0||)^{2^{n+1}} \tag{2.1.51}$$

and this concludes the induction argument and the proof of the proposition. □

The last step that allows us to prove Theorem 2.1.4 consists in showing the rapid decay of the solution u of our initial problem (2.1.1) to the fixed stationary trajectory φ_1.

Proposition 2.1.14. *Let* $\theta \in (0, 1)$ *and* $||v_0|| \leq \frac{\theta}{K_T}$. *Then, under the hypotheses* (1.0.1), (2.1.3), (2.1.4) *and* $\lambda_1 = 0$, *there exists a constant* $C_T > 0$ *such that*

$$||u(t) - \varphi_1|| \leq \frac{C_T}{K_T} \theta^{2^{t/T-1}} \qquad \forall t \geq 0. \tag{2.1.52}$$

Proof. We have supposed that $||v_0|| \leq \frac{\theta}{K_T}$, with $\theta \in (0, 1)$. Thus, (2.1.46) becomes

$$||v(nT)|| \leq \frac{\theta^{2^n}}{K_T}. \tag{2.1.53}$$

Consider now the time interval $[nT, (n+1)T]$. From estimate (2.1.8) for the solution of the control system in the time interval $[nT, (n+1)T]$ and from the bound (2.1.49) for the control p, we deduce that there exists a constant $C_T > 0$ such that

$$||v(t)|| \leq C_T ||v(nT)||, \qquad t \in [nT, (n+1)T]. \tag{2.1.54}$$

Therefore, using (2.1.46) in (2.1.54), we obtain that

$$||v(t)|| \leq C_T ||v(nT)|| \leq \frac{C_T}{K_T} \theta^{2^n} = \frac{C_T}{K_T} \left(\theta^{2^{(n+1)}} \right)^{1/2}. \tag{2.1.55}$$

Since $n \leq \frac{t}{T} \leq (n+1)$ and $\theta \in (0, 1)$, it holds that

$$||v(t)|| \leq \frac{C_T}{K_T} \left(\theta^{2^{(n+1)}} \right)^{1/2} \leq \frac{C_T}{K_T} \left(\theta^{2^{t/T}} \right)^{1/2} = \frac{C_T}{K_T} \theta^{2^{t/T-1}}. \tag{2.1.56}$$

By definition, $v(t) = u(t) - \varphi_1$. So, we get

$$||u(t) - \varphi_1|| \leq \frac{C_T}{K_T} \theta^{2^{t/T-1}}, \qquad t \geq 0. \tag{2.1.57}$$

□

We are ready to prove Theorem 2.1.4.

Proof of Theorem 2.1.4. We first consider the case in which the first eigenvalue of A is zero. Let $\theta \in (0,1)$ and let $\rho > 0$ be the value for which $\theta = e^{-2\rho}$. Then, from Proposition 2.1.14, there exist a constant $R_\rho > 0$ such that if $||u_0 - \varphi_1|| \le R_\rho$, then

$$||u(t) - \varphi_1|| \le M_T e^{-\rho e^{\omega_T t}}, \forall t \ge 0.$$

where $M_T, \omega_T > 0$ are constants that depend only on T. With the notation of the previous propositions, we have that

$$R_\rho := \frac{e^{-2\rho}}{K_T}, \qquad M_T := \frac{C_T}{K_T}, \qquad \omega_T := \frac{\log 2}{T}. \tag{2.1.58}$$

Now, in order to deal with a general operator A satisfying (1.0.1), we introduce the operator

$$A_1 := A - \lambda_1 I. \tag{2.1.59}$$

We observe that $A_1 : D(A_1) \subset X \to X$ is self-adjoint, accretive and $-A_1$ generates a strongly continuous analytic semigroup of contraction. Its eigenvalues are given by

$$\mu_k = \lambda_k - \lambda_1, \qquad \forall k \in \mathbb{N}^* \tag{2.1.60}$$

(in particular, $\mu_1 = 0$) and it has the same eigenfunctions as A, $\{\varphi_k\}_{k\in\mathbb{N}^*}$. Moreover, the family $\{\mu_k\}_{k\in\mathbb{N}^*}$ satisfies the same gap condition (2.1.3) that is satisfied by the eigenvalues of A. Indeed, it holds that

$$\sqrt{\mu_{k+1}} - \sqrt{\mu_k} = \frac{\lambda_{k+1} - \lambda_k}{\sqrt{\mu_{k+1}} + \sqrt{\mu_k}} \ge \frac{\lambda_{k+1} - \lambda_k}{\sqrt{\lambda_{k+1}} + \sqrt{\lambda_k}} = \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \ge \alpha, \qquad \forall k \in \mathbb{N}^*.$$

Thus, the operator A_1 satisfies the hypotheses that are required in Theorem 2.1.4.
We observe that if we introduce the function $z(t) = e^{\lambda_1 t} u(t)$, where u is the solution of (2.1.1), then z solves

$$\begin{cases} z'(t) + A_1 z(t) + p(t) B z(t) = 0, & t > 0, \\ z(0) = u_0. \end{cases} \tag{2.1.61}$$

So, we can apply the previous analysis to this problem and deduce that there exist $M_T, \omega_T > 0$ such that, for all $\rho > 0$ there exists $R_\rho > 0$ such that, if $||u_0 - \varphi_1|| \le R_\rho$, then

$$||z(t) - \varphi_1|| \le M_T e^{-\rho e^{\omega_T t}}, \qquad \forall t \ge 0. \tag{2.1.62}$$

We claim that the local superexponetial stabilizability of z to the stationary trajectory φ_1 implies the same property of u to the ground state solution ψ_1. Indeed, it holds that

$$||u(t) - \psi_1(t)|| = ||e^{-\lambda_1 t} z(t) - e^{-\lambda_1 t}\varphi_1|| = e^{-\lambda_1 t}||z(t) - \varphi_1|| \le M_T e^{-(\rho e^{\omega_T t} + \lambda_1 t)}, \quad \forall t \ge 0$$

and this concludes the proof also in the case of a strictly accretive operator A. □

Remark 2.1.15. *Even in the case when $A : D(A) \subseteq X \to X$ has a finite number of negative eigenvalues, we can define the operator $A_1 := A - \lambda_1 I$. A_1 has nonnegative eigenvalues and we can perform the proof of Theorem 2.1.4 and deduce the superexponential stabilizability of the solution u of the problem with diffusion operator A to the ground state solution. In this case $\psi_1(t) = e^{\lambda_1 t}\varphi_1$ blows up as $t \to \infty$ since $\lambda_1 < 0$, and the same occurs for the controlled solution u.*

2.2 Applications

In this section we discuss examples of bilinear control systems to which we can apply Theorem 2.1.4. The first problems we study are 1D parabolic equations of the form

$$u_t(t,x) - u_{xx}(t,x) + p(t)Bu(t,x) = 0, \quad (t,x) \in [0,T] \times (0,1)$$

in the state space $X = L^2(0,1)$, with Dirichlet or Neumann boundary conditions and with B the following multiplication operators:

$$Bu(t,x) = \mu(x)u(t,x).$$

Then, we prove the superexponential stabilizability of the following one dimensional equation with variable coefficients

$$u_t(t,x) - ((1+x)^2 u_x(t,x))_x + p(t)Bu(t,x) = 0$$

with Dirichlet boundary condition.
Moreover, we apply Theorem 2.1.4 to the following parabolic equation

$$u_t(t,x) - \Delta u(t,x) + p(t)Bu(t,x) = 0, \quad (t,x) \in [0,T] \times B^3$$

for radial data in the 3D unit ball B^3.
Finally, we use the abstract result to prove stabilizability for a class of degenerate parabolic equations of the form

$$u_t(t,x) - (x^\alpha u_x(t,x))_x + p(t)Bu(t,x) = 0, \quad (t,x) \in [0,T] \times (0,1),$$

with $\alpha \in [0,3/2)$.
In each example, we will denote by $\{\lambda_k\}_{k\in\mathbb{N}^*}$ and $\{\varphi_k\}_{k\in\mathbb{N}^*}$, respectively the eigenvalues and eigenfunctions of the second order operator associated with the problem under investigation. We will take $(\bar{u},\bar{p}) = (\psi_1, 0)$ as reference trajectory-control pair, where $\psi_1 = e^{-\lambda_1 t}\varphi_1$ is the solution of the uncontrolled problem with initial condition the ground state $u(0,x) = \varphi_1$.

2.2.1 Diffusion equation with Dirichlet boundary conditions.

Let $\Omega = (0,1)$, $X = L^2(\Omega)$ and consider the problem

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\mu(x)u(t,x) = 0 & x \in \Omega, t > 0 \\ u = 0 & x \in \partial\Omega, t > 0 \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases} \tag{2.2.1}$$

where $p \in L^2(0,T)$ is the control function, u the state variable, and μ is a function in $H^3(\Omega)$. We denote by A the operator defined by

$$D(A) = H^2 \cap H^1_0(\Omega), \quad A\varphi = -\frac{d^2\varphi}{dx^2}. \tag{2.2.2}$$

A satisfies all the properties in (1.0.1): in particular, it is strictly accretive and its eigenvalues and eigenvectors have the following explicit expressions

$$\lambda_k = (k\pi)^2, \quad \varphi_k(x) = \sqrt{2}\sin(k\pi x), \quad \forall k \in \mathbb{N}^*.$$

It is straightforward to prove that the eigenvalues fulfill the required gap property. Indeed,

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k+1)\pi - k\pi = \pi, \qquad \forall k \in \mathbb{N}^*.$$

So, (2.1.3) is satisfied.
In order to apply Theorem 2.1.4 to system (2.2.1) and deduce the superexponential stabilizability to the trajectory ψ_1, we need to prove that there exists $\tau > 0$ such that:

- $\langle B\varphi_1, \varphi_k\rangle \neq 0$, for all $k \in \mathbb{N}^*$,
- the series
$$\sum_{k\in\mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k\rangle|^2}$$
is finite.

For this purpose, let us compute the scalar product $\langle B_0\varphi_1, \varphi_k\rangle = \langle \mu\varphi_1, \varphi_k\rangle$

$$\begin{aligned}
\langle \mu\varphi_1, \varphi_k\rangle &= \sqrt{2}\int_0^1 \mu(x)\varphi_1(x)\sin(k\pi x)dx \\
&= \sqrt{2}\left(-(\mu(x)\varphi_1(x))\frac{\cos(k\pi x)}{k\pi}\Big|_0^1 + \int_0^1 (\mu\varphi_1)_x(x)\frac{\cos(k\pi x)}{k\pi}dx \right) \\
&= \sqrt{2}\left((\mu\varphi_1)_x(x)\frac{\sin(k\pi x)}{(k\pi)^2}\Big|_0^1 - \int_0^1 (\mu\varphi_1)_{xx}(x)\frac{\sin(k\pi x)}{(k\pi)^2}dx \right) \\
&= \sqrt{2}\left((\mu\varphi_1)_{xx}(x)\frac{\cos(k\pi x)}{(k\pi)^3}\Big|_0^1 - \int_0^1 (\mu\varphi_1)_{xxx}(x)\frac{\cos(k\pi x)}{(k\pi)^3}dx \right) \\
&= \frac{4}{k^3\pi^2}\left[\mu_x(1)(-1)^{k+1} - \mu_x(0)\right] - \frac{\sqrt{2}}{(k\pi)^3}\int_0^1 (\mu\varphi_1)_{xxx}\cos(k\pi x)dx.
\end{aligned}$$

Observe that the last integral term above represents the k^{th}-Fourier coefficient of the integrable function $(\mu\varphi_1)_{xxx}(x)$ and thus, it converges to zero as k goes to infinity. Therefore, if we assume

$$\mu_x(1) \pm \mu_x(0) \neq 0 \quad \text{and} \quad \langle \mu\varphi_1, \varphi_k\rangle \neq 0 \quad \forall k \in \mathbb{N}^* \tag{2.2.3}$$

then, we deduce that $\langle \mu\varphi_1, \varphi_k\rangle$ is of order $1/k^3$ as $k \to \infty$.

Remark 2.2.1. An example of a function which satisfies (2.2.3) is $\mu(x) = x^2$. Indeed, in this case

$$\langle x^2\varphi_1, \varphi_k\rangle = \begin{cases} \frac{4k(-1)^k}{(k^2-1)^2}, & k \geq 2, \\ \\ \frac{2\pi^2-3}{6\pi^2}, & k = 1 \end{cases}$$

and so $\langle x^2\varphi_1, \varphi_k\rangle \neq 0$ for all $k \in \mathbb{N}^*$ and furthermore

$$|\langle x^2\varphi_1, \varphi_k\rangle| \geq \frac{2\pi^2-3}{6\pi^2}\frac{1}{k^3} = \frac{\pi(2\pi^2-3)}{6}\frac{1}{\lambda_k^{3/2}}, \quad \forall k \in \mathbb{N}^*.$$

We conclude that, under assumption (2.2.3),

$$\exists\, C > 0 \text{ such that } |\langle B\varphi_1, \varphi_k\rangle| \geq ck^{-3} = C\lambda_k^{-3/2}, \quad \forall k \in \mathbb{N}^* \tag{2.2.4}$$

and thanks to the polynomial behavior of the bound, the series

$$\sum_{k\in\mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k\rangle|^2}$$

converges for all $\tau > 0$.
Therefore, all the hypotheses of Theorem 2.1.4 are satisfied and system (2.2.1) is superexponentially stabilizable to the trajectory ψ_1.

Remark 2.2.2. Assumption (2.2.4) for problem (2.2.1) is not too restrictive. In fact, it is possible to prove that the set of functions in $H^3(\Omega)$ for which (2.2.4) holds is dense in $H^3(\Omega)$. For a proof of this fact, see Appendix A in [11].

2.2.2 Diffusion equation with Neumann boundary conditions

Now we look at an example with Neumann boundary conditions: let $\Omega = (0,1)$ and consider the following bilinear stabilzability problem

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\mu(x)u(t,x) = 0 & x \in \Omega, t > 0 \\ u_x = 0 & x \in \partial\Omega, t > 0 \\ u(0,x) = u_0(x). & x \in \Omega \end{cases} \tag{2.2.5}$$

Let $X = L^2(\Omega)$. When we rewrite (2.2.5) in abstract form, the operators A and B are defined by

$$D(A) = \{\varphi \in H^2(0,1) : \varphi_x = 0 \text{ on } \partial\Omega\}, \quad A\varphi = -\varphi_{xx}$$
$$D(B) = X, \quad B\varphi = \mu\varphi.$$

where μ is a real-valued function in $H^2(\Omega)$.
Operator A satisfies the assumptions in (1.0.1) and it is possible to compute explicitly its eigenvalues and eigenvectors:

$$\begin{array}{ll} \lambda_0 = 0, & \varphi_0 = 1 \\ \lambda_k = (k\pi)^2, & \varphi_k(x) = \sqrt{2}\cos(k\pi x), \quad \forall k \geq 1. \end{array}$$

Since the eigenvalues are the same of those in Example 2.2.1 for $k \geq 1$, the gap condition is satisfied for all $k \geq 0$.
Let us compute the scalar product $\langle \mu\varphi_0, \varphi_k\rangle$ to find, if it is possible, a lower bound of the Fourier coefficients of $B\varphi_0$:

$$\begin{aligned} \langle \mu\varphi_0, \varphi_k\rangle &= \sqrt{2}\int_0^1 \mu(x)\cos(k\pi x)dx \\ &= \sqrt{2}\left(\mu(x)\frac{\sin(k\pi x)}{k\pi}\Big|_0^1 - \int_0^1 \mu_x(x)\frac{\sin(k\pi x)}{k\pi}dx \right) \\ &= \sqrt{2}\left(\mu_x(x)\frac{\cos(k\pi x)}{(k\pi)^2}\Big|_0^1 - \int_0^1 \mu_{xx}(x)\frac{\cos(k\pi x)}{(k\pi)^2}dx \right) \\ &= \frac{\sqrt{2}}{(k\pi)^2}\left(\mu_x(1)(-1)^k - \mu_x(0)\right) - \frac{\sqrt{2}}{(k\pi)^2}\int_0^1 \mu_{xx}(x)\cos(k\pi x)dx. \end{aligned}$$

Thus, reasoning as Example 2.2.1, if $\langle B\varphi_0, \varphi_k\rangle \neq 0 \ \forall k \in \mathbb{N}$ and $\mu_x(1) \pm \mu_x(0) \neq 0$, then we have that

$$\exists\, C > 0 \text{ such that } |\langle B\varphi_0, \varphi_k\rangle| \geq Ck^{-2} = C\lambda_k^{-1}, \quad \forall k \in \mathbb{N}^* \tag{2.2.6}$$

and therefore the series in (2.1.4) is finite for all $\tau > 0$.

Remark 2.2.3. An example of a suitable function μ for problem (2.2.5) that satisfies the above hypothesis, is $\mu(x) = x^2$, for which

$$\langle x^2\varphi_0, \varphi_k\rangle = \begin{cases} \frac{2\sqrt{2}(-1)^k}{(k\pi)^2}, & k \geq 1, \\ \\ \frac{1}{3}, & k = 0. \end{cases}$$

Applying Theorem 2.1.4, it follows that system (2.2.5) is superexponentially stabillizable to ψ_1.

2.2.3 Variable coefficient parabolic equation with Dirichlet boundary conditions

In this example, we analyze the superexponential stabilizability of a parabolic equation in divergence form with nonconstant coefficients in the second order term.
Let $\Omega = (0,1)$, $X = L^2(\Omega)$ and consider the problem

$$\begin{cases} u_t(t,x) - ((1+x)^2 u_x(t,x))_x + p(t)\mu(x)u(t,x) = 0 & x \in \Omega, t > 0 \\ u(t,0) = 0, \quad u(t,1) = 0, & t > 0 \\ u(0,x) = u_0(x) & x \in \Omega \end{cases} \tag{2.2.7}$$

where $p \in L^2(0,T)$ is the control and μ is a function in $H^2(\Omega)$ with some properties to be specified later.
We denote by A the operator

$$A : D(A) \subset X \to X, \qquad Au = -((1+x)^2 u_x)_x$$

where $D(A) = H^2 \cap H^1_0(\Omega)$ and it is possible to prove that A satisfies the properties in (1.0.1). The eigenvalues and eigenvectors of A are computed as follows

$$\lambda_k = \frac{1}{4} + \left(\frac{k\pi}{\ln 2}\right)^2, \qquad \varphi_k = \sqrt{\frac{2}{\ln 2}}(1+x)^{-1/2} \sin\left(\frac{k\pi}{\ln 2}\ln(1+x)\right).$$

The gap condition holds true because

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \frac{\pi}{\ln 2}, \quad \forall k \in \mathbb{N}^*.$$

Now, we check the hypotheses on the operator $B\varphi = \mu\varphi$ needed to apply Theorem 2.1.4. We recall that we want to prove that:

- $\langle B\varphi_1, \varphi_k\rangle \neq 0$, for all $k \in \mathbb{N}^*$,
- there exists $\tau > 0$ such that the series

  $$\sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k\rangle|^2} \tag{2.2.8}$$

  is finite.

Let us compute the Fourier coefficients of $B\varphi_1$:

$$
\begin{aligned}
\langle \mu\varphi_1, \varphi_k\rangle &= \sqrt{\frac{2}{\ln 2}}\int_0^1 \mu(x)\varphi_1(x)(1+x)^{-1/2}\sin\left(\frac{k\pi}{\ln 2}\ln(1+x)\right)dx\\
&= \sqrt{\frac{2}{\ln 2}}\frac{\ln 2}{k\pi}\left(-\mu(x)\varphi_1(x)(1+x)^{1/2}\cos\left(\frac{k\pi}{\ln 2}\ln(1+x)\right)\Big|_0^1 + \right.\\
&\qquad\left. + \int_0^1 \left((1+x)^{1/2}\mu\varphi_1\right)_x(x)\cos\left(\frac{k\pi}{\ln 2}\ln(1+x)\right)dx\right)\\
&= \sqrt{\frac{2}{\ln 2}}\left(\frac{\ln 2}{k\pi}\right)^2\left(\left((1+x)^{1/2}\mu\varphi_1\right)_x(x)(1+x)\sin\left(\frac{k\pi}{\ln 2}\ln(1+x)\right)\Big|_0^1 + \right.\\
&\qquad\left. - \int_0^1 \left(\left((1+x)^{1/2}\mu\varphi_1\right)_x(1+x)\right)_x(x)\sin\left(\frac{k\pi}{\ln 2}\ln(1+x)\right)dx\right)\\
&= \sqrt{\frac{2}{\ln 2}}\left(\frac{\ln 2}{k\pi}\right)^3\left(\left(\left((1+x)^{1/2}\mu\varphi_1\right)_x(1+x)\right)_x(x)(1+x)\cos\left(\frac{k\pi}{\ln 2}\ln(1+x)\right)\Big|_0^1 + \right.\\
&\qquad\left. - \int_0^1 \left(\left(\left((1+x)^{1/2}\mu\varphi_1\right)_x(1+x)\right)_x(1+x)\right)_x(x)\cos\left(\frac{k\pi}{\ln 2}\ln(1+x)\right)dx\right)\\
&= \sqrt{\frac{2}{\ln 2}}\left(\frac{\ln 2}{k\pi}\right)^3\left(\sqrt{\frac{2}{\ln 2}}\frac{2\pi}{\ln 2}\left(-2\mu_x(1)(-1)^k - \mu_x(0)\right) + \right.\\
&\qquad\left. - \int_0^1 \left(\left(\left((1+x)^{1/2}\mu\varphi_1\right)_x(1+x)\right)_x(1+x)\right)_x(x)\cos\left(\frac{k\pi}{\ln 2}\ln(1+x)\right)dx\right)
\end{aligned}
$$

Observe that, for the same reason of Example 2.2.1, if $2\mu_x(1)\pm\mu_x(0)\neq 0$ and $\langle\mu\varphi_1,\varphi_k\rangle\neq 0$, $\forall k\in\mathbb{N}^*$ then, there exists a constant $C>0$ such that $|\langle B_0\varphi,\varphi_k\rangle|$ is bounded from below by $C\lambda_k^{-3/2}$, for all $k\in\mathbb{N}^*$. Thus, series (2.2.8) is finite for all $\tau>0$.

Remark 2.2.4. As an example of a function μ that verifies the lower bound $|\langle B\varphi,\varphi_k\rangle|\geq C\lambda_k^{-3/2}$, one can consider again $\mu(x)=x$: indeed, it satisfies the sufficient condition $2\mu_x(1)\pm\mu_x(0)\neq 0$ and the Fourier coefficients of $B\varphi_1 = x\varphi_1$ are all different from zero:

$$
\langle x\varphi_1,\varphi_k\rangle = \begin{cases} \frac{2(2(-1)^{k+1}-1)}{(k^2-1)^2\left(1+\frac{(k+1)^2\pi^2}{(\ln 2)^2}\right)\left(1+\frac{(k-1)^2\pi^2}{(\ln 2)^2}\right)}\left(4k^3+k+1+2k(k^2-1)^2\frac{\pi}{(\ln 2)^2}\right), & k\geq 2\\ \frac{1}{\ln 2}\left(\frac{(1-\ln 2)\left(\frac{2\pi}{\ln 2}\right)^3-\frac{2\pi}{\ln 2}}{1+\left(\frac{2\pi}{\ln 2}\right)^3}\right), & k=1 \end{cases}
$$

This concludes the verification of the hypotheses of Theorem 2.1.4, that imply the superexponential stabilizability of (2.2.7) to ψ_1.

2.2.4 Diffusion equation in a $3D$ ball with radial data

In this example we consider an evolution equation in the three dimensional unit ball B^3 for radial data. The bilinear stabilizability problem is the following

$$
\begin{cases} u_t(t,r)-\Delta u(t,r)+p(t)\mu(r)u(t,r)=0 & r\in[0,1], t>0\\ u(t,1)=0, & t>0\\ u(0,r)=u_0(r) & r\in[0,1] \end{cases} \tag{2.2.9}
$$

where the Laplacian in polar coordinates for radial data has the form

$$\Delta\varphi(r) = \partial_r^2\varphi(r) + \frac{2}{r}\partial_r\varphi(r).$$

The function μ is a radial function as well in the space $H_r^3(B^3)$, where the spaces $H_r^k(B^3)$ are defined as follows

$$X := L_r^2(B^3) = \left\{\varphi \in L^2(B^3) \,|\, \exists\psi : \mathbb{R} \to \mathbb{R}, \varphi(x) = \psi(|x|)\right\}$$

$$H_r^k(B^3) := H^k(B^3) \cap L_r^2(B^3).$$

The domain of the Dirichlet Laplacian $A := -\Delta$ in X is $D(A) = H_r^2 \cap H_0^1(B^3)$. We observe that A satisfies the hypotheses required to apply Theorem 2.1.4. We denote by $\{\lambda_k\}_{k\in\mathbb{N}^*}$ and $\{\varphi_k\}_{k\in\mathbb{N}^*}$ the families of eigenvalues and eigenvectors of A, $A\varphi_k = \lambda_k\varphi_k$, namely

$$\varphi_k = \frac{\sin(k\pi r)}{\sqrt{2\pi}r}, \qquad \lambda_k = (k\pi)^2 \tag{2.2.10}$$

$\forall k \in \mathbb{N}^*$, see [48], section 8.14. The family $\{\varphi_k\}_{k\in\mathbb{N}^*}$ forms an orthonormal basis of X.
In order to prove a superexponential stabilizability result to the trajectory ψ_1, we need to verify the remaining hypotheses in Theorem 2.1.4 regarding the gap condition of the eigenvalues of A and the properties of the operator $B : X \mapsto X$, $B\varphi = \mu\varphi$.
Since the Laplacian in the $3D$ ball for radial data behaves as a one dimensional operator, the analysis is very similar to the previous cases. Indeed, since the eigenvalues of the operator A are actually the same of the $1D$ Dirichlet Laplacian, we have

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = \pi, \quad \forall k \in \mathbb{N}^*.$$

In order to compute a suitable lower bound for the Fourier coefficients of $B\varphi_1$, we recall the following property of radial symmetric functions $f = f(r)$: the integral over the unit ball $B^n \subset \mathbb{R}^n$ of $f = f(r)$ reduces to

$$\int_{B^n} f\,dV = |S^{n-1}| \int_0^1 f(r)r^{n-1}dr \tag{2.2.11}$$

where $|S^{n-1}|$ is the measure of the surface of the sphere S^{n-1}.
Therefore,

$$\begin{aligned}
\langle\mu\varphi_1, \varphi_k\rangle &= \int_{B^3} \frac{1}{2\pi}\mu(r)\frac{\sin(\pi r)}{r}\frac{\sin(k\pi r)}{r}dV \\
&= 4\pi\int_0^1 \frac{1}{2\pi}\mu(r)\frac{\sin(\pi r)}{r}\frac{\sin(k\pi r)}{r}r^2dr \\
&= \int_0^1 2\mu(r)\sin(\pi r)\sin(k\pi r)dr \\
&= -\frac{4}{k^3\pi^2}\left(\partial_r\mu(1)(-1)^k + \partial_r\mu(0)\right) + \\
&\quad -\frac{2}{(k\pi)^3}\int_0^1 \partial_r^3(\mu\sin(\pi r))(r)\cos(k\pi r)dr.
\end{aligned} \tag{2.2.12}$$

Following the same argument as in Example 2.2.1, if all the coefficients $\langle\mu\varphi_1,\varphi_k\rangle$ are different from zero and, moreover, $\partial_r\mu(1)\pm\partial_r\mu(0)\neq 0$ then, there exists a constant $C>0$ such that

$$|\langle\mu\varphi_1,\varphi_k\rangle|\geq C\lambda_k^{-3/2},\qquad \forall k\in\mathbb{N}^*$$

and thus the series in (2.2.8) is finite also in this case, for all $\tau>0$.

Remark 2.2.5. An example of a function $\mu\in H_r^3(B^3)$ with the aforementioned properties is $\mu(r)=r^2$. In this case the Fourier coefficients of $B\varphi_1$ are defined by

$$\langle B\varphi_1,\varphi_k\rangle=\begin{cases}\frac{8(-1)^{k+1}k}{(k^2-1)^2\pi^2}, & k\geq 2\\ \\ \frac{2\pi^2-3}{6\pi^2}, & k=1\end{cases}$$

Finally, applying Theorem 2.1.4, we deduce that, fixed $T>0$, there exist constants $M_T,\omega_T>0$ such that, for all $\rho>0$, there exists $R_\rho>0$ such that, if the initial condition u_0 satisfies $||u_0-\varphi_1||\leq R_\rho$, then

$$||u(t)-\psi_1(t)||\leq M_T e^{-(\rho e^{\omega_T t}+\pi^2 t)},\qquad \forall t>0.$$

2.2.5 Degenerate parabolic equation

Let $I=(0,1)$, $X=L^2(I)$ and consider the following degenerate parabolic equation

$$\begin{cases}u_t-(x^\alpha u_x)_x+p(t)x^{2-\alpha}u=0, & (t,x)\in(0,\infty)\times(0,1)\\ u(t,1)=0,\quad \begin{cases}u(t,0)=0, & \text{if } \alpha\in[0,1),\\ (x^\alpha u_x)(t,0)=0, & \text{if } \alpha\in[1,3/2),\end{cases} & \\ u(0,x)=u_0(x). & \end{cases} \tag{2.2.13}$$

where p is the bilinear control function and the parameter $\alpha\in[0,2)$ describes the degeneracy magnitude. In particular, the problem is called weakly degenerate for $\alpha\in[0,1)$, and strongly degenerate for $\alpha\in[1,2)$.
Depending on the type of degeneracy, it is customary to assign different boundary conditions to the problem and therefore the spectral analysis of the second order degenerate operator will be different. We refer to Appendix A for a more detailed discussion.
The natural spaces for the well-posedness of degenerate problems are weighted Sobolev spaces, that we will indicate by $H^s_\alpha(I)$, and that differ in the weak and strong degeneracy settings.
We introduce the linear operator

$$\begin{aligned}A:D(A)\subset X&\to X\\ u&\mapsto-(x^\alpha u_x)_x\end{aligned}$$

that can be shown to be a densely defined, self-adjoint, accretive operator with compact resolvent for all $\alpha\in[0,2)$.
For any $\alpha\in[0,2)$, let

$$\nu_\alpha:=\frac{|1-\alpha|}{2-\alpha},\qquad k_\alpha:=\frac{2-\alpha}{2}.$$

Given $\nu \geq 0$, we denote by J_ν the Bessel function of the first kind and order ν and by $j_{\nu,1} < j_{\nu,2} < \dots < j_{\nu,k} < \dots$ the sequence of all positive zeros of J_ν. It is possible to prove that the eigenvalue $\{\lambda_{\alpha,k}\}_{k\in\mathbb{N}^*}$ and the corresponding eigenfunction $\{\varphi_{\alpha,k}\}_{k\in\mathbb{N}^*}$ related to the operator A are given by

$$\lambda_{\alpha,k} = k_\alpha^2 j_{\alpha,k}^2, \tag{2.2.14}$$

$$\varphi_{\alpha,k}(x) = \frac{\sqrt{2k_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,k})|} x^{(1-\alpha)/2} J_{\nu_\alpha}\left(j_{\nu_\alpha,k} x^{k_\alpha}\right) \tag{2.2.15}$$

for every $k \in \mathbb{N}^*$. Moreover, the family $\left(\varphi_{\alpha,k}\right)_{k\in\mathbb{N}^*}$ is an orthonormal basis of X, see [41].

To apply Theorem 2.1.4 to problem (2.2.13), we have to prove the validity of the gap condition (2.1.3) and of hypothesis (2.1.4) for the multiplication operator

$$\begin{aligned} B : X &\to X \\ u &\mapsto \mu u, \end{aligned}$$

with $\mu(x) = x^{2-\alpha}$.

Concerning the gap condition, it has been proved (see [47], page 135) that

- if $\alpha \in [0,1)$, $\nu_\alpha = \frac{1-\alpha}{2-\alpha} \in \left(0, \frac{1}{2}\right]$, the sequence $\left(j_{\nu_\alpha,k+1} - j_{\nu_\alpha,k}\right)_{k\in\mathbb{N}^*}$ is nondecreasing and converges to π. Therefore,

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = k_\alpha\left(j_{\nu_\alpha,k+1} - j_{\nu_\alpha,k}\right) \geq k_\alpha\left(j_{\nu_\alpha,2} - j_{\nu_\alpha,1}\right) \geq \frac{7}{16}\pi,$$

- if $\nu_\alpha \geq \frac{1}{2}$, the sequence $\left(j_{\nu_\alpha,k+1} - j_{\nu_\alpha,k}\right)_{k\in\mathbb{N}^*}$ is nonincreasing and converges to π. Thus,

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = k_\alpha\left(j_{\nu_\alpha,k+1} - j_{\nu_\alpha,k}\right) \geq k_\alpha \pi \geq \frac{\pi}{2}.$$

Therefore, (2.1.3) is satisfied in both weak and strong degenerate problems with different constants.

The operator B is linear and bounded in I. What remains to prove is that there exists $\tau > 0$ such that

$$\begin{aligned} &\langle \mu\varphi_{\alpha,1}, \varphi_{\alpha,k}\rangle \neq 0, \qquad \forall k \in \mathbb{N}^*, \\ &\sum_{k\in\mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle \mu\varphi_{\alpha,1}, \varphi_{\alpha,k}\rangle|^2} < +\infty. \end{aligned} \tag{2.2.16}$$

We compute the scalar product $\langle \mu\varphi_{\alpha,1}, \varphi_{\alpha,k}\rangle$ for $k \neq 1$ and, from now on, we write φ_k instead of $\varphi_{\alpha,k}$ to lighten the notation:

$$\begin{aligned}
\langle \mu\varphi_1, \varphi_k\rangle &= \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x) = -\frac{1}{\lambda_k}\int_0^1 \mu(x)\varphi_1(x)\left(x^\alpha(\varphi_k)_x\right)_x(x)dx \\
&= -\frac{1}{\lambda_k}\left(\mu(x)\varphi_1(x)x^\alpha(\varphi_k)_x(x)|_0^1 - \int_0^1 (\mu\varphi_1)_x(x)x^\alpha(\varphi_k)_x(x)dx\right) \\
&= \frac{1}{\lambda_k}\left(\int_0^1 \mu_x(x)\varphi_1(x)x^\alpha(\varphi_k)_x(x)dx + \int_0^1 \mu(x)(\varphi_1)_x(x)x^\alpha(\varphi_k)_x(x)dx\right) \\
&= \frac{1}{\lambda_k}\left(\int_0^1 \mu_x(x)\varphi_1(x)x^\alpha(\varphi_k)_x(x)dx + \mu(x)(\varphi_1)_x(x)x^\alpha\varphi_k(x)|_0^1\right. \\
&\qquad\qquad \left. - \int_0^1 (\mu(\varphi_1)_x x^\alpha)_x(x)\varphi_k(x)dx\right) \\
&= \frac{1}{\lambda_k}\left(\int_0^1 \mu_x(x)\varphi_1(x)x^\alpha(\varphi_k)_x(x)dx - \int_0^1 \mu_x(x)(\varphi_1)_x(x)x^\alpha\varphi_k(x)dx\right. \\
&\qquad\qquad \left. - \int_0^1 \mu(x)(x^\alpha(\varphi_1)_x)_x(x)\varphi_k(x)dx\right) \\
&= \frac{1}{\lambda_k}\left(\int_0^1 \mu_x(x)x^\alpha\left[\varphi_1(x)(\varphi_k)_x(x) - (\varphi_1)_x(x)\varphi_k(x)\right]dx\right. \\
&\qquad\qquad \left. + \lambda_1\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx\right).
\end{aligned} \tag{2.2.17}$$

We observe that in the weakly degenerate case, thanks to the Dirichlet conditions in both extrema, the boundary terms vanish. We can deduce the same vanishing property at $x = 0$ for the strong degenerate case thanks to the first item of Proposition A.1.1 and to (A.1.12). Moving the last term of (2.2.17) to the left-hand side, we get

$$\left(1 - \frac{\lambda_1}{\lambda_k}\right)\langle \mu\varphi_1, \varphi_k\rangle = \frac{1}{\lambda_k}\int_0^1 \mu_x(x)x^\alpha\varphi_1^2(x)\left(\frac{\varphi_k}{\varphi_1}\right)_x(x)dx \tag{2.2.18}$$

and therefore, integrating by parts we obtain

$$\begin{aligned}
\langle \mu\varphi_1, \varphi_k\rangle &= \frac{1}{\lambda_k - \lambda_1}\left(\mu_x(x)x^\alpha\varphi_1^2(x)\frac{\varphi_k(x)}{\varphi_1(x)}\Big|_0^1 - \int_0^1 \left(\mu_x x^\alpha\varphi_1^2\right)_x(x)\frac{\varphi_k(x)}{\varphi_1(x)}dx\right) \\
&= \frac{-1}{\lambda_k - \lambda_1}\left(\int_0^1 (\mu_x x^\alpha)_x(x)\varphi_1^2(x)\frac{\varphi_k(x)}{\varphi_1(x)}dx\right. \\
&\qquad\qquad \left. + 2\int_0^1 \mu_x(x)x^\alpha\varphi_1(x)(\varphi_1)_x(x)\frac{\varphi_k(x)}{\varphi_1(x)}dx\right) \qquad (2.2.19) \\
&= -\frac{1}{\lambda_k - \lambda_1}\left(\int_0^1 (\mu_x x^\alpha)_x(x)\varphi_1(x)\varphi_k(x)dx\right. \\
&\qquad\qquad \left. + 2\int_0^1 \mu_x(x)x^\alpha(\varphi_1)_x(x)\varphi_k(x)dx\right).
\end{aligned}$$

The boundary terms vanish for the Dirichet conditions if $\alpha \in [0,1)$ and thanks to the second item in Proposition A.1.1 for $\alpha \in [1,3/2)$.

Recalling that $\mu(x) = x^{2-\alpha}$, we have that

$$
\begin{aligned}
\langle \mu\varphi_1, \varphi_k \rangle &= -\frac{2(2-\alpha)}{\lambda_k - \lambda_1} \int_0^1 x(\varphi_1)_x(x)\varphi_k(x)dx \\
&= \frac{2(2-\alpha)}{\lambda_k(\lambda_k - \lambda_1)} \int_0^1 x(\varphi_1)_x(x)\left(x^\alpha(\varphi_k)_x\right)_x(x)dx \\
&= \frac{2(2-\alpha)}{\lambda_k(\lambda_k - \lambda_1)} \left(x(\varphi_1)_x(x)x^\alpha(\varphi_k)_x(x)\Big|_0^1 - \int_0^1 \left(x(\varphi_1)_x\right)_x(x)x^\alpha(\varphi_k)_x(x)dx \right) \\
&= \frac{2(2-\alpha)}{\lambda_k(\lambda_k - \lambda_1)} \Big(x^{1+\alpha}(\varphi_1)_x(x)(\varphi_k)_x(x)\Big|_0^1 - \left(x(\varphi_1)_x\right)_x(x)x^\alpha\varphi_k(x)\Big|_0^1 \\
&\qquad\qquad + \int_0^1 \left(\left(x(\varphi_1)_x\right)_x(x)x^\alpha\right)_x \varphi_k(x)dx \Big) \\
&= \frac{2(2-\alpha)}{\lambda_k(\lambda_k - \lambda_1)} \Big(x^{1+\alpha}(\varphi_1)_x(x)(\varphi_k)_x(x)\Big|_0^1 \\
&\qquad\qquad + \int_0^1 \left(\left((\varphi_1)_x + x(\varphi_1)_{xx}\right)x^\alpha\right)_x(x)\varphi_k(x)dx \Big) \\
&= \frac{2(2-\alpha)}{\lambda_k(\lambda_k - \lambda_1)} \Big(x^{1+\alpha}(\varphi_1)_x(x)(\varphi_k)_x(x)\Big|_0^1 - \lambda_1 \int_0^1 \varphi_1(x)\varphi_k(x)dx \\
&\qquad\qquad + \int_0^1 \left(x^{1+\alpha}(\varphi_1)_{xx}\right)_x(x)\varphi_k(x)dx \Big) \\
&= \frac{2(2-\alpha)}{\lambda_k(\lambda_k - \lambda_1)} \left(x^{1+\alpha}(\varphi_1)_x(x)(\varphi_k)_x(x)\Big|_0^1 + \int_0^1 \left(x^{1+\alpha}(\varphi_1)_{xx}\right)_x(x)\varphi_k(x)dx \right)
\end{aligned}
\tag{2.2.20}
$$

where we have used the fact that, for $\alpha \in [1,3/2)$, $\left(x(\varphi_1)_x\right)_x(x)x^\alpha\varphi_k(x)\big|_0^1$ vanishes in view of Proposition A.1.1.

Since φ_k is an eigenfunction of A for all $k \in \mathbb{N}^*$, it satisfies the equation

$$
-\left(\alpha x^{\alpha-1}(\varphi_k)_x(x) + x^\alpha(\varphi_k)_{xx}(x)\right) = \lambda_k\varphi_k(x), \tag{2.2.21}
$$

then we can rewrite the expression of $(\varphi_k)_{xx}(x)$ in (2.2.20) using (2.2.21):

$$
\begin{aligned}
\langle \mu\varphi_1, \varphi_k\rangle &= \frac{2(2-\alpha)}{\lambda_k(\lambda_k-\lambda_1)}\left(x^{1+\alpha}(\varphi_1)_x(x)(\varphi_k)_x(x)\Big|_0^1 + \int_0^1 \left(x^{1+\alpha}(\varphi_1)_{xx}\right)_x(x)\varphi_k(x)dx\right)\\
&= \frac{2(2-\alpha)}{\lambda_k(\lambda_k-\lambda_1)}\Big(x^{1+\alpha}(\varphi_1)_x(x)(\varphi_k)_x(x)\Big|_0^1\\
&\qquad\qquad - \int_0^1 (\lambda_1 x\varphi_1 + \alpha x^{\alpha}(\varphi_1)_x)_x(x)\varphi_k(x)dx\Big)\\
&= \frac{2(2-\alpha)}{\lambda_k(\lambda_k-\lambda_1)}\Big(x^{1+\alpha}(\varphi_1)_x(x)(\varphi_k)_x(x)\Big|_0^1 - \lambda_1\int_0^1 x(\varphi_1)_x(x)\varphi_k(x)dx\\
&\quad -\lambda_1\int_0^1 \varphi_1(x)\varphi_k(x)dx - \alpha\int_0^1 \underbrace{(x^{\alpha}(\varphi_1)_x)_x(x)}_{-\lambda_1\varphi_1(x)}\varphi_k(x)dx\Big).
\end{aligned}
\tag{2.2.22}
$$

Recalling that $\{\varphi_k\}_{k\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(0,1)$, the last two terms on the right-hand side of the above equality are zero.
Thus, from the first equality of (2.2.20) and the last one of (2.2.22), we obtain that

$$
-\frac{2(2-\alpha)}{\lambda_k-\lambda_1}\left(1-\frac{\lambda_1}{\lambda_k}\right)\int_0^1 x(\varphi_1)_x(x)\varphi_k(x)dx = \frac{2(2-\alpha)}{\lambda_k(\lambda_k-\lambda_1)}\, x^{1+\alpha}(\varphi_1)_x(x)(\varphi_k)_x(x)\Big|_0^1
\tag{2.2.23}
$$

that implies

$$
\langle \mu\varphi_1, \varphi_k\rangle = -\frac{2(2-\alpha)}{\lambda_k-\lambda_1}\int_0^1 x(\varphi_1)_x(x)\varphi_k(x)dx = \frac{2(2-\alpha)}{(\lambda_k-\lambda_1)^2}\, x^{1+\alpha}(\varphi_1)_x(x)(\varphi_k)_x(x)\Big|_0^1
\tag{2.2.24}
$$

Recalling that the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}^*}$ of A are defined by (2.2.14) where $\nu_\alpha = |1-\alpha|/(2-\alpha)$, and the eigenfunctions, $\{\varphi_k\}_{k\in\mathbb{N}^*}$, by (2.2.15), we compute the right-hand side of (2.2.24):

$$
\begin{aligned}
&x^{1+\alpha}(\varphi_1)_x(x)(\varphi_k)_x(x) =\\
&\quad= \frac{2(2-\alpha)k_\alpha x^{1+\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,1})||J'_{\nu_\alpha}(j_{\nu_\alpha,k})|}\Big(\frac{1-\alpha}{2}x^{-(1+\alpha)/2}J_{\nu_\alpha}(j_{\nu_\alpha,1}x^{k_\alpha})\\
&\qquad\qquad + j_{\nu_\alpha,1}k_\alpha x^{(1-2\alpha)/2}J'_{\nu_\alpha}(j_{\nu_\alpha,1}x^{k_\alpha})\Big)\\
&\quad\cdot\left(\frac{1-\alpha}{2}x^{-(1+\alpha)/2}J_{\nu_\alpha}(j_{\nu_\alpha,k}x^{k_\alpha}) + j_{\nu_\alpha,k}k_\alpha x^{(1-2\alpha)/2}J'_{\nu_\alpha}(j_{\nu_\alpha,k}x^{k_\alpha})\right).
\end{aligned}
\tag{2.2.25}
$$

Therefore

$$
x^{1+\alpha}(\varphi_1)_x(x)(\varphi_k)_x(x)\ \Big|_0^1 = (\varphi_1)_x(1)(\varphi_k)_x(1) = \frac{2k_\alpha^3 j_{\nu_\alpha,1}j_{\nu_\alpha,k}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,1})||J'_{\nu_\alpha}(j_{\nu_\alpha,k})|}J'_{\nu_\alpha}(j_{\nu_\alpha,1})J'_{\nu_\alpha}(j_{\nu_\alpha,k}).
\tag{2.2.26}
$$

Now, we recall that the zeros of J'_{ν_α}, $j'_{\nu_\alpha,k}$, satisfy $\nu_\alpha < j'_{\nu_\alpha,1} < j_{\nu_\alpha,1} < j'_{\nu_\alpha,2} < j_{\nu_\alpha,2}\ldots$, to conclude that the right-hand side of (2.2.26) does not vanish.
From (2.2.24) and (2.2.26) we deduce that there exists a constant C such that

$$
|\langle \mu\varphi_1, \varphi_k\rangle| \ge \frac{C}{\lambda_k^{3/2}}, \quad \forall k\in\mathbb{N}^*,\ k\neq 1.
\tag{2.2.27}
$$

For $k=1$, we have

$$\begin{aligned}\langle \mu\varphi_1, \varphi_1\rangle &= \frac{2k_\alpha}{|J'_{\nu_\alpha}(j_{\nu_\alpha,1})|^2}\int_0^1 x^{2-\alpha}x^{1-\alpha}J^2_{\nu_\alpha}(j_{\nu_\alpha,1}x^{k_\alpha})dx \\ &= \frac{4k_\alpha j^4_{\nu_\alpha,1}}{(2-\alpha)|J'_{\nu_\alpha}(j_{\nu_\alpha,1})|^2}\int_0^{j_{\nu_\alpha,1}} z^3 J^2_{\nu_\alpha}(z)dz.\end{aligned} \tag{2.2.28}$$

We now appeal to the identity

$$\begin{aligned}(\sigma+2)\int^z t^{\sigma+2}J_\nu(t)dt &= (\sigma+1)\left\{\nu^2-\frac{1}{4}(\sigma+1)^2\right\}\int^z t^\sigma J^2_\nu(t)dt \\ &+\frac{1}{2}z^{\sigma+1}\left[\left\{zJ'_\nu(z)-\frac{1}{2}(\sigma+1)J_\nu(z)\right\}^2+\left\{z^2-\nu^2+\frac{1}{4}(\sigma+1)^2\right\}J^2_\nu(z)\right]\end{aligned} \tag{2.2.29}$$

with $\sigma=1$ (see [51], equation (17) page 256) to turn (2.2.29) into

$$\begin{aligned}\langle \mu\varphi_1,\varphi_1\rangle &= \frac{4k_\alpha j^4_{\nu_\alpha,1}}{(2-\alpha)|J'_{\nu_\alpha}(j_{\nu_\alpha,1})|^2}\frac{2}{3}\left\{\nu^2_\alpha-1\right\}\int_0^{j_{\nu_\alpha,1}} zJ^2_{\nu_\alpha}(z)dz \\ &+\frac{1}{6}j^3_{\nu_\alpha,1}\left[\left\{j_{\nu_\alpha,1}J'_{\nu_\alpha}(j_{\nu_\alpha,1})-J_{\nu_\alpha}(j_{\nu_\alpha,1})\right\}^2+\left\{j^2_{\nu_\alpha,1}-\nu^2_\alpha+1\right\}J^2_{\nu_\alpha}(j_{\nu_\alpha,1})\right].\end{aligned} \tag{2.2.30}$$

Using Lommel's integral

$$\int_0^c zJ_\nu(az)^2dz = \frac{c^2}{2}\left[J^2_\nu(ac)-J_{\nu-1}(ac)J_{\nu+1}(ac)\right] \tag{2.2.31}$$

in (2.2.30), we obtain

$$\begin{aligned}\langle \mu\varphi_1,\varphi_1\rangle &= \\ &= \frac{4k_\alpha j^4_{\nu_\alpha,1}}{(2-\alpha)|J'_{\nu_\alpha}(j_{\nu_\alpha,1})|^2}\left[\frac{2}{3}\left\{\nu^2_\alpha-1\right\}\left(\frac{j^2_{\nu_\alpha,1}}{2}\left(J^2_{\nu_\alpha}(j_{\nu_\alpha,1})-J_{\nu_\alpha-1}(j_{\nu_\alpha,1})J_{\nu_\alpha+1}(j_{\nu_\alpha,1})\right)\right)\right. \\ &\qquad\left.+\frac{1}{6}j^3_{\nu_\alpha,1}\left(j_{\nu_\alpha,1}J'_{\nu_\alpha}(j_{\nu_\alpha,1})\right)^2\right] \\ &= \frac{4k_\alpha j^4_{\nu_\alpha,1}}{(2-\alpha)|J'_{\nu_\alpha}(j_{\nu_\alpha,1})|^2}\left(-\frac{1}{3}j^2_{\nu_\alpha,1}\left\{\nu^2_\alpha-1\right\}J_{\nu_\alpha-1}(j_{\nu_\alpha,1})J_{\nu_\alpha+1}(j_{\nu_\alpha,1})\right. \\ &\qquad\left.+\frac{1}{24}j^5_{\nu_\alpha,1}\left(J_{\nu_\alpha-1}(j_{\nu_\alpha,1})-J_{\nu_\alpha+1}(j_{\nu_\alpha,1})\right)^2\right) \\ &= \frac{4k_\alpha j^4_{\nu_\alpha,1}}{(2-\alpha)|J'_{\nu_\alpha}(j_{\nu_\alpha,1})|^2}\left(\frac{1}{24}j^5_{\nu_\alpha,1}\left(J^2_{\nu_\alpha-1}(j_{\nu_\alpha,1})+J^2_{\nu_\alpha+1}(j_{\nu_\alpha,1})\right)\right. \\ &\qquad\left.-\left(\frac{1}{3}j^2_{\nu_\alpha,1}\left\{\nu^2_\alpha-1\right\}+\frac{j^5_{\nu_\alpha,1}}{12}\right)J_{\nu_\alpha-1}(j_{\nu_\alpha,1})J_{\nu_\alpha+1}(j_{\nu_\alpha,1})\right) \\ &\geq \frac{4k_\alpha j^4_{\nu_\alpha,1}}{(2-\alpha)|J'_{\nu_\alpha}(j_{\nu_\alpha,1})|^2}\left(\frac{1}{24}j^5_{\nu_\alpha,1}\left(J^2_{\nu_\alpha-1}(j_{\nu_\alpha,1})+J^2_{\nu_\alpha+1}(j_{\nu_\alpha,1})\right)\right. \\ &\qquad\left.-\left(\frac{1}{3}j^2_{\nu_\alpha,1}\left\{\nu^2_\alpha-1\right\}+\frac{j^5_{\nu_\alpha,1}}{12}\right)\frac{1}{2}\left(J^2_{\nu_\alpha-1}(j_{\nu_\alpha,1})+J^2_{\nu_\alpha+1}(j_{\nu_\alpha,1})\right)\right).\end{aligned}$$

Thus, $\langle \mu\varphi_1,\varphi_1\rangle > 0$ if

$$\frac{1}{24}j^5_{\nu_\alpha,1} > \frac{1}{2}\left(\frac{1}{3}j^2_{\nu_\alpha,1}\left\{\nu^2_\alpha-1\right\}+\frac{j^5_{\nu_\alpha,1}}{12}\right). \tag{2.2.32}$$

Since

$$\alpha \in [0,1) \;\Rightarrow\; \nu_\alpha \in (0,1/2],$$
$$\alpha \in [1,3/2) \;\Rightarrow \nu_\alpha \in [0,1),$$

equation (2.2.32) holds true for both weak and strong degeneracy.
Hence, since $\langle \mu\varphi_1, \varphi_k \rangle \neq 0$ for every $k \in \mathbb{N}^*$ and (2.2.27) is valid, the series (2.2.16) converges for every $\tau > 0$.
We have checked that every hypothesis of Theorem 2.1.4 holds for problem (2.2.13) if $\alpha \in [0,3/2)$. Therefore, we conclude that, for any $\rho > 0$, if the initial condition u_0 is close enough to φ_1, the system is superexponentially stabilizable to the ground state solution ψ_1.

CHAPTER 3

Exact contrallability to trajectories

In this chapter we present results of local and global controllability to a target trajectory that are based on [3].
In a separable Hilbert space X, we study the linear evolution equation

$$u'(t) + Au(t) + p(t)Bu(t) = 0,$$

where A is an accretive self-adjoint linear operator, B is a bounded linear operator on X, and $p \in L^2_{loc}(0, +\infty)$ is a bilinear control.
We give sufficient conditions in order for the above control system to be locally controllable to the ground state solution, that is, the solution of the free equation ($p \equiv 0$) starting from the ground state of A. Such a property, that is obviously stronger than superexponential stabilizability, holds true in more restrictive settings than those considered in Theorem 2.1.4. Nevertheless, the result we present in this chapter apply to all the examples of parabolic problems we have treated in chapter 2.
We also derive global controllability results in large time.

3.1 Main result

In a separable Hilbert space X, consider the control system

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, \quad t > 0 \\ \\ u(0) = u_0. \end{cases} \tag{3.1.1}$$

where $A : D(A) \subset X \to X$ is a linear self-adjoint maximal accretive operator on X, B belongs to $\mathscr{L}(X)$, the space of all bounded linear operators on X, and $p(t)$ is a scalar function representing a bilinear control.
In chapter 2, we have studied the stabilizability of (3.1.1) along the ground state solution of the free equation ($p \equiv 0$). More precisely, we have given sufficient conditions on A and B to ensure the superexponential stabilizability of (3.1.1) along ψ_1: for all u_0 in some neighborhood of φ_1 there exists a control $p \in L^2_{loc}([0, +\infty))$ such that the corresponding solution u of (3.1.1) satisfies

$$||u(t) - \psi_1(t)|| \leq Me^{-(e^{\omega t} + \lambda_1 t)}, \qquad \forall\, t \geq 0 \tag{3.1.2}$$

for some constants $\omega, M > 0$.
In this chapter, we address the related, more delicate, issue of the exact controllability of (3.1.1) to the ground state solution ψ_1 via a bilinear control.

Theorem 3.1.1. *Let $A : D(A) \subset X \to X$ be a densely defined linear operator such that* (1.0.1) *holds and suppose that there exists a constant $\alpha > 0$ for which the eigenvalues of A fulfill the gap condition*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \alpha, \quad \forall\, k \in \mathbb{N}^*. \tag{3.1.3}$$

Let $B : X \to X$ be a bounded linear operator such that there exist $b, q > 0$ for which

$$\langle B\varphi_1, \varphi_1 \rangle \neq 0, \quad \textit{and} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall\, k > 1. \tag{3.1.4}$$

Then, for any $T > 0$, there exists a constant $R_T > 0$ such that, for any $u_0 \in B_{R_T}(\varphi_1)$, there exists a control $p \in L^2(0,T)$ for which system (3.1.1) *is controllable to the ground state solution in time T. Furthermore, the following estimate holds*

$$||p||_{L^2(0,T)} \leq \frac{e^{-\pi^2 C_K / T_f}}{e^{2\pi^2 C_K/(3T_f)} - 1}, \tag{3.1.5}$$

where

$$T_f := \min\{T, T_\alpha\}, \qquad T_\alpha := \frac{\pi^2}{6} \min\left\{1, 1/\alpha^2\right\} \tag{3.1.6}$$

and C_K is a suitable positive constant.

The main idea of the proof consists of applying the stability estimates of [4] on a suitable sequence of time intervals of decreasing length T_j, such that $\sum_{j=1}^{\infty} T_j < \infty$. Such a sequence, however, has to be suitably chosen in order to fit the error estimates that we take from [4].
From the above local exact controllability property we deduce two global controllability results. In the first one, Theorem 3.1.2 below, we prove that all initial states lying in a suitable strip, i.e., satisfying $|\langle u_0, \varphi_1 \rangle - 1| < r_1$, can be steered to the ground state solution (see Figure 3.1). Moreover, we give a uniform estimate for the controllability time.

Theorem 3.1.2. *Let A and B satisfy hypotheses* (1.0.1), (3.1.3), *and* (3.1.4). *Then there exists a constant $r_1 > 0$ such that for any $R > 0$ there exists $T_R > 0$ such that for all $u_0 \in X$ that satisfy*

$$\begin{aligned} &|\langle u_0, \varphi_1 \rangle - 1| < r_1, \\ &||u_0 - \langle u_0, \varphi_1 \rangle \varphi_1|| \leq R, \end{aligned} \tag{3.1.7}$$

problem (3.1.1) *is exactly controllable to the ground state solution $\psi_1(t) = e^{-\lambda_1 t}\varphi_1$ in time T_R.*

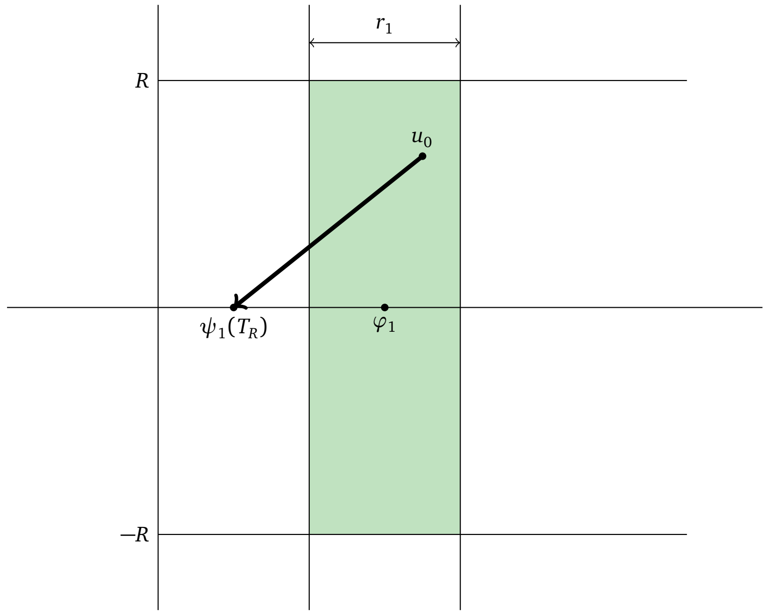

Figure 3.1: the colored region represents the set of initial conditions that can be steered to the ground state solution in time T_R.

Our second global result, Theorem 3.1.3 below, ensures the exact controllability of all initial states $u_0 \in X \setminus \varphi_1^{\perp}$ to the evolution of their orthogonal projection along the ground state solution defined by

$$\phi_1(t) = \langle u_0, \varphi_1 \rangle \psi_1(t), \quad \forall\, t \geq 0, \tag{3.1.8}$$

where ψ_1 is the ground state solution.

Theorem 3.1.3. *Let A and B satisfy hypotheses* (1.0.1)*,* (3.1.3) *and* (3.1.4)*. Then, for any $R > 0$ there exists $T_R > 0$ such that for all $u_0 \in X$ satisfying*

$$||u_0 - \langle u_0, \varphi_1 \rangle \varphi_1|| \leq R |\langle u_0, \varphi_1 \rangle| \tag{3.1.9}$$

system (3.1.1) *is exactly controllable to ϕ_1, defined in* (3.1.8)*, in time T_R.*

Notice that, denoting by θ the angle between the half-lines $\mathbb{R}_+\varphi_1$ and $\mathbb{R}_+u_0$, condition (3.1.9) is equivalent to

$$|\tan\theta| \leq R,$$

which defines a closed cone, say Q_R, with vertex at 0 and axis equal to $\mathbb{R}\varphi_1$ (see Figure 3.2). Therefore, Theorem 3.1.3 ensures a uniform controllability time for all initial conditions lying in Q_R. We observe that, since R is any arbitrary positive constant, all initial conditions $u_0 \in X \setminus \varphi_1^{\perp}$ can be steered to the corresponding projection to the ground state solution. Indeed, for any $u_0 \in X \setminus \varphi_1^{\perp}$, we define

$$R_0 := \left\| \frac{u_0}{\langle u_0, \varphi_1 \rangle} - \varphi_1 \right\|.$$

Then, for any $R \geq R_0$ condition (3.1.9) is fulfilled:

$$\frac{1}{|\langle u_0, \varphi_1 \rangle|} ||u_0 - \langle u_0, \varphi_1 \rangle \varphi_1|| = R_0 \leq R.$$

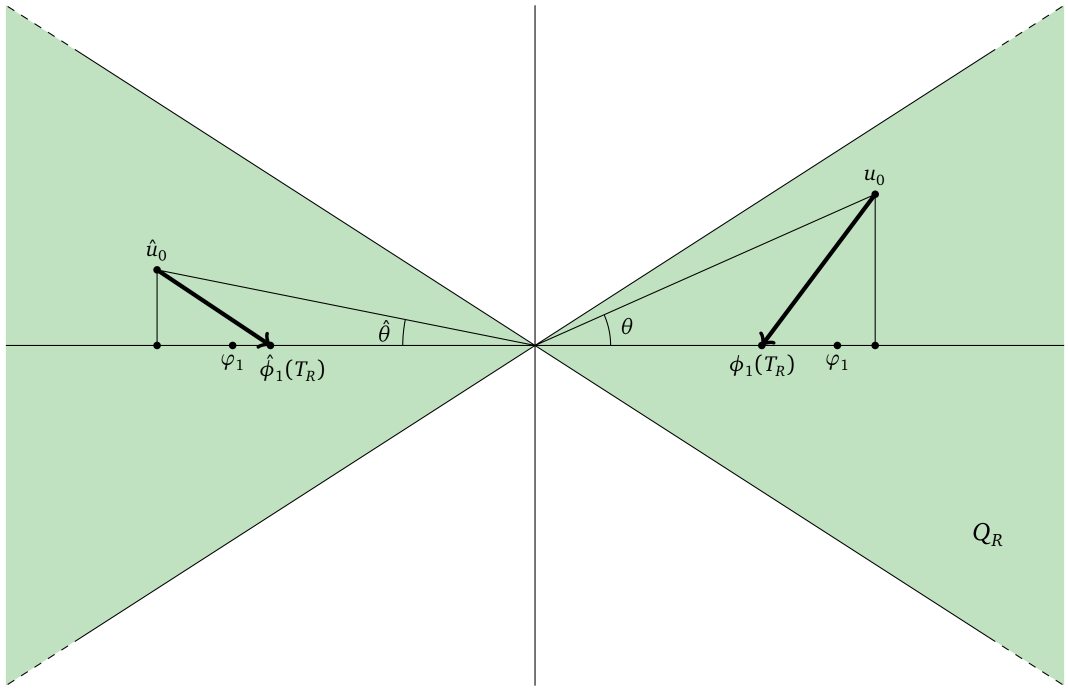

Figure 3.2: fixed any $R > 0$, the set of initial conditions exactly controllable in time T_R to their projection along the ground state solution is indicated by the colored cone Q_R.

3.2 Proof of Theorem 3.1.1

First, we recall some results from chapter 2 that are necessary for the construction of the proof of Theorem 3.1.1. Fixed $T > 0$, consider the following bilinear control problem

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) + f(t) = 0, \quad t \in [0, T] \\ \\ u(0) = u_0, \end{cases} \tag{3.2.1}$$

with $u_0 \in X$, $p \in L^2(0, T)$ and $f \in L^2(0, T; X)$. Proposition 2.1.5 ensures the existence of a unique mild solution $u \in C([0, T]; X)$ of (3.2.1) for which the following estimate holds

$$||u||_{\infty,0} \le C_1(T)(||u_0|| + ||f||_{2,0}). \tag{3.2.2}$$

Our aim is to show the controllability of the following system

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, \quad t \in [0, T] \\ \\ u(0) = u_0, \end{cases} \tag{3.2.3}$$

to the ground state solution $\psi_1 = e^{-\lambda_1 t}\varphi_1$, that is the solution of (3.2.3) when $p = 0$ and $u_0 = \varphi_1$. We first consider the case $\lambda_1 = 0$ and prove the controllability result to the corresponding ground state solution $\psi_1 = \varphi_1$. Then, we recover the result also for the case $\lambda_1 > 0$.
Set $v := u - \varphi_1$, then v is the solution of the following Cauchy problem

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \quad t \in [0, T] \\ \\ v(0) = v_0 = u_0 - \varphi_1. \end{cases} \tag{3.2.4}$$

We observe that the controllability of u to φ_1 is equivalent to the null controllabiliy of (3.2.4). In order to prove this latter result, we consider the following linearized system

$$\begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\varphi_1 = 0, & t \in [0,T] \\ \\ \bar{v}(0) = v_0. \end{cases} \tag{3.2.5}$$

and we recall the definition the constant Λ_T

$$\Lambda_T := \left(\sum_{k\in\mathbb{N}^*} \frac{e^{-2\lambda_k T} e^{\bar{C}\sqrt{\lambda_k}/\alpha}}{|\langle B\varphi_1, \varphi_k\rangle|^2} \right)^{1/2}. \tag{3.2.6}$$

where α is the constant in (3.1.3). We observe that, thanks to assumption (3.1.4), Λ_T converges for any $T > 0$.
Thanks to Proposition 2.1.7, we are able to build a control $p \in L^2(0,T)$

$$p(t) = \sum_{k\in\mathbb{N}^*} \frac{\langle v_0, \varphi_k\rangle}{\langle B\varphi_1, \varphi_k\rangle} \sigma_k(t) \tag{3.2.7}$$

that steers the solution of (3.2.5) to 0 in time T. Such a control p satisfies the following bound

$$||p||_{L^2(0,T)} \leq C_\alpha(T)\Lambda_T ||v_0|| \tag{3.2.8}$$

where Λ_T is defined in (3.2.6) and α is the constant in (3.1.3).

Remark 3.2.1. *The behavior of $C_\alpha(\cdot)$ with respect to its argument has been studied in [24] and is given by*

$$C_\alpha^2(T) = \bar{C} \cdot \begin{cases} \left(\frac{1}{T} + \frac{1}{T^2\alpha^2}\right) e^{\bar{C}/(T\alpha^2)}, & T \leq \frac{1}{\alpha^2} \\ \\ \bar{C}\alpha^2, & T \geq \frac{1}{\alpha^2}, \end{cases} \tag{3.2.9}$$

where $\bar{C} > 0$ is a constant independent of T and α.

By using the control p built in Proposition 2.1.7 also in the nonlinear system (3.2.4) and we have proved in chapter 2, Proposition 2.1.11 that the solution v of (3.2.4) satisfies

$$\sup_{t\in[0,T]} ||v(t)||^2 \leq e^{C_3(T)\Lambda_T ||v_0|| + C_B T}(1 + C_4(T)\Lambda_T^2)||v_0||^2 \tag{3.2.10}$$

where $C_B \geq 1$ is the norm of the operator B, $C_3(T) := 2\sqrt{T} C_B C_\alpha(T)$, and $C_4(T) := C_B C_\alpha^2(T)$.
We introduce the function $w(t) := v(t) - \bar{v}(t)$ that satisfies the following Cauchy problem

$$\begin{cases} w'(t) + Aw(t) + p(t)Bv(t) = 0, & t \in [0,T] \\ \\ w(0) = 0. \end{cases} \tag{3.2.11}$$

We define the function K on $(0,\infty)$ by

$$K^2(T) := C_B e^{2C_B\sqrt{T} + (C_B+1)T} C_4(T)\Lambda_T^2(1 + C_4(T)\Lambda_T^2). \tag{3.2.12}$$

In the following Proposition we estimate how close we are able to steer v to 0 in time T by means of the control p defined in (3.2.7).

Proposition 3.2.2. *Let A and B satisfy hypotheses* (1.0.1), (3.1.3), (3.1.4), *and, furthermore, we assume $\lambda_1 = 0$. Let $T > 0$, p be defined by* (3.2.7), *and let $v_0 \in X$ be such that*

$$C_\alpha(T)\Lambda_T ||v_0|| \le 1. \tag{3.2.13}$$

Then, it holds that

$$||w(T)|| = ||v(T)|| \le K(T)||v_0||^2. \tag{3.2.14}$$

Proof. Observe that $w \in C([0,T];X)$ is the mild solution of (3.2.11). Moreover $w \in H^1(0,T;X) \cap L^2(0,T;D(A))$ and thus w satisfies the equality

$$w'(t) + Aw(t) + p(t)Bv(t) = 0 \tag{3.2.15}$$

for almost every $t \in [0,T]$.
We multiply equation (3.2.15) by $w(t)$ and we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||w(t)||^2 &\le |p(t)|||Bv(t)||||w(t)|| \\ &\le \frac{1}{2}||w(t)||^2 + C_B^2 \frac{1}{2}|p(t)|^2||v(t)||^2.\end{aligned} \tag{3.2.16}$$

Therefore, applying Gronwall′s inequality, taking the supremum over $[0,T]$ and using (3.2.10) and (3.2.8), we get

$$\begin{aligned}\sup_{t\in[0,T]} ||w(t)||^2 &\le C_B^2 e^T ||p||^2_{L^2(0,T)} \sup_{t\in[0,T]} ||v(t)||^2 \\ &\le C_B^2 e^{C_3(T)\Lambda_T||v_0|| + C_B T + T}(1 + C_4(T)\Lambda_T^2)||v_0||^2||p||^2_{L^2(0,T)} \\ &\le C_B^2 C_\alpha^2(T)\Lambda_T^2 e^{C_3(T)\Lambda_T||v_0|| + (C_B+1)T}(1 + C_4(T)\Lambda_T^2)||v_0||^4.\end{aligned} \tag{3.2.17}$$

We can suppose, without loss of generality, that $C_\alpha(T) \ge 1$. Thus, thanks to (3.2.13), we obtain

$$\sup_{t\in[0,T]} ||w(t)||^2 \le C_B^2 C_\alpha^2(T)\Lambda_T^2 e^{2C_B\sqrt{T} + (C_B+1)T}(1 + C_4(T)\Lambda_T^2)||v_0||^4$$

that is equivalent to

$$\sup_{t\in[0,T]} ||w(t)||^2 \le K(T)^2||v_0||^4.$$

By the last inequality we infer that

$$||w(T)|| \le K(T)||v_0||^2. \tag{3.2.18}$$

□

Fixed $0 < T \le \min\left\{1, 1/\alpha^2\right\}$, we define the sequence $\{T_j\}_{j\in\mathbb{N}^*}$ by

$$T_j := T/j^2, \tag{3.2.19}$$

and the time steps

$$\tau_n = \sum_{j=1}^{n} T_j, \qquad \forall n \in \mathbb{N}, \tag{3.2.20}$$

with the convention that $\sum_{j=1}^{0} T_j = 0$. Notice that $\sum_{j=1}^{\infty} T_j = \frac{\pi^2}{6}T$.

The proof of our result relies on the construction of the solution v of (3.2.4) in consecutive intervals of the form $[\tau_n, \tau_{n+1}]$ for which we are able to perform an iterate estimate of (3.2.14).
First, through the following Lemma, we study the behavior of the constant $K(T)$ with respect to T.
We define the function

$$G_M(T) := \frac{M}{T^2} e^{M/T} \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k T + M\sqrt{\lambda_k}}}{|\langle B\varphi_1, \varphi_k\rangle|^2}, \qquad 0 < T \le 1 \tag{3.2.21}$$

where M is a positive constant.

Lemma 3.2.3. *Let $A : D(A) \subset X \to X$ be such that* (1.0.1) *and* (3.1.3) *hold and $B : X \to X$ be such that* (3.1.4) *holds. Then, there exists a suitable positive constant C_M such that*

$$G_M(T) \le e^{C_M/T}, \qquad \forall\ 0 < T \le 1. \tag{3.2.22}$$

Proof. Thanks to assumption (3.1.4), we have that

$$\begin{aligned} G_M(T) &= \frac{M}{T^2} e^{M/T} \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k T + M\sqrt{\lambda_k}}}{|\langle B\varphi_1, \varphi_k\rangle|^2} \\ &\le \frac{M}{T^2} e^{M/T} \left[\frac{e^{M^2/(8T)}}{|\langle B\varphi_1, \varphi_1\rangle|^2} + \frac{1}{b^2} \sum_{k=2}^{\infty} \left(\lambda_k^{2q} e^{-\lambda_k T}\right) e^{-\lambda_k T + M\sqrt{\lambda_k}} \right]. \end{aligned} \tag{3.2.23}$$

For any $\lambda \ge 0$ we set $f(\lambda) = e^{-\lambda T + M\sqrt{\lambda}}$. The maximum value of f is attained at $\lambda = \left(\frac{M}{2T}\right)^2$. So, we can bound $G_M(T)$ as follows

$$G_M(T) \le \frac{M}{T^2} e^{M/T} \left[\frac{e^{M^2/(8T)}}{|\langle B\varphi_1, \varphi_1\rangle|^2} + \frac{e^{M^2/(4T)}}{b^2} \sum_{k=1}^{\infty} \lambda_k^{2q} e^{-\lambda_k T} \right]. \tag{3.2.24}$$

Now, for any $\lambda \ge 0$ we define the function $g(\lambda) = \lambda^{2q} e^{-\lambda T}$. Its derivative is given by

$$g'(\lambda) = (2q - \lambda T)\lambda^{2q-1} e^{-\lambda T}$$

and therefore we deduce that

$$g(\lambda) \text{ is } \begin{cases} \text{increasing} & \text{if } 0 \le \lambda < (2q)/T \\ \\ \text{decreasing} & \text{if } \lambda \ge (2q)/T \end{cases}$$

and g has a maximum at $\lambda = (2q)/T$. We define the following index:

$$k_1 := k_1(T) = \sup\left\{ k \in \mathbb{N}^* \,:\, \lambda_k \le \frac{2q}{T} \right\}$$

Note that $k_1(T)$ goes to ∞ as T converges to 0. We can rewrite the sum in (3.2.24) as follows

$$\sum_{k=1}^{\infty} \lambda_k^{2q} e^{-\lambda_k T} = \sum_{k \le k_1 - 1} \lambda_k^{2q} e^{-\lambda_k T} + \sum_{k_1 \le k \le k_1 + 1} \lambda_k^{2q} e^{-\lambda_k T} + \sum_{k \ge k_1 + 2} \lambda_k^{2q} e^{-\lambda_k T}. \tag{3.2.25}$$

For any $k \le k_1 - 1$, we have

$$\int_{\lambda_k}^{\lambda_{k+1}} \lambda^{2q} e^{-\lambda T} d\lambda \ge (\lambda_{k+1} - \lambda_k)\lambda_k^{2q} e^{-\lambda_k T} \ge \alpha(\sqrt{\lambda_2} + \sqrt{\lambda_1})\lambda_k^{2q} e^{-\lambda_k T} \tag{3.2.26}$$

and for any $k \geq k_1 + 2$

$$\int_{\lambda_{k-1}}^{\lambda_k} \lambda^{2q} e^{-\lambda T} d\lambda \geq (\lambda_k - \lambda_{k-1})\lambda_k^{2q} e^{-\lambda_k T} \geq \alpha(\sqrt{\lambda_2} + \sqrt{\lambda_1})\lambda_k^{2q} e^{-\lambda_k T}. \tag{3.2.27}$$

So, by using estimates (3.2.26) and (3.2.27), (3.2.25) becomes

$$\sum_{k=1}^{\infty} \lambda_k^{2q} e^{-\lambda_k T} = \frac{2}{\alpha(\sqrt{\lambda_2} + \sqrt{\lambda_1})} \int_0^{\infty} \lambda^{2q} e^{-\lambda T} d\lambda + \sum_{k_1 \leq k \leq k_1+1} \lambda_k^{2q} e^{-\lambda_k T}. \tag{3.2.28}$$

Furthermore, recalling that g has a maximum for $\lambda = 2q/T$, it holds that

$$k = k_1, k_1 + 1 \quad \Rightarrow \quad \lambda_k^{2q} e^{-\lambda_k T} \leq (2q/T)^{2q} e^{-2q}. \tag{3.2.29}$$

Finally, the integral term of (3.2.28) can be rewritten as

$$\int_0^{\infty} \lambda^{2q} e^{-\lambda T} d\lambda = \frac{1}{T} \int_0^{\infty} \left(\frac{s}{T}\right)^{2q} e^{-s} ds = \frac{1}{T^{1+2q}} \int_0^{\infty} s^{2q} e^{-s} ds = \frac{\Gamma(2q+1)}{T^{1+2q}}, \tag{3.2.30}$$

where by $\Gamma(\cdot)$ we indicate the Euler integral of the second kind.
Therefore, we conclude from (3.2.29) and (3.2.30) that there exist two constants $C_q, C_{q,\alpha} > 0$ such that

$$\sum_{k=1}^{\infty} \lambda_k^{2q} e^{-\lambda_k T} \leq \frac{C_q}{T^{2q}} + \frac{C_{\alpha,q}}{T^{1+2q}}. \tag{3.2.31}$$

We use this last bound to prove that there exists $C_M > 0$ such that

$$G_M(T) \leq \frac{M}{T^2} e^{M/T} \left[\frac{e^{M^2/(8T)}}{|\langle B\varphi_1, \varphi_1 \rangle|^2} + \frac{e^{M^2/(4T)}}{b^2} \left(\frac{C_q}{T^{2q}} + \frac{C_{\alpha,q}}{T^{1+2q}} \right) \right] \leq e^{C_M/T}, \qquad 0 < T \leq 1$$

as claimed. □

Remark 3.2.4. *We recall that $K(\cdot)$ is defined by*

$$K^2(T) := C_B^2 e^{2C_B\sqrt{T} + (C_B+1)T} C_\alpha^2(T) \Lambda_T^2 (1 + C_B C_\alpha^2(T) \Lambda_T^2).$$

For any $0 < T \leq \min\{1, 1/\alpha^2\}$, $C_\alpha^2(\cdot)$ is given by

$$C_\alpha^2(T) = \bar{C} \left(\frac{1}{T} + \frac{1}{T^2 \alpha^2} \right) e^{\bar{C}/(\alpha^2 T)}.$$

Thus, we have the following bound for $K(\cdot)$

$$K(T)^2 \leq C_B^2 e^{2C_B\sqrt{T} + (C_B+1)T} G_M(T) (1 + C_B G_M(T)), \tag{3.2.32}$$

where $G_M(\cdot)$ is defined by (3.2.21) and the subscribed M is given by $M = \bar{C}\left(1 + \frac{1}{\alpha^2}\right)$.
Thanks to Lemma 3.2.3, we infer that there exists a suitable constant $C_K > 0$ such that $C_K > C_M$ and

$$K(T) \leq e^{C_K/T}, \qquad \forall\, T \in (0, 1]. \tag{3.2.33}$$

In the following Proposition we prove that it is possible to iterate the construction of v in consecutive time intervals of the form $[\tau_{n-1}, \tau_n]$.

Proposition 3.2.5. *Let $0 < T \leq \min\{1, 1/\alpha^2\}$, and consider the sequence $(T_j)_{j\in\mathbb{N}^*}$ defined by* (3.2.19). *Let $v_0 \in X$ for which $\|v_0\| < e^{-6C_K/T}$, let $A : D(A) \subset X \to X$ be such that* (1.0.1) *and* (3.1.3) *hold and let $B : X \to X$ satisfies* (3.1.4). *Moreover, we assume that $\lambda_1 = 0$. Then, for every $n \in \mathbb{N}^*$, problem*

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \quad t \in [\tau_{n-1}, \tau_n] \\ \\ v(\tau_{n-1}) = v_{n-1}, \end{cases} \tag{3.2.34}$$

where v_{n-1} is determined by induction from the previous steps, $p \in L^2(\tau_{n-1}, \tau_n)$ is given by

$$p(t) = \sum_{k=1}^{\infty} \frac{\langle v_{n-1}, \varphi_k\rangle}{\langle B\varphi_1, \varphi_k\rangle} \sigma_k(t - \tau_{n-1}), \tag{3.2.35}$$

admits a unique mild solution $v \in C([\tau_{n-1}, \tau_n], X)$ that satisfies

$$\|v(\tau_n)\| \leq e^{\left(\sum_{j=1}^{n} 2^{n-j} j^2 - 2^n 6\right) C_K/T}, \tag{3.2.36}$$

where the time steps $\{\tau_n\}_{n\in\mathbb{N}}$ are defined by (3.2.20).

Proof. To prove the result, we proceed by induction on n. For $n = 1$, by Proposition 3.2.2, the hypothesis on v_0 and Remark 3.2.4, v it satisfies

$$\|v(T)\| \leq K(T)\|v_0\|^2 \leq e^{-11C_K/T}.$$

Now, suppose the statement is true for all indices $k \leq n-1$, we show the validity for index n. Therefore, by inductive hypothesis, the solution v has been constructed in consecutive intervals until $[\tau_{n-2}, \tau_{n-1}]$ and it satisfies

$$\|v(\tau_{n-1})\| \leq e^{\left(\sum_{j=1}^{n-1} 2^{n-1-j} j^2 - 2^{n-1} 6\right) C_K/T}.$$

Hence,

$$\begin{aligned} C_\alpha(T_n)\Lambda_{T_n} \|v(\tau_{n-1})\| &\leq e^{C_M n^2/T} e^{\left(\sum_{j=1}^{n-1} 2^{n-1-j} j^2 - 2^{n-1} 6\right) C_K/T} \\ &\leq e^{(n^2 + (-(n-1)^2 - 4(n-1) + 2^{n-1}6 - 6 - 2^{n-1}6) C_K/T} \\ &= e^{-(2n+3)C_K/T}, \end{aligned} \tag{3.2.37}$$

where we have used that $C_M < C_K$ and the identity

$$\sum_{j=0}^{n} \frac{j^2}{2^j} = 2^{-n}(-n^2 - 4n + 6(2^n - 1)), \qquad n \geq 0, \tag{3.2.38}$$

which can be easily checked by induction.
Consider problem (3.2.34) with v_{n-1} the solution built in the previous interval, evaluated at τ_{n-1}. By the change of variables $s = t - \tau_{n-1}$, we shift (3.2.34) into the interval $[0, T_n]$. We introduce the functions $\tilde{v}(s) = v(s + \tau_{n-1})$ and $\tilde{p}(s) = p(s + \tau_{n-1})$ and we rewrite (3.2.34) as

$$\begin{cases} \tilde{v}'(s) + A\tilde{v}(s) + \tilde{p}(s)B\tilde{v}(s) + \tilde{p}(s)B\varphi_1 = 0, \quad s \in [0, T_n] \\ \\ \tilde{v}(0) = v_{n-1}. \end{cases} \tag{3.2.39}$$

From (3.2.37) it follows that $C_\alpha(T_n)\Lambda_{T_n}\,||v(\tau_{n-1})|| < 1$ and thus we can apply Proposition 3.2.2 to problem (3.2.39), obtaining

$$||\tilde{v}(T_n)|| \leq K(T_n)||v_{n-1}||^2. \tag{3.2.40}$$

We shift back the problem into the original interval $[\tau_{n-1}, \tau_n]$ and we get

$$||v(\tau_n)|| \leq K(T_n)||v_{n-1}||^2. \tag{3.2.41}$$

By inductive hypothesis, we can estimate $||v(\tau_n)||$ as follows

$$||v(\tau_n)|| \leq e^{C_K n^2/T}\left[e^{\left(\sum_{j=1}^{n-1}2^{n-1-j}j^2-2^{n-1}6\right)C_K/T}\right]^2 = e^{\left(\sum_{j=1}^{n}2^{n-j}j^2-2^n 6\right)C_K/T}. \tag{3.2.42}$$

□

Proposition 3.2.6. *Let* $0 < T \leq \min\left\{1, 1/\alpha^2\right\}$ *and consider the sequence* $(T_j)_{j\in\mathbb{N}^*}$ *defined by* (3.2.19). *Let* $v_0 \in X$ *be such that* $||v_0|| < e^{-6C_K/T}$, *let* $A : D(A) \subset X \to X$ *be such that* (1.0.1) *and* (3.1.3) *hold and let* $B : X \to X$ *satisfies* (3.1.4). *Let* $p \in L^2(\tau_{n-1}, \tau_n)$ *be defined by* (3.2.35). *Moreover, we assume that* $\lambda_1 = 0$. *Then, the solution of* (3.2.34) *satisfies*

$$||v(\tau_n)|| \leq \prod_{j=1}^{n} K(T_j)^{2^{n-j}}||v_0||^{2^n}, \tag{3.2.43}$$

for all $n \in \mathbb{N}^*$.

Proof. We prove formula (3.2.43) by induction on n. The case $n = 1$ follows from Proposition 3.2.2, thanks to the assumption $||v_0|| < e^{-6C_K/T}$. Now, suppose the formula holds for all the indices less than or equal to $n-1$. We prove it for n as follows. We consider problem (3.2.34) and in order to shift it in the interval $[0, T_n]$, we introduce the variable $s = t - \tau_{n-1}$ as before and the functions $\tilde{v}(s) = v(s + \tau_{n-1})$ and $\tilde{p}(s) = p(s + \tau_{n-1})$. Thus, (3.2.34) can be rewritten as

$$\begin{cases} \tilde{v}'(s) + A\tilde{v}(s) + \tilde{p}(s)B\tilde{v}(s) + \tilde{p}(s)B\varphi_1 = 0, \quad s \in [0, T_n] \\ \\ \tilde{v}(0) = v_{n-1}. \end{cases} \tag{3.2.44}$$

By Proposition 3.2.5, it holds that $C_\alpha(T_n)\Lambda_{T_n}||v_{n-1}|| \leq 1$ and hence, we can apply Proposition 3.2.2 considering as final time T_n (instead of T), obtaining that

$$||v(\tau_n)|| = ||\tilde{v}(T_n)|| \leq K(T_n)||v_{n-1}||^2. \tag{3.2.45}$$

Finally, by inductive hypothesis, we conclude that

$$||v(\tau_n)|| \leq K(T_n)||v_{n-1}||^2 \leq K(T_n)\left[\prod_{j=1}^{n-1}K(T_j)^{2^{n-1-j}}||v_0||^{2^{n-1}}\right]^2 \tag{3.2.46}$$

that is equivalent to formula (3.2.43). □

We are now ready to prove our main result.

Proof of Theorem 3.1.1. We start the proof by considering the case in which $\lambda_1 = 0$. Let $T > 0$ and let T_α and T_f be defined by (3.1.6). We define $\tilde{T} = \frac{6}{\pi^2} T_f$ and $R_T := e^{-\pi^2 C_K / T_f}$. Observe that $0 < \tilde{T} \leq 1$ and we define the time steps $\{\tau_n\}_{n\in\mathbb{N}}$ as in (3.2.20) with $T_j := \tilde{T}/j^2$. Fixed $v_0 \in B_{R_T}(0)$, we apply (3.2.43) to obtain

$$\begin{aligned} \|v(\tau_n)\| &\leq \prod_{j=1}^{n} K(T_j)^{2^{n-j}} \|v_0\|^{2^n} \\ &\leq \prod_{j=1}^{n} \left(e^{C_K j^2/\tilde{T}}\right)^{2^{n-j}} \|v_0\|^{2^n} \\ &= e^{C_K 2^n/\tilde{T} \sum_{j=1}^{n} j^2/2^j} \|v_0\|^{2^n} \\ &\leq e^{C_K 2^n/\tilde{T} \sum_{j=1}^{\infty} j^2/2^j} \|v_0\|^{2^n} \\ &\leq \left(e^{6C_K/\tilde{T}} \|v_0\|\right)^{2^n} \end{aligned} \tag{3.2.47}$$

where we have used that $\sum_{j=1}^{\infty} j^2/2^j = 6$. We take the limit as $n \to \infty$ of (3.2.47) and we get

$$\left\| u\left(\frac{\pi^2}{6}\tilde{T}\right) - \varphi_1 \right\| = \left\| v\left(\frac{\pi^2}{6}\tilde{T}\right) \right\| = \|v(T_f)\| \leq 0 \tag{3.2.48}$$

since $\|v_0\| < e^{-\pi^2 C_K/T_f} = e^{-6C_K/\tilde{T}}$. This means that, we have built a control $p \in L^2_{loc}([0,\infty))$, defined by

$$p(t) = \begin{cases} \sum_{n=0}^{\infty} p_n(t) \chi_{[\tau_n, \tau_{n+1}]}(t), & t \in \left(0, T_f\right], \\ 0, & t \in (T_f, +\infty) \end{cases} \tag{3.2.49}$$

where

$$p_n(t) = \sum_{k=1}^{\infty} \frac{\langle v(\tau_n), \varphi_k \rangle}{\langle B\varphi_1, \varphi_k \rangle} \sigma_k(t - \tau_n), \qquad \forall n \in \mathbb{N}, \tag{3.2.50}$$

such that the solution u of (3.1.1) reaches the ground state solution φ_1 in time T, and stays on it forever.

Observe that, thanks to (3.2.8) and (3.2.37), we are able to yield a bound for the L^2-norm of such a control:

$$\begin{aligned} \|p\|^2_{L^2(0,T)} &= \sum_{n=0}^{\infty} \|p_n\|^2_{L^2(\tau_n, \tau_{n+1})} \\ &\leq \sum_{n=0}^{\infty} \left(C_\alpha(T_{n+1}) \Lambda_{T_{n+1}} \|v(\tau_n)\|\right)^2 \\ &\leq \sum_{n=0}^{\infty} e^{-2(2(n+1)+3)C_K/\tilde{T}} \\ &= \frac{e^{-6C_K/\tilde{T}}}{e^{4C_K/\tilde{T}} - 1} \\ &= \frac{e^{-\pi^2 C_K/T_f}}{e^{2\pi^2 C_K/(3T_f)} - 1} \end{aligned} \tag{3.2.51}$$

Now we face the case $\lambda_1 > 0$. We define the operator

$$A_1 := A - \lambda_1 I.$$

It is possible to check that A_1 satisfies (1.0.1) and moreover it has the same eigenfuctions, $\{\varphi_k\}_{k\in\mathbb{N}^*}$, of A, while the eigenvalues are given by

$$\mu_k = \lambda_k - \lambda_1, \qquad \forall k \in \mathbb{N}^*. \tag{3.2.52}$$

In particular, $\mu_1 = 0$ and furthermore, $\{\mu_k\}_{k\in\mathbb{N}^*}$ satisfy the same gap condition (3.1.3) fulfilled by $\{\lambda_k\}_{k\in\mathbb{N}^*}$.
We define the function $z(t) = e^{\lambda_1 t}u(t)$, where u is the solution of (3.1.1). Then, z solves the following problem

$$\begin{cases} z'(t) + A_1 z(t) + p(t)Bz(t) = 0, & t > 0, \\ \\ z(0) = u_0. \end{cases} \tag{3.2.53}$$

For any $T > 0$, we define T_f as in (3.1.6) and the constant $R_T := e^{-\pi^2 C_K/T_f}$. We deduce from the previous analysis that, if $u_0 \in B_{R_T}(\varphi_1)$, then there exists a control $p \in L^2([0,+\infty))$ that steers the solution z to the ground state solution φ_1 in time $T_f \le T$. This implies the exact controllability of u to the ground state solution $\psi_1(t) = e^{-\lambda_1 t}\varphi_1$: indeed,

$$\left|\left|u\left(T_f\right) - \psi_1\left(T_f\right)\right|\right| = \left|\left|e^{-\lambda_1 T_f} z\left(T_f\right) - e^{-\lambda_1 T_f}\varphi_1\right|\right| = e^{-\lambda_1 T_f}\left|\left|z\left(T_f\right) - \varphi_1\right|\right| = 0.$$

This concludes the proof of our Theorem. □

Remark 3.2.7. *We observe that, from* (3.2.51)*, it follows that* $||p||_{L^2(0,T_f)} \to 0$ *as* $T_f \to 0$*. This fact is not surprising because as* T_f *approaches* 0*, also the size of the neighborhood where the initial condition can be chosen goes to zero.*

3.3 Proof of Theorems 3.1.2 and 3.1.3

Before proving Theorem 3.1.2, let us show a preliminary result that demonstrates the statement in the case of a strictly accretive operator.

Lemma 3.3.1. *Let A and B satisfy hypotheses* (1.0.1)*,* (3.1.3) *and* (3.1.4)*. Furthermore, we assume $\lambda_1 = 0$. Then, there exists a constant $r_1 > 0$ such that for any $R > 0$ there exists $T_R > 0$ such that for all $v_0 \in X$ that satisfy*

$$\begin{aligned} &|\langle v_0, \varphi_1\rangle| < r_1, \\ &||v_0 - \langle v_0, \varphi_1\rangle\varphi_1|| \le R, \end{aligned} \tag{3.3.1}$$

problem (3.2.4) *is null controllable in time T_R.*

Proof. **First step.** We fix $T = 1$. Thanks to Theorem 3.1.1, there exists a constant $r_1 > 0$ such that if $||u_1(0) - \varphi_1|| < \sqrt{2}r_1$ then there exists a control $p_1 \in L^2(0,1)$ for which the solution u_1 of (3.1.1) on $[0,1]$ with p replaced by p_1, satisfies $u_1(1) = \varphi_1$. We set $v_1 = u_1 - \varphi_1$ on $[0,1]$. We deduce that if $||v_1(0)|| < \sqrt{2}r_1$ then there exists a control $p_1 \in L^2(0,1)$ for which the solution v_1 of (3.2.4) on $[0,1]$ with p replaced by p_1, satisfies $v_1(1) = 0$.

Second step. Let $v_0 \in X$ be the initial condition of (3.2.4). We decompose v_0 as follows

$$v_0 = \langle v_0, \varphi_1\rangle\varphi_1 + v_{0,1},$$

where $v_{0,1} \in \varphi_1^\perp$ and we suppose that $|\langle v_0, \varphi_1\rangle| < r_1$. We define t_R as

$$t_R := \frac{1}{2\lambda_2} \log\left(\frac{R^2}{r_1^2}\right) \tag{3.3.2}$$

and in the time interval $[0, t_R]$ we take the control $p \equiv 0$. Then, for all $t \in [0, t_R]$, we have that

$$\|v(t)\|^2 \leq \left\|e^{-tA}\left(\langle v_0, \varphi_1\rangle\varphi_1 + v_{0,1}\right)\right\|^2 \leq |\langle v_0, \varphi_1\rangle|^2 + e^{-2\lambda_2 t}\left\|v_{0,1}\right\|^2 < r_1^2 + e^{-2\lambda_2 t}R^2.$$

In particular, for $t = t_R$, it holds that $\|v(t_R)\|^2 < 2r_1^2$.
Now, we define $T_R := t_R + 1$ and set $v_1(0) = v(t_R)$. Thanks to the first step of the proof, there exists a control $p_1 \in L^2(0,1)$, such that $v_1(1) = 0$, where v_1 is the solution of (3.2.4) on $[0,1]$ with p replaced by p_1.
Then $v(t) = v_1(t - t_R)$ solves (3.2.4) in the time interval $(t_R, T_R]$ with the control $p_1(t - t_R)$ that steers the solution v to 0 at T_R. □

Proof (of Theorem 3.1.2). We start with the case $\lambda_1 = 0$. Let $u_0 \in X$ that satisfies (3.1.7). Set $v(t) := u(t) - \varphi_1$, then v satisfies (3.2.4) and moreover $v_0 := v(0) = u_0 - \varphi_0$ fulfills (3.3.1). Thus, by Lemma 3.3.1, problem (3.1.1) is exactly controllable to the ground state solution $\psi_1 \equiv \varphi_1$ in time T_R.
Now, we consider the case $\lambda_1 > 0$. As in the proof of Theorem 3.1.1, we introduce the variable $z(t) = e^{\lambda_1 t}u(t)$ that solves problem (3.2.53). For such a system, since the first eigenvalue of A_1 is equal 0, we have the exact controllability to φ_1 in time T_R. Namely $z(T_R) = \varphi_1$, that is equivalent to the exact controllability of u to ψ_1:

$$z(T_R) = \varphi_1 \quad \Longleftrightarrow \quad e^{\lambda_1 T_R}u(T_R) = \varphi_1 \quad \Longleftrightarrow \quad u(T_R) = \psi_1(T_R). \tag{3.3.3}$$

The proof is thus complete. □

The proof of Theorem 3.1.3 easily follows from Theorem (3.1.2).

Proof (of Theorem 3.1.3). Suppose that $\gamma := \langle u_0, \varphi_1\rangle \neq 0$. We decompose u_0 as $u_0 = \gamma\varphi_1 + \zeta_1$, with $\zeta_1 := u_0 - \langle u_0, \varphi_1\rangle\varphi_1 \in \varphi_1^\perp$ and define $\tilde{u}(t) := u(t)/\gamma$. Hence, $\tilde{u}$ solves

$$\begin{cases} \tilde{u}'(t) + A\tilde{u}(t) + p(t)B\tilde{u}(t) = 0, \quad t > 0 \\ \tilde{u}(0) = \varphi_1 + \tilde{\zeta_1}, \end{cases} \tag{3.3.4}$$

where $\tilde{\zeta_1} := \zeta_1/\gamma$.
We apply Theorem 3.1.2 to (3.3.4) to deduce the existence of $T_R > 0$ such that $\tilde{u}(T_R) = \psi_1(T_R)$. Therefore, the solution of (3.1.1) with initial condition $u_0 \in X$ that do not vanish along the direction φ_1 can be exactly controlled in time T_R to the trajectory $\langle u_0, \varphi_1\rangle\psi_1(\cdot)$.
Note that if $u_0 \in X$ satisfies both $u_0 \in \varphi_1^\perp$ and (3.1.9), then we have trivially that $u_0 \equiv 0$. We then choose $p \equiv 0$, so that the solution of (3.1.1) remains constantly equal to $\phi_1 \equiv 0$. □

3.4 Applications

In this section we present some examples of parabolic equations for which Theorem 3.1.1 can be applied. The hypotheses (1.0.1), (3.1.3) and (3.1.4) have been verified in Chapter 2, section 2.2, to which we refer for more details. Furthermore, we observe that also the global results Theorem 3.1.2 and Theorem 3.1.3 can be applied to any example.

3.4.1 Diffusion equation with Dirichlet boundary conditions

Let $I=(0,1)$ and $X=L^2(0,1)$. Consider the following problem

$$\begin{cases} u_t(t,x)-u_{xx}(t,x)+p(t)\mu(x)u(t,x)=0 & x\in I, t>0\\ u(t,0)=0,\ u(t,1)=0, & t>0\\ u(0,x)=u_0(x) & x\in I. \end{cases} \tag{3.4.1}$$

We denote by A the operator defined by

$$D(A)=H^2\cap H_0^1(I),\quad A\varphi=-\frac{d^2\varphi}{dx^2}.$$

and it can be checked that A satisfies (1.0.1). We indicate by $\{\lambda_k\}_{k\in\mathbb{N}^*}$ and $\{\varphi_k\}_{k\in\mathbb{N}^*}$ the families of eigenvalues and eigenfunctions of A, respectively:

$$\lambda_k=(k\pi)^2,\quad \varphi_k(x)=\sqrt{2}\sin(k\pi x),\quad \forall k\in\mathbb{N}^*.$$

It is easy to see that (3.1.3) holds true:

$$\sqrt{\lambda_{k+1}}-\sqrt{\lambda_k}=\pi,\qquad \forall k\in\mathbb{N}^*.$$

Let $B:X\to X$ be the operator

$$B\varphi=\mu\varphi$$

with $\mu\in H^3(I)$ such that

$$\mu'(1)\pm\mu'(0)\neq 0\quad \text{and}\quad \langle\mu\varphi_1,\varphi_k\rangle\neq 0\quad \forall k\in\mathbb{N}^*. \tag{3.4.2}$$

Then, there exists $b>0$ such that

$$\lambda_k^{3/2}|\langle\mu\varphi_1,\varphi_k\rangle|\geq b,\qquad \forall k\in\mathbb{N}^*.$$

For instance, a suitable function that satisfies (3.4.2) is $\mu(x)=x^2$, for which $b=\frac{2\pi^2-3}{6\pi^2}$.
For any $T>0$, we define T_f as in (3.1.6). Then, there exists a constant $R_{T_f}>0$ such that the solution u of (3.4.1), with $u_0\in B_{R_{T_f}}(\varphi_1)$, reaches the ground state solution $\psi_1(t,x)=\sqrt{2}\sin(\pi x)e^{-\pi^2 t}$ in time T_f and stays on it forever.

3.4.2 Diffusion equation with Neumann boundary conditions

Let $I=(0,1)$, $X=L^2(I)$ and consider the Cauchy problem

$$\begin{cases} u_t(t,x)-u_{xx}(t,x)+p(t)\mu(x)u(t,x)=0 & x\in I, t>0\\ u_x(t,0)=0,\ u_x(t,1)=0, & t>0\\ u(0,x)=u_0(x). & x\in I. \end{cases} \tag{3.4.3}$$

The operator A, defined by

$$D(A) = \{\varphi \in H^2(0,1) : \varphi'(0) = 0,\ \varphi'(1) = 0\}, \quad A\varphi = -\frac{d^2\varphi}{dx^2}$$

satisfies (1.0.1) and has the following eigenvalues and eigenfunctions

$$\begin{aligned} &\lambda_0 = 0, && \varphi_0 = 1 \\ &\lambda_k = (k\pi)^2, && \varphi_k(x) = \sqrt{2}\cos(k\pi x), \quad \forall k \geq 1. \end{aligned}$$

Thus, the gap condition (3.1.3) is fulfilled with $\alpha = \pi$. The ground state solution is just the stationary function $\psi_1(x) = \varphi_1(x) = 1$.
We define $B : X \to X$ as the multiplication operator by a function $\mu \in H^2(I)$, $B\varphi = \mu\varphi$, such that

$$\mu'(1) \pm \mu'(0) \neq 0 \quad \text{and} \quad \langle \mu, \varphi_k \rangle \neq 0 \quad \forall k \in \mathbb{N}. \tag{3.4.4}$$

It can be proved that, there exists $b > 0$ such that

$$\lambda_k |\langle \mu\varphi_0, \varphi_k \rangle| \geq b, \qquad \forall k \in \mathbb{N}^*. \tag{3.4.5}$$

For example, $\mu(x) = x^2$ satisfies (3.4.5) with $b = 2\sqrt{2}$.
Therefore, equation (3.4.3) is controllable to the ground state solution $\psi_1 = 1$ in any time $T > 0$ as long as $u_0 \in B_{R_T}(1)$, with $R_T > 0$ a suitable constant.

3.4.3 Variable coefficient parabolic equation

Let $I = (0,1)$, $X = L^2(I)$ and consider the problem

$$\begin{cases} u_t(t,x) - ((1+x)^2 u_x(t,x))_x + p(t)\mu(x)u(t,x) = 0 & x \in I, t > 0 \\ u(t,0) = 0, \quad u(t,1) = 0, & t > 0 \\ u(0,x) = u_0(x) & x \in I. \end{cases} \tag{3.4.6}$$

We denote by $A : D(A) \subset X \to X$ the following operator

$$D(A) = H^2 \cap H^1_0(I), \qquad A\varphi = -((1+x)^2\varphi_x)_x.$$

It can be checked that A satisfies (1.0.1) and that the eigenvalues and eigenfunctions have the following expression

$$\lambda_k = \frac{1}{4} + \left(\frac{k\pi}{\ln 2}\right)^2, \qquad \varphi_k = \sqrt{\frac{2}{\ln 2}}(1+x)^{-1/2} \sin\left(\frac{k\pi}{\ln 2}\ln(1+x)\right).$$

Furthermore, $\{\lambda_k\}_{k \in \mathbb{N}^*}$ verifies the gap condition (3.1.3) with $\alpha = \pi/\ln 2$.
We define the operator $B : X \to X$ by $B\varphi = \mu\varphi$, where $\mu \in H^2(I)$ is such that

$$2\mu'(1) \pm \mu'(0) \neq 0, \quad \text{and} \quad \langle \mu\varphi_1, \varphi_k \rangle \neq 0 \quad \forall k \in \mathbb{N}^*. \tag{3.4.7}$$

Hence, thanks to (3.4.7), (3.1.4) is fulfilled with $q = 3/2$. An example of a suitable function μ that satisfies (3.4.7) is $\mu(x) = x$, see Chapter 2, section 2.2.2 for the verification.
Thus, from Theorem 3.1.1, we deduce that, for any $T > 0$, system (3.4.6) is controllable to the ground state solution if the initial condition u_0 is close enough to φ_1.

3.4.4 Diffusion equation in a $3D$ ball with radial data

In this example, we study the controllability of an evolution equation in the three dimensional unit ball B^3 for radial data. The bilinear control problem is the following

$$\begin{cases} u_t(t,r) - \Delta u(t,r) + p(t)\mu(r)u(t,r) = 0 & r \in [0,1], t > 0 \\ u(t,1) = 0, & t > 0 \\ u(0,r) = u_0(r) & r \in [0,1] \end{cases} \tag{3.4.8}$$

where the Laplacian in polar coordinates for radial data is given by the following expression

$$\Delta\varphi(r) = \partial_r^2\varphi(r) + \frac{2}{r}\partial_r\varphi(r).$$

The function μ is a radial function as well in the space $H_r^3(B^3)$, where the spaces $H_r^k(B^3)$ are defined as follows

$$X := L_r^2(B^3) = \left\{\varphi \in L^2(B^3) \,|\, \exists\psi : \mathbb{R} \to \mathbb{R}, \varphi(x) = \psi(|x|)\right\}$$

$$H_r^k(B^3) := H^k(B^3) \cap L_r^2(B^3).$$

The domain of the Dirichlet Laplacian $A := -\Delta$ in X is $D(A) = H_r^2 \cap H_0^1(B^3)$. We observe that A satisfies hypothesis (1.0.1). We denote by $\{\lambda_k\}_{k\in\mathbb{N}^*}$ and $\{\varphi_k\}_{k\in\mathbb{N}^*}$ the families of eigenvalues and eigenfunctions of A, $A\varphi_k = \lambda_k\varphi_k$, namely

$$\varphi_k = \frac{\sin(k\pi r)}{\sqrt{2\pi}r}, \qquad \lambda_k = (k\pi)^2 \tag{3.4.9}$$

$\forall k \in \mathbb{N}^*$, see [48, Section 8.14]. Since the eigenvalues of A are actually the same of the Dirichlet $1D$ Laplacian, (3.1.3) is satisfied, as we have seen in Example 3.4.1.
Let $B : X \to X$ be the multiplication operator $Bu(t,r) = \mu(r)u(t,r)$, with μ be such that

$$\mu'(1) \pm \mu'(0) \neq 0, \quad \text{and} \quad \langle\mu\varphi_1, \varphi_k\rangle \neq 0 \quad \forall k \in \mathbb{N}^*. \tag{3.4.10}$$

Then, it can be proved that

$$\lambda_k^{3/2}|\langle\mu\varphi_1, \varphi_k\rangle| \geq b, \qquad \forall k \in \mathbb{N}^*, \tag{3.4.11}$$

with b a positive constant. For instance, $\mu(x) = x^2$ verifies (3.4.10) and (3.4.11) with $b = \frac{2\pi^2-3}{6\pi^2}$.
Therefore, by applying Theorem 3.1.1, we conclude that for any $T > 0$, the exists a suitable constant $R_T > 0$ such that, if $u_0 \in B_{R_T}(\varphi_1)$, problem (3.4.8) is exactly controllable to the ground state ψ_1 in time T.

3.4.5 Degenerate parabolic equation

In this last section we want to address an example of a control problem for a degenerate evolution equation of the form

$$\begin{cases} u_t - (x^\gamma u_x)_x + p(t)x^{2-\gamma}u = 0, & (t,x) \in (0,+\infty)\times(0,1) \\ u(t,1) = 0, \quad \begin{cases} u(t,0) = 0, & \text{if } \gamma \in [0,1), \\ (x^\gamma u_x)(t,0) = 0, & \text{if } \gamma \in [1,3/2), \end{cases} \\ u(0,x) = u_0(x). \end{cases} \tag{3.4.12}$$

where $\gamma \in [0,3/2)$ describes the degeneracy magnitude, for which Theorem 3.1.1 applies.
If $\gamma \in [0,1)$ problem (3.4.12) is called weakly degenerate and the natural spaces for the well-posedness are the following weighted Sobolev spaces. Let $I = (0,1)$ and $X = L^2(I)$, we define

$$H^1_\gamma(I) = \left\{u \in X : u \text{ is absolutely continuous on } [0,1], x^{\gamma/2}u_x \in X\right\}$$

$$H^1_{\gamma,0}(I) = \left\{u \in H^1_\gamma(I) : \ u(0) = 0, \ u(1) = 0\right\}$$

$$H^2_\gamma(I) = \left\{u \in H^1_\gamma(I) : x^\gamma u_x \in H^1(I)\right\}.$$

We denote by $A : D(A) \subset X \to X$ the linear degenerate second order operator

$$\begin{cases} \forall u \in D(A), \quad Au := -(x^\gamma u_x)_x, \\ D(A) := \{u \in H^1_{\gamma,0}(I), \ x^\gamma u_x \in H^1(I)\}. \end{cases} \tag{3.4.13}$$

It is possible to prove that A satisfies (1.0.1) (see, for instance [16]) and furthermore, if we denote by $\{\lambda_k\}_{k\in\mathbb{N}^*}$ the eigenvalues and by $\{\varphi_k\}_{k\in\mathbb{N}^*}$ the corresponding eigenfunctions, it turns out that the gap condition (3.1.3) is fulfilled with $\alpha = \frac{7}{16}\pi$ (see [47], page 135).
If $\gamma \in [1,3/2)$, problem (3.4.12) is called strong degenerate and the corresponding weighted Sobolev space are described as follows: given $I = (0,1)$ and $X = L^2(I)$, we define

$$H^1_\gamma(I) = \left\{u \in X : u \text{ is absolutely continuous on } (0,1], \ x^{\gamma/2}u_x \in X\right\}$$

$$H^1_{\gamma,0}(I) := \left\{u \in H^1_\gamma(I) : \ u(1) = 0\right\},$$

$$H^2_\gamma(I) = \left\{u \in H^1_\gamma(I) : \ x^\gamma u_x \in H^1(I)\right\}.$$

In this case the operator $A : D(A) \subset X \to X$ is defined by

$$\begin{cases} \forall u \in D(A), \quad Au := -(x^\gamma u_x)_x, \\ D(A) := \left\{u \in H^1_{\gamma,0}(I) : \ x^\gamma u_x \in H^1(I)\right\} \\ \qquad = \left\{u \in X : \ u \text{ is absolutely continuous in } (0,1]\,, \ x^\gamma u \in H^1_0(I), \right. \\ \qquad\qquad \left. x^\gamma u_x \in H^1(I) \text{ and } (x^\gamma u_x)(0) = 0\right\} \end{cases}$$

and it has been proved that (1.0.1) holds true (see, for instance [21]) and that (3.1.3) is satisfied for $\alpha = \frac{\pi}{2}$ (see [47]).
For all $\gamma \in [0, 3/2)$, we define the linear operator $B : X \to X$ by $Bu(t,x) = x^{2-\gamma}u(t,x)$ and in Chapter 2, section 2.2.5 we have proved that there exists a constant $b > 0$ such that

$$\lambda_k^{3/2} |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k \in \mathbb{N}^*.$$

Finally, by applying Theorem 3.1.1, we ensure the exact controllability of problem (3.4.12) to the ground state solution, for both weakly and strongly degenerate problems.

CHAPTER 4

Exact controllability of degenerate wave equation

In this chapter we consider the linear degenerate wave equation

$$w_{tt} - (x^\alpha w_x)_x = p(t)\mu(x)w, \quad x \in (0,1)$$

controlled by means of a bilinear control p and subject to Neumann boundary conditions. We study the controllability of such an equation locally around the ground state solution. We prove that, generically with respect to μ, any target state close to the ground state solution in the $H^3 \times H^2$ topology (suitably adapted to the underlying degenerate operator) is reachable in time $T > \frac{4}{2-\alpha}$, with controls in $L^2((0,T),\mathbb{R})$.
The content of the chapter is based on [20] in which we extend to the degenerate case the work of Beauchard [7] concerning the bilinear control of the classical wave equation ($\alpha = 0$), and adapt to bilinear controls the work of Alabau-Boussouira, Cannarsa and Leugering [2] on the degenerate wave equation where additive control are considered.
It is worth noting that one the main difficulties when dealing with degenerate operators is the study of the associated spectral problem. Since it requires a long and technical analysis, we discuss this topic in Section A.2 of the Appendix A.

4.1 Main result

This chapter is devoted to the study of the controllability property of the following degenerate control system

$$\begin{cases} w_{tt} - (x^\alpha w_x)_x = p(t)\mu(x)w, & x \in (0,1), t \in (0,T), \\ (x^\alpha w_x)(0,t) = 0, & t \in (0,T), \\ w_x(1,t) = 0, & t \in (0,T), \\ w(x,0) = w_0(x), & x \in (0,1), \\ w_t(x,0) = w_1(x), & x \in (0,1), \end{cases} \tag{4.1.1}$$

where T is a positive constant, $\alpha \in [0,2)$ is the degereracy parameter ($\alpha = 0$ for the classical wave equation and $\alpha \in (0,2)$ in the degenerate case), $p \in L^2(0,T)$ is the bilinear control and μ is an admissible potential.
We recall that when $\alpha \in [0,1)$ the problem is said to be weakly degenerate, while when $\alpha \in [1,2)$ we have a strongly degenerate problem (see chapter A). For $\alpha \in [0,1)$ we consider the following Hilbert spaces

$$H^1_\alpha(0,1) := \left\{u \in L^2(0,1),\, u \text{ absolutely continuous in } [0,1],\, x^{\alpha/2}u_x \in L^2(0,1)\right\}, \tag{4.1.2}$$

and

$$H^2_\alpha(0,1) := \left\{u \in H^1_\alpha(0,1), x^\alpha u_x \in H^1(0,1)\right\}. \tag{4.1.3}$$

We define the operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ by

$$\begin{cases} \forall\, u \in D(A), \quad Au := -(x^\alpha u_x)_x, \\ D(A) := \{u \in H^2_\alpha(0,1), (x^\alpha u_x)(0) = 0, u_x(1) = 0\}. \end{cases} \tag{4.1.4}$$

For the strongly degenerate problem, we introduce the Hilbert spaces

$$H^1_\alpha(0,1) := \left\{u \in L^2(0,1),\, u \text{ locally absolutely continuous in } (0,1],\, x^{\alpha/2} u_x \in L^2(0,1)\right\}, \tag{4.1.5}$$

$$H^2_\alpha(0,1) := \{u \in H^1_\alpha(0,1) \ \mid x^\alpha u_x \in H^1(0,1)\}. \tag{4.1.6}$$

and the operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$

$$\begin{cases} \forall\, u \in D(A), \quad Au := -(x^\alpha u_x)_x, \\ D(A) := \{u \in H^2_\alpha(0,1), (x^\alpha u_x)(0) = 0, u_x(1) = 0\}. \end{cases} \tag{4.1.7}$$

For any $\alpha \in [0,2)$ we have proved in Propositions A.2.1 and A.2.5 that the operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ defined by (4.1.4) and (4.1.7) (for the weakly and strongly degenerate setting, respectively) is self-adjoint, accretive and with dense domain. Thus, $-A$ is the infinitesimal generator of an analytic semigroup of contraction e^{-tA} on $L^2(0,1)$
Furthermore, from Propositions A.2.3 and A.2.7 it follows that the eigenvalues and eigenfunctions of A are given by:

- for $\alpha \in [0,1)$:
$$\lambda_{\alpha,0} = 0, \quad \varphi_{\alpha,0}(x) = 1 \tag{4.1.8}$$
and for all $m \geq 1$
$$\lambda_{\alpha,m} = \kappa_\alpha^2\, j^2_{-\nu_\alpha - 1,m}, \tag{4.1.9}$$
$$\varphi_{\alpha,m}(x) = K_{\alpha,m} x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\left(j_{-\nu_\alpha-1,m} x^{\frac{2-\alpha}{2}}\right), \tag{4.1.10}$$
where
$$\kappa_\alpha := \frac{2-\alpha}{2}, \quad \nu_\alpha := \frac{1-\alpha}{2-\alpha},$$
$J_{-\nu_\alpha}$ is the Bessel's function of order $-\nu_\alpha$, $(j_{-\nu_\alpha-1,m})_{m\geq 1}$ are the positive zeros of the Bessel's function $J_{-\nu_\alpha-1}$ and $K_{\alpha,m}$ are positive constants,

- for $\alpha \in [1,2)$:
$$\lambda_{\alpha,0} = 0, \quad \varphi_{\alpha,0}(x) = 1 \tag{4.1.11}$$
and for all $m \geq 1$
$$\lambda_{\alpha,m} = \kappa_\alpha^2\, j^2_{\nu_\alpha+1,m}, \tag{4.1.12}$$
$$\varphi_{\alpha,m}(x) = K_{\alpha,m} x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\left(j_{\nu_\alpha+1,m}\, x^{\frac{2-\alpha}{2}}\right), \tag{4.1.13}$$
where
$$\kappa_\alpha := \frac{2-\alpha}{2}, \quad \nu_\alpha := \frac{\alpha-1}{2-\alpha},$$
J_{ν_α} is the Bessel's function of order ν_α, $(j_{\nu_\alpha+1,m})_{m\geq 1}$ are the positive zeros of the Bessel's function $J_{\nu_\alpha+1}$ and $K_{\alpha,m}$ are positive constants.

To avoid possible problems generated by the eigenvalue 0, we denote

$$\lambda^*_{\alpha,n} := \max(1, \lambda_{\alpha,n}) = \begin{cases} 1 & \text{for } n = 0, \\ \lambda_{\alpha,n} & \text{for } n \geq 1. \end{cases} \tag{4.1.14}$$

We recall that since the operator A satisfies hypothesis (1.0.1), for any $s \geq 0$ the fractional powers $A^s : D(A^s) \subset X \to X$ are characterized by (1.0.3) and the fractional Sobolev spaces $H^s_{(0)}(0,1) := D(A^{s/2})$ are defined by

$$H^s_{(0)}(0,1) := \left\{ \psi \in L^2(0,1), \sum_{k=0}^{\infty} (\lambda^*_{\alpha,k})^s \langle \psi, \varphi_{\alpha,k} \rangle^2_{L^2(0,1)} < \infty \right\}, \tag{4.1.15}$$

equipped with the norm

$$\|\psi\|_{H^s_{(0)}(0,1)} := \left(\sum_{k=0}^{\infty} (\lambda^*_{\alpha,k})^s \langle \psi, \varphi_{\alpha,k} \rangle^2_{L^2(0,1)} \right)^{1/2}.$$

We also introduce the following spaces, that are related to the potential μ:

$$V^{(2,\infty)}_\alpha(0,1) := \{\mu \in H^2_\alpha(0,1),\ x^{\alpha/2}\mu_x \in L^\infty(0,1)\}, \tag{4.1.16}$$

$$V^{(2,\infty,\infty)}_\alpha(0,1) := \{\mu \in H^2_\alpha(0,1),\ x^{\alpha/2}\mu_x \in L^\infty(0,1),\ (x^\alpha \mu_x)_x \in L^\infty(0,1)\}, \tag{4.1.17}$$

$$V^2_\alpha(0,1) := \begin{cases} V^{(2,\infty)}_\alpha(0,1) & \text{if } \alpha \in [0,1), \\ V^{(2,\infty,\infty)}_\alpha(0,1) & \text{if } \alpha \in [1,2), \end{cases} \tag{4.1.18}$$

and also the following closed subspace of $H^2_\alpha(0,1)$

$$V^{(2,0)}_\alpha(0,1) := \{w \in H^2_\alpha(0,1),\ (x^\alpha w_x)(0) = 0\}. \tag{4.1.19}$$

Given $(w_0, w_1) \in H^1_\alpha(0,1) \times L^2(0,1)$ and $p \in L^2(0,T)$, we will denote $w^{(w_0,w_1;p)}$ the solution of (4.1.1). When $(w_0, w_1) = (1,0)$, that is when the initial condition of the problem is the ground state, and $p = 0$, we observe that

$$w^{(1,0;0)} \equiv 1$$

solves (4.1.1). We are interested in studying the controllability of problem (4.1.1), with initial condition $(w_0, w_1) = (1,0)$, along the ground state solution, that is the stationary trajectory $w^{(1,0;0)}$.
Thus, we consider the following control problem

$$\begin{cases} w_{tt} - (x^\alpha w_x)_x = p(t)\mu(x)w, & x \in (0,1), t \in (0,T), \\ (x^\alpha w_x)(0,t) = 0, & t \in (0,T), \\ w_x(1,t) = 0, & t \in (0,T), \\ w(x,0) = 1, & x \in (0,1), \\ w_t(x,0) = 0, & x \in (0,1). \end{cases} \tag{4.1.20}$$

The solution of (4.1.20) will be denoted by $w^{(1,0;p)}$ or, more simply, by $w^{(p)}$.
The main result of this chapter is contained in the following theorem.

Theorem 4.1.1. *For any $\alpha \in [0,2)$, let*

$$T > T_0 := \frac{4}{2-\alpha}, \tag{4.1.21}$$

and let $\mu \in V^2_\alpha(0,1)$ be such that

$$\exists\, c > 0 : \quad |\langle \mu, \varphi_{\alpha,n}\rangle_{L^2(0,1)}| \geq \frac{c}{\lambda^*_{\alpha,n}}, \quad \forall\, n \geq 0. \tag{4.1.22}$$

Then, there exists a neighborhood $\mathscr{V}(1,0)$ of $(1,0)$ in $H^3_{(0)}(0,1) \times D(A)$ and a C^1-map

$$\Gamma_{\alpha,T} : \mathscr{V}(1,0) \to L^2(0,T)$$

such that, for all $(w^f_0, w^f_1) \in \mathscr{V}(1,0)$, the solution of (4.1.20) *with $p = p^f := \Gamma_{\alpha,T}(w^f_0, w^f_1)$ satisfies*

$$(w^{(p^f)}(T), w^{(p^f)}_t(T)) = (w^f_0, w^f_1).$$

Remark 4.1.2. *Let us immediately note that there the set of functions μ satisfying* (4.1.22) *is not empty. For instance*

$$\mu(x) = x^{2-\alpha},$$

verifies (4.1.22).
Furthermore, the set of functions in $V^2_\alpha(0,1)$ that fulfill (4.1.22) *is dense in $V^2_\alpha(0,1)$, see section 4.2.4.*

4.2 Proof of Theorem 4.1.1

The proof of Theorem 4.1.1 is built through a series of preliminary results. The first one is the well–posedness of our control system.

4.2.1 Well–posedness

Let $T > 0$ and consider the nonhomogeneous problem

$$\begin{cases} w_{tt} - (x^\alpha w_x)_x = p(t)\mu(x)w + f(x,t), & x \in (0,1), t \in (0,T), \\ (x^\alpha w_x)(0,t) = 0, & t \in (0,T), \\ w_x(1,t) = 0, & t \in (0,T), \\ w(x,0) = w_0(x), & x \in (0,1), \\ w_t(x,0) = w_1(x), & x \in (0,1). \end{cases} \tag{4.2.1}$$

We recast it into the a first order problem: introducing

$$\mathscr{W} := \begin{pmatrix} w \\ w_t \end{pmatrix}, \quad \mathscr{W}_0 := \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}, \quad \mathscr{F}(x,t) := \begin{pmatrix} 0 \\ f(x,t) \end{pmatrix},$$

the state space

$$\mathscr{X} := H^1_\alpha(0,1) \times L^2(0,1),$$

and the operators

$$\mathscr{A} := \begin{pmatrix} 0 & \mathrm{Id} \\ -A & 0 \end{pmatrix}, \quad D(\mathscr{A}) := D(A) \times H^1_\alpha(0,1), \tag{4.2.2}$$

and

$$\mathscr{B} := \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}, \quad D(\mathscr{B}) := H^1_\alpha(0,1) \times L^2(0,1), \tag{4.2.3}$$

then, problem (4.2.1) can be rewritten as

$$\begin{cases} \mathscr{W}'(x,t) = \mathscr{A}\mathscr{W}(x,t) + p(t)\mathscr{B}\mathscr{W}(x,t) + \mathscr{F}(x,t), \\ \mathscr{W}(x,0) = \mathscr{W}_0(x). \end{cases} \tag{4.2.4}$$

We also introduce the space

$$V^{(1,\infty)}_\alpha(0,1) := \{\mu \in H^1_\alpha(0,1),\ x^{\alpha/2}\mu_x \in L^\infty(0,1)\}. \tag{4.2.5}$$

Proposition 4.2.1. *Let $T > 0$, $p \in L^2(0,T)$ and $f \in L^2((0,T), H^1_\alpha(0,1))$. Assume that*

$$\mu \in V^1_\alpha(0,1) := \begin{cases} H^1_\alpha(0,1) & \text{if } \alpha \in [0,1), \\ V^{(1,\infty)}_\alpha & \text{if } \alpha \in [1,2). \end{cases} \tag{4.2.6}$$

Then, for all $(w_0, w_1) \in D(A) \times H^1_\alpha(0,1)$, problem (4.1.1) *has a unique classical solution of* (4.2.1)*, i.e. a function*

$$(w, w_t) \in C^0([0,T], D(A) \times H^1_\alpha(0,1)),$$

such that the following equality holds in $D(A) \times H^1_\alpha(0,1)$: for every $t \in [0,T]$,

$$\mathscr{W}(t) = e^{t\mathscr{A}}\mathscr{W}_0 + \int_0^t e^{(t-s)\mathscr{A}}(\mathscr{B}\mathscr{W}(s) + \mathscr{F}(s))\,ds. \tag{4.2.7}$$

Moreover, there exists $C = C(\alpha, T, p) > 0$ such that $\mathscr{W}$ satisfies

$$\|\mathscr{W}\|_{C^0([0,T],D(\mathscr{A}))} \le C \left(\|\mathscr{W}_0\|_{D(\mathscr{A})} + \|\mathscr{F}\|_{L^2(0,T;D(\mathscr{A}))}\right). \tag{4.2.8}$$

To prove the above proposition we need first to prove the following lemma.

Lemma 4.2.2. *Let $\mu \in V^1_\alpha(0,1)$. Then, the operator $\mathscr{B}$ defined in* (4.2.3) *satisfies*

$$\mathscr{B} \in \mathscr{L}_c(D(\mathscr{A}), D(\mathscr{A})).$$

Proof. We have to prove that

$$z \in D(A) \implies \mu z \in H^1_\alpha(0,1)$$

and that there exists $C > 0$ such that

$$\|\mu z\|_{H^1_\alpha(0,1)} \le C\|z\|_{D(A)} \quad \forall\, z \in D(A). \tag{4.2.9}$$

We distinguish the cases $\alpha \in [0,1)$ and $\alpha \in [1,2)$.
$\boldsymbol{\alpha \in [0,1)}$

For any $z \in D(A)$, by definition we have that

$$z \in H^2_\alpha(0,1) \Rightarrow z \in H^1_\alpha(0,1) \Rightarrow x^{\alpha/2} z_x \in L^2(0,1).$$

Moreover, we can express z_x as $z_x = (x^{\alpha/2} z_x)(x^{-\alpha/2})$ and this implies that $z_x \in L^1(0,1)$ and thus $z \in L^\infty(0,1)$. The same holds for μ because $V^1_\alpha(0,1) = H^1_\alpha(0,1)$ when $\alpha \in [0,1)$. Hence, $(\mu z)_x = \mu_x z + \mu z_x \in L^1(0,1)$ and therefore μz is absolutely continuous in $[0,1]$. Furthermore, we have that $x^{\alpha/2}(\mu z)_x = (x^{\alpha/2}\mu_x)z + \mu(x^{\alpha/2} z_x) \in L^2(0,1)$ and we deduce that $z\mu \in H^1_\alpha(0,1)$. Finally, there exists $C > 0$ such that

$$\forall w \in H^1_\alpha(0,1), \quad \|w\|_{L^\infty(0,1)} \leq C\|w\|_{H^1_\alpha(0,1)},$$

and this implies that (4.2.9) holds.

$\boldsymbol{\alpha \in [1,2)}$.

First we note that $\mu \in V^{1,\infty}_\alpha(0,1)$ implies that $|\mu_x| \leq \frac{C}{x^{\alpha/2}}$. Therefore we get that $\mu_x \in L^1(0,1)$. So, $\mu \in L^\infty(0,1)$ and $\mu z \in L^2(0,1)$. Moreover, $x^{\alpha/2}(\mu z)_x = (x^{\alpha/2}\mu_x)z + (x^{\alpha/2} z_x)\mu$, and since $x^{\alpha/2}\mu_x \in L^\infty(0,1)$ and $z \in L^2(0,1)$, we have $(x^{\alpha/2}\mu_x)z \in L^2(0,1)$. Furthermore, since $x^{\alpha/2} z_x \in L^2(0,1)$ and $\mu \in L^\infty(0,1)$, we have $(x^{\alpha/2} z_x)\mu \in L^2(0,1)$, hence $x^{\alpha/2}(\mu z)_x \in L^2(0,1)$. By reasoning as in the case $\alpha \in [0,1)$, we deduce that also (4.2.9) is verified. □

Proof of Proposition 4.2.1. We prove the existence and uniqueness of the solution of problem (4.2.4) by a fixed point argument. We consider the map

$$\mathscr{K} : C^0([0,T], D(\mathscr{A})) \to C^0([0,T], D(\mathscr{A}))$$

defined by

$$\forall\, t \in [0,T], \quad \mathscr{K}(\mathscr{W})(t) := e^{t\mathscr{A}}\mathscr{W}_0 + \int_0^t e^{(t-s)\mathscr{A}}(p(s)\mathscr{B}\mathscr{W}(s) + \mathscr{F}(s))\,ds. \tag{4.2.10}$$

We first prove that $\mathscr{K}$ is well-defined, which means that it maps $C^0([0,T], D(\mathscr{A}))$ into itself. We observe that, thanks to Lemma 4.2.2, for any $\mathscr{W} \in C^0([0,T], D(\mathscr{A}))$, $\mathscr{B}\mathscr{W} \in C^0([0,T], D(\mathscr{A}))$ and thus $p\mathscr{B}\mathscr{W} \in L^2((0,T), D(\mathscr{A}))$. Hence, it is possible to apply the classical result of existence of strict solutions (see Proposition 1.0.5) and deduce that $\mathscr{K}(\mathscr{W}) \in C^0([0,T], D(\mathscr{A}))$. Moreover, for any $\mathscr{W}_1, \mathscr{W}_2 \in C^0([0,T], D(\mathscr{A}))$, it holds that

$$\begin{aligned}
\|\mathscr{K}(\mathscr{W}_1)(t) - \mathscr{K}(\mathscr{W}_2)(t)\|_{D(\mathscr{A})} &= \left\| \int_0^t e^{(t-s)\mathscr{A}} p(s)\mathscr{B}(\mathscr{W}_1(s) - \mathscr{W}_2(s))\,ds \right\|_{D(\mathscr{A})} \\
&\leq \int_0^t |p(s)| \|e^{(t-s)\mathscr{A}}\mathscr{B}(\mathscr{W}_1(s) - \mathscr{W}_2(s))\|_{D(\mathscr{A})} ds \\
&\leq C_1 \int_0^t |p(s)| \|\mathscr{B}(\mathscr{W}_1(s) - \mathscr{W}_2(s))\|_{D(\mathscr{A})} ds \\
&\leq C_1 C_{\mathscr{B}} \|p\|_{L^1(0,T)} \|\mathscr{W}_1 - \mathscr{W}_2\|_{C^0([0,T],D(\mathscr{A}))}.
\end{aligned}$$

Suppose $C_1 C_{\mathscr{B}}\|p\|_{L^1(0,T)} < 1$. Then, $\mathscr{K}$ is a contraction and therefore it has a unique fixed point. Furthermore, we have that

$$\begin{aligned}
\|\mathscr{W}\|_{C^0([0,T],D(\mathscr{A}))} &\le \sup_{t\in[0,T]} \left\| e^{t\mathscr{A}}\mathscr{W}_0 + \int_0^t e^{(t-s)\mathscr{A}}(p(s)\mathscr{B}\mathscr{W}(s)+\mathscr{F}(s))ds \right\|_{D(\mathscr{A})} \\
&\le C_1\left(\|\mathscr{W}_0\|_{D(\mathscr{A})} + \int_0^T |p(s)|\|\mathscr{B}\mathscr{W}(s)\|_{D(\mathscr{A})} + \|\mathscr{F}(s)\|_{D(\mathscr{A})}ds \right) \\
&\le C_1\left(\|\mathscr{W}_0\|_{D(\mathscr{A})} + C_{\mathscr{B}}\|\mathscr{W}\|_{C^0([0,T],D(\mathscr{A}))}\|p\|_{L^1(0,T)} + \sqrt{T}\|\mathscr{F}\|_{L^2(0,T;D(\mathscr{A}))}\right).
\end{aligned}$$

Hence

$$\|\mathscr{W}\|_{C^0([0,T],D(\mathscr{A}))} \le \frac{C_1}{1-C_1C_{\mathscr{B}}\|p\|_{L^1(0,T)}}(\|\mathscr{W}_0\|_{D(\mathscr{A})} + \sqrt{T}\|\mathscr{F}\|_{L^2(0,T;D(\mathscr{A}))}). \tag{4.2.11}$$

Thus, we have obtained the conclusion under the extra hypothesis that p satisfies $C_1C_{\mathscr{B}}\|p\|_{L^1(0,T)} < 1$. In the general case, it is sufficient to represent $[0,T]$ as the union of a finite family of sufficiently small subintervals on each of which we can repeat the above argument. □

Equivalently, (4.2.8) can be proved by Gronwall's Lemma, obtaining:

$$\|\mathscr{W}\|_{C^0([0,T],D(\mathscr{A}))} \le C_1\left(\|\mathscr{W}_0\|_{D(\mathscr{A})} + \sqrt{T}\|\mathscr{F}\|_{L^2(0,T;D(\mathscr{A}))}\right) e^{C_1\|p\|_{L^1(0,T)}}. \tag{4.2.12}$$

4.2.2 Controllability of the linearized problem

In this section we prove that the solution of (4.1.20) is more regular than expected (extending [7, Theorem 3] to the degenerate case). So, we can introduce the endpoint map

$$\begin{aligned}
\Theta_T : L^2(0,T) &\to H^3_{(0)}(0,1)\times D(A) \\
p &\mapsto (w^{(p)}(T), w_t^{(p)}(T)).
\end{aligned}$$

Our aim is to apply the inverse mapping theorem to Θ_T. This would mean that, chosen any target state (w_0^f, w_1^f) in a suitable subspace of the image of Θ_T, we are able to provide a control $p \in L^2(0,T)$ such that the solution of our control problem (4.1.20) satisfies $(w^{(p)}(T), w_t^{(p)}(T)) = (w_0^f, w_1^f)$. Namely, (4.1.20) is exactly controllable.

Proposition 4.2.3. *Let $\mu \in V^2_\alpha(0,1)$ (the space defined in (4.1.18)). Then,*
a) for all $p \in L^2(0,T)$, the solution $w^{(p)}$ of (4.1.20) has the following additional regularity

$$(w^{(p)}(T), w_t^{(p)}(T)) \in H^3_{(0)}(0,1)\times D(A), \tag{4.2.13}$$

b) given $p \in L^2(0,T)$, Θ_T is differentiable at p, and $D\Theta_T(p) : L^2(0,T) \to H^3_{(0)}(0,1)\times D(A)$ is a continuous linear application, that satisfies

$$D\Theta_T(p)\cdot q = (W^{(p,q)}(T), W_t^{(p,q)}(T)),$$

where $W^{(p,q)}$ is the solution of

$$\begin{cases}
W_{tt}^{(p,q)} - (x^\alpha W_x^{(p,q)})_x = p(t)\mu(x)W^{(p,q)} + q(t)\mu(x)w^{(p)}, & x\in(0,1), t\in(0,T),\\
(x^\alpha W_x^{(p,q)})(x=0,t) = 0, & t\in(0,T),\\
W_x^{(p,q)}(x=1,t) = 0, & t\in(0,T),\\
W^{(p,q)}(x,0) = 0, & x\in(0,1),\\
W_t^{(p,q)}(x,0) = 0, & x\in(0,1),
\end{cases} \tag{4.2.14}$$

c) moreover, the map

$$\Theta_T : L^2(0,T) \to H^3_{(0)}(0,1) \times D(A), \quad \Theta_T(p) := (w^{(p)}(T), w_t^{(p)}(T)) \tag{4.2.15}$$

is of class C^1.

The proof of Proposition 4.2.3 is based on several steps, the first one consists of analyzing the eigenvalues and eigenfunctions of the operator $\mathscr{A}$. Thus, we first solve the problem

$$\mathscr{A}\Psi = \tilde{\omega}\Psi \quad \text{with} \quad \Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \in D(\mathscr{A}). \tag{4.2.16}$$

Lemma 4.2.4. *The solutions $\tilde{\omega}$ of problem* (4.2.16) *form a sequence* $(i\omega_{\alpha,n})_{n\in\mathbb{Z}}$ *with*

$$\begin{cases} \omega_{\alpha,n} = -\sqrt{\lambda_{\alpha,|n|}}, & n \leq -1, \\ \omega_{\alpha,0} = 0, & n = 0, \\ \omega_{\alpha,n} = \sqrt{\lambda_{\alpha,n}}, & n \geq 1, \end{cases} \tag{4.2.17}$$

associated with the eigenfunctions

$$\begin{cases} \Psi_{\alpha,n} = \begin{pmatrix} \varphi_{\alpha,|n|} \\ -i\sqrt{\lambda_{\alpha,|n|}}\varphi_{\alpha,|n|} \end{pmatrix}, & n \leq -1, \\[2ex] \Psi_{\alpha,0} = \begin{pmatrix} \varphi_{\alpha,0} = 1 \\ 0 \end{pmatrix}, & n = 0, \\[2ex] \Psi_{\alpha,n} = \begin{pmatrix} \varphi_{\alpha,n} \\ i\sqrt{\lambda_{\alpha,n}}\varphi_{\alpha,n} \end{pmatrix}, & n \geq 1, \end{cases} \tag{4.2.18}$$

where $\{\lambda_{\alpha,n}\}_{n\in\mathbb{N}}$ and $\{\varphi_{\alpha,n}\}_{n\in\mathbb{N}}$ are the eigenvalues and eigenfunctions of A, respectively.

Proof of Lemma 4.2.4. The spectral problem (4.2.16) can be explicitly written as

$$\begin{cases} \psi_2(x) = \tilde{\omega}\psi_1(x), \\ (x^\alpha \psi_1')' = \tilde{\omega}\psi_2(x), \end{cases}$$

hence, ψ_1 solves

$$(x^\alpha \psi_1')' = \tilde{\omega}^2 \psi_1(x), \quad \text{with } \psi_1 \in D(A).$$

Using Propositions A.2.3 and A.2.7 concerning the eigenvalues of A, we obtain that

$$\exists\, n \geq 0, \quad \text{such that} \quad \tilde{\omega}^2 = -\lambda_{\alpha,n} \quad \text{and} \quad \psi_1 = \varphi_{\alpha,n}.$$

Vice–versa, given $n \in \mathbb{Z}$, let $\tilde{\omega} = \pm i\sqrt{\lambda_{\alpha,|n|}}$, and $\psi_1 = \varphi_{\alpha,|n|}$, and $\psi_2 = \tilde{\omega}\psi_1$. Then, it can be checked that $\tilde{\omega}$ solves of the eigenvalue problem. Therefore, the eigenvalues of $\mathscr{A}$ form the sequence $(i\omega_{\alpha,n})_{n\in\mathbb{Z}}$ defined in (4.2.17), associated with the eigenfunctions (4.2.18). □

Since the family $\{\varphi_{\alpha,n}\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(0,1)$, we can decompose the solution $w^{(p)}$ of (4.1.20) under the form

$$w^{(p)}(x,t) = \sum_{n=0}^{\infty} w_n(t)\varphi_{\alpha,n}(x).$$

We decompose in the same way the nonlinear term

$$r(x,t) := p(t)\mu(x)w^{(p)}(x,t) = \sum_{n=0}^{\infty} r_n(t)\varphi_{\alpha,n}(x)$$

with

$$r_n(t) = \langle p(t)\mu(\cdot)w^{(p)}(\cdot,t), \varphi_{\alpha,n}\rangle_{L^2(0,1)}.$$

So, (4.1.20) implies that the sequence $(w_n(t))_{n\geq 0}$ satisfies

$$\begin{cases} w_0''(t) = r_0(t), \\ w_0(0) = 1, \\ w_0'(0) = 0, \end{cases} \quad \text{and} \quad \forall\, n \geq 1, \quad \begin{cases} w_n''(t) + \lambda_{\alpha,n} w_n(t) = r_n(t), \\ w_n(0) = 0, \\ w_n'(0) = 0. \end{cases}$$

We obtain that

$$w_0(t) = 1 + \int_0^t r_0(s)(t-s)ds \quad \text{and} \quad w_n(t) = \int_0^t r_n(s)\frac{\sin\sqrt{\lambda_{\alpha,n}}(t-s)}{\sqrt{\lambda_{\alpha,n}}}ds.$$

Hence, the solution of (4.1.20) can be written as

$$w^{(p)}(x,t) = \left(1 + \int_0^t r_0(s)(t-s)ds\right) + \sum_{n=1}^{\infty}\left(\int_0^t r_n(s)\frac{\sin\sqrt{\lambda_{\alpha,n}}(t-s)}{\sqrt{\lambda_{\alpha,n}}}ds\right)\varphi_{\alpha,n}(x), \quad (4.2.19)$$

and

$$w_t^{(p)}(x,t) = \left(\int_0^t r_0(s)ds\right) + \sum_{n=1}^{\infty}\left(\sqrt{\lambda_{\alpha,n}}\int_0^t r_n(s)\frac{\cos\sqrt{\lambda_{\alpha,n}}(t-s)}{\sqrt{\lambda_{\alpha,n}}}ds\right)\varphi_{\alpha,n}(x), \quad (4.2.20)$$

or, equivalently,

$$\begin{pmatrix} w^{(p)}(x,t) \\ w_t^{(p)}(x,t) \end{pmatrix} = \begin{pmatrix} w_0(t) \\ w_0'(t) \end{pmatrix} + \sum_{n=1}^{\infty}\begin{pmatrix} \left(\int_0^t r_n(s)\frac{\sin\sqrt{\lambda_{\alpha,n}}(t-s)}{\sqrt{\lambda_{\alpha,n}}}ds\right)\varphi_{\alpha,n}(x) \\ \left(\sqrt{\lambda_{\alpha,n}}\int_0^t r_n(s)\frac{\cos\sqrt{\lambda_{\alpha,n}}(t-s)}{\sqrt{\lambda_{\alpha,n}}}ds\right)\varphi_{\alpha,n}(x) \end{pmatrix}.$$

Now, by manipulating the above formula and we get

$$\begin{aligned}
&\begin{pmatrix} w^{(p)}(x,t) \\ w_t^{(p)}(x,t) \end{pmatrix} - \begin{pmatrix} w_0(t) \\ w_0'(t) \end{pmatrix} = \\
&= \sum_{n=1}^{\infty}\frac{1}{2i\sqrt{\lambda_{\alpha,n}}}\begin{pmatrix} \left(\int_0^t r_n(s)(e^{i\sqrt{\lambda_{\alpha,n}}(t-s)} - e^{-i\sqrt{\lambda_{\alpha,n}}(t-s)})ds\right)\varphi_{\alpha,n}(x) \\ \left(\int_0^t r_n(s)(e^{i\sqrt{\lambda_{\alpha,n}}(t-s)} + e^{-i\sqrt{\lambda_{\alpha,n}}(t-s)})ds\right)i\sqrt{\lambda_{\alpha,n}}\varphi_{\alpha,n}(x) \end{pmatrix} \\
&= \sum_{n=1}^{\infty}\frac{1}{2i\sqrt{\lambda_{\alpha,n}}}\left(\int_0^t r_n(s)e^{-i\sqrt{\lambda_{\alpha,n}}s}ds\right)\begin{pmatrix} \varphi_{\alpha,n}(x) \\ i\sqrt{\lambda_{\alpha,n}}\varphi_{\alpha,n}(x) \end{pmatrix}e^{i\sqrt{\lambda_{\alpha,n}}t} \\
&\quad - \frac{1}{2i\sqrt{\lambda_{\alpha,n}}}\left(\int_0^t r_n(s)e^{i\sqrt{\lambda_{\alpha,n}}s}ds\right)\begin{pmatrix} \varphi_{\alpha,n}(x) \\ -i\sqrt{\lambda_{\alpha,n}}\varphi_{\alpha,n}(x) \end{pmatrix}e^{-i\sqrt{\lambda_{\alpha,n}}t} \\
&= \sum_{n\in\mathbb{Z}, n\neq 0}\frac{1}{2i\omega_{\alpha,n}}\left(\int_0^t r_n(s)e^{-i\omega_{\alpha,n}s}ds\right)\begin{pmatrix} \varphi_{\alpha,|n|}(x) \\ i\omega_{\alpha,n}\varphi_{\alpha,|n|}(x) \end{pmatrix}e^{i\omega_{\alpha,n}t},
\end{aligned}$$

that can be expressed more compactly by

$$\begin{pmatrix} w^{(p)}(x,t) \\ w_t^{(p)}(x,t) \end{pmatrix} = \begin{pmatrix} 1+\int_0^t r_0(s)(t-s)\,ds \\ \int_0^t r_0(s)ds \end{pmatrix} + \sum_{n\in\mathbb{Z},n\neq 0} \frac{1}{2i\omega_{\alpha,n}} \left(\int_0^t r_n(s) e^{-i\omega_{\alpha,n}(t-s)} ds \right) \Psi_{\alpha,n}(x) e^{i\omega_{\alpha,n}t}. \tag{4.2.21}$$

To lighten the notation, we rewrite (4.2.21) as

$$\begin{pmatrix} w^{(p)}(x,T) \\ w_t^{(p)}(x,T) \end{pmatrix} = \Gamma_0^{(p)}(T) + \sum_{n\in\mathbb{Z}^*} \frac{1}{2i\omega_{\alpha,n}} \gamma_n^{(p)}(T)\Psi_{\alpha,n}(x)e^{i\omega_{\alpha,n}T}, \tag{4.2.22}$$

with

$$\Gamma_0^{(p)}(T) = \begin{pmatrix} 1+\int_0^T r_0(s)(T-s)ds \\ \int_0^T r_0(s)ds \end{pmatrix} = \begin{pmatrix} \gamma_{00}^{(p)}(T) \\ \gamma_{01}^{(p)}(T) \end{pmatrix}, \tag{4.2.23}$$

and

$$\forall\, n\in\mathbb{Z}^*, \quad \gamma_n^{(p)}(T) = \int_0^T r_n(s)e^{-i\omega_{\alpha,n}s}\,ds, \tag{4.2.24}$$

where we recall that $r_n(\cdot)$ is defined by

$$\forall\, n\in\mathbb{Z}, \quad r_n(s) = \langle p(s)\mu(\cdot)w^{(p)}(\cdot,s), \varphi_{\alpha,|n|}\rangle_{L^2(0,1)}.$$

From Proposition 4.2.1, we already know that $(w^{(p)}(T), w_t^{(p)}(T)) \in D(A)\times H_\alpha^1(0,1)$. To prove the hidden regularity result, it is useful to consider expression (4.2.22). We have that

$$w^{(p)}(T) - \gamma_{00}^{(p)}(T) = \sum_{n=1}^{\infty} \frac{1}{2i\omega_{\alpha,n}} \left(\gamma_n^{(p)}(T)\, e^{i\omega_{\alpha,n}T} - \gamma_{-n}^{(p)}(T)\, e^{-i\omega_{\alpha,n}T} \right) \varphi_{\alpha,n}(x),$$

hence, $w^{(p)}(T) \in H^3_{(0)}(0,1)$ if and only if

$$\sum_{n=1}^{\infty} \lambda_{\alpha,n}^3 \left| \frac{1}{2i\omega_{\alpha,n}} \left(\gamma_n^{(p)}(T)\, e^{i\omega_{\alpha,n}T} - \gamma_{-n}^{(p)}(T)\, e^{-i\omega_{\alpha,n}T} \right) \right|^2 < \infty.$$

Moreover,

$$w_t^{(p)}(T) - \gamma_{01}^{(p)}(T) = \sum_{n=1}^{\infty} \frac{1}{2} \left(\gamma_n^{(p)}(T)\, e^{i\omega_{\alpha,n}T} + \gamma_{-n}^{(p)}(T)\, e^{-i\omega_{\alpha,n}T} \right) \varphi_{\alpha,n}(x),$$

thus, $w_t^{(p)}(T) \in H^2_{(0)}(0,1)$ if and only if

$$\sum_{n=1}^{\infty} \lambda_{\alpha,n}^2 \left| \frac{1}{2} \left(\gamma_n^{(p)}(T)\, e^{i\omega_{\alpha,n}T} + \gamma_{-n}^{(p)}(T)\, e^{-i\omega_{\alpha,n}T} \right) \right|^2 < \infty.$$

Therefore,

$$\sum_{n\in\mathbb{Z}} \lambda_{\alpha,|n|}^2 \,|\gamma_n^{(p)}(T)|^2 < \infty \quad \Longrightarrow \quad (w^{(p)}(T), w_t^{(p)}(T)) \in H^3_{(0)}(0,1)\times D(A). \tag{4.2.25}$$

In what follows we prove that

$$\sum_{n\in\mathbb{Z}} \lambda_{\alpha,|n|}^2 \,|\gamma_n^{(p)}(T)|^2 < \infty. \tag{4.2.26}$$

Lemma 4.2.5. *Let $T > 0$, $p \in L^2(0,T)$, $g \in C^0([0,T], V_\alpha^{(2,0)}(0,1))$. Consider the sequence $(S_n^{(p,g)})_{n\geq 1}$ defined by*

$$\forall\, n \geq 1, \quad S_n^{(p,g)} = \int_0^T p(s)\, \langle g(\cdot,s), \varphi_{\alpha,n}\rangle_{L^2(0,1)} e^{i\sqrt{\lambda_{\alpha,n}}s} ds. \tag{4.2.27}$$

Then $(S_n^{(p,g)})_{n\geq 1}$ satisfies

$$\sum_{n=1}^{\infty} \lambda_{\alpha,n}^2\, |S_n^{(p,g)}|^2 < \infty, \tag{4.2.28}$$

and moreover, there exists a constant $C(\alpha,T) > 0$ independent of $p \in L^2(0,T)$ and of $g \in C^0([0,T], V_\alpha^{2,0}(0,1))$ such that

$$\left(\sum_{n=1}^{\infty} \lambda_{\alpha,n}^2\, |S_n^{(p,g)}|^2\right)^{1/2} \leq C(\alpha,T)\, \|p\|_{L^2(0,T)}\, \|g\|_{C^0([0,T],V_\alpha^{(2,0)}(0,1))}. \tag{4.2.29}$$

Proof. We proceed as in [7], but the properties of the space $V_\alpha^{(2,0)}(0,1)$ will help us to overcome some new difficulties. (Note that $V_\alpha^{(2,0)}(0,1) = H_\alpha^2(0,1)$ if $\alpha \in [1,2)$.)
First, we observe that

$$\begin{aligned} S_n^{(p,g)} &= \int_0^T p(s) \langle g(\cdot,s), \varphi_{\alpha,n}\rangle_{L^2(0,1)} e^{i\sqrt{\lambda_{\alpha,n}}s} ds \\ &= \int_0^T p(s) \langle g(\cdot,s), \frac{1}{\lambda_{\alpha,n}} (A\varphi_{\alpha,n})\rangle_{L^2(0,1)} e^{i\sqrt{\lambda_{\alpha,n}}s} ds \\ &= \frac{-1}{\lambda_{\alpha,n}} \int_0^T p(s) \langle g(\cdot,s), (x^\alpha \varphi_{\alpha,n}')'\rangle_{L^2(0,1)} e^{i\sqrt{\lambda_{\alpha,n}}s} ds. \end{aligned} \tag{4.2.30}$$

Next, integrating by parts, we have

$$\begin{aligned} \langle g, (x^\alpha \varphi_{\alpha,n}')'\rangle_{L^2(0,1)} &= \int_0^1 g(x)(x^\alpha \varphi_{\alpha,n}')'(x) dx \\ &= [g(x) x^\alpha \varphi_{\alpha,n}'(x)]_0^1 - \int_0^1 g'(x) x^\alpha \varphi_{\alpha,n}'(x) dx \\ &= [g(x) x^\alpha \varphi_{\alpha,n}'(x)]_0^1 - [x^\alpha g'(x) \varphi_{\alpha,n}(x)]_0^1 + \int_0^1 (x^\alpha g')'(x) \varphi_{\alpha,n}(x) dx. \end{aligned}$$

Using the above expression of the scalar product in (4.2.30), we get

$$-\lambda_{\alpha,n} S_n^{(p,g)} = S_n^{(1)} - S_n^{(2)} + S_n^{(3)}, \tag{4.2.31}$$

with

$$\forall\, i \in \{1,2,3\}, \quad S_n^{(i)} = \int_0^T h_n^{(i)}(s)\, e^{i\sqrt{\lambda_{\alpha,n}}s} ds, \tag{4.2.32}$$

and the associated functions

$$h_n^{(1)}(s) = p(s)[g(x,s) x^\alpha \varphi_{\alpha,n}'(x)]_{x=0}^{x=1}, \tag{4.2.33}$$

$$h_n^{(2)}(s) = p(s)[x^\alpha g_x(x,s) \varphi_{\alpha,n}(x)]_{x=0}^{x=1}, \tag{4.2.34}$$

$$h_n^{(3)}(s) = p(s) \langle (x^\alpha g_x)_x, \varphi_{\alpha,n}\rangle_{L^2(0,1)}. \tag{4.2.35}$$

To conclude the proof we appeal to the following results.

Lemma 4.2.6. *Let $T > 0$, $p \in L^2(0,T)$ and $g \in C^0([0,T], V_\alpha^{(2,0)}(0,1))$. Then function $h_n^{(1)}$ defined in* (4.2.33) *satisfies*

$$\forall s \in [0,T], \quad h_n^{(1)}(s) = 0.$$

Proof of Lemma 4.2.6. The proof follows from regularity properties: since $g(\cdot,s) \in H_\alpha^2(0,1)$, then $g(\cdot,s) \in H^1(\frac{1}{2},1)$ and has a finite limit as $x \to 1$. Hence, thanks to the Neumann boundary condition at $x = 1$ for $\varphi_{\alpha,n}$, we have

$$g(x,s)x^\alpha \varphi'_{\alpha,n}(x) \to 0, \quad \text{as } x \to 1.$$

When $x \to 0$, we have to distinguish the cases of weak and strong degeneracy:
$\boldsymbol{\alpha \in [0,1)}$
First, we note that $g(\cdot,s)$ has a finite limit as $x \to 0$: indeed,

$$g_x(x,s) = (x^{\alpha/2} g_x(x,s)) x^{-\alpha/2},$$

and since $x \mapsto x^{\alpha/2} g_x(x,s)$ and $x \mapsto x^{-\alpha/2}$ belong to $L^2(0,1)$, then $g_x(\cdot,s) \in L^1(0,1)$, which implies that $g(\cdot,s)$ has a finite limit as $x \to 0$. Therefore,

$$g(x,s)x^\alpha \varphi'_{\alpha,n}(x) \to 0, \quad \text{as } x \to 0,$$

$\boldsymbol{\alpha \in [1,2)}$
Observe that g can be unbounded as $x \to 0$. However, the series of $\varphi_{\alpha,n}$ obtained thanks to (A.2.37) gives that

$$\exists\, C_{\alpha,n}, \quad |x^\alpha \varphi'_{\alpha,n}(x)| \leq C_{\alpha,n} x \quad \forall\, x \in (0,1).$$

We claim that $x \mapsto x g(x,s)$ has a finite limit as $x \to 0$. Indeed,

$$(x g(x,s))_x = g(x,s) + x g_x(x,s) = g(x,s) + x^{\alpha/2} g_x(x,s) x^{1-\alpha/2},$$

and since $g(\cdot,s) \in L^2(0,1)$, $x \mapsto x^{\alpha/2} g_x(x,s) \in L^2(0,1)$ and $x \mapsto x^{1-\alpha/2} \in L^\infty(0,1)$, we have that $(x g(x,s))_x \in L^1(0,1)$. Therefore $x \mapsto x g(x,s)$ has a finite limit as $x \to 0$:

$$\exists\, \ell^{(s)}, \quad x g(x,s) \to \ell^{(s)}, \quad \text{as } x \to 0.$$

However, since $g(\cdot,s) \in L^2(0,1)$ we get that $x \mapsto \frac{\ell^{(s)}}{x} \in L^2(0,1)$, which is possible only if $\ell^{(s)} = 0$. Therefore,

$$x g(x,s) \to 0, \quad \text{as } x \to 0,$$

and so

$$g(x,s)x^\alpha \varphi'_{\alpha,n}(x) \to 0, \quad \text{as } x \to 0.$$

□

Lemma 4.2.7. *Let $T > 0$, $p \in L^2(0,T)$ and $g \in C^0([0,T], V_\alpha^{(2,0)}(0,1))$. Then, $h_n^{(2)}$ defined in* (4.2.34) *belongs to $L^2(0,T)$ and there exists $C(\alpha,T) > 0$ independent of $p \in L^2(0,T)$ and of $g \in C^0([0,T], V_\alpha^{(2,0)}(0,1))$ and of $n \geq 1$ such that*

$$\|h_n^{(2)}\|_{L^2(0,T)} \leq C(\alpha,T) \|p\|_{L^2(0,T)} \|g\|_{C^0([0,T],V_\alpha^{2,0}(0,1))}. \tag{4.2.36}$$

Furthermore,

$$\sum_{n=1}^{\infty} |S_n^{(2)}|^2 < \infty, \tag{4.2.37}$$

and there exists $C_2(\alpha,T)>0$ independent of $p\in L^2(0,T)$ and of $g\in C^0([0,T],V_\alpha^{(2,0)}(0,1))$ such that

$$\sum_{n=1}^{\infty}|S_n^{(2)}|^2\le C_2(\alpha,T)^2\|p\|^2_{L^2(0,T)}\|g\|^2_{C^0([0,T],V_\alpha^{(2,0)}(0,1))}. \tag{4.2.38}$$

Proof of Lemma 4.2.7. We recall that

$$h_n^{(2)}(s)=p(s)\left(x^\alpha g_x(x,s)\varphi_{\alpha,n}(x)\right)\Big|_{x=1}-p(s)\left(x^\alpha g_x(x,s)\varphi_{\alpha,n}(x)\right)\Big|_{x=0}. \tag{4.2.39}$$

Using the definition of $V_\alpha^{(2,0)}(0,1)$ and respectively (A.2.23) when $\alpha\in[0,1)$ and (A.2.47) when $\alpha\in[1,2)$ in (4.2.39), we obtain that

$$\left(x^\alpha g_x(x,s)\varphi_{\alpha,n}(x)\right)\Big|_{x=0}=0.$$

Therefore from (A.2.22) and (A.2.46), we have

$$\left|h_n^{(2)}(s)\right|=\left|p(s)(x^\alpha g_x(x,s)\varphi_{\alpha,n}(x))\big|_{x=1}\right|=\sqrt{2-\alpha}\,\left|p(s)(x^\alpha g_x(x,s)\big|_{x=1}\right|.$$

Moreover, since $g(\cdot,s)\in H_\alpha^2(0,1)$, then $x\mapsto x^\alpha g_x(x,s)$ belongs to $H^1(0,1)$. By the continuous injection of $H^1(0,1)$ into $L^\infty(0,1)$ (and hence of $H_\alpha^2(0,1)$ into $L^\infty(0,1)$), there exists a positive constant C_∞ such that

$$\left|(x^\alpha g_x(x,s))\big|_{x=1}\right|\le C_\infty\|g(\cdot,s)\|_{H_\alpha^2(0,1)}\le C_\infty\|g\|_{C^0([0,T],V_\alpha^{(2,0)}(0,1))}.$$

Therefore, we get

$$\forall\, n\ge 1,\quad |h_n^{(2)}(s)|\le C_\infty\sqrt{2-\alpha}\,|p(s)|\|g\|_{C^0([0,T],V_\alpha^{(2,0)}(0,1))},$$

hence, $h_n^{(2)}\in L^2(0,T)$ and

$$\exists\, C'_\infty>0,\ \text{such that}\ \|h_n^{(2)}\|_{L^2(0,T)}\le C'_\infty\|p\|_{L^2(0,T)}\|g\|_{C^0([0,T],V_\alpha^{(2,0)}(0,1))},\ \forall\, n\ge 1. \tag{4.2.40}$$

This proves (4.2.36).

Now, we prove (4.2.37) and (4.2.38). These results follow from (4.2.36) and from classical results of Ingham type (we refer, in particular, to [11, Proposition 19, Theorem 6 and Corollary 4]).

We have seen in Proposition A.2.3, when $\alpha\in[0,1)$, and in Proposition A.2.7, when $\alpha\in[1,2)$, that

$$\sqrt{\lambda_{\alpha,n+1}}-\sqrt{\lambda_{\alpha,n}}\to\frac{2-\alpha}{2}\pi\quad \text{as } n\to\infty.$$

Furthermore, a stronger gap condition holds

$$\forall\,\alpha\in[0,2),\quad \sqrt{\lambda_{\alpha,n+1}}-\sqrt{\lambda_{\alpha,n}}\ge\frac{2-\alpha}{2}\pi.$$

Hence we are allowed to apply a general result of Ingham (see, e.g., [47, Theorem 4.3], generalized by Haraux [42], see also [11, Theorem 6]), and we derive that given

$$\forall\, T_1>T_0:=\frac{2\pi}{\frac{2-\alpha}{2}\pi}=\frac{4}{2-\alpha},$$

there exist $C_1(\alpha,T_1),C_2(\alpha,T_1)>0$ such that, for for every sequence $(c_n)_{n\ge1}$ with finite support and complex values, it holds that

$$C_1(\alpha,T_1)\sum_{n=1}^{\infty}|c_n|^2\le\int_0^{T_1}\left|\sum_{n=1}^{\infty}c_n e^{i\sqrt{\lambda_{\alpha,n}}t}\right|^2dt\le C_2(\alpha,T_1)\sum_{n=1}^{\infty}|c_n|^2. \tag{4.2.41}$$

Therefore, if $T_1 > T_0$, (4.2.41) implies that the sequence $(e^{i\sqrt{\lambda_{\alpha,n}}t})_{n\geq 1}$ is a Riesz basis of Vect $\{e^{i\sqrt{\lambda_{\alpha,n}}t}, n\geq 1\} \subset L^2(0,T_1)$ (see [11, Proposition 19, point (2)]).
So, for all $T>0$, there exists a positive constant $C_I(\alpha,T)$ such that

$$\forall f \in L^2(0,T), \quad \sum_{n=1}^{\infty} \left| \int_0^T f(t) e^{i\sqrt{\lambda_{\alpha,n}}t} dt \right|^2 \leq C_I(\alpha,T) \|f\|^2_{L^2(0,T)} \tag{4.2.42}$$

(by applying [11, Proposition 19, point (3)], if $T > T_0$, or extending f by 0 on (T,T_0) if $T\leq T_0$, see also [11, Corollary 4]).
And this is what we need to conclude the proof of Lemma 4.2.7. First, we note from (4.2.39) that

$$\begin{aligned}\left|\int_0^T h_n^{(2)}(t) e^{i\sqrt{\lambda_{\alpha,n}}t} dt\right| &= \left|\int_0^T p(t)(x^\alpha g_x(x,t)\varphi_{\alpha,n}(x))(x=1) e^{i\sqrt{\lambda_{\alpha,n}}t} dt\right| \\ &= \sqrt{2-\alpha}\left|\int_0^T p(t) g_x(1,t) e^{i\sqrt{\lambda_{\alpha,n}}t} dt\right|,\end{aligned}$$

and then we can apply (4.2.42) to the function $t\mapsto p(t)g_x(1,t)$ (which is independent of n), and we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty} |S_n^{(2)}|^2 &= \sum_{n=1}^{\infty} \left|\int_0^T h_n^{(2)}(s) e^{i\sqrt{\lambda_{\alpha,n}}s} ds\right|^2 \\ &= (2-\alpha)\sum_{n=1}^{\infty} \left|\int_0^T p(t) g_x(1,t) e^{i\sqrt{\lambda_{\alpha,n}}t} dt\right|^2 \\ &\leq (2-\alpha) C_I(\alpha,T) \|p(\cdot) g_x(1,\cdot)\|^2_{L^2(0,T)} \\ &\leq (2-\alpha) C_I(\alpha,T) C_\infty^2 \|p\|^2_{L^2(0,T)} \|g\|^2_{C^0([0,T],V_\alpha^{(2,0)}(0,1))}.\end{aligned}$$

This concludes the proof of Lemma 4.2.7. □

Lemma 4.2.8. *Let $T>0$, $p\in L^2(0,T)$ and $g\in C^0([0,T],V_\alpha^{(2,0)}(0,1))$. Then, the sequence $(S_n^{(3)})_{n\geq 1}$ satisfies*

$$\sum_{n=1}^{\infty} |S_n^{(3)}|^2 < \infty, \tag{4.2.43}$$

and there exists $C_3(\alpha,T)>0$ independent of $p\in L^2(0,T)$ and of $g\in C^0([0,T],V_\alpha^{(2,0)}(0,1))$ such that

$$\sum_{n=1}^{\infty} |S_n^{(3)}|^2 \leq C_3(\alpha,T)^2 \|p\|^2_{L^2(0,T)} \|g\|^2_{C^0([0,T],V_\alpha^{(2,0)}(0,1))}. \tag{4.2.44}$$

Proof of Lemma 4.2.8. First, let us prove that $\sum_n |S_n^{(3)}|^2 < \infty$. Notice that

$$|S_n^{(3)}|^2 \leq \left(\int_0^T |h_n^{(3)}(s)| ds\right)^2 \leq \|p\|^2_{L^2(0,T)} \left(\int_0^T |\langle (x^\alpha g_x)_x, \varphi_{\alpha,n}\rangle_{L^2(0,1)}|^2 ds\right),$$

hence

$$\sum_{n=1}^{\infty}|S_n^{(3)}|^2 \le \|p\|^2_{L^2(0,T)}\left(\int_0^T \sum_{n=1}^{\infty}|\langle (x^\alpha g_x)_x, \varphi_{\alpha,n}\rangle_{L^2(0,1)}|^2 ds\right)$$
$$\le \|p\|^2_{L^2(0,T)}\left(\int_0^T \|(x^\alpha g_x)_x\|^2_{L^2(0,1)} ds\right)$$
$$\le \|p\|^2_{L^2(0,T)} T \|g\|^2_{C^0([0,T],V_\alpha^{(2,0)}(0,1))}.$$

This concludes the proof of Lemma 4.2.8. □

Then, the proof of Lemma 4.2.5 follows directly from (4.2.31) and Lemmas 4.2.6, 4.2.7 and 4.2.8. □

In what follows we prove that Lemma 4.2.5 implies that (4.2.26) holds true, and then from (4.2.25) we deduce that $(w^{(p)}(T), w_t^{(p)}(T)) \in H^3_{(0)}(0,1)\times D(A)$, which is the aim in point a) of Proposition 4.2.3.
Let us prove the following regularity result:

Lemma 4.2.9. *If $\mu \in V_\alpha^2(0,1)$ and $w \in C^0([0,T],D(A))$, then $\mu w \in C^0([0,T],V_\alpha^{(2,0)}(0,1))$. Moreover, there exists $C(\alpha,T)>0$, independent of $\mu \in V_\alpha^2(0,1)$ and $w \in C^0([0,T],D(A))$, such that*

$$\|\mu w\|_{C^0([0,T],V_\alpha^{(2,0)}(0,1))} \le C(\alpha,T)\|\mu\|_{V_\alpha^2(0,1)}\|w\|_{C^0([0,T],D(A))}. \tag{4.2.45}$$

Proof. We separately treat the case of weak and strong degeneracy.
$\boldsymbol{\alpha \in [0,1)}$.
Let $\mu \in V_\alpha^{(2,\infty)}(0,1)$ and $w \in V_\alpha^{(2,0)}(0,1)$. As we have already shown,

$$\mu \in H^2_\alpha(0,1) \quad \Rightarrow \quad \mu \in L^\infty(0,1) \quad \Rightarrow \quad \mu w \in L^2(0,1).$$

Moreover,

$$(\mu w)_x = \mu_x w + \mu w_x \ \in L^1(0,1)$$

because $\mu, w \in L^\infty(0,1)$, $\mu_x = (x^{\alpha/2}\mu_x)x^{-\alpha/2} \in L^1(0,1)$ and $w_x = (x^{\alpha/2}w_x)x^{-\alpha/2} \in L^1(0,1)$. Thus μw is absolutely continuous on $[0,1]$.
Furthermore,

$$x^{\alpha/2}(\mu w)_x = (x^{\alpha/2}\mu_x)w + (x^{\alpha/2}w_x)\mu \in L^2(0,1)$$

because $x^{\alpha/2}\mu_x, x^{\alpha/2}w_x \in L^2(0,1)$ and $w, \mu \in L^\infty(0,1)$.
We observe that

$$(x^\alpha(\mu w)_x)_x = (x^\alpha \mu_x)_x w + (x^\alpha w_x)_x \mu + 2x^\alpha \mu_x w_x.$$

Since $(x^\alpha\mu_x)_x, (x^\alpha w_x)_x \in L^2(0,1)$ and $w, \mu \in L^\infty(0,1)$, then we deduce that $(x^\alpha\mu_x)_x w + (x^\alpha w_x)_x \mu \in L^2(0,1)$. Concerning the last term of the above identity, we note that

$$\mu \in V_\alpha^{(2,\infty)}(0,1) \quad \Rightarrow \quad |x^\alpha \mu_x w_x| \le C x^{\alpha/2}|w_x|,$$

and since $x^{\alpha/2}w_x \in L^2(0,1)$, we obtain that $\mu w \in H^2_\alpha(0,1)$.
It remains to check the condition at $x=0$. We have that

$$x^\alpha(\mu w)_x = x^\alpha \mu_x w + x^\alpha w_x \mu.$$

Since $\mu \in V_\alpha^{(2,\infty)}(0,1)$ and $w \in L^\infty(0,1)$, it holds that

$$x^\alpha \mu_x w \to 0 \quad \text{as} \quad x \to 0,$$

and moreover, since $w \in V_\alpha^{2,0}(0,1)$ and $\mu \in L^\infty(0,1)$, then

$$x^\alpha w_x \mu \to 0 \quad \text{as} \quad x \to 0.$$

Thus $\mu w \in V_\alpha^{(2,0)}(0,1)$.
We conclude that $\mu \in V_\alpha^{(2,\infty)}(0,1)$ and $w \in C^0([0,T],D(A))$, then $\mu w \in C^0([0,T],V_\alpha^{(2,0)}(0,1))$.
$\boldsymbol{\alpha \in [1,2)}$.
We observe that

$$\mu \in V_\alpha^{(2,\infty,\infty)}(0,1) \quad \Rightarrow \quad |\mu_x| \le \frac{C}{x^{\alpha/2}} \quad \Rightarrow \quad \mu_x \in L^1(0,1) \quad \Rightarrow \quad \mu \in L^\infty(0,1).$$

This implies that $\mu w \in L^2(0,1)$.
Moreover

$$x^{\alpha/2}(\mu w)_x = (x^{\alpha/2}\mu_x)w + (x^{\alpha/2}w_x)\mu \in L^2(0,1)$$

because $x^{\alpha/2}\mu_x \in L^\infty(0,1)$, $w \in L^2(0,1)$ (thus, $(x^{\alpha/2}\mu_x)w \in L^2(0,1)$) and $x^{\alpha/2}w_x \in L^2(0,1)$, $\mu \in L^\infty(0,1)$ (hence, $(x^{\alpha/2}w_x)\mu \in L^2(0,1)$).
Now, we consider

$$(x^\alpha(\mu w)_x)_x = (x^\alpha \mu_x)_x w + (x^\alpha w_x)_x \mu + 2x^\alpha \mu_x w_x.$$

Since $(x^\alpha \mu_x)_x \in L^\infty(0,1)$ and $w \in L^2(0,1)$, we have that $(x^\alpha \mu_x)_x w \in L^2(0,1)$. Furthermore, since $(x^\alpha w_x)_x \in L^2(0,1)$ and $\mu \in L^\infty(0,1)$, it holds that $(x^\alpha w_x)_x \mu \in L^2(0,1)$. Concerning the last term of the above identity, we note that

$$\mu \in V_\alpha^{(2,\infty,\infty)}(0,1) \quad \Rightarrow \quad |x^\alpha \mu_x w_x| \le C x^{\alpha/2}|w_x| \in L^2(0,1),$$

therefore $\mu w \in H_\alpha^2(0,1)$. Finally, $\mu w \in H_\alpha^2(0,1)$, with $\alpha \in [1,2)$, imply that $x^\alpha(\mu w)_x \to 0$ as $x \to 0$.
So, we have proved that $\mu w \in V_\alpha^{(2,0)}(0,1)$. And, if $\mu \in V_\alpha^{(2,\infty,\infty)}(0,1)$ and $w \in C^0([0,T],D(A))$, then $\mu w \in C^0([0,T],V_\alpha^{(2,0)}(0,1)$. □

We now proceed to prove Proposition 4.2.3.

Proof of Proposition 4.2.3. The first result to prove is that the solution $(w^{(p)}, w_t^{(p)})$ of (4.1.20) fulfills the regularity property (4.2.13). By using formula (4.2.24), Lemma 4.2.9 (with $w = w^{(p)}$) and Lemma 4.2.5, we obtain that (4.2.26) holds true and then (4.2.25) shows that $(w^{(p)}(T), w_t^{(p)}(T)) \in H^3_{(0)}(0,1) \times D(A)$.
Now we show that Θ_T is differentiable at every $p \in L^2(0,T)$. Let $p_0, q \in L^2(0,T)$. Then, consider $w^{(p_0)}$, solution of (4.1.20) with $p = p_0$, and $w^{(p_0+q)}$, solution of (4.1.20) with $p = p_0 + q$.
Formally, let us write a limited development of $w^{(p_0+q)}$ with respect to q:

$$w^{(p_0+q)} = w^{(p_0)} + W_1(q) + \cdots$$

We use this development in (4.1.20) to find the equation satisfied by the supposed first order term $W_1(q)$: denoting

$$Pw := w_{tt} - (x^\alpha w_x)_x,$$

we have

$$P\left(w^{(p_0)} + W_1(q) + \cdots\right) = \left(p_0(t) + q(t)\right)\left(\mu(x)w^{(p_0)} + \mu(x)W_1(q) + \cdots\right),$$

hence we deduce that $W_1(q)$ is (probably) solution of

$$PW_1(q) = p_0(t)\mu(x)W_1(q) + q(t)\mu(x)w^{(p_0)},$$

which is the motivation in taking $W_1(q)$ as the solution of (4.2.14) with $p = p_0$, or in other words:

$$W_1(q) = W^{(p_0,q)}.$$

This is the motivation to introduce

$$v^{(p_0,q)} := w^{(p_0+q)} - w^{(p_0)} - W^{(p_0,q)}, \tag{4.2.46}$$

which allows us to write

$$\Theta_T(p_0 + q) = \Theta_T(p_0) + (W^{(p_0,q)}(T), W_t^{(p_0,q)}(T)) + (v^{(p_0,q)}(T), v_t^{(p_0,q)}(T)).$$

We are going to prove the following Lemmas:

Lemma 4.2.10. *The application*

$$\begin{aligned} L^2(0,T) &\to H^3_{(0)}(0,1) \times D(A) \\ q &\mapsto (W^{(p_0,q)}(T), W_t^{(p_0,q)}(T)) \end{aligned}$$

is well-defined, linear and continuous.

and

Lemma 4.2.11. *The application*

$$\begin{aligned} L^2(0,T) &\to H^3_{(0)}(0,1) \times D(A) \\ q &\mapsto (v^{(p_0,q)}(T), v_t^{(p_0,q)}(T)) \end{aligned}$$

is well-defined, and satisfies

$$\frac{\|(v^{(p_0,q)}(T), v_t^{(p_0,q)}(T))\|_{H^3_{(0)}(0,1)\times D(A)}}{\|q\|_{L^2(0,T)}} \to 0, \quad \text{as} \quad \|q\|_{L^2(0,T)} \to 0. \tag{4.2.47}$$

Then, we conclude that Θ_T is differentiable at p_0 and that

$$D\theta_T(p_0) \cdot q = (W^{(p_0,q)}(T), W_t^{(p_0,q)}(T)).$$

Proof of Lemma 4.2.10. First, we prove that

$$(W^{(p_0,q)}(T), W_t^{(p_0,q)}(T)) \in H^3_{(0)}(0,1) \times D(A).$$

We observe that problem (4.2.14) is well-posed. Indeed, from Proposition 4.2.1 we deduce that $w^{(p_0)} \in C^0([0,T], D(A))$, and by applying Lemma 4.2.9 we obtain that $\mu w^{(p_0)} \in$

$C^0([0,T],V_\alpha^{(2,0)}(0,1))$ because $\mu \in V_\alpha^2(0,1)$. Therefore $\mu w^{(p_0)} \in C^0([0,T],H_\alpha^1(0,1))$. Multiplying $\mu w^{(p_0)}$ by q, we get that $q\mu w^{(p_0)} \in L^2(0,T;H_\alpha^1(0,1))$. Thus, we can apply Proposition 4.2.1 to (4.2.14) (taking $f = q\mu w^{(p_0)}$), and we deduce that

$$(W^{(p_0,q)}, W_t^{(p_0,q)}) \in C^0([0,T], D(A) \times H_\alpha^1(0,1)).$$

Furthermore, (4.2.8) gives that

$$\begin{aligned}\|W^{(p_0,q)}\|_{C^0([0,T],D(A))} + \|W_t^{(p_0,q)}\|_{C^0([0,T],H_\alpha^1(0,1))} \\ &\leq C(T)\|q\mu w^{(p_0)}\|_{L^2(0,T;H_\alpha^1(0,1)))} \\ &\leq C(T)\|q\|_{L^2(0,T)}\|\mu w^{(p_0)}\|_{C^0([0,T],H_\alpha^1(0,1)))}.\end{aligned} \tag{4.2.48}$$

We now decompose $(W^{(p_0,q)}(T), W_t^{(p_0,q)}(T))$ as follows: denoting

$$R_n(s) = \langle p_0(s)\mu(\cdot)W^{(p_0,q)}(\cdot,s) + q(s)\mu(\cdot)w^{(p_0)}(\cdot,s), \varphi_{\alpha,|n|}\rangle_{L^2(0,1)}, \tag{4.2.49}$$

we have

$$\begin{pmatrix} W^{(p_0,q)}(x,T) \\ W_t^{(p_0,q)}(x,T) \end{pmatrix} = \Gamma_0^{(p_0,q)}(T) + \sum_{n\in\mathbb{Z}^*} \frac{1}{2i\omega_{\alpha,n}} \gamma_n^{(p_0,q)}(T)\Psi_{\alpha,n}(x)e^{i\omega_{\alpha,n}T}, \tag{4.2.50}$$

with

$$\Gamma_0^{(p_0,q)}(T) = \begin{pmatrix} \int_0^T R_0(s)(T-s)ds \\ \int_0^T R_0(s)ds \end{pmatrix} = \begin{pmatrix} \gamma_{00}^{(p_0,q)}(T) \\ \gamma_{01}^{(p_0,q)}(T) \end{pmatrix}, \tag{4.2.51}$$

and

$$\forall\, n \in \mathbb{Z}^*, \quad \gamma_n^{(p_0,q)}(T) = \int_0^T R_n(s)e^{-i\omega_{\alpha,n}s}ds. \tag{4.2.52}$$

Moreover, the following implication holds true

$$\sum_{n\in\mathbb{Z}} \lambda_{\alpha,|n|}^2 |\gamma_n^{(p_0,q)}(T)|^2 < \infty \quad \Longrightarrow \quad (W^{(p_0,q)}(T), W_t^{(p_0,q)}(T)) \in H_{(0)}^3(0,1) \times D(A). \tag{4.2.53}$$

Therefore, to prove that

$$(W^{(p_0,q)}(T), W_t^{(p_0,q)}(T)) \in H_{(0)}^3(0,1) \times D(A)$$

we have to prove the convergence of the above series. We decompose as follows: $\forall\, n \neq 0$

$$\begin{aligned}\gamma_n^{(p_0,q)}(T) &= \int_0^T p_0(s)\langle \mu(\cdot)W^{(p_0,q)}(\cdot,s), \varphi_{\alpha,|n|}\rangle_{L^2(0,1)} e^{-i\omega_{\alpha,n}s}ds \\ &\quad + \int_0^T q(s)\langle \mu(\cdot)w^{(p_0)}(\cdot,s), \varphi_{\alpha,|n|}\rangle_{L^2(0,1)} e^{-i\omega_{\alpha,n}s}ds \\ &=: \gamma_n^{(p_0,\mu,W^{(p_0,q)})}(T) + \gamma_n^{(q,\mu,w^{(p_0)})}(T).\end{aligned} \tag{4.2.54}$$

We apply Lemma 4.2.5 first choosing $p = p_0$ and $g = \mu W^{(p_0,q)}$. Since $p_0 \in L^2(0,T)$ and $\mu W^{(p_0,q)} \in C^0([0,T],V_\alpha^{(2,0)}(0,1)$ we obtain

$$\sum_{n\in\mathbb{Z}^*} \lambda_{\alpha,|n|}^2 |\gamma_n^{(p_0,\mu,W^{(p_0,q)})}(T)|^2 < \infty, \tag{4.2.55}$$

and furthermore

$$\begin{aligned}\Big(\sum_{n\in\mathbb{Z}^*}\lambda_{\alpha,|n|}^2|\gamma_n^{(p_0,\mu,W^{(p_0,q)})}(T)|^2\Big)^{1/2}&\\ \le C(\alpha,T)\|p_0\|_{L^2(0,T)}\|\mu W^{(p_0,q)}\|_{C^0([0,T],V_\alpha^{(2,0)}(0,1))}&\\ \le C'(\alpha,T)\|p_0\|_{L^2(0,T)}\|\mu\|_{V_\alpha^2}\|W^{(p_0,q)}\|_{C^0([0,T],D(A))}&\\ \le C''(\alpha,T)\|p_0\|_{L^2(0,T)}\|\mu\|_{V_\alpha^2}\|q\|_{L^2(0,T)}\|\mu w^{(p_0)}\|_{C^0([0,T],H_\alpha^1(0,1)))}&\\ \le C'''(\alpha,T)\|p_0\|_{L^2(0,T)}\|q\|_{L^2(0,T)}\|\mu\|_{V_\alpha^2}^2\|w^{(p_0)}\|_{C^0([0,T],D(A))},&\end{aligned} \tag{4.2.56}$$

In the same way, we apply Lemma 4.2.5 with $p=q$ and $g=\mu w^{(p_0)}$ and we get

$$\sum_{n\in\mathbb{Z}^*}\lambda_{\alpha,|n|}^2|\gamma_n^{(q,\mu,w^{(p_0)})}(T)|^2<\infty. \tag{4.2.57}$$

Moreover,

$$\begin{aligned}\Big(\sum_{n\in\mathbb{Z}^*}\lambda_{\alpha,|n|}^2|\gamma_n^{(q,\mu,w^{(p_0)})}(T)|^2\Big)^{1/2}&\\ \le C(\alpha,T)\|q\|_{L^2(0,T)}\|\mu w^{(p_0)}\|_{C^0([0,T],V_\alpha^{2,0}(0,1))}&\\ \le C'(\alpha,T)\|q\|_{L^2(0,T)}\|\mu\|_{V_\alpha^2}\|w^{(p_0)}\|_{C^0([0,T],D(A))}.&\end{aligned} \tag{4.2.58}$$

We have proved that

$$\sum_{n\in\mathbb{Z}^*}\lambda_{\alpha,|n|}^2|\gamma_n^{(p_0,q)}(T)|^2<\infty.$$

So, from (4.2.53) we have that

$$(W^{(p_0,q)}(T),W_t^{(p_0,q)}(T))\in H^3_{(0)}(0,1)\times D(A),$$

and furthermore

$$\begin{aligned}\Big\|\begin{pmatrix}W^{(p_0,q)}(\cdot,T)\\ W_t^{(p_0,q)}(\cdot,T)\end{pmatrix}-\Gamma_0^{(p_0,q)}(T)\Big\|_{H^3_{(0)}(0,1)\times D(A)}&\\ \le C\Big(\sum_{n\in\mathbb{Z}^*}\lambda_{\alpha,|n|}^2|\gamma_n^{(p_0,q)}(T)|^2\Big)^{1/2}&\\ \le C'''(\alpha,T)\|p_0\|_{L^2(0,T)}\|q\|_{L^2(0,T)}\|\mu\|_{V_\alpha^2(0,1)}^2\|w^{(p_0)}\|_{C^0([0,T],D(A))}&\\ +C'(\alpha,T)\|q\|_{L^2(0,T)}\|\mu\|_{V_\alpha^2(0,1)}\|w^{(p_0)}\|_{C^0([0,T],D(A))}.&\end{aligned}$$

However, $\Gamma_0^{(p_0,q)}(T)$ is independent of x, hence

$$\|\Gamma_0^{(p_0,q)}(T)\|_{H^3_{(0)}(0,1)\times D(A)}=\|\Gamma_0^{(p_0,q)}(T)\|_{L^2(0,1)\times L^2(0,1)}\le C\int_0^T|R_0(s)|ds,$$

and

$$\begin{aligned}|R_0(s)|&\le|p_0(s)||\langle\mu W^{(p_0,q)}(s),\varphi_{\alpha,0}\rangle_{L^2(0,1)}|+|q(s)||\langle\mu w^{(p_0)}(s),\varphi_{\alpha,0}\rangle_{L^2(0,1)}|\\ &\le|p_0(s)|\|\mu W^{(p_0,q)}(s)\|_{L^2(0,1)}+|q(s)|\|\mu w^{(p_0)}(s)\|_{L^2(0,1)}\\ &\le C|p_0(s)|\|\mu\|_{V_\alpha^2(0,1)}\|W^{(p_0,q)}\|_{C^0([0,T],D(A))}\\ &\qquad+C|q(s)|\|\mu\|_{V_\alpha^2(0,1)}\|w^{(p_0)}\|_{C^0([0,T],D(A))}\\ &\le C'|p_0(s)|\|\mu\|_{V_\alpha^2(0,1)}^2\|q\|_{L^2(0,T)}\|w^{(p_0)}\|_{C^0([0,T],D(A))}\\ &\qquad+C|q(s)|\|\mu\|_{V_\alpha^2(0,1)}\|w^{(p_0)}\|_{C^0([0,T],D(A))}.\end{aligned}$$

Thus

$$\|R_0\|_{L^1(0,T)} \le C(\alpha,T,\mu,w^{(p_0)})\|q\|_{L^2(0,T)},$$

and therefore

$$\left\|\begin{pmatrix} W^{(p_0,q)}(\cdot,T) \\ W_t^{(p_0,q)}(\cdot,T)\end{pmatrix}\right\|_{H^3_{(0)}(0,1)\times D(A)} \le C(\alpha,T,\mu,w^{(p_0)})\|q\|_{L^2(0,T)}.$$

This concludes the proof of Lemma 4.2.10. □

Proof of Lemma 4.2.11. The function $v^{(p_0,q)}$, defined in (4.2.46), is the classical solution of

$$\begin{cases} v_{tt}^{(p_0,q)} - (x^\alpha v_x^{(p_0,q)})_x = p_0(t)\mu(x)v^{(p_0,q)} + q(t)\mu(x)(w^{(p_0+q)} - w^{(p_0)}), \\ (x^\alpha v_x^{(p_0,q)})(0,t) = 0, \\ v_x^{(p_0,q)}(1,t) = 0, \\ v^{(p_0,q)}(x,0) = 0, \\ v_t^{(p_0,q)}(x,0) = 0, \end{cases} \tag{4.2.59}$$

that is actually a problem similar to (4.2.14) with $p = p_0$ and $w^{(p_0+q)} - w^{(p_0)}$ that replaces w^p. The linear control system (4.2.59) is well-posed, and

$$(v^{(p_0,q)}, v_t^{(p_0,q)}) \in C^0([0,T], D(A)\times H^1_\alpha(0,1)).$$

So, (4.2.8) gives that

$$\begin{aligned} \|v^{(p_0,q)}\|_{C^0([0,T],D(A))} + \|v_t^{(p_0,q)}\|_{C^0([0,T],H^1_\alpha(0,1))} \\ &\le C\|q\mu(w^{(p_0+q)} - w^{(p_0)})\|_{L^2(0,T;H^1_\alpha(0,1)))} \\ &\le C\|q\|_{L^2(0,T)}\|\mu(w^{(p_0+q)} - w^{(p_0)})\|_{C^0([0,T],H^1_\alpha(0,1)))} \\ &\le C\|q\|_{L^2(0,T)}\|\mu\|_{V^2_\alpha(0,1)}\|(w^{(p_0+q)} - w^{(p_0)})\|_{C^0([0,T],H^1_\alpha(0,1)))}; \end{aligned} \tag{4.2.60}$$

We decompose $(v^{(p_0,q)}(T), v_t^{(p_0,q)}(T))$ as follows: denoting

$$z_n(s) = \langle p_0(s)\mu(\cdot)v^{(p_0,q)}(\cdot,s) + q(s)\mu(\cdot)(w^{(p_0+q)} - w^{(p_0)})(\cdot,s), \varphi_{\alpha,|n|}\rangle_{L^2(0,1)}, \tag{4.2.61}$$

we have

$$\begin{pmatrix} v^{(p_0,q)}(x,T) \\ v_t^{(p_0,q)}(x,T)\end{pmatrix} = E_0^{(p_0,q)}(T) + \sum_{n\in\mathbb{Z}^*} \frac{1}{2i\omega_{\alpha,n}}\varepsilon_n^{(p_0,q)}(T)\Psi_{\alpha,n}(x)\,e^{i\omega_{\alpha,n}T}, \tag{4.2.62}$$

with

$$E_0^{(p_0,q)}(T) = \begin{pmatrix} \int_0^T z_0(s)(T-s)ds \\ \int_0^T z_0(s)ds \end{pmatrix}, \tag{4.2.63}$$

and

$$\forall\, n\in\mathbb{Z}^*, \quad \varepsilon_n^{(p_0,q)}(T) = \int_0^T z_n(s)e^{-i\omega_{\alpha,n}s}ds. \tag{4.2.64}$$

As showed by (4.2.25), we have that

$$\sum_{n\in\mathbb{Z}} \lambda^2_{\alpha,|n|}|\varepsilon_n^{(p_0,q)}(T)|^2 < \infty \quad\Longrightarrow\quad (v^{(p_0,q)}(T), v_t^{(p_0,q)}(T)) \in H^3_{(0)}(0,1)\times D(A). \tag{4.2.65}$$

Thus, we have to prove the convergence of the series on the left-hand side of (4.2.65). We observe that

$$p_0(t)\mu(x)v^{(p_0,q)} + q(t)\mu(x)(w^{(p_0+q)} - w^{(p_0)}) = (p_0(t)+q(t))\mu(x)v^{(p_0,q)} + q(t)\mu(x)W^{(p_0,q)},$$

therefore, we decompose $\varepsilon_n^{(p_0,q)}(T)$ as follows: $\forall\, n \neq 0$

$$\begin{aligned}\varepsilon_n^{(p_0,q)}(T) &= \int_0^T (p_0(s)+q(s))\langle \mu(\cdot)v^{(p_0,q)}(\cdot,s), \varphi_{\alpha,|n|}\rangle_{L^2(0,1)} e^{-i\omega_{\alpha,n}s} ds \\ &\quad + \int_0^T q(s)\langle \mu(\cdot)W^{(p_0,q)}(\cdot,s), \varphi_{\alpha,|n|}\rangle_{L^2(0,1)} e^{-i\omega_{\alpha,n}s} ds \\ &= \gamma_n^{(p_0+q,\mu,v^{(p_0,q)})}(T) + \gamma_n^{(q,\mu,W^{(p_0,q)})}(T).\end{aligned} \tag{4.2.66}$$

Applying twice Lemma 4.2.5, we obtain

$$\begin{aligned}&\Big(\sum_{n=1}^{\infty} \lambda_{\alpha,n}^2 |\gamma_n^{(p_0+q,\mu,v^{(p_0,q)})}(T)|^2\Big)^{1/2} \\ &\qquad \le C(\alpha,T)\|p_0+q\|_{L^2(0,T)}\|\mu v^{(p_0,q)}\|_{C^0([0,T],V_\alpha^{2,0}(0,1))} \\ &\qquad \le C'(\alpha,T)\|p_0+q\|_{L^2(0,T)}\|\mu\|_{V_\alpha^2(0,1)}\|v^{(p_0,q)}\|_{C^0([0,T],D(A))} \\ &\qquad \le C''(\alpha,T)\|p_0+q\|_{L^2(0,T)}\|q\|_{L^2(0,T)}\|\mu\|^2_{V_\alpha^2(0,1)}\|w^{(p_0+q)} - w^{(p_0)}\|_{C^0([0,T],D(A))},\end{aligned} \tag{4.2.67}$$

and

$$\begin{aligned}\Big(\sum_{n=1}^{\infty} \lambda_{\alpha,n}^2 |\gamma_n^{(q,\mu,W^{(p_0,q)})}(T)|^2\Big)^{1/2} &\le C(\alpha,T)\|q\|_{L^2(0,T)}\|\mu W^{(p_0,q)}\|_{C^0([0,T],V_\alpha^{2,0}(0,1))} \\ &\le C'(\alpha,T)\|q\|_{L^2(0,T)}\|\mu\|_{V_\alpha^2(0,1)}\|W^{(p_0,q)}\|_{C^0([0,T],D(A))} \\ &\le C''(\alpha,T)\|q\|^2_{L^2(0,T)}\|\mu\|^2_{V_\alpha^2(0,1)}\|w^{(p_0)}\|_{C^0([0,T],D(A))}.\end{aligned} \tag{4.2.68}$$

Thus, we have proved that

$$\sum_{n\in\mathbb{Z}^*} \lambda_{\alpha,|n|}^2 |\varepsilon_n^{(p_0,q)}(T)|^2 < \infty,$$

which implies that

$$(v^{(p_0,q)}(T), v_t^{(p_0,q)}(T)) \in H^3_{(0)}(0,1)\times D(A).$$

Furthermore,

$$\begin{aligned}&\left\| \begin{pmatrix} v^{(p_0,q)}(\cdot,T) \\ v_t^{(p_0,q)}(\cdot,T)\end{pmatrix} - E_0^{(p_0,q)}(T)\right\|_{H^3_{(0)}(0,1)\times D(A)} \\ &\qquad \le C\Big(\sum_{n\in\mathbb{Z}} \lambda_{\alpha,|n|}^2 |\varepsilon_n^{(p_0,q)}(T)|^2\Big)^{1/2} \\ &\qquad \le C''(\alpha,T)\|p_0+q\|_{L^2(0,T)}\|q\|_{L^2(0,T)}\|\mu\|^2_{V_\alpha^2(0,1)}\|w^{(p_0+q)} - w^{(p_0)}\|_{C^0([0,T],D(A))} \\ &\qquad\quad + C''(\alpha,T)\|q\|^2_{L^2(0,T)}\|\mu\|^2_{V_\alpha^2(0,1)}\|w^{(p_0)}\|_{C^0([0,T],D(A))}.\end{aligned}$$

To conclude, we observe that

$$u^{(p_0,q)} := w^{(p_0+q)} - w^{(p_0)} \tag{4.2.69}$$

is solution of

$$\begin{cases} u_{tt}^{(p_0,q)} - (x^\alpha u_x^{(p_0,q)})_x = p_0(t)\mu(x)u^{(p_0,q)} + q(t)\mu(x)w^{(p_0+q)}, \\ (x^\alpha u_x^{(p_0,q)})(0,t) = 0, \\ u_x^{(p_0,q)}(1,t) = 0, \\ u^{(p_0,q)}(x,0) = 0, \\ u_t^{(p_0,q)}(x,0) = 0, \end{cases} \tag{4.2.70}$$

hence (4.2.8) implies that

$$\begin{aligned} \|w^{(p_0+q)} - w^{(p_0)}\|_{C^0([0,T],D(A))} &= \|u^{(p_0,q)}\|_{C^0([0,T],D(A))} \\ &\le C\|q\mu w^{(p_0+q)}\|_{L^2(0,T;H^1_\alpha(0,1)))} \\ &\le C\|q\|_{L^2(0,T)}\|\mu\|_{V^2_\alpha}\|w^{(p_0+q)}\|_{C^0([0,T],D(A))}. \end{aligned}$$

So, we get

$$\begin{aligned} &\left\| \begin{pmatrix} v^{(p_0,q)}(\cdot,T) \\ v_t^{(p_0,q)}(\cdot,T) \end{pmatrix} - E_0^{(p_0,q)}(T) \right\|_{H^3_{(0)}(0,1)\times D(A)} \\ &\quad \le C''(\alpha,T)\|q\|^2_{L^2(0,T)}\Big(\|p_0+q\|_{L^2(0,T)}\|\mu\|^3_{V^2_\alpha(0,1)}\|w^{(p_0+q)}\|_{C^0([0,T],D(A))} \\ &\qquad + \|\mu\|^2_{V^2_\alpha(0,1)}\|w^{(p_0)}\|_{C^0([0,T],D(A))}\Big). \end{aligned}$$

However, $E_0^{(p_0,q)}(T)$ is independent of x, hence

$$\|E_0^{(p_0,q)}(T)\|_{H^3_{(0)}(0,1)\times D(A)} = \|E_0^{(p_0,q)}(T)\|_{L^2(0,1)\times L^2(0,1)} \le C\int_0^T |z_0(s)|ds,$$

and

$$\begin{aligned} |z_0(s)| &\le |p_0(s)||\langle \mu v^{(p_0,q)}(s), \varphi_{\alpha,0}\rangle_{L^2(0,1)}| + |q(s)||\langle \mu(w^{(p_0+q)} - w^{(p_0)})(s), \varphi_{\alpha,0}\rangle_{L^2(0,1)}| \\ &\le |p_0(s)|\|\mu v^{(p_0,q)}(s)\|_{L^2(0,1)} + |q(s)|\|\mu(w^{(p_0+q)} - w^{(p_0)})(s)\|_{L^2(0,1)} \\ &\le C|p_0(s)|\|\mu\|_{V^2_\alpha(0,1)}\|v^{(p_0,q)}\|_{C^0([0,T],D(A))} \\ &\quad + C|q(s)|\|\mu\|_{V^2_\alpha(0,1)}\|w^{(p_0+q)} - w^{(p_0)}\|_{C^0([0,T],D(A))} \\ &\le C'|p_0(s)|\|\mu\|^2_{V^2_\alpha(0,1)}\|q\|_{L^2(0,T)}\|(w^{(p_0+q)} - w^{(p_0)})\|_{C^0([0,T],H^1_\alpha(0,1)))} \\ &\quad + C|q(s)|\|\mu\|^2_{V^2_\alpha(0,1)}\|q\|_{L^2(0,T)}\|w^{(p_0+q)}\|_{C^0([0,T],D(A))} \\ &\le C'|p_0(s)|\|\mu\|^3_{V^2_\alpha(0,1)}\|q\|^2_{L^2(0,T)}\|w^{(p_0+q)}\|_{C^0([0,T],D(A))} \\ &\quad + C|q(s)|\|\mu\|^2_{V^2_\alpha(0,1)}\|q\|_{L^2(0,T)}\|w^{(p_0+q)}\|_{C^0([0,T],D(A))}. \end{aligned}$$

Hence, we have showed that

$$\|z_0\|_{L^1(0,T)} \le C(\alpha,T,\mu,w^{(p_0)})\|q\|^2_{L^2(0,T)},$$

and therefore

$$\left\| \begin{pmatrix} v^{(p_0,q)}(\cdot,T) \\ v_t^{(p_0,q)}(\cdot,T) \end{pmatrix} \right\|_{H^3_{(0)}(0,1)\times D(A)} \le C(\alpha,T,\mu,w^{(p_0)})\|q\|^2_{L^2(0,T)}.$$

This concludes the proof of Lemma 4.2.11. □

It remains to prove that Θ_T is of class C^1. To this purpose, we have to prove that the application $D\Theta_T$ is continuous from $L^2(0,T)$ into $\mathscr{L}_c(L^2(0,T), H^3_{(0)}(0,1)\times D(A))$, namely

$$|||D\Theta_T(p_0+\tilde{p}) - D\Theta_T(p_0)|||_{\mathscr{L}_c(L^2(0,T),H^3_{(0)}(0,1)\times D(A))} \to 0, \quad \text{as } \|\tilde{p}\|_{L^2(0,T)} \to 0.$$

Proceeding as in the proof of Lemma, 4.2.10 it is easy to verify that there exists $C(\alpha, T, \mu, p_0) > 0$ such that for any $\tilde{p}$, $q \in L^2(0,T)$

$$\begin{aligned} \|D\Theta_T(p_0+\tilde{p})\cdot q - D\Theta_T(p_0)\cdot q\|_{H^3_{(0)}(0,1)\times D(A)} &= \|W^{(p_0+\tilde{p},q)}(T) - W^{(p_0,q)}(T)\|_{H^3_{(0)}(0,1)\times D(A)} \\ &\leq C(\alpha, T, \mu, p_0)\|\tilde{p}\|_{L^2(0,T)}\|q\|_{L^2(0,T)}, \end{aligned} \tag{4.2.71}$$

which implies that

$$\forall\, \tilde{p}, \quad |||D\Theta_T(p_0+\tilde{p}) - D\Theta_T(p_0)|||_{\mathscr{L}_c(L^2(0,T),H^3_{(0)}(0,1)\times D(A))} \leq C(\alpha, T, \mu, p_0)\|\tilde{p}\|_{L^2(0,T)}.$$

Thus, $D\Theta_T$ is continuous and this concludes the proof of Proposition 4.2.3. □

The proof of Theorem (4.1.1) follows from the application of the classical inverse mapping theorem to the function $\Theta_T : L^2(0,T) \to H^3_{(0)}(0,1)\times D(A)$ at the point $p_0 = 0$. We recall that $\Theta_T(p_0 = 0) = (1,0)$.
The key point of the proof is represented by the following Lemma.

Lemma 4.2.12. *The linear application*

$$\begin{aligned} D\Theta_T(0) : L^2(0,T) &\to H^3_{(0)}(0,1)\times D(A) \\ q &\mapsto (W^{(0,q)}(T), W^{(0,q)}_t(T)) \end{aligned}$$

is surjective, and

$$D\Theta_T(0) : \overline{Vect\ \{1, t, \cos\sqrt{\lambda_{\alpha,n}}t, \sin\sqrt{\lambda_{\alpha,n}}t, n \geq 1\}} \to H^3_{(0)}(0,1)\times D(A)$$

is invertible.

Proof. Since $w^{(0)} = 1$, (4.2.14) implies that $W^{(0,q)}$ is solution of the following linear problem

$$\begin{cases} W^{(0,q)}_{tt} - (x^\alpha W^{(0,q)}_x)_x = q(t)\mu(x), & x \in (0,1), t \in (0,T), \\ (x^\alpha W^{(0,q)}_x)(0,t) = 0, & t \in (0,T), \\ W^{(0,q)}_x(1,t) = 0, & t \in (0,T), \\ W^{(0,q)}(x,0) = 0, & x \in (0,1), \\ W^{(0,q)}_t(x,0) = 0, & x \in (0,1). \end{cases} \tag{4.2.72}$$

Following the procedure presented in the proof of Proposition (4.2.3) , we introduce

$$r_n(s) = \langle q(s)\mu, \varphi_{\alpha,n}\rangle_{L^2(0,1)} = \mu_{\alpha,n}q(s) \tag{4.2.73}$$

with $\mu_{\alpha,n} = \langle \mu, \varphi_{\alpha,n}\rangle_{L^2(0,1)}$, and we have

$$W^{(0,q)}(x,T) = \int_0^T r_0(s)(T-s)ds + \sum_{n=1}^{\infty}\left(\int_0^T r_n(s)\frac{\sin\sqrt{\lambda_{\alpha,n}}(T-s)}{\sqrt{\lambda_{\alpha,n}}}ds\right)\varphi_{\alpha,n}(x), \tag{4.2.74}$$

and

$$W_t^{(0,q)}(x,T) = \int_0^T r_0(s)ds + \sum_{n=1}^{\infty}\left(\sqrt{\lambda_{\alpha,n}}\int_0^T r_n(s)\frac{\cos\sqrt{\lambda_{\alpha,n}}(T-s)}{\sqrt{\lambda_{\alpha,n}}}ds\right)\varphi_{\alpha,n}(x). \quad (4.2.75)$$

To prove the surjectivity of $D\Theta_T(0)$, we choose any pair $(Y^f, Z^f) \in H^3_{(0)}(0,1)\times D(A)$, and we want to show that there exists $q \in L^2(0,T)$ such that

$$(W^{(0,q)}(T), W_t^{(0,q)}(T)) = (Y^f, Z^f). \quad (4.2.76)$$

Introducing the Fourier coefficients of the target state

$$Y^f_{\alpha,n} = \langle Y^f, \varphi_{\alpha,n}\rangle_{L^2(0,1)}, \quad \text{and} \quad Z^f_{\alpha,n} = \langle Z^f, \varphi_{\alpha,n}\rangle_{L^2(0,1)}$$

we can decompose (Y^f, Z^f) as follows

$$Y^f(x) = \sum_{n=0}^{\infty} Y^f_{\alpha,n}\varphi_{\alpha,n}(x) = Y^f_{\alpha,0} + \sum_{n=1}^{\infty} Y^f_{\alpha,n}\varphi_{\alpha,n}(x)$$

and

$$Z^f(x) = \sum_{n=0}^{\infty} Z^f_{\alpha,n}\varphi_{\alpha,n}(x) = Z^f_{\alpha,0} + \sum_{n=1}^{\infty} Z^f_{\alpha,n}\varphi_{\alpha,n}(x).$$

We derive from (4.2.74) and (4.2.75) that (4.2.76) is satisfied if and only if

$$\begin{cases} \int_0^T r_0(s)ds = Z^f_{\alpha,0}, & \\ \sqrt{\lambda_{\alpha,n}}\int_0^T r_n(s)\frac{\cos\sqrt{\lambda_{\alpha,n}}(T-s)}{\sqrt{\lambda_{\alpha,n}}}ds = Z^f_{\alpha,n}, & \text{for all } n \geq 1, \\ \int_0^T r_n(s)\frac{\sin\sqrt{\lambda_{\alpha,n}}(T-s)}{\sqrt{\lambda_{\alpha,n}}}ds = Y^f_{\alpha,n}, & \text{for all } n \geq 1, \\ \int_0^T r_0(s)(T-s)ds = Y^f_{\alpha,0}. & \end{cases} \quad (4.2.77)$$

Introducing

$$Q(s) := q(T-s),$$

(4.2.77) becomes

$$\begin{cases} \mu_{\alpha,0}\int_0^T Q(t)dt = Z^f_{\alpha,0}, & \\ \mu_{\alpha,n}\int_0^T Q(t)\cos\sqrt{\lambda_{\alpha,n}}t\,dt = Z^f_{\alpha,n}, & \text{for all } n \geq 1, \\ \mu_{\alpha,n}\int_0^T Q(t)\sin\sqrt{\lambda_{\alpha,n}}t\,dt = \sqrt{\lambda_{\alpha,n}}Y^f_{\alpha,n}, & \text{for all } n \geq 1, \\ \mu_{\alpha,0}\int_0^T Q(t)t\,dt = Y^f_{\alpha,0}. & \end{cases} \quad (4.2.78)$$

System (4.2.78) is usually called moment problem. Observe that (4.1.22) implies that the coefficients $\mu_{\alpha,n}$ are all different from 0 for all $n \geq 0$, which is necessary for solving (4.2.78). Let us introduce

$$\begin{cases} A^f_{\alpha,0} := \frac{Z^f_{\alpha,0}}{\mu_{\alpha,0}}, & \\ A^f_{\alpha,n} := \frac{Z^f_{\alpha,n}}{\mu_{\alpha,n}}, & \text{for all } n \geq 1, \\ B^f_{\alpha,n} := \frac{\sqrt{\lambda_{\alpha,n}}Y^f_{\alpha,n}}{\mu_{\alpha,n}}, & \text{for all } n \geq 1, \\ B^f_{\alpha,0} := \frac{Y^f_{\alpha,0}}{\mu_{\alpha,0}}, & \end{cases} \quad (4.2.79)$$

and

$$\begin{cases} c_{\alpha,0} : t \in (0,T) \mapsto 1, \\ c_{\alpha,n} : t \in (0,T) \mapsto \cos \sqrt{\lambda_{\alpha,n}}t, \quad \text{for all } n \geq 1, \\ s_{\alpha,n} : t \in (0,T) \mapsto \sin \sqrt{\lambda_{\alpha,n}}t, \quad \text{for all } n \geq 1, \\ s_{\alpha,0} : t \in (0,T) \mapsto t, \end{cases} \tag{4.2.80}$$

so that (4.2.78) can be written as follows

$$\begin{cases} \langle Q, c_{\alpha,0}\rangle_{L^2(0,T)} = A^f_{\alpha,0}, \\ \langle Q, c_{\alpha,n}\rangle_{L^2(0,T)} = A^f_{\alpha,n}, \quad \text{for all } n \geq 1, \\ \langle Q, s_{\alpha,n}\rangle_{L^2(0,T)} = B^f_{\alpha,n}, \quad \text{for all } n \geq 1, \\ \langle Q, s_{\alpha,0}\rangle_{L^2(0,T)} = B^f_{\alpha,0}. \end{cases} \tag{4.2.81}$$

Finally, we define the space

$$E_\alpha := \overline{\text{Vect}\,\{c_{\alpha,0}, c_{\alpha,n}, s_{\alpha,n},\, n \geq 1\}},$$

which is a closed subspace of $L^2(0,T)$.
To solve (4.2.81) we use the following characterization of the Riesz basis (see [11, Prop. 19] or also [32]):
the family $\{c_{\alpha,0}, c_{\alpha,n}, s_{\alpha,n}, n \geq 1\}$ is a Riesz basis of E_α if and only if there exist $C_1(\alpha,T), C_2(\alpha,T) > 0$ such that, for all $N \geq 1$ and for any $(a_n)_{0\leq n\leq N}$, $(b_n)_{1\leq n\leq N}$ it holds that

$$C_1(\alpha,T)\left(a_0^2 + \sum_{n=1}^N a_n^2 + b_n^2\right) \leq \int_0^T \left|S^{(a,b)}(t)\right|^2 dt \leq C_2(\alpha,T)\left(a_0^2 + \sum_{n=1}^N a_n^2 + b_n^2\right), \tag{4.2.82}$$

where

$$S^{(a,b)}(t) = a_0 c_{\alpha,0}(t) + \sum_{n=1}^N a_n c_{\alpha,n}(t) + b_n s_{\alpha,n}(t). \tag{4.2.83}$$

We observe that (4.2.82) holds as a consequence of Ingham theory. Indeed, by expressing $\cos y$ and $\sin y$ as

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}, \quad \text{and} \quad \sin y = \frac{e^{iy} - e^{-iy}}{2i},$$

we have

$$\begin{aligned} S^{(a,b)}(t) &= a_0 e^{i\omega_{\alpha,0}t} + \sum_{n=1}^N a_n \frac{e^{i\omega_{\alpha,n}t} + e^{-i\omega_{\alpha,n}t}}{2} + b_n \frac{e^{i\omega_{\alpha,n}t} - e^{-i\omega_{\alpha,n}t}}{2i} \\ &= \sum_{n=-N}^N d_n e^{i\omega_{\alpha,n}t}, \end{aligned}$$

with

$$\begin{cases} d_0 = a_0, \\ d_n = \frac{a_n}{2} + \frac{b_n}{2i}, \quad \text{for } n \geq 1, \\ d_n = \frac{a_{-n}}{2} - \frac{b_{-n}}{2i}, \quad \text{for } n \leq -1. \end{cases}$$

Since $\omega_{\alpha,n+1} - \omega_{\alpha,n} > 0$ for all $n \in \mathbb{Z}$ and

$$\forall\, |n| \geq 2, \quad \omega_{\alpha,n+1} - \omega_{\alpha,n} \geq \frac{2-\alpha}{2}\pi,$$

we can apply a general result of Haraux [42] (see also [11, Theorem 6]) that ensures that if

$$T > \frac{2\pi}{\frac{2-\alpha}{2}\pi} = \frac{4}{2-\alpha},$$

then there exist $C_1^{(I)}, C_2^{(I)} > 0$ independent of N, and coefficients $(d_n)_{-N\leq n\leq N}$ such that

$$C_1^{(I)} \sum_{n=-N}^{N} |d_n|^2 \leq \int_0^T \Big| \sum_{n=-N}^{N} d_n e^{i\omega_{\alpha,n} t} \Big|^2 dt \leq C_2^{(I)} \sum_{n=-N}^{N} |d_n|^2. \tag{4.2.84}$$

Since

$$\sum_{n=-N}^{N} |d_n|^2 = a_0^2 + 2\sum_{n=1}^{N} \frac{a_n^2 + b_n^2}{4},$$

(4.2.84) implies that (4.2.82) is verified and so $\{c_{\alpha,0}, c_{\alpha,n}, s_{\alpha,n}, n \geq 1\}$ is a Riesz basis.
We can now use [11, Proposition 20] that ensure that if $\{c_{\alpha,0}, c_{\alpha,n}, s_{\alpha,n}, n \geq 1\}$ is a Riesz basis, then the application $\mathscr{F} : E_\alpha \to \ell^2(\mathbb{N})$:

$$\mathscr{F}(f) = (\langle f, c_{\alpha,0}\rangle_{L^2(0,T)}, \langle f, c_{\alpha,1}\rangle_{L^2(0,T)}, \langle f, s_{\alpha,1}\rangle_{L^2(0,T)}, \langle f, c_{\alpha,2}\rangle_{L^2(0,T)}, \cdots)$$

is an isomorphism.
We note that

$$Y^f \in H^3_{(0)}(0,1) \quad \Longrightarrow \quad \sum_{n=0}^{\infty} \lambda_{\alpha,n}^3 |Y_n^f|^2 < \infty,$$

and

$$Z^f \in D(A) \quad \Longrightarrow \quad \sum_{n=0}^{\infty} \lambda_{\alpha,n}^2 |Z_n^f|^2 < \infty,$$

and then (4.1.22) ensures us that

$$|A^f_{\alpha,0}|^2 + \sum_{n=1}^{\infty} |A^f_{\alpha,n}|^2 + |B^f_{\alpha,n}|^2 < \infty,$$

and therefore there exists a unique $Q_\alpha \in E_\alpha$ such that

$$\mathscr{F}(Q_\alpha) = (A^f_{\alpha,0}, A^f_{\alpha,1}, B^f_{\alpha,1}, A^f_{\alpha,2}, \cdots).$$

Thus,

$$\begin{cases} \langle Q_\alpha, c_{\alpha,0}\rangle_{L^2(0,T)} = A^f_{\alpha,0}, \\ \langle Q_\alpha, c_{\alpha,n}\rangle_{L^2(0,T)} = A^f_{\alpha,n} \quad \text{for all } n \geq 1, \\ \langle Q_\alpha, s_{\alpha,n}\rangle_{L^2(0,T)} = B^f_{\alpha,n} \quad \text{for all } n \geq 1; \end{cases} \tag{4.2.85}$$

and, moreover, the application

$$l^2(\mathbb{N}) \to E_\alpha,$$
$$(A^f_{\alpha,0}, A^f_{\alpha,1}, B^f_{\alpha,1}, A^f_{\alpha,2}, \cdots) \mapsto Q_\alpha$$

is continuous.
Finally, we claim that $s_{\alpha,0} \notin E_\alpha$: indeed, if $t \mapsto t$ was the limit of a sequence of linear combinations of $c_{\alpha,0}$, $c_{\alpha,n}$ and $s_{\alpha,n}$, so it would be $t \mapsto t^2$ by integration. Then, by integrating further, also $t \mapsto t^3$ would be the limit of a sequence of linear combinations of $c_{\alpha,0}$, $c_{\alpha,n}$ and $s_{\alpha,n}$.

Thus, by iterating this procedure, we deduce that all the polynomials could be written in this form. Therefore, $L^2(0,T)$ would be equal to E_α and (4.2.85) would have a unique solution. However, this is not the case: define $T_0 := \frac{4}{2-\alpha}$, and choose q_α smooth, compactly supported in $(0, \frac{T-T_0}{2})$ and different from Q_α on that interval. Now, consider the following problem

$$\begin{cases} \langle \tilde{Q}_\alpha, c_{\alpha,0}\rangle_{L^2(\frac{T-T_0}{2},T)} = A^f_{\alpha,0} - \langle \tilde{q}_\alpha, c_{\alpha,0}\rangle_{L^2(0,\frac{T-T_0}{2})}, \\ \langle \tilde{Q}_\alpha, c_{\alpha,n}\rangle_{L^2(\frac{T-T_0}{2},T)} = A^f_{\alpha,n} - \langle \tilde{q}_\alpha, c_{\alpha,n}\rangle_{L^2(0,\frac{T-T_0}{2})} \quad \text{for all } n \geq 1, \\ \langle \tilde{Q}_\alpha, s_{\alpha,n}\rangle_{L^2(\frac{T-T_0}{2},T)} = B^f_{\alpha,n} - \langle \tilde{q}_\alpha, s_{\alpha,n}\rangle_{L^2(0,\frac{T-T_0}{2})} \quad \text{for all } n \geq 1. \end{cases} \tag{4.2.86}$$

Since $T - \frac{T-T_0}{2} = \frac{T+T_0}{2} > T_0$, and the sequences $(\langle \tilde{q}_\alpha, c_{\alpha,n}\rangle_{L^2(0,\frac{T-T_0}{2})})_n$ and $(\langle \tilde{q}_\alpha, s_{\alpha,n}\rangle_{L^2(0,\frac{T-T_0}{2})})_n$ are square-integrable (by integration by parts), there exists a solution $\tilde{Q} \in L^2(\frac{T-T_0}{2}, T)$ of (4.2.86). So, the function

$$Q^*_\alpha := \begin{cases} q_\alpha \text{ on } (0, \frac{T-T_0}{2}), \\ \tilde{Q}_\alpha \text{ on } (\frac{T-T_0}{2}, T) \end{cases}$$

solves (4.2.85) and it is different from Q_α. Hence $s_{\alpha,0} \notin E_\alpha$, and if we denote $p^\perp_{\alpha,0}$ the orthogonal projection of $s_{\alpha,0}$ on E_α, then $s_{\alpha,0} - p^\perp_{\alpha,0} \neq 0$, and

$$Q^\perp_\alpha := \frac{s_{\alpha,0} - p^\perp_{\alpha,0}}{\|s_{\alpha,0} - p^\perp_{\alpha,0}\|^2_{L^2(0,T)}}$$

is orthogonal to E_α, and furthermore

$$\langle Q^\perp_\alpha, s_{\alpha,0}\rangle_{L^2(0,T)} = 1.$$

Thus,

$$Q := Q_\alpha + B_{\alpha,0} Q^\perp_\alpha$$

solves (4.2.81). Moreover,

$$\|Q\|^2_{L^2(0,T)} = \|Q_\alpha\|^2_{L^2(0,T)} + \|B^f_{\alpha,0} Q^\perp_\alpha\|^2_{L^2(0,T)} \leq C\Big(\sum_{n=0}^{\infty} |A^f_{\alpha,n}|^2 + |B^f_{\alpha,n}|^2\Big),$$

which completes the proof of Lemma 4.2.12. □

4.2.3 Inverse mapping theorem

Proof of Theorem 4.1.1. We have proved in Proposition 4.2.1 that problem (4.1.20) is well defined and the solution $(w^{(p)}, w^{(p)}_t)$ is of class $C^0([0,T]; D(A) \times H^1_\alpha(0,1))$. Moreover, from Proposition 4.2.3 we know that $(w^{(p)}(T), w^{(p)}_t(T))$ is even more regular

$$(w^{(p)}(T), w^{(p)}_t(T)) \in H^3_\alpha(0,1) \times D(A).$$

Therefore, the end point map

$$\Theta_T : L^2(0,T) \to H^3_{(0)}(0,1) \times D(A), \quad p \to (w^{(p)}(T), w^{(p)}_t(T))$$

is well defined. Furthermore, in Proposition 4.2.3 we have showed that Θ_T is of class C^1 and the action of the differential $D\Theta_T$ can be represented by

$$D\Theta_T(p)\cdot q = \left(W^{(p,q)}(T), W_t^{(p,q)}(T)\right),$$

where $W^{(p,q)}$ is the solution of (4.2.14).
Now, since we want to apply the inverse mapping theorem to Θ_T, we have proved in Lemma (4.2.12) that

$$D\Theta_T(0) : L^2(0,T) \to H^3_{(0)}(0,1)\times D(A), \quad q \to (W^{(0,q)}(T), W_t^{(0,q)}(T))$$

is surjective. Moreover, if we define the space

$$F_\alpha := \overline{\text{Vect }\{1, t, \cos\sqrt{\lambda_{\alpha,n}}t, \sin\sqrt{\lambda_{\alpha,n}}t, n\geq 1\}}.$$

then, the restriction of Θ_T to F_α

$$\begin{aligned}\Theta_{\alpha,T} : F_\alpha &\to H^3_{(0)}(0,1)\times D(A),\\ p &\mapsto \Theta_{\alpha,T}(p) := \Theta_T(p)\end{aligned}$$

is C^1 (Proposition 4.2.3) and $D\Theta_{\alpha,T}(0)$ is invertible (Lemma 4.2.12). Thus, the inverse mapping theorem ensures that there exists a neighborhood $\mathscr{V}(0)\subset F_\alpha$ and a neighborhood $\mathscr{V}(1,0)\subset H^3_{(0)}(0,1)\times D(A)$ such that

$$\Theta_{\alpha,T} : \mathscr{V}(0)\to\mathscr{V}(1,0)$$

is a C^1-diffeomorphism. Hence, given $(w_0^f, w_1^f)\in\mathscr{V}(1,0)$, we choose $p^f := \Theta^{-1}_{\alpha,T}(w_0^f, w_1^f)$, and so the solution of (4.1.20) with $p = p^f$ satisfies

$$(w(T), w_t(T)) = \Theta_T(p^f) = \Theta_T(\Theta^{-1}_{\alpha,T}(w_0^f, w_1^f)) = (w_0^f, w_1^f).$$

Therefore, we have proved that, starting from the first eigenfunction $\varphi_{\alpha,0}\equiv 1$, the solution of the control system (4.1.20) reaches a neighborhood of the trajectory $w^{(1,0;0)}\equiv 1$ in time $T > T_0$. Hence, by time reversibility, we have proved the local exact controllablility of (4.1.20) along the ground state solution $w^{(1,0;0)}$ in any time $T > 2T_0$. □

4.2.4 Proof of Remark 4.1.2

First we check that

$$\mu(x) = x^{2-\alpha}$$

satisfies all the regularity assumptions.
$\alpha\in[0,1)$
We observe that

$$\mu'(x) = (2-\alpha)x^{1-\alpha}\in L^1(0,1).$$

Hence μ is absolutely continuous on $[0,1]$. Moreover,

$$x^{\alpha/2}\mu'(x) = (2-\alpha)x^{1-\frac{\alpha}{2}}\in L^2(0,1).$$

Thus, $\mu \in H^1_\alpha(0,1)$. Furthermore,

$$x^\alpha \mu'(x) = (2-\alpha)x \in H^1(0,1)$$

and therefore $\mu \in H^2_\alpha(0,1)$. Finally,

$$x^{\alpha/2}\mu'(x) = (2-\alpha)x^{1-\frac{\alpha}{2}} \in L^\infty(0,1)$$

that implies $\mu \in V_\alpha^{(2,\infty)}(0,1)$.
$\boldsymbol{\alpha \in [1,2)}$
It easy to check that $\mu \in L^2(0,1)$. Moreover,

$$x^{\alpha/2}\mu'(x) = (2-\alpha)x^{1-\frac{\alpha}{2}} \in L^2(0,1)$$

and therefore $\mu \in H^1_\alpha(0,1)$. Furthermore,

$$x^\alpha \mu'(x) = (2-\alpha)x \in H^1(0,1).$$

Thus $\mu \in H^2_\alpha(0,1)$. Finally,

$$x^{\alpha/2}\mu'(x) = (2-\alpha)x^{1-\frac{\alpha}{2}} \in L^\infty(0,1),$$

and

$$(x^\alpha \mu')'(x) = 2-\alpha \in L^\infty(0,1).$$

Hence $\mu \in V_\alpha^{(2,\infty,\infty)}(0,1)$.
We have showed that the regularity assumptions are satisfied.
It remains to check the validity of (4.1.22). We have that

$$\langle \mu, \varphi_{\alpha,0}\rangle_{L^2(0,1)} = \int_0^1 x^{2-\alpha}dx = \frac{1}{3-\alpha},$$

and, for all $n \geq 1$, we develop the scalar product as follows

$$\begin{aligned}
\langle \mu, \varphi_{\alpha,n}\rangle_{L^2(0,1)} &= \int_0^1 \mu(x)\varphi_{\alpha,n}(x)dx \\
&= \frac{1}{\lambda_{\alpha,n}}\int_0^1 \mu(x)\lambda_{\alpha,n}\varphi_{\alpha,n}(x)dx \\
&= \frac{1}{\lambda_{\alpha,n}}\int_0^1 \mu(x)(-x^\alpha\varphi'_{\alpha,n})'(x)dx \\
&= \frac{1}{\lambda_{\alpha,n}}\left([-x^\alpha\mu(x)\varphi'_{\alpha,n}(x)]_0^1 + \int_0^1 x^\alpha\mu'(x)\varphi'_{\alpha,n}(x)\right).
\end{aligned}$$

Recalling that $\mu(x) = x^{2-\alpha}$, we obtain

$$\begin{aligned}
\int_0^1 x^\alpha\mu'(x)\varphi'_{\alpha,n}(x) &= (2-\alpha)\int_0^1 x\varphi'_{\alpha,n}(x) \\
&= (2-\alpha)[x\varphi_{\alpha,n}(x)]_0^1 - (2-\alpha)\int_0^1 \varphi_{\alpha,n}(x)dx \\
&= (2-\alpha)[x\varphi_{\alpha,n}(x)]_0^1 - (2-\alpha)\langle\varphi_{\alpha,0},\varphi_{\alpha,n}\rangle_{L^2(0,1)}.
\end{aligned}$$

Since the eigenfunctions are orthogonal, we have that

$$\langle \varphi_{\alpha,0}, \varphi_{\alpha,n} \rangle_{L^2(0,1)} = 0,$$

hence

$$\langle \mu, \varphi_{\alpha,n} \rangle_{L^2(0,1)} = \frac{1}{\lambda_{\alpha,n}} \left([-x^2 \varphi'_{\alpha,n}(x)]_0^1 + (2-\alpha)[x\varphi_{\alpha,n}(x)]_0^1 \right).$$

From the Neumann boundary conditions satisfied by $\varphi_{\alpha,n}$, we know that $x\varphi'_{\alpha,n}(x) \to 0$ as $x \to 0$ and $x \to 1$, thus

$$[-x^2 \varphi'_{\alpha,n}(x)]_0^1 = 0.$$

We have also proved (Lemmas A.2.4 and A.2.8) that $\varphi_{\alpha,n}$ has a finite limit as $x \to 0$, therefore

$$x\varphi_{\alpha,n}(x) \to 0, \qquad \text{as } x \to 0.$$

Finally, once again from Lemmas A.2.4 and A.2.8 we have that $|\varphi_{\alpha,n}(1)| = \sqrt{2-\alpha}$ that yields

$$|(2-\alpha)[x\varphi_{\alpha,n}(x)]_0^1| = (2-\alpha)^{3/2},$$

and

$$|\langle \mu, \varphi_{\alpha,n} \rangle_{L^2(0,1)}| = \frac{(2-\alpha)^{3/2}}{\lambda_{\alpha,n}}.$$

Hence, (4.1.22) is satisfied.
Now, let us prove that the set of functions μ satisfying (4.1.22) is dense in V_α^2. By integrating by part, we get

$$\begin{aligned}\langle \mu, \varphi_{\alpha,n} \rangle_{L^2(0,1)} &= \frac{1}{\lambda_{\alpha,n}} \left([-x^\alpha \mu(x) \varphi'_{\alpha,n}(x)]_0^1 + \int_0^1 x^\alpha \mu'(x) \varphi'_{\alpha,n}(x) dx \right) \\ &= \frac{1}{\lambda_{\alpha,n}} \left([-x^\alpha \mu(x) \varphi'_{\alpha,n}(x)]_0^1 + [x^\alpha \mu'(x) \varphi_{\alpha,n}(x)]_0^1 - \int_0^1 (x^\alpha \mu')'(x) \varphi_{\alpha,n}(x) dx \right).\end{aligned}$$

Then, since $\mu \in L^\infty(0,1)$, we have

$$[-x^\alpha \mu(x) \varphi'_{\alpha,n}(x)]_0^1 = 0.$$

Moreover, since $x^{\alpha/2}\mu' \in L^\infty(0,1)$ and $\varphi_{\alpha,n}$ has a finite limit as $x \to 0$, we deduce

$$x^\alpha \mu'(x) \varphi_{\alpha,n}(x) \to 0, \qquad \text{as } x \to 0.$$

Thus, we obtain

$$[x^\alpha \mu'(x) \varphi_{\alpha,n}(x)]_0^1 = \mu'(1) \varphi_{\alpha,n}(1),$$

and we recall that $|\varphi_{\alpha,n}(1)| = \sqrt{2-\alpha}$.
Finally, since $(x^\alpha \mu')'(x) \in L^2(0,1)$, we get

$$\int_0^1 (x^\alpha \mu')'(x) \varphi_{\alpha,n}(x) dx = \langle (x^\alpha \mu')'(x), \varphi_{\alpha,n} \rangle_{L^2(0,1)} \to 0, \qquad \text{as } n \to \infty.$$

So,

$$\mu \in V_\alpha^2(0,1) \quad \Longrightarrow \quad |\lambda_{\alpha,n} \langle \mu, \varphi_{\alpha,n} \rangle_{L^2(0,1)}| \to \sqrt{2-\alpha}|\mu'(1)|, \qquad \text{as } n \to \infty.$$

We define the spaces

$$\mathscr{V}_n := \begin{cases} \{\mu \in V^2_\alpha(0,1),\ \langle \mu, \varphi_{\alpha,n}\rangle_{L^2(0,1)} \neq 0\} & \text{for } n \geq 0, \\ \{\mu \in V^2_\alpha(0,1),\ \mu'(1) \neq 0\} & \text{for } n = -1, \end{cases}$$

and

$$\mathscr{V}^2_\alpha := \cap_{n=-1}^{\infty} \mathscr{V}_n.$$

Clearly every $\mathscr{V}_n$ is open in V^2_α, and they are also dense. Indeed, consider $\tilde{\mu} \in V^2_\alpha(0,1)$ such that $\tilde{\mu} \notin \mathscr{V}_n$ for some $n \geq -1$, and define

$$\tilde{\mu}_\varepsilon(x) := \tilde{\mu}(x) + \varepsilon x^{2-\alpha}$$

where $\varepsilon \in \mathbb{R}^*$. Then, if $n \geq 0$, we have

$$\langle \tilde{\mu}_\varepsilon, \varphi_{\alpha,n}\rangle_{L^2(0,1)} = \varepsilon \langle x^{2-\alpha}, \varphi_{\alpha,n}\rangle_{L^2(0,1)} \neq 0,$$

and if $n = -1$, we have

$$\tilde{\mu}'_\varepsilon(1) = \varepsilon(2-\alpha) \neq 0.$$

Therefore $\tilde{\mu}_\varepsilon \in \mathscr{V}_n$ and it is close to $\tilde{\mu}$ in V^2_α if ε is sufficiently small. This means that $\mathscr{V}_n$ is dense in V^2_α. Thus $\mathscr{V}^2_\alpha$ is the intersection of a sequence of open and dense subsets and, thanks to Baire Theorem, it is dense in V^2_α.

CHAPTER 5

A constructive algorithm for building mixing coupling real valued potentials

Given an unbounded linear operator A on a separable Hilber space $(X, \langle\cdot,\cdot\rangle)$, such that its eigenfunctions $\{\varphi_k\}$ form an orthonormal basis of X, we are interested in characterizing the functions μ such that

$$\langle \mu\varphi_1, \varphi_k\rangle \neq 0, \quad \forall\, k \in \mathbb{N}^*. \tag{5.0.1}$$

In particular, in this Chapter we will consider the Laplacian operator on the space $X = L^2(0,1)$ and we provide an algorithm to build polynomials of ant degree $q \in \mathbb{N}^*$ that fulfill the non-vanishing property (5.0.1).
Furthermore, we will explain the importance of such kind of functions in the context of control theory by giving examples of applications to different types of problems as the bilinear controllability of the Schrödinger equation with mixed boundary conditions as well as the stabilizability and the controllability of the heat equation with mixed boundary conditions via bilinear control.
The content of this Chapter is based on [5].

5.1 Introduction

As we have seen in the introduction, bilinear controls are well-suited to describe processes capable of modifying some of their physical characteristics in presence of the control. A well-known example of bilinear control system is given by the description of the motion of a quantum particle in an electric field. The corresponding model is given by the Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = -\partial_x^2 u(t,x) - p(t)\mu(x)u(t,x), & (t,x) \in (0,T)\times(0,1), \\ u(t,0) = 0,\ u(t,1) = 0, \\ u(0,\cdot) = u_0(\cdot) \in (0,1), \end{cases} \tag{5.1.1}$$

where u is the wave function of the particle, $p \in L^2(0,T;\mathbb{R})$ is the control and represents the magnitude of the electric field and μ is a real valued function called dipolar moment of the particle. Denote by A the operator defined by:

$$D(A) := H^2 \cap H_0^1((0,1);\mathbb{C})\ , A\varphi = -\frac{d^2\varphi}{dx^2}, \tag{5.1.2}$$

then, its eigenvalues and eigenfunctions are given by:

$$\lambda_k := (k\pi)^2\,, \varphi_k(x) := \sqrt{2}\sin(k\pi x)\,, \quad \forall\ k \in \mathbb{N}^*. \tag{5.1.3}$$

Beauchard and Laurent in [11] proved a local controllability result along the ground state solution $\psi_1(t) = e^{-i\lambda_k t}\varphi_1$ for system (5.1.1) (in a smoother space than the natural one for the well-posedness) provided that $\mu \in H^3(0,1)$ satisfies

$$\exists\, c > 0 \text{ such that } \left|\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx\right| \geqslant \frac{c}{\lambda_k^{3/2}} \qquad \forall k \in \mathbb{N}^*. \tag{5.1.4}$$

They also prove that this condition holds generically in $H^3(0,1)$. Observe that a necessary condition for (5.1.4) to hold is that

$$\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx \neq 0 \qquad \forall k \in \mathbb{N}^*. \tag{5.1.5}$$

On the other hand, Beauchard and Morancey in [13] proved that if condition (5.1.5) is violated for an index $k \in \mathbb{N}^*$, then there exists a minimal time such that the Schrödinger equation (5.1.1) is controllable along the ground state solution.
A further example of bilinear control problem is represented by the heat equation with a controlled potential

$$\begin{cases} \partial_t u(t,x) - \partial_x^2 u(t,x) + p(t)\mu(x)u(t,x) = 0, \quad (t,x) \in (0,T)\times(0,1), \\ u(t,0) = 0,\ u(t,1) = 0, \\ u(0,\cdot) = u_0(\cdot) \in (0,1), \end{cases} \tag{5.1.6}$$

where $p \in L^2(0,T;\mathbb{R})$ is the control and stands for the temperature and μ is an admissible potential. More generally, one can consider a parabolic control system of the form

$$\begin{cases} \partial_t u + Au + p(t)Bu = 0, \quad t \in (0,T), \\ u(0,\cdot) = u_0(\cdot), \end{cases} \tag{5.1.7}$$

where A is a self-adjoint accretive operator on a Hilbert space and $p(\cdot)$ is the control function. Alabau-Boussouira, Cannarsa and Urbani proved in [4] (see Chapter 2) a result of superexponential stabilizability to the ground state solution of (5.1.7) when $B : X \to X$ is a linear bounded operator such that there exists $\tau > 0$ for which

$$\begin{gathered} \langle B\varphi_1, \varphi_k\rangle \neq 0, \qquad \forall k \in \mathbb{N}^*, \\ \sum_{k\in\mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1,\varphi_k\rangle|^2} < +\infty. \end{gathered} \tag{5.1.8}$$

Moreover, we proved in [3] (see Chapter 3) a result of exact controllability for (5.1.7) to the ground state solution in X, under the following condition on the linear bounded operator $B : X \to X$

$$\langle B\varphi_1, \varphi_1\rangle \neq 0, \quad \text{and } \exists\ b, q > 0, \text{ such that } \quad \lambda_k^q |\langle B\varphi_1,\varphi_k\rangle| \geq b \quad \forall\, k > 1. \tag{5.1.9}$$

When B is defined as $Bu = \mu u$ for all $u \in X$ (under suitable assumptions on μ), several examples of applications of these two abstract results are also given in [4, 25, 3].
Observe that, in both [4] and [3] the weaker condition

$$\langle B\varphi_1, \varphi_k\rangle \neq 0, \qquad \forall k \in \mathbb{N}^*, \tag{5.1.10}$$

is necessary to have (5.1.8) and (5.1.9).
In practice:

- even though conditions (5.1.4), (5.1.8) and (5.1.9) are satisfied generically, it is not so easy to exhibit a large explicit class of real valued potential μ satisfying them and only few examples of suitable potentials are available in the existing literature,

- these examples also are based on the knowledge of the explicit form of the eigenvalues and eigenfunctions. However, if one changes, for instance, the boundary conditions from Dirichlet-Dirichlet to Dirichlet-Robin, the eigenvalues cannot longer be explicitly represented.

Therefore, some natural questions that raise are:

1. Is it possible to exhibit large classes of functions μ satisfying (5.1.10)?

2. Can we build a general constructive algorithm to build such functions μ?

3. Is it possible to extend Beauchard and Laurent controllability results for Schrödinger equation and Alabau-Boussouira, Cannarsa and Urbani superexponential stabilization [4] and controllability [3] results for parabolic equations, and further existing results for other equations to more general boundary conditions?

The purpose of our work has been to give positive answers to these questions and, in particular, to give a general algorithm to provide a large (infinite) class of explicit real valued potential μ satisfying (5.1.10).
Let us describe the general framework that can be considered for bilinear control systems by using spectral properties of the eigenvalues and eigenfunctions associated to the infinitesimal generator of the semigroup $-A$.
Let X be a separable complex Hilbert space equipped with a scalar product denoted by $\langle \cdot, \cdot \rangle_X$ and the corresponding norm $|| \cdot ||_X$.
Fix $T > 0$ and consider the following bilinear control problem associated to the pair (A, B)

$$\begin{cases} u\prime(t) + \beta_1 Au(t) + \beta_2 p(t)Bu(t) = 0, \quad t \in [0, T] \\ \\ u(0) = u_0, \end{cases} \tag{5.1.11}$$

where β_1, β_2 are given suitable complex numbers.
The control operator B is defined as follows. We consider real valued potentials $\mu \in Y$ where Y is a suitable subspace of X so that the multiplication operator B defined by

$$Bv = \mu v, \quad \forall \ v \in X, \tag{5.1.12}$$

is well-defined on X and such that $B \in \mathscr{L}(X)$.
Let $A : D(A) \subset X \mapsto X$ be a given unbounded linear operator acting on X. The operators that we consider for applications are differential operators. We shall denote in the sequel by A_0 the unbounded operator when we do not precise the boundary conditions to which it is associated.
For the sake of simplicity, we fix the spatial domain as $\Omega = (0, 1)$. We set $X = L^2(\Omega)$, $A = -\Delta$, that is the Laplacian operator with some admissible boundary conditions denoted by (BC) (so that the associated elliptic problem is well-posed) and B the multiplication operator defined in (5.1.12).

Observe that, taking $\beta_1 = \beta_2 = -i$, we recover from (5.1.11) the bilinear control problem (5.1.1) for the Schrödinger equation, while by choosing $\beta_1 = \beta_2 = 1$ we obtain (5.1.6) from (5.1.11).
Examples of boundary conditions (BC) that can be imposed to $u \in D(A)$ are Dirichlet-Dirichlet boundary conditions denoted by (DD), Dirichlet-Robin boundary conditions denoted by (DR), Dirichlet-Neumann boundary conditions denoted by (DN), Neumann-Neumann boundary conditions denoted by (NN), ...:

$$\begin{cases} (DD) & u(0)=0, \quad u(1)=0, \\ (DR) & u(0)=0, \quad u'(1)+u(1)=0, \\ (DN) & u(0)=0, \quad u'(1)=0, \\ (NN) & u'(0)=0, \quad u'(1)=0, \\ \quad\vdots & \end{cases} \tag{5.1.13}$$

Note that the domain of A is defined with respect to the chosen (BC) among the above ones. The method that we present is valid for different boundary conditions and allows to prove new controllability results for bilinear Schrödinger equations as well as for parabolic equations. It also gives a large variety of explicit classes of real valued potentials μ for which the existing controllability results are valid.
Our goal is thus to find and explicit algorithm to select functions μ in certain classes of functions that satisfy

$$\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx \neq 0, \qquad \forall k \in \mathbb{N}^*. \tag{5.1.14}$$

The more refined asymptotic behavior (5.1.4) required as sufficient condition in [11] for Schrödinger equation with (DD) boundary conditions, or of the form (5.1.8) and (5.1.9) for parabolic bilinear control systems in [4, 3] respectively, can be proved more easily than the above (necessary) nonvanishing condition. This will be shown in the sequel.

5.2 Main results

We present the following key result based only on the property that (φ_k, λ_k) are the eigenfunctions and eigenvalues of the Laplacian operator (without specifying the boundary conditions). Let $A_{0,Lap}$ be the second order differential operator defined by

$$A_{0,Lap} = -\frac{d^2}{dx^2}, \tag{5.2.1}$$

and let $(\varphi_k, \lambda_k)_{k\in\mathbb{N}^*}$ be any pair that solves

$$A_{0,Lap}\,\varphi_k = \lambda_k\varphi_k, ||\varphi_k||_X \neq 0. \tag{5.2.2}$$

Moreover, we define the positive constants

$$\alpha_k := 2(\lambda_1 + \lambda_k). \tag{5.2.3}$$

Theorem 5.2.1. *For any pair* (φ_k, λ_k) *that solves* (5.2.2)*, for any function* $\mu \in H^4(0,1)$ *and for any* $k \geqslant 2$ *the following relation holds*

$$(\lambda_k - \lambda_1)^2 \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = \int_0^1 T_k(\mu)(x)\varphi_1(x)\varphi_k(x)dx + B_{G,k}(\mu). \tag{5.2.4}$$

where $T_k : H^4(0,1) \mapsto L^2(0,1)$ are the linear operators defined by

$$T_k(\mu) = -\left[\mu^{(4)} + \alpha_k \mu''\right], \quad \forall \mu \in H^4(0,1), \quad \forall k \in \mathbb{N}^*, \tag{5.2.5}$$

and $B_{G,k} : H^4(0,1) \mapsto \mathbb{R}$ are the linear operators defined by

$$\begin{aligned} B_{G,k}(\mu) = \\ &-\mu''(1)[\varphi_1(1)\varphi_k'(1) + \varphi_1'(1)\varphi_k(1)] + \mu^{(3)}(1)\varphi_1(1)\varphi_k(1) + \mu''(0)[\varphi_1(0)\varphi_k'(0) + \varphi_1'(0)\varphi_k(0)] \\ &-\mu^{(3)}(0)\varphi_1(0)\varphi_k(0) + \mu'(1)[(\lambda_1 + \lambda_k)\varphi_1(1)\varphi_k(1) + 2\varphi_1'(1)\varphi_k'(1)] \\ &-\mu'(0)[(\lambda_1 + \lambda_k)\varphi_1(0)\varphi_k(0) + 2\varphi_1'(0)\varphi_k'(0)] + \mu(1)(\lambda_k - \lambda_1)[\varphi_1'(1)\varphi_k(1) - \varphi_1(1)\varphi_k'(1)] \\ &+\mu(0)(\lambda_k - \lambda_1)[\varphi_1(0)\varphi_k'(0) - \varphi_1'(0)\varphi_k(0)], \end{aligned} \tag{5.2.6}$$

for all $\mu \in H^4(0,1)$ and $k \in \mathbb{N}^*$.

Proof. Thanks to (5.2.2) we have the following identities:

$$\begin{aligned} \lambda_k \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx &= \int_0^1 (\mu'(x)\varphi_1(x) + \mu(x)\varphi_1'(x))\,\varphi_k'(x)dx - \mu(x)\varphi_1(x)\varphi_k'(x)|_0^1 \\ \lambda_1 \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx &= \int_0^1 (\mu'(x)\varphi_k(x) + \mu(x)\varphi_k'(x))\,\varphi_1'(x)dx - \mu(x)\varphi_1'(x)\varphi_k(x)|_0^1. \end{aligned} \tag{5.2.7}$$

Taking the difference between the first and the second equation in (5.2.7), we obtain

$$\begin{aligned} (\lambda_k - \lambda_1)\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = &\int_0^1 \mu'(x)[\varphi_1(x)\varphi_k'(x) - \varphi_1'(x)\varphi_k(x)]\,dx \\ &- \mu(x)\,[\varphi_1(x)\varphi_k'(x) - \varphi_1'(x)\varphi_k(x)]|_0^1. \end{aligned} \tag{5.2.8}$$

Now, recalling that $\varphi_k''(x) = -\lambda_k\varphi_k(x)$, for all $k \in \mathbb{N}^*$, we compute the following equalities

$$\begin{aligned} \lambda_k \int_0^1 \mu'(x)\varphi_1'(x)\varphi_k(x)dx &= -\int_0^1 \mu'(x)\varphi_1'(x)\varphi_k''(x)dx \\ &= \int_0^1 (\mu'\varphi_1')'(x)\varphi_k'(x)dx - \mu'(x)\varphi_1'(x)\varphi_k'(x)|_0^1 \\ &= \int_0^1 \left[\mu''(x)\varphi_1'(x) - \lambda_1\mu'(x)\varphi_1(x)\right]\varphi_k'(x)dx - \mu'(x)\varphi_1'(x)\varphi_k'(x)|_0^1. \end{aligned} \tag{5.2.9}$$

By exchanging the indices 1 and k we have that

$$\lambda_1 \int_0^1 \mu'(x)\varphi_1(x)\varphi_k'(x)dx = \int_0^1 \left[\mu''(x)\varphi_k'(x) - \lambda_k\mu'(x)\varphi_k(x)\right]\varphi_1'(x)dx - \mu'(x)\varphi_1'(x)\varphi_k'(x)|_0^1. \tag{5.2.10}$$

We consider the left-hand side of (5.2.9) and we integrate by parts:

$$\begin{aligned} \lambda_k \int_0^1 \mu'(x)\varphi_1'(x)\varphi_k(x)dx &= \lambda_k\left(\mu'(x)\varphi_1(x)\varphi_k(x)|_0^1 - \int_0^1 (\mu'\varphi_k)'(x)\varphi_1(x)dx\right) \\ &= \lambda_k\left(\mu'(x)\varphi_1(x)\varphi_k(x)|_0^1 - \int_0^1 \left(\mu''(x)\varphi_1(x)\varphi_k(x) + \mu'(x)\varphi_1(x)\varphi_k'(x)\right)dx\right). \end{aligned} \tag{5.2.11}$$

Using (5.2.11) inside (5.2.9), we get

$$\lambda_k \mu'(x)\varphi_1(x)\varphi_k(x)|_0^1 - \lambda_k \int_0^1 \mu''(x)\varphi_1(x)\varphi_k(x)dx - \lambda_k \int_0^1 \mu'(x)\varphi_1(x)\varphi_k'(x)dx$$
$$= \int_0^1 \mu''(x)\varphi_1'(x)\varphi_k'(x)dx - \lambda_1 \int_0^1 \mu'(x)\varphi_1(x)\varphi_k'(x) - \mu'(x)\varphi_1'(x)\varphi_k'(x)|_0^1 \tag{5.2.12}$$

and therefore, recasting the terms (5.2.12) becomes

$$(\lambda_k - \lambda_1) \int_0^1 \mu'(x)\varphi_1(x)\varphi_k'(x)dx = -\int_0^1 \mu''(x)\varphi_1'(x)\varphi_k'(x)dx - \lambda_k \int_0^1 \mu''(x)\varphi_1(x)\varphi_k(x)dx$$
$$+ \mu'(x)\varphi_1'(x)\varphi_k'(x)|_0^1 + \lambda_k \mu'(x)\varphi_1(x)\varphi_k(x)|_0^1. \tag{5.2.13}$$

By exchanging the indices 1 and k in (5.2.13), we obtain

$$-(\lambda_k - \lambda_1) \int_0^1 \mu'(x)\varphi_1'(x)\varphi_k(x)dx = -\int_0^1 \mu''(x)\varphi_1'(x)\varphi_k'(x)dx - \lambda_1 \int_0^1 \mu''(x)\varphi_1(x)\varphi_k(x)dx$$
$$+ \mu'(x)\varphi_1'(x)\varphi_k'(x)|_0^1 + \lambda_1 \mu'(x)\varphi_1(x)\varphi_k(x)|_0^1. \tag{5.2.14}$$

Adding (5.2.13) to (5.2.14), we get

$$(\lambda_k - \lambda_1) \int_0^1 \mu'(x)\left(\varphi_1(x)\varphi_k'(x) - \varphi_1'(x)\varphi_k(x)\right)dx = -2\int_0^1 \mu''(x)\varphi_1'(x)\varphi_k'(x)dx$$
$$-(\lambda_k + \lambda_1) \int_0^1 \mu''(x)\varphi_1(x)\varphi_k(x)dx$$
$$+ (\lambda_1 + \lambda_k)\mu'(x)\varphi_1(x)\varphi_k(x)|_0^1$$
$$+ 2\mu'(x)\varphi_1'(x)\varphi_k'(x)|_0^1, \tag{5.2.15}$$

and using (5.2.15) in (5.2.8), we obtain

$$(\lambda_k - \lambda_1)^2 \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = -2\int_0^1 \mu''(x)\varphi_1'(x)\varphi_k'(x)dx - (\lambda_k + \lambda_1) \int_0^1 \mu''(x)\varphi_1(x)\varphi_k(x)dx$$
$$+ (\lambda_1 + \lambda_k)\mu'(x)\varphi_1(x)\varphi_k(x)|_0^1 + 2\mu'(x)\varphi_1'(x)\varphi_k'(x)|_0^1$$
$$-(\lambda_k - \lambda_1)\mu(x)\left[\varphi_1(x)\varphi_k'(x) - \varphi_1'(x)\varphi_k(x)\right]|_0^1. \tag{5.2.16}$$

On the other hand, we have that

$$2\varphi_1'(x)\varphi_k'(x) = (\varphi_1\varphi_k)''(x) + (\lambda_1 + \lambda_k)\varphi_1(x)\varphi_k(x). \tag{5.2.17}$$

We use expression (5.2.17) in the first term on the right-hand side of (5.2.16) and we get that

$$
\begin{aligned}
2\int_0^1 \mu''(x)\varphi_1'(x)\varphi_k'(x)dx &= \int_0^1 \mu''(x)(\varphi_1\varphi_k)''(x)dx + (\lambda_1+\lambda_k)\int_0^1 \mu''(x)\varphi_1(x)\varphi_k(x)dx \\
&= \mu''(x)(\varphi_1\varphi_k)'(x)\Big|_0^1 - \int_0^1 \mu'''(x)(\varphi_1\varphi_k)'(x)dx \\
&\quad + (\lambda_1+\lambda_k)\int_0^1 \mu''(x)\varphi_1(x)\varphi_k(x)dx \\
&= \mu''(x)(\varphi_1\varphi_k)'(x)\Big|_0^1 - \mu'''(x)\varphi_1(x)\varphi_k(x)\Big|_0^1 \\
&\quad + \int_0^1 \mu^{(4)}(x)\varphi_1(x)\varphi_k(x)dx + (\lambda_1+\lambda_k)\int_0^1 \mu''(x)\varphi_1(x)\varphi_k(x)dx.
\end{aligned}
\tag{5.2.18}
$$

Thus, thanks to (5.2.18), (5.2.16) becomes

$$
\begin{aligned}
(\lambda_k-\lambda_1)^2\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = &-\int_0^1 \mu^{(4)}(x)\varphi_1(x)\varphi_k(x)dx - 2(\lambda_1+\lambda_k)\int_0^1 \mu''(x)\varphi_1(x)\varphi_k(x)dx \\
&- \mu''(x)(\varphi_1\varphi_k)'(x)\Big|_0^1 + \mu'''(x)\varphi_1(x)\varphi_k(x)\Big|_0^1 \\
&+ (\lambda_1+\lambda_k)\mu'(x)\varphi_1(x)\varphi_k(x)|_0^1 + 2\mu'(x)\varphi_1'(x)\varphi_k'(x)|_0^1 \\
&- (\lambda_k-\lambda_1)\mu(x)\left[\varphi_1(x)\varphi_k'(x) - \varphi_1'(x)\varphi_k(x)\right]|_0^1
\end{aligned}
\tag{5.2.19}
$$

that can be rewritten as (5.2.4). □

In the following Theorem we derive a relation to express $\langle \mu\varphi_1, \varphi_k\rangle$ through higher order derivative of μ. For this, we introduce for any $n \in \mathbb{N}^*$ and any $k \geqslant 2$, the following inductive boundary linear operator $\mathscr{R}_{G,k,n}$ on $H^{4n}(0,1)$ as follows

Definition 5.2.2. *Let* $(\varphi_k, \lambda_k)_{k\in\mathbb{N}^*}$ *be any pair that solves* (5.2.2) *and let* α_k *be the constant in* (5.2.3). *We define, for any* $n \in \mathbb{N}^*$ *and any* $k \geqslant 2$*,*

$$
\mathscr{R}_{G,k,n}(\mu) = \sum_{p=0}^{n-1}(\lambda_k-\lambda_1)^{2p}(-1)^{n-p-1}\sum_{l=0}^{n-p-1} C_{n-p-1}^{l}\alpha_k^l\left[B_{G,k}(\mu^{(4(n-p-1)-2l)})\right], \mu \in H^{4n}(0,1).
\tag{5.2.20}
$$

Remark 5.2.3. *Note that this definition is valid for any boundary conditions that can be associated to the Laplacian operator* $A_{0,Lap}$ *(in such a way that the corresponding elliptic problem is well-posed in a suitable Sobolev space for its corresponding variational form).*

Remark 5.2.4. *Observe that the operator* $\mathscr{R}_{G,k,n}$ *involves only derivatives of* μ *(as well as derivatives of the eigenfunctions* φ_k *and* φ_1*) at the boundaries of the interval* $(0,1)$*. This is the reason why we called it* boundary operator *with respect to* μ*.*

We denote by $\lceil\cdot\rceil : \mathbb{R} \to \mathbb{Z}$ the function that associates to every real number x the smallest integer greater or equal to x.

Theorem 5.2.5. *Let* $(\varphi_k, \lambda_k)_{k\in\mathbb{N}^*}$ *be any pair that solves* (5.2.2) *and* α_k *be defined by* (5.2.3). *For any function* $\mu \in H^{4n}(0,1)$*, for any* $n \in \mathbb{N}^*$ *and any* $k \geqslant 2$*, we have the following inductive formulas:*

(i)

$$(\lambda_k-\lambda_1)^{2n}\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = \int_0^1 T_k^n(\mu)(x)\varphi_1(x)\varphi_k(x)dx + \sum_{p=0}^{n-1}(\lambda_k-\lambda_1)^{2p}\left[B_{G,k}(T_k^{n-p-1}(\mu))\right], \tag{5.2.21}$$

where $T_k^n(\mu) = \underbrace{(T_k\circ\cdots\circ T_k)}_{n}(\mu)$ *and* $T_k^0(\mu) = Id$,

(ii)

$$T_k^p(\mu) = (-1)^p\sum_{l=0}^{p} C_p^l \alpha_k^l \mu^{(4p-2l)}, \qquad \forall\, p\in\mathbb{N}^*, \tag{5.2.22}$$

where the notation C_p^l *stands for the binomial coefficient* $C_p^l = \binom{p}{l} = \frac{p!}{l!(p-l)!}$, *for all* $0\leq l\leq p$,

(iii)

$$(\lambda_k-\lambda_1)^{2n}\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = (-1)^n\sum_{l=0}^{n} C_n^l\alpha_k^l\int_0^1 \mu^{(4n-2l)}(x)\varphi_1(x)\varphi_k(x)dx + \mathscr{R}_{G,k,n}(\mu), \tag{5.2.23}$$

where for any function $\mu\in H^{4n}(0,1)$, *any* $n\in\mathbb{N}^*$ *and any* $k\geqslant 2$, *the following identity*

$$\mathscr{R}_{G,k,n}(\mu) = \sum_{r=0}^{2(n-1)} B_{G,k}(\mu^{(2r)})\sum_{j=\lceil \frac{r}{2}\rceil}^{\min(r,n-1)}(-1)^j C_j^{r-j}\alpha_k^{2j-r}(\lambda_k-\lambda_1)^{2(n-j-1)} \tag{5.2.24}$$

holds.

Proof. (i) We proceed by induction on the index $n\in\mathbb{N}^*$. For $n=1$, we have proved the validity of formula (5.2.4) in Theorem 5.2.1. Suppose (5.2.21) holds true up to index n. We shall prove that it also holds for $n+1$.

Consider equation (5.2.21) and multiply it by $(\lambda_k-\lambda_1)^2$:

$$(\lambda_k-\lambda_1)^{2(n+1)}\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = (\lambda_k-\lambda_1)^2\int_0^1 T_k^n(\mu)(x)\varphi_1(x)\varphi_k(x)dx + \sum_{p=0}^{n-1}(\lambda_k-\lambda_1)^{2(p+1)}\left[B_{G,k}(T_k^{n-p-1}(\mu))\right].$$

Then, for the integral term on the right-hand side, we use the identity (5.2.4) applied to the function $T_k^n(\mu)$ and we obtain

$$
\begin{aligned}
(\lambda_k-\lambda_1)^{2(n+1)}\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx &= \int_0^1 T_k(T_k^n(\mu))(x)\varphi_1(x)\varphi_k(x)dx + B_{G,k}(T_k^n(\mu)) \\
&\quad +\sum_{p=0}^{n-1}(\lambda_k-\lambda_1)^{2(p+1)}\left[B_{G,k}(T_k^{n-p-1}(\mu))\right] \\
&= \int_0^1 T_k^{n+1}(\mu)(x)\varphi_1(x)\varphi_k(x)dx + B_{G,k}(T_k^n(\mu)) \\
&\quad +\sum_{p=1}^{n}(\lambda_k-\lambda_1)^{2p}\left[B_{G,k}(T_k^{n-p}(\mu))\right] \\
&= \int_0^1 T_k^{n+1}(\mu)(x)\varphi_1(x)\varphi_k(x)dx \\
&\quad +\sum_{p=0}^{n}(\lambda_k-\lambda_1)^{2p}\left[B_{G,k}(T_k^{(n+1)-p-1}(\mu))\right],
\end{aligned}
$$

which is exactly equation (5.2.21) with index $n+1$. This concludes the induction argument.

(ii) To prove formula (5.2.22), we use again an induction argument on the index $p \in \mathbb{N}^*$. If $p=1$ we have

$$
\begin{aligned}
T_k(\mu) &= (-1)\sum_{l=0}^{1} C_1^l \alpha_k^l \mu^{(4-2l)} \\
&= -\mu^{(4)} - \alpha_k \mu''
\end{aligned}
\tag{5.2.25}
$$

that is the definition of the operator T_k. We suppose that (5.2.22) holds till the index p and we prove it for $p+1$:

$$
\begin{aligned}
T_k(T_k^p(\mu)) &= T_k\left((-1)^p\sum_{l=0}^{p} C_p^l \alpha_k^l \mu^{(4p-2l)}\right) \\
&= (-1)^p\sum_{l=0}^{p} C_p^l \alpha_k^l T_k(\mu^{(4p-2l)}) \\
&= (-1)^p\sum_{l=0}^{p} C_p^l \alpha_k^l (-\mu^{(4p-2l+4)} - \alpha_k\mu^{(4p-2l+2)}) \\
&= (-1)^{p+1}\sum_{l=0}^{p+1} C_{p+1}^l \alpha_k^l \mu^{(4(p+1)-2l)},
\end{aligned}
\tag{5.2.26}
$$

where we used the relation for the binomial coefficients $C_p^l + C_p^{l-1} = C_{p+1}^l$. Hence (5.2.22) also holds for $p+1$. This concludes the induction argument to prove (5.2.22).

We deduce the inductive formula (5.2.23) using (5.2.22) in (5.2.21) together with the linearity of the operators T_k^n and $B_{G,k}$.

(iii) We now prove (5.2.24) as follows. We make the change of index $r = 2(n-p-1)-l$ in the second sum of (5.2.20) defining $\mathscr{R}_{G,k,n}(\mu)$. This gives

$$
\mathscr{R}_{G,k,n}(\mu) := \sum_{p=0}^{n-1}(\lambda_k-\lambda_1)^{2p}(-1)^{n-p-1}\sum_{r=n-p-1}^{2(n-p-1)} C_{n-p-1}^{2(n-p-1)-r}\alpha_k^{2(n-p-1)-r} B_{G,k}(\mu^{(2r)}).
$$

Then, we perform another change of indices, replacing p by $j = n - p - 1$. We obtain

$$\mathscr{R}_{G,k,n}(\mu) := \sum_{j=0}^{n-1} (\lambda_k - \lambda_1)^{2(n-j-1)} (-1)^j \sum_{r=j}^{2j} C_j^{2j-r} \alpha_k^{2j-r} B_{G,k}(\mu^{(2r)}). \tag{5.2.27}$$

We can rewrite (5.2.27) as

$$\mathscr{R}_{G,k,n}(\mu) = \sum_{j=0}^{n-1} \beta_{k,j} \sum_{r=j}^{2j} \gamma_{k,j,r} \sigma_{k,r}, \tag{5.2.28}$$

with

$$\beta_{k,j} = (\lambda_k - \lambda_1)^{2(n-j-1)} (-1)^j$$

$$\gamma_{k,r,j} = C_j^{2j-r} \alpha_k^{2j-r}$$

$$\sigma_{k,r} = B_{G,k}(\mu^{(2r)}).$$

Therefore, we have

$$\begin{aligned}
\sum_{j=0}^{n-1} \beta_{k,j} \sum_{r=j}^{2j} \gamma_{k,j,r} \sigma_{k,r} &= \sum_{j=0}^{n-1} \sum_{r=j}^{2j} \beta_{k,j} \gamma_{k,j,r} \sigma_{k,r} \\
&= \sum_{r=0}^{2(n-1)} \sum_{j=\lceil \frac{r}{2} \rceil}^{\min(r,n-1)} \beta_{k,j} \gamma_{k,j,r} \sigma_{k,r} \\
&= \sum_{r=0}^{2(n-1)} \sigma_{k,r} \sum_{j=\lceil \frac{r}{2} \rceil}^{\min(r,n-1)} \gamma_{k,j,r} \beta_{k,j}.
\end{aligned}$$

We conclude thanks to the equality $C_j^{2j-r} = C_j^{r-j}$:

$$C_j^{2j-r} = \frac{j!}{(2j-r)!(j-r)!}, \qquad C_j^{r-j} = \frac{j!}{(r-j)!(2j-r)!}.$$

□

From now on we denote by $\mathscr{P}_q(\mathbb{R})$ the space of real valued polynomials of degree q on $\mathbb{R}$, and by $\mathscr{P}(\mathbb{R})$ the space of real valued polynomials on $\mathbb{R}$.

Definition 5.2.6. *We define the function $R_r(\cdot) : \mathbb{R} \to \mathbb{R}$ as*

$$R_r(\alpha) := \sum_{j=\lceil \frac{r}{2} \rceil}^{r} (-1)^j \alpha^j C_j^{r-j}, \quad \forall\, \alpha \in \mathbb{R}. \tag{5.2.29}$$

Corollary 5.2.7. *Let $\mu \in \mathscr{P}_q(\mathbb{R})$ and n be such that $2n > q$. Then, for any $k \geqslant 2$, the following inductive formula holds*

$$(\lambda_k - \lambda_1)^{2n} \int_0^1 \mu(x) \varphi_1(x) \varphi_k(x) dx = \sum_{r=0}^{n-1} B_{G,k}(\mu^{(2r)}) \frac{(\lambda_k - \lambda_1)^{2(n-1)}}{\alpha_k^r} R_r \left(\left(\frac{\alpha_k}{\lambda_k - \lambda_1} \right)^2 \right) \tag{5.2.30}$$

where (φ_k, λ_k), for all $k \in \mathbb{N}^*$, are the eigenfunctions and eigenvalues associated to $A_{0,Lap}$ and α_k is given by (5.2.3)*. Moreover, the following inequality holds*

$$\left(\frac{\alpha_k}{\lambda_k - \lambda_1} \right)^2 > 4, \quad \forall\, k \geqslant 2. \tag{5.2.31}$$

Proof. We use the inductive formula (5.2.23) for $n > 2q$. Looking at the first term on the right-hand side of (5.2.23), we observe that the involved derivative of μ with the lowest order is $\mu^{(2n)}$. Thus, since μ is a polynomial of degree q, with $q < 2n$, then all the terms of $\sum_{l=0}^{n} \int_0^1 \mu^{(4n-2l)}(x)\varphi_1(x)\varphi_k(x)dx$ vanish. The second term on the right hand side is $\mathscr{R}_{G,k,n}(\mu)$ which is given by (5.2.24). Thus for $r \geqslant n$, we have $2r \geqslant 2n > q$, so that $\mu^{(2r)} \equiv 0$ and $B_{G,k}(\mu^{(2r)}) = 0$ for all $r \in [n, 2(n-1)]$. Using this property in (5.2.24), we have

$$\mathscr{R}_{G,k,n}(\mu) = \sum_{r=0}^{n-1} B_{G,k}(\mu^{(2r)}) \sum_{j=\lceil \frac{r}{2} \rceil}^{r} (-1)^j C_j^{r-j} \alpha_k^{2j-r} (\lambda_k - \lambda_1)^{2(n-j-1)}. \tag{5.2.32}$$

Therefore, we can easily conclude by using the definition of R_r and α_k. □

We concentrate now on the operator $R_r(\alpha)$, with $\alpha > 4$ arbitrary. We shall prove that it has a precise sign depending on the parity of r.
If r is odd, $r = 2m+1$, $m \geq 0$, we have

$$R_{2m+1}(\alpha) = \sum_{j=m+1}^{2m+1} (-\alpha)^j C_j^{2m+1-j},$$

and we perform the following change of variable $s = j - (m+1)$ obtaining

$$R_{2m+1}(\alpha) = \sum_{s=0}^{m} (-\alpha)^{s+m+1} C_{s+m+1}^{m-s}. \tag{5.2.33}$$

If r is even, $r = 2m$, $m \geq 0$, the expression of R_r is

$$R_{2m}(\alpha) = \sum_{j=m}^{2m} (-\alpha)^j C_j^{2m-j},$$

and if we introduce the variable $s = j - m$, we get

$$R_{2m}(\alpha) = \sum_{s=0}^{m} (-\alpha)^{s+m} C_{s+m}^{m-s}. \tag{5.2.34}$$

In the following Lemma we prove a relation between three consecutive elements of the sequence $\{R_r(\alpha)\}_r$, where $\alpha > 0$ is fixed.

Lemma 5.2.8. *Let $\alpha > 0$. Then, the sequence $\{R_r(\alpha)\}_r$, with $R_r(\cdot)$ defined by* (5.2.29)*, satisfies*

$$R_{2m}(\alpha) = -\alpha \left(R_{2m-2}(\alpha) + R_{2m-1}(\alpha) \right), \tag{5.2.35}$$

and

$$R_{2m+1}(\alpha) = -\alpha (R_{2m-1}(\alpha) + R_{2m}(\alpha)) \tag{5.2.36}$$

for any $m \geq 1$.

Proof. We can compute the expression of R_{2m+2} using (5.2.34):

$$R_{2m+2}(\alpha) = (-\alpha)^{m+1} \sum_{s=0}^{m+1} (-\alpha)^s C_{m+1+s}^{m+1-s}. \tag{5.2.37}$$

Moreover using in (5.2.37) Pascal's formula for the binomial coefficients, that is: $C_n^k + C_n^{k+1} = C_{n+1}^{k+1}$ for all $n \geq 0$ and for all $k = 0, \dots, n-1$, we have

$$
\begin{aligned}
R_{2m+2}(\alpha) &= (-\alpha)^{m+1}\left[1 + \sum_{s=1}^{m}(-\alpha)^s C_{m+1+s}^{m+1-s} + (-\alpha)^{m+1}\right] \\
&= (-\alpha)^{m+1}\left[1 + \sum_{s=1}^{m}(-\alpha)^s (C_{m+s}^{m-s} + C_{m+s}^{m-s+1}) + (-\alpha)^{m+1}\right] \\
&= \left[(-\alpha)^{m+1} + (-\alpha)^{m+1}\sum_{s=1}^{m}(-\alpha)^s C_{m+s}^{m-s}\right] + (-\alpha)^{m+1}\left[\sum_{s=1}^{m}(-\alpha)^s C_{m+s}^{m-s+1} + (-\alpha)^{m+1}\right] \\
&= (-\alpha)^{m+1}\sum_{s=0}^{m}(-\alpha)^s C_{m+s}^{m-s} + (-\alpha)^{m+1}\left[\sum_{s=0}^{m-1}(-\alpha)^{s+1} C_{m+s+1}^{m-s} + (-\alpha)^{m+1}\right] \\
&= (-\alpha)^{m+1}\sum_{s=0}^{m}(-\alpha)^s C_{m+s}^{m-s} + (-\alpha)^{m+2}\sum_{s=0}^{m}(-\alpha)^s C_{m+s+1}^{m-s} \\
&= -\alpha R_{2m}(\alpha) - \alpha R_{2m+1}(\alpha),
\end{aligned}
$$

and therefore

$$R_{2m+2}(\alpha) = -\alpha(R_{2m}(\alpha) + R_{2m+1}(\alpha)).$$

Thus, since we have prove the relation for R_r when r is even, we can recover formula (5.2.35)

$$R_{2m}(\alpha) = -\alpha\left(R_{2m-2}(\alpha) + R_{2m-1}(\alpha)\right).$$

Let us show what relation R_r satisfies for r odd:

$$
\begin{aligned}
R_{2m+1}(\alpha) &= (-\alpha)^{m+1}\sum_{s=0}^{m}(-\alpha)^s C_{m+s+1}^{m-s} \\
&= (-\alpha)^{m+1}\left[\sum_{s=0}^{m-1}(-\alpha)^s C_{m+s+1}^{m-s} + (-\alpha)^m\right] \\
&= (-\alpha)^{m+1}\left[\sum_{s=0}^{m-1}(-\alpha)^s \left(C_{m+s}^{m-s-1} + C_{m+s}^{m-s}\right) + (-\alpha)^m\right] \\
&= (-\alpha)^{m+1}\sum_{s=0}^{m-1}(-\alpha)^s C_{m+s}^{m-s-1} + (-\alpha)^{m+1}\left[\sum_{s=0}^{m-1}(-\alpha)^s C_{m+s}^{m-s} + (-\alpha)^m\right] \\
&= -\alpha R_{2m-1}(\alpha) - \alpha R_{2m}(\alpha)
\end{aligned}
$$

that implies formula (5.2.36). □

Lemma 5.2.9. *Let $R_r(\cdot)$ be defined by* (5.2.29)*. Then, it holds that*

$$R_{2m}(\alpha) > 0, \qquad \forall\, \alpha > 4,\ \forall\, m \geq 0, \tag{5.2.38}$$

$$R_{2m+1}(\alpha) < 0, \qquad \forall\, \alpha > 4,\ \forall\, m \geq 0. \tag{5.2.39}$$

Proof. We define the following vectors

$$U_m(\alpha) = \begin{pmatrix} R_{2m}(\alpha) \\ R_{2m+1}(\alpha) \end{pmatrix}, \qquad U_{m-1}(\alpha) = \begin{pmatrix} R_{2m-2}(\alpha) \\ R_{2m-1}(\alpha) \end{pmatrix} \tag{5.2.40}$$

for all $m \geq 1$, and thanks to (5.2.35) and (5.2.36) we can write the following relation

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} R_{2m}(\alpha) \\ R_{2m+1}(\alpha) \end{pmatrix} = -\alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_{2m-2}(\alpha) \\ R_{2m-1}(\alpha) \end{pmatrix}. \tag{5.2.41}$$

We set

$$M_\alpha := \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \qquad N := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then, (5.2.41) becomes

$$M_\alpha U_m(\alpha) = -\alpha N U_{m-1}(\alpha), \qquad \forall m \geq 1. \tag{5.2.42}$$

Observe that $\det(M_\alpha) = 1$ for all α and therefore M_α is invertible and we can rewrite (5.2.42) as follows

$$U_m(\alpha) = -\alpha M_\alpha^{-1} N U_{m-1}(\alpha). \tag{5.2.43}$$

Let us set $A_\alpha := -\alpha \hat{C}_\alpha$ with $\hat{C}_\alpha := M_\alpha^{-1} N = \begin{pmatrix} 1 & 1 \\ -\alpha & 1-\alpha \end{pmatrix}$. Hence, we have $U_m(\alpha) = A_\alpha U_{m-1}(\alpha)$ for all $m \geq 1$ and it holds that

$$U_m(\alpha) = A_\alpha U_{m-1}(\alpha) = A_\alpha(A_\alpha U_{m-2}(\alpha)) = \cdots = A_\alpha^m U_0(\alpha) = (-\alpha)^m \hat{C}_\alpha^m U_0(\alpha) \tag{5.2.44}$$

where by definition of $R_r(\alpha)$, we have

$$U_0(\alpha) = \begin{pmatrix} R_0(\alpha) \\ R_1(\alpha) \end{pmatrix} = \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} \tag{5.2.45}$$

To compute $\hat{C}_\alpha^m$ for any $m \in \mathbb{N}$, we prove below that we can diagonale the matrix $\hat{C}_\alpha$. For this, we look for its eigenvalues. Its characteristic polynomial Π is given by $\Pi(\lambda) = \lambda^2 + (\alpha - 2)\lambda + 1$ for all $\lambda \in \mathbb{R}$. The eigenvalues of $\hat{C}_\alpha$ are real and distinct if and only if

$$\theta_\alpha := \alpha(\alpha - 4)$$

is such that $\theta_\alpha > 0$. This holds in particular for all $\alpha > 4$.
We recall that, for our analysis, we are interested in studying $R_r(\alpha)$ for $\alpha = \left(\frac{\alpha_k}{\lambda_k - \lambda_1}\right)^2$ for all $k \geq 2$, and where the coefficients α_k are given by (5.2.3). Thanks to the inequality (5.2.31) proved in Corollary 5.2.7, we have $\left(\frac{\alpha_k}{\lambda_k - \lambda_1}\right)^2 > 4$ for all $k \geq 2$. This implies that we only need to consider the matrices $\hat{C}_\alpha$ for parameters $\alpha > 4$. For all these cases, the matrices $\hat{C}_\alpha$ have thus always two distinct real eigenvalues given by

$$\lambda_{\hat{C}_\alpha}^+ = \frac{2 - \alpha + \sqrt{\theta_\alpha}}{2}, \qquad \lambda_{\hat{C}_\alpha}^- = \frac{2 - \alpha - \sqrt{\theta_\alpha}}{2} \tag{5.2.46}$$

and corresponding eigenvectors given by

$$f_{\hat{C}_\alpha}^+ = \begin{pmatrix} 1 \\ \lambda_{\hat{C}_\alpha}^+ - 1 \end{pmatrix}, \qquad f_{\hat{C}_\alpha}^- = \begin{pmatrix} 1 \\ \lambda_{\hat{C}_\alpha}^- - 1 \end{pmatrix}. \tag{5.2.47}$$

The diagonalization of the matrix $\hat{C}_\alpha$ is given by $\hat{C}_\alpha = Q_\alpha D_\alpha Q_\alpha^{-1}$ where D_α is the diagonal matrix formed by the eigenvalues, and Q_α is the matrix of the eigenvectors $f_{\hat{C}_\alpha}^+$ and $f_{\hat{C}_\alpha}^-$, so that

$$D_\alpha = \begin{pmatrix} \lambda^+_{\hat{C}_\alpha} & 0 \\ 0 & \lambda^-_{\hat{C}_\alpha} \end{pmatrix}, \qquad Q_\alpha = \begin{pmatrix} 1 & 1 \\ \lambda^+_{\hat{C}_\alpha} - 1 & \lambda^-_{\hat{C}_\alpha} - 1 \end{pmatrix},$$

We can easily get that

$$Q_\alpha^{-1} = -\frac{1}{\sqrt{\theta_\alpha}} \begin{pmatrix} \lambda^-_{\hat{C}_\alpha} - 1 & -1 \\ -(\lambda^+_{\hat{C}_\alpha} - 1) & 1 \end{pmatrix}$$

Thus, we have for any $m \in \mathbb{N}$ $\hat{C}_\alpha^m = Q_\alpha D_\alpha^m Q_\alpha^{-1}$. This implies that for or any $m \in \mathbb{N}$, we have

$$\hat{C}_\alpha^m = \frac{1}{\sqrt{\theta_\alpha}} \begin{pmatrix} (\lambda^+_{\hat{C}_\alpha})^m(1-\lambda^-_{\hat{C}_\alpha}) + (\lambda^-_{\hat{C}_\alpha})^m(\lambda^+_{\hat{C}_\alpha} - 1) & (\lambda^+_{\hat{C}_\alpha})^m - (\lambda^-_{\hat{C}_\alpha})^m \\ -(\lambda^+_{\hat{C}_\alpha} - 1)(\lambda^-_{\hat{C}_\alpha} - 1)((\lambda^+_{\hat{C}_\alpha})^m - (\lambda^-_{\hat{C}_\alpha})^m) & (\lambda^+_{\hat{C}_\alpha})^m(\lambda^+_{\hat{C}_\alpha} - 1) - (\lambda^-_{\hat{C}_\alpha})^m(\lambda^-_{\hat{C}_\alpha} - 1) \end{pmatrix}. \tag{5.2.48}$$

It is possible to simplify the above expression of $\hat{C}_\alpha^m$. Observing that

$$\lambda^+_{\hat{C}_\alpha} - 1 = \frac{-\alpha + \sqrt{\theta_\alpha}}{2}, \qquad \lambda^-_{\hat{C}_\alpha} - 1 = \frac{-\alpha - \sqrt{\theta_\alpha}}{2}$$

we have $(\lambda^+_{\hat{C}_\alpha} - 1)(\lambda^-_{\hat{C}_\alpha} - 1) = \alpha$. Moreover, we also have

$$\lambda^+_{\hat{C}_\alpha} \lambda^-_{\hat{C}_\alpha} = 1.$$

Thus, we obtain

$$\begin{aligned} (\lambda^-_{\hat{C}_\alpha})^m(\lambda^+_{\hat{C}_\alpha} - 1) - (\lambda^+_{\hat{C}_\alpha})^m(\lambda^-_{\hat{C}_\alpha} - 1) &= (\lambda^-_{\hat{C}_\alpha})^{m-1}(\lambda^-_{\hat{C}_\alpha}\lambda^+_{\hat{C}_\alpha} - \lambda^-_{\hat{C}_\alpha}) - (\lambda^+_{\hat{C}_\alpha})^{m-1}(\lambda^+_{\hat{C}_\alpha}\lambda^-_{\hat{C}_\alpha} - \lambda^+_{\hat{C}_\alpha}) \\ &= (\lambda^-_{\hat{C}_\alpha})^{m-1}(1-\lambda^-_{\hat{C}_\alpha}) - (\lambda^+_{\hat{C}_\alpha})^{m-1}(1-\lambda^+_{\hat{C}_\alpha}). \end{aligned}$$

Thus, (5.2.48) is equivalent to

$$\hat{C}_\alpha^m = \frac{-1}{\sqrt{\theta_\alpha}} \begin{pmatrix} (\lambda^+_{\hat{C}_\alpha})^{m-1}(1-\lambda^+_{\hat{C}_\alpha}) - (\lambda^-_{\hat{C}_\alpha})^{m-1}(1-\lambda^-_{\hat{C}_\alpha}) & (\lambda^-_{\hat{C}_\alpha})^m - (\lambda^+_{\hat{C}_\alpha})^m \\ \alpha((\lambda^+_{\hat{C}_\alpha})^m - (\lambda^-_{\hat{C}_\alpha})^m) & (\lambda^-_{\hat{C}_\alpha})^m(\lambda^-_{\hat{C}_\alpha} - 1) - (\lambda^+_{\hat{C}_\alpha})^m(\lambda^+_{\hat{C}_\alpha} - 1) \end{pmatrix}. \tag{5.2.49}$$

So, we can rewrite $U_m(\alpha)$ as

$$\begin{aligned} U_m(\alpha) &= \frac{-(-\alpha)^m}{\sqrt{\theta_\alpha}} \hat{C}_\alpha^m \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} \\ &= \frac{-(-\alpha)^m}{\sqrt{\theta_\alpha}} \begin{pmatrix} (\lambda^+_{\hat{C}_\alpha})^{m-1}(1-\lambda^+_{\hat{C}_\alpha}) - (\lambda^-_{\hat{C}_\alpha})^{m-1}(1-\lambda^-_{\hat{C}_\alpha}) - \alpha((\lambda^-_{\hat{C}_\alpha})^m - (\lambda^+_{\hat{C}_\alpha})^m) \\ \alpha((\lambda^+_{\hat{C}_\alpha})^{m+1} - (\lambda^-_{\hat{C}_\alpha})^{m+1}) \end{pmatrix}. \end{aligned} \tag{5.2.50}$$

Furthermore, since both the eigenvalues are strictly negative, we can use that

$$\lambda^i_{\hat{C}_\alpha} = -|\lambda^i_{\hat{C}_\alpha}|, \qquad i = +, -$$

inside (5.2.50), obtaining

$$\begin{aligned} U_m(\alpha) &= \frac{-(-\alpha)^m(-1)^{m-1}}{\sqrt{\theta_\alpha}} \begin{pmatrix} |\lambda^+_{\hat{C}_\alpha}|^{m-1}(1+|\lambda^+_{\hat{C}_\alpha}|) - |\lambda^-_{\hat{C}_\alpha}|^{m-1}(1+|\lambda^-_{\hat{C}_\alpha}|) + \alpha(|\lambda^-_{\hat{C}_\alpha}|^m - |\lambda^+_{\hat{C}_\alpha}|^m) \\ \alpha(|\lambda^+_{\hat{C}_\alpha}|^{m+1} - |\lambda^-_{\hat{C}_\alpha}|^{m+1}) \end{pmatrix} \\ &= \frac{\alpha^m}{\sqrt{\theta_\alpha}} \begin{pmatrix} |\lambda^+_{\hat{C}_\alpha}|^{m-1}(1+|\lambda^+_{\hat{C}_\alpha}| - \alpha|\lambda^+_{\hat{C}_\alpha}|) - |\lambda^-_{\hat{C}_\alpha}|^{m-1}(1+|\lambda^-_{\hat{C}_\alpha}| - \alpha|\lambda^-_{\hat{C}_\alpha}|) \\ \alpha(|\lambda^+_{\hat{C}_\alpha}|^{m+1} - |\lambda^-_{\hat{C}_\alpha}|^{m+1}) \end{pmatrix}. \end{aligned} \tag{5.2.51}$$

Now, we introduce the function $f_m(x) = x^{m-1}(1 + x - \alpha x)$, $m \geq 1$, $\alpha > 4$, and we observe that (5.2.51) is equivalent to the following formula

$$U_m(\alpha) = \begin{pmatrix} R_{2m}(\alpha) \\ R_{2m+1}(\alpha) \end{pmatrix} = \frac{\alpha^m}{\sqrt{\theta_\alpha}} \begin{pmatrix} f_m(|\lambda^+_{\hat{C}_\alpha}|) - f_m(|\lambda^-_{\hat{C}_\alpha}|) \\ \alpha(|\lambda^+_{\hat{C}_\alpha}|^{m+1} - |\lambda^-_{\hat{C}_\alpha}|^{m+1}) \end{pmatrix}. \tag{5.2.52}$$

To deduce the sign of the first component of $U_m(\alpha)$, we study the variation of f_m:

$$f\prime_m(x) = (m-1)x^{m-2} + mx^{m-1}(1-\alpha).$$

Thus, f_m has a maximum at $x = \frac{m-1}{m(\alpha-1)}$. Moreover, since $0 \leq \frac{m-1}{m} < 1$ and $\alpha > 4$, we deduce that $0 \leq \frac{m-1}{m(\alpha-1)} < \frac{1}{\alpha-1}$. We claim that

$$0 \leq \frac{m-1}{m(\alpha-1)} < \frac{1}{\alpha-1} < |\lambda^+_{\hat{C}_\alpha}| < |\lambda^-_{\hat{C}_\alpha}|, \qquad \forall\, m \geq 1, \forall\, \alpha > 4. \tag{5.2.53}$$

Indeed, it is easy to check

$$\frac{1}{\alpha-1} < \frac{\alpha - 2 - \sqrt{\theta_\alpha}}{2}, \quad \forall\, \alpha > 4.$$

Moreover,

$$|\lambda^+_{\hat{C}_\alpha}| = \frac{\alpha - 2 - \sqrt{\theta_\alpha}}{2} < \frac{\alpha - 2 + \sqrt{\theta_\alpha}}{2} = |\lambda^-_{\hat{C}_\alpha}|.$$

Hence, since f_m is strictly decreasing in $\left[\frac{m-1}{m(\alpha-1)}, +\infty\right)$, and thanks to the above inequalities, we get that $f_m(|\lambda^+_{\hat{C}_\alpha}|) > f_m(|\lambda^-_{\hat{C}_\alpha}|)$. Thus, we have

$$\begin{aligned} R_{2m}(\alpha) &= \frac{\alpha^m}{\sqrt{\theta_\alpha}} \Big(f_m(|\lambda^+_{\hat{C}_\alpha}|) - f_m(|\lambda^+_{\hat{C}_\alpha}|) \Big) > 0, \quad \forall\, m \geq 1, \forall\, \alpha > 4, \\ R_{2m+1}(\alpha) &= \frac{\alpha^{m+1}}{\sqrt{\theta_\alpha}} \Big(|\lambda^+_{\hat{C}_\alpha}|^{m+1} - |\lambda^-_{\hat{C}_\alpha}|^{m+1} \Big) < 0, \quad \forall\, m \geq 1, \forall\, \alpha > 4. \end{aligned} \tag{5.2.54}$$

Note that thanks to (5.2.45), we also have $R_0(\alpha) > 0$ and $R_1(\alpha)) < 0$ for all $\alpha > 4$. □

5.3 Example of application with respect to the choice of boundary conditions: Dirichlet-Robin boundary conditions

In this section we consider the Laplacian operator $A_{0,Lap}$ associated with mixed boundary conditions of (DR) type. The eigenvalues and eigenfunctions are now precisely defined by the spectral problem

$$\begin{cases} -\varphi_k''(x) = \lambda_k \varphi_k(x), & x \in (0,1), \\ \varphi_k(0) = 0, \quad \varphi_k(1) + \varphi_k\prime(1) = 0, & \end{cases} \tag{5.3.1}$$

for all $k \in \mathbb{N}^*$, and are given by

$$\lambda_k = r_k^2, \qquad \varphi_k(x) = \eta_k \sin(r_k x), \tag{5.3.2}$$

where r_k are the positive solutions of

$$\sin r_k + r_k \cos r_k = 0, \tag{5.3.3}$$

and $\eta_k > 0$ are defined by

$$\eta_k = \sqrt{2}\frac{r_k}{\sqrt{r_k^2 + \sin^2 r_k}}, \quad \forall\, k \in \mathbb{N}^*. \tag{5.3.4}$$

Proposition 5.3.1. *The following properties hold*

$$r_k \in \left(\frac{\pi}{2} + (k-1)\pi, \frac{3\pi}{4} + (k-1)\pi\right), \forall\, k \in \mathbb{N}^*, \tag{5.3.5}$$

and

$$\sin r_k = (-1)^{k-1}|\sin r_k| = (-1)^{k-1}\frac{r_k}{\sqrt{r_k^2+1}}. \tag{5.3.6}$$

Proof. Thanks to (5.3.3), we know that $\cos r_k \neq 0$ for all $k \in \mathbb{N}^*$, thus (5.3.3) is equivalent to the equation $\tan(r_k) + r_k = 0$ for all $k \in \mathbb{N}^*$. Observing that $\tan\left(\frac{3\pi}{4} + (k-1)\pi\right) + \frac{3\pi}{4} + (k-1)\pi = -1 + \frac{3\pi}{4} + (k-1)\pi > 0$ for all $k \in \mathbb{N}^*$, we easily deduce (5.3.5). On the other hand, using once again (5.3.3), we obtain

$$\sin^2 r_k = r_k^2 \cos^2 r_k = r_k^2(1 - \sin^2 r_k),$$

so that

$$|\sin r_k| = \frac{r_k}{\sqrt{r_k^2+1}}. \tag{5.3.7}$$

We deduce (5.3.6) thanks to (5.3.5). □

We set

$$\gamma_k := \frac{1}{\sqrt{r_1^2+1}\sqrt{r_k^2+1}}, \quad \forall\, k \geqslant 1. \tag{5.3.8}$$

From Theorem 5.2.1, we deduce the following formula for the Fourier coefficients of $\mu\varphi_1$.

Corollary 5.3.2. *Let* $(\varphi_k, \lambda_k)_{k\in\mathbb{N}^*}$ *be any pair that solves* (5.3.1)*. Then, it holds that*

$$B_{G,k}(\mu) = B_k(\mu) + C_k(\mu), \quad \forall\, \mu \in H^4(0,1), \quad \forall\, k \geqslant 2, \tag{5.3.9}$$

where the linear operator T_k *is defined in* (5.2.5) *and the linear operators* $B_k : H^4(0,1) \mapsto \mathbb{R}$ *and* $C_k : H^4(0,1) \mapsto \mathbb{R}$ *are respectively defined for all* $k \geqslant 2$ *by*

$$B_k(\mu) = \eta_1\eta_k r_1 r_k (-1)^{k-1}\gamma_k\left[(\lambda_k + \lambda_1 + 2)\,\mu\prime(1) + 2\mu''(1) + \mu^{(3)}(1)\right], \quad \forall\, \mu \in H^4(0,1). \tag{5.3.10}$$

$$C_k(\mu) = -2\eta_1\eta_k r_1 r_k \mu\prime(0), \quad \forall\, \mu \in H^4(0,1). \tag{5.3.11}$$

Moreover, the following identity holds

$$(\lambda_k - \lambda_1)^2 \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = \int_0^1 T_k(\mu)(x)\varphi_1(x)\varphi_k(x)dx + B_k(\mu) + C_k(\mu). \tag{5.3.12}$$

Proof. We use identity (5.2.4) proved in Theorem 2.1. Thanks to the (DR) boundary conditions given in (5.3.1) satisfied by φ_k and the expression of the eigenfunctions (5.3.2), we easily deduce that (5.3.9) holds. By using the expression (5.3.9) for $B_{G,k}$ in (5.2.4), we obtain (5.3.12). □

Proposition 5.3.3. *For any function $\mu \in H^{4n}(0,1)$ and for any $n \in \mathbb{N}^*$, and any $k \geqslant 2$, we have*

$$(\lambda_k - \lambda_1)^{2n} \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = (-1)^n \sum_{l=0}^{n} C_n^l \alpha_k^l \int_0^1 \mu^{(4n-2l)}(x)\varphi_1(x)\varphi_k(x)dx + \eta_1\eta_k r_1 r_k D_{k,n}(\mu), \tag{5.3.13}$$

where $D_{k,n}(\cdot)$ is the linear operator defined on $H^{4n}(0,1)$ by

$$D_{k,n}(\mu) = \sum_{r=0}^{2(n-1)} \left[(-1)^{k-1}\gamma_k \left((\lambda_k + \lambda_1 + 2)\mu^{(2r+1)}(1) + 2\mu^{(2r+2)}(1) + \mu^{(2r+3)}(1)\right) - 2\mu^{(2r+1)}(0)\right] \cdot \sum_{j=\lceil \frac{r}{2} \rceil}^{\min(r,n-1)} (-1)^j C_j^{r-j} \alpha_k^{2j-r} (\lambda_k - \lambda_1)^{2(n-j-1)}, \tag{5.3.14}$$

where γ_k is defined in (5.3.8).

Proof. We use the formulas (5.2.23) and (5.2.24) of Theorem 5.2.5 together with the formulas (5.3.9), (5.3.10) and (5.3.11). This concludes the proof. ☐

From now on, we drop the argument μ in $D_{k,n}$ to shorten the notation.
We consider the even and odd terms with respect to the index k of $D_{k,n}$ and we define

$$\begin{aligned} A_{i,n} &:= -D_{2i,n} \\ &= \sum_{r=0}^{2(n-1)} \left[\gamma_{2i} \left((\lambda_{2i} + \lambda_1 + 2)\mu^{(2r+1)}(1) + 2\mu^{(2r+2)}(1) + \mu^{(2r+3)}(1)\right) + 2\mu^{(2r+1)}(0)\right] \cdot \\ &\qquad \cdot \sum_{j=\lceil \frac{r}{2} \rceil}^{\min(r,n-1)} (-1)^j C_j^{r-j} \alpha_{2i}^{2j-r} (\lambda_{2i} - \lambda_1)^{2(n-j-1)}, \end{aligned} \tag{5.3.15}$$

for all $i \geq 1$, and

$$\begin{aligned} B_{i,n} &:= D_{2i+1,n} \\ &= \sum_{r=0}^{2(n-1)} \left[\gamma_{2i+1} \left((\lambda_{2i+1} + \lambda_1 + 2)\mu^{(2r+1)}(1) + 2\mu^{(2r+2)}(1) + \mu^{(2r+3)}(1)\right) - 2\mu^{(2r+1)}(0)\right] \cdot \\ &\qquad \cdot \sum_{j=\lceil \frac{r}{2} \rceil}^{\min(r,n-1)} (-1)^j C_j^{r-j} \alpha_{2i+1}^{2j-r} (\lambda_{2i+1} - \lambda_1)^{2(n-j-1)}, \end{aligned} \tag{5.3.16}$$

for all $i \geq 1$.

Lemma 5.3.4. *For any $\mu \in \mathscr{P}_q(\mathbb{R})$, and any $n \in \mathbb{N}^*$ such that $2n > q$, it holds that*

$$\begin{aligned} A_{i,n} = \sum_{r=0}^{n-1} &\left[\gamma_{2i} \left((\lambda_{2i} + \lambda_1 + 2)\mu^{(2r+1)}(1) + 2\mu^{(2r+2)}(1) + \mu^{(2r+3)}(1)\right) + 2\mu^{(2r+1)}(0)\right] \cdot \\ &\cdot \frac{(\lambda_{2i} - \lambda_1)^{2(n-1)}}{\alpha_{2i}^r} R_r \left(\left(\frac{\alpha_{2i}}{(\lambda_{2i} - \lambda_1)}\right)^2\right), \end{aligned} \tag{5.3.17}$$

for all $i \geq 1$, and

$$B_{i,n} = \sum_{r=0}^{n-1} \left[\gamma_{2i+1} \left((\lambda_{2i+1} + \lambda_1 + 2)\mu^{(2r+1)}(1) + 2\mu^{(2r+2)}(1) + \mu^{(2r+3)}(1) \right) - 2\mu^{(2r+1)}(0) \right] \cdot \cdot \frac{(\lambda_{2i+1} - \lambda_1)^{2(n-1)}}{\alpha_{2i+1}^r} R_r \left(\left(\frac{\alpha_{2i+1}}{(\lambda_{2i+1} - \lambda_1)} \right)^2 \right). \tag{5.3.18}$$

for all $i \geq 1$.

Proof. We apply the formula (5.2.30) of Corollary 5.2.7 for $k = 2i$ and $k = 2i + 1$. □

Since we have proved in Lemma 5.2.9 that the sign of $R_r(\cdot)$ depends only on the parity of r, we are able to give sufficient conditions to build a polynomial $\mu(\cdot)$ that satisfies $\langle \mu\varphi_1, \varphi_k \rangle \neq 0$, $\forall\, k > 1$.
For any $q \in \mathbb{N}^*$, we now choose the smallest n so that the assumption $2n > q$ of Lemma 5.3.4 holds, as follows

Definition 5.3.5.

$$\begin{aligned} &\textit{if } q \textit{ is even, } q = 2l, && \textit{then we set} \quad n := \tfrac{q}{2} + 1 = l + 1, \\ &\textit{if } q \textit{ is odd, } q = 2l + 1, && \textit{then we set} \quad n := \tfrac{q-1}{2} + 1 = l + 1. \end{aligned} \tag{5.3.19}$$

This choice is now fixed in all the sequel. We set

$$\begin{aligned} a_r &:= \mu^{(2r+3)}(1) + 2\mu^{(2r+2)}(1), && r = 0, 1, \dots, l-1 \\ b_r &:= \mu^{(2r+1)}(1), && r = 0, 1, \dots, l-1 \ (r = 0, 1, \dots, l \text{ if } q \text{ is odd}), \\ c_r &:= 2\mu^{(2r+1)}(0), && r = 0, 1, \dots, l-1 \ (r = 0, 1, \dots, l \text{ if } q \text{ is odd}), \end{aligned} \tag{5.3.20}$$

and define the function

$$g_r(x) := \frac{1}{\sqrt{x+1}} \left(a_r + (x + \lambda_1 + 2) b_r \right), \quad \forall\, x \in (\lambda_1, +\infty). \tag{5.3.21}$$

Remark 5.3.6. *Note that if $q = 2l$ and defining n as in Definition 5.3.5, then it is still meaningful to write a_{n-1}, b_{n-1} and c_{n-1} (or similarly a_l, b_l and c_l) but one has in this case $a_{n-1} = b_{n-1} = c_{n-1} = 0$ (or similarly $a_l = b_l = c_l = 0$). We may use the upper bound $n-1$ (or similarly l) even in the case $q = 2l$ for shortening some statements as for instance the next Corollary.*

Corollary 5.3.7. *For any $\mu \in \mathscr{P}_q(\mathbb{R})$, choosing n with respect to q as in Definition 5.3.5, it holds that for all $i \geq 1$*

$$A_{i,n} = \sum_{r=0}^{n-1} \left[\frac{1}{\sqrt{\lambda_1 + 1}} g_r(\lambda_{2i}) + c_r \right] \frac{(\lambda_{2i} - \lambda_1)^{2(n-1)}}{\alpha_{2i}^r} R_r(\tau_{2i}), \tag{5.3.22}$$

and

$$B_{i,n} = \sum_{r=0}^{n-1} \left[\frac{1}{\sqrt{\lambda_1 + 1}} g_r(\lambda_{2i+1}) - c_r \right] \frac{(\lambda_{2i+1} - \lambda_1)^{2(n-1)}}{\alpha_{2i+1}^r} R_r(\tau_{2i+1}), \tag{5.3.23}$$

where for all $k \geq 2$

$$\tau_k := 4\left(\frac{\lambda_k + \lambda_1}{\lambda_k - \lambda_1}\right)^2, \tag{5.3.24}$$

and α_k *is given in* (5.2.3). *Moreover if* μ *is such that*

$$A_{i,n} \neq 0,\ B_{i,n} \neq 0, \qquad \forall\, i \in \mathbb{N}^*. \tag{5.3.25}$$

then

$$\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x) \neq 0, \forall\, k \geq 2. \tag{5.3.26}$$

Our aim is to give an algorithm to build an infinite set of polynomials $\mu \in \mathscr{P}_q(\mathbb{R})$ such that, choosing n as in Definition 5.3.5, we have (5.3.25). Notice that such polynomials are completely determined by the knowledge of $\mu^{(k)}$ for all $k \in \{0,\dots,q\}$, thanks to the (finite) Taylor expansion of μ at 1:

$$\mu(x) = \sum_{k=0}^{q} \frac{\mu^{(k)}(1)}{k!}(x-1)^k.$$

We shall prove that knowing the coefficients $\mu^{(k)}(1)$, $k = 1,\dots,q$ is equivalent to knowing the coefficients (a_r) and (b_r) for a suitable range of indices r (depending on the parity of q, see (5.3.19), (5.3.20)).

Remark 5.3.8. *Observe that knowing the coefficients* $\mu^{(k)}(1)$*,* $k = 1,\dots,q$ *determines the polynomial* μ *up to a constant, namely* $\mu(1)$ *is completely free at this stage.*

For the sake of clarity, let us introduce the linear operator P_0 defined on the set $\mathscr{P}(\mathbb{R})$ by

$$P_0(\mu) = \mu(1), \quad \forall\, \mu \in \mathscr{P}(\mathbb{R}). \tag{5.3.27}$$

We set

$$Q_0 = Id - P_0, \quad E_0 = P_0\mathscr{P}(\mathbb{R}), \quad F_0 = Q_0\mathscr{P}(\mathbb{R}), \tag{5.3.28}$$

and for any $q \in \mathbb{N}^*$

$$E_{0,q} = P_0\mathscr{P}_q(\mathbb{R}), \quad F_{0,q} = Q_0\mathscr{P}_q(\mathbb{R}), \tag{5.3.29}$$

where Id denotes the identity operator on $\mathscr{P}(\mathbb{R})$.

Lemma 5.3.9. *Let* $q \in \mathbb{N}^*$ *be given. We have the following properties*

(i) If q *is an even integer, that is* $q = 2l$ *with* $l \geq 1$*, then*

for all $((a_r)_{0\leq r\leq l-1},(b_r)_{0\leq r\leq l-1})$*, there exists a unique* $\mu \in F_{0,2l}$ *such that for all* $r \in \{0,\dots,l-1\}$

$$\begin{cases} a_r := \mu^{(2r+3)}(1) + 2\mu^{(2r+2)}(1), \\ \\ b_r := \mu^{(2r+1)}(1), \end{cases} \tag{5.3.30}$$

and μ *is determined by the relations*

$$\begin{cases} \mu^{(2r)}(1) = \frac{1}{2}\left(a_{r-1} - b_r\right), & \forall\ r \in \{1,\dots,l-1\}, \\ \\ \mu^{(2r+1)}(1) = b_r, & \forall\ r \in \{0,\dots,l-1\}. \end{cases} \tag{5.3.31}$$

Conversely, $\left(\mu^{(k)}(1)\right)_{k=1,\dots,2l}$ *determines uniquely the coefficients* $((a_r)_{0\leq r\leq l-1},(b_r)_{0\leq r\leq l-1})$.

(ii) If q is odd, that is $q = 2l + 1$ with $l \geq 0$, then

for all $((a_r)_{0\leq r\leq l-1}, (b_r)_{0\leq r\leq l})$, there exists a unique $\mu \in F_{0,2l+1}$ determined by the relations

$$\begin{cases} \mu^{(2r)}(1) = \dfrac{1}{2}\left(a_{r-1} - b_r\right), & \forall\ r \in \{1, \ldots, l-1\}, \\ \mu^{(2r+1)}(1) = b_r, & \forall\ r \in \{0, \ldots, l\}. \end{cases} \tag{5.3.32}$$

Conversely, $\left(\mu^{(k)}(1)\right)_{k=1,\ldots,2l+1}$ determines uniquely the coefficients $((a_r)_{0\leq r\leq l-1}, (b_r)_{0\leq r\leq l})$, thanks to (5.3.30).

Remark 5.3.10. *Note that for $q = 2l$, the second relation in* (5.3.30) *for $r = l$ is still meaningful, since it can be written as $b_l = 0$ even though we are only interested in coefficients $(b_r)_{0\leq r\leq l-1})$. This is due to the fact that μ is assumed to be a polynomial of degree $2l$ in that case. In a similar way, for $q = 2l+1$, the first relation in* (5.3.30) *for $r = l$ is still meaningful, since it can be written as $a_l = 0$ even though we are only interested in coefficients $(a_r)_{0\leq r\leq l-1})$. This is due to the fact that μ is assumed to be a polynomial of degree $2l + 1$ in that case.*

Proof. Let $q \in \mathbb{N}^*$ be even, $q = 2l$. Then, the second relation in (5.3.30) implies that

$$\mu^{(2r+1)}(1) = b_r, \forall\ r \in \{0, \ldots, l-1\}.$$

Choosing $r = l - 1$ in the first relation of (5.3.30) and since μ is a polynomial of degree $2l$, we have

$$\mu^{(2l)}(1) = \frac{1}{2} a_{l-1}.$$

Hence, changing r into $r - 1$ in the first relation of (5.3.30), we deduce that

$$\mu^{(2r)}(1) = \frac{1}{2}\left(a_{r-1} - b_r\right), \forall\ r \in \{1, \ldots, l-1\}.$$

Therefore, $\mu^{(k)}(1)$, $\forall\, k = 1, \ldots, 2l$ can be uniquely determined from the coefficients $((a_r)_{0\leq r\leq l-1}, (b_r)_{0\leq r\leq l-1})$.
Let $q \in \mathbb{N}^*$ be odd, $q = 2l + 1$. Then, the second relation in (5.3.30) implies that

$$\mu^{(2r+1)}(1) = b_r, \quad \forall\ r \in \{0, \ldots, l\}.$$

Hence, changing r into $r - 1$ in the first relation of (5.3.30), we deduce that

$$\mu^{(2r)}(1) = \frac{1}{2}\left(a_{r-1} - b_r\right), \quad \forall\ r \in \{1, \ldots, l-1\}.$$

Namely, we recover every $\mu^{(k)}(1)$, $k = 1, \ldots, 2l+1$ from the coefficients $((a_r)_{0\leq r\leq l-1}, (b_r)_{0\leq r\leq l})$. □

Thus, in the sequel, instead of working directly on the coefficients $\left(\mu^{(k)}(1)\right)_{k=1,\ldots,q}$, we shall determine sufficient conditions on $((a_r), (b_r))$ with the suitable range of indices r (depending on the parity of q), which will ensure that

$$\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx \neq 0, \quad \forall\ k \geq 2.$$

Remark 5.3.11. *We observe that the coefficients* $c_r = 2\mu^{(2r+1)}(0)$ *can be completely determined from* $(\mu^{(k)}(1))_{1\le k\le q}$ *since we have:*

$$\mu^{(k)}(0) := \sum_{i=k}^{q} \frac{\mu^{(i)}(1)}{(i-k)!}(-1)^{i-k}. \tag{5.3.33}$$

Therefore, we can explicit the coefficients c_r in terms of the coefficients $((a_r),(b_r))$ with the suitable range of indices r (depending on the parity of q) as explained in the following Proposition.

Proposition 5.3.12. *Let* $q = 2l$ *with* $l \ge 1$*, and* $((a_r)_{0\le r\le l-1},(b_r)_{0\le r\le l-1})$ *be a given set of coefficients. Then, we have*

$$c_r = \sum_{k=0}^{(l-1)-r} \left[\frac{2(k+1)}{(2k)!} b_{r+k} - \frac{1}{(2k+1)!} a_{r+k} \right], \quad r = 0,1\dots,l-1, \tag{5.3.34}$$

Let $q = 2l+1$ *with* $l \ge 0$*, and* $((a_r)_{0\le r\le l-1},(b_r)_{0\le r\le l})$ *be given. Then, we have*

$$c_r = \sum_{k=0}^{l-r} \frac{2(k+1)}{(2k)!} b_{r+k} - \sum_{k=0}^{(l-1)-r} \frac{1}{(2k+1)!} a_{r+k}, \quad r = 0,1,\dots,l, \tag{5.3.35}$$

with the convention that $\sum_{k=0}^{-1} \equiv 0$.

Proof. Let $q = 2l$ with $l \ge 1$. We multiply by 2 (5.3.33) and set $i = 2r+1$. On the left hand side of the resulting equation we have c_r, on the right hand side we separate the sum between odd and even indices, and use (5.3.31). We easily get (5.3.34). For $q = 2l+1$, we proceed in a similar way to get (5.3.35) using (5.3.32). □

From Lemma 5.2.9, we know the sign of $R_r(\tau_k)$ for all $k \ge 2$. Hence, looking at formulas (5.3.22) and (5.3.23), we find out that a sufficient condition to guarantee the non-vanishing property of $A_{i,n}, B_{i,n}$ will come from the choice of the sign of the factors in (5.3.22) and (5.3.23), respectively, where n is chosen with respect to q as in Definition 5.3.5.
Assume that either

$$\begin{cases} \frac{1}{\sqrt{\lambda_1+1}} g_r(\lambda_{2i}) + c_r \ge 0, \ \forall\, i \ge 1, \ \forall\, r \text{ even }, \ r \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q = 2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} \\ \exists\, r^* \text{ even }, \ r^* \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q = 2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} \ : \ \frac{1}{\sqrt{\lambda_1+1}} g_{r^*}(\lambda_{2i}) + c_{r^*} > 0, \ \forall\, i \ge 1 \\ \text{and} \\ \frac{1}{\sqrt{\lambda_1+1}} g_r(\lambda_{2i}) + c_r \le 0, \ \forall\, i \ge 1, \ \forall\, r \text{ odd }, \ r \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q = 2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} \\ \exists\, r^{**} \text{ odd }, \ r^{**} \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q = 2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} \ : \ \frac{1}{\sqrt{\lambda_1+1}} g_{r^{**}}(\lambda_{2i}) + c_{r^{**}} < 0, \ \forall\, i \ge 1 \end{cases} \tag{5.3.36}$$

or that

$$\begin{cases} \frac{1}{\sqrt{\lambda_1+1}} g_r(\lambda_{2i}) + c_r \leq 0, \ \forall\, i \geq 1, \ \forall\, r \text{ even }, \ r \in \begin{cases} \{0,1,\ldots,l-1\} \text{ if } q = 2l, \\ \{0,1,\ldots,l\} \text{ if } q = 2l+1, \end{cases} , \\ \exists\, r_* \text{ even }, \ r_* \in \begin{cases} \{0,1,\ldots,l-1\} \text{ if } q = 2l, \\ \{0,1,\ldots,l\} \text{ if } q = 2l+1, \end{cases} \ : \ \frac{1}{\sqrt{\lambda_1+1}} g_{r_*}(\lambda_{2i}) + c_{r_*} < 0, \ \forall\, i \geq 1 \\ \text{and} \\ \frac{1}{\sqrt{\lambda_1+1}} g_r(\lambda_{2i}) + c_r \geq 0, \ \forall\, i \geq 1, \ \forall\, r \text{ odd }, \ r \in \begin{cases} \{0,1,\ldots,l-1\} \text{ if } q = 2l, \\ \{0,1,\ldots,l\} \text{ if } q = 2l+1, \end{cases} \\ \exists\, r_{**} \text{ odd }, \ r_{**} \in \begin{cases} \{0,1,\ldots,l-1\} \text{ if } q = 2l, \\ \{0,1,\ldots,l\} \text{ if } q = 2l+1, \end{cases} \ : \ \frac{1}{\sqrt{\lambda_1+1}} g_{r_{**}}(\lambda_{2i}) + c_{r_{**}} > 0, \ \forall\, i \geq 1. \end{cases} \tag{5.3.37}$$

Defining n with respect to q as in Definition 5.3.5, then, we observe that

$$(5.3.36) \Rightarrow A_{i,n} > 0, \ \forall\, i \geq 1,$$

$$(5.3.37) \Rightarrow A_{i,n} < 0, \ \forall\, i \geq 1.$$

Furthermore, assume that either

$$\begin{cases} \frac{1}{\sqrt{\lambda_1+1}} g_r(\lambda_{2i+1}) - c_r \geq 0, \ \forall\, i \geq 1, \ \forall\, r \text{ even }, \ r \in \begin{cases} \{0,1,\ldots,l-1\} \text{ if } q = 2l, \\ \{0,1,\ldots,l\} \text{ if } q = 2l+1, \end{cases} , \\ \exists\, \hat{r}^* \text{ even }, \ \hat{r}^* \in \begin{cases} \{0,1,\ldots,l-1\} \text{ if } q = 2l, \\ \{0,1,\ldots,l\} \text{ if } q = 2l+1, \end{cases} \ : \ \frac{1}{\sqrt{\lambda_1+1}} g_{\hat{r}^*}(\lambda_{2i+1}) - c_{\hat{r}^*} > 0, \ \forall\, i \geq 1 \\ \text{and} \\ \frac{1}{\sqrt{\lambda_1+1}} g_r(\lambda_{2i+1}) - c_r \leq 0, \ \forall\, i \geq 1, \ \forall\, r \text{ odd }, \ r \in \begin{cases} \{0,1,\ldots,l-1\} \text{ if } q = 2l, \\ \{0,1,\ldots,l\} \text{ if } q = 2l+1, \end{cases} , \\ \exists\, \hat{r}^{**} \text{ odd }, \ \hat{r}^{**} \in \begin{cases} \{0,1,\ldots,l-1\} \text{ if } q = 2l, \\ \{0,1,\ldots,l\} \text{ if } q = 2l+1, \end{cases} \ : \ \frac{1}{\sqrt{\lambda_1+1}} g_{\hat{r}^{**}}(\lambda_{2i+1}) - c_{\hat{r}^{**}} < 0, \ \forall\, i \geq 1 \end{cases} \tag{5.3.38}$$

or that

$$\begin{cases} \frac{1}{\sqrt{\lambda_1+1}} g_r(\lambda_{2i+1}) - c_r \leq 0, \, \forall\, i \geq 1, \, \forall\, r \text{ even }, \; r \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q=2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} , \\ \exists\, \hat{r}_* \text{ even }, \; \hat{r}_* \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q=2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} \quad : \; \frac{1}{\sqrt{\lambda_1+1}} g_{\hat{r}_*}(\lambda_{2i+1}) - c_{\hat{r}_*} < 0, \, \forall\, i \geq 1 \\ \text{and} \\ \frac{1}{\sqrt{\lambda_1+1}} g_r(\lambda_{2i+1}) - c_r \geq 0, \, \forall\, i \geq 1, \, \forall\, r \text{ odd }, \; r \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q=2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} , \\ \exists\, \hat{r}_{**} \text{ odd }, \; \hat{r}_{**} \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q=2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} \quad : \; \frac{1}{\sqrt{\lambda_1+1}} g_{\hat{r}_{**}}(\lambda_{2i+1}) - c_{\hat{r}_{**}} > 0, \, \forall\, i \geq 1. \end{cases} \tag{5.3.39}$$

Defining n with respect to q as in Definition 5.3.5, then, we note that

$$(5.3.38) \Rightarrow B_{i,n} > 0, \, \forall\, i \geq 1,$$

$$(5.3.39) \Rightarrow B_{i,n} < 0, \, \forall\, i \geq 1.$$

Hence, keeping in mind that n is defined with respect to q as in Definition 5.3.5, we have to check that there exist coefficients (a_r, b_r, c_r) such that

$$A_{i,n} > 0 \text{ and } B_{i,n} > 0, \, \forall\, i \geq 1, \tag{5.3.40}$$

or

$$A_{i,n} > 0 \text{ and } B_{i,n} < 0, \, \forall\, i \geq 1, \tag{5.3.41}$$

or

$$A_{i,n} < 0 \text{ and } B_{i,n} > 0, \, \forall\, i \geq 1, \tag{5.3.42}$$

or

$$A_{i,n} < 0 \text{ and } B_{i,n} < 0, \, \forall\, i \geq 1. \tag{5.3.43}$$

If one of the options (5.3.40), (5.3.41), (5.3.42) or (5.3.43) holds, then (5.3.25) is verified and it implies $\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx \neq 0, \, \forall\, k \geq 2$. For the sake of brevity, we will analyze in detail only choice (5.3.40), which holds true as soon as (5.3.36) and (5.3.38) are satisfied. For this purpose, we introduce the following definitions:

- we say that property $(\mathscr{P}_{1,r})$ holds if

$$-\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}} \leq c_r \leq \frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}}, \quad \forall i \geq 1, \tag{5.3.44}$$

- we say that property $(\mathscr{P}_{2,r})$ holds if

$$-\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}} < c_r, \quad \forall i \geq 1, \tag{5.3.45}$$

- we say that property $(\mathscr{P}_{3,r})$ holds if

$$c_r < \frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}}, \quad \forall i \geq 1, \tag{5.3.46}$$

- we say that property $(\mathscr{Q}_{1,r})$ holds if

$$\frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}} \leq c_r \leq -\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}}, \quad \forall i \geq 1, \tag{5.3.47}$$

- we say that property $(\mathscr{Q}_{2,r})$ holds if

$$c_r < -\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}}, \quad \forall i \geq 1, \tag{5.3.48}$$

- we say that property $(\mathscr{Q}_{3,r})$ holds if

$$\frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}} < c_r, \quad \forall i \geq 1. \tag{5.3.49}$$

Assume that $(\mathscr{P}_{1,r})$ holds for every r even, $r \in \{0,1,\dots,l-1\}$ if $q=2l$ and $r \in \{0,1,\dots,l\}$ if $q=2l+1$, and that there exist $r^*, \hat{r}^*$ even, where $r^*, \hat{r}^*$ are in $\{0,1,\dots,l-1\}$ if $q=2l$ and are in $\{0,1,\dots,l\}$ if $q=2l+1$, such that $(\mathscr{P}_{2,r^*})$ and $(\mathscr{P}_{3,\hat{r}^*})$ hold respectively, then the first properties in (5.3.36) and (5.3.38) are satisfied.
Moreover, if $(\mathscr{Q}_{1,r})$ is fulfilled for every r odd, $r \in \{0,1,\dots,l-1\}$ if $q=2l$ and $r \in \{0,1,\dots,l\}$ if $q=2l+1$, and there exist $r^{**}, \hat{r}^{**}$ odd, where $r^{**}, \hat{r}^{**}$ are in $\{0,1,\dots,l-1\}$ if $q=2l$ and are in $\{0,1,\dots,l\}$ if $q=2l+1$, such that $(\mathscr{Q}_{2,r^{**}})$ and $(\mathscr{Q}_{3,\hat{r}^{**}})$ hold respectively, then the second properties in (5.3.36) and (5.3.38) are satisfied.
For the sake of simplicity and shortness, we will give conditions on (a_r, b_r) (with a suitable range of indices r) to satisfy (5.3.44) and (5.3.47) with strict inequalities. Thus, to guarantee that $A_{i,n} > 0$ and $B_{i,n} > 0$, $\forall\, i \geq 0$ (where n and l are defined with respect to q in Definition 5.3.5), we should give sufficient conditions on (a_r, b_r) to fulfill

$$\begin{cases} -\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}} < c_r < \frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}}, \quad \forall\, i \geq 1, \forall r \text{ even }, r \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q=2l, \\ \{0,1,\dots,l\} \text{ if } q=2l+1, \end{cases} \\ \\ \frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}} < c_r < -\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}}, \quad \forall\, i \geq 1, \forall r \text{ odd }, r \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q=2l, \\ \{0,1,\dots,l\} \text{ if } q=2l+1, \end{cases} \end{cases} . \tag{5.3.50}$$

For every $k \geq 1$ we define

$$\mu_k := \lambda_k + 1. \tag{5.3.51}$$

Set

$$K_i := \mu_1 + \sqrt{\mu_{2i}}\sqrt{\mu_{2i+1}}, \forall\, i \geq 1. \tag{5.3.52}$$

Lemma 5.3.13. *A necessary condition for* (5.3.50) *to hold is*

$$\begin{cases} a_r > -K_i b_r, \quad \forall\, i \geq 1, \forall\, r \textit{ even }, r \in \begin{cases} \{0,1,\dots,l-1\} \textit{ if } q=2l, \\ \{0,1,\dots,l\} \textit{ if } q=2l+1, \end{cases} \\ \\ a_r < -K_i b_r, \quad \forall\, i \geq 1, \forall\, r \textit{ odd }, r \in \begin{cases} \{0,1,\dots,l-1\} \textit{ if } q=2l, \\ \{0,1,\dots,l\} \textit{ if } q=2l+1, \end{cases} \end{cases} . \tag{5.3.53}$$

Proof. From (5.3.50), the following compatibility conditions must hold

$$\begin{cases} g_r(\lambda_{2i}) + g_r(\lambda_{2i+1}) > 0, \quad \forall\, i \geq 1, \forall\, r \text{ even }, r \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q = 2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} \\ \\ g_r(\lambda_{2i}) + g_r(\lambda_{2i+1}) < 0, \quad \forall\, i \geq 1, \forall\, r \text{ odd }, r \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q = 2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} \end{cases} . \tag{5.3.54}$$

Recalling the definition (5.3.21) of $g_r(\cdot)$, we have

$$\begin{aligned} g_r(\lambda_{2i}) + g_r(\lambda_{2i+1}) &= a_r \left(\frac{1}{\sqrt{\mu_{2i}}} + \frac{1}{\sqrt{\mu_{2i+1}}} \right) + b_r \left(\frac{\mu_{2i} + \mu_1}{\sqrt{\mu_{2i}}} + \frac{\mu_{2i+1} + \mu_1}{\sqrt{\mu_{2i+1}}} \right) \\ &= \left(\frac{1}{\sqrt{\mu_{2i}}} + \frac{1}{\sqrt{\mu_{2i+1}}} \right) \left(a_r + \left(\frac{\sqrt{\mu_{2i}} + \sqrt{\mu_{2i+1}}}{\frac{1}{\sqrt{\mu_{2i}}} + \frac{1}{\sqrt{\mu_{2i+1}}}} + \mu_1 \right) b_r \right) \\ &= \left(\frac{1}{\sqrt{\mu_{2i}}} + \frac{1}{\sqrt{\mu_{2i+1}}} \right) \big(a_r + (\mu_1 + \sqrt{\mu_{2i}}\sqrt{\mu_{2i+1}}) b_r \big) \\ &= \left(\frac{1}{\sqrt{\mu_{2i}}} + \frac{1}{\sqrt{\mu_{2i+1}}} \right) (a_r + K_i b_r)\,. \end{aligned}$$

Hence, (5.3.54) is equivalent to

$$\begin{cases} a_r > -K_i b_r, \quad \forall\, i \geq 1, \forall\, r \text{ even }, r \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q = 2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} \\ \\ a_r < -K_i b_r, \quad \forall\, i \geq 1, \forall\, r \text{ odd }, r \in \begin{cases} \{0,1,\dots,l-1\} \text{ if } q = 2l, \\ \{0,1,\dots,l\} \text{ if } q = 2l+1, \end{cases} \end{cases}$$

□

Remark 5.3.14. $(K_i)_{i\geq 1}$ *is a strictly increasing sequence and* $\lim_{i\to+\infty} K_i = +\infty$.

Lemma 5.3.15. *Conditions* (5.3.53) *imply*

$$\begin{cases} b_r \geq 0, \quad \forall\, r \textit{ even }, r \in \begin{cases} \{0,1,\dots,l-1\} \textit{ if } q = 2l, \\ \{0,1,\dots,l\} \textit{ if } q = 2l+1, \end{cases} \\ \\ b_r \leq 0, \quad \forall\, r \textit{ odd }, r \in \begin{cases} \{0,1,\dots,l-1\} \textit{ if } q = 2l, \\ \{0,1,\dots,l\} \textit{ if } q = 2l+1, \end{cases} \end{cases} . \tag{5.3.55}$$

Proof. We proceed by contradiction. Let r be even, in $\{0,1,\dots,l-1\}$ if $q = 2l$ and in $\{0,1,\dots,l\}$ if $q = 2l+1$, be such that $b_r < 0$. From Lemma 5.3.13 condition $0 < a_r + K_i b_r$ must be satisfied $\forall i \geq 1$. Taking the limit $i \to +\infty$ in this inequality, we find $0 \leq \lim_{i\to+\infty} (a_r + K_i b_r) = -\infty$. Thus, the coefficients b_r should be necessarily nonnegative, for every r even, $r \in \{0,1,\dots,l-1\}$ if $q = 2l$ and $r \in \{0,1,\dots,l\}$ if $q = 2l+1$.
Analogously, let $r \in \{0,1,\dots,l-1\}$ if $q = 2l$ and $r \in \{0,1,\dots,l\}$ if $q = 2l+1$, be odd and such that $b_r > 0$. From the second condition of Lemma 5.3.13, we get that $a_r + K_i b_r < 0$, for all $i \geq$

1. Therefore, taking the limit $i \to +\infty$ in this inequality, we obtain $0 \geq \lim_{i\to+\infty}(a_r + K_i b_r) = +\infty$. Hence, b_r should be necessarily nonpositive, for every r odd, $r \in \{0, 1, \dots, l-1\}$ if $q = 2l$ and $r \in \{0, 1, \dots, l\}$ if $q = 2l + 1$. □

Lemma 5.3.16. *The first condition in* (5.3.53) *holds if and only if*

$$-K_1 b_r < a_r, \quad \forall\, r \textit{ even} , \; r \in \begin{cases} \{0, 1, \dots, l-1\} \textit{ if } q = 2l, \\ \{0, 1, \dots, l\} \textit{ if } q = 2l + 1, \end{cases} . \tag{5.3.56}$$

The second condition in (5.3.53) *holds if and only if*

$$a_r < -K_1 b_r, \quad \forall\, r \textit{ odd} , \; r \in \begin{cases} \{0, 1, \dots, l-1\} \textit{ if } q = 2l, \\ \{0, 1, \dots, l\} \textit{ if } q = 2l + 1, \end{cases} . \tag{5.3.57}$$

Proof. If $a_r > -K_i b_r$, $\forall\, i \geq 1$, $\forall\, r$ even, $r \in \{0, 1, \dots, l-1\}$ if $q = 2l$ and $r \in \{0, 1, \dots, l\}$ if $q = 2l + 1$, then (5.3.56) trivially holds. Vice versa, if (5.3.56) is verified, then, since $(K_i)_{i\geq 1}$ is strictly increasing and from Lemma (5.3.15) $b_r \geq 0$, we get

$$-K_i b_r < -K_{i-1} b_r < \dots < -K_1 b_r < a_r, \quad \forall\, i \geq 1, \forall\, r \text{ even} , \; r \in \begin{cases} \{0, 1, \dots, l-1\} \text{ if } q = 2l, \\ \{0, 1, \dots, l\} \text{ if } q = 2l + 1, \end{cases} .$$

If $a_r < -K_i b_r$, $\forall\, i \geq 1$, $\forall\, r$ odd, $r \in \{0, 1, \dots, l-1\}$ if $q = 2l$ and $r \in \{0, 1, \dots, l\}$ if $q = 2l + 1$, then (5.3.57) trivially holds. Vice versa, if (5.3.57) is satisfied, since $(K_i)_{i\geq 1}$ is strictly increasing and from Lemma (5.3.15) $b_r \leq 0$, we have

$$a_r < -K_1 b_r < -K_2 b_r < \dots < -K_i b_r, \quad \forall\, i \geq 1, \forall\, r \text{ odd} , \; r \in \begin{cases} \{0, 1, \dots, l-1\} \text{ if } q = 2l, \\ \{0, 1, \dots, l\} \text{ if } q = 2l + 1, \end{cases} .$$

□

In summary, necessary conditions for (5.3.50) to hold are

$$\begin{cases} \begin{cases} b_r \geq 0 \\ -K_1 b_r < a_r, \end{cases} \quad \forall\, r \text{ even} , \; r \in \begin{cases} \{0, 1, \dots, l-1\} \text{ if } q = 2l, \\ \{0, 1, \dots, l\} \text{ if } q = 2l + 1, \end{cases} , \\ \text{and} \\ \begin{cases} b_r \leq 0 \\ a_r < -K_1 b_r, \end{cases} \quad \forall\, r \text{ odd} , \; r \in \begin{cases} \{0, 1, \dots, l-1\} \text{ if } q = 2l, \\ \{0, 1, \dots, l\} \text{ if } q = 2l + 1, \end{cases} . \end{cases} \tag{5.3.58}$$

We define the following functions

$$
\begin{aligned}
&J_i := \varphi(\sqrt{\mu_{2i+1}}), \quad \forall\, i \geq 1, && \varphi(x) := \tfrac{(x-\sqrt{\mu_1})^2}{x\sqrt{\mu_1}+1},\\
&L_i = \psi(\sqrt{\mu_{2i}}), \quad \forall\, i \geq 1, && \psi(x) := \tfrac{(x+\sqrt{\mu_1})^2}{x\sqrt{\mu_1}-1},\\
&T_i := \sqrt{\mu_1}\zeta(\sqrt{\mu_{2i+1}}), \quad \forall\, i \geq 1, && \zeta(x) := \tfrac{x}{x\sqrt{\mu_1}+1},\\
&Q_i := \sqrt{\mu_1}\xi(\sqrt{\mu_{2i}}), \quad \forall\, i \geq 1, && \xi(x) := \tfrac{x}{x\sqrt{\mu_1}-1},
\end{aligned}
\tag{5.3.59}
$$

and the quantities

$$
G_{r,l} = \sum_{k=1}^{(l-1)-r} \left[\frac{2(k+1)}{(2k)!} b_{r+k} - \frac{1}{(2k+1)!} a_{r+k} \right], \quad r = 0, 1 \ldots, l-1, \text{ when } q = 2l
$$

$$
M_{r,l} := \frac{2(l-r+1)}{(2(l-r))!} b_l + G_{r,l}, \qquad r = 0, 1 \ldots, l-1 \text{ when } q = 2l+1.
$$

Lemma 5.3.17. *The following properties hold true:*

- $\{J_i\}_{i\geq 1}$ *is a positive, strictly increasing sequence,*
- $\dfrac{4(1+\mu_1)}{\mu_1} \leq \min_{i\geq 1} \psi(\sqrt{\mu_{2i}}),$
- $\{T_i\}_{i\geq 1}$ *is a positive, increasing sequence and* $T_i < 1$ *for all* $i \geq 1$*,*
- $\{Q_i\}_{i\geq 1}$ *is a positive, decreasing sequence and* $Q_i > 1$ *for all* $i \geq 1$*.*
-
$$
\begin{cases}
c_r = 2b_r - a_r + G_{r,l}, \forall\, r = 0, 1 \ldots, l-1, \text{ when } q = 2l,\\
\\
c_r = 2b_r - a_r + M_{r,l}, \forall\, r = 0, 1 \ldots, l-1, c_l = 2b_l, \text{ when } q = 2l+1.
\end{cases}
\tag{5.3.60}
$$

Proof.

- It is easy to check that $\varphi\prime(x) = \frac{(x-\sqrt{\mu_1})(x\sqrt{\mu_1}+2+\mu_1)}{(x\sqrt{\mu_1}+1)^2} > 0$ for all $x > \sqrt{\mu_1}$. Hence,

$$
0 < J_1 = \varphi(\sqrt{\mu_3}) < J_i, \quad \forall\, i > 1. \tag{5.3.61}
$$

- It can be easily proved that $\psi(\cdot)$ has a minimum over $(1/\sqrt{\mu_1}, \infty)$ at $x = \frac{2}{\sqrt{\mu_1}} + \sqrt{\mu_1}$. Moreover we have $(1/\sqrt{\mu_1}) < 1 < \sqrt{\mu_{2i}} < \sqrt{\mu_{2i+1}}$ for all $i \geq 1$. In addition, ψ is strictly increasing for $x > \frac{2}{\sqrt{\mu_1}} + \sqrt{\mu_1}$. Therefore,

$$
\frac{4(1+\mu_1)}{\mu_1} = \min_{x>1/\sqrt{\mu_1}} \psi(x) < \min_{i\geq 1} \psi(\sqrt{\mu_{2i}}) = \min_{i\geq 1} L_i. \tag{5.3.62}
$$

- The derivative of ζ is given by $\zeta\prime(x) = \frac{1}{(x\sqrt{\mu_1}+1)^2} > 0$. Thus,

$$
\frac{\sqrt{\mu_1}\sqrt{\mu_3}}{\sqrt{\mu_1}\sqrt{\mu_3}+1} < \frac{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}+1} < 1, \quad \forall\, i > 1. \tag{5.3.63}
$$

- It is easy to prove that $\xi\prime(x) = \frac{-1}{(x\sqrt{\mu_1}-1)^2} < 0$ that implies

$$1 < \frac{\sqrt{\mu_1}\sqrt{\mu_{2i}}}{\sqrt{\mu_1}\sqrt{\mu_{2i}}-1} < \frac{\sqrt{\mu_1}\sqrt{\mu_2}}{\sqrt{\mu_1}\sqrt{\mu_2}-1}, \quad \forall\, i > 1. \tag{5.3.64}$$

□

Theorem 5.3.18. *Let $(A_{i,n}, B_{i,n})_{i\in\mathbb{N}^*}$ be defined as in Corollary 5.3.7 with respect to the coefficients (a_r, b_r) (with r in a suitable range of indices, depending on the parity of q). We present the algorithm that allows to find coefficients (a_r, b_r) such that $A_{i,n} > 0$ and $B_{i,n} > 0$. This gives in particular sufficient conditions so that (5.3.26) holds for all $k \geq 2$. The algorithm is built as follows*

Algorithm 1:

if $q = 2l$ **then**

  $r = l - 1;$

  if r *is even* **then**

$$\begin{cases} b_r > 0 \\ -\min\{K_1, J_1\}\, b_r < a_r < \frac{4(1+\mu_1)}{\mu_1} b_r \end{cases};$$

  else

$$\begin{cases} b_r < 0 \\ \frac{4(1+\mu_1)}{\mu_1} b_r < a_r < -\min\{K_1, J_1\}\, b_r \end{cases};$$

  for $r = l - 2, \ldots, 0$ **do**

    if r *is even* **then**

$$\begin{cases} b_r > \max\left\{ \frac{Q_1|G_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+K_1}, \frac{(1+Q_1)|G_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+J_1} \right\} \\ \max\{-K_1 b_r, -J_1 b_r + |G_{r,l}|\} < a_r < \frac{4(1+\mu_1)}{\mu_1} b_r - Q_1|G_{r,l}| \end{cases};$$

    else

$$\begin{cases} b_r < \min\left\{ \frac{-Q_1|G_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+K_1}, \frac{-(1+Q_1)|G_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+J_1} \right\} \\ \frac{4(1+\mu_1)}{\mu_1} b_r + Q_1|G_{r,l}| < a_r < \min\{-K_1 b_r, -J_1 b_r - |G_{r,l}|\} \end{cases};$$

else

  $r = l;$

  if r *is even* **then**

    $b_r > 0,\ a_r = 0;$

  else

    $b_r < 0,\ a_r = 0;$

  for $r = l - 1, \ldots, 0$ **do**

    if r *is even* **then**

$$\begin{cases} b_r > \max\left\{ \frac{Q_1|M_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+K_1}, \frac{(1+Q_1)|M_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+J_1} \right\} \\ \max\{-K_1 b_r, -J_1 b_r + |M_{r,l}|\} < a_r < \frac{4(1+\mu_1)}{\mu_1} b_r - Q_1|M_{r,l}| \end{cases};$$

    else

$$\begin{cases} b_r < \min\left\{ \frac{-Q_1|M_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+K_1}, \frac{-(1+Q_1)|M_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+J_1} \right\} \\ \frac{4(1+\mu_1)}{\mu_1} b_r + Q_1|M_{r,l}| < a_r < \min\{-K_1 b_r, -J_1 b_r - |M_{r,l}|\} \end{cases};$$

Remark 5.3.19. *Note that the above algorithm provides easily sufficient conditions on the coefficients* (a_r, b_r) *such that* $A_{i,n} < 0$ *and* $B_{i,n} < 0$ *for all* $i \geq 1$*. It is sufficient to remark that the sign of* $(A_{i,n})_{i\geq 1}$ *and of* $(B_{i,n})_{i\geq 1}$ *is uniformly changed in the opposite sign when the coefficients* (a_r, b_r) *are all changed into* $(-a_r, -b_r)$ *(or equivalently when* μ *is changed into* $-\mu$*). Indeed by applying the above algorithm to the coefficients* $(-a_r, -b_r)$*, we derive the resulting sufficient conditions on the coefficients* (a_r, b_r)*. These conditions are then sufficient conditions for* $A_{i,n} < 0$ *and* $B_{i,n} < 0$ *for all* $i \geq 1$ *to hold.*

Thus we just need to explore two cases, that is the above case for which $A_{i,n} > 0$ *and* $B_{i,n} > 0$ *for all* $i \geq 1$ *and the case which consists in producing the second algorithm that leads to coefficients* (a_r, b_r) *such that* $A_{i,n} > 0$ *and* $B_{i,n} < 0$ *for all* $i \geq 1$*. This latter case will also allow to derive the algorithm to produce coefficients* (a_r, b_r) *such that* $A_{i,n} < 0$ *and* $B_{i,n} > 0$ *for all* $i \geq 1$ *by to changing the coefficients* (a_r, b_r) *into* $(-a_r, -b_r)$*, use the second algorithm, and finally changing back* $(-a_r, -b_r)$ *into* (a_r, b_r)*. The second algorithm will be presented in [5].*

Proof. The proof consist of showing that following the instructions of the algorithm, we produce coefficients (a_r, b_r) that fulfilled (5.3.50) with strict inequalities. Indeed, we have seen that conditions (5.3.50) guarantee $A_{i,n} > 0$ and $B_{i,n} > 0$, for all $i \in \mathbb{N}$.

We recall that, necessary conditions for (5.3.50) to be well defined, are given by (5.3.58).

First case: q **is even, that is** $q = 2l$ **with** $l \geq 1$

Let $r = l - 1$ be even. We prove that

$$\begin{cases} b_{l-1} > 0, \\ -\min\{K_1, J_1\}\, b_{l-1} < a_{l-1} < \frac{4(1+\mu_1)}{\mu_1} b_{l-1}, \end{cases} \Longrightarrow \begin{cases} -\frac{g_{l-1}(\lambda_{2i})}{\sqrt{\lambda_1+1}} < c_{l-1} < \frac{g_{l-1}(\lambda_{2i+1})}{\sqrt{\lambda_1+1}}, \quad \forall\, i \geq 1, \\ b_{l-1} \geq 0 \\ -K_1\, b_{l-1} < a_{l-1}. \end{cases}$$

Recalling the definition (5.3.21) of $g_r(\cdot)$ and the identity (5.3.34) for c_r for $r = l - 1$, we obtain

$$\begin{cases} -\frac{g_{l-1}(\lambda_{2i})}{\sqrt{\lambda_1+1}} < c_{l-1}, \quad \forall\, i \geq 1, \\ c_{l-1} < \frac{g_{l-1}(\lambda_{2i+1})}{\sqrt{\lambda_1+1}}, \quad \forall\, i \geq 1, \\ b_{l-1} \geq 0, \\ -K_1 b_{l-1} < a_{l-1}, \end{cases} \Leftrightarrow \begin{cases} -\frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}(a_{l-1} + (\mu_{2i} + \mu_1)b_{l-1}) < 2b_{l-1} - a_{l-1}, \quad \forall\, i \geq 1, \\ 2b_{l-1} - a_{l-1} < \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}(a_{l-1} + (\mu_{2i+1} + \mu_1)b_{l-1}), \quad \forall\, i \geq 1, \\ -K_1 b_{l-1} < a_{l-1}, \quad b_{l-1} \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(1 - \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}\right) a_{l-1} < \left(2 + \frac{\mu_{2i}+\mu_1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}\right) b_{l-1}, \quad \forall\, i \geq 1, \\ \left(2 - \frac{\mu_{2i+1}+\mu_1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}\right) b_{l-1} < \left(1 + \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}\right) a_{l-1}, \quad \forall\, i \geq 1, \\ -K_1 b_{l-1} < a_{l-1}, \quad b_{l-1} \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} a_{l-1} < \frac{(\sqrt{\mu_{2i}}+\sqrt{\mu_1})^2}{\sqrt{\mu_1}\sqrt{\mu_{2i}}-1} b_{l-1}, \quad \forall\, i \geq 1, \\ -\frac{(\sqrt{\mu_{2i+1}}-\sqrt{\mu_1})^2}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}+1} b_{l-1} < a_{l-1}, \quad \forall\, i \geq 1, \\ -K_1 b_{l-1} < a_{l-1}, \quad b_{l-1} \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} a_{l-1} < \min_{i\geq 1} \left\{ \frac{(\sqrt{\mu_{2i}}+\sqrt{\mu_1})^2}{\sqrt{\mu_1}\sqrt{\mu_{2i}}-1} \right\} b_{l-1}, \\ \max_{i\geq 1} \left\{ -\frac{(\sqrt{\mu_{2i+1}}-\sqrt{\mu_1})^2}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}+1} \right\} b_{l-1} < a_{l-1}, \\ -K_1 b_{l-1} < a_{l-1}, \quad b_{l-1} \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} a_{l-1} < \min_{i\geq 1} \{L_i\}\, b_{l-1}, \\ -\min\left\{K_1, \min_{i\geq 1}\{J_i\}\right\} b_{l-1} < a_{l-1}, \\ b_{l-1} \geq 0. \end{cases}$$

Therefore, using the properties of L_i and J_1 of Lemma 5.3.17 it can be checked that

$$\begin{cases} b_{l-1} > 0, \\ -\min\{K_1, J_1\}\, b_{l-1} < a_{l-1} < \frac{4(1+\mu_1)}{\mu_1} b_{l-1}, \end{cases} \implies \begin{cases} a_{l-1} < \min_{i\geq 1} \{L_i\}\, b_{l-1}, \\ -\min\left\{K_1, \min_{i\geq 1}\{J_i\}\right\} b_{l-1} < a_{l-1}, \\ b_{l-1} \geq 0. \end{cases}$$

If $r = l-1$ is odd, we have to prove that

$$\begin{cases} b_{l-1} < 0 \\ \frac{4(1+\mu_1)}{\mu_1} b_{l-1} < a_{l-1} < -\min\{K_1, J_1\} b_{l-1} \end{cases} \Longrightarrow \begin{cases} \frac{g_{l-1}(\lambda_{2i+1})}{\sqrt{\lambda_1+1}} < c_{l-1} < -\frac{g_{l-1}(\lambda_{2i})}{\sqrt{\lambda_1+1}}, \quad \forall\, i \geq 1, \\ b_{l-1} \leq 0 \\ a_{l-1} < -K_1 b_{l-1}. \end{cases}$$

We observe that

$$\begin{cases} \frac{g_{l-1}(\lambda_{2i+1})}{\sqrt{\lambda_1+1}} < c_{l-1}, \quad \forall\, i \geq 1, \\ c_{l-1} < -\frac{g_{l-1}(\lambda_{2i})}{\sqrt{\lambda_1+1}}, \quad \forall\, i \geq 1, \\ b_{l-1} \leq 0 \\ a_{l-1} < -K_1 b_{l-1}. \end{cases} \Leftrightarrow \begin{cases} \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}(a_{l-1} + (\mu_{2i+1}+\mu_1) b_{l-1}) < 2b_{l-1} - a_{l-1}, \quad \forall\, i \geq 1, \\ 2b_{l-1} - a_{l-1} < -\frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}(a_{l-1} + (\mu_{2i}+\mu_1) b_{l-1}), \quad \forall\, i \geq 1, \\ a_{l-1} < -K_1 b_{l-1}, \quad b_{l-1} \leq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(1 + \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}\right) a_{l-1} < \left(2 - \frac{\mu_{2i+1}+\mu_1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}\right) b_{l-1}, \quad \forall\, i \geq 1, \\ \left(2 + \frac{\mu_{2i}+\mu_1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}\right) b_{l-1} < \left(1 - \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}\right) a_{l-1}, \quad \forall\, i \geq 1, \\ a_{l-1} < -K_1 b_{l-1}, \quad b_{l-1} \leq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} a_{l-1} < -\frac{(\sqrt{\mu_{2i+1}} - \sqrt{\mu_1})^2}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}+1} b_{l-1}, \quad \forall\, i \geq 1, \\ \frac{(\sqrt{\mu_{2i}} + \sqrt{\mu_1})^2}{\sqrt{\mu_1}\sqrt{\mu_{2i}}-1} b_{l-1} < a_{l-1}, \quad \forall\, i \geq 1, \\ a_{l-1} < -K_1 b_{l-1}, \quad b_{l-1} \leq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} a_{l-1} < \min\left\{K_1, \min_{i\geq 1}\{J_i\}\right\}(-b_{l-1}), \\ \min_{i\geq 1}\{L_i\} b_{l-1} < a_{l-1}, \\ b_{l-1} \leq 0. \end{cases}$$

Since $b_{l-1} \leq 0$, from (5.3.62), we have

$$\frac{4(1+\mu_1)}{\mu_1} b_{l-1} \geq \psi(\sqrt{\mu_{2i}}) b_{l-1} = L_i b_{l-1}, \quad \forall\, i \geq 1,$$

and thus

$$\frac{4(1+\mu_1)}{\mu_1} b_{l-1} \geq \min_{i\geq 1}\{L_i\} b_{l-1}.$$

Hence, it follows that

$$\begin{cases} b_{l-1} < 0 \\ \frac{4(1+\mu_1)}{\mu_1} b_{l-1} < a_{l-1} < -\min\{K_1, J_1\} b_{l-1} \end{cases} \Longrightarrow \begin{cases} a_{l-1} < \min\left\{K_1, \min_{i\geq 1}\{J_i\}\right\}(-b_{l-1}), \\ \min_{i\geq 1}\{L_i\} b_{l-1} < a_{l-1}, \\ b_{l-1} \leq 0. \end{cases}$$

For $r = l-2, \dots, 0$, we should prove that the algorithm yields coefficients (a_r, b_r) such that

$$\begin{cases} -\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}} < c_r < \frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}}, \quad \forall\, i \geq 1, \\ b_r \geq 0, \quad -K_1 b_r < a_r, \end{cases} \qquad \forall\, r \text{ even }, \ r \in \{0, 1, \dots, l-2\}$$

$$\begin{cases} \frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}} < c_r < -\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}}, \quad \forall\, i \geq 1, \\ b_r \leq 0, \quad a_r < -K_1 b_r. \end{cases} \qquad \forall\, r \text{ odd }, \ r \in \{0, 1, \dots, l-2\}$$

Using (5.3.60), we notice that for any r even, we have

$$\begin{aligned}
\begin{cases} -\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}} < c_r, \quad \forall\, i \geq 1, \\ c_r < \frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}}, \quad \forall\, i \geq 1, \\ b_r \geq 0, \quad -K_1 b_r < a_r, \end{cases}
&\Leftrightarrow \begin{cases} -\frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}(a_r + (\mu_{2i} + \mu_1) b_r) < 2b_r - a_r + G_{r,l}, \qquad \forall\, i \geq 1, \\ 2b_r - a_r + G_{r,l} < \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}(a_r + (\mu_{2i+1} + \mu_1) b_r), \quad \forall\, i \geq 1, \\ -K_1 b_r < a_r, \quad b_r \geq 0, \end{cases} \\
&\Leftrightarrow \begin{cases} \left(1 - \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}\right) a_r < \left(2 + \frac{\mu_{2i}+\mu_1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}\right) b_r + G_{r,l}, \qquad \forall\, i \geq 1, \\ \left(2 - \frac{\mu_{2i+1}+\mu_1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}\right) b_r + G_{r,l} < \left(1 + \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}\right) a_r, \quad \forall\, i \geq 1, \\ -K_1 b_r < a_r, \quad b_r \geq 0, \end{cases} \\
&\Leftrightarrow \begin{cases} a_r < \frac{(\sqrt{\mu_{2i}}+\sqrt{\mu_1})^2}{\sqrt{\mu_1}\sqrt{\mu_{2i}}-1} b_r + Q_i G_{r,l}, \qquad \forall\, i \geq 1, \\ -\frac{(\sqrt{\mu_{2i+1}}-\sqrt{\mu_1})^2}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}+1} b_r + T_i G_{r,l} < a_r, \quad \forall\, i \geq 1, \\ -K_1 b_r < a_r, \quad b_r \geq 0, \end{cases} \\
&\Leftrightarrow \begin{cases} a_r < \min\limits_{i \geq 1} \left\{ L_i b_r + Q_i G_{r,l} \right\}, \\ \max\left\{ -K_1 b_r, \max\limits_{i \geq 1} \left\{ -J_i b_r + T_i G_{r,l} \right\} \right\} < a_r, \\ b_r \geq 0. \end{cases}
\end{aligned}$$

where T_i and Q_i are defined in (5.3.59).

The quantities $G_{r,l}$ do not have a prescribed sign, however from Lemma 5.3.17 we deduce the following bounds

$$Q_i G_{r,l} \geq \left\{ \begin{array}{ll} G_{r,l} & \text{if } G_{r,l} > 0 \\ \\ Q_1 G_{r,l} & \text{if } G_{r,l} < 0 \\ \\ 0 & \text{if } G_{r,l} = 0 \end{array} \right\} \geq -Q_1 |G_{r,l}|, \quad \forall\, i \geq 1,$$

$$T_i G_{r,l} \leq \left\{ \begin{array}{ll} G_{r,l} & \text{if } G_{r,l} > 0 \\ \\ T_1 G_{r,l} & \text{if } G_{r,l} < 0 \\ \\ 0 & \text{if } G_{r,l} = 0 \end{array} \right\} \leq |G_{r,l}|, \qquad \forall\, i \geq 1.$$

Hence, it is easy to check that

$$\left\{ \begin{array}{l} b_r > \max\left\{ \frac{Q_1|G_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+K_1}, \frac{(1+Q_1)|G_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+J_1} \right\}, \\ \\ \max\{-K_1 b_r, -J_1 b_r + |G_{r,l}|\} < a_r, \\ \\ a_r < \frac{4(1+\mu_1)}{\mu_1} b_r - Q_1|G_{r,l}| \end{array} \right. \implies \left\{ \begin{array}{l} a_r < \min_{i\geq 1} \left\{ L_i b_r + Q_i G_{r,l} \right\}, \\ \\ \max\left\{ -K_1 b_r, \max_{i \geq 1} \left\{ -J_i b_r + T_i G_{r,l} \right\} \right\} < a_r, \\ \\ b_r \geq 0. \end{array} \right. \tag{5.3.65}$$

Observe that the constraint that appears in the algorithm on b_r ensures that the set of definition of a_r is nonempty, or equivalently,

$$\max\{-K_1 b_r, -J_1 b_r + |G_{r,l}|\} < \frac{4(1+\mu_1)}{\mu_1} b_r - Q_1|G_{r,l}|.$$

Thanks to (5.3.60), we notice that for any r odd, we have

$$\begin{cases} \frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}} < c_r, & \forall\, i \geq 1, \\ c_r < -\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}}, & \forall\, i \geq 1, \\ b_r \leq 0, \\ a_r < -K_1 b_r \end{cases} \Leftrightarrow \begin{cases} \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}(a_r + (\mu_{2i+1}+\mu_1)b_r) < 2b_r - a_r + G_{r,l}, & \forall\, i \geq 1, \\ 2b_r - a_r + G_{r,l} < -\frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}(a_r + (\mu_{2i}+\mu_1)b_r), & \forall\, i \geq 1, \\ b_r \leq 0, \\ a_r < -K_1 b_r, \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(1 + \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}}\right) a_r < -\left(\frac{\mu_{2i+1}+\mu_1}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}} - 2\right) b_r + G_{r,l}, & \forall\, i \geq 1, \\ \left(2 + \frac{\mu_{2i}+\mu_1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}\right) b_r + G_{r,l} < \left(1 - \frac{1}{\sqrt{\mu_1}\sqrt{\mu_{2i}}}\right) a_r, & \forall\, i \geq 1, \\ b_r \leq 0, \\ a_r < -K_1 b_r, \end{cases}$$

$$\Leftrightarrow \begin{cases} a_r < -\frac{(\sqrt{\mu_{2i+1}}-\sqrt{\mu_1})^2}{\sqrt{\mu_1}\sqrt{\mu_{2i+1}}+1} b_r + T_i G_{r,l}, & \forall\, i \geq 1, \\ \frac{(\sqrt{\mu_{2i}}+\sqrt{\mu_1})^2}{\sqrt{\mu_1}\sqrt{\mu_{2i}}-1} b_r + Q_i G_{r,l} < a_r, & \forall\, i \geq 1, \\ b_r \leq 0, \\ a_r < -K_1 b_r, \end{cases}$$

$$\Leftrightarrow \begin{cases} a_r < \min\left\{-K_1 b_r, \min_{i\geq 1}\left\{-J_i b_r + T_i G_{r,l}\right\}\right\} \\ \max_{i \geq 1}\left\{L_i b_r + Q_i G_{r,l}\right\} < a_r \\ b_r \leq 0. \end{cases}$$

Thanks to the following bounds

$$Q_i G_{r,l} \leq \left\{\begin{array}{ll} Q_1 G_{r,l} & \text{if } G_{r,l} > 0 \\ Q_i |G_{r,l}| & \text{if } G_{r,l} < 0 \\ 0 & \text{if } G_{r,l} = 0 \end{array}\right\} \leq Q_1 |G_{r,l}|, \quad \forall\, i \geq 1,$$

$$T_i G_{r,l} \geq \left\{\begin{array}{ll} T_1 G_{r,l} & \text{if } G_{r,l} > 0 \\ -T_i |G_{r,l}| & \text{if } G_{r,l} < 0 \\ 0 & \text{if } G_{r,l} = 0 \end{array}\right\} \geq -|G_{r,l}|, \quad \forall\, i \geq 1,$$

it easily follows that

$$\begin{cases} b_r < \min\left\{\frac{-Q_1|G_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+K_1}, \frac{-(1+Q_1)|G_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+J_1}\right\} \\ \frac{4(1+\mu_1)}{\mu_1} b_r + Q_1|G_{r,l}| < a_r \\ a_r < \min\{-K_1 b_r, -J_1 b_r - |G_{r,l}|\} \end{cases} \Longrightarrow \begin{cases} a_r < \min\left\{-K_1 b_r, \min_{i\geq 1}\left\{-J_i b_r + T_i G_{r,l}\right\}\right\} \\ \max_{i\geq 1}\left\{L_i b_r + Q_i G_{r,l}\right\} < a_r \\ b_r \leq 0. \end{cases}$$

Notice that, the constraint on the coefficients b_r ensures that

$$\frac{4(1+\mu_1)}{\mu_1} b_r + Q_1|G_{r,l}| < \min\{-K_1 b_r, -J_1 b_r - |G_{r,l}|\},$$

thus, the set where to choose a_r is nonempty.

Second case: q is odd, that is $q = 2l+1$ with $l \geq 0$.

We recall that thanks to (5.3.60) and since q is odd, the coefficients c_r are given by

$$c_r = \begin{cases} 2b_r, & r = l, \\ (2b_r - a_r) + M_{r,l}, & r = 0, 1, \ldots, l-1. \end{cases}$$

Let $r = l$ be even. We recall that $a_l = 0$. Then, we should prove that the algorithm implies

$$\begin{cases} -\frac{g_l(\lambda_{2i})}{\sqrt{\lambda_1+1}} < c_l < \frac{g_l(\lambda_{2i+1})}{\sqrt{\lambda_1+1}}, & \forall\, i \geq 1, \\ b_l \geq 0, \quad -K_1 b_l < 0, \end{cases} \tag{5.3.66}$$

which is equivalent to

$$\begin{cases} -\frac{(\mu_{2i}+\mu_1)b_l}{\sqrt{\mu_{2i}}\sqrt{\mu_1}} < 2b_l < \frac{(\mu_{2i+1}+\mu_1)b_l}{\sqrt{\mu_{2i+1}}\sqrt{\mu_1}}, \\ b_l \geq 0, \quad -K_1 b_l < 0. \end{cases}$$

Hence, it is sufficient to choose $b_l > 0$ to verify (5.3.66).
Analogously, if $r = l$ is odd, it easy to check that

$$b_l < 0 \quad \Longrightarrow \quad \begin{cases} \frac{g_l(\lambda_{2i+1})}{\sqrt{\lambda_1+1}} < c_l < -\frac{g_l(\lambda_{2i})}{\sqrt{\lambda_1+1}}, & \forall\, i \geq 1, \\ b_l \leq 0, \quad -K_1 b_l > 0. \end{cases}$$

For any $r = l-1, \ldots, 0$, we should prove that, if r is even,

$$\begin{cases} b_r > \max\left\{\frac{Q_1|M_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+K_1}, \frac{(1+Q_1)|M_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+J_1}\right\}, \\ \max\{-K_1 b_r, -J_1 b_r + |M_{r,l}|\} < a_r, \\ a_r < \frac{4(1+\mu_1)}{\mu_1} b_r - Q_1|M_{r,l}| \end{cases} \Longrightarrow \begin{cases} -\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}} \leq c_r, & \forall\, i \geq 1, \\ c_r \leq \frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}}, & \forall\, i \geq 1 \\ b_{l-1} \geq 0, \\ -K_1 b_{l-1} < a_{l-1}, \end{cases} \tag{5.3.67}$$

and if r is odd, then

$$\begin{cases} b_r < \min\left\{ \frac{-Q_1|M_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+K_1}, \frac{-(1+Q_1)|M_{r,l}|}{\frac{4(1+\mu_1)}{\mu_1}+J_1} \right\}, \\ \frac{4(1+\mu_1)}{\mu_1} b_r + Q_1|M_{r,l}| < a_r, \\ a_r <\in \{-K_1 b_r, -J_1 b_r - |M_{r,l}|\} \end{cases} \Longrightarrow \begin{cases} \frac{g_r(\lambda_{2i+1})}{\sqrt{\lambda_1+1}} < c_r, \quad \forall\, i \geq 1, \\ c_r < -\frac{g_r(\lambda_{2i})}{\sqrt{\lambda_1+1}}, \quad \forall\, i \geq 1, \\ b_r \leq 0, \\ a_r < -K_1 b_r. \end{cases} \tag{5.3.68}$$

The case $q = 2l + 1$ can now be analyzed similarly to the first case $q = 2l$ noticing that it is sufficient to replace $G_{r,l}$ by $M_{r,l}$ (due to (5.3.60)) along the proof of the first case.
This concludes the proof of the Theorem 5.3.18. □

Let $q \geq 1$ be given arbitrarily, n be as in Definition 5.3.5 and $A_{i,n} > 0$, $B_{i,n} > 0$ for all $i \geq 1$ being defined in (5.3.15). Then the above algorithm allows us to produce sets of coefficients (a_r, b_r) (r varying in a suitable finite range) such that we have $A_{i,n} > 0$, $B_{i,n} > 0$ for all $i \geq 1$. Thanks to the bijection between the sets of coefficients (a_r, b_r) and the subspace of polynomials $F_{0,q}$ (defined in (5.3.29)), when q varies in $\mathbb{N}^*$, this algorithm provides an infinite class of polynomials $\tilde{\mu} \in \cup_{q=1} F_{0,q}$ such that the condition $\int_0^1 \mu\varphi_1\varphi_k dx \neq 0$ holds for all $k \geq 2$. It remains to provide polynomials $\mu \in \mathscr{P}_q(\mathbb{R})$ satisfying the condition $\int_0^1 \mu\varphi_1\varphi_k dx \neq 0$ for all $k \geq 1$. By definition, for any $\tilde{\mu} \in F_{0,q}$, we have $\tilde{\mu}(1) = 0$. The idea is to use the polynomials $\tilde{\mu}$ built thanks to the above algorithm and to determine suitable associated polynomials μ of degree q using just the degree of freedom we have with $\mu(1)$, which is for the moment free, to guarantee that $\int_0^1 \mu(x)\varphi_1^2(x)dx \neq 0$, once chosen the higher order coefficients. Denote by $\tilde{\mu}$ a polynomial belonging to the sets of polynomials we have built in Theorem 5.3.18 (using the bijection above mentioned). We recall that

$$\tilde{\mu}(x) = \sum_{k=1}^{q} \frac{\tilde{\mu}^{(k)}(1)}{k!}(x-1)^k.$$

We set

$$\mu(x) = K + \tilde{\mu}(x). \tag{5.3.69}$$

Note that $K = \mu(1)$. Then, the following result holds.

Proposition 5.3.20. *Let $q \in \mathbb{N}^*$, and $\tilde{\mu} \in \mathscr{P}_q(\mathbb{R}) \in F_{0,q}$ be any polynomial built as in Theorem 5.3.18 and let $\mu \in \mathscr{P}_q(\mathbb{R})$ be defined by* (5.3.69)*. Let K be any real number such that*

$$K \neq -\int_0^1 \tilde{\mu}(x)\varphi_1^2(x)dx, \tag{5.3.70}$$

then, μ satisfies

$$\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx \neq 0, \quad \forall\, k \geq 1. \tag{5.3.71}$$

Proof. The polynomial $\tilde{\mu} \in \mathscr{P}_q(\mathbb{R})$ built in Theorem 5.3.18 satisfies by construction that

$$\int_0^1 \tilde{\mu}(x)\varphi_1(x)\varphi_k(x)dx \neq 0, \quad \forall\, k \geq 2.$$

Therefore we have

$$\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = K\int_0^1 \varphi_1(x)\varphi_k(x)dx + \int_0^1 \tilde{\mu}(x)\varphi_1(x)\varphi_k(x)dx$$
$$= \int_0^1 \tilde{\mu}(x)\varphi_1(x)\varphi_k(x)dx \neq 0,$$

for all $k \geq 2$. Furthermore, for $k = 1$ we obtain

$$\int_0^1 \mu(x)\varphi_1^2(x)(x)dx = K\int_0^1 \varphi_1^2(x)dx + \int_0^1 \tilde{\mu}(x)\varphi_1^2(x)dx$$
$$= K + \int_0^1 \tilde{\mu}(x)\varphi_1^2(x)dx \neq 0,$$

by hypothesis (5.3.70). □

Remark 5.3.21. *Hence, Theorem 5.3.18 and Proposition 5.3.20 allow us to build an infinite class of polynomials of any degree, such that the non vanishing condition* (5.3.26) *holds. We will see in the next section how the constructive ideas and algorithm provided here lead to many applications for the bilinear control of PDEs.*

5.4 Applications

In this section we present extensions of the works [11], [4] and [3] by considering control systems subject to mixed boundary conditions. In particular, we will study the controllability through bilinear control of the Schrödinger equation with (DR) boundary conditions, super-exponential stabilizability and exact controllability to the ground state solution of the heat equation with a controlled potential and (DR) boundary condition.

5.4.1 Bilinear controllabilty of Schrödinger equation with mixed boundary conditions

We consider the motion of a quantum particle that is influenced by the presence of an electric field. Fixed $T > 0$, the wave function of the particle is described by the following Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = -\partial_x^2 u(t,x) - p(t)\mu(x)u(t,x), & (t,x) \in (0,T)\times(0,1) \\ u(t,0) = 0, \ u(t,1) + \partial_x u(t,1) = 0 \end{cases} \tag{5.4.1}$$

where $p \in L^2(0,T;\mathbb{R})$ is the control function and represents the magnitude of the electric field. The function μ is the dipolar moment of the particle.
Let $X = L^2(0,1;\mathbb{C})$ and define the linear operator A by

$$D(A) = \left\{\varphi \in H^2(0,1;\mathbb{C}) : \varphi(0) = 0, \partial_x\varphi(1) + \varphi(1) = 0\right\} \subset X, \qquad A\varphi = -\partial_x^2\varphi. \tag{5.4.2}$$

The eigenvalues and eigenfunctions of A are given by

$$\lambda_k = r_k^2, \qquad \varphi_k(x) = \eta_k \sin(r_k x), \qquad \forall k \in \mathbb{N}^*, \tag{5.4.3}$$

where the elements of the family $\{r_k\}_{k\in\mathbb{N}^*}$ are solutions of the equation

$$r_k \cos(r_k) + \sin(r_k) = 0, \tag{5.4.4}$$

and η_k are normalization constants

$$\eta_k = \frac{\sqrt{2} r_k}{\sqrt{r_k^2 + \sin^2(r_k)}}, \qquad \forall\, k \in \mathbb{N}^*. \tag{5.4.5}$$

The family $\{\varphi_k\}_{k\in\mathbb{N}^*}$ forms an orthonormal basis of X. The operator $-iA$ generates a group of isometries, e^{-iAt}, defined by

$$e^{-iAt}\varphi = \sum_{k=1}^{\infty} \langle \varphi, \varphi_k \rangle e^{-i\lambda_k t} \varphi_k, \qquad \forall\, \varphi \in L^2(0,1), \tag{5.4.6}$$

where we denote by $\langle \cdot, \cdot \rangle$ the standard $L^2(0,1;\mathbb{C})$ scalar product.
The solution of system (5.4.1) with $p = 0$ and initial condition φ_1, that is $\psi_1(t,x) = e^{-i\lambda_1 t}\varphi_1(x)$, is tipically called *ground state solution*. Our aim is to prove the local controllability of (5.4.1) along ψ_1.
For all $s > 0$, we define the spaces

$$\begin{aligned} &H^s_{(0)}(0,1;\mathbb{C}) = D(A^{s/2}), \\ &h^s(\mathbb{N}^*,\mathbb{C}) := \left\{ a = (a_k)_{k\in\mathbb{N}^*} \,:\, \textstyle\sum_{k=1}^{\infty} |\lambda_k^{s/2} a_k|^2 < +\infty \right\}, \end{aligned} \tag{5.4.7}$$

equipped, respectively, with the norms

$$\begin{aligned} &\|\varphi\|_{H^s_{(0)}} = \left(\textstyle\sum_{k\in\mathbb{N}^*} |\lambda_k^{s/2} \langle \varphi, \varphi_k \rangle|^2 \right)^{1/2}, \\ &\|a\|_{h^s} := \left(\textstyle\sum_{k\in\mathbb{N}^*} |\lambda_k^{s/2} a_k|^2 \right)^{1/2}. \end{aligned} \tag{5.4.8}$$

In the result that follows we characterize the reachable set from the first eigenstate φ_1:

Theorem 5.4.1. *Let $T > 0$ and $\mu \in H^2(0,1;\mathbb{R})$ be such that*

$$\exists\, C > 0 \text{ such that } |\langle \mu\varphi_1, \varphi_k \rangle| \geq \frac{C}{\lambda_k}, \quad \forall\, k \in \mathbb{N}^*. \tag{5.4.9}$$

Then, there exists $\delta > 0$ for which the reachable set from φ_1, with $p \in L^2(0,T;\mathbb{R})$, is defined by

$$\mathscr{R}_T := \left\{ u_f \in \mathscr{S} \cap H^2_{(0)}(0,1;\mathbb{C}) \,:\, \left\| u_f - \psi_1(T) \right\|_{H^2} < \delta \right\}, \tag{5.4.10}$$

where $\mathscr{S}$ denote the unit $L^2(0,1;\mathbb{C})$-sphere.

In the proposition that follows we establish the well-posedness of the following problem

$$\begin{cases} i\partial_t u = -\partial_x^2 u - p(t)\mu(x)u - f(t,x), & (t,x) \in (0,T) \times (0,1) \\ u(t,0) = 0, \ u(t,1) + \partial_x u(t,1) = 0 \\ u(0,x) = u_0(x). \end{cases} \tag{5.4.11}$$

Proposition 5.4.2. *Let* $T > 0$*,* $\mu \in H^2(0,1;\mathbb{R})$*,* $u_0 \in H^2_{(0)}(0,1;\mathbb{C})$*,* $p \in L^2(0,T;\mathbb{R})$ *and* $f \in L^2(0,T;(H^2 \cap H^1_{(0)})(0,1;\mathbb{C}))$*. Then, there exists a unique mild solution* $u \in C^0([0,T],H^2_{(0)}(0,1;\mathbb{C}))$ *of* (5.4.11)*, defined by*

$$u(t) = e^{-iAt}u_0 + i\int_0^t e^{-iA(t-s)}[p(s)\mu u(s) + f(s)]ds. \tag{5.4.12}$$

Moreover, there exists a constant $C = C(T) > 0$ *such that*

$$\|u\|_{C^0([0,T];H^2_{(0)})} \leq C\left(\|u_0\|_{H^2_{(0)}} + \|f\|_{L^2(0,T;H^2\cap H^1_{(0)})}\right). \tag{5.4.13}$$

The proof relies on a fixed point argument. To use this strategy, a crucial point is played by the regularizing effect due to the action of the Schrödingr group. This result is contained in the following Lemma.

Lemma 5.4.3. *Let* $T > 0$ *and* $f \in L^2(0,T;(H^2 \cap H^1_{(0)})(0,1))$*. Then, the function*

$$F(t) := t \mapsto \int_0^t e^{iAs}f(s)ds$$

belongs to $C^0([0,T],H^2_{(0)}(0,1))$ *and furthermore the following inequality holds*

$$\|F\|_{L^\infty(0,T;H^2_{(0)})} \leq C_1(T)\|f\|_{L^2(0,T;H^2\cap H^1_{(0)})}, \tag{5.4.14}$$

where $C_1(T) > 0$ *is uniformly bounded in bounded intervals with respect to* T*.*

Proof. From the definition of the group generated by $-iA$, we rewrite $F(\cdot)$ as

$$F(t) = \sum_{k=1}^{\infty}\left(\int_0^t \langle f(s),\varphi_k\rangle e^{i\lambda_k s}ds\right)\varphi_k.$$

We observe that the scalar product $\langle f(s,\cdot),\varphi_k\rangle$ can be expressed by

$$\begin{aligned}\langle f(s,\cdot),\varphi_k\rangle &= -\frac{1}{\lambda_k}\int_0^1 f(s,x)\partial_x^2\varphi_k(x)dx = -\frac{1}{\lambda_k}\left(f(s,x)\partial_x\varphi_k(x)|_0^1 - \int_0^1 \partial_x f(s,x)\partial_x\varphi_k(x)dx\right)\\ &= -\frac{1}{\lambda_k}\left(f(s,1)\partial_x\varphi_k(1) - \partial_x f(s,x)\varphi_k(x)|_0^1 + \int_0^1 \partial_x^2 f(s,x)\varphi_k(x)dx\right)\\ &= -\frac{1}{\lambda_k}\left(f(s,1)\eta_k r_k\cos(r_k) - \partial_x f(s,1)\eta_k\sin(r_k) + \int_0^1 \partial_x^2 f(s,x)\varphi_k(x)dx\right)\end{aligned}$$

and recalling relation (5.4.4), we have

$$\langle f(s),\varphi_k\rangle = \frac{1}{\lambda_k}\left(\left(f(s,1) + \partial_x f(s,1)\right)\eta_k\sin(r_k) - \int_0^1 \partial_x^2 f(s,x)\varphi_k(x)dx\right).$$

Therefore, using this last expression, we get

$$
\begin{aligned}
\|F(t)\|_{H^2_{(0)}} &= \left(\sum_{k=1}^{\infty} |\lambda_k \langle F(t), \varphi_k\rangle|^2\right)^{1/2} \\
&= \left(\sum_{k=1}^{\infty} \left|\lambda_k \left\langle \sum_{j=1}^{\infty} \left(\int_0^t \langle f(s), \varphi_k\rangle e^{i\lambda_k s} ds\right) \varphi_j, \varphi_k \right\rangle\right|^2\right)^{1/2} \\
&= \left(\sum_{k=1}^{\infty} \left|\lambda_k \int_0^t \langle f(s), \varphi_k\rangle e^{i\lambda_k s} ds\right|^2\right)^{1/2} \\
&= \left\|\int_0^t \langle f(s), \varphi_k\rangle e^{i\lambda_k s} ds\right\|_{h^2} \\
&= \left\|\int_0^t \frac{1}{\lambda_k}\left((f(s,1)+\partial_x f(s,1))\eta_k \sin(r_k) - \langle \partial_x^2 f(s), \varphi_k\rangle\right) e^{i\lambda_k s} ds\right\|_{h^2} \\
&\leq \left\|\int_0^t (f(s,1)+\partial_x f(s,1))\eta_k \sin(r_k) e^{i\lambda_k s} ds\right\|_{l^2} + \left\|\int_0^t \langle \partial_x^2 f(s), \varphi_k\rangle e^{i\lambda_k s} ds\right\|_{l^2} \\
&=: \|F_1(t)\|_{l^2} + \|F_2(t)\|_{l^2}.
\end{aligned}
$$

We estimate $F_2(\cdot)$ as follows:

$$
\begin{aligned}
\|F_2(t)\|_{l^2} &= \left(\sum_{k=1}^{\infty} \left|\int_0^t \langle \partial_x^2 f(s), \varphi_k\rangle e^{i\lambda_k s} ds\right|^2\right)^{1/2} \\
&\leq \left(\sum_{k=1}^{\infty} t \int_0^t |\langle \partial_x^2 f(s), \varphi_k\rangle|^2 ds\right)^{1/2} \\
&\leq \sqrt{t}\left(\int_0^t \|\partial_x^2 f(s)\|^2_{L^2(0,t)} ds\right)^{1/2} \\
&\leq \sqrt{t}\|f\|_{L^2(0,t;H^2)}.
\end{aligned}
$$

To bound $F_1(\cdot)$ we appeal to [11, Corollary 4], obtaining

$$
\|F_1(t)\|_{l^2} \leq \eta_k c(t)\left(\|f(\cdot,1)\|_{L^2(0,t)} + \|\partial_x f(\cdot,1)\|_{L^2(0,t)}\right).
$$

Thus, by trace Thorem we get

$$
\|F(t)\|_{H^2_{(0)}} \leq C_1(t)\|f\|_{L^2(0,t;H^2\cap H^1_{(0)})},
$$

where $C_1(t)$ is uniformly bounded for t lying in bounded intervals. Hence, we have proved that $F(t)$ is in $H^2_{(0)}(0,1)$ and furthermore that F is continuous at $t=0$. It is possible to prove the continuity at any $t \in (0,T)$. □

Now, we prove the well-posedness of problem (5.4.11).

Proof of Proposition 5.4.2. We prove the existence of a solution for problem (5.4.11) through a fix point argument. Consider the map

$$
\begin{aligned}
&\Phi : C^0([0,T]; H^2_{(0)}(0,1)) \to C^0([0,T]; H^2_{(0)}(0,1)) \\
&\qquad u(t) \mapsto \Phi(u)(t) = e^{-iAt}u_0 + i\int_0^t e^{-iA(t-s)}\left(p(s)\mu u(s) + f(s)\right) ds.
\end{aligned}
$$

For any $u(\cdot) \in C^0([0,T]; H^2_{(0)}(0,1))$, function $p(\cdot)\mu u(\cdot)$ belongs to $L^2(0,T;(H^2 \cap H^1_{(0)})(0,1))$ since $p \in L^2(0,T)$, $\mu \in H^2(0,1)$ and furthermore we have used that, in dimension 1, H^s is an algebra for any $s > 1/2$. Then, Lemma 5.4.3 ensures that Φ maps $C^0([0,T]; H^2_{(0)}(0,1))$ into itself.
We show that Φ is a contraction: for any $u_1, u_2 \in C^0([0,T]; H^2_{(0)}(0,1))$ thanks to (5.4.14) we have

$$\begin{aligned}\|\Phi(u_1)(t) - \Phi(u_2)(t)\|_{H^2_{(0)}} &= \left\| \int_0^t e^{iAs} p(s)\mu(u_1(t) - u_2(t))ds \right\|_{H^2_{(0)}} \\ &\leq C_1(t)\|p(t)\mu(u_1 - u_2)\|_{L^2(0,t;H^2 \cap H^1_{(0)})} \\ &\leq C_1(t)\|p\|_{L^2(0,t)}\|\mu(u_1 - u_2)\|_{L^\infty(0,t;H^2 \cap H^1_{(0)})} \\ &\leq C_2(t,\mu)\|p\|_{L^2(0,t)}\|(u_1 - u_2)\|_{L^\infty(0,t;H^2_{(0)})}\end{aligned}$$

that implies

$$\|\Phi(u_1) - \Phi(u_2)\|_{L^\infty(0,T;H^2_{(0)})} \leq C_2(T,\mu)\|p\|_{L^2(0,T)}\|(u_1 - u_2)\|_{L^\infty(0,T;H^2_{(0)})}.$$

Hence, if $C_2(T,\mu)\|p\|_{L^2(0,T)} \leq 1/2$, Φ is a contraction and it has a fixed point u such that

$$\|u\|_{L^\infty(0,T;H^2_{(0)})} \leq 2\left(\|u_0\|_{H^2_{(0)}} + C_1(T)\|f\|_{L^2(0,T;H^2 \cap H^1_{(0)})}\right).$$

If $C_2(T,\mu)\|p\|_{L^2(0,T)} > 1/2$, we divide the time interval in subintervals of the form $[T_k, T_{k+1}]$, with $0 = T_1 < T_2 < \cdots < T_n = T$, in which $\|p\|_{L^2(T_k,T_{k+1})}$ is small enough for every $k = 1, 2, \ldots n-1$. We perform a fixed point strategy in each interval and eventually we glue the solution. □

Let $T > 0$, we introduce the tangent space of $\mathscr{S}$, the unit $L^2(0,1;\mathbb{C})$-sphere, to the ground state solution at time T, $\psi_1(T)$,

$$\mathscr{T}_{\psi_1(T)} := \left\{\xi \in L^2(0,1) : \mathscr{R}\langle \xi, \psi_1(T)\rangle = 0\right\}, \tag{5.4.15}$$

and the projection onto this space:

$$\begin{aligned}\mathscr{P}_{\mathscr{T}_{\psi_1(T)}} : L^2(0,1) &\to \mathscr{T}_{\psi_1(T)} \\ \xi &\mapsto \xi - \frac{\mathscr{R}\langle \xi, \psi_1(T)\rangle}{\|\psi_1(T)\|^2}\psi_1(T)\end{aligned} \tag{5.4.16}$$

In the Lemma that follows we prove some properties of the set $\mathscr{T}_{\psi_1(T)}$ and the map $\mathscr{P}_{\mathscr{T}_{\psi_1(T)}}$.

Lemma 5.4.4. *$\mathscr{T}_{\psi_1(T)}$ is a closed convex subset of $L^2(0,1)$. Moreover, it holds that*

$$\mathscr{P}_{\mathscr{T}_{\psi_1(T)}}(H^2_{(0)}(0,1)) \subset H^2_{(0)}(0,1).$$

Proof. It easy to check that $\mathscr{T}_{\psi_1(T)}$ is stable under sum and multiplication by a real number. However, it is not the case for multiplication by a complex number. Let $\xi \in L^2(0,1)$, we have

$$2\mathscr{R}\langle i\xi, \psi_1(T)\rangle = i\left[\langle \xi, \psi_1(T)\rangle - \overline{\langle \xi, \psi_1(T)\rangle}\right],$$

that does not necessarily vanish. Moreover, it can be showed that $\mathscr{T}_{\psi_1(T)}$ is closed and for any $\sigma \in [0,1]$ and for any $\xi \in \mathscr{T}_{\psi_1(T)}$, the function $\sigma\xi + (1-\sigma)\xi \in \mathscr{T}_{\psi_1(T)}$. Thus, $\mathscr{T}_{\psi_1(T)}$ is a closed convex subset of $L^2(0,1)$.

It can be proved that $\mathscr{P}_{\mathscr{T}_{\psi_1(T)}}$ is a projection onto the set $\mathscr{T}_{\psi_1(T)}$ and furthermore for every $\xi \in H^2_{(0)}(0,1)$, the projection is defined by

$$\mathscr{P}_{\mathscr{T}_{\psi_1(T)}}(\xi) = \xi - \frac{\mathscr{R}\langle \xi, \psi_1(T)\rangle}{\|\psi_1(T)\|^2}\psi_1(T),$$

and since $\psi_1(T) = e^{-i\lambda_1 T}\varphi_1$ is in $H^2_{(0)}(0,1)$, then

$$\mathscr{P}_{\mathscr{T}_{\psi_1(T)}}(H^2_{(0)}(0,1)) \subset H^2_{(0)}(0,1).$$

□

We define the end-point map

$$\begin{aligned} \Theta_T : L^2(0,T;\mathbb{R}) &\to \mathscr{T}_{\psi_1(T)} \cap H^2_{(0)}(0,1) \\ p &\mapsto \mathscr{P}_{\mathscr{T}_{\psi_1(T)}}(u(T)) \end{aligned}$$

where u is the solution of (5.4.1) with initial condition $u(0,x) = \varphi_1(x)$.
The results that follows can be proved with the same strategy of [11].

Proposition 5.4.5. *Let $T > 0$ and $\mu \in H^2(0,1;\mathbb{R})$. The map Θ_T is of class C^1. Moreover, for any $p,q \in L^2(0,T;\mathbb{R})$, the differential of the end-point map satisfies*

$$d\Theta_T(p)\cdot q = \mathscr{P}_{\mathscr{T}_{\psi_1(T)}}[U(T)],$$

where U is the mild solution of the linearized system

$$\begin{cases} i\partial_t U = -\partial_x^2 U - p(t)\mu(x)U - q(t)\mu(x)u, & (t,x) \in (0,T)\times(0,1), \\ U(t,0) = 0, \quad U(t,1) + \partial_x U(t,1) = 0, \\ U(0,x) = 0, \end{cases}$$

and u is the mild solution of (5.4.1) *with the ground state as initial condition.*

Proposition 5.4.6. *Let $T > 0$ and $\mu \in H^2(0,1;\mathbb{R})$ be such that* (5.4.9) *is fulfilled. Then, the linear map*

$$d\Theta_T(0) : L^2(0,T;\mathbb{R}) \to \mathscr{T}_{\psi_1(T)} \cap H^2_{(0)}(0,1)$$

has a continuous right inverse $d\Theta_T(0)^{-1} : \mathscr{T}_{\psi_1(T)} \cap H^2_{(0)}(0,1) \to L^2(0,T;\mathbb{R})$.

Thanks to Proposition 5.4.6 it is possible to apply the inverse mapping theorem to the map Θ_T in a neighborhood of $p = 0$, proving that the Schrödinger equation (5.4.1) with initial condition $u(0,x) = \varphi_1$ is exactly controllable locally along the ground state solution ψ_1. We refer to [11] for the strategy of the proof of Theorem 5.4.1.
We want to point out that since the eigenvalues and eigenfunctions of the Laplacian with (DR) boundary conditions are not explicit, it is even more difficult to provide examples of potentials μ that fulfill hypothesis (5.4.9), and in particular such that

$$\langle \mu\varphi_1, \varphi_k\rangle \neq 0, \quad \forall k \in \mathbb{N}^*. \tag{5.4.17}$$

However, following the algorithm presented in Theorem 5.3.18 it is possible to construct polynomials of any degree $q \in \mathbb{N}^*$ that verifies the aforementioned non-vanishing condition of the Fourier coefficients of $\mu\varphi_1$. Furthermore, we will show that actually, in this setting, property (5.4.17) implies (5.4.9).
The first step consists in proving the following Lemma.

Lemma 5.4.7. *Let* $(\lambda_k,\varphi_k)_{k\in\mathbb{N}^*}$ *be the eigenvalues and eigenfunctions of the operator* (5.4.2) *and let* $\mu\in H^2(0,1)$.
Then, it holds that

$$\lambda_k\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx=\mu'(1)\varphi_1(1)\varphi_k(1)-\int_0^1 (\mu\varphi_1)''(x)\varphi_k(x)dx,\qquad \forall\, k\geq 1. \tag{5.4.18}$$

Proof. Using the equation satisfied by φ_k, and the boundary condition, we have that

$$\begin{aligned}\lambda_k\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx&=-\int_0^1 \mu(x)\varphi_1(x)\varphi_k''(x)dx\\&=-\left[\mu(x)\varphi_1(x)\varphi_k'(x)\right]_0^1+\int_0^1 (\mu\varphi_1)'(x)\varphi_k'(x)dx\\&=\left[\mu'(x)\varphi_1(x)\varphi_k(x)+\mu(x)\varphi_1'(x)\varphi_k(x)-\mu(x)\varphi_1(x)\varphi_k'(x)\right]_0^1\\&\quad-\int_0^1 (\mu\varphi_1)''(x)\varphi_k(x)dx\\&=\mu'(1)\varphi_1(1)\varphi_k(1)-\int_0^1 (\mu\varphi_1)''(x)\varphi_k(x)dx,\end{aligned}$$

as it was claimed. □

Now, we show the asymptotic behavior of $\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx$.

Lemma 5.4.8. *Let* $(\lambda_k,\varphi_k)_{k\in\mathbb{N}^*}$ *be the eigenvalues and eigenfunctions of the operator* (5.4.2) *and let* $\mu\in H^2(0,1)$ *be such that* $\mu'(1)\neq 0$.
Then, there exists $k_0\in\mathbb{N}^*$ *and* $c_\mu>0$ *such that*

$$\left|\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx\right|\geq\frac{c_\mu}{\lambda_k},\qquad \forall\, k>k_0.$$

Proof. Recalling the expression of the eigenfunctions (5.4.3) and noticing that $1\leq\eta_k\leq\sqrt{2}$, $\forall k\in\mathbb{N}^*$, and that

$$\eta_k\to\sqrt{2},\qquad \text{as } k\to\infty,$$

from (5.4.18) we obtain the following estimate

$$\begin{aligned}\left|\int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx\right|&\geq\frac{1}{\lambda_k}|\mu'(1)|\frac{r_1r_k}{\sqrt{r_1^2+1}\sqrt{r_k^2+1}}\eta_1\eta_k-\frac{\eta_k}{\lambda_k}\left|\int_0^1 (\mu\varphi_1)''(x)\sin(r_kx)dx\right|\\&\geq\frac{1}{\lambda_k}|\mu'(1)|\frac{r_1^2}{r_1^2+1}-\frac{\sqrt{2}}{\lambda_k}\left|\int_0^1 (\mu\varphi_1)''(x)\sin(r_kx)dx\right|\end{aligned} \tag{5.4.19}$$

for all $k\geq 1$. Thanks to Riemann-Lebesgue Lemma, since $\mu\in H^2(0,1)$, we deduce that

$$\int_0^1 (\mu\varphi_1)''(x)\sin(r_kx)dx\to 0,\quad \text{as } k\to\infty,$$

which implies that, fixed $\varepsilon := \frac{|\mu'(1)|}{2\sqrt{2}} \frac{r_1^2}{r_1^2+1}$, there exists $k_0 \in \mathbb{N}^*$ such that

$$\left| \int_0^1 (\mu\varphi_1)''(x)\sin(r_k x)dx \right| < \varepsilon, \quad \forall\, k > k_0.$$

Thus, using the latter bound in (5.4.19), we get

$$\left| \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx \right| \geq \frac{c_\mu}{\lambda_k}, \qquad \forall k > k_0,$$

where $c_\mu := \frac{|\mu'(1)|}{2} \frac{r_1^2}{r_1^2+1}$. □

Finally, we prove that condition (5.4.17) yields to (5.4.9).

Lemma 5.4.9. *Let* $(\lambda_k, \varphi_k)_{k\in\mathbb{N}^*}$ *be the eigenvalues and eigenfunctions of the operator* (5.4.2) *and let* $\mu \in H^2(0,1)$ *be such that* (5.4.17) *is verified and* $\mu'(1) \neq 0$.
Then, condition (5.4.9) *holds true.*

Proof. We set $I_k(\mu) := \left| \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx \right|$. From the hypotheses on μ, we have that

$$|I_k(\mu)| = \frac{1}{\lambda_k}\,|\lambda_k I_k(\mu)| \geq \frac{\lambda_1}{\lambda_k}\min\{|I_1(\mu)|, |I_2(\mu)|, \ldots, |I_{k_0}(\mu)|\} > 0, \qquad \forall\, 1 \leq k \leq k_0.$$

Thus,

$$|I_k(\mu)| \geq \frac{1}{\lambda_k}\min\left\{c_\mu, \min_{1\leq \ell \leq k_0} |I_\ell(\mu)|\right\} = \frac{C_\mu}{\lambda_k}, \qquad \forall\, k \geq 1.$$

□

Therefore, by choosing a polynomial μ of any degree $q \in \mathbb{N}^*$ that satisfies Theorem 5.3.18 and Proposition 5.3.20, we have that

$$\mu'(1) = b_0 \neq 0.$$

Hence Lemma 5.4.9 implies (5.4.9) for any function μ built through our algorithm.

5.4.2 Superexponential stabilizability of the heat equation with potential

Consider the bilinear control problem

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\mu(x)u(t,x) = 0, & t > 0,\ x \in (0,1) \\ u(t,0) = 0, \quad u'(t,1) + u(t,1) = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{5.4.20}$$

Definition 5.4.10. *Given an initial condition* $\bar{u}_0 \in X$ *and a control* $\bar{p} \in L^2_{loc}([0,+\infty))$*, we say that the control system* (5.4.20) *is* locally superexponentially stabilizable *to* $\bar{u}(\cdot;\bar{u}_0,\bar{p})$ *if for any* $\rho > 0$ *there exists* $R(\rho) > 0$ *such that, for every* $u_0 \in B_{R(\rho)}(\bar{u}_0)$*, there exists a control* $p \in L^2_{loc}([0,+\infty))$ *such that*

$$||u(t;u_0,p) - \bar{u}(t;\bar{u}_0,\bar{p})|| \leq Me^{-\rho e^{\omega t}}, \qquad \forall t > 0,$$

where $M, \omega > 0$ *are suitable constants.*

From [4, Theorem 3.4] we deduce that

Theorem 5.4.11. *Let $\tau > 0$ and $\mu \in H^2(0,1;\mathbb{R})$ be such that*

$$\begin{aligned} &\langle \mu\varphi_1, \varphi_k \rangle \neq 0, \qquad \forall k \in \mathbb{N}^*, \\ &\sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle \mu\varphi_1, \varphi_k \rangle|^2} < +\infty. \end{aligned} \tag{5.4.21}$$

Then, system (5.4.20) *is superexponentially stabilizable to ψ_1.*
Moreover, for every $\rho > 0$ there exists $R_\rho > 0$ such that any $u_0 \in B_{R_\rho}(\varphi_1)$ admits a control $p \in L^2_{loc}([0,+\infty))$ such that the corresponding solution $u(\cdot; u_0, p)$ of (5.4.20) *satisfies*

$$||u(t) - \psi_1(t)|| \leq M e^{-(\rho e^{\omega t} + \lambda_1 t)}, \qquad \forall t \geq 0, \tag{5.4.22}$$

where M and ω are positive constants.

We observe that, reasoning as in the previous section, choosing any μ that fulfills Theorem 5.3.18 and Proposition 5.3.20, such function verifies (5.4.9). Hence the series in (5.4.21) converges for all $\tau > 0$.
Thus, is possible to exhibit an infinite class of polynomials μ that satisfy (5.4.21) thanks to the algorithm described in Theorem 5.3.18.

5.4.3 Exact controllability to the ground state solution of the heat equation with potential

Let $T > 0$ and consider the bilinear control system

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\mu(x)u(t,x) = 0, & (t,x) \in (0,T) \times (0,1) \\ u(t,0) = 0, \quad u'(t,1) + u(t,1) = 0, & t \in (0,T) \\ u(0,x) = u_0(x) & x \in (0,1). \end{cases} \tag{5.4.23}$$

Thanks to [3, Theorem 1.1] we deduce that

Theorem 5.4.12. *Let $\mu \in H^2(0,1;\mathbb{R})$ be such that there exist $b, q > 0$ for which*

$$\langle \mu\varphi_1, \varphi_1 \rangle \neq 0, \quad \textit{and} \quad \lambda_k^q |\langle \mu\varphi_1, \varphi_k \rangle| \geq b \quad \forall\, k > 1. \tag{5.4.24}$$

Then, for any $T > 0$, there exists a constant $R_T > 0$ such that, for any $u_0 \in B_{R_T}(\varphi_1)$, there exists a control $p \in L^2(0,T)$ for which system (5.4.23) *is controllable to the ground state solution in time T.*

Therefore, it is enough to choose μ that verifies Theorem 5.3.18 and Proposition 5.3.20 to have (5.4.24) fulfilled with $q = 1$, as showed in Lemma 5.4.9.

APPENDIX A

Spectral properties of degenerate operators

In this appendix we present a class of degenerate operators and we study their spectral properties. We revise some known feature of this operators and show new results contained in [25] and [20].
Let $I = (0,1)$, $X = L^2(I)$ and consider the degenerate operator

$$Au = -(a(x)u_x)_x \tag{A.0.1}$$

where $a(x)$ is the degenerate coefficient. We now recall the definition of two different kind of degenerate operators. Let

$$a \in C^0([0,1]) \cap C^1((0,1]),\ a > 0 \text{ on } (0,1] \text{ and } a(0) = 0. \tag{A.0.2}$$

Definition A.0.1. *If* (A.0.2) *holds and moreover*

$$\frac{1}{a} \in L^1(I) \tag{A.0.3}$$

we say that the operator A defined in (A.0.1) *is* weakly degenerate.

Definition A.0.2. *If* (A.0.2) *holds and moreover*

$$a \in C^1([0,1]) \textit{ and } \frac{1}{\sqrt{a}} \in L^1(I) \tag{A.0.4}$$

we say that the operator A defined in (A.0.1) *is* strongly degenerate.

In particular, we will be interested in treating the degenerate coefficient

$$a(x) = x^\alpha.$$

Following the above definitions, we have a weakly degenerate operator for $\alpha \in [0,1)$ and a strongly degenerate one for $\alpha \in [1,2)$.

A.1 Dirichlet boundary conditions at $x = 1$

Consider the degenerate operator

$$Au = -(x^\alpha u_x)_x, \tag{A.1.1}$$

with Dirichlet boundary condition at $x = 1$. Depending on the type of degeneracy, it is customary to assign different boundary conditions at $x = 0$.

A.1.1 Weak degeneracy

Let $\alpha \in [0,1)$ and consider the degenerate operator (A.1.1) applied to a class of functions that satisfy Dirichlet boundary conditions at both extrema. The natural spaces to define the domains such operators are weighted Sobolev spaces. Let $X = L^2(I)$, we define the spaces

$$H^1_\alpha(I) = \left\{u \in X : u \text{ is absolutely continuous on } [0,1],\ x^{\alpha/2}u_x \in X\right\} \tag{A.1.2}$$

endowed with the natural scalar product

$$(f,g) = \int_0^1 (x^\alpha f_x g_x + fg)\,dx, \quad \forall f,g \in H^1_\alpha(0,1),$$

and

$$\begin{aligned} H^1_{\alpha,0}(I) &= \left\{u \in H^1_\alpha(I) :\ u(0)=0,\ u(1)=0\right\}, \\ H^2_\alpha(I) &= \left\{u \in H^1_\alpha(I) : x^\alpha u_x \in H^1(I)\right\}. \end{aligned} \tag{A.1.3}$$

The domain of the linear operator (A.1.1) is defined by

$$D(A) := \{u \in H^1_{\alpha,0}(I),\ x^\alpha u_x \in H^1(I)\}. \tag{A.1.4}$$

It is possible to prove that $D(A)$ is dense in X and $A : D(A) \subset X \to X$ is a self-adjoint accretive operator (see, for instance, [16]). Therefore $-A$ is the infinitesimal generator of an analytic C^0-semigroup of contraction e^{-tA} on X.
To determine the spectrum of A, we need to solve the eigenvalue problem

$$\begin{cases} -(x^\alpha \varphi_x)_x(x) = \lambda\varphi(x), & x \in I \\ \varphi(0) = 0, \quad \varphi(1) = 0, \end{cases} \tag{A.1.5}$$

and it turns out that Bessel functions play a fundamental role in this circumstance. Indeed, let us define the function ψ by

$$\varphi(x) = x^{\frac{1-\alpha}{2}}\psi\left(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\right),$$

then, ψ solves the following boundary problem

$$\begin{cases} y^2\psi''(y) + y\psi'(y) + \left(y^2 - \left(\frac{\alpha-1}{2-\alpha}\right)^2\right)\psi(y) = 0, & y \in \left(0, \frac{2\sqrt{\lambda}}{2-\alpha}\right), \\ y^{\frac{1-\alpha}{2-\alpha}}\psi(y) \to 0, \text{ as } y \to 0, \\ \psi\left(\frac{2\sqrt{\lambda}}{2-\alpha}\right) = 0. \end{cases}$$

For $\alpha \in [0,1)$ let

$$\nu_\alpha := \frac{1-\alpha}{2-\alpha}, \qquad k_\alpha := \frac{2-\alpha}{2}. \tag{A.1.6}$$

We can rewrite the above differential equation as

$$y^2\psi''(y) + y\psi'(y) + \left(y^2 - \nu_\alpha^2\right)\psi(y) = 0$$

that is usually called *Bessel's equation for functions of order* ν_α. The solutions of the Bessel's equation generate a vector space of dimension 2 and it can be checked (see [58], pag. 43) that the functions

$$J_{\nu_\alpha}(y) := \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m+\nu_\alpha+1)}\left(\frac{y}{2}\right)^{2m+\nu_\alpha} = \sum_{m=0}^\infty c^+_{\nu_\alpha,m} y^{2m+\nu_\alpha},$$

and

$$J_{-\nu_\alpha}(y) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\nu_\alpha+1)} \left(\frac{y}{2}\right)^{2m-\nu_\alpha} = \sum_{m=0}^{\infty} c^-_{\nu_\alpha,m} y^{2m-\nu_\alpha},$$

where Γ is the Gamma function, are well-defined on $\mathbb{R}^*_+$ and are a fundamental system of solutions of the Bessel′s equation. Hence, any other solution ψ is a linear combination of J_{ν_α} and $J_{-\nu_\alpha}$:

$$\psi(y) = C_+ J_{\nu_\alpha}(y) + C_- J_{-\nu_\alpha}(y), \quad \forall\, y \in \left(0, \frac{2\sqrt{\lambda}}{2-\alpha}\right).$$

Coming back to the variable φ, we obtain that any solution of the differential equation in (A.1.5) is defined by

$$\varphi(x) = C_+ x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\right) + C_- x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\right).$$

Using the series expression of J_{ν_α} and $J_{-\nu_\alpha}$, it is possible to prove that $\varphi \in H^1_\alpha(0,1)$ (see, for instance, [24, pag. 182]) and, moreover, by imposing the boundary conditions we get that

$$\varphi(0) = 0 \quad \Longrightarrow \quad C_- = 0,$$

and

$$\varphi(1) = 0 \quad \Longrightarrow \quad J_{\nu_\alpha}\left(\frac{2\sqrt{\lambda}}{2-\alpha}\right) = 0.$$

For any $\nu \geq 0$, the Bessel function J_ν of the first kind and order ν has an infinite number of real zeros which are simple with the possible exception of $x = 0$ (see [58, pag. 478-479]). Let $j_{\nu_\alpha,1} < j_{\nu_\alpha,2} < \cdots < j_{\nu_\alpha,k} < \ldots$ be the sequence of all positive zeros of J_{ν_α}, then the boundary condition $\varphi(1) = 0$ implies that

$$\frac{2\sqrt{\lambda_{\alpha,k}}}{2-\alpha} = j_{\nu_\alpha,k} \quad \Longleftrightarrow \quad \sqrt{\lambda_{\alpha,k}} = \frac{2-\alpha}{2} j_{\nu_\alpha,k} \quad \Longleftrightarrow \quad \lambda_{\alpha,k} = \left(\frac{2-\alpha}{2}\right)^2 j^2_{\nu_\alpha,k}, \quad \forall\, k \in \mathbb{N}^*.$$

Therefore, we have proved that the pairs eigenvalue/eigenfunction $(\lambda_{\alpha,k}, \varphi_{\alpha,k})$ that satisfy (A.1.5) are given by

$$\lambda_{\alpha,k} = k^2_\alpha j^2_{\nu_\alpha,k}, \tag{A.1.7}$$

$$\varphi_{\alpha,k}(x) = \frac{\sqrt{2k_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,k})|} x^{(1-\alpha)/2} J_{\nu_\alpha}\left(j_{\nu_\alpha,k} x^{k_\alpha}\right) \tag{A.1.8}$$

for every $k \in \mathbb{N}^*$. Moreover, the family $\left(\varphi_{\alpha,k}\right)_{k\in\mathbb{N}^*}$ is an orthonormal basis of X, see [41].

A.1.2 Strong degeneracy

In the case of strong degeneracy, that is, when $\alpha \in [1,2)$, we apply the operator A to a class of functions satisfying a Neumann condition at the extremum where degeneracy occurs, $x = 0$, and a Dirichlet condition at $x = 1$.
We define the Sobolev spaces

$$\begin{aligned} &H^1_\alpha(I) = \left\{u \in X : u \text{ is absolutely continuous on } (0,1],\ x^{\alpha/2}u_x \in X\right\} \\ &H^1_{\alpha,0}(I) = \left\{u \in H^1_\alpha(I) :\ u(1) = 0\right\}, \\ &H^2_\alpha(I) = \left\{u \in H^1_\alpha(I) :\ x^\alpha u_x \in H^1(I)\right\} \end{aligned} \tag{A.1.9}$$

Thus, the domain of A is defined as

$$\begin{aligned} D(A) := &\left\{u \in H^1_{\alpha,0}(I) : \; x^\alpha u_x \in H^1(I)\right\} \\ = &\left\{u \in X : \; u \text{ is absolutely continuous in } (0,1]\,, \; x^\alpha u \in H^1_0(I), \right. \\ &\left. \qquad x^\alpha u_x \in H^1(I) \text{ and } (x^\alpha u_x)(0) = 0\right\}. \end{aligned} \tag{A.1.10}$$

It can be proved that $D(A)$ is dense in X and that A is self-adjoint and accretive (see, for instance, [21]) and thus $-A$ is the infinitesimal generator of an analytic semigroup of contractions e^{tA} on X.
To compute the eigenvalues and eigenfunctions of A, we should solve the eigenvalue problem

$$\begin{cases} -(x^\alpha \varphi_x(x))_x = \lambda \varphi(x), \quad x \in I \\ (x^\alpha \varphi_x)(0) = 0, \\ \varphi(1) = 0. \end{cases} \tag{A.1.11}$$

First of all, we observe that $\lambda > 0$. Indeed, multiplying the equation in (A.1.11) by φ and integrating by parts, we obtain

$$\lambda \int_0^1 \varphi^2(x)dx = \int_0^1 x^\alpha \varphi_x^2(x)dx$$

that implies $\lambda \geq 0$. Moreover, if $\lambda = 0$, then $\varphi(x) \equiv c$ and by imposing the boundary conditions we get $\varphi \equiv 0$. Thus, $\lambda = 0$ is not an admissible eigenvalue.
As for the weakly degenerate spectral problem, we introduce the function ψ, implicitly defined by

$$\varphi(x) = x^{\frac{1-\alpha}{2}} \psi\left(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\right),$$

that is the solution of the following boundary value problem

$$\begin{cases} y^2\psi''(y) + y\psi'(y) + \left(y^2 - \left(\frac{\alpha-1}{2-\alpha}\right)^2\right)\psi(y) = 0, \qquad y \in \left(0, \frac{2\sqrt{\lambda}}{2-\alpha}\right), \\ (2-\alpha)y^{\frac{1}{2-\alpha}}\psi'(y) - (\alpha-1)y^{\frac{\alpha-1}{2-\alpha}}\psi(y) \to 0, \text{ as } y \to 0, \\ \psi\left(\frac{2\sqrt{\lambda}}{2-\alpha}\right) = 0. \end{cases}$$

For every $\nu_\alpha \notin \mathbb{N}$, every solution of the above Bessel's equation can be expressed as a linear combination of the fundamental system $(J_{\nu_\alpha}, J_{-\nu_\alpha})$:

$$\psi(y) = C_+ J_{\nu_\alpha}(y) + C_- J_{-\nu_\alpha}(y), \quad \forall\, y \in \left(0, \frac{2\sqrt{\lambda}}{2-\alpha}\right),$$

or, equivalently,

$$\varphi(x) = C_+ x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\right) + C_- x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\right), \quad \forall\, x \in (0,1).$$

On the other hand, when $\nu_\alpha \in \mathbb{N}$, J_{ν_α} and $J_{-\nu_\alpha}$ are linearly dependent. Indeed, for any $n \in \mathbb{N}$, it holds that $J_n(y) = (-1)^n J_n(y)$ (see, for instance, [58, pag. 43]). Therefore, to determine a fundamental system of solutions, we introduce the *Bessel's functions of order ν of second kind* Y_ν, defined by

$$\begin{cases} \forall\, \nu \notin \mathbb{N}, \quad Y_\nu(y) := \frac{J_\nu(y)\cos(\nu\pi) - J_{-\nu}(y)}{\sin(\nu\pi)}, \\ \forall\, n \in \mathbb{N}, \quad Y_n(y) := \lim_{\nu \to n} Y_\nu(y). \end{cases}$$

For any $\nu \in \mathbb{R}_+$, the functions J_ν and Y_ν are linearly independent (see [58, pag. 76]) and, in particular, for any $n \in \mathbb{N}$ the pair (J_n, Y_n) forms a fundamental system of solutions of the Bessel′s equation.
Therefore, if $\nu_\alpha \in \mathbb{N}$, any solution of the Bessel′s equation can be expressed in terms of J_{ν_α} and Y_{ν_α}:

$$\psi(y) = C_+ J_{\nu_\alpha}(y) + C_- Y_{\nu_\alpha}(y), \quad \forall\, y \in \left(0, \frac{2\sqrt{\lambda}}{2-\alpha}\right),$$

or, back to the function φ:

$$\varphi(x) = C_+ x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\right) + C_- x^{\frac{1-\alpha}{2}} Y_{-\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\right), \quad \forall\, x \in (0,1).$$

In both cases, $\nu_\alpha \notin \mathbb{N}$ and $\nu_\alpha = n_\alpha \in \mathbb{N}$, it is possible to prove that $\varphi \in H^1_\alpha(0,1)$ if and only if $C_- = 0$ (see [23, pag- 13-15]). Hence, $\varphi(x) = C_+ x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\right) \in H^1_0(0,1)$. Furthermore, from the series expression of J_{ν_α} it can be shown that

$$x^\alpha \left(x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\right)\right)_x \to 0 \quad \text{as } x \to 0,$$

thus the boundary condition at $x = 0$ is automatically satisfied.
Similarly to the weakly degenerate case, from the boundary condition at $x = 1$ we find that

$$\lambda_{k,\alpha} = k_\alpha^2 j^2_{\nu_\alpha,k}, \quad \forall\, k \in \mathbb{N}^*,$$

where $\{j_{\nu_\alpha,k}\}_{j\in\mathbb{N}^*}$ are the positive zeros of the Bessel′s function J_{ν_α}.
To sum up, for any $\alpha \in [1,2)$, if we define the quantities

$$\nu_\alpha := \frac{\alpha-1}{2-\alpha}, \qquad k_\alpha := \frac{2-\alpha}{2},$$

the eigenvalues and eigenfunctions that solve (A.1.11) are

$$\lambda_{\alpha,k} = k_\alpha^2 j^2_{\nu_\alpha,k},$$

$$\varphi_{\alpha,k}(x) = \frac{\sqrt{2k_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,k})|} x^{(1-\alpha)/2} J_{\nu_\alpha}\left(j_{\nu_\alpha,k} x^{k_\alpha}\right)$$

for every $k \in \mathbb{N}^*$, and the family $\left(\varphi_{\alpha,k}\right)_{k\in\mathbb{N}^*}$ is an orthonormal basis of X.
In the following Proposition (from [25]) we present some properties enjoyed by the functions in $D(A)$, for $\alpha \in [1,2)$, which will be useful to study the degenerate control problem of section 2.2.5 in chapter 2.

Proposition A.1.1. *Let $\alpha \in [1,2)$. The following properties holds true:*

1. $|v(x)| \le \frac{2\|v\|_{D(A)}}{\alpha-1} x^{1-\alpha}$, $\forall\, v \in D(A)$,

2. $|x^\alpha v(x)| \le C\sqrt{x}$, $\forall\, v \in D(A)$,

3. *for $\alpha \in [1, 3/2)$ it holds that*

$$\lim_{x\to 0} x^2 v(x) w_x(x) = 0, \quad \forall v, w \in D(A),$$

4. *for* $\alpha \in [1, 3/2)$ *it holds that*

$$\lim_{x\to 0} xv(x)w(x) = 0, \quad \forall v, w \in D(A),$$

5. *let* $\{\varphi_{\alpha,k}\}_{k\in\mathbb{N}^*}$ *be the family of eigenfunctions of A. For* $\alpha \in [1, 3/2)$ *and for every* $k, j \in \mathbb{N}^*$*, it holds that*

$$\lim_{x\to 0} \big(x(\varphi_{\alpha,j})_x(x)\big)_x \, x^\alpha \varphi_{\alpha,k}(x) = 0$$

.

Proof. 1. For all $v \in D(A)$ and $y \in I$, we have

$$\begin{aligned} |v(1) - v(y)| &= \left| \int_y^1 v_x(x)dx \right| = \left| \int_y^1 (x^\alpha v_x(x)) \frac{1}{x^\alpha} dx \right| \\ &\leq \sup_{0<x<1} |x^\alpha v_x(x)| \frac{\left|1 - y^{1-\alpha}\right|}{\alpha - 1} \\ &\leq \frac{2||v||_{D(A)}}{\alpha - 1} y^{1-\alpha} \end{aligned}$$

where in the last inequality we have used that, for all $v \in D(A)$, it holds that

$$|a(y)v_x(y)| = \left| \int_0^y (av_x)_x(x)dx \right| \leq ||(av_x)_x||_X \sqrt{y} \tag{A.1.12}$$

with $a(y) = y^\alpha$. Finally, recalling that $v(1) = 0$, we obtain the desired formula.

2. For every $v \in D(A)$ and $y \in I$, we have

$$|y^\alpha v(y)| \leq \left| \int_0^y (x^\alpha v)_x(x)dx \right| \leq ||(av)_x||_X \sqrt{y}.$$

3. Let $v, w \in D(A)$. We can rewrite $x^2 v(x)w_x(x)$ as

$$x^{2-\alpha} v(x) x^\alpha w_x(x). \tag{A.1.13}$$

Thanks to (A.1.12), there exists a constant $C > 0$ such that

$$|x^\alpha w_x(x)| \leq C x^{1/2}. \tag{A.1.14}$$

Thus, using the first item and (A.1.14) we obtain that

$$|x^{2-\alpha} v(x) x^\alpha w_x(x)| \leq C x^{2-\alpha} x^{1-\alpha} x^{1/2} \tag{A.1.15}$$

and therefore the right-hand side tends to 0 as x goes to 0 for $\alpha < 3/2$.

4. Let $v \in D(A)$. It is sufficient to prove that $\lim_{x\to 0} x^{1/2} v(x) = 0$.

For this purpose, we observe that the function $x^{1/2}v(x)$ is integrable in I: indeed, using again the first point of the Proposition, we get

$$|x^{1/2} v(x)| \leq C x^{1/2+1-\alpha}$$

that is integrable in I. Moreover, the derivative of $x^{1/2}v(x)$ is integrable in I:

$$\left(x^{1/2}v(x)\right)_x = x^{1/2}v_x(x) + \frac{1}{2}x^{-1/2}v(x) \tag{A.1.16}$$

and we can bound the two terms on the right by

$$|x^{1/2}v_x(x)| \le |x^{\alpha}v_x(x)x^{1/2-\alpha}| \le Cx^{1-\alpha}$$

that is integrable for any $\alpha \in [1,2)$ and by

$$|x^{-1/2}v(x)| \le Cx^{1/2-\alpha}$$

that is integrable for $\alpha \in [1,3/2)$.

Thus, we can deduce that the function $x^{1/2}v(x)$ is absolutely continuous in I for $\alpha \in [1,3/2)$. So, the limit

$$\lim_{x\to 0^+} x^{1/2}v(x) = L \tag{A.1.17}$$

does exist. If $L \neq 0$, then $v(x)$ would be of the same order as $\frac{1}{x^{1/2}}$ near 0. This contradicts the fact that $v \in X$. Thus, $L = 0$.

5. Recalling that $(x^{\alpha}(\varphi_{\alpha,k})_x)_x(x) = -\lambda_k\varphi_{\alpha,k}(x)$, we have

$$\begin{aligned}(x(\varphi_{\alpha,j})_x)_x(x)x^{\alpha}\varphi_{\alpha,k}(x) &= (x^{\alpha}(\varphi_{\alpha,j})_x x^{1-\alpha})_x(x)x^{\alpha}\varphi_{\alpha,k}(x)\\ &= (x^{\alpha}(\varphi_{\alpha,j})_x)_x(x)x^{1-\alpha}x^{\alpha}\varphi_{\alpha,k}(x)\\ &\quad + (1-\alpha)x^{\alpha}(\varphi_{\alpha,j})_x(x)x^{-\alpha}x^{\alpha}\varphi_{\alpha,k}(x)\\ &= -\lambda_j x\varphi_{\alpha,j}(x)\varphi_{\alpha,k}(x) + (1-\alpha)x^{\alpha}(\varphi_{\alpha,j})_x(x)\varphi_{\alpha,k}(x).\end{aligned}$$

The first of the two terms in the last equation on the right-hand side of the above formula goes to 0 as $x \to 0$, for $\alpha < 3/2$, by the previous item. Moreover, we have

$$|x^{\alpha}(\varphi_{\alpha,j})_x(x)\varphi_{\alpha,k}(x)| \le Cx^{1/2}x^{1-\alpha}.$$

Therefore,

$$\lim_{x\to 0}\left(x(\varphi_{\alpha,j})_x\right)_x(x)x^{\alpha}\varphi_{\alpha,k}(x) = 0$$

for $\alpha \in [1,3/2)$, as it was claimed.

□

A.2 Neumann boundary conditions

In this section (mostly based on [20]) we study the degenerate operator A, defined in (A.1.1) applied to functions that satisfy Neumann boundary conditions.

A.2.1 Weak degeneracy

Let $\alpha \in [0,1)$, $I = (0,1)$, $X = L^2(I)$ and consider the weighted Sobolev spaces $H^1_{\alpha}(I)$ and $H^2_{\alpha}(I)$ defined in (A.1.2) and (A.1.3) respectively.

We define the linear operator $A : D(A) \subset X \to X$ by

$$\begin{cases} \forall u \in D(A), \quad Au := -(x^\alpha u_x)_x, \\ D(A) := \left\{u \in H^2_\alpha(I),\, (x^\alpha u_x)(0) = 0,\, u_x(1) = 0\right\}. \end{cases} \tag{A.2.1}$$

Proposition A.2.1. *Let $\alpha \in [0,1)$, then $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ is a self-adjoint accretive operator with dense domain.*

Before proving Proposition A.2.1, let us show the following integration by parts formula.

Lemma A.2.2. *Let $\alpha \in [0,1)$, then*

$$\int_0^1 (x^\alpha f')'(x) g(x) dx = -\int_0^1 x^\alpha f'(x) g'(x) dx, \quad \forall f, g \in H^2_\alpha(I) \tag{A.2.2}$$

Proof. If $f \in H^2_\alpha(I)$, then

$$F(x) := x^\alpha f'(x) \in H^1(I).$$

Let $g \in H^2_\alpha(I)$, and $\varepsilon \in (0,1)$. Decompose

$$\int_0^1 F'(x) g(x) dx = \int_0^\varepsilon F'(x) g(x) dx \int_\varepsilon^1 F'(x) g(x) dx.$$

Then, since $g \in H^2_\alpha(I) \subset H^1(\varepsilon, 1)$, the usual integration by parts formula gives

$$\begin{aligned} \int_\varepsilon^1 F'(x) g(x) dx = [F(x)g(x)]_\varepsilon^1 - \int_\varepsilon^1 F(x) g'(x) dx \\ = [F(x)g(x)]_\varepsilon^1 - \int_\varepsilon^1 (x^{\alpha/2} f'(x))(x^{\alpha/2} g'(x)) dx. \end{aligned}$$

Now, since $x^{\alpha/2} f'$ and $x^{\alpha/2} g'$ belong to $L^2(I)$, we have

$$\int_\varepsilon^1 F(x) g'(x) dx \to \int_0^1 F(x) g'(x) dx, \quad \text{as } \varepsilon \to 0,$$

and since F' and g belong to $L^2(I)$, we get

$$\int_0^\varepsilon F'(x) g(x) dx \to 0, \quad \text{as } \varepsilon \to 0.$$

It remains to study the boundary terms. First, because of Neumann boundary condition at $x = 1$, we have

$$[F(x)g(x)]_\varepsilon^1 = -F(\varepsilon) g(\varepsilon).$$

We note that $F(\varepsilon) \to 0$ as $\varepsilon \to 0$, and g is absolutely continuous on $[0,1]$, hence

$$[F(x)g(x)]_\varepsilon^1 = -F(\varepsilon) g(\varepsilon) \to 0, \text{ as } \varepsilon \to 0,$$

and this concludes the integration by parts formula. □

Proof of Proposition A.2.1. First, we note that $D(A)$ is dense in X, since it contains all the functions of class C^∞, compactly supported in I.
We derive from Lemma A.2.2 that

$$\langle Af, f\rangle = -\int_0^1 \left(x^\alpha f'\right)'(x) f(x)dx = \int_0^1 x^\alpha f'(x)^2 dx \geq 0, \quad \forall f \in D(A),$$

therefore A is accretive.
In order to show that A is symmetric, we apply Lemma A.2.2 twice to obtain that

$$\begin{aligned}\forall f,g \in D(A), \quad \langle Af, g\rangle &= -\int_0^1 (x^\alpha f')'(x)g(x)dx = \int_0^1 x^\alpha f'(x)g'(x)dx \\ &= \int_0^1 (x^\alpha g'(x))f'(x)dx = -\int_0^1 (x^\alpha g')'(x)f(x)dx = \langle f, Ag\rangle.\end{aligned}$$

Finally, we check that $I+A$ is surjective. Let $f \in L^2(I)$. Then, by Riesz theorem, there exists one and only one element $u \in H^1_\alpha(I)$ such that

$$\forall v \in H^1_\alpha(I), \quad \int_0^1 \left(uv + x^\alpha u'v'\right) = \int_0^1 fv.$$

In particular, the above relation holds true for all v of class C^∞, compactly supported in I. Thus, $x \mapsto x^\alpha u'$ has a weak derivative given by

$$-\left(x^\alpha u'\right)' = f - u.$$

Since $f - u \in L^2(I)$, we obtain that $(x^\alpha u')' \in L^2(I)$. Hence, $u \in H^2_\alpha(I)$. Now, choosing first v of class C^∞ compactly supported in $[\frac{1}{2}, 1]$, but not equal to 0 at the point $x = 1$, we derive that

$$\begin{aligned}\int_0^1 fv &= \int_0^1 \left(uv + x^\alpha u'v'\right) \\ &= \int_0^1 uv + [x^\alpha u'v]_0^1 - \int_0^1 (x^\alpha u')'v = [x^\alpha u'v]_0^1 + \int_0^1 (u - (x^\alpha u')')v,\end{aligned}$$

therefore $u'(1)v(1) = 0$ that implies $u'(1) = 0$. In the same way, by choosing v of class C^∞ compactly supported in $[0, \frac{1}{2}]$, but not equal to 0 at the point $x = 0$, we obtain that $(x^\alpha u')(0) = 0$. Thus, $u \in D(A)$ and $(I+A)u = f$. So, the operator $I+A$ is surjective. This concludes the proof of Proposition A.2.1. □

We now investigate the eigenvalues and eigenfunctions of the operator $A : D(A) \subset X \to X$ by looking for solutions (λ, φ) of the following eigenvalue problem

$$\begin{cases} -(x^\alpha \varphi_x)_x(x) = \lambda\varphi, & x \in I, \\ (x^\alpha \varphi_x)(0) = 0, \\ \varphi_x(1) = 0. \end{cases} \tag{A.2.3}$$

Proposition A.2.3. *Given $\alpha \in [0,1)$, let*

$$\kappa_\alpha := \frac{2-\alpha}{2}, \quad \nu_\alpha := \frac{1-\alpha}{2-\alpha},$$

and consider the Bessel function $J_{-\nu_\alpha}$ of negative order $-\nu_\alpha$, and the positive zeros $(j_{-\nu_\alpha-1,m})_{m\geq 1}$ of the Bessel function $J_{-\nu_\alpha-1}$.
Then, the solutions of problem (A.2.3) *are*

$$\lambda_{\alpha,0} = 0, \quad \varphi_{\alpha,0}(x) = 1 \tag{A.2.4}$$

and for all $m \geq 1$

$$\lambda_{\alpha,m} = \kappa_\alpha^2 \, j^2_{-\nu_\alpha-1,m}, \tag{A.2.5}$$

$$\varphi_{\alpha,m}(x) = K_{\alpha,m} x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\Big(j_{-\nu_\alpha-1,m} x^{\frac{2-\alpha}{2}}\Big), \tag{A.2.6}$$

where the positive constant $K_{\alpha,m}$ is such that $\|\varphi_{\alpha,m}\|_{L^2(0,1)} = 1$. Moreover, the sequence $(\varphi_{\alpha,m})_{m\geq 0}$ forms an orthonormal basis of $L^2(0,1)$.
Furthermore, the following property holds true: the sequence $(\sqrt{\lambda_{\alpha,m+1}} - \sqrt{\lambda_{\alpha,m}})_{m\geq 1}$ is decreasing and

$$\sqrt{\lambda_{\alpha,m+1}} - \sqrt{\lambda_{\alpha,m}} \to \frac{2-\alpha}{2}\pi, \quad \text{as } m \to \infty. \tag{A.2.7}$$

Proof. First, we note that if (λ,φ) solves (A.2.3) then $\lambda \geq 0$: indeed, for any $\alpha \in [0,1)$, multiplying by φ, we obtain

$$\lambda \int_0^1 \varphi^2 = \int_0^1 -(x^\alpha \varphi')'\varphi = [-(x^\alpha\varphi')\varphi]_0^1 + \int_0^1 x^\alpha(\varphi')^2 = \int_0^1 x^\alpha (\varphi')^2.$$

If $\lambda = 0$, then $x \mapsto x^\alpha \varphi'$ is constant and, by imposing to the boundary conditions, we find that it is actually equal to 0. Thus, the constant functions are the ones and only ones associated to the eigenvalue $\lambda = 0$.
We now investigate the positive eigenvalues: if $\lambda > 0$, we introduce the function ψ defined by the relation

$$\varphi(x) = x^{\frac{1-\alpha}{2}} \psi\Big(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\Big),$$

and the associated new space variable

$$y = \frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}.$$

After some classical computations, we obtain that ψ satisfies the following problem:

$$\begin{cases} y^2\psi''(y) + y\psi'(y) + \Big(y^2 - (\frac{1-\alpha}{2-\alpha})^2\Big)\psi(y) = 0, \quad y \in (0, \frac{2}{2-\alpha}\sqrt{\lambda}), \\ y^{\frac{1}{2-\alpha}}\psi'(y) + \frac{1-\alpha}{2-\alpha} y^{\frac{\alpha-1}{2-\alpha}}\psi(y) \to 0 \text{ as } y \to 0, \\ \sqrt{\lambda}\psi'(\frac{2}{2-\alpha}\sqrt{\lambda}) + \frac{1-\alpha}{2}\psi'(\frac{2}{2-\alpha}\sqrt{\lambda}) = 0. \end{cases} \tag{A.2.8}$$

The first equation in (A.2.8) is the Bessel equation of order

$$\nu_\alpha := \frac{1-\alpha}{2-\alpha} \in \Big(0, \frac{1}{2}\Big].$$

Then, the ODE we need to solve can be rewritten as

$$y^2\psi''(y) + y\psi'(y) + (y^2 - \nu_\alpha^2)\psi(y) = 0. \tag{A.2.9}$$

The fundamental theory of ordinary differential equations establishes that the solutions of (A.2.9) generate a vector space of dimension 2. Looking for solutions of (A.2.9) of the form of series of ascending powers of y, we find that the Bessel functions of order ν_α and $-\nu_\alpha$ solve the equation

$$J_{\nu_\alpha}(y) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu_\alpha+1)} \left(\frac{y}{2}\right)^{2m+\nu_\alpha} = \sum_{m=0}^{\infty} c^+_{\nu_\alpha,m} y^{2m+\nu_\alpha}, \tag{A.2.10}$$

$$J_{-\nu_\alpha}(y) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m-\nu_\alpha+1)} \left(\frac{y}{2}\right)^{2m-\nu_\alpha} = \sum_{m=0}^{\infty} c^-_{\nu_\alpha,m} y^{2m-\nu_\alpha}. \tag{A.2.11}$$

When $\nu_\alpha \notin \mathbb{N}$, the two functions J_{ν_α} and $J_{-\nu_\alpha}$ are linearly independent and therefore the pair $(J_{\nu_\alpha}, J_{-\nu_\alpha})$ forms a fundamental system of solutions of (A.2.9), (see [58, section 3.1, eq. (8), p. 40], [58, section 3.12, eq. (2), p. 43] or [48, eq. (5.3.2), p. 102])): hence

$$\begin{cases} y^2\psi''(y) + y\psi'(y) + (y^2 - \nu_\alpha^2)\psi(y) = 0, \\ y \in I \end{cases}$$
$$\implies \quad \exists\, C_+, C_- \in \mathbb{R}, \quad \begin{cases} \psi(y) = C_+ J_{\nu_\alpha}(y) + C_- J_{-\nu_\alpha}(y), \\ y \in I. \end{cases} \tag{A.2.12}$$

Thus, going back to the original variables, we obtain that

$$\begin{cases} -(x^\alpha \varphi')'(x) = \lambda\varphi(x), \\ x \in I \end{cases} \quad \implies \quad \exists\, C_+, C_- \in \mathbb{R}, \quad \begin{cases} \varphi(x) = C_+\varphi_+(x) + C_-\varphi_-(x), \\ x \in I, \end{cases} \tag{A.2.13}$$

with

$$\begin{aligned} \varphi_+(x) &= x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\Big) = x^{\frac{1-\alpha}{2}} \sum_{m=0}^{\infty} c^+_{\nu_\alpha,m} \Big(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\Big)^{2m+\nu_\alpha} \\ &= \sum_{m=0}^{\infty} c^+_{\nu_\alpha,m} \Big(\frac{2}{2-\alpha}\sqrt{\lambda}\Big)^{2m+\nu_\alpha} x^{1-\alpha+(2-\alpha)m} = \sum_{m=0}^{\infty} \tilde{c}^+_{\alpha,\lambda,m} x^{1-\alpha+(2-\alpha)m} \end{aligned} \tag{A.2.14}$$

and

$$\begin{aligned} \varphi_-(x) &= x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\Big) = x^{\frac{1-\alpha}{2}} \sum_{m=0}^{\infty} c^-_{\nu_\alpha,m} \Big(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\Big)^{2m-\nu_\alpha} \\ &= \sum_{m=0}^{\infty} c^-_{\nu_\alpha,m} \Big(\frac{2}{2-\alpha}\sqrt{\lambda}\Big)^{2m-\nu_\alpha} x^{(2-\alpha)m} = \sum_{m=0}^{\infty} \tilde{c}^-_{\alpha,\lambda,m} x^{(2-\alpha)m}. \end{aligned} \tag{A.2.15}$$

Note that

$$\varphi_+(x) \to 0, \quad \text{as } x \to 0^+,$$

hence $\varphi_+ \in L^2(I)$. Moreover,

$$\varphi'_+(x) \sim \tilde{c}^+_{\alpha,\lambda,0} \frac{1-\alpha}{x^\alpha}, \quad \text{as } x \to 0^+,$$

therefore φ_+ is absolutely continuous on $[0,1]$ and multiplying φ'_+ by $x^{\alpha/2}$ we obtain

$$x^{\alpha/2}\varphi'_+(x) \sim \tilde{c}^+_{\alpha,\lambda,0} \frac{1-\alpha}{x^{\alpha/2}}, \quad \text{as } x \to 0^+,$$

so, $\varphi_+ \in H^1_\alpha(I)$. Finally,

$$(x^\alpha \varphi_+')'(x) \to 0, \quad \text{as } x \to 0^+,$$

and we deduce that $\varphi_+ \in H^2_\alpha(I)$. With the same procedure, one easily checks that $\varphi_- \in H^2_\alpha(I)$. Since the eigenfunctions, in addition, have to satisfy that $x^\alpha \varphi'(x) \to 0$ as $x \to 0$, we have

$$x^\alpha \varphi_-'(x) \to 0, \quad \text{as } x \to 0,$$

while this is not the case for φ_+:

$$x^\alpha \varphi_+'(x) \to \tilde{c}^+_{\alpha,\lambda,0}(1-\alpha) \neq 0, \quad \text{as } x \to 0.$$

Therefore,

$$\begin{cases} -(x^\alpha \varphi')'(x) = \lambda \varphi(x), & x \in I \\ (x^\alpha \varphi')(0) = 0 \end{cases} \quad \Longrightarrow \quad \exists\, C_- \in \mathbb{R}, \quad \begin{cases} \varphi(x) = C_- \varphi_-(x), \\ x \in I, \end{cases}$$

Furthermore, φ has to fulfill the second boundary condition: $\varphi'(1) = 0$. The expression of the derivative of φ_- is given by

$$\varphi_-'(x) = \frac{1-\alpha}{2} x^{\frac{-1-\alpha}{2}} J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\Big) + x^{\frac{1-\alpha}{2}} \sqrt{\lambda} x^{-\alpha/2} J'_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\Big),$$

and since $C_- \neq 0$, the condition $\varphi_-'(1) = 0$ is equivalent to require

$$\frac{1-\alpha}{2} J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}\Big) + \sqrt{\lambda} J'_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}\Big) = 0. \tag{A.2.16}$$

This is the equation that characterizes the eigenvalues λ. Multiplying by $\frac{2}{2-\alpha}$, (A.2.16) becomes

$$\frac{2}{2-\alpha}\frac{1-\alpha}{2} J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}\Big) + \frac{2}{2-\alpha}\sqrt{\lambda} J'_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}\Big) = 0. \tag{A.2.17}$$

Introducing

$$X_\lambda = \frac{2}{2-\alpha}\sqrt{\lambda},$$

(A.2.17) can be rewritten as

$$\nu_\alpha J_{-\nu_\alpha}(X_\lambda) + X_\lambda J'_{-\nu_\alpha}(X_\lambda) = 0. \tag{A.2.18}$$

This is a known formula, see [58] p. 45, formula (3):

$$\nu J_\nu(z) + z J'_\nu(z) = z J_{\nu-1}(z). \tag{A.2.19}$$

Hence, we get that

$$X_\lambda J_{-\nu_\alpha - 1}(X_\lambda) = 0, \tag{A.2.20}$$

which implies

$$J_{-\nu_\alpha - 1}(X_\lambda) = 0. \tag{A.2.21}$$

Thus, the possible values for X_λ are the positive zeros of $J_{-\nu_\alpha-1}$:

$$\frac{2}{2-\alpha}\sqrt{\lambda} = X_\lambda = j_{-\nu_\alpha-1,m}.$$

We obtain that the eigenvalues of (A.2.3) have the following form:

$$\lambda = \kappa_\alpha^2 j^2_{-\nu_\alpha-1,m}.$$

Vice–versa, given $m \geq 1$, consider

$$\lambda_m := \kappa_\alpha^2 \, j^2_{-\nu_\alpha-1,m} \quad \text{and} \quad \varphi_m(x) = x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\Big(j_{-\nu_\alpha-1,m} x^{\frac{2-\alpha}{2}}\Big).$$

From the previous argument, we deduce that $\varphi_m \in H^2_\alpha(I)$ and that (λ_m, φ_m) solves (A.2.3). Finally, the proof of (A.2.7) follows directly from [47] p. 135. Since $-\nu_\alpha - 1 \leq -1 < -\frac{1}{2}$, the sequence $(j_{-\nu_\alpha-1,m+1} - j_{-\nu_\alpha-1,m})_{m\geq 1}$ is decreasing and moreover

$$j_{-\nu_\alpha-1,m+1} - j_{-\nu_\alpha-1,m} \to \pi \quad \text{as } m \to \infty.$$

This concludes the proof of Proposition A.2.3. □

Lemma A.2.4. *Given $\alpha \in [0,1)$, the eigenvalues $\varphi_{\alpha,n}$ satisfies*

$$|\varphi_{\alpha,n}(1)| = \sqrt{2-\alpha}, \quad \forall\, n \geq 1, \tag{A.2.22}$$

and

$$\varphi_{\alpha,n}(0) \sim c^-_{\nu_\alpha,0} \sqrt{\frac{(2-\alpha)\pi}{2}} (j_{-\nu_\alpha-1,n})^{\frac{1}{2}-\nu_\alpha}, \quad \text{as } n \to +\infty, \tag{A.2.23}$$

where the coefficient $c^-_{\nu_\alpha,0}$ is defined in (A.2.11)*. In particular, the sequence $(\varphi_{\alpha,n}(0))_{n\geq 1}$ is bounded if and only if $\alpha = 0$.*

Proof. First we note that $j_{-\nu_\alpha-1,n}$ is not a zero of $J_{-\nu_\alpha}$:

$$\forall\, \alpha \in [0,1), \forall\, n \geq 1, \quad J_{-\nu_\alpha}(j_{-\nu_\alpha-1,n}) \neq 0. \tag{A.2.24}$$

Indeed, if $J_{-\nu_\alpha}(j_{-\nu_\alpha-1,n}) = 0$, we derive from (A.2.19) that $J'_{-\nu_\alpha}(j_{-\nu_\alpha-1,n}) = 0$, and then the Cauchy problem satisfied by $J_{-\nu_\alpha}$ would imply that $J_{-\nu_\alpha}$ is constantly equal to zero.
We also deduce from (A.2.19) that

$$J'_{-\nu_\alpha}(j_{-\nu_\alpha-1,n}) = -\frac{\nu_\alpha}{j_{-\nu_\alpha-1,n}} J_{-\nu_\alpha}(j_{-\nu_\alpha-1,n}). \tag{A.2.25}$$

We compute the value of the constants $K_{\alpha,n}$ that appear in (A.2.6): we have

$$1 = K^2_{\alpha,n} \int_0^1 x^{1-\alpha} J_{-\nu_\alpha}\Big(j_{-\nu_\alpha-1,n} x^{\frac{2-\alpha}{2}}\Big)^2 dx.$$

Thanks to the change of variables $y = x^{\frac{2-\alpha}{2}}$, we get

$$1 = K^2_{\alpha,n} \frac{2}{2-\alpha} \int_0^1 y J_{-\nu_\alpha}\Big(j_{-\nu_\alpha-1,n} y\Big)^2 dy,$$

and applying formula (5.14.5) p.129 in [48], we obtain

$$1 = K^2_{\alpha,n} \frac{1}{2-\alpha} \left(J'_{-\nu_\alpha}\left(j_{-\nu_\alpha-1,n}\right)^2 + \left(1 - \frac{\nu_\alpha^2}{j^2_{-\nu_\alpha-1,n}}\right) J_{-\nu_\alpha}\left(j_{-\nu_\alpha-1,n}\right)^2 \right).$$

Therefore

$$\forall\, \alpha \in [0,1), \forall\, n \geq 1, \quad K_{\alpha,n} = \left(\frac{2-\alpha}{J'_{-\nu_\alpha}\left(j_{-\nu_\alpha-1,n}\right)^2 + \left(1 - \frac{\nu_\alpha^2}{j^2_{-\nu_\alpha-1,n}}\right) J_{-\nu_\alpha}\left(j_{-\nu_\alpha-1,n}\right)^2} \right)^{1/2},$$

and using (A.2.25), we obtain a simple expression for $K_{\alpha,n}$:

$$\forall\, \alpha \in [0,1), \forall\, n \geq 1, \quad K_{\alpha,n} = \frac{\sqrt{2-\alpha}}{|J_{-\nu_\alpha}(j_{-\nu_\alpha-1,n})|}. \tag{A.2.26}$$

Thus, from (A.2.6) we deduce the value of $|\varphi_{\alpha,n}(1)|$ given in (A.2.22), and the value of $\varphi_{\alpha,n}(0)$. Indeed, from (A.2.11), we have

$$\varphi_{\alpha,n}(0) = \frac{\sqrt{2-\alpha}}{|J_{-\nu_\alpha}(j_{-\nu_\alpha-1,n})|} c^-_{\nu_\alpha,0} (j_{-\nu_\alpha-1,n})^{-\nu_\alpha}. \tag{A.2.27}$$

In particular, function $\varphi_{\alpha,n}$ has a finite limit as $x \to 0$. Moreover, using the classical asymptotic development (see, for instance, [48, formula (5.11.6) p. 122]):

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \Big[\cos(z - \frac{\nu\pi}{2} - \frac{\pi}{4})(1 + O(\frac{1}{z^2})) + O(\frac{1}{z}) \Big], \quad \text{as } z \to \infty, \tag{A.2.28}$$

we obtain that

$$J_\nu(z)^2 = \frac{2}{\pi z} \cos^2(z - \frac{\nu\pi}{2} - \frac{\pi}{4}) + O(\frac{1}{z^2}), \quad \text{as } z \to \infty \tag{A.2.29}$$

Applying this latter formula with $\nu+1$, we get

$$z J_{\nu+1}(z)^2 = \frac{2}{\pi} \cos^2\left(z - \frac{(\nu+1)\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{z}\right) = \frac{2}{\pi} \sin^2\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{z}\right).$$

Therefore

$$z J_\nu(z)^2 + z J_{\nu+1}(z)^2 = \frac{2}{\pi} + O\left(\frac{1}{z}\right),$$

which gives that

$$z J_\nu(z)^2 + z J_{\nu+1}(z)^2 \to \frac{2}{\pi}, \quad \text{as } z \to +\infty. \tag{A.2.30}$$

This implies that

$$j_{-\nu_\alpha-1,n} J_{-\nu_\alpha-1}(j_{-\nu_\alpha-1,n})^2 + j_{-\nu_\alpha-1,n} J_{-\nu_\alpha}(j_{-\nu_\alpha-1,n})^2 \to \frac{2}{\pi}, \quad \text{as } n \to +\infty.$$

Hence

$$J_{-\nu_\alpha}(j_{-\nu_\alpha-1,n})^2 \sim \frac{2}{\pi j_{-\nu_\alpha-1,n}}, \quad \text{as } n \to +\infty, \tag{A.2.31}$$

and then, combining with (A.2.27) we obtain (A.2.23). □

A.2.2 Strong degeneracy

In this section we study the properties of the strongly degenerate operator

$$Au = -(x^\alpha u_x)_x,$$

that is, when $\alpha \in [1,2)$. Consider the weighted Sobolev spaces $H^1_\alpha(I)$ and $H^2_\alpha(I)$ introduced in (A.1.9). We define the domain of $A : D(A) \subset X \to X$ by

$$D(A) := \left\{u \in H^2_\alpha(I),\, (x^\alpha u_x)(0) = 0,\, u_x(1) = 0\right\}.$$

Then, the following result holds true.

Proposition A.2.5. *Let $\alpha \in [1,2)$, then $A : D(A) \subset X \to X$ is a self-adjoint accretive operator with dense domain.*

Therefore, also in the strongly degenerate setting, A is the infinitesimal generator of an analytic semigroup of contractions e^{tA} on X.
To prove the above Proposition, the following integration by parts formula will be necessary.

Lemma A.2.6. *Let $\alpha \in [1,2)$, then*

$$\forall f, g \in H^2_\alpha(I), \quad \int_0^1 (x^\alpha f')'(x) g(x) dx = -\int_0^1 x^\alpha f'(x) g'(x) dx. \tag{A.2.32}$$

Proof. If $f \in H^2_\alpha(I)$, then

$$F(x) := x^\alpha f'(x) \in H^1(I).$$

Let $g \in H^2_\alpha(I)$, and $\varepsilon \in (0,1)$. Decompose

$$\int_0^1 F'(x) g(x) dx = \int_0^\varepsilon F'(x) g(x) dx + \int_\varepsilon^1 F'(x) g(x) dx.$$

Since $g \in H^2_\alpha(I) \subset H^1(\varepsilon, 1)$, the classical integration by parts formula gives

$$\int_0^1 F'(x) g(x) dx = \int_0^\varepsilon F'(x) g(x) dx + [F(x) g(x)]_\varepsilon^1 - \int_\varepsilon^1 F(x) g'(x) dx.$$

To prove equation (A.2.32), we have to let $\varepsilon \to 0$ in this identity. First, we note that

$$\int_\varepsilon^1 F(x) g'(x) dx = \int_\varepsilon^1 (x^\alpha f'(x)) g'(x) dx = \int_\varepsilon^1 (x^{\alpha/2} f'(x))(x^{\alpha/2} g'(x)) dx,$$

and since $x \mapsto x^{\alpha/2} f'(x)$ and $x \mapsto x^{\alpha/2} g'(x)$ belong to $L^2(I)$, we have that

$$\int_\varepsilon^1 (x^{\alpha/2} f'(x))(x^{\alpha/2} g'(x)) dx \to \int_0^1 (x^{\alpha/2} f'(x))(x^{\alpha/2} g'(x)) dx, \quad \text{as } \varepsilon \to 0.$$

Therefore,

$$\int_\varepsilon^1 F(x) g'(x) dx \to \int_0^1 F(x) g'(x) dx, \quad \text{as } \varepsilon \to 0.$$

Moreover, since F' and g belong to $L^2(I)$, we get that

$$\int_0^\varepsilon F'(x) g(x)\, dx \to 0, \quad \text{as } \varepsilon \to 0.$$

It remains to study the boundary terms: first, because of Neumann boundary conditions at $x = 1$, we have $F(1) = 0$ and since g has a finite limit as $x \to 1$, we obtain

$$[F(x) g(x)]_\varepsilon^1 = -F(\varepsilon) g(\varepsilon).$$

Now, we note that

$$\forall x \in (0,1), \quad (F(x)g(x))' = F'(x)g(x) + F(x)g'(x) \\ = F'(x)g(x) + (x^{\alpha/2}f'(x))(x^{\alpha/2}g'(x)),$$

and therefore $(Fg)' \in L^1(0,1)$ because F', g, $x^{\alpha/2}f'$, $x^{\alpha/2}g'$ belong to $L^2(0,1)$. Thus, Fg is absolutely continuous on $(0,1]$ and it has a limit as $x \to 0$. This means that there exists L such that

$$F(x)g(x) \to L, \text{ as } x \to 0^+.$$

We claim that $L = 0$. Indeed, the function $x \mapsto x^\alpha f'(x)$ belongs to $H^1(I)$, hence it has a limit as $x \to 0^+$:

$$x^\alpha f'(x) \to \ell, \quad \text{as } x \to 0^+.$$

If $\ell \neq 0$,

$$x^{\alpha/2}f'(x) \sim \frac{\ell}{x^{\alpha/2}}, \quad \text{as } x \to 0^+.$$

However, since $\alpha \geq 1$, we have that $\frac{\ell}{x^{\alpha/2}} \notin L^2(I)$, so $\ell = 0$. Moreover,

$$\forall x \in (0,1), \quad x^\alpha f'(x) = \int_0^x (s^\alpha f')'(s)ds,$$

and using the Cauchy-Schwartz inequality, we obtain

$$\forall x \in (0,1), \quad |x^\alpha f'(x)| \leq C\sqrt{x}.$$

Finally,

$$\forall x \in (0,1), \quad |x^\alpha f'(x)g(x)| \leq C\sqrt{x}|g(x)|,$$

thus

$$\forall x \in (0,1), \quad |F(x)g(x)| \leq C\sqrt{x}|g(x)|.$$

If $L \neq 0$, then for x sufficiently close to 0 we have

$$|g(x)| \geq \frac{CL}{2\sqrt{x}},$$

which is in contradiction with the fact that $g \in L^2(I)$. Therefore, $L = 0$.
This implies that

$$[F(x)g(x)]_\varepsilon^1 = -F(\varepsilon)g(\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0^+.$$

This concludes the proof of Lemma A.2.6. □

Proof of Proposition A.2.5. The strategy of the proof is similar to the one of Proposition A.2.1 and relies on the integration by parts formula given by Lemma A.2.6. It is possible to prove, as in Proposition A.2.1, that $D(A)$ is dense in X and furthermore that A is accretive and symmetric. To prove the surjectivity of $I + A$, we have already noted that for any $f \in L^2(I)$ there exists a unique $u \in H^1_\alpha(I)$ such that

$$\forall v \in H^1_\alpha(I), \quad \int_0^1 \left(uv + x^\alpha u'v'\right) = \int_0^1 fv.$$

Actually, we have proved that $u \in H^2_\alpha(0,1)$. This implies that $x^\alpha u'(x) \to 0$, as $x \to 0$. Hence, the boundary condition is satisfied at $x = 0$. Taking now v of class C^∞, but not equal to 0 at the point $x = 1$, we derive that

$$\begin{aligned}\int_0^1 f v &= \int_0^1 \big(uv + x^\alpha u'v'\big)\\ &= \int_0^1 uv + [x^\alpha u'v]_0^1 - \int_0^1 (x^\alpha u')'v = [x^\alpha u'v]_0^1 + \int_0^1 (u - (x^\alpha u')')v,\end{aligned}$$

thus $u'(1)v(1) = 0$, and therefore $u'(1) = 0$. We obtain that $u \in D(A)$ and $(I + A)u = f$. So, the operator $I + A$ is surjective. Therefore $A : D(A) \subset X \to X$ is self-adjoint. □

We now analyze the spectral property of the strongly degenerate operator $A : D(A) \subset X \to X$. Thus, we want to solve (A.2.3) for $\alpha \in [1,2)$.

Proposition A.2.7. *For any $\alpha \in [1,2)$, let*

$$\kappa_\alpha := \frac{2-\alpha}{2}, \quad \nu_\alpha := \frac{\alpha - 1}{2-\alpha},$$

and consider the Bessel function J_{ν_α} of positive order ν_α, and the positive zeros $(j_{\nu_\alpha+1,m})_{m\geq 1}$ of the Bessel function $J_{\nu_\alpha+1}$.
Then, the solutions of problem (A.2.3) *are*

$$\lambda_{\alpha,0} = 0, \quad \varphi_{\alpha,0}(x) = 1 \tag{A.2.33}$$

and for all $m \geq 1$

$$\lambda_{\alpha,m} = \kappa_\alpha^2 \, j^2_{\nu_\alpha+1,m}, \tag{A.2.34}$$

$$\varphi_{\alpha,m}(x) = K_{\alpha,m} x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\Big(j_{\nu_\alpha+1,m}\, x^{\frac{2-\alpha}{2}}\Big), \tag{A.2.35}$$

where the positive constant constant $K_{\alpha,m}$ is such that $\|\varphi_{\alpha,m}\|_{L^2(0,1)} = 1$. Moreover, the sequence $(\varphi_{\alpha,m})_{m\geq 0}$ forms an orthonormal basis of $L^2(0,1)$.
Furthermore, the following property holds true: the sequence $(\sqrt{\lambda_{\alpha,m+1}} - \sqrt{\lambda_{\alpha,m}})_{m\geq 1}$ is decreasing and

$$\sqrt{\lambda_{\alpha,m+1}} - \sqrt{\lambda_{\alpha,m}} \to \frac{2-\alpha}{2}\pi, \quad \textit{as } m \to \infty. \tag{A.2.36}$$

Proof. First, we note that if (λ, φ) solves (A.2.3) with $\alpha \in [1,2)$, then $\lambda \geq 0$: indeed, for any $\alpha \in [1,2)$, multiplying by φ, we obtain

$$\lambda \int_0^1 \varphi^2 = \int_0^1 -(x^\alpha \varphi')'\varphi = [-(x^\alpha \varphi')\varphi]_0^1 + \int_0^1 x^\alpha (\varphi')^2 = \int_0^1 x^\alpha (\varphi')^2.$$

If $\lambda = 0$, then $x \mapsto x^\alpha \varphi'$ is constant and, by imposing to the boundary conditions, we find that it is actually equal to 0. Thus, the constant functions are the ones and only ones associated to the eigenvalue $\lambda = 0$.
We now investigate the positive eigenvalues. Following the same strategy of the proof of Proposition (A.2.3), we introduce the function ψ defined by the relation

$$\varphi(x) = x^{\frac{1-\alpha}{2}} \psi\Big(\frac{2}{2-\alpha}\sqrt{\lambda} x^{\frac{2-\alpha}{2}}\Big),$$

and the associated new space variable

$$y = \frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}.$$

It turns out that ψ satisfies problem (A.2.8). The first equation in (A.2.8) is the Bessel equation of order

$$\nu_\alpha := \frac{\alpha-1}{2-\alpha} \in \left(0, \frac{1}{2}\right].$$

Then, the ODE we need to solve can be rewritten as (A.2.9).
As recalled previously, when $\nu_\alpha \notin \mathbb{N}$, J_{ν_α} and $J_{-\nu_\alpha}$ form a fundamental system of solutions of (A.2.9). Hence (A.2.12) and (A.2.13) still hold. However, the difference lies in the functions φ_+ and φ_-: here we have

$$\begin{aligned}
\varphi_+(x) &= x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\Big) \\
&= x^{\frac{1-\alpha}{2}} \sum_{m=0}^{\infty} c^+_{\nu_\alpha,m}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\Big)^{2m+\nu_\alpha} \\
&= \sum_{m=0}^{\infty} c^+_{\nu_\alpha,m}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}\Big)^{2m+\nu_\alpha} x^{(2-\alpha)m} \\
&= \sum_{m=0}^{\infty} \tilde{c}^+_{\alpha,\lambda,m} x^{(2-\alpha)m},
\end{aligned} \tag{A.2.37}$$

and

$$\begin{aligned}
\varphi_-(x) &= x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\Big) \\
&= x^{\frac{1-\alpha}{2}} \sum_{m=0}^{\infty} c^-_{\nu_\alpha,m}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\Big)^{2m-\nu_\alpha} \\
&= \sum_{m=0}^{\infty} c^-_{\nu_\alpha,m}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}\Big)^{2m-\nu_\alpha} x^{(2-\alpha)m} \\
&= \sum_{m=0}^{\infty} \tilde{c}^-_{\alpha,\lambda,m} x^{1-\alpha+(2-\alpha)m}.
\end{aligned} \tag{A.2.38}$$

We note that

$$\varphi_+(x) \to \tilde{c}^+_{\alpha,\lambda,0}, \quad \text{as } x \to 0^+,$$

hence $\varphi_+ \in L^2(I)$. Moreover,

$$x^{\alpha/2}\varphi_+'(x) \sim \tilde{c}^+_{\alpha,\lambda,1}(2-\alpha)x^{1-\frac{\alpha}{2}}, \quad \text{as } x \to 0^+,$$

that implies $\varphi_+ \in H^1_\alpha(I)$. Furthermore,

$$(x^\alpha\varphi_+')'(x) \to \tilde{c}^+_{\alpha,\lambda,1}(2-\alpha), \quad \text{as } x \to 0^+,$$

thus $\varphi_+ \in H^2_\alpha(I)$. However, for φ_- it holds that

$$x^{\alpha/2}\varphi_-'(x) \sim \tilde{c}^-_{\alpha,\lambda,1}(1-\alpha)x^{-\alpha/2}, \quad \text{as } x \to 0^+,$$

and we deduce that $\varphi_- \notin H^1_\alpha(I)$, and, in particular, $\varphi_- \notin H^2_\alpha(I)$.

Therefore $C_- = 0$ and (A.2.13) becomes

$$\begin{cases} -(x^\alpha \varphi')' = \lambda\varphi, \\ x \in I \end{cases} \implies \exists\, C_+ \in \mathbb{R}, \quad \begin{cases} \varphi(x) = C_+\varphi_+(x), \\ x \in I. \end{cases} \tag{A.2.39}$$

Observe that $(x^\alpha \varphi_+')(0) = 0$ and therefore the boundary condition at $x = 0$ is automatically satisfied.
Additionally, to be an eigenfunction, φ has to solve the second boundary condition $\varphi'(1) = 0$. We recall that

$$\varphi_+'(x) = \frac{1-\alpha}{2} x^{\frac{-1-\alpha}{2}} J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\right) + x^{\frac{1-\alpha}{2}}\sqrt{\lambda}x^{-\alpha/2} J_{\nu_\alpha}'\left(\frac{2}{2-\alpha}\sqrt{\lambda}x^{\frac{2-\alpha}{2}}\right).$$

Hence, if φ is an eigenfunction, $C_+ \neq 0$ and $\varphi_+'(1) = 0$ that yields the following relation

$$\frac{1-\alpha}{2} J_{\nu_\alpha}\left(\frac{2}{2-\alpha}\sqrt{\lambda}\right) + \sqrt{\lambda} J_{\nu_\alpha}'\left(\frac{2}{2-\alpha}\sqrt{\lambda}\right) = 0. \tag{A.2.40}$$

This is the equation that characterizes the eigenvalues λ. Multiplying by $\frac{2}{2-\alpha}$, (A.2.40) becomes

$$\frac{2}{2-\alpha}\frac{1-\alpha}{2} J_{\nu_\alpha}\Big(\frac{2}{2-\alpha}\sqrt{\lambda}\Big) + \frac{2}{2-\alpha}\sqrt{\lambda} J_{\nu_\alpha}'\Big(\frac{2}{2-\alpha}\sqrt{\lambda}\Big) = 0. \tag{A.2.41}$$

Introducing once again

$$X_\lambda = \frac{2}{2-\alpha}\sqrt{\lambda},$$

equation (A.2.41) can be rewritten as

$$-\nu_\alpha J_{\nu_\alpha}(X_\lambda) + X_\lambda J_{\nu_\alpha}'(X_\lambda) = 0. \tag{A.2.42}$$

This is a known formula, see [58, p. 45, formula (4)]:

$$z J_\nu'(z) - \nu J_\nu(z) = z J_{\nu+1}(z). \tag{A.2.43}$$

that in our case would be the following relation

$$-\nu_\alpha J_{\nu_\alpha}(X_\lambda) + X_\lambda J_{\nu_\alpha}'(X_\lambda) = X_\lambda J_{\nu_\alpha+1}(X_\lambda).$$

Thus, (A.2.41) implies

$$X_\lambda J_{\nu_\alpha+1}(X_\lambda) = 0, \tag{A.2.44}$$

or, equivalently,

$$J_{\nu_\alpha+1}(X_\lambda) = 0. \tag{A.2.45}$$

The possible values for X_λ are the positive zeros of $J_{\nu_\alpha+1}$:

$$\frac{2}{2-\alpha}\sqrt{\lambda} = X_\lambda = j_{\nu_\alpha+1,m}.$$

This identity provides the following expression for the eigenvalues

$$\lambda = \kappa_\alpha^2 j_{\nu_\alpha+1,m}^2.$$

Vice–versa, given $m \geq 1$, consider

$$\lambda_m := \kappa_\alpha^2 j_{\nu_\alpha+1,m}^2 \quad \text{and} \quad \varphi_m(x) = x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}\Big(j_{\nu_\alpha+1,m} x^{\frac{2-\alpha}{2}}\Big).$$

It is clear from the previous analysis that $\varphi_m \in H^2_\alpha(I)$ and that the pair (λ_m, φ_m) solves (A.2.3). Finally, the proof of (A.2.36) follows directly from [47, p. 135]. Since $\nu_\alpha + 1 \geq 1 > \frac{1}{2}$, the sequence $(j_{\nu_\alpha+1,m+1} - j_{\nu_\alpha+1,m})_{m\geq 1}$ is decreasing

$$j_{\nu_\alpha+1,m+1} - j_{\nu_\alpha+1,m} \to \pi \quad \text{as } m \to \infty.$$

Now, we cope with the case $\nu_\alpha = \frac{\alpha-1}{2-\alpha} \in \mathbb{N}$. It has been proved in [23] that (A.2.39) remains true (with φ_+ defined in (A.2.37), the only difference is that the fundamental system of the solutions of (A.2.9) now involve J_{ν_α} and Y_{ν_α}, the Bessel's function of order ν_α and of second kind (see [58, section 3.54, eq. (1)-(2), p. 64] or [48, eq. (5.4.5)-(5.4.6), p. 104]). Thus, one can conclude by reasoning as in the case $\nu_\alpha \notin \mathbb{N}$.
Note that there is a hidden continuity property concerning the eigenvalues as $\alpha \to 1$: if $\alpha \in [0,1)$, then (A.2.5) gives that

$$\sqrt{\lambda_m(\alpha)} = \kappa_\alpha\, j_{-\nu_\alpha-1,m} \to \frac{1}{2} j_{-1,m}, \quad \text{as } \alpha \to 1^-,$$

and if $\alpha \in [1,2)$, then (A.2.34) gives that

$$\sqrt{\lambda_m(\alpha)} = \kappa_\alpha\, j_{-\nu_\alpha-1,m} \to \frac{1}{2} j_{1,m} = \sqrt{\lambda_m(1)} \quad \text{as } \alpha \to 1^+.$$

From [58, p. 45 formula (1)]

$$J_{-1}(x) + J_1(x) = 0,$$

we deduce that J_{-1} and J_1 have the same zeros, and therefore

$$\sqrt{\lambda_m(\alpha)} \to \sqrt{\lambda_m(1)} \quad \text{as } \alpha \to 1^-.$$

□

The following result shows that the eigenvalues of the degenerate operator are unbounded as $x \to 0$.

Lemma A.2.8. *Given $\alpha \in [1,2)$, function $\varphi_{\alpha,n}$ satisfies*

$$|\varphi_{\alpha,n}(1)| = \sqrt{2-\alpha}, \quad \forall\, n \geq 1, \tag{A.2.46}$$

and

$$\varphi_{\alpha,n}(0) \sim c^+_{\nu_\alpha,0} \sqrt{\frac{(2-\alpha)\pi}{2}} \left(j_{\nu_\alpha+1,n}\right)^{\frac{1}{2}+\nu_\alpha}, \quad \text{as } n \to +\infty \tag{A.2.47}$$

where the coefficient $c^+_{\nu_\alpha,0}$ is defined in (A.2.10). *In particular, the sequence $(\varphi_{\alpha,n}(0))_{n\geq 1}$ is unbounded.*

Proof. First we note that $j_{\nu_\alpha+1,n}$ is not a zero of J_{ν_α}:

$$\forall\, \alpha \in [1,2), \forall\, n \geq 1, \quad J_{\nu_\alpha}(j_{\nu_\alpha+1,n}) \neq 0. \tag{A.2.48}$$

Indeed, if $J_{\nu_\alpha}(j_{\nu_\alpha+1,n}) = 0$, we derive from (A.2.43) that $J'_{\nu_\alpha}(j_{\nu_\alpha+1,n}) = 0$, and then the Cauchy problem satisfied by J_{ν_α} would imply that J_{ν_α} is constantly equal to zero.
We also deduce from (A.2.43) that

$$J'_{\nu_\alpha}(j_{\nu_\alpha+1,n}) = \frac{\nu_\alpha}{j_{\nu_\alpha+1,n}} J_{\nu_\alpha}(j_{\nu_\alpha+1,n}). \tag{A.2.49}$$

With the same strategy of Lemma A.2.4, we compute the value of $K_{\alpha,n}$ that appears in (A.2.35), and we find that

$$\forall\,\alpha \in [1,2), \forall\, n \geq 1, \quad K_{\alpha,n} = \frac{\sqrt{2-\alpha}}{|J_{\nu_\alpha}(j_{\nu_\alpha+1,n})|}. \tag{A.2.50}$$

Therefore, we obtain from (A.2.35) the value given in (A.2.46) of $|\varphi_{\alpha,n}(1)|$, and the value of $\varphi_{\alpha,n}(0)$. Indeed, using (A.2.10), we have

$$\varphi_{\alpha,n}(0) = \frac{\sqrt{2-\alpha}}{|J_{\nu_\alpha}(j_{\nu_\alpha+1,n})|} c^+_{\nu_\alpha,0} (j_{\nu_\alpha+1,n})^{\nu_\alpha}, \tag{A.2.51}$$

and, in particular, the function $\varphi_{\alpha,n}$ has a finite limit as $x \to 0$. Moreover, using once again (A.2.30), we have

$$j_{\nu_\alpha+1,n} J_{\nu_\alpha}(j_{\nu_\alpha+1,n})^2 + j_{\nu_\alpha+1,n} J_{\nu_\alpha+1}(j_{\nu_\alpha+1,n})^2 \to \frac{2}{\pi} \quad \text{as } n \to +\infty,$$

and hence

$$J_{\nu_\alpha}(j_{\nu_\alpha+1,n})^2 \sim \frac{2}{\pi j_{\nu_\alpha+1,n}}, \quad \text{as } n \to +\infty. \tag{A.2.52}$$

Finally, combining (A.2.52) and (A.2.27), we obtain (A.2.47). □

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