Gran Sasso Science Institute

MATHEMATICS OF NATURAL, SOCIAL AND LIFE SCIENCES DOCTORAL PROGRAMME

Cycle XXXI - AY 2017/2018

Local Geometry of Random Spherical Harmonics

PHD CANDIDATE

Anna Paola Todino

PhD Thesis Submitted
November 24, 2019

ADVISOR

Domenico Marinucci
Università di Roma Tor Vergata

GRAN SASSO SCIENCE INSTITUTE
SCHOOL OF ADVANCED STUDIES
Scuola Universitaria Superiore

SISSA
Scuola Internazionale Superiore di Studi Avanzati

A Silvia e Alessio

A Gennaro, Lars e Roberto

*Home is behind, the world ahead,
And there are many paths to tread
Through shadows to the edge of night,
Until the stars are all alight.*

J.R.R. TOLKIEN

Contents

Introduction

The main purpose of this thesis is to extend and generalize some findings concerning random fields indexed by the two-dimensional unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ (see for example [50], [52], [51], [47], [46], [23], [21]), to proper subsets of the sphere, i.e., spherical caps. Indeed, considerable interest has been drawn by the analysis of geometric functionals for the excursion sets of random eigenfunctions T_ℓ on the unit sphere (spherical harmonics), defined, for $z \in \mathbb{R}$, as

$$A_z(\ell) := A_z(T_\ell; \mathbb{S}^2) := \{x \in \mathbb{S}^2 : T_\ell(x) \geq z\}. \tag{0.0.1}$$

Hence, to describe these regions, the so-called Lipschitz Killing Curvatures (LKC) ([3]) have been investigated. In dimension 2, they correspond to the area, half of the boundary length and the Euler-Poincaré characteristic. Since they are random features, the interest is to compute their mean, their variance and eventually to establish a Limit Theorem. These stochastic results are strongly motivated by cosmological applications, in particular in connection with the analysis of the Cosmic Microwave Background (CMB) radiation data [37], [53], [45]. Some examples can be found in [63] and [62].

As stated, the thesis investigates the excursion sets on a spherical cap rather than the full sphere. In particular, it focuses on the asymptotic behavior of the excursion area, in the case of the level sets $z \neq 0$, and of the nodal lines ($z = 0$) for random spherical harmonics restricted to shrinking domains, in the two-dimensional case. Moreover, a Central Limit Theorem (CLT) is established in both cases. Here below we explain how the framework is organized focussing in particular on its main results.

The first chapter gives the background of this field of research; i.e., some general definitions on random fields, the Kac-Rice Formula ([2], [3], [27], [6]), the Lipschitz Killing Curvatures, and the Gaussian Kinematic Formula (GKF) ([3]). Moreover, the Wiener chaoses expansion and the Stein-Malliavin Method ([58], [57]) are briefly discussed, in order to enable us to exploit the Fourth Moment Theorem (see for example [57]) along the thesis and to establish some Limit Theorems. Furthermore, we focus on the case of the sphere $\mathbb{S}^2$ and hence on the spherical harmonics ([45]), proving the Spectral Representation Theorem ([45] and [43]). Finally, the excursion sets of random eigenfunctions on the unit sphere are also introduced and an overview about the previous findings on the analysis of the LKC of the excursion regions in the sphere is given (some references are [50], [52], [51], [47], [46], [23], [21], [77], [11] and [24]). The nodal case is also briefly discussed, where the so-called "Berry's cancellation" phenomenon [13] occurs, namely the asymptotic order of the functionals becomes lower than in the general case $z \neq 0$. Thus, after discussing the area, the boundary length and the Euler-Poincaré characteristic for the full sphere, it is natural to investigate what happens in subregions of the sphere; in this regards, some findings on the nodal length on subdomains are summarised ([77], [11]).

At this point, in Chapter 2, we consider the spherical cap and we focus on the area of the excursion regions of eigenfunctions over this subdomain, establishing a Quantitative Central Limit Theorem (QCLT), the first main result of the thesis. More precisely, let us consider B a symmetric spherical cap of radius $r < \pi$, which we can suppose, without loss of generality, to be centered around the North Pole $N = (0, 0)$, i.e.,

$$B = \left\{ x \in \mathbb{S}^2 : 0 \leq \theta_x \leq r,\ 0 \leq \varphi_x \leq 2\pi \right\}. \tag{0.0.2}$$

We shall then consider the excursion set

$$A_z(T_\ell, B) = \{x \in B : T_\ell(x) > z\}, \tag{0.0.3}$$

and in particular the excursion area, which can be written as

$$S_\ell(B, z) = \int_B 1_{\{T_\ell(x) > z\}}(T_\ell(x))\, dx.$$

Our result is a Quantitative Central Limit Theorem of the form

Theorem 0.0.1. *For every $z \neq 0$, as $\ell \to \infty$, we have that*

$$d_W\left(\frac{S_\ell(B,z) - \mathbb{E}[S_\ell(B,z)]}{\sqrt{\operatorname{Var}(S_\ell(B,z))}}, Z \right) = O\left(\frac{1}{\sqrt{\ell}} \right),$$

where $Z \sim \mathcal{N}(0, 1)$, and d_W denote the Wasserstein distance (see 1.2.2).

Our main ideas are broadly similar to those exploited in the case of the full sphere: namely, the proof of Theorem 0.0.1 is based on the chaotic expansion of Wiener chaoses for non-linear functionals of Gaussian fields, and on a careful analysis of this expansion. It results that the second-order term is the dominating one, as in the full sphere. Along these similarities, we stress however that there exist as well very important differences, which we list below as follows:

- While the first-order chaos term is identically zero in the case of the full sphere (see (1.4.6)), this result does no longer hold on subdomains and a careful analysis is needed to show that the corresponding term is of lower stochastic order. Here we shall also require the properties of a smooth approximation for the indicator function of the spherical cap, whose construction is of some independent interest (Section 2.2).

- The second-order chaos term is still the leading one in the L^2 expansion, and it decays to zero with the same rate ℓ^{-1} as in the full spherical case. However, the normalizing constants are different, and they can be given a natural interpretation as the relative area of the region under consideration.

- It is still possible to show that a (Quantitative) Central Limit Theorem holds. However the proof is entirely different from the one exploited in the case of the full sphere, and indeed much more challenging. In fact, due to Parseval's identity, in the case of the full sphere the second-order chaos boils down to a simple sum of independent and identically distributed random variables, so that the Central Limit Theorem, even in its Quantitative version, is almost immediate. Here, on the contrary, these identities no longer hold, and it thus becomes necessary to exploit the full power of Stein-Malliavin results (see [57] and [58]) by means of a careful computation of fourth-order cumulants. In particular, the latter result requires the investigation of complex cross-sums of the so-called Clebsch-Gordan coefficients (see [75], [45]), which arise from integrals of multiple products of spherical harmonics. Finally, it is remarkable that the asymptotic rate of convergence in the Quantitative Central Limit Theorem turns out to be identical to the full spherical case.

- It remains true that the leading term in the variance expansion vanishes in the "nodal" case $z = 0$, i.e., some form of the Berry's cancellation phenomenon (see [14], [66], [77]) applies to subdomains of the sphere as well.

In Chapter 3, the second main finding of the thesis is discussed. We investigate the behavior of the nodal length for random spherical harmonics evaluated in a shrinking ball on the sphere. As the previous case, without loss of generality, we can consider spherical caps centered in the North Pole N. Hence, let $B_{r_\ell} \subset \mathbb{S}^2$ be a shrinking spherical cap of radius r_ℓ centered in N such that

$$r_\ell \ell \to \infty, \tag{0.0.4}$$

as $\ell \to \infty$ (meaning that the support is not shrinking too rapidly). We define the nodal set as

$$\{x \in \mathbb{S}^2 \cap B_{r_\ell} : T_\ell(x) = 0\},$$

hence we denote its nodal length by

$$\mathscr{L}_{\ell,r_\ell} := \mathscr{L}^{B_{r_\ell}}(T_\ell) = \operatorname{len}(\{x \in \mathbb{S}^2 \cap B_{r_\ell} : T_\ell(x) = 0\}). \tag{0.0.5}$$

Our first non-trivial result concerns the asymptotic variance and it is the following.

Theorem 0.0.2. *Let $\mathscr{L}_{\ell,r_\ell}$ be the nodal length defined in (0.0.5), then its variance is given by*

$$\operatorname{Var}(\mathscr{L}_{\ell,r_\ell}) = \frac{1}{256} \cdot r_\ell^2 \log(r_\ell \ell) + O(r_\ell^2), \tag{0.0.6}$$

as $\ell \to \infty$.

Hence the variance of the nodal length is logarithmic in the high energy limit; moreover, it is asymptotically fully equivalent, in the L^2-sense, to the "local sample trispectrum", namely, the integral on the ball of the fourth-order Hermite polynomial. As a consequence a Central Limit Theorem is established for the nodal length on the shrinking spherical cap.

Theorem 0.0.3. *Let $\mathscr{L}_{\ell,r_\ell}$ defined in (0.0.5), then, as $\ell \to \infty$, we have that*

$$\frac{\mathscr{L}_{\ell,r_\ell} - \mathbb{E}[\mathscr{L}_{\ell,r_\ell}]}{\sqrt{\operatorname{Var}(\mathscr{L}_{\ell,r_\ell})}} \to_d Z,$$

where $\to_d$ denote the convergence in distribution and $Z \sim \mathcal{N}(0,1)$.

Theorem 0.0.3 follows by exploiting the Fourth Moment Theorem (Theorem 5.2.7 in [57]) for the fourth chaotic component, after lengthy computations of the fourth cumulant of this chaotic projection.
It is natural to compare our results with the one obtained for the shrinking ball in the torus in [11].

- In contrast to the torus, where a full correlation between the nodal length in shrinking domains and the one in the total manifold has been proved in [11], in the sphere, the following proposition holds.

  Proposition 0.0.4. *Let define $\mathscr{Z}(T_\ell) := \text{len}(\{x \in \mathbb{S}^2 : T_\ell(x) = 0\})$ and $\mathscr{Z}_{\ell,r_\ell}$ as in (0.0.5); then, the correlation between $\mathscr{Z}_{\ell,r_\ell}$ and $\mathscr{Z}(T_\ell)$, as $\ell \to \infty$, is given by*

  $$\text{Corr}(\mathscr{Z}_{\ell,r_\ell}; \mathscr{Z}(T_\ell)) = O\left(r_\ell \sqrt{\frac{\log \ell}{\log r_\ell \ell}} \right).$$

  Proposition 0.0.4 entails on the contrary that the correlation between the "local" and "global" nodal length is zero, in the high frequency limit. The discrepancy between these two results can be heuristically explained as follows: in the case of the torus, local integrals for products of four eigenfunctions have the same form, whatever the centre of the disc on which they are computed (see [11]). This is not the case when integral of the products of four spherical harmonics is computed on a disc; this integral has different values depending on the centre of the disc and because of this full correlation cannot be expected.

- In the case of the torus, the full correlation result allows to establish immediately the (nonCentral) Limit Theorem for the nodal length in the shrinking set; indeed, the "local" limiting distribution is the same as the "global" one, up to a different scaling constant. On the contrary, to establish a (Central) Limit Theorem for the spherical cap, a different proof is required; indeed we need to apply Theorem 5.2.7 in [57] and hence to compute the fourth cumulant of the leading chaos projection of the nodal length. In passing we stress that the limiting in distribution is Gaussian in the present framework, while it is a linear combinations of chi-square random variables in the torus [46].

- In both the manifolds and their subregions, the fourth chaotic component is the leading term of the chaos expansion of the nodal length and the "Berry's cancellation" phenomenon occurs. However, only in the sphere and in its subdomains, the dominant component is asymptotic to the sample trispectrum, i.e. it has a much simpler form as the integral of the fourth Hermite polynomial, computed only on the eigenfunctions themselves.

Finally, for completeness, Appendix A collects some further background materials, as orthogonal polynomials, some properties of spherical harmonics ([45]), Hilb's asymptotic formula ([77]) and Clebsch-Gordan coefficients ([45] and [75]).

0.0.1 Notation

In the sequel, given any two positive sequences a_n, b_n, we shall write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$. We will use $A \ll B$ and $A = O(B)$ in the same way when A/B is asymptotically bounded. O_φ means that the constants involved depend on the function φ. Given a set $F \subset \mathbb{S}^2$, we denote its area $m(F)$ and for a smooth curve $C \subset \mathbb{S}^2$, $\text{len}(C)$ its length.

CHAPTER 1

Background: Random Fields and Excursion Sets of Spherical Harmonics

The first chapter gives some background material on random fields and spherical harmonics. First, in Section 1.1 some standard notions of random fields on a generic manifold $\mathbb{M}$ are discussed. Then, we define particular regions, namely, the excursion sets of a random field, whose geometry can be described by the so-called Lipschitz Killing Curvatures. In order to compute the mean of these random functionals, the Kac-Rice formula and the so-called Expectation Metatheorem (see [3]) are discussed; we then state the Gaussian Kinematic Formula [3]. We mainly refer to [3], [2] and [6] for further details. Moreover, in Section 1.2, the Stein-Malliavin method is briefly summarized. Actually, in the recent years it has become a very important tool for the investigation of the speed of convergence to the limiting distribution of random processes. Indeed, the Central Limit Theorem, one of the most important results of probability and statistics, supplies asymptotic law of sequences of random variables but it does not give information on the speed of convergence to the limiting distribution. Along this line of research some developments have been achieved by the exploitation of the Stein Malliavin calculus ([57], [58], [61]). These results aim at the investigation of the asymptotic behavior for various probability metrics, such as Kolmogorov, Total Variation and Wasserstein distances. Hence the definitions of Wiener chaoses and the Fourth Moment Theorem will be recalled. Since we will deal with the case of the sphere, i.e. $\mathbb{M} = \mathbb{S}^2$, Section 1.3 discusses this particular case, introducing the spherical harmonics (see [45]) and proving the Spectral Representation Theorem ([43]). In Section 1.4, the previous results known about the analysis of the area, the boundary length and the Euler-Poincaré characteristic for the excursion sets of random eigenfunctions T_ℓ on the unit sphere, defined, for $z \in \mathbb{R}$, as in (0.0.1), are summarized. Briefly, the means were computed by the Gaussian Kinematic Formula, whereas the asymptotic variances were obtained exploiting the projection in Wiener chaoses components. It is noted that all these geometrical features are dominated by a single term, the second (for $z \neq 0$), and then, in view of the Fourth Moment Theorem, Central Limit Theorems have been established. The nodal case $z = 0$ is also discussed and in the last subsection, namely Subsection 1.4.4, in order to address our research towards the case of subregions, some findings on the nodal length in subdomains of the sphere and of the torus are collected. The differences between these two cases are stressed in the Introduction.

1.1 The Geometry of Random Fields

Let us start by introducing some definitions about random fields (see for example [2], [3], [45]).

Definition 1.1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and T a topological space. Then a measurable mapping $f : T \times \Omega \to \mathbb{R}$ is called a real-valued random field.*

Definition 1.2. *Given $(\Omega, \mathscr{F}, \mathbb{P})$, f and T as in Definition 1.1, a real-valued (random) field f on a parameter set T is Gaussian if the (finite-dimensional) distributions of $(f_{t_1}, \dots, f_{t_n})$ are multivariate Gaussian for each $1 \leq n < \infty$ and each $(t_1, \dots, t_n) \in T^n$. The functions*

$$m(t) \equiv m_t = \mathbb{E}[f_t]$$

and

$$C(s,t) = \mathbb{E}[(f_s - m_s)(f_t - m_t)]$$

are called the mean and the covariance functions of f, respectively.

Note that given any set T, a function $m : T \to \mathbb{R}$, and a nonnegative definite function $C : T \times T \to \mathbb{R}$, there exists a unique Gaussian process on T with mean function m and covariance function C. Hence, for a Gaussian process, everything is determined by the mean and the covariance functions.

We can pass now to consider random fields on manifolds. Hence the following definitions are given.

Definition 1.3. *We say that $\mathbb{M}$ is a $topological$ n-$manifold$ if it is a locally compact Hausdorff space such that for each $t \in \mathbb{M}$, there exist an open neighborhood $U \subset \mathbb{M}$ of t, an open set $\tilde{U} \subset \mathbb{R}^n$ and a homeomorphism $\varphi : U \to \tilde{U}$.*

We recall that a $chart$ on $\mathbb{M}$ is a pair (φ, U), where, as above, $U \subset \mathbb{M}$ is open and $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorphism. A collection $\{\varphi_i : U_i \to \mathbb{R}^n\}_{i \in I}$ of charts is said to be C^k $compatible$ if

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j\left(U_i \cap U_j\right) \to \varphi_i\left(U_i \cap U_j\right)$$

is a C^k diffeomorphism for every $i, j \in I$ for which $U_i \cap U_j$ is not empty. If a collection of $charts$ is such that $\bigcup_{i \in I} U_i = \mathbb{M}$ then it is called a C^k-$atlas$. Finally, the component functions $x_1, \dots, x_n$ of a chart (φ, U), defined by $\varphi(t) = (x_1(t), \dots, x_n(t))$, are called *local coordinates* of U. Moreover, a C^k manifold $\mathbb{M}$ is said to be $orientable$ if there is an atlas $\{U_i, \varphi_i\}_{i \in I}$ of $\mathbb{M}$ such that for any pair of charts (U_i, φ_i), $\left(U_j, \varphi_j\right)$ with $U_i \cap U_j \neq \emptyset$, the Jacobian of the map $\varphi_i \circ \varphi_j^{-1}$ has a positive determinant.

Now, in order to define random fields on manifolds, the following definitions and the tangent space of $\mathbb{M}$ at t are introduced.

Definition 1.4. *A function $f : \mathbb{M} \to \mathbb{R}$ is said to be of class C^k if $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is of class C^k, in the Euclidean sense, for every chart in the atlas. We identify with $C^k(\mathbb{M})$ the space of such functions.*

Let us denote by $T_t\mathbb{M}$ the *tangent space* of $\mathbb{M}$ at t. Its elements X_t, while being differential operators, are called the *tangent vectors* at t. Then, Given two manifolds $\mathbb{M}$ and $\mathbb{N}$ and any function $g \in C^1$, $g : \mathbb{M} \to \mathbb{N}$, the *push-$forward$* or $differential$ of g in t is the linear map $g_{*,t} : T_t\mathbb{M} \to T_{g(t)}\mathbb{N}$ defined on charts (U,φ) and (V,ψ) of $\mathbb{M}$ and $\mathbb{N}$, for which $g(U)\cap V \neq \emptyset$, as

$$\left(g_{*,t}(X_t)\right) f = X_t (f \circ g) \tag{1.1.1}$$

for every $f \in C^1(\mathbb{N})$.

Definition 1.5. *A C^k-vector field on a manifold $\mathbb{M}$ is a C^k map that assigns, to each $t \in \mathbb{M}$, a tangent vector $X_t \in T_t\mathbb{M}$.*

In order to understand the assumptions of the so-called Expectation Metatheorem (see [3]), given in the next section, we need to introduce some more notions, in particular the definition of Riemannian metric on a manifold. Then, for $k \geq 0$, we denote $Sym\left(\mathcal{T}_0^k(\mathbb{M})\right)$ the space of symmetric covariant k-tensors on $\mathbb{M}$ (see [3]) and we give the following definitions.

Definition 1.6. *Let $\mathbb{M}$ be a manifold and g be a section of $Sym\left(\mathcal{T}_0^2(\mathbb{M})\right)$, i.e. a map $g : \mathbb{M} \to Sym\left(\mathcal{T}_0^2(\mathbb{M})\right)$ such that $\pi \circ g = id_{\mathbb{M}}$, where π is the usual projection, and for each $t \in \mathbb{M}$, g_t is positive definite; that is, $g_t(X_t, X_t) \geq 0$ for every $t \in \mathbb{M}$ and $X_t \in T_t(\mathbb{M})$, with the equality if and only if $X_t = 0$. Then, a such function, is said to be a Riemmanian metric on $\mathbb{M}$.*

Definition 1.7. *If $\mathbb{M}$ is a C^k manifold and g a C^{k-1} Riemannian metric on $\mathbb{M}$, the couple ($\mathbb{M}$, g) is called a C^k Riemannian Manifold.*

Note that a Riemannian metric is not a true metric on $\mathbb{M}$. However, it does induce a metric τ_g on $\mathbb{M}$. Since g determines the length of a tangent vector, we can define the length of a C^1 curve $c : [0,1] \to \mathbb{M}$ by

$$L(c) = \int_{[0,1]} \sqrt{g_t(c'_t, c'_t)}\, dt$$

and define the metric τ_g by

$$\tau_g(s,t) = \inf_{c \in D^1([0,1];\mathbb{M})_{(s,t)}} L(c),$$

where $D^1([0,1];\mathbb{M})_{(s,t)}$ is the set of all piecewise C^1 maps $c : [0,1] \to \mathbb{M}$ with $c(0) = s, c(1) = t$.

Definition 1.8. *Given $f \in C^1(\mathbb{M}, g)$ we call $gradient$ of f the unique continuous vector field on $\mathbb{M}$ such that*

$$g_t(\nabla f_t, X_t) = X_t f \tag{1.1.2}$$

for every vector field X.

Let f be a random field on a manifold $\mathbb{M}$, not necessary Gaussian. Assume that it has zero mean and, with probability one, is C^2 over $\mathbb{M}$. Then we define

$$g_t(X_t, Y_t) := \mathbb{E}[(X_t f) \cdot (Y_t f)], \tag{1.1.3}$$

where $X_t, Y_t \in T_t\mathbb{M}$. We note that g is closely related to the covariance function $C(s,t) = \mathbb{E}[f_s f_t]$. In particular it follows from (1.1.3) that

$$g_t(X_t, Y_t) = Y_s X_t C(s,t)|_{s=t}. \tag{1.1.4}$$

We recall also the notion of *orthonormal frame bundle*, denoted by $\mathcal{O}(\mathbb{M})$, which is the set of all collections of N unit tangent vectors $(X_t^1, \dots, X_t^N)$ of $\mathbb{M}$ such that $(X_t^1, \dots, X_t^N)$ form an orthonormal basis for $T_t\mathbb{M}$. Then, a C^k*−orthonormal frame field* $\{E_i\}_{1\leq i\leq N}$ is a C^k section of the orthonormal (with respect to g) frame bundle $\mathcal{O}(\mathbb{M})$ (see [3] for details).

1.1.1 The Kac-Rice Formula

We now give in this section the statement of a theorem about the expected number of points at which a vector valued random field takes values in some set, whose proof is given in Adler and Taylor [3]. Before stating properly this theorem, we give an heuristics meaning of it, presented by the same authors in [3]; the hypothesis which will be assumed in Theorem 1.1.1 make this argument rigorous.

Let f be a real-valued stochastic process on the interval $[0, T]$, and $u \in \mathbb{R}$. We denote by

$$N_u^+(0,T) = \#\{t \in [0,T] : f(t) = u, f'(t) > 0\}, \tag{1.1.5}$$

the number of *upcrossings* by f of the level u in the interval $[0, T]$. Here we are going to assume that $N_u^+(0,T)$ is well defined and finite, without worrying about conditions that ensure this.
First, we would like to compute $\mathbb{E}[N_u^+(0,T)]$. To this end, let δ_x be the Dirac delta function at x, hence, for any reasonable test function g,

$$\int_{\mathbb{R}} \delta_x(y)g(y)\,dy = g(x).$$

Suppose that the upcrossing points of f, i.e., those $t \in [0, T]$ at which $f(t) = u$ and $f'(t) > 0$, are isolated, so that each one can be covered by a small interval $[-\varepsilon, \varepsilon]$, $\forall \varepsilon > 0$, in which there are no other upcrossings and throughout which $f' > 0$. Then, treating δ as if it were a smooth function, a change of variable argument gives that

$$1 = \int_{\mathbb{R}} \delta_u(y)\,dy = \int_{-\varepsilon}^{\varepsilon} \delta_u(f(t)) \cdot f'(t)\,dt.$$

Concatenating all such intervals $[-\varepsilon, \varepsilon]$, and noting that there is no contribution to the following integral from outside of them, we obtain

$$N_u^+(0,T) = \int_0^T \delta_u(f(t)) \cdot 1_{[0,\infty]}(f'(t)) \cdot f'(t)\,dt. \tag{1.1.6}$$

Now taking the expectation in both sides of (1.1.6), exchanging orders of integration and assuming that the pairs of random variables $(f(t), f'(t))$ have joint probability densities p_t, we get that

$$\begin{aligned}\mathbb{E}[N_u^+(0,T)] &= \int_0^T dt \int_{-\infty}^{\infty} dx \int_0^{\infty} dy\, y\, \delta_u(x) p_t(x,y) \\ &= \int_0^T \int_0^{\infty} y\, p_t(u,y)\,dy\,dt.\end{aligned} \tag{1.1.7}$$

That is the most basic form of the Kac-Rice formula, and it holds for all processes on $\mathbb{R}$ for which the various operations above are justifiable. Note, however, that it requires no specific distributional assumptions on the process f. However, it turns out to be quite difficult to compute the integral in (1.1.7) if f is not Gaussian or a function of Gaussian processes.

Now we give the proper statement of the theorem. For some $n, k \geq 1$ let $f = \left(f^1, \ldots, f^n\right)$ and $g = \left(g^1, \ldots, g^k\right)$, respectively, be $\mathbb{R}^n$- and $\mathbb{R}^k$-valued random fields. We need two sets, $T \subset \mathbb{R}^n$ a compact parameter set such that its boundary ∂T has dimension $n-1$, and $B \subset \mathbb{R}^k$ an open set such that ∂B has dimension $k-1$. As usual, ∇f denotes the gradient of f. Since f takes values in $\mathbb{R}^n$, this is now an $n \times n$ matrix of first order partial derivatives of f, i.e.

$$(\nabla f)(t) \equiv \nabla f(t) \equiv \left(f^i_j(t)\right)_{i,j=1,\ldots,n} \equiv \left(\frac{\partial f^i(t)}{\partial t_j}\right)_{i,j=1,\ldots,n}$$

All the derivatives here are assumed to exist in an almost sure sense.

Theorem 1.1.1 (Kac-Rice Theorem [3], [6]). *Let f, g, T and B as above. Assume that the following conditions are satisfied for some $u_0 \in \mathbb{R}^n$:*

1. *All components of $f, \nabla f$ and g are $a.s.$ continuous and have finite variances over T.*
2. *For all $t \in T$, the marginal densities $p_t(x)$ of f (implicitly assumed to exist) are continuous at $x = u_0$.*
3. *The conditional densities $p_t(x|\nabla f(t), g(t))$ of $f(t)$ given $g(t)$ and $\nabla f(t)$ (implicitly assumed to exist) are bounded above and continuous at $x = u_0$, uniformly in $t \in T$.*
4. *The conditional densities $p_t(z|f(t) = x)$ of* $\det(\nabla f(t))$ *given $f(t) = x$, are continuous for z and x in neighborhoods of 0 and u_0, respectively, uniformly in $t \in T$.*
5. *The conditional densities $p_t(z|f(t) = x)$ of $g(t)$ given $f(t) = x$, are continuous for all z and x in neighborhoods of u_0, uniformly in $t \in T$.*
6. *The following moment condition holds*
$$\sup_{t\in T} \max_{1\leq i,j\leq n} \mathbb{E}\left[\left|f^i_j(t)\right|^n\right] < \infty. \tag{1.1.8}$$
7. *The moduli of continuity with respect to the usual Euclidean norm of each of the components of f, ∇f and g satisfy*
$$\mathbb{P}(\omega(\nu) > \varepsilon) = o(\nu^n) \quad \text{as } \nu \to 0 \tag{1.1.9}$$
for any $\varepsilon > 0$.

Then, if

$$N_{u_0} \equiv N_{u_0}(T) \equiv N_{u_0}(f, g; T, B)$$

denotes the number of points in T for which

$$f(t) = u_0 \in \mathbb{R}^n, \quad g(t) \in B \subset \mathbb{R}^k$$

and $p_t(x, \nabla y, v)$ denotes the joint density of $(f_t, \nabla f_t, g_t)$, we have, with $d := \frac{n(n+1)}{2} + k$

$$\mathbb{E}\left[N_{u_0}\right] = \int_T \int_{\mathbb{R}^d} |\det \nabla y| \, \mathbf{1}_B(g(t)) p_t(u_0, \nabla y, v) \, d(\nabla y) dv dt. \tag{1.1.10}$$

It is sometimes more convenient to write this as

$$\mathbb{E}\left[N_{u_0}\right] = \int_T \mathbb{E}\left[|\det \nabla f(t)| \, \mathbf{1}_B(g(t)) | f(t) = u_0\right] p_t(u_0) dt, \tag{1.1.11}$$

where p_t here is the density of $f(t)$.

Corollary 1.1.2. *Let f and g be centered Gaussian fields over T that satisfy the conditions of theorem* 1.1.1. *If for each $t \in T$, the joint distributions of $(f(t), \nabla f(t), g(t))$ are nondegenerate, then (1.1.10) and (1.1.11) hold.*

Now, we move from an Euclidean setting to a manifold one. Here we always follow Adler and Taylor in [3].

Let $(\mathbb{M}, g)$ be an n-dimensional Riemannian manifold, and let $f : \mathbb{M} \to \mathbb{R}^n$ be C^1. Fix an orthonormal frame field E. Then ∇f_E denotes the vector field whose coordinates are given by

$$(\nabla f_E)_i \equiv \nabla f_{E_i} \doteq (\nabla f)(E_i) \equiv E_i f \tag{1.1.12}$$

If $f = (f^1, \dots, f^n)$ takes values in $\mathbb{R}^n$ then ∇f_E is an $n \times n$ matrix with elements $\nabla f^j_{E_i}$.

Theorem 1.1.3. *Let $\mathbb{M}$ be a compact, oriented, n-dimensional C^1 manifold with a C^1 Riemannian metric g. Let $f = (f^1, \dots, f^n) : \mathbb{M} \to \mathbb{R}^n$ and $h = (h^1, \dots, h^k) : \mathbb{M} \to \mathbb{R}^k$ be random fields on $\mathbb{M}$. For an open set $B \subset \mathbb{R}^k$ for which ∂B has dimension $k-1$, and a point $u_0 \in \mathbb{R}^n$, let*

$$N_{u_0} \equiv N_{u_0}(\mathbb{M}) \equiv N_{u_0}(f, h; \mathbb{M}, B)$$

denote the number of points $t \in \mathbb{M}$ for which

$$f(t) = u_0, \quad h(t) \in B.$$

Assume that the following conditions are satisfied for some orthonormal frame field E:

1. *All components of $f, \nabla f_E$ and g are a.s. continuous and have finite variances over $\mathbb{M}$.*
2. *For all $t \in \mathbb{M}$, the marginal densities $p_t(x)$ of $f(t)$ (implicitly assumed to exist) are continuous at $x = u_0$.*
3. *The conditional densities $p_t(x|\nabla f_E(t), h(t))$ of $f(t)$ given $h(t)$ and $\nabla f_E(t)$ (implicitly assumed to exist) are bounded above and continuous at $x = u_0$, uniformly in $t \in \mathbb{M}$.*
4. *The conditional densities $p_t(z|f(t) = x)$ of $\det(\nabla f^i_{E_j}(t))$ given $f(t) = x$, are continuous for z and x in neighborhoods of 0 and u_0, respectively, uniformly in $t \in \mathbb{M}$.*
5. *The conditional densities $p_t(z|f(t) = x)$ of $h(t)$ given $f(t) = x$, are continuous for all z and x in neighborhoods of u_0, uniformly in $t \in \mathbb{M}$.*
6. *The following moment condition holds*

$$\sup_{t \in \mathbb{M}} \max_{1 \le i,j \le n} \mathbb{E}\left[\left|\nabla f^j_{E_i}(t)\right|^n\right] < \infty. \tag{1.1.13}$$

7. *The moduli of continuity with respect to the metric induced by g of each components of h, each component of f and each $\nabla f^j_{E_i}$ all satisfy, for any $\varepsilon > 0$,*

$$\mathbb{P}(\omega(\nu) > \varepsilon) = o(\nu^n) \quad \text{as } \nu \to 0 \tag{1.1.14}$$

for any $\varepsilon > 0$.

Then

$$\mathbb{E}[N_{u_0}] = \int_{\mathbb{M}} \mathbb{E}[|\det(\nabla f_E)| \mathbf{1}_B(h)| f = u_0] p(u_0) dVol_g \tag{1.1.15}$$

where $p\,(= p_t)$ is the density of $f\,(= f_t)$ and $dVol_g$ is the volume element on $\mathbb{M}$ induced by g.

The conditions of the Theorem do not depend on the choice of orthonormal frame field. Hence, they hold for all orthonormal frame fields and also for any bounded vector field X. Furthermore, since N_{u_0} does not depend on the metric g, the same is true for its expectation. Consequently, even if we need the metric g in order to evaluate the right hand side of equation (1.1.15), the final result is metric-independent.
As in the Euclidean case, the above Theorem assume a much easier form if the random field is Gaussian.

Corollary 1.1.4. *Let $(\mathbb{M}, g)$ be a Riemannian manifold satisfying the conditions of Theorem 1.1.3. Let f and h be centered Gaussian random fields over $\mathbb{M}$. Then if f, h and ∇f_E are a.s. continuous over $\mathbb{M}$, and if for every $t \in \mathbb{M}$, the joint distributions of $(f(t), \nabla f_E(t), h(t))$ are nondegenerate, then (1.1.15) holds.*

1.1.2 Lipschitz Killing Curvatures

In this section we introduce the so-called *intrinsic volumes* or *Lipschitz Killing Curvatures* (LKC), some functionals which give a complete characterization of some "regular" sets. These geometrical features are a class of $n+1$ additional position and rotation invariant functionals, denoted by $\{\mathscr{L}_j\}_{j=0}^n$, that are also additive in the sense that

$$\mathscr{L}_j(A \cup B) = \mathscr{L}_j(A) + \mathscr{L}_j(B) - \mathscr{L}_j(A \cap B), \tag{1.1.16}$$

where $A, B \subset \mathbb{R}^n$, and scale with dimensionality, i.e.

$$\mathscr{L}_j(\lambda A) = \lambda^j \mathscr{L}_j(A), \quad \lambda > 0, \tag{1.1.17}$$

where $\lambda A = \{t : t = \lambda s, s \in A\}$. In particular, $\mathscr{L}_0$ is the Euler-Poincaré characteristic, which actually enjoys these conditions. Indeed, one of the way it can be defined is as an integer-valued functional χ satisfying (1.1.16) and (1.1.17), such that

$$\chi(A) = \begin{cases} 0, & \text{if } A = \emptyset \\ 1, & \text{if } A \neq \emptyset \text{ is basic.} \end{cases}$$

A full description of the LKC and a proper definition of basic convexes, can be found in [3]. One way to define them is through the *Tube formula* or *Steiner formula*. For $\mathbb{M} \subset \mathbb{R}^n$ convex and $\rho > 0$, let

$$Tube(\mathbb{M}, \rho) = \{x \in \mathbb{R}^n : d(x, \mathbb{M}) \leq \rho\}$$

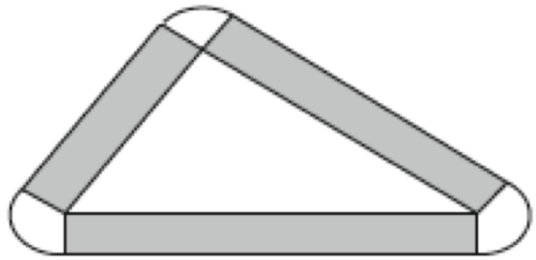


Figure 1.1: The tube around a triangle (figure of [3]).

be the Tube of radius ρ around $\mathbb{M}$, where

$$d(x,\mathbb{M}) = \inf_{y\in\mathbb{M}} |x-y|$$

is the usual Euclidean distance from the point x to the set $\mathbb{M}$. We have the following result.

Theorem 1.1.5. *Let us suppose $\mathbb{M} \subset \mathbb{R}^n$ is a C^2, locally convex manifold. For ρ small enough,*

$$\mu_n(Tube(\mathbb{M},\rho)) = \sum_{i=0}^{n} \rho^{n-i}\omega_{n-i}\mathscr{L}_i(\mathbb{M})$$

where μ_n is the Lebesgue measure on $\mathbb{M}$, $\mathscr{L}_0(\mathbb{M})$ is the Euler-Poincarè characteristic of $\mathbb{M}$, ω_k is the Lebesgue measure of the ball of radius 1 in $\mathbb{R}^k$, i.e., $\omega_k = \mu_k(B_1(0)) = \frac{\pi^{k/2}}{\Gamma(\frac{k}{2}+1)}$, where $\Gamma(\cdot)$ is, as usual, the Gamma function.

LKC depend on the Riemaniann metric and they are a measure of $k-$dimension size of the Riemaniann manifold $\mathbb{M}$. Note that when $\mathbb{M}$ is a C^2 domain in $\mathbb{R}^2$, the LKC are $\mathscr{L}_2(\mathbb{M}) = \text{Area}(\mathbb{M})$, $\mathscr{L}_1(\mathbb{M}) = \dfrac{\text{length}(\partial\mathbb{M})}{2}$, $\mathscr{L}_0(\mathbb{M})$ is the Euler-Poincaré characteristic. For instance, in the picture 1.1 ([3]), to find the area of the enlarged triangle, one needs only to sum three terms: the area of the inner triangle; the area of the three rectangles (i.e. the perimeter multiplied by ρ); the area of the three corner sectors (the union of these sectors will always give a disk of Euler characteristic 1 and radius ρ). This means that,

$$\text{area}(Tube(\mathbb{M},\rho)) = \pi\rho^2\chi(\mathbb{M}) + \rho\,\text{perimeter}(\mathbb{M}) + \text{area}(\mathbb{M}).$$

Remark 1.1.6. *The Lipschitz Killing Curvatures are equivalent to the so-called Minkowski functionals, up to the labelling of the indexes and constants. They are defined as*

$$\mathscr{M}_j(\mathbb{M}) = (j!\omega_j)\mathscr{L}_{n-j}(\mathbb{M}).$$

Then, Steiner's formula can be written as

$$\mu_n(Tube(\mathbb{M},\rho)) = \sum_{j=0}^{n} \frac{\rho^j}{j!}\mathscr{M}_j(\mathbb{M}).$$

Although the $\mathscr{L}_j$ are independent of the ambient space in which they are, this is not true for the Minkowski functionals; this is due to the reversed numbering system and the choice of constants.

1.1.3 Gaussian Kinematic Formula

Let us now introduce, in this section, some particular subsets of a given manifold $\mathbb{M}$, related with the random field f; the so-called *excursion sets* for f over $\mathbb{M}$ (as defined in (0.0.1) for $\mathbb{M} = \mathbb{S}^2$):

$$A_z(f;\mathbb{M}) := \{x \in \mathbb{M} : f(x) \geq z\}. \tag{1.1.18}$$

As we mentioned in Section 1.1.2, the LKC supply a complete characterization of the geometry of these sets. Since they are random functionals, the first step is the investigation of their means; the Gaussian Kinematic Formula (GKF) gives a way to compute them. The first result was established for the Euler-Poincaré characteristic and it is the following.

Theorem 1.1.7. *Let f be a centered, unit-variance Gaussian random field on an n-dimensional C^2 manifold $\mathbb{M}$, such that the sample of f are, with probability one, Morse functions* [1] *over $\mathbb{M}$. Then*

$$\mathbb{E}[\chi(A_z(f,\mathbb{M}))] = \sum_{j=0}^{n} \mathscr{L}_j^f(\mathbb{M})\rho_j(z), \tag{1.1.19}$$

where χ is the Euler-Poincaré characteristic, the ρ_j are given by

$$\rho_j(z) = (2\pi)^{-\frac{j+1}{2}} H_{j-1}(z) e^{-\frac{z^2}{2}}, \quad j \geq 0, \tag{1.1.20}$$

H_j is the j-th Hermite polynomial, defined as $H_0 \equiv 1$, for $j \geq 1$, (see for instance [57])

$$H_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}},\ x \in \mathbb{R} \tag{1.1.21}$$

and

$$H_{-1}(x)e^{-x^2/2} := \sqrt{2\pi}(1-\Phi(x)),$$

where $\Phi(x)$ is the distribution function of a standard Gaussian random variable, and $\mathscr{L}_j^f(\mathbb{M})$ are the Lipschitz Killing curvatures of $\mathbb{M}$ (computed with respect to the metric induced by f).

Thus, Theorem 1.1.7 gives the expectation of the Euler-Poincaré characteristic of the excursion sets. For the other Lipschitz Killing Curvatures we have the theorem below.

Theorem 1.1.8 (Gaussian Kinematic Formula [3])**.** *Suppose that $\mathbb{M} \subset \mathbb{R}^n$ is a C^2 manifold and that f is a centered, unit-variance, Gaussian process on $\mathbb{M}$ such that every sample of f is, with probability one, a Morse function over $\mathbb{M}$. Then for every $0 \leq j \leq dim(\mathbb{M})$,*

$$\mathbb{E}\left[\mathscr{L}_j^f(A_z(f,\mathbb{M}))\right] = \sum_{l=0}^{dim\mathbb{M}-j} \begin{bmatrix} j+l \\ l \end{bmatrix} \rho_l(z)\mathscr{L}_{j+l}^f(\mathbb{M}) \tag{1.1.22}$$

[1] Recall that a function $f = \mathbb{R}^n \to \mathbb{R}$ is a Morse function over a rectangle $T \subset \mathbb{R}^n$ if

1. f is C^2 on an open neighborhood of T.
2. The critical points of $f_{|\partial_k T}$ are nondegenerate for all $k = 0, \dots, n$.
3. $f_{|\partial_k T}$ has no critical points on $\bigcup_{j=0}^{k} \partial_j T$ for all $k = 0, \dots, n$.

Using a chart-argument this definition can be extended to an abstract manifold. Moreover in the Gaussian scenario, see [3], the following condition is sufficient for f to be a Morse function:

$$\max_{i,j} \left| C_{f_{ij}}(t,t) + C_{f_{ij}}(s,s) - 2C_{f_{ij}}(s,t) \right| \leq K \left| \ln|t-s| \right|^{-(1+\alpha)}$$

for some constant K and α and all $s, t \in \mathbb{M}$.

where

$$\begin{bmatrix} N \\ j \end{bmatrix} = \frac{[N]!}{[N-j]![j]!} = \binom{N}{j} \frac{\omega_N}{\omega_{N-j}\omega_j}, \tag{1.1.23}$$

$[N]! = N!\omega_N$ *and the* $\mathscr{L}_k^f$ *on both sides of this equation are computed with respect to the metric induced by* f. *The* ρ_l *remain as in (*1.1.20*).*

Note that, in (1.1.22) the quantity in the left-hand side depends on the process f, the threshold z and the manifold $\mathbb{M}$. On the other hand, the right-hand side is easier to compute: ρ depends only on the threshold z, the functionals are not evaluated on the excursion sets (which depend on z) but on the entire manifold and they depend only on the covariance function. In particular, in the isotropic case (see below), the metric is simply proportional to the Euclidean case; the scaling factor λ equals the square root of the derivatives of the covariance function at the origin. In Section 1.3.2 we will consider the example of the Sphere and the role of the scaling factor λ will be clearer.

1.2 Stein-Malliavin Normal approximations

The Malliavin-Stein approach [57] is a powerful technique that combines ideas from Malliavin Calculus and Stein's method to produce strong bounds on the distance to Gaussinity for random sequences belonging to so-called Wiener chaoses. We briefly discuss this topic in this section in order to deal with the Fourth Moment Theorem, which will be used along the thesis. For a complete treatment we refer to [57] (see also [25]).

Let H be a real separable Hilbert space, with inner product $\langle \cdot, \cdot \rangle_H$.

Definition 1.9. *An isonormal Gaussian process over* H *is a collection* $X = \{X(h) : h \in H\}$ *of jointly Gaussian random variables defined on some probability space* $(\Omega, \mathscr{F}, \mathbb{P})$*, such that* $\mathbb{E}[X(h)X(g)] = \langle h, g \rangle_H$ *for every* $h, g \in H$.

We assume that $\mathscr{F}$ is generated by X and let us consider the measure space $(A, \mathscr{A}, \mu)$, where A is a Polish space [2], $\mathscr{A}$ the associated $\sigma-$field and μ a positive, $\sigma-$finite and non-atomic measure. The space $H = L^2(A, \mathscr{A}, \mu)$ is then a real separable Hilbert space with inner product $\langle f, g \rangle_H = \int_A g(a)h(a)\,\mu(da)$. For every $h \in H$, the isonormal Gaussian process

$$X(h) = \int_A h(a)W(da)$$

is defined as the Wiener-Itô integral of h with respect to the Gaussian family $W = \{W(B) : B \in \mathscr{A}, \mu(B) < \infty\}$ such that for every $B, C \in \mathscr{A}$ of finite $\mu-$measure $\mathbb{E}[W(B)W(C)] = \mu(B \cap C)$.

Let us now introduce the Wiener chaoses. We define $C_0 := \mathbb{R} \subset L^2(\Omega)$ the space of constants, and for $q \geq 1$, C_q be the closure in $L^2(\Omega)$ of the linear subspace generated by random variables of the form

$$H_q(X(f)) \;\; f \in H, \; ||f||_H = 1,$$

where H_q is the $q-$th Hermite polynomial, defined in (1.1.21). C_q is called the $q-$th Wiener chaos. It is important to recall the following property.

[2] A Polish space is a separable completely metrizable topological space; that is, a space homeomorphic to a complete metric space that has a countable dense subset.

Proposition 1.2.1. *Let $Z_1, Z_2 \sim \mathcal{N}(0,1)$ be jointly Gaussian. Then, for all $n, m \geq 0$*

$$\mathbb{E}[H_{q_1}(Z_1)H_{q_2}(Z_2)] = q_1!\{\mathbb{E}[Z_1Z_2]\}^{q_1}\delta_{q_1}^{q_2}. \tag{1.2.1}$$

For the proof see [57].
A significant fact is that each random variable $F \in L^2(\Omega)$ admits a unique decomposition in the $L^2(\Omega)$—sense of the form

$$F = \sum_{q=0}^{\infty} J_q(F), \tag{1.2.2}$$

where $J_q : L^2(\Omega) \to C_q$ is the orthogonal projection operator. Equivalently, the chaotic Wiener-Ito expansion holds, namely

$$L^2(\Omega) = \bigoplus_{q=0}^{+\infty} C_q, \tag{1.2.3}$$

the above sum being orthogonal from (1.2.1). Note that $J_0(F) = \mathbb{E}[F]$. Let $H^{\otimes q}$ and $H^{\odot q}$ be the q—th tensor product and the q—th symmetric tensor product of H respectively: $H^{\otimes q} = L^2(\Omega^q, \mathscr{F}, \mu^q)$ and $H^{\odot q} = L^2_s(\Omega^q, \mathscr{F}^q, \mu^q)$; where L^2_s denotes the space of square integrable and symmetric functions. For $(x_1, x_2, \ldots, x_q) \in \Omega^q$ and $f \in H$, we have

$$f^{\otimes q}(x_1, x_2, \ldots, x_q) = f(x_1)f(x_2)\ldots f(x_q).$$

Moreover, for $f \in H$ and $q \geq 1$, the map $I_q(f^{\otimes q}) := H_q(X(f))$ can be extended to a linear isometry between $H^{\odot q}$, with the modified norm $\sqrt{q!}||\cdot||_{H^{\odot q}}$, and the q—th Wiener chaos C_q. For $q = 0$, we set $I_0(c) = c \in \mathbb{R}$ and the relation in (1.2.2) becomes

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{1.2.4}$$

where $f_0 = \mathbb{E}[F]$ and for $q \geq 1$, $f_q \in H^{\odot q}$ are uniquely determined. In this context, it is known that for $h \in H^{\odot q}$, $I_q(h)$ and the multiple Wiener-Ito integral of h with respect to the Gaussian measure W coincide, i.e.

$$I_q(h) = \int_{\Omega^q} h(x_1, x_2, \ldots, x_q)dW(x_1)dW(x_2)\ldots dW(x_q) \tag{1.2.5}$$

and so, in words, F in (1.2.4) can be seen as a series of (multiple) stochastic integrals. For every $p, q \geq 1, f \in H^{\otimes p}, g \in H^{\otimes q}$ and $r = 1, 2, \ldots p \wedge q$, the so-called *contraction* of f and g of order r is the element $f \otimes_r g \in H^{\otimes p+q-2r}$ defined as

$$\begin{aligned}&(f \otimes_r g)(x_1, \ldots, x_{p+q-2r}) = \\ &\int_{\Omega^r} f(x_1, \ldots, x_{p-r}, y_1, \ldots, y_r)g(x_{p-r+1}, \ldots, x_{p+q-2r}, y_1, \ldots, y_r)d\mu(y_1)\ldots d\mu(y_r).\end{aligned} \tag{1.2.6}$$

For $p = q = r$, we have $f \otimes_r g = \langle f, g\rangle_{H^{\otimes r}}$ and for $r = 0, f \otimes_0 g = f \otimes g$. Denoting by $f\tilde{\otimes}_r g$ the canonical symmetrization of $f \otimes_r g$, the following multiplication formula is well-known. For $p, q = 1, 2, \ldots f \in H^{\odot q}, g \in H^{\odot q}$, we have

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r}I_{p+q-2r}(f\tilde{\otimes}_r g).$$

1.2.1 Malliavin Derivative

Let us define now the Malliavin Derivative. We consider $F = I_q(f) \in C_q$ a random variable, where $f \in H^{\odot q}$, then for $q, r \geq 1$, the $r-$th Malliavin derivative is the element $D^r F : \Omega \to H^{\odot r}$ given by

$$D^r F := \frac{q!}{(q-r)!} I_{q-r}(f),$$

for $r \leq q$, and $D^r F = 0$ for $r > q$. So that, the $r-$th Malliavin derivative of the random variable F in (1.2.4) could be written as

$$D^r F = \sum_{q=r}^{+\infty} \frac{q!}{(q-r)!} I_{q-r}(f_q).$$

Definition 1.10. *We say that F as in (1.2.4) belongs to $\mathbb{D}^{r,q}$ if*

$$||F||_{\mathbb{D}^{r,q}} := (\mathbb{E}[|F|^q] + \ldots \mathbb{E}[||D^r F||^q_{H^{\odot r}}])^{\frac{1}{q}} < +\infty.$$

It is easy to check that $F \in \mathbb{D}^{r,2}$ if and only if

$$\sum_{q=r}^{\infty} q^r q! ||f_q||^2_{H^{\odot q}} < +\infty.$$

We introduce also the generator of the Ornstein-Uhlenbeck semigroup, defined as

$$L := -\sum_{q=1}^{+\infty} q J_q.$$

The domain of L consists of $F \in L^2(\mathbb{P})$ such that

$$\sum_{q=1}^{+\infty} q^2 ||J_q(F)||^2_{L^2(\mathbb{P})} < +\infty.$$

Then, it is possible to define the pseudo-inverse operator of L as

$$L^{-1} = -\sum_{q=1}^{\infty} \frac{1}{q} J_q$$

and for each $F \in L^2(\mathbb{P})$, it satisfies

$$LL^{-1}F = F - \mathbb{E}[F].$$

1.2.2 Fourth Moment Theorem

In this subsection we finally report the Fourth Moment Theorem, which allows us to establish Quantitative Central Limit Theorems. First, let us introduce the following definitions.

Definition 1.11. *Let $\mathscr{H}$ be a class of functions such that $\mathbb{E}[|h(X)|] < \infty$ for all $h \in \mathscr{H}$ and $\mathbb{E}h(X) = \mathbb{E}h(Y)$ for all $h \in \mathscr{H}$ if and only if $X \stackrel{d}{=} Y$ (meaning that X and Y have the same distribution law). Then $\mathscr{H}$ is called a separating class, and we can introduce the probability metric*

$$d_{\mathscr{H}}(X,Y) := \sup_{h \in \mathscr{H}} |\mathbb{E}h(X) - \mathbb{E}h(Y)|.$$

Taking

$$\mathcal{H}_{Kol} := \{h(\cdot) = 1_{(-\infty,x]}(\cdot), x \in \mathbb{R}\},$$

where $1_A(\cdot)$ is, as usual, the indicator function of the set A, which takes value one if the condition in the argument is satisfied, zero otherwise, yields the so-called *Kolmogorov* distance

$$d_{Kol}(X,Y) := \sup_{z\in\mathbb{R}} |F_X(z) - F_Y(z)|;$$

whereas considering

$$\mathcal{H}_{TV} := \{h(\cdot) = 1_A(\cdot), A \in \mathcal{B}(\mathbb{R})\},$$

one has the so-called *total variation* distance

$$d_{TV}(X,Y) := \sup_{A\in\mathcal{B}(\mathbb{R})} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|;$$

finally for

$$\mathcal{H}_W := \{h : ||h||_{Lip} \leq 1\}$$

one obtains the so-called *Wasserstein* distance

$$d_W(X,Y) := \sup_{h:||h||_{Lip\leq 1}} |\mathbb{E}h(X) - \mathbb{E}h(Y)|.$$

Remark 1.2.2. *It is easy to see that the convergence in any of these three metrics implies convergence in distribution. Moreover, convergence in total variation implies convergence in Kolmogorov distance, but no other implication holds in general.*

To establish convergence with these probability metrics, the Stein's equation is exploited. Actually, note first that for $Z \sim \mathcal{N}(0,1)$

$$\mathbb{E}Xf(X) - \mathbb{E}f(X) = 0;$$

it is possible to verify this relation by checking it on polynomials, recalling that for n odd one has

$$\mathbb{E}Z^{n+1} = n\mathbb{E}Z^{n-1}.$$

For n even, both sides are seen to be zero. The idea is that for arbitrary random variables W, if $\mathbb{E}Wf(W) - \mathbb{E}f'(W)$ is close to zero, the distribution will be close to Gaussian. The following result summarizes the connection between stochastic calculus and probability metrics.

Proposition 1.2.3 (Theorem 5.1.3 [57])**.** *Let $F \in \mathbb{D}^{1,2}$ such that $\mathbb{E}[F] = 0, \mathbb{E}[F^2] = \sigma^2 < +\infty$. Then we have, for $Z \sim \mathcal{N}(0,\sigma^2)$,*

$$d_W(F,Z) \leq \sqrt{\frac{2}{\sigma^2\pi}}\mathbb{E}[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H|].$$

Also, assuming in addition that F has a density,

$$d_{TV}(F,Z) \leq \frac{2}{\sigma^2}\mathbb{E}[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H|],$$

$$d_K(F,Z) \leq \frac{1}{\sigma^2}\mathbb{E}[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H|].$$

Moreover if $F \in \mathbb{D}^{1,4}$, we have also

$$\mathbb{E}[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H|] \leq \sqrt{\mathrm{Var}[\langle DF, -DL^{-1}F\rangle_H]}.$$

When $F = I_q(f)$ for $f \in H^{\odot q}, q \geq 2$ then from [57] Theorem 5.2.6,

$$\mathbb{E}[|\sigma^2 - \langle DF, -DL^{-1}F\rangle_H|] \leq \sqrt{\mathrm{Var}\left(\frac{1}{q}||DF||_H^2\right)}.$$

Furthermore, it is possible to prove that: for $q \geq 2$

$$\mathrm{Var}\left(\frac{1}{q}||DF||_H^2\right) \leq \frac{q-1}{3q} cum_4(F) \leq (q-1)\mathrm{Var}\left(\frac{1}{q}||DF||_H^2\right),$$

where

$$cum_4(F) := \mathbb{E}[F^4] - 3(\sigma^2)^2 \tag{1.2.7}$$

is the *fourth cumulant* of F. As a consequence the following result holds.

Corollary 1.2.4 (Fourth Moment Theorem, Theorem 5.2.7 [57]). *Let $F_n, n \geq 1$, be a sequence of random variables belonging to the q—th Wiener chaos, for some fixed integer $q \geq 2$. Then we have the following bound: for $\mathscr{D} \in \{K, TV, W\}$*

$$d_{\mathscr{D}}\left(\frac{F_n}{\sqrt{\mathrm{Var}(F_n)}}, Z\right) \leq C_{\mathscr{D}(q)} \sqrt{\frac{cum_4(F_n)}{\mathrm{Var}(F_n)^2}}, \tag{1.2.8}$$

where $Z \sim \mathscr{N}(0,1)$, for some constant $C_{\mathscr{D}}(q) > 0$. In particular, if the right hand side in (1.2.8) vanishes for $n \to +\infty$, then

$$\frac{F_n}{\sqrt{\mathrm{Var}(F_n)}} \to_d Z, \tag{1.2.9}$$

where $\to_d$ denotes convergence in distribution.

Remark 1.2.5. *For zero mean random variables, $\mathbb{E}\left[\frac{W}{\sqrt{\mathrm{Var}(W)}}\right]^4 - 3$ is equal to the fourth-order cumulant of $\frac{W}{\sqrt{\mathrm{Var}(W)}}$. The classical method of moments/cumulants allows us to establish a CLT by proving that all cumulants of order larger than 2 converge to zero.*

1.3 Spherical Harmonics

Since the thesis focuses on the two-dimensional sphere $\mathbb{S}^2$, from this section on, we turn our attention to this particular manifold, hence, we take $\mathbb{M} = \mathbb{S}^2$.

Let $T(x, \omega)$, $x \in \mathbb{S}^2, \omega \in \Omega$, a random field on the sphere, i.e. T is a measurable application from $\mathbb{S}^2 \times \Omega \to \mathbb{R}$ for the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We assume that the field is isotropic ([45], [9]), meaning that $T(g\cdot) \stackrel{d}{=} T(\cdot)$ for all $g \in SO(3)$ (the special group of rotations in $\mathbb{R}^3$), that it has zero mean $\mathbb{E}T(x) = \int T(x,\omega)d\mathbb{P}(\omega) = 0$ and that it has finite variance $\mathbb{E}T^2(x) = \int T^2(x,\omega)d\mathbb{P}(\omega) < \infty$. Under these conditions, the field can be shown to be mean-square continuous ([27], [44]), meaning that

$$\lim_{x_n \to x} \mathbb{E}[|T(x_n) - T(x)|^2] = 0, \text{ for all } x \in \mathbb{S}^2.$$

We denote with Γ the covariance function

$$\Gamma(x, y) := \mathbb{E}[T(x)T(y)],\ x, y \in \mathbb{S}^2;$$

by isotropy, we have that

$$\Gamma(x,y) = \Gamma(x',y'),$$

for all pairs $\{(x,y),(x',y')\}$ such that $\langle x,y\rangle = \langle x',y'\rangle$.
Moreover, let us also consider the Helmholtz equation

$$\Delta_{\mathbb{S}^2} T_\ell + \lambda_\ell T_\ell = 0, \quad T_\ell : \mathbb{S}^2 \to \mathbb{R}, \tag{1.3.1}$$

where $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator on $\mathbb{S}^2$, defined as usual as

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\right\} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2},\ 0 \le \theta \le \pi,\ 0 \le \varphi < 2\pi,$$

and $\lambda_\ell := \ell(\ell+1),\ \ell = 0,1,\dots$. For a given eigenvalue λ_ℓ, the corresponding eigenspace is the $(2\ell+1)$–dimensional space of spherical harmonics of degree ℓ. A standard, complex-valued L^2 basis $\{Y_{\ell m}(\cdot)\}_{m=-\ell,\dots,\ell}$ can be defined as (see [45] page 64)

$$Y_{\ell m}(\theta,\varphi) := \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}P_{\ell m}(\cos\theta)\exp(im\varphi)\text{ , for } m \ge 0\text{ ,} \tag{1.3.2}$$

$$Y_{\ell m}(\theta,\varphi) := (-1)^m \overline{Y}_{\ell,-m}(\theta,\varphi)\text{ , for } m < 0\text{ ,} \tag{1.3.3}$$

where $P_{\ell m}(\cdot)$ denotes the associated Legendre functions (see Appendix A.1.2). We can hence consider random eigenfunctions of the form

$$T_\ell(x) = \sqrt{\frac{4\pi}{2\ell+1}}\sum_{m=-\ell}^{\ell} a_{\ell m}Y_{\ell m}(x), \tag{1.3.4}$$

where the coefficients $\{a_{\ell m}\}$ are independent, safe for the condition $a_{\ell m} = (-1)^m\overline{a}_{\ell,-m}$; for $m \neq 0$ they are standard complex-valued Gaussian variables, while $a_{\ell 0}$ is a standard real-valued Gaussian variable. The random fields $\{T_\ell(x), x \in \mathbb{S}^2\}$ are Gaussian and isotropic. Also, we have that

$$\begin{aligned}\mathbb{E}[T_\ell(x)] &= 0,\text{ and } \mathbb{E}[T_\ell(x)^2] = 1,\\ \mathbb{E}[T_\ell(x)T_\ell(y)] &= P_\ell(\cos d(x,y)),\end{aligned}$$

where P_ℓ are the Legendre polynomials (see Appendix A.1.1) and $d(x,y)$ is the spherical geodesic distance between x and y, i.e.

$$d(x,y) = \arccos(\langle x,y\rangle).$$

1.3.1 The Spectral Representation Theorem on the Sphere

In this subsection we see the Spectral Representation Theorem on the Sphere, which in [45] is given as a special case of the Stochastic Peter-Weyl Theorem. We give here a more direct proof (see also [43]), which does not involve the Group Representation Theory. Before proving it we need to recall the following result.

Theorem 1.3.1 (Schoenberg (1942) [71]). *Assume $\Gamma(x)$ is a positive definite continuous function $\Gamma : [-1,1] \to \mathbb{R}$. Then there exists a sequence of positive weights $\{C_\ell\}$ such that for all x we have*

$$\Gamma(x) = \sum_{\ell=1}^{\infty}\frac{2\ell+1}{4\pi}C_\ell P_\ell(x).$$

We can also write,

$$\Gamma(x,y)=\Gamma(\langle x,y\rangle)=\sum_{\ell}\frac{2\ell+1}{4\pi}C_\ell P_\ell(\langle x,y\rangle). \tag{1.3.5}$$

Then, the Spectral Representation Theorem is the following.

Theorem 1.3.2 (The Spectral Representation Theorem on $\mathbb{S}^2$). *Let $T(x,\omega)$ a random field on the sphere (i.e. a measurable application $T:\mathbb{S}^2\times\Omega\to\mathbb{R}$), isotropic with zero mean and $\mathbb{E}T(x)^2<\infty$. Then, for all $x\in\mathbb{S}^2$, we have that*

$$T(x,\omega)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}a_{\ell m}(\omega)Y_{\ell m}(x)$$

the equality holding in the mean-square and almost sure sense, i.e.

$$\lim_{L\to\infty}\mathbb{E}\Big[T(x,\omega)-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}a_{\ell m}(\omega)Y_{\ell m}(x)\Big]^2=0.$$

The random coefficients $\{a_{\ell m}\}_{\ell=1,2,\dots,m=-\ell,\dots,\ell}$ are such that

$$\mathbb{E}a_{\ell m}\bar{a}_{\ell' m'}=C_\ell\delta_\ell^{\ell'}\delta_m^{m'},\ell=1,2,\dots,m=-\ell,\dots,\ell.$$

Moreover, if $T(x,\omega)$ is Gaussian or if $C_\ell>0$ for all ℓ, one has also the representation

$$T(x)=\int_{\mathbb{S}^2}\sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\sqrt{C_\ell}P_\ell(\langle x,y\rangle)W(dy),$$

where $\mathbb{E}[W(A)\bar{W}(B)]=\mu(A\cap B)$, $\mu(\cdot)$ denoting Lebesgue measure on the sphere.

Remark 1.3.3. *For simplicity, through the thesis we shall assume*

$$\int_{\mathbb{S}^2}T(x)\,dx=0,\ \ \textit{which implies}\ \ \frac{a_{00}}{2\pi}=0.$$

Proof. We define the linear operator $J:L^2(\Omega,\mathbb{P})\to L^2(\mathbb{S}^2)$ as

$$J(T(x))=\sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\sqrt{C_\ell}P_\ell(<x,\cdot>);$$

we have that

$$\begin{aligned}
&<J(T(x)),J(T(y))>_{L^2(\mathbb{S}^2)}=\\
&<\sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\sqrt{C_\ell}P_\ell(<x,\cdot>),\sum_{\ell'=0}^{\infty}\frac{2\ell'+1}{4\pi}\sqrt{C_{\ell'}}P_{\ell'}(<y,\cdot>)>_{L^2(\mathbb{S}^2)}=\\
&\int_{\mathbb{S}^2}\sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\sqrt{C_\ell}P_\ell(<x,z>)\sum_{\ell'=0}^{\infty}\frac{2\ell'+1}{4\pi}\sqrt{C_{\ell'}}P_{\ell'}(<y,z>)\,dz=\\
&\sum_{\ell\ell'}\sqrt{C_\ell}\sqrt{C_{\ell'}}\int_{\mathbb{S}^2}\frac{2\ell+1}{4\pi}P_\ell(<x,z>)\frac{2\ell'+1}{4\pi}P_{\ell'}(<z,y>)\,dz
\end{aligned} \tag{1.3.6}$$

and for Duplication property (see Appendix A.1.10), (1.3.6) is equal to

$$=\sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}C_\ell P_\ell(<x,y>)=\mathbb{E}[T(x)T(y)].$$

It follows that J is an isometry and hence it is injective. The image space is made by the closure of the span of all functions which have the form

$$\sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(< x, \cdot >). \tag{1.3.7}$$

Moreover, all the functions in (1.3.7) are indeed in $L^2(\mathbb{S}^2)$. Actually,

$$\begin{aligned} &\left\| \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(< x, \cdot >) \right\|^2_{L^2(\mathbb{S}^2)} = \\ &\int_{\mathbb{S}^2} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(< x, y >) \sum_{\ell'=0}^{\infty} \frac{2\ell'+1}{4\pi} \sqrt{C_{\ell'}} P_{\ell'}(< x, y >) dy \end{aligned} \tag{1.3.8}$$

and, again by the Duplication property, we get that (1.3.8) is equal to

$$\sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} C_\ell P_\ell(< x, x >) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} C_\ell < \infty.$$

Now we define $a_{\ell m}(\omega) := \int_{\mathbb{S}^2} T(x) \bar{Y}_{\ell m}(x)\, dx$ and we verify that it is well defined as an element of $L^2(\Omega)$. To this aim, we consider the sequence

$$a_{\ell m}(j) = \sum_{k \in N_j} T(x_{jk}) \bar{Y}_{\ell m}(x_{jk}) \mu(V_{jk}),$$

where V_{jk} is a family of (exhaustive and disjoint) Voronoi cells, so that there exist spherical caps and constant $0 < c < c'$ such that (see i.e. [8])

$$B_{\epsilon/2}(x_{jk}) \subset V_{jk} \subset B_\epsilon(x_{jk}),\ c2^{-j} \leq \epsilon \leq c'2^{-j},\ \text{for all } j, k.$$

It is readily seen that $\{a_{\ell m}(j)\}_j$ is a Cauchy sequence, in fact considering refining partitions $V_{j'k}$, we have that

$$\begin{aligned} &\mathbb{E}|a_{\ell m}(j) - a_{\ell m}(j')|^2 = \mathbb{E}\Big[\sum_{k \in N_j} \{ T(x_{jk}) \bar{Y}_{\ell m}(x_{jk}) - T(x_{j'k}) \bar{Y}_{\ell m}(x_{j'k}) \} \mu(V_{jk}) \Big]^2 \\ &= \mathbb{E}\Big[\sum_{k \in N_j} \{ (T(x_{jk}) - T(x_{j'k})) \bar{Y}_{\ell m}(x_{jk}) + T(x_{j'k}) (\bar{Y}_{\ell m}(x_{jk}) - \bar{Y}_{\ell m}(x_{j'k})) \} \mu(V_{jk}) \Big]^2 \\ &= \mathbb{E}\Big[\sum_{k \in N_j} \sum_{k' \in N_j} (T(x_{jk}) - T(x_{j'k}))(T(x_{jk'}) - T(x_{jk'})) \bar{Y}_{\ell m}(x_{jk}) \bar{Y}_{\ell m}(x_{jk'}) \mu(V_{jk}) \mu(V_{jk}) \\ &+ T(x_{j'k}) T(x_{j'k'}) (\bar{Y}_{\ell m}(x_{jk}) - \bar{Y}_{\ell m}(x_{j'k})) (\bar{Y}_{\ell m}(x_{jk'}) - \bar{Y}_{\ell m}(x_{j'k'})) \mu(V_{jk}) \mu(V_{jk'}) \\ &+ (T(x_{jk}) - T(x_{j'k})) \bar{Y}_{\ell m}(x_{jk}) T(x_{j'k'}) (\bar{Y}_{\ell m}(x_{jk'}) - \bar{Y}_{\ell m}(x_{jk'})) \mu(V_{jk}) \mu(V_{jk'}) \\ &+ T(x_{j'k}) (\bar{Y}_{\ell m}(x_{jk}) - \bar{Y}_{\ell m}(x_{j'k})) (T(x_{jk'}) - T(x_{j'k'})) (\bar{Y}_{\ell m}(x_{jk'})) \mu(V_{jk}) \mu(V_{jk'}) \Big] \\ &\leq 2 \sum_{k \in N_j} \sum_{k' \in N_j} \mathbb{E}\Big[(T(x_{jk}) - T(x_{j'k}))(T(x_{jk'}) - T(x_{j'k'})) \bar{Y}_{\ell m}(x_{jk}) \bar{Y}_{\ell m}(x_{jk'}) \mu(V_{jk}) \mu(V_{jk'}) \Big] \\ &+ 2 \sum_{k \in N_j} \sum_{k' \in N_j} \mathbb{E}\Big[T(x_{jk}) T(x_{j'k}) (\bar{Y}_{\ell m}(x_{jk}) - \bar{Y}_{\ell m}(x_{j'k})) (\bar{Y}_{\ell m}(x_{jk'}) - \bar{Y}_{\ell m}(x_{j'k'})) \mu(V_{jk}) \mu(V_{jk'}) \Big]. \end{aligned} \tag{1.3.9}$$

The first summand in (1.3.9) is bounded by

$$
\begin{aligned}
&2\sup \mathbb{E}[|T(x_{jk})-T(x_{j'k})|^2]\Big\{\sum_{k\in N_j}\bar{Y}_{\ell m}(x_{jk})\mu(V_{jk})\Big\}^2\\
&\leq 2\sup \mathbb{E}|T(x_{jk})-T(x_{jk})|^2 \sup|\bar{Y}_{\ell m}(x_{jk})|^2\Big\{\sum_{k\in N_j}\mu(V_{jk})\Big\}^2.
\end{aligned}
$$

Since V_{jk} are disjointed, the sum $\sum_{k\in N_j}\mu(V_{jk})$ is smaller than the surface of the sphere and so it is finite. It follows from the mean-square continuity of T that the first term in (1.3.9) converges to zero as $j\to\infty$; whereas, the second is bounded by

$$
\begin{aligned}
&2\sup \mathbb{E}[T(x)^2]\Big\{\sum_{k\in N_j}\Big(\bar{Y}_{\ell m}(x_{jk})-\bar{Y}_{\ell m}(x_{j'k})\Big)\mu(V_{jk})\Big\}^2\\
\leq&2\sup \mathbb{E}[T(x)^2]\sup\Big|\bar{Y}_{\ell m}(x_{jk})-\bar{Y}_{\ell m}(x_{j'k})\Big|^2\Big\{\sum_{k\in N_j}\mu(V_{jk})\Big\}^2
\end{aligned}
$$

and so it converges to zero and then we proved that $\{a_{\ell m}(j)\}$ is a Cauchy sequence. Note that

$$
\begin{aligned}
J(a_{\ell m}) =&J\Big(\int_{\mathbb{S}^2}T(x)\bar{Y}_{\ell m}(x)dx\Big)=\int_{\mathbb{S}^2}J(T(x))\bar{Y}_{\ell m}(x)dx=\\
&\int_{\mathbb{S}^2}\sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\sqrt{C_\ell}P_\ell(<x,\cdot>)\bar{Y}_{\ell m}(x)dx
\end{aligned}
\tag{1.3.10}
$$

and for the Addition formula (Appendix A.1.9) and the fact that $\int_{\mathbb{S}^2}Y_{\ell'm'}(y)\bar{Y}_{\ell m}(y)=\delta_\ell^{\ell'}\delta_m^{m'}$, (1.3.10) is equal to

$$
\sqrt{C_\ell}\bar{Y}_{\ell m}(\cdot).
$$

Remark 1.3.4. $\mathbb{E}[a_{\ell m}\bar{a}_{\ell'm'}]=\langle\sqrt{C_\ell}\bar{Y}_{\ell m}(\cdot),\sqrt{C_\ell}\bar{Y}_{\ell'm'}(\cdot)\rangle_{L^2(\mathbb{S}^2)}=\int_{\mathbb{S}^2}\sqrt{C_\ell C_{\ell'}}\bar{Y}_{\ell m}(x)\bar{Y}_{\ell m}(x)dx=C_\ell\delta_\ell^{\ell'}\delta_m^{m'}$.

Since J is an isometry, it is injective and $J^{-1}=T$ is well defined.
It follows immediately that

$$
\begin{aligned}
T(x)=J^{-1}\Big[\sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\sqrt{C_\ell}P_\ell(\langle x,\cdot\rangle)\Big]&=J^{-1}\Big[\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell m}(x)\bar{Y}_{\ell m}(\cdot)\sqrt{C_\ell}\Big]\\
=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell m}(x)J^{-1}\Big[\sqrt{C_\ell}\bar{Y}_{\ell m}(\cdot)\Big]&=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell m}(x)a_{\ell m};
\end{aligned}
\tag{1.3.11}
$$

hence the first part of the theorem is proved.
Assume now that $C_\ell>0$ for all $\ell=0,1,\ldots$.
The space

$$
\overline{span}\Big\{\sum_\ell\frac{2\ell+1}{4\pi}\sqrt{C_\ell}P_\ell(\langle x,\cdot\rangle)\Big\}
$$

is the image space of J. We have seen in (1.3.10) that

$$
\int_{\mathbb{S}^2}\sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\sqrt{C_\ell}P_\ell(<x,\cdot>)\bar{Y}_{\ell m}(x)dx=\sqrt{C_\ell}Y_{\ell m}(x)\text{ for all }\ell,m
$$

and it is known that

$$L^2(\mathbb{S}^2) = \oplus_{\ell=0}^{\infty} H_\ell,$$

where

$$H_\ell = span\{Y_{\ell m}(\cdot),\ m = -\ell \ldots, \ell\}.$$

Then, it follows that the system $\{\sqrt{C_\ell} Y_{\ell m}\}$ gives an orthogonal basis for $L^2(\mathbb{S}^2)$, assuming $C_\ell > 0$. In order to establish the second request of the theorem, we need to show that there exists a random measure $W(\cdot)$ such that

$$T(x) = \int_{\mathbb{S}^2} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(\langle x, z\rangle) W(dz)$$

in $L^2(\Omega)$.

The function $1_A(\cdot) \in L^2(\mathbb{S}^2)$ for all $A \subset \mathbb{S}^2$, so we have that

$$\lim_{j\to\infty} \sum_k \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(\langle x, x_{jk}\rangle) 1_{V_{jk}}(\cdot) \to_{L^2(\mathbb{S}^2)} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(\langle x, \cdot\rangle).$$

By injectivity, we have that

$$\begin{aligned} T(x) &= \lim_{j\to\infty} J^{-1}\Big[\sum_k \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(\langle x, x_{jk}\rangle) 1_{V_{jk}}(\cdot)\Big] \\ &= \lim_{j\to\infty} \sum_k \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(\langle x, x_{jk}\rangle) J^{-1}[1_{V_{jk}}(\cdot)] \end{aligned} \tag{1.3.12}$$

in $L^2(\Omega)$ sense. $J^{-1}[1_{V_{jk}}(\cdot)]$ is the element of $L^2(\Omega)$ such that $\mathbb{E}[J^{-1}[1_A(\cdot)]J^{-1}[1_B(\cdot)]] = \mu(A \cap B)$. By the standard construction of stochastic integrals

$$J^{-1}\Big[\sum_k \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(\langle x, x_{jk}\rangle) 1_{V_{jk}}(\cdot)\Big] \to_{L^2(\Omega)} \int_{\mathbb{S}^2} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(\langle x, y\rangle) W(dy)$$

and because of the injectivity, we get that

$$\begin{aligned} T(x) &= \lim_{j\to\infty} J^{-1}\Big[\sum_k \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(\langle x, x_{jk}\rangle) 1_{V_{jk}}(\cdot)\Big] \\ &= \int_{\mathbb{S}^2} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} \sqrt{C_\ell} P_\ell(\langle x, y\rangle) W(dy) \end{aligned} \tag{1.3.13}$$

in $L^2(\Omega)$ sense and the proof is completed. □

Remark 1.3.5. *(The analogy with the time series case). Let X_t, $t \in \mathbb{Z}$, be a mean zero second-order stationary random process with real values. If the covariance function is summable and $f(\lambda)$ is the spectral density, one has that*

$$\Gamma(\tau) = \mathbb{E}[X_t X_{t-\tau}] = \int_{[-\pi,\pi]} \exp(i\lambda\tau) f(\lambda)\, d\lambda,$$

for $\tau \in \mathbb{R}$. It is then possible to see an analogy with the classical result for stationary processes (see for instance [27], [18]). In particular, consider the class of functions $h_t(\lambda) := \sqrt{f(\lambda)} \exp(i\lambda t)$, and the isometry

$$\mathscr{I}(X_t) = h_t(\cdot);$$

it is easily seen that

$$\mathbb{E}[X_t X_s] =: \langle X_t, X_s \rangle_{L^2(\Omega)} = \langle h_t(\cdot), h_s(\cdot) \rangle_{L^2(-\pi,\pi)}$$

and with the same argument we followed in the proof, one finds also that

$$X_t = \mathscr{I}^{-1}(h_t(\cdot)) = \int_{-\pi}^{\pi} \sqrt{f(\lambda)} \exp(i\lambda t) W(d\lambda),$$

where $W(\cdot)$ is now a white noise measure on $[-\pi, \pi]$. Note in particular that the indicator function of the subsets of $[-\pi, \pi]$ always belongs to $L^2(F)$ if $F(\cdot)$ is the distribution function of a nonatomic measure on $[-\pi, \pi]$.

Remark 1.3.6. *There are two important differences with the time series case:*

- *The Spectral Representation Theorem requires an integral rather than a series. This is due to the fact that the sphere is a compact space, whereas the integers $\mathbb{Z}$ are not. This has very much to do with group-theoretic results, in particular the fact that representation of compact groups are countable, while representations of noncompact groups such as $\mathbb{Z}$ are uncountably many. Note that the sphere by itself is not a group, but it can be realized as quotient space $\mathbb{S}^2 = SO(3)/SO(2)$, see [45] for more details.*
- *In the time series case there is a single deterministic component $\sqrt{f(\lambda)} \exp(i\lambda t)$ corresponding to each frequency (i.e., inverse scale) λ, while in the spherical case there are $2\ell + 1$ spherical harmonics corresponding to a single frequency ℓ. Again, this has a simple explanation in group theoretic terms: in fact, second order stationary processes on $\mathbb{Z}$ enjoy some form of invariance with respect to the action of a commutative group ($X_t \to X_{t+\tau}$), while isotropy implies invariance in distribution with respect to the action of the noncommutative group $SO(3)$. It is a standard fact of group representation theory that noncommutative groups of multiple representations of the same dimensions, and this generates the multiple spherical harmonics appearing at the same scale ℓ. Note that the complex exponentials are indeed eigenfunctions $\frac{\partial^2}{\partial t^2} \exp(i\lambda t) = -\lambda^2 \exp(i\lambda t)$, in perfect analogy with spherical harmonics.*

The analysis of random eigenfunctions on the sphere or on other compact manifolds (such as the torus) has been recently considered in many papers, due to strong motivations arising from Cosmology and Quantum Mechanics, see i.e., [45], [37], [53], [68], [70] and [69]. We will summarize in the last section of this chapter some of the previous results. For more detailed treatments on spherical harmonics, we refer for instance to [4] and [5].

1.3.2 The Gaussian Kinematic Formula on $\mathbb{S}^2$

Let us consider $\mathbb{M} = \mathbb{S}^2$, the real valued eigenfunctions $T_\ell : \mathbb{S}^2 \to \mathbb{R}$ of the Laplace-Beltrami operator in (1.3.1) and the z-excursion sets of T_ℓ defined, for $z \in \mathbb{R}$, as in (0.0.1). As we have already stressed, the interest is investigate the geometry of these regions; many papers have focussed on it, for instance [51], [50], [52], [66]. More precisely, as we have seen in Section 1.1.2 and Section 1.1.3, to characterize the geometry of $\left\{A_z(T_\ell, \mathbb{S}^2)\right\}$, we need to investigate the Lipschitz Killing Curvatures (Section 1.1.2, see also [3]). Hence, in order to obtain their expected values, we exploit the Gaussian Kinematic Formula in (1.1.22).

First of all, let us compute the metric induced by the process. Thus, according to (1.1.4), we calculate the derivatives of the covariance function at the origin

$$\nabla_x \nabla_y \Gamma(x,y)\Big|_{x=y.}$$

Let us use the spherical coordinates (θ_x, φ_x) for a point $x = (x_1, x_2, x_3) \in \mathbb{S}^2$, then

$$\begin{cases} x_1 = \sin\theta_x \cos\varphi_x \\ x_2 = \sin\theta_x \sin\varphi_x \\ x_3 = \cos\theta_x; \end{cases} \tag{1.3.14}$$

and we have that

$$\begin{aligned} \langle x, y\rangle &= x_1 y_1 + x_2 y_2 + x_3 y_3 = \sin\theta_x \sin\theta_y \{\cos\varphi_x \cos\varphi_y + \sin\varphi_x \sin\varphi_y\} + \cos\theta_x \cos\theta_y \\ &= \sin\theta_x \sin\theta_y \{\cos(\varphi_x - \varphi_y)\} + \cos\theta_x \cos\theta_y. \end{aligned} \tag{1.3.15}$$

Now, $P_\ell(\langle x, y\rangle) = P_\ell(\sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) + \cos\theta_x \cos\theta_y)$, and its partial derivatives are given by

$$\begin{aligned} \frac{\partial}{\partial\theta_y} P_\ell(\sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) + \cos\theta_x \cos\theta_y) &= P'_\ell(\sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) + \cos\theta_x \cos\theta_y) \\ &\quad \times [\sin\theta_x \cos\theta_y \cos(\varphi_x - \varphi_y) - \cos\theta_x \sin\theta_y] \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial\varphi_y} P_\ell(\sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) + \cos\theta_x \cos\theta_y) &= P'_\ell(\sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) + \cos\theta_x \cos\theta_y) \\ &\quad \times [\sin\theta_x \sin\theta_y \sin(\varphi_x - \varphi_y)]. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial\theta_x}\frac{\partial}{\partial\theta_y} P_\ell(\sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) + \cos\theta_x \cos\theta_y) &= P''_\ell(\sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) \\ &+ \cos\theta_x \cos\theta_y)[\cos\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) - \sin\theta_x \cos\theta_y] \\ &\times [\sin\theta_x \cos\theta_y \cos(\varphi_x - \varphi_y) - \cos\theta_x \sin\theta_y] \\ &+ P'_\ell(\sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) + \cos\theta_x \cos\theta_y)[\cos\theta_x \cos\theta_y \cos(\varphi_x - \varphi_y) + \sin\theta_x \sin\theta_y] \end{aligned} \tag{1.3.16}$$

which in $x = y$ gives $P'_\ell(1)$. Similarly, we get

$$\begin{aligned} \frac{\partial}{\partial\varphi_x}\frac{\partial}{\partial\theta_y} P_\ell(\sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) + \cos\theta_x \cos\theta_y) &= P''_\ell(\sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) \\ &+ \cos\theta_x \cos\theta_y)[-\sin\theta_x \sin\theta_y \sin(\varphi_x - \varphi_y)] \times [\sin\theta_x \cos\theta_y \cos(\varphi_x - \varphi_y) - \cos\theta_x \sin\theta_y] \\ &+ P'_\ell(\sin\theta_x \sin\theta_y \cos(\varphi_x - \varphi_y) + \cos\theta_x \cos\theta_y)[-\sin\theta_x \cos\theta_y \sin(\varphi_x - \varphi_y)], \end{aligned} \tag{1.3.17}$$

which is 0 for $x = y$. Likewise,

$$\frac{\partial}{\partial\theta_x}\frac{\partial}{\partial\varphi_y}P_\ell(\sin\theta_x\sin\theta_y\cos(\varphi_x-\varphi_y)+\cos\theta_x\cos\theta_y)\Big|_{x=y} = P_\ell''(\sin\theta_x\sin\theta_y\cos(\varphi_x-\varphi_y)$$
$$+\cos\theta_x\cos\theta_y)\times[\sin\theta_x\sin\theta_y\sin(\varphi_x-\varphi_y)][\sin\theta_x\cos\theta_y\cos(\varphi_x-\varphi_y)-\cos\theta_x\sin\theta_y]$$
$$+P_\ell'(\sin\theta_x\sin\theta_y\cos(\varphi_x-\varphi_y)+\cos\theta_x\cos\theta_y)[\cos\theta_x\sin\theta_y\sin(\varphi_x-\varphi_y)]\Big|_{x=y} = 0 \tag{1.3.18}$$

and finally,

$$\frac{\partial}{\partial\varphi_x}\frac{\partial}{\partial\varphi_y}P_\ell(\sin\theta_x\sin\theta_y\cos(\varphi_x-\varphi_y)+\cos\theta_x\cos\theta_y)\Big|_{x=y} = P_\ell''(\sin\theta_x\sin\theta_y\cos(\varphi_x-\varphi_y)$$
$$+\cos\theta_x\cos\theta_y)\times[-\sin\theta_x\sin\theta_y\sin(\varphi_x-\varphi_y)][\sin\theta_x\sin\theta_y\sin(\varphi_x-\varphi_y)]$$
$$+P_\ell'(\sin\theta_x\sin\theta_y\cos(\varphi_x-\varphi_y)+\cos\theta_x\cos\theta_y)[\sin\theta_x\sin\theta_y\cos(\varphi_x-\varphi_y)]\Big|_{x=y}$$
$$= P_\ell'(1)\sin^2\theta_x. \tag{1.3.19}$$

Since $\Gamma(x,y) = P_\ell(\langle x,y\rangle)$ and the operator we are considering is $\nabla_x = \left(\frac{\partial}{\partial\theta_x}, \frac{1}{\sin\theta_x}\frac{\partial}{\partial\varphi_x}\right)$, the Riemannian metric, denoted by g, is given by

$$g = \begin{bmatrix} P_\ell'(1) & 0 \\ 0 & P_\ell'(1) \end{bmatrix}, \tag{1.3.20}$$

and we recall that $P_\ell'(1) = \frac{\ell(\ell+1)}{2}$.

In order to compute the GKF (1.1.22), we evaluate the LKC on $\mathbb{S}^2$, which we stress again that in this case they correspond to the area of $A_z(\ell)$ (which we shall write as $\mathcal{L}_2(A_z(T_\ell,\mathbb{S}^2))$, (half) the boundary length $\partial A_z(\ell)$ (i.e., the length of level curves $T_\ell^{-1}(z)$, written $\mathcal{L}_1(A_z(T_\ell,\mathbb{S}^2))$), and their Euler-Poincaré characteristic, (written $\mathcal{L}_0(A_z(T_\ell,\mathbb{S}^2))$). Note that since the Euler-Poincaré characteristic (which in the two-dimensional case is the difference between the number of connected regions and the number of "holes") is a geometric invariant, it does not depend on the metric induced by the process, so that $\mathcal{L}_0^{T_\ell}(\mathbb{S}^2) = 2$. Moreover, in view of the fact that the sphere is compact, the boundary length of the entire manifold is $\mathcal{L}_1^{T_\ell}(\mathbb{S}^2) = 0$. Finally, as far as the area is concerned, we have that

$$\mathcal{L}_2^{T_\ell}(\mathbb{S}^2) = \int_{\mathbb{S}^2}\sqrt{\det g}\,dx = 4\pi\frac{\ell(\ell+1)}{2}.$$

At this stage we are in the condition to apply (1.1.22). Then, denoting $\lambda_\ell = \ell(\ell+1)$ as in (1.3.1) and since

$$\rho_0(z) = (2\pi)^{-1/2}H_{-1}(z)e^{-z^2/2} = 1-\Phi(z)$$
$$\rho_1(z) = (2\pi)^{-1}H_0(z)e^{-z^2/2},$$
$$\rho_2(z) = (2\pi)^{-3/2}H_1(z)e^{-z^2/2},$$

one obtains, respectively,

$$\mathbb{E}[\mathscr{L}_0(A_z(T_\ell,\mathbb{S}^2))] = \begin{bmatrix}0\\0\end{bmatrix}\rho_0(z)\mathscr{L}_0^{T_\ell}(\mathbb{S}^2) + \begin{bmatrix}1\\1\end{bmatrix}\rho_1(z)\mathscr{L}_1^{T_\ell}(\mathbb{S}^2) + \begin{bmatrix}2\\2\end{bmatrix}\rho_2(z)\mathscr{L}_2^{T_\ell}(\mathbb{S}^2)$$

$$= 2(1-\Phi(z)) + 0 + (2\pi)^{-3/2} z e^{-z^2/2}\frac{4\pi(\ell(\ell+1))}{2} \tag{1.3.21}$$

$$= 2\{1-\Phi(z)\} + \frac{\lambda_\ell}{2}\frac{z e^{-z^2/2}}{\sqrt{(2\pi)^3}}4\pi,$$

for the Euler-Poincaré characteristic,

$$\mathbb{E}[\mathscr{L}_1^{T_\ell}(A_z(T_\ell,\mathbb{S}^2))] = \begin{bmatrix}1\\0\end{bmatrix}\rho_0(z)\mathscr{L}_1^{T_\ell}(\mathbb{S}^2) + \begin{bmatrix}2\\1\end{bmatrix}\rho_1(z)\mathscr{L}_2^{T_\ell}(\mathbb{S}^2) = 2\left[\frac{\sqrt{\pi}}{2}\right]^2\frac{e^{-z^2/2}}{2\pi}\mathscr{L}_2^{T_\ell}(\mathbb{S}^2)$$

$$= \pi \times e^{-z^2/2}\frac{\ell(\ell+1)}{2},$$

which implies that

$$\sqrt{\frac{\ell(\ell+1)}{2}}\mathbb{E}[\mathscr{L}_1(A_z(T_\ell,\mathbb{S}^2))] = \pi \times \frac{\ell(\ell+1)}{2}e^{-z^2/2}$$

and then

$$\mathbb{E}[\mathscr{L}_1(A_z(T_\ell,\mathbb{S}^2))] = \pi \times \sqrt{\frac{\lambda_\ell}{2}}e^{-z^2/2}, \tag{1.3.22}$$

for (half) the boundary length, and finally

$$\mathbb{E}[\mathscr{L}_2^{T_\ell}(A_z(T_\ell,\mathbb{S}^2))] = \begin{bmatrix}2\\0\end{bmatrix}\rho_0(z)\mathscr{L}_2^{T_\ell}(\mathbb{S}^2) = H_{-1}(z)e^{-z^2/2}4\pi\frac{\ell(\ell+1)}{2}$$

which leads to

$$\frac{\ell(\ell+1)}{2}\mathbb{E}[\mathscr{L}_2(A_z(T_\ell,\mathbb{S}^2))] = 4\pi \times \{1-\Phi(z)\}\frac{\ell(\ell+1)}{2},$$

so that

$$\mathbb{E}[\mathscr{L}_2(A_z(T_\ell,\mathbb{S}^2))] = 4\pi \times \{1-\Phi(z)\}, \tag{1.3.23}$$

for the area.

1.4 Previous Works

In this section we give a background of the results which have already been established about the analysis of geometric functionals for the excursion sets of random eigenfunctions on $\mathbb{S}^2$. Some of these findings will be extended to the spherical cap in the next chapters.
We are always considering the real valued eigenfunctions of the Laplace-Beltrami operator in (1.3.1), $T_\ell : \mathbb{S}^2 \to \mathbb{R}$, $\forall \ell > 0$ and the excursion sets defined in (0.0.1). In Subsection 1.3.2, we computed the expected value of the LCK; then, the next step is the derivation of their

variances and their limiting distributions. A crucial step to achieve these results is to note that all these statistics can be written as nonlinear functionals of the random fields itself and their spatial derivatives. For instance the excursion area can be expressed by

$$S_\ell(z) = \int_{\mathbb{S}^2} 1_{(z,+\infty)}(T_\ell(x))\,dx; \tag{1.4.1}$$

likewise, using a Kac-Rice argument (see [6], [3]) the length of level curves can be written as

$$\mathscr{L}_\ell(z) = \int_{\mathbb{S}^2} \delta_z(T_\ell(x))\|\nabla T_\ell(x)\|\,dx,$$

and a related formula can be given for the Euler-Poincaré characteristic (see [21]). Starting from these expressions, it is possible to compute explicitly the expansion of Lipschitz Killing Curvatures into the orthonormal system generated by Hermite polynomials. We start with the area here below.

1.4.1 The Area of the Excursion Sets

Let us consider the excursion area in (1.4.1); it can be readily shown that (see [51], [50], [47])

$$S_\ell(z) = \sum_{q=0}^{\infty} \frac{J_q(z)}{q!} \int_{\mathbb{S}^2} H_q(T_\ell(x))\,dx, \tag{1.4.2}$$

the equality holding in the $L^2(\Omega)$-sense. The coefficients $\{J_q(\cdot)\}$ have the analytic expressions $J_0(z) = \Phi(z)$, $J_1(z) = -\phi(z)$, $J_2(z) = z\phi(z)$, $J_3(z) = (1-z^2)\phi(z)$ and in general

$$J_q(z) = -H_{q-1}(z)\phi(z),\ q = 1,2,3... \tag{1.4.3}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density function and the distribution function of a standard Gaussian variable ([51], [50]). As in [50], we define

$$h_{\ell,q} := \int_{\mathbb{S}^2} H_q(T_\ell(x))\,dx \quad q = 1,2,\dots, \tag{1.4.4}$$

and we can hence write

$$S_\ell(z) = \sum_{q=0}^{\infty} \frac{J_q(z)}{q!} h_{\ell,q} \text{ in } L^2(\Omega). \tag{1.4.5}$$

It can be readily verified that the term corresponding to $q = 1$ in (1.4.5) is identically equal to zero for every $\ell \geq 1$; indeed we have that

$$h_{\ell,1} := \int_{\mathbb{S}^2} \sqrt{\frac{4\pi}{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x)dx = \sqrt{\frac{4\pi}{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell m} \int_{\mathbb{S}^2} Y_{\ell m}(x)dx = 0. \tag{1.4.6}$$

The crucial step in [51], [47] is then to show that a single term, corresponding to $q = 2$, has asymptotically (in the high-energy regime $\ell \to \infty$) a dominating role in the expansion, i.e.,

$$\mathrm{Var}(S_\ell) = \left\{\frac{J_2(z)}{2}\right\}^2 \mathrm{Var}(h_{\ell;2}) + o(\mathrm{Var}(S_\ell)),\ \text{as } \ell \to \infty,$$

so that both the asymptotic variance and the Central Limit Theorem (CLT) can be established by simply considering the behavior of this single term ([51], [52], [50]). It should be noted that the functionals $\{h_{\ell;q}\}$ belong to the Wiener chaoses of order q, as such, they belong to the domain of application for the Stein-Malliavin method, leading to very neat characterizations for Quantitative Central Limit Theorems (QCLT) (see i.e., [58], [57]). More precisely, thanks to Corollary 1.2.4 (i.e. Theorem 5.2.7, p. 99 [57]), we have that

$$d_W\left(\frac{h_{\ell;q}}{\sqrt{\mathrm{Var}(h_{\ell;q})}}, Z\right) \le 2\sqrt{\frac{q-1}{3q}\left(\frac{cum_4(h_{\ell;q})}{\mathrm{Var}^2(h_{\ell;q})}\right)}, \tag{1.4.7}$$

where $Z \sim \mathcal{N}(0,1)$ and cum_4 is the fourth-order cumulant defined in (1.2.7). In words, this means that in these circumstances to prove a Quantitative Central Limit Theorem for standardized sequences it is enough to show that their fourth-order moment goes to 3. This approach was used to establish Quantitative Central Limit Theorems in [51],[47] (see also [50], [48], [21], [46]), i.e., for $z \neq 0$,

$$d_W\left(\frac{S_\ell(z) - \mathbb{E}[S_\ell(z)]}{\sqrt{\mathrm{Var}(S_\ell(z))}}, Z\right) = O\left(\ell^{-1/2}\right), \tag{1.4.8}$$

as $\ell \to \infty$, entailing as a Corollary that

$$\frac{S_\ell(z) - \mathbb{E}[S_\ell(z)]}{\sqrt{\mathrm{Var}(S_\ell(z))}} \to_d Z, \; z \neq 0.$$

In [47] or [66] it is shown that these results hold also in the m—dimensional sphere. The authors considered nonlinear functionals of Gaussian eigenfunctions (denoted with (T_ℓ^m), $\ell \in \mathbb{N}$), on the m—dimensional unit sphere $\mathbb{S}^m$, $m \ge 2$. The ℓ—th Gaussian eigenfunction T_ℓ^m on $\mathbb{S}^m$ satisfies

$$\Delta_{\mathbb{S}^m} T_\ell^m + \ell(\ell + m - 1) T_\ell^m = 0 \text{ a.s}, \tag{1.4.9}$$

where $\Delta_{\mathbb{S}^m}$ is the Laplace-Beltrami operator on $\mathbb{S}^m$, and it is a centered isotropic Gaussian field with covariance function

$$\mathbb{E}[T_\ell^m(x) T_\ell^m(y)] = G_{\ell;m}(\cos d(x,y)),$$

where $G_{\ell;m}$ is the normalized Gegenbauer polynomial [72] and d the usual distance on the m—sphere.

A common features of all these statistics is the disappearance of the leading term at the zero level $z = 0$ (the so-called Berry's cancellation phenomenon [14], [13], investigated for instance in [77], [52], [78]). In the case of the excursion area, for instance, at $z = 0$ all the odd-order chaoses become relevant, and the Central Limit Theorem can be established as in [48], [21], [46]. To be clear, the nodal case, $z = 0$, corresponds to the Defect D_ℓ, which is defined as the difference between the measure of the positive and negative regions, i.e.,

$$D_\ell = \int_{\mathbb{S}^m} 1_{(0,+\infty)}(T_\ell(x))\, dx - \int_{\mathbb{S}^m} 1_{(-\infty,0)}(T_\ell(x))\, dx. \tag{1.4.10}$$

Note that, in two dimension, $D_\ell = 2S_\ell(0) - 4\pi$; the mean is $\mathbb{E}[D_\ell] = 0$ and the variance is computed in [51] to be

$$\mathrm{Var}(D_\ell) = \frac{C}{\ell^2}(1 + o(1)), \; \ell \to +\infty, \tag{1.4.11}$$

for some $C > \frac{32}{\sqrt{27}}$ and in [50] the following Central Limit Theorem is proved, i.e.,

$$\frac{D_\ell}{\sqrt{\mathrm{Var}(D_\ell)}} \to_d Z,$$

where, as before, $Z \sim \mathcal{N}(0,1)$. In [65] the nodal case, for $m \geq 2$ is discussed. Actually, it is seen that, as $\ell \to +\infty$,

$$\mathrm{Var}(D_\ell) = \frac{C_m}{\ell^m}(1+o(1)),$$

where C_m is a constant depending on the dimension m. In addition, it is also proved the CLT, in the high-energy limit, and the QCLT in Wasserstein distance, which gives

$$d_W\left(\frac{D_\ell}{\sqrt{\mathrm{Var}(D_\ell)}}, Z\right) = O\left(\frac{1}{\sqrt[4]{\log \ell}}\right)$$

(see [47] for the case $m = 2$ and $m > 5$, and [65] for the remaining cases $m = 3, 4, 5$).

1.4.2 The Boundary Lengths

In this subsection we consider the length of the boundary $\mathscr{L}_\ell(z)$, defined as

$$\mathscr{L}_\ell(z) := \mathrm{len}(\{x \in \mathbb{S}^2 : T_\ell(x) = z\}). \tag{1.4.12}$$

The expected value given in (1.3.22) is computed in [77], [76], whereas the variance is studied in [76] and if $z \neq 0$ it is

$$\mathrm{Var}(\mathscr{L}_\ell(z)) \sim Ce^{-z^2} z^4 \cdot \ell, \ \ell \to +\infty,$$

for some $C > 0$. The asymptotic distribution of the length of the level curves is investigated in [46]. The authors manage to show an exact formula for the second chaotic component, i.e.,

$$Proj(\mathscr{L}_\ell(z)|C_2) = \sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4}e^{-z^2/2}z^2 \int_{\mathbb{S}^2} H_2(T_\ell(x))\,dx = \sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4}e^{-z^2/2}z^2 h_{\ell,2}, \tag{1.4.13}$$

where $Proj(\mathscr{L}_\ell(z)|C_2)$ is the projection on the second component of the chaotic expansion. Furthermore, it is seen that the boundary length has the same asymptotic distribution of the second chaotic projection and as a consequence, as $\ell \to \infty$, if $z \neq 0$, it results that

$$\frac{\mathscr{L}_\ell(z) - \mathbb{E}[\mathscr{L}_\ell(z)]}{\sqrt{\mathrm{Var}(\mathscr{L}_\ell(z))}} \to_d Z.$$

Similarly to what happens for the excursion area, the second component vanishes if and only if $z = 0$. Let us focus now on this latter case; i.e., we consider the nodal set of T_ℓ, given by $T_\ell^{-1}(0) := \{x \in \mathbb{S}^2 : T_\ell(x) = 0\}$ and hence its volume by replacing $z = 0$ in (1.4.12), i.e., we define

$$\mathscr{Z}(T_\ell) := \mathrm{len}(\{x \in \mathbb{S}^2 : T_\ell(x) = 0\}). \tag{1.4.14}$$

These sets have been analysed by many authors, see i.e. [26], [60], [80], [79], [29], [17], [19]. As a consequence of the general Yau's conjecture ([80], [79]) for eigenfunctions on compact manifolds (proved in [29] for real analytic metrics and by [39], [38] and [40] for the smooth case) we know that, in the high energy limit, the volume of the nodal set is bounded by

$$c_1\sqrt{\ell(\ell+1)} \leq \text{len}(T_\ell^{-1}(0)) \leq c_2\sqrt{\ell(\ell+1)},$$

where $c_1, c_2 > 0$. In the case of Gaussian random eigenfunctions, some sharper probabilistic bounds can be given. The asymptotic behavior of the expected value was given in [12]; for any dimension m, with $m \geq 2$, if we denote with $\mathscr{Z}(T_\ell^m)$ the volume of the nodal set $\{x \in \mathbb{S}^m : T_\ell^m(x) = 0\}$, where T_ℓ^m are the eigenfunctions in (1.4.9), the authors in [12] obtained

$$\mathbb{E}[\mathscr{Z}(T_\ell^m)] = c_m\sqrt{\ell(\ell+m-1)},$$

where $c_m = \dfrac{2\pi^{m/2}}{\sqrt{m}\Gamma(\frac{m}{2})}$ (see also [56] and [78]). As far as the variance is concerned, Neuheisel [56] gave an upper bound which was later improved in [78] and [77], where it was computed to be

$$\text{Var}(\mathscr{Z}(T_\ell)) = \frac{1}{32}\log\ell + O(1),$$

as $\ell \to \infty$. As a consequence, the variance of the nodal volume $\mathscr{Z}(T_\ell)$ has smaller order $O(\log\ell)$, in the high energy limit, with respect to the variance of boundary lengths at thresholds different from zero, which has been shown to be $O(\ell)$ (see for instance [65]). This is again the "Berry's cancellation" [14]; it is known to occur on the torus [33] and on other geometric functionals of random eigenfunctions, see i.e., [23], [24], [21]. More precisely, as far as the torus is concerned, Rudnick and Wigman in [68] and Krishnapur, Kurlberg and Wigman in [33] studied the volume of the nodal line (denoted with $\mathscr{L}_\ell$) of random eigenfunctions ("arithmetic random waves") $\mathscr{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The expected length was evaluated in [68] (Proposition 4.1) with the Kac-Rice formula (1.1.3),

$$\mathbb{E}[\mathscr{L}_\ell] = \frac{1}{2\sqrt{2}}\sqrt{4\pi^2\ell},$$

and the asymptotic behavior of the variance was established in [33]; it holds that

$$\text{Var}(\mathscr{L}_\ell) = c_\ell \cdot \frac{4\pi^2\ell}{\mathscr{N}_\ell^2}\left(1 + O\left(\frac{1}{\mathscr{N}_\ell^{1/2}}\right)\right), \tag{1.4.15}$$

where $\mathscr{N}_\ell$ is the size of the lattice points lying on the radius-$\sqrt{\ell}$ circle [33] and c_ℓ is the leading coefficient, depending on the distribution of the lattice points on the circle. Hence, as mentioned before, the "Berry's cancellation" phenomenon [14] takes place also for the toral nodal length. The distribution of $\mathscr{L}_\ell$ was investigated in [46], where the authors established a nonCentral Limit Theorem.

Also in this case a general interpretation of these results can be given quickly again by means the chaotic expansion (we refer to [46], [48], [21] for more discussions and details). If we expand the nodal length $\mathscr{L}_\ell$, in the L^2–sense, in terms of its q-th order chaotic components, i.e.,

$$\mathscr{L}_\ell - \mathbb{E}[\mathscr{L}_\ell] = \sum_q Proj[\mathscr{L}_\ell|q],$$

$Proj[\mathscr{L}_\ell|q]$ denoting the projection on the q-component, it can be shown that, in the case of functionals evaluated on the full sphere or torus, the projection on the first component vanishes identically; in the nodal case, $Proj[\mathscr{L}_\ell|2]$ vanishes as well, and the whole series is dominated simply by the term $Proj[\mathscr{L}_\ell|4]$, i.e., the so-called fourth-order chaos, which has indeed logarithmic variance. More explicitly, the variance of this single term is asymptotically equivalent to the variance of the full series, and its asymptotic distribution (Gaussian in the spherical case, nonGaussian for the torus, see [68]) gives also the limiting behavior of the nodal fluctuations. It should also be noted that, in the case of the sphere, $Proj[\mathscr{L}_\ell|4]$ takes a very simple form, because it is proportional to the so-called sample trispectrum of T_ℓ, $\int_{\mathbb{S}^2} H_4(T_\ell(x))\,dx$: this is to some extent unexpected, because the fourth-order chaotic term should in general be given by a complicated linear combination of polynomials involving also the gradient of the eigenfunctions (as it happens for arithmetic random waves on the torus, see [46]).

See also [67] for nodal intersections, [19] for the number of nodal domains, [32] and [36] for the Planck-scale mass equidistribution, [28] for the total number of phase singularities, [70] for nodal intersections on the 3-dimensional torus. Berry's random planar wave model was also considered (see [59]), both in the real and complex case; see also [10] for percolation of random nodal lines.

1.4.3 The Euler-Poincaré Characteristic

The Euler-Poincaré characteristic of z—excursion set $\chi(A_\ell(z))$ for random spherical harmonics has been investigated in [23]. An application of the Gaussian Kinematic Formula gives (1.3.21) (for a proof see, for example, [49], Corollary 5). The asymptotic variance is computed in [24] and it is equal to

$$\lim_{\ell\to\infty} \ell^{-3}\,\mathrm{Var}(\chi(A_\ell(z))) = \frac{1}{4}(z^3-z)^2\phi(z)^2,$$

as $\ell\to\infty$. Even in this case the variance of the nodal case has a smaller order than the case $z\neq 0$ and the Quantitative Central Limit Theorem is established in [21]. It is noted that also here, the high frequency behavior is dominated by the second term.

In conclusion, it can be noted that all the findings concerning the asymptotic variance of the LKC can be written as an asymptotic second-order Gaussian Kinematic Formula for the excursion sets of Gaussian spherical harmonics. Actually, exploiting similar expansions to (1.4.2), for the boundary length and the Euler-Poincaré characteristic (see [21], [48], [77], [19]), the following asymptotic expressions for the variances have been derived (see [21], [50], [47], [66]):

$$\begin{aligned}
&\lim_{\ell\to\infty} \ell^{-3}\,\mathrm{Var}(\mathscr{L}_0(A_z(T_\ell,\mathbb{S}^2))) = \frac{1}{4}(H_3(z)+H_2'(z))^2\phi(z)^2,\\
&\lim_{\ell\to\infty} \ell^{-1}\mathrm{Var}(\mathscr{L}_1(A_z(T_\ell,\mathbb{S}^2))) = \frac{\pi^3}{2}(H_2(z)+H_1'(z))^2\phi(z)^2\\
&\lim_{\ell\to\infty} \ell\mathrm{Var}[\mathscr{L}_2(A_z(T_\ell,\mathbb{S}^2))] = 4\pi^2(H_1(z)+H_0'(z))^2\phi(z)^2.
\end{aligned}$$

See also [33], [78], [47], [77], [20], [60], [59], [46], [28] for related results on the torus and on the plane, and [48], [11], [21], [47], [51], [68], [70] for other works concerning the geometry of random eigenfunctions on compact manifolds. It is conjectured that similar results will hold for Gaussian eigenfunctions on general compact manifolds, as the dimension of the eigenspaces diverges to infinity. For more references on this line of research, see for instance [66], [22], [23], [47], [50].

Remark 1.4.1. *It would be also interesting to consider finite average of spherical harmonics, so-called energy windows; these fields are connected to spherical wavelets/needlets, which can be important for statistical applications. For papers which have considered continuous version of the needlet transform, see for instance [7], [22], [30], [31], [49]. We leave these issues for future research.*

1.4.4 On the Subdomains

At this stage, we would like to generalize the results known for the sphere, to subdomains; in this line, nodal lengths for subregions in the sphere have been investigated in [77] (see instead [11] for arithmetic random waves). Precisely, the nodal volume inside a "nice" domain $F \subset \mathbb{S}^2$ of the sphere, is defined as

$$\mathscr{Z}^F(T_\ell) := \mathrm{len}(\{T_\ell = 0\} \cap F). \tag{1.4.16}$$

In [77], to address this issue, the so-called *linear statistics* of the nodal set are introduced; more precisely, let $\varphi : \mathbb{S}^2 \to \mathbb{R}$ be a smooth function, and define the random variable $\mathscr{Z}^\varphi(T_\ell)$ as

$$\mathscr{Z}^\varphi(T_\ell) := \int_{T_\ell^{-1}(0)} \varphi(x)\, d\, \mathrm{len}_{T_\ell^{-1}(0)}(x). \tag{1.4.17}$$

Apparently this definition is well-posed only for continuous test function $\varphi \in C(\mathbb{S}^2)$; nevertheless, it was shown in [77] that bounded variation functions $BV(\mathbb{S}^2)$ can be considered: indeed, it is possible to prove that, for $\varphi \in BV(\mathbb{S}^2) \cap L^\infty(\mathbb{S}^2)$ a not identically vanishing function, as $\ell \to \infty$, the variance satisfies

$$\mathrm{Var}(\mathscr{Z}^\varphi(T_\ell)) = \frac{||\varphi||_{L^2(\mathbb{S}^2)}}{128\pi} \cdot \log \ell + O_\varphi(1). \tag{1.4.18}$$

These results allow to cover indicator functions; indeed, if we substitute $\varphi(x) = 1_F(x)$ in (1.4.17), we obtain (1.4.16), i.e., we have that $\mathscr{Z}^{1_F}(T_\ell) = \mathrm{len}(\{x \in \mathbb{S}^2 \cap F : T_\ell(x) = 0\})$. As a consequence of (1.4.18), for $F \subset \mathbb{S}^2$ a submanifold of the sphere with C^2 boundary, it was proved, see [77], that, as $\ell \to \infty$, the asymptotic variance of (1.4.16) is given by:

$$\mathrm{Var}(\mathscr{Z}^F(T_\ell)) = \frac{m(F)}{128\pi} \cdot \log \ell + O_F(1),$$

i.e., logarithmic behavior occurs also in subdomains.

As far as the torus is concerned, the nodal length of arithmetic random waves restricted to shrinking balls (denoted with $\mathscr{L}_{\ell,r_\ell}$, where r_ℓ is the radius of the ball) was investigated in [11] under the condition $r_\ell > \ell^{-1/2}$. The mean was easily obtained by means of the Kac-Rice formula ([3], [6])

$$\mathbb{E}[\mathscr{L}_{\ell,r_\ell}] = \frac{1}{2\sqrt{2}}(\pi r_\ell^2) \cdot \sqrt{4\pi^2 \ell},$$

whereas the variance was shown to be proportional to the variance of the toral nodal length in (1.4.15), i.e.,

$$\mathrm{Var}(\mathscr{L}_{\ell,r_\ell}) = c_\ell \cdot (\pi r_\ell^2)^2 \cdot \frac{4\pi^2\ell}{\mathcal{N}_\ell^2}\left(1 + O\left(\frac{1}{\mathcal{N}_\ell^{1/2}}\right)\right).$$

More surprisingly, it was shown that, asymptotically, the local and global nodal lengths are fully correlated. This result entails also that, up to a scaling factor, the same limiting nonGaussian distribution holds in both cases.

CHAPTER 2

A Quantitative Central Limit Theorem for the Excursion Area of Random Spherical Harmonics over Subdomains of $\mathbb{S}^2$

In this chapter we consider the case of the excursion area evaluated on a spherical cap rather than the full sphere. Hence, we prove Theorem 0.0.1 (see also [73]), which extends the result in (1.4.8) to this subdomain. More precisely, we derive the Wiener chaos expansion for the excursion area and, through a careful analysis of the terms of development, we show that its asymptotic behavior is dominated by the second-order chaotic component. Then, we exploit this result to establish a Quantitative Central Limit Theorem, in the high energy limit. Note that the results are equivalent to the one obtained in the full sphere, reported in Section 1.4.1, where the leading term is the second and the same rate of convergence holds (namely $\frac{1}{\sqrt{\ell}}$). However, in our case, the proof is different and requires more sophisticated techniques, for example a careful analysis (of some independent interest) for smooth approximations (Section 2.2) of the indicator function for spherical caps subsets.
Moreover, while in the case of the full sphere the first chaotic component is zero (see (1.4.6)), this is not true in the spherical cap, nevertheless, it is proved that its order is still lower than the one of the second projection.
Finally, it can be noted that in subregions, as in the full sphere, the Berry's cancellation phenomenon ([14], [13]) occurs, i.e., if the level set is $z = 0$, the second chaos vanishes and then the asymptotic order of the area is lower than the one obtained for the level sets $z \neq 0$.
The Chapter is organized as follows. In Section 2.1 we briefly explain the ideas of the proof of Theorem 0.0.1, while Section 2.2 discusses the construction of a smooth approximation to the indicator function and its asymptotic properties. Finally, the proof of the Central Limit Theorem is given in Section 2.3 after investigating the asymptotic behavior of the Chaos components. Further technical computations are collected in Section 2.4 and finally, the last section reports some numerical evidence. More precisely, along the proof of Theorem 0.0.1, we come across the bounds of the chaotic components of the Wiener expansion of the excursion area in the case of the full sphere ([51], [52]). Hence, we propose numerical computations of these bounds.

2.1 On the proof of Theorem 0.0.1

From now on B will denote the spherical cap defined in (0.0.2).
We consider the excursion sets given in (0.0.3); similar to the case of the full sphere (see equation (1.4.2)), in order to study the excursion area we start by writing it as a functional, i.e., in this case, one has

$$S_\ell(B,z) = \int_B 1_{(T_\ell(x)>z)}(T_\ell(x))dx$$

and then, exploiting the L^2-expansion into Wiener Chaoses (Section 1.2, equation (1.2.3)), we have

$$1_{(T_\ell(x)\leq z)}(T_\ell(x)) = \sum_{q=0}^{\infty} \frac{J_q(z)}{q!} H_q(T_\ell(x)), \tag{2.1.1}$$

meaning that

$$\lim_{Q\to\infty} \mathbb{E}\Big[\Big|\sum_{q=0}^{Q} \frac{J_q(z)}{q!} H_q(T_\ell(x)) - 1_{(T_\ell(x)\leq z)}(T_\ell(x))\Big|^2\Big] = 0.$$

Because of the linearity of the integral and Jensen inequality, one has

$$\int_B 1_{(T_\ell(x)\geq z)}(T_\ell(x))\,dx = \lim_{Q\to\infty} \sum_{q=0}^{Q} \frac{J_q(z)}{q!} \int_B H_q(T_\ell(x))\,dx = \lim_{Q\to\infty} \sum_{q=0}^{Q} \frac{J_q(z)}{q!} h_{\ell;q}(B)$$

where

$$h_{\ell;q}(B) := \int_B H_q(T_\ell(x))\,dx \tag{2.1.2}$$

(in agreement with (1.4.4)); indeed,

$$\mathbb{E}\Big[\Big|\sum_{q=0}^{Q} \frac{J_q(z)}{q!} \int_B H_q(T_\ell(x))\,dx - \int_B 1_{(T_\ell(x)\leq z)}(T_\ell(x))\,dx\Big|^2\Big]$$
$$\leq \mathbb{E}\Big[\int_B \Big|\sum_{q=0}^{Q} \frac{J_q(z)}{q!} H_q(T_\ell(x)) - 1_{(T_\ell(x)\leq z)}(T_\ell(x))\Big|^2 dx\Big],$$

which goes to zero thanks to (2.1.1). Since the coefficients J_q are the same as defined in (1.4.3), we can hence write

$$\int_B 1_{(T_\ell(x)>z)}(T_\ell(x))\,dx = \int_B (1-\Phi(z))\,dx + \int_B \phi(z)H_1(T_\ell(x))dx + \int_B z\phi(z)\frac{1}{2}H_2(T_\ell(x))dx + \int_B \sum_{q=3}^{\infty} \frac{J_q(z)}{q!} H_q(T_\ell(x))\,dx, \tag{2.1.3}$$

in the $L^2(\Omega)$—convergence sense. The same holds for the variance thanks to the continuity of the norm. Indeed

$$\begin{aligned} \mathrm{Var}\Big(\int_B 1_{(T_\ell(x)>z)}(T_\ell(x))\,dx\Big) &= \mathbb{E}\Big[\Big(\int_B 1_{(T_\ell(x)>z)}(T_\ell(x))\,dx\Big)^2\Big] = \\ &= \langle \int_B 1_{(T_\ell(x)>z)}(T_\ell(x))\,dx, \int_B 1_{(T_\ell(x)>z)}(T_\ell(x))\,dx \rangle_{L^2(\Omega)} = \\ &= \lim_{Q\to\infty} \langle \sum_{q=0}^{Q} \frac{J_q(z)}{q!} \int_B H_q(T_\ell(x))\,dx, \sum_{q=0}^{Q} \frac{J_q(z)}{q!} \int_B H_q(T_\ell(x))\,dx \rangle. \end{aligned} \tag{2.1.4}$$

Hence the following expansion holds in $L^2(\Omega)$ sense:

$$\mathrm{Var}\Big(\int_B 1_{(T_\ell(x)>z)}(T_\ell(x))dx\Big) = 0 + \phi(z)^2\mathrm{Var}\Big(\int_B H_1(T_\ell(x))dx\Big) + \frac{z^2\phi(z)^2}{4}\mathrm{Var}\Big(\int_B H_2(T_\ell(x))dx\Big) + \mathrm{Var}\Big(\int_B \sum_{q=3}^{\infty} \frac{J_q(z)}{q!} H_q(T_\ell(x))dx\Big). \tag{2.1.5}$$

Note also that the mean value of the excursion area is simply given by

$$\mathbb{E}[S_\ell(B,z)] = \int_B \mathbb{E}[1_{(T_\ell(x)>z)}(T_\ell(x))]\,dx = (1-\Phi(z))m(B),$$

where we recall $m(B)$ denotes the measure of B (see Section 0.0.1).

The Quantitative Central Limit Theorem is established by the analysis of the asymptotic behavior for each of the terms in (2.1.5); in this section we give a summary of the results we obtained for these singular components.

In the sequel, we shall need a continuous differentiable function, which we denote as $1_{B,\varepsilon}(x)$, for $\varepsilon > 0$, converging to the indicator function in $L^1(\mathbb{S}^2)$, as $\varepsilon \to 0$.

Remark 2.1.1. *Along the framework we will refer to a particular smooth function, constructed in Section 2.2. However, we would like to stress that the specific choice of the mollifier function is not important; more precisely, the fundamental issue is the behavior of its Fourier coefficients: indeed, we need them to go to zero quite "fast", in order to exchange integrals and series and to work with absolute convergent series. In Section 2.2 we give just an example of such a possible function.*

Here below, we summarize the conditions which ε has to satisfy in order to prove Theorem 0.0.1. Note that we take $\varepsilon := \varepsilon_\ell$ as a sequence depending on ℓ; nevertheless, we drop the ℓ whenever possible for notational simplicity.

Assumption 1. *Let us consider $1_{B,\varepsilon}(x)$ the smooth function constructed in Section 2.2, then $\varepsilon = \varepsilon_\ell$ is such that*

$$\ell^{\frac{3-M}{2M+1}} < \varepsilon_\ell < \ell^{-\frac{1}{3}}. \tag{2.1.6}$$

The parameter M will be fixed below (see Section 2.2).

The condition on the left in (2.1.6) ensures the convergence of the Fourier coefficients $b_{\ell;\varepsilon}$ of the Fourier expansion of $1_{B;\varepsilon}(x)$, namely

$$1_{B;\varepsilon}(x) = \sum_\ell b_{\ell;\varepsilon} Y_{\ell 0}(x), \tag{2.1.7}$$

and the absolutely convergence of (2.1.7) (see the bound in (2.2.10)). Whereas, the bound on the right hand side of (2.1.6) makes the first chaotic component smaller than the second (see the proof of Proposition 2.1.3).

Example 2.1.2. *If we set for instance $\varepsilon = \varepsilon_\ell = \dfrac{1}{\ell^\alpha}$, with $\alpha > 0, \alpha \in \mathbb{R}$, the lower bound in (2.1.6) implies*

$$\alpha < \frac{M-3}{2M+1}$$

and the upper bound

$$\alpha > \frac{1}{3};$$

hence, a sequence ε_ℓ satisfying Assumption 1 exists taking $M > 10$.

Since $H_1(T_\ell(x)) = T_\ell(x)$, we obtain, for the first chaotic component, the following proposition.

Proposition 2.1.3. *Let B be the spherical cap defined in (0.0.2), then the variance of the first chaotic component of (2.1.5) is*

$$\mathrm{Var}\left(\int_{\mathbb{S}^2} 1_B(x) T_\ell(x)\, dx \right) = o\left(\frac{1}{\ell} \right)$$

as $\ell \to \infty$.

To establish this result, we write the variance as

$$\begin{aligned}\mathrm{Var}\left(\int_{\mathbb{S}^2} 1_B(x) T_\ell(x)\, dx \right) = \mathrm{Var}\left(\int_{\mathbb{S}^2} (1_B(x) - 1_{B,\varepsilon}(x)) T_\ell(x)\, dx \right) + \mathrm{Var}\left(\int_{\mathbb{S}^2} 1_{B,\varepsilon}(x) T_\ell(x)\, dx \right) + \\ + \mathbb{E}\left[\int_{\mathbb{S}^2} (1_{B,\varepsilon}(x) - 1_B(x)) 1_{B;\varepsilon}(y) T_\ell(x) T_\ell(y)\, dx dy \right],\end{aligned} \tag{2.1.8}$$

where $1_{B;\varepsilon}(\cdot)$ is the mollifier function converging in $L^1(\mathbb{S}^2)$ and satisfying Assumption 1. The second integral in (2.1.8) will be computed to be

$$\mathrm{Var}\left(\int_{\mathbb{S}^2} 1_{B,\varepsilon}(x) T_\ell(x)\, dx \right) = \frac{4\pi}{2\ell + 1} b_{\ell,\varepsilon}^2,$$

where, we have already said, $b_{\ell,\varepsilon}$ are the Fourier coefficients of $1_{B,\varepsilon}(x)$, given in Theorem 2.2.5. The former and the latter terms in (2.1.8) will be proved to be of order $\frac{1}{\sqrt{\ell}} \varepsilon^{3/2}$, for $\varepsilon \to 0$. Hence, the right hand side of Assumption 1 implies the thesis of Proposition 2.1.3.

As far as the second chaotic component is concerned, the following proposition will be proved.

Proposition 2.1.4. *Let us consider $1_{B,\varepsilon}(x)$, for $\varepsilon > 0$, the continuous function constructed in Section 2.2, converging to $1_B(x)$ in $L^1(\mathbb{S}^2)$, with ε satisfying Assumption 1. It can be proved (see Lemma 2.3.2) that*

$$\mathrm{Var}\left(\int_{\mathbb{S}^2} 1_{B,\varepsilon}(x) H_2(T_\ell(x))\, dx \right) = 8\pi \sum_{\ell_1} b_{\ell_1;\varepsilon}^2 \frac{1}{2\ell_1 + 1} \left(C_{\ell 0 \ell 0}^{\ell_1 0} \right)^2, \tag{2.1.9}$$

where $\{C_{\ell 0 \ell 0}^{\ell_1 0}\}$ are the Clebsch-Gordan coefficients ([75], Chapter 8 or Appendix A.2). Then, the variance of the second chaotic projection of the excursion area in (2.1.5) is

$$\mathrm{Var}\left(\int_B H_2(T_\ell(x))\, dx \right) = 8\pi \sum_{\ell_1} b_{\ell_1;\varepsilon}^2 \frac{1}{2\ell_1 + 1} \left(C_{\ell 0 \ell 0}^{\ell_1 0} \right)^2 + o\left(\frac{1}{\ell} \right), \tag{2.1.10}$$

as $\ell \to \infty$, where the bound is uniformly in ε.

Remark 2.1.5. *It is easy to see that* $\sum_\ell b^2_{\ell;\varepsilon} = ||1_{B;\varepsilon}||^2_{L^2(\mathbb{S}^2)} \le m(\mathbb{S}^2) = 4\pi$, *indeed*

$$\begin{aligned} ||1_{B;\varepsilon}||^2_{L^2(\mathbb{S}^2)} &= \int_{\mathbb{S}^2} [1_{B;\varepsilon}(x)]^2\, dx = \int_{\mathbb{S}^2} \sum_{\ell=0}^{\infty} b_{\ell;\varepsilon} Y_{\ell 0}(x) \sum_{\ell'=0}^{\infty} b_{\ell';\varepsilon} Y_{\ell' 0}(x)\, dx \\ &= \sum_{\ell=0}^{\infty} b_{\ell;\varepsilon} \sum_{\ell'=0}^{\infty} b_{\ell';\varepsilon} \int_{\mathbb{S}^2} Y_{\ell 0}(x) Y_{\ell' 0}(x)\, dx \end{aligned} \tag{2.1.11}$$

and the orthogonality condition (A.1.8) implies

$$||1_{B;\varepsilon}||^2_{L^2(\mathbb{S}^2)} = \sum_{\ell=0}^{\infty} b_{\ell;\varepsilon} \sum_{\ell'=0}^{\infty} b_{\ell';\varepsilon} \delta_\ell^{\ell'} = \sum_\ell b^2_{\ell;\varepsilon}. \tag{2.1.12}$$

Remark 2.1.6. *It is interesting to compare the results in Proposition 2.1.4 with the one in the case of the full sphere. Then, let us consider* $B = \mathbb{S}^2$, *i.e.* $1_B(\cdot) = 1_{\mathbb{S}^2}(\cdot)$; *in this case the approximating function* $1_{B;\varepsilon}(\cdot)$ *is not necessary. Indeed, the only term of the Fourier expansion of the indicator function* $1_{\mathbb{S}^2}(\cdot)$ *is* $\ell_1 = 0$, *moreover, for (A.2.15)*

$$C^{00}_{\ell 0 \ell 0} = \frac{1}{\sqrt{2\ell+1}},$$

$$\text{and} \quad b_0 = 2\pi \int_0^\pi \frac{1}{\sqrt{4\pi}} \sin\theta\, d\theta = \frac{4\pi}{\sqrt{4\pi}} = \sqrt{4\pi},$$

so that

$$8\pi \sum_{\ell_1=0}^{\infty} b^2_{\ell_1;\varepsilon} \frac{1}{2\ell_1+1} (C^{\ell_1 0}_{\ell 0 \ell 0})^2 = 8\pi b_0^2 (C^{00}_{\ell 0 \ell 0})^2 = 32\pi^2 \frac{1}{2\ell+1},$$

hence, the variance is

$$\mathrm{Var}\left(\int_B H_2(T_\ell(x))\, dx \right) = 32\pi^2 \frac{1}{2\ell+1} \sim 16\pi^2 \frac{1}{\ell},$$

that is exactly the value obtained (using (1.2.1)) in [51], Proposition 2.1.

The idea of the proof of Proposition 2.1.4 is similar to the one given in Proposition 2.1.3. More precisely, we write

$$\begin{aligned} \mathrm{Var}\left(\int_{\mathbb{S}^2} H_2(T_\ell(x))\, dx \right) &= \mathrm{Var}\left(\int_{\mathbb{S}^2} (1_B(x) - 1_{B,\varepsilon}(x)) H_2(T_\ell(x))\, dx \right) \\ &+ \mathrm{Var}\left(\int_{\mathbb{S}^2} 1_{B,\varepsilon}(x) H_2(T_\ell(x))\, dx \right) \\ &+ 2\mathbb{E}\left[\int_{\mathbb{S}^2\times\mathbb{S}^2} 1_{B,\varepsilon}(x) \Big(1_B(y) - 1_{B,\varepsilon}(y) \Big) H_2(T_\ell(x)) H_2(T_\ell(y))\, dx dy \right]. \end{aligned} \tag{2.1.13}$$

The first integral in (2.1.13) can be shown to be smaller than $\dfrac{Const \cdot \varepsilon}{2\ell+1}$ (see below (2.3.14)), which is a $o\left(\dfrac{1}{\ell}\right)$ since $\varepsilon \to 0$; the same bound holds for the third integral in (2.1.13), in view of the Cauchy-Schwarz inequality. Whereas, for the second integral in (2.1.13), the validity of (2.1.9) can be proved (see Lemma 2.3.2) and it is seen that its asymptotic order is $\dfrac{1}{\ell}$ (see

Lemma 2.3.3). The proof of (2.1.9) is based on manipulations of spherical harmonics and their integrals. More precisely, we make use of the addition formula (A.1.9) (see for example [45], eq. (3.42) p. 66)

$$\sum_{m=-\ell}^{\ell} \overline{Y}_{\ell m}(x) Y_{\ell m}(y) = \frac{2\ell+1}{4\pi} P_\ell(\langle x, y \rangle) \; ; \tag{2.1.14}$$

moreover, recalling that

$$Y_{\ell 0}(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta), \tag{2.1.15}$$

using the expansion

$$1_{B;\varepsilon}(x) = \sum_{\ell=0}^{\infty} b_{\ell;\varepsilon} Y_{\ell 0}(x) \tag{2.1.16}$$

and replacing these formulae in the left hand side in (2.1.9), we obtain the so-called Gaunt integral of spherical harmonics ([45] eq. (3.64), p. 81) which can be computed by the following relation:

$$\int_{\mathbb{S}^2} Y_{\ell_1 m_1}(x) Y_{\ell_2 m_2}(x) \overline{Y}_{\ell_3 m_3}(x) d\sigma(x) = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_3+1)}} C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} C^{\ell_3 0}_{\ell_1 0 \ell_2 0}, \tag{2.1.17}$$

for all ℓ_1, ℓ_2, ℓ_3 (Proposition A.2.1). Finally, the proof is completed by a careful analysis of properties for the Clebsch-Gordan coefficients, most of which are reported in the Appendix A.2.

The next important step in our argument is to establish the Quantitative Central Limit Theorem. This argument requires two steps; first we need to show that the variance of all higher-order chaoses for $q \geq 3$ is of smaller order; this can be done quite simply by some rather easy majorizations, which allow to show that all these terms are of order $o\left(\frac{1}{\ell}\right)$. On the other hand, since the second term is the leading component, we would like to compute its fourth-order cumulant to apply Corollary 1.2.4 (Theorem 5.2.7, [57]) and hence to establish asymptotic Gaussianity. However, the difficulty to handle computations with the indicator function leads us to consider the fourth cumulant of

$$h^*_{\ell,2}(B) := \int_{\mathbb{S}^2} 1_{B,\varepsilon}(x) H_2(T_\ell(x))\, dx; \tag{2.1.18}$$

more precisely, we shall show that

Proposition 2.1.7. *Under the assumptions of Proposition 2.1.4, the fourth cumulant of (2.1.18) satisfies*

$$cum_4(h^*_{\ell;2}(B)) = O\left(\frac{1}{\ell^3}\right),$$

as $\ell \to \infty$.

Our approach in Proposition 2.1.7 is different from the one used in related circumstances by for instance [52], [24], [65]; indeed these papers use an approximation of Legendre polynomials known as Hilb's asymptotics (see Appendix A.1.3): however this approximation turned out not to be efficient enough in the present framework. Hence, we need to exploit a different argument, i.e., we compute the exact values of the multiple integrals for spherical harmonics by means of Gaunt integrals (A.2.1) (see [45]) and Clebsch-Gordan coefficients.

At this stage, from Corollary 1.2.4 (Theorem 5.2.7 [57]), the following bound holds

$$d_W\left(\frac{h^*_{\ell;2}(B)}{\sqrt{\mathrm{Var}(h^*_{\ell;2}(B))}}, Z\right) \leq \sqrt{\frac{1}{6}\left(\frac{cum_4(h^*_{\ell;2}(B))}{\mathrm{Var}(h^*_{\ell;2}(B))^2}\right)},$$

then, CLT can be proved for $h^*_{\ell,2}(B)$. Now, in view of the fact that

$$\mathbb{E}\left[\int_{\mathbb{S}^2} 1_{B,\varepsilon}(x) H_2(T_\ell(x))\,dx - \int_{\mathbb{S}^2} 1_B(x) H_2(T_\ell(x))\,dx\right]^2 = o\left(\frac{1}{\ell}\right) \text{ as } \varepsilon \to 0, \qquad (2.1.19)$$

we prove Theorem 0.0.1 exploiting the triangular inequality for the Wasserstein distance (see the last subsection of Section 2.3). Note that, the result of Theorem 0.0.1 is the same obtained for the sphere in (1.4.8) (see also [47]).

2.2 Construction of a mollifier for the characteristic function

This section can be considered of some independent interest; it describes a method to construct an approximation of the indicator function, i.e., it gives an explicit expression for the function $1_{B,\varepsilon}(\cdot)$, already mentioned, converging to the indicator function $1_B(\cdot)$ in $L^1(\mathbb{S}^2)$.

For any fixed $M > 0, M \in \mathbb{N}$, a general method to construct a function $\phi(\cdot) \in C^M$, can be given by the B-splines approach (see [45], p. 250), as follows. First of all, recall that the Bernstein polynomials are defined as

$$B_i^{(n)}(t) := \binom{n}{i} t^i (1-t)^{n-i},$$

where $t \in [0,1]$, $i = 0, \dots, n$ and $n = 1, 2, \dots$. Then, we can define polynomials

$$q_{2k+1}(t) := \sum_{i=0}^{k} B_i^{(2k+1)}(t);$$

one has that $q_{2k+1}(0) = 1$ and $q_{2k+1}(1) = 0$. Moreover,

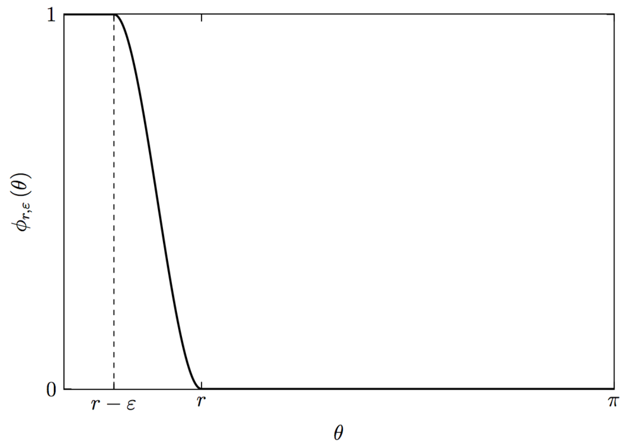
$$q^{(m)}_{2k+1}(1) = q^{(m)}_{2k+1}(0) = 0 \text{ for } m = 1, \dots, k.$$

Hence, let $r \in (0, \pi)$ and $\theta \in [0, \pi)$, for any $\varepsilon > 0$ we set

$$t := \frac{\theta - (r-\varepsilon)}{r - (r-\varepsilon)} \in [0,1]$$

and define the function

$$\phi_{r,\varepsilon}(\theta) := \begin{cases} 1 & \text{if } \theta \in [0, r-\varepsilon) \\ q_{2k+1}(t) = q_{2k+1}(\frac{\theta - r + \varepsilon}{\varepsilon}) & \text{if } \theta \in [r-\varepsilon, r] \\ 0 & \text{if } \theta \in [r, \pi) \end{cases} \qquad (2.2.1)$$



with $\theta \in (0,\pi)$.
The function $\phi_{r,\varepsilon}(\theta)$ is a $2k+1$-degree polynomial, so $\phi_{r,\varepsilon} \in C^M$ for $M < k+1/2$ and $\phi_{r,\varepsilon}(r-\varepsilon) = q_{2k+1}(0) = 1$.

Remark 2.2.1. *The indicator function $1_B(x), x \in \mathbb{S}^2$ can be written in spherical coordinates as $1_B(\theta,\varphi)$, with $\theta \in [0,\pi)$ and $\varphi \in [0,2\pi]$, but indeed it only depends on the angle θ, namely,*

$$1_B(\theta,\varphi) = \begin{cases} 1 & \theta \le r \\ 0 & otherwise \end{cases} = 1_B(\theta). \tag{2.2.2}$$

Defining $1_{B,\varepsilon}(\theta) := \phi_{r,\varepsilon}(\theta)$, it is easily to see that, as $\varepsilon \to 0$, $1_{B,\varepsilon}(\cdot) \to 1_B(\cdot)$ in $L^1(\mathbb{S}^2)$. In fact,

$$\begin{aligned}\int_{\mathbb{S}^2} |1_B(x) - 1_{B,\varepsilon}(x)| dx &= 2\pi \int_0^\pi |1_B(\theta) - 1_{B,\varepsilon}(\theta)| \sin\theta d\theta \\ &\le 2\pi \int_{r-\varepsilon}^r \left| q_{2k+1}\left(\frac{\theta - \cos r + \varepsilon}{\varepsilon}\right)\right| \sin\theta d\theta \le 2\pi\varepsilon \to 0,\end{aligned}$$

as $\varepsilon \to 0$.

Now we focus on the function $\phi_{r,\varepsilon}(\cdot)$. As denoted in [35], we define $k_{r,\varepsilon}(\mu) := \phi_{r,\varepsilon}(\arccos\mu)$ with $\mu \in [-1,1]$. Now recall that any function $u \in L^2(-1,1)$ can be expanded in the $L^2(-1,1)$ convergent Fourier-Legendre series as

$$u = \sum_{\ell=0}^{\infty} u_\ell \frac{2\ell+1}{2} P_\ell = \sum u_\ell \frac{4\pi}{2} \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{2\ell+1}{4\pi}} P_\ell = \sum 2\pi u_\ell \sqrt{\frac{2\ell+1}{4\pi}} Y_\ell = \sum b_\ell Y_\ell,$$

with

$$u_\ell = \int_{-1}^1 u(x) P_\ell(x)\, dx$$

and hence

$$b_\ell = 2\pi u_\ell \sqrt{\frac{2\ell+1}{4\pi}} = \int_{-1}^1 u(x) Y_\ell(x)\, dx;$$

thus we can expand $k_{r,\varepsilon}$ in such a series and its Fourier coefficients are

$$b^r_{\ell,\varepsilon} = \sqrt{\frac{2\ell+1}{4\pi}} \int_{-1}^{1} k_{r,\varepsilon}(\mu) P_\ell(\mu)\, d\mu.$$

Remark 2.2.2. *For $\ell = 0$, it is easy to see that $b^r_{0,\varepsilon}$ is bounded above and below by two positive constants. Actually, by definition,*

$$b^r_{0,\varepsilon} = \sqrt{\frac{1}{4\pi}} \int_{-1}^{1} k_{r,\varepsilon}(\theta)\, d\theta = \frac{1}{\sqrt{4\pi}} \int_{-1}^{1} \phi_{r,\varepsilon}(\arccos\theta)\, d\theta;$$

changing coordinates $\arccos\theta = x$, *one has*

$$\begin{aligned} b^r_{0,\varepsilon} &= \frac{1}{\sqrt{4\pi}} \int_0^\pi \phi_{r,\varepsilon}(x) \sin x\, dx \\ &= \frac{1}{\sqrt{4\pi}} \int_0^{r-\varepsilon} \sin x\, dx + \frac{1}{\sqrt{4\pi}} \int_{r-\varepsilon}^{r} q_{2k+1}(x) \sin x\, dx \qquad (2.2.3) \\ &\geq \frac{1}{\sqrt{4\pi}} \int_0^{r-\varepsilon} \sin x\, dx = \frac{1}{\sqrt{4\pi}}(1 - \arccos(r-\varepsilon)) \geq \frac{1-r+\varepsilon}{\sqrt{4\pi}} > \frac{1-r}{\sqrt{4\pi}} \end{aligned}$$

and since

$$|\phi_{r,\varepsilon}(\theta)| \leq 1,$$

it is immediate to conclude that

$$\frac{1-r}{\sqrt{4\pi}} \leq b^r_{0,\varepsilon} \leq \frac{1}{\sqrt{\pi}}.$$

The main result of this section is given in the proposition below, which yields a bound for the Fourier coefficients $b^r_{\ell;\varepsilon}$.

Proposition 2.2.3. *For any fixed $M \in \mathbb{N}$ and $r \in (0,\pi)$, there exists a constant $K_{M,r}$ such that*

$$|b^r_{\ell,\varepsilon}| \leq \min\left\{ b^r_{0,\varepsilon}, \frac{K_{M,r}}{\ell^{M-\frac{1}{2}}\varepsilon^{2M+1}} \right\}.$$

In order to prove Proposition 2.2.3, we get a bound for the $M-$derivative of $k_{r,\varepsilon}$. Since $k_{r,\varepsilon}(\mu)$ is a composite function, Faà di Bruno's formula implies:

$$D^M(\phi_{r,\varepsilon}(\arccos\mu)) = M! \sum_{\nu=1}^{M} \frac{(D^\nu \phi_{r,\varepsilon})(\arccos\mu)}{\nu!} \sum_{h_1+\dots+h_\nu = M} \frac{D^{h_1}\arccos\mu}{h_1!} \cdots \frac{D^{h_\nu}\arccos\mu}{h_{\nu!}}, \qquad (2.2.4)$$

where the second sum is computed on all the possible integer values of $h_1, \dots, h_\nu \geq 1$ with sum equal to M. We note that this sum is bounded by a constant which depends on r; indeed, the arccos is a C^∞ function in each compact subset of $(-1,1)$ and since outside $[r-\varepsilon, r]$ all the derivatives of ϕ are zero and $r \neq \pi$, μ is always different from $+1$ and -1; hence the second sum of (2.2.4) is bounded away from -1 and 1. As far as the first sum is concerned in (2.2.4), it is possible to compute it explicitly

$$\begin{aligned}
&\sum_{\nu=1}^{M}\frac{(D^{\nu}\phi_{r,\varepsilon})(\arccos\mu)}{\nu!}=\sum_{\nu=1}^{M}\frac{1}{\nu!}D^{\nu}\Big(\phi_{r,\varepsilon}\Big(\frac{\arccos\mu-r+\varepsilon}{\varepsilon}\Big)\Big)\\
&=\sum_{\nu=1}^{M}\frac{1}{\nu!}\Big[D^{\nu}q_{2k+1}\Big(\frac{\arccos\mu-r+\varepsilon}{\varepsilon}\Big)\Big]1_{[r-\varepsilon,r]}=\sum_{\nu=1}^{M}\frac{1}{\nu!}\Big[D^{\nu}\sum_{i=0}^{k}B_i^{2k+1}\Big(\frac{\arccos\mu-r+\varepsilon}{\varepsilon}\Big)\Big]1_{[r-\varepsilon,r]}\\
&=\sum_{\nu=1}^{M}\frac{1}{\nu!}\Big[D^{\nu}\sum_{i=0}^{k}\binom{2k+1}{i}\Big(\frac{\arccos\mu-r+\varepsilon}{\varepsilon}\Big)^{i}\Big(1-\frac{\arccos\mu-r+\varepsilon}{\varepsilon}\Big)^{2k+1-i}\Big]1_{[r-\varepsilon,r]}\\
&=\sum_{\nu=1}^{M}\frac{1}{\nu!}\frac{1}{\varepsilon^{2k+1}}\Big[D^{\nu}\sum_{i=0}^{k}\binom{2k+1}{i}(\arccos\mu-r+\varepsilon)^{i}(r-\arccos\mu)^{2k+1-i}\Big]1_{[r-\varepsilon,r]}.
\end{aligned}\tag{2.2.5}$$

Since $\sum_{i=0}^{k}\binom{2k+1}{i}(\arccos\mu-r+\varepsilon)^{i}(r-\arccos\mu)^{2k+1-i}$ is a polynomial in the compact domain $[r-\varepsilon,r]$, we can bound (2.2.5) by $\frac{C_{M,r}}{\varepsilon^{2M+1}}$, where $C_{M,r}$ is a constant depending on r and M. The absolute value of (2.2.4) satisfies then

$$\Big|D^{M}\phi_{r,\varepsilon}(\arccos\mu)\Big|\leq\frac{M!C_{M,r}}{\varepsilon^{2M+1}}.\tag{2.2.6}$$

We are hence in the position to prove Proposition 2.2.3.

Proof of Proposition 2.2.3. We recall the following property of the Legendre polynomials (see for instance [1], Chapter 22, formula 22.7 and formula 22.8 combined with the Legendre differential equation)

$$(2\ell+1)P_\ell(x)=\frac{d}{dx}\Big[P_{\ell+1}(x)-P_{\ell-1}(x)\Big],\tag{2.2.7}$$

and we substitute it in the definition of $b^{r}_{\ell;\varepsilon}$ to obtain, integrating by parts,

$$\begin{aligned}
\int_{-1}^{1}k_{\varepsilon,r}(x)P_\ell(x)\,dx&=\Big[k_{\varepsilon,r}(x)\frac{P_{\ell+1}(x)-P_{\ell-1}(x)}{2\ell+1}\Big|_{-1}^{1}-\int_{-1}^{1}\frac{d}{dx}k_{\varepsilon,r}(x)\frac{P_{\ell+1}(x)-P_{\ell-1}(x)}{2\ell+1}\,dx\Big]\\
&=\frac{1}{2\ell+1}\int_{-1}^{1}\frac{d}{dx}k_{\varepsilon,r}(x)P_{\ell+1}(x)\,dx-\frac{1}{2\ell+1}\int_{-1}^{1}\frac{d}{dx}k_{\varepsilon,r}(x)P_{\ell-1}(x)\,dx.
\end{aligned}\tag{2.2.8}$$

Applying again (2.2.7) to $P_{\ell+1}$ and to $P_{\ell-1}$ in the place of P_ℓ and integrating by parts, one has that (2.2.8) is equal to

$$\begin{aligned}
&=\frac{1}{2\ell+1}\frac{1}{2\ell+3}\int_{-1}^{1}\frac{d^2}{dx^2}k_{\varepsilon,r}(x)(P_{\ell+2}(x)-P_\ell(x))\,dx+\\
&-\frac{1}{(2\ell+1)}\frac{1}{2\ell-1}\int_{-1}^{1}\frac{d^2}{dx^2}k_{\varepsilon,r}(x)(P_\ell(x)-P_{\ell-2}(x))\,dx\\
&=\frac{1}{2\ell+1}\frac{1}{2\ell+3}\int_{-1}^{1}\frac{d^2}{dx^2}k_{\varepsilon,r}(x)P_{\ell+2}(x)\,dx+\frac{1}{(2\ell+1)}\frac{1}{2\ell-1}\int_{-1}^{1}\frac{d^2}{dx^2}k_{\varepsilon,r}(x)P_{\ell-2}(x)\,dx\\
&-\frac{1}{2\ell+1}\frac{1}{2\ell+3}\int_{-1}^{1}\frac{d^2}{dx^2}k_{\varepsilon,r}(x)P_\ell(x)\,dx-\frac{1}{2\ell+1}\frac{1}{2\ell-1}\int_{-1}^{1}\frac{d^2}{dx^2}k_{\varepsilon,r}(x)P_\ell(x)\,dx.
\end{aligned}\tag{2.2.9}$$

Iterating M times, taking the absolute value, using (2.2.6) and the fact that $|P_\ell(x)| \le 1$ in $[-1,1]$ $\forall \ell$, one has that $|\int_{-1}^{1} k_{\varepsilon,r}(x)P_\ell(x)\,dx|$ is bounded by 2^M terms times

$$\frac{C}{\ell^M}\frac{M!C_{M,r}}{\varepsilon^{2M+1}}.$$

Consequently, for $\ell \ge 1$

$$|b^r_{\ell;\varepsilon}| \le \sqrt{\frac{2\ell+1}{4\pi}}\frac{C2^M M!C_{M,r}}{\varepsilon^{2M+1}\ell^M} \le \frac{K_{M,r}}{\ell^{M-1/2}\varepsilon^{2M+1}}, \tag{2.2.10}$$

where $K_{M,r} = \sqrt{\frac{3}{4\pi}}M!2^M C C_{M,r}$.

□

Remark 2.2.4. *Note that for the coefficients $b^r_{\ell;\varepsilon}$ to go to zero, the condition*

$$\ell^{M-1/2}\varepsilon^{2M+1} \to \infty,$$

as $\ell \to \infty$ and $\varepsilon \to 0$, has to be satisfied; Assumption 1 ensures it.

In conclusion, this section can be summarized in the theorem below.

Theorem 2.2.5. *Let $B \subset \mathbb{S}^2$ be a spherical cap of radius $r \in (0,\pi)$, parametrized by $\theta \in [0,r], \varphi \in [0,2\pi]$. For any $M > 0 \in \mathbb{N}$ and $\varepsilon > 0$ there exists a function $1_{B,\varepsilon} \in C^M$ which converges to the indicator function $1_B(x)$ in $L^1(\mathbb{S}^2)$, as $\varepsilon \to 0$, such that the coefficients $b^r_{\ell,\varepsilon}$ of the Fourier expansion*

$$1_{B,\varepsilon}(\theta) = \sum_{\ell=0}^{\infty} b^r_{\ell;\varepsilon}\sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta), \quad b^r_{\ell;\varepsilon} = \sqrt{\frac{2\ell+1}{4\pi}}\int_{-1}^{1} 1_{B,\varepsilon}(\arccos x)Y_\ell(x)\,dx \tag{2.2.11}$$

satisfy the condition

$$|b^r_{\ell;\varepsilon}| \le \min\left\{b^r_{0,\varepsilon}, \frac{K_{M,r}}{\ell^{M-\frac{1}{2}}\varepsilon^{2M+1}}\right\} \tag{2.2.12}$$

as $\ell \to \infty$, where

$$K_{M,r} = \sqrt{\frac{3}{4\pi}}M!2^M C C_{M,r}.$$

Example 2.2.6. *Let us consider $k = 1$, then $n = 2k+1 = 3$, $M = 1$ and $B_0(t) = (1-t)^3$, $B_1(t) = 3t(1-t)^2$. It follows that*

$$q(t) = B_0(t) + B_1(t) = 2t^3 - 3t^2 + 1$$

and

$$q'(t) = 6t^2 - 6t.$$

Hence, the first derivative of $k_{\varepsilon,r}(\mu), \mu \in [-1,1]$ is

$$\begin{aligned}
\frac{d}{d\mu}k(\mu) &= \frac{d}{d\mu}\phi(\arccos\mu) = \phi'(\arccos\mu)\frac{-1}{\sqrt{1-\mu^2}} \\
&= \left[\frac{6}{\varepsilon^3}(\arccos\mu - r + \varepsilon)^2 - \frac{6}{\varepsilon^3}(\arccos\mu - r + \varepsilon)\right]\frac{-1}{\sqrt{1-\mu^2}} \\
&= \frac{6}{\varepsilon^3}(\arccos\mu - r + \varepsilon)(\arccos\mu - r)\frac{-1}{\sqrt{1-\mu^2}}.
\end{aligned} \tag{2.2.13}$$

Accordingly, for $b^r_{\ell;\varepsilon}$ one obtains

$$|b^r_{\ell,\varepsilon}| \leq \left| \frac{1}{2\ell+1}\sqrt{\frac{2\ell+1}{4\pi}} \int_{-1}^{1} \frac{d}{d\mu} k(\mu)(P_{\ell+1}(x) - P_{\ell-1}(x))\, dx \right| \leq \frac{C_r}{\ell^{1/2}\varepsilon^3}.$$

We give some values of $b^r_{\ell,\varepsilon}$ in figure 2.1; the graphic was realized choosing the parameters as $\varepsilon = \frac{1}{2}$ and $r = \frac{\pi}{4}$.

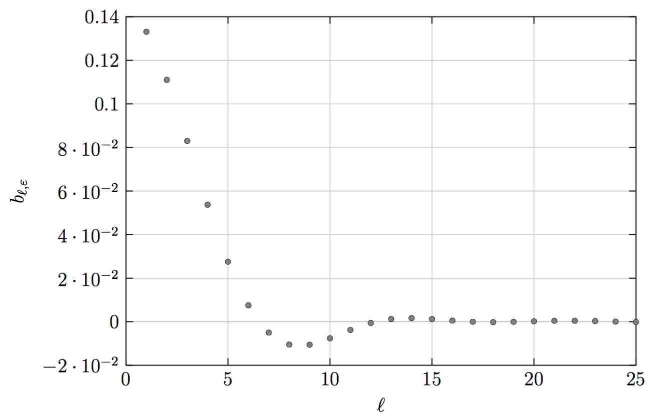


Figure 2.1: First values of $b^{\pi/4}_{\ell,1/2}$ varying ℓ.

Example 2.2.7. *Choosing $k = 2$, one has $n = 5$, $M = 2$ and $B_0(t) = (1-t)^5$, $B_1(t) = 5t(1-t)^4$ and $B_2(t) = 10t^2(1-t)^3$. One finds that*

$$q(t) = -6t^5 + 15t^4 - 10t^3 + 1,$$

$$q'(t) = -30t^4 + 60t^3 - 30t^2$$

and

$$q''(t) = -120t^3 + 180t^2 - 60t.$$

Then, the first and the second derivatives of $k_{\varepsilon,r}(\mu)$ are respectively

$$\frac{d}{d\mu}k(\mu)=\Big[-30(\frac{\arccos\mu-r+\varepsilon}{\varepsilon})^4+60(\frac{\arccos\mu-r+\varepsilon}{\varepsilon})^3-30(\frac{\arccos\mu-r+\varepsilon}{\varepsilon})^2\Big]\frac{1}{\varepsilon}\frac{-1}{\sqrt{1-\mu^2}};$$

$$\begin{aligned}\frac{d^2}{d\mu^2}k(\mu)=&\Big[-120(\frac{\arccos\mu-r+\varepsilon}{\varepsilon})^3+180(\frac{\arccos\mu-r+\varepsilon}{\varepsilon})^2-60\frac{\arccos\mu-r+\varepsilon}{\varepsilon}\Big]\times\\ &\times\frac{1}{\varepsilon^2}\frac{1}{1-\mu^2}\\ &+\Big[-30(\frac{\arccos\mu-r+\varepsilon}{\varepsilon})^4+60(\frac{\arccos\mu-r+\varepsilon}{\varepsilon})^3-30(\frac{\arccos\mu-r+\varepsilon}{\varepsilon})^2\Big]\times\\ &\times\frac{1}{\varepsilon}\mu\sqrt{1-\mu^2}\frac{-1}{1-\mu^2}\\ &=\frac{\arccos\mu-r+\varepsilon}{\varepsilon^5}\Big[-120(\arccos\mu-r+\varepsilon)^2+180(\arccos\mu-r+\varepsilon)\varepsilon-60\varepsilon^2\Big]\times\\ &\times\frac{1}{1-\mu^2}+\\ &+\frac{\arccos\mu-r+\varepsilon}{\varepsilon^5}\Big[-30(\arccos\mu-r+\varepsilon)^2+60(\arccos\mu-r+\varepsilon)\varepsilon-30\varepsilon^2\Big]\times\\ &\times\frac{1}{1-\mu^2}\Big(\frac{-\mu}{\sqrt{1-\mu^2}}\Big).\end{aligned}$$

Hence

$$\left|\frac{d^2}{d\mu^2}k(\mu)\right|\le\frac{C_r}{\varepsilon^5}$$

and

$$|b^r_{\ell,\varepsilon}|\le\frac{C_r}{\ell^{3/2}\varepsilon^5}.$$

Remark 2.2.8. *In the table and in Figure 2.2, we compare $b^r_{\ell;\varepsilon}:=b_{\ell;\varepsilon}$, for $\ell=1,2,3,4,5$, for different values of ε and k, n, M as in Example 2.2.7.*

ℓ	$b_{\ell;1/2}$	$b_{\ell,1/4}$	$b_{\ell,1/8}$	$b_{\ell,1/10}$
1	0.132269	0.188425	0.218866	0.225059
2	0.111278	0.147981	0.163897	0.166747
3	0.0843363	0.0983641	0.0987674	0.0982093
4	0.0557163	0.0493925	0.0381274	0.0352638
5	0.0294925	0.00959262	-0.00638063	-0.0097985

We note that the decay of the coefficients $b_{\ell;\varepsilon}$ is actually faster than the one given by our upper bound.

Remark 2.2.9. *It is quite natural to compare our result with the work of Lang and Schwab in [35]. We report here, briefly, their findings. Hence, they define the space $V^n(-1,1)$ as the closures of $H^n(-1,1)$, where $H^n(-1,1)$ is the standard Sobolev spaces, with respect to the weighted norms $||u||^2_{V^n(-1,1)}:=\sum_{j=0}^n|u|^2_{V^j(-1,1)}$, where for $j\in\mathbb{N}_0$,*

$$|u|^2_{V^j(-1,1)}:=\int_{-1}^1\left|\frac{\partial^j}{\partial\mu^j}u(\mu)\right|^2(1-\mu^2)^j\,d\mu,$$

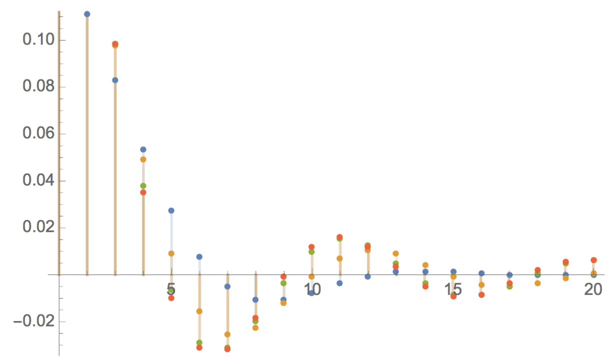
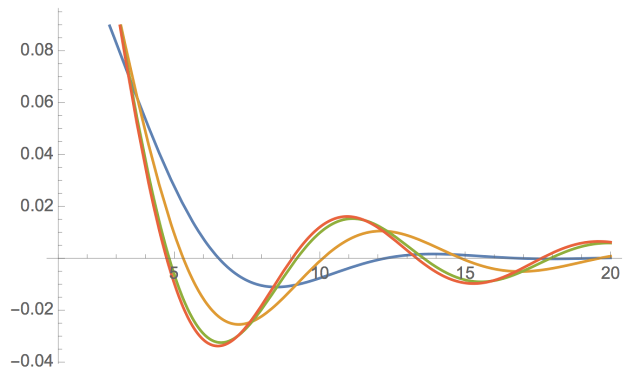


Figure 2.2: $b^r_{\ell;\varepsilon}$ varying ℓ, for $\varepsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{10}$.



is a seminorn. Denoted as $(\frac{2\ell+1}{2}(1+\ell^{2n}), \ell \in \mathbb{N}_0)$ *the sequence of weights, the authors in [35] show an isomorphism between the spaces* $V^n(-1,1)$ *and the spaces of the weights* $\ell_n := \ell^2((\frac{2\ell+1}{2}(1+\ell^{2n}), \ell \in \mathbb{N})$*. Precisely, they proved that, for* $u(\mu) \in V^n(-1,1), n \in \mathbb{N}_0$*, the sequence* $(\ell^{n+1/2}A_\ell, \ell \geq n)$*, with* $A_\ell = 2\pi u_\ell$*, is in* $\ell^2(\mathbb{N}_0)$ *if and only if* $(1-\mu^2)^{n/2}\frac{\partial^n}{\partial \mu^n}u(\mu)$ *is in* $L^2(-1,1)$*; namely,*

$$\frac{1}{(4\pi)^2}\sum_{\ell\geq n} A_\ell^2 \frac{2\ell+1}{2}\ell^{2n} < +\infty$$

if and only if

$$\int_{-1}^{1}\left|\frac{\partial^n}{\partial \mu^n}u(\mu)\right|^2 (1-\mu^2)^n \, d\mu < \infty.$$

More explicitly, in their proof (p. 13 [35]) they get that

$$\int_{-1}^{1}\left|\frac{\partial^n}{\partial \mu^n}u(\mu)\right|^2 (1-\mu^2)^n \, d\mu = \sum_{\ell\geq n} A_\ell^2 \frac{2\ell+1}{2(4\pi)^2}\frac{(\ell+n)!}{(\ell-n)!} \tag{2.2.14}$$

and

$$c_1(n)\ell^{2n} \leq \frac{(\ell+n)!}{(\ell-n)!} \leq c_2(n)\ell^{2n}.$$

Although it is possible to compute explicitly the integral on the left hand side of (2.2.14), this would be sufficient only for a bound on the tail behavior of the series in the right hand side of (2.2.14), while, in our situation, we require a full control on any term A_ℓ^2.

Remark 2.2.10. *We refer to [34] for the broadly similar construction of a "spherical bump function". Also, our proposal is in some sense symmetric to so-called needlets (see i.e., [55], [54], [7] and Chapter 10 of [45]). Indeed, in the standard needlet construction one considers spherical functions with compact support in the harmonic domain and nearly-exponential decay in the real domain, whereas here the converse is studied: functions with compact support in the real domain and polynomial decays in the harmonic space.*

2.3 Proof of Theorem 0.0.1

Here we finally prove Theorem 0.0.1; as stated at the beginning of the paragraph, we do that studying each single term of the chaotic projection in (2.1.5) separately. We divide in small different subsections the results obtained for these components.

From now on, $1_{B,\varepsilon}(x)$ is the function given in Remark 2.2.1, satisfying (2.2.12) and Assumption 1.

First chaotic component

The variance of the first chaotic component, i.e., Proposition 2.1.3 follows as a corollary of the lemma below.

Lemma 2.3.1. *For any $\varepsilon > 0$, satisfying Assumption 1,*

$$\mathrm{Var}\left(\int_{\mathbb{S}^2} 1_B(x)T_\ell(x)\,dx\right) = \frac{4\pi}{2\ell+1} b_{\ell;\varepsilon}^2 + O(\ell^{-1/2}\varepsilon^{3/2}), \tag{2.3.1}$$

as $\ell \to 0$, where $b_{\ell;\varepsilon}$ are the Fourier coefficients of $1_{B,\varepsilon}(x)$, given by (2.2.11).

Proof of Lemma 2.3.1. The first chaotic projection can be written as

$$\int_B T_\ell(x)\,dx = \int_{\mathbb{S}^2} [1_B(x) - 1_{B;\varepsilon}(x)]T_\ell(x)\,dx + \int_{\mathbb{S}^2} 1_{B;\varepsilon}(x)T_\ell(x)\,dx$$

and consequently, its variance as

$$\begin{aligned}\mathrm{Var}\left(\int_B T_\ell(x)\,dx\right) &= \mathrm{Var}\left(\int_{\mathbb{S}^2} [1_B(x) - 1_{B;\varepsilon}(x)]T_\ell(x)\,dx\right) + \mathrm{Var}\left(\int_{\mathbb{S}^2} 1_{B;\varepsilon}(x)T_\ell(x)\,dx\right) \\ &\quad + 2\mathbb{E}\left[\int_{\mathbb{S}^2\times\mathbb{S}^2} (1_B(x) - 1_{B;\varepsilon}(x))1_{B;\varepsilon}(y)T_\ell(x)T_\ell(y)\,dx\,dy\right].\end{aligned} \tag{2.3.2}$$

For the first variance of (2.3.2) it holds that

$$\begin{aligned}\mathrm{Var}\Big(\int_{\mathbb{S}^2}(1_B(x)-1_{B;\varepsilon}(x))T_\ell(x)\,dx\Big) &= \int_{\mathbb{S}^2\times\mathbb{S}^2}(1_B(x)-1_{B;\varepsilon}(x))(1_B(y)-1_{B;\varepsilon}(y))\mathbb{E}[T_\ell(x)T_\ell(y)]\,dxdy \\ &\le \int_{\mathbb{S}^2}|1_B(x)-1_{B;\varepsilon}(x)|\Big(\int_{\mathbb{S}^2}|1_B(y)-1_{B;\varepsilon}(y)|P_\ell(\langle x,y\rangle)|\,dy\Big)dx\end{aligned} \tag{2.3.3}$$

and applying the Cauchy-Schwarz inequality to the second integral, (2.3.3) is bounded by

$$\begin{aligned}&\le \int_{\mathbb{S}^2}|1_B(x)-1_{B;\varepsilon}(x)|\Big(\int_{\mathbb{S}^2}|1_B(y)-1_{B;\varepsilon}(y)|^2\,dy\Big)^{1/2}\Big(\int_{\mathbb{S}^2}|P_\ell(\langle x,y\rangle)|^2\,dy\Big)^{1/2}dx \\ &\le \sqrt{\frac{2}{2\ell+1}}2\pi\sqrt{2\pi}\sqrt{2\pi\varepsilon}\varepsilon;\end{aligned} \tag{2.3.4}$$

the third term in (2.3.2) is as small as this one by Cauchy-Schwarz inequality. Concerning the second variance in (2.3.2), one has

$$\begin{aligned}&\mathrm{Var}\Big(\int_{\mathbb{S}^2}1_{B;\varepsilon}(x)T_\ell(x)\,dx\Big) = \mathbb{E}\Big[\Big(\int_{\mathbb{S}^2}1_{B;\varepsilon}(x)T_\ell(x)\,dx\Big)^2\Big] \\ &= \int_{\mathbb{S}^2\times\mathbb{S}^2}1_{B;\varepsilon}(x)1_{B;\varepsilon}(y)\mathbb{E}[T_\ell(x)T_\ell(y)]\,dx\,dy = \int_{\mathbb{S}^2\times\mathbb{S}^2}1_{B;\varepsilon}(x)1_{B;\varepsilon}(y)P_\ell(\langle x,y\rangle)\,dx\,dy.\end{aligned} \tag{2.3.5}$$

Through the addition formula (A.1.9) (see [45]) and the expansion

$$1_{B,\varepsilon}(x) = \sum_{\ell=0}^{\infty} b_{\ell;\varepsilon}Y_{\ell 0}(x), \tag{2.3.6}$$

it is possible to write (2.3.5) as

$$\int_{\mathbb{S}^2\times\mathbb{S}^2}\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}\overline{Y_{\ell m}(x)}Y_{\ell m}(y)\sum_{\ell_1=0}^{\infty}b_{\ell_1;\varepsilon}Y_{\ell_1 0}(x)\sum_{\ell_2=0}^{\infty}b_{\ell_2;\varepsilon}Y_{\ell_2 0}(y)\,dx\,dy. \tag{2.3.7}$$

Condition (2.1.6) implies that the series $\sum_{\ell=0}b_{\ell;\varepsilon}Y_{\ell 0}(x)$ is absolutely convergent; indeed

$$\sum|b^r_{\ell;\varepsilon}||Y_\ell(x)|\sim\sum|b^r_{\ell;\varepsilon}|\sqrt{\ell}<\sum\frac{1}{\ell^2}<\infty,$$

and so we can exchange the series with the integral to derive that (2.3.7) equals to

$$\begin{aligned}&\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}\sum_{\ell_1=0}^{\infty}b_{\ell_1;\varepsilon}\sum_{\ell_2=0}^{\infty}b_{\ell_2;\varepsilon}\int_{\mathbb{S}^2\times\mathbb{S}^2}\overline{Y_{\ell m}(x)}Y_{\ell m}(y)Y_{\ell_1 0}(x)Y_{\ell_2 0}(y)\,dx\,dy = \\ &=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}\sum_{\ell_1=0}^{\infty}b_{\ell_1;\varepsilon}\sum_{\ell_2=0}^{\infty}b_{\ell_2;\varepsilon}\int_{\mathbb{S}^2}\overline{Y_{\ell m}(x)}Y_{\ell_1 0}(x)\,dx\int_{\mathbb{S}^2}Y_{\ell m}(y)Y_{\ell_2 0}(y)\,dy.\end{aligned} \tag{2.3.8}$$

The orthogonality condition (A.1.8) ([45] eq. (3.39), p. 66)

$$\int_{\mathbb{S}^2}\overline{Y_{\ell m}(x)}Y_{\ell' m'}(x)\,dx = \delta^{\ell}_{\ell'}\delta^{m}_{m'}, \tag{2.3.9}$$

reduces (2.3.8) to

$$\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}\sum_{\ell_1=0}^{\infty} b_{\ell_1,\varepsilon}\sum_{\ell_2=0}^{\infty} b_{\ell_2;\varepsilon}\delta_{\ell_1}^{\ell}\delta_0^m\delta_{\ell_2}^{\ell}\delta_0^m = \frac{4\pi}{2\ell+1}b_{\ell;\varepsilon}^2$$

and then the variance (2.3.5) is

$$\mathrm{Var}\left(\int_{\mathbb{S}^2} 1_{B;\varepsilon}(x)T_\ell(x)\,dx\right) = \frac{4\pi}{2\ell+1}b_{\ell;\varepsilon}^2; \tag{2.3.10}$$

thus, (2.3.10) and (2.3.3) lead to the thesis of the lemma. □

Now, Proposition 2.1.3 follows from the choice of $\varepsilon = \varepsilon_\ell$ satisfying Assumption 1.

Second chaotic component

In this subsection we prove Proposition 2.1.4; to this aim we introduce the two lemmas below, whose proofs can be found in Section 2.4.

Lemma 2.3.2. *Under the assumptions of Proposition 2.1.4, one has that*

$$\mathrm{Var}\left(\int_{\mathbb{S}^2} 1_{B;\varepsilon}(x)H_2(T_\ell(x))\,dx\right) = 8\pi\sum_{\ell_1} b_{\ell_1;\varepsilon}^2\frac{1}{2\ell_1+1}\left(C_{\ell 0\ell 0}^{\ell_1 0}\right)^2, \tag{2.3.11}$$

where $\{C_{\ell 0\ell 0}^{\ell_1 0}\}$ *are the Clebsch-Gordan coefficients (see [75] or the Appendix).*

Lemma 2.3.3. *There exist two strictly positive constants* c_1 *and* c_2 *such that*

$$\frac{c_1}{\ell} \le \mathrm{Var}\left(\int_{\mathbb{S}^2} 1_{B;\varepsilon}(x)H_2(T_\ell(x))\,dx\right) = 8\pi\sum_{\ell_1} b_{\ell_1;\varepsilon}^2\frac{1}{(2\ell+1)}\left(C_{\ell 0\ell 0}^{\ell_1 0}\right)^2 \le \frac{c_2}{\ell}. \tag{2.3.12}$$

as $\ell\to\infty$.

Proof of Proposition 2.1.4. The variance of the second chaotic component can be written as

$$\begin{aligned}\mathrm{Var}\left[\int_{\mathbb{S}^2} 1_B(x)H_2(T_\ell(x))\,dx\right] &= \mathbb{E}\left[\int_{\mathbb{S}^2}(1_B(x)-1_{B;\varepsilon}(x))H_2(T_\ell(x))\,dx\right]^2+\\ &+\mathrm{Var}\left[\int_{\mathbb{S}^2} 1_{B,\varepsilon}(x)H_2(T_\ell(x))\,dx\right]\\ &+2\mathbb{E}\left[\int_{\mathbb{S}^2\times\mathbb{S}^2} 1_{B,\varepsilon}(x)\Big(1_B(y)-1_{B,\varepsilon}(y)\Big)H_2(T_\ell(x))H_2(T_\ell(y))\,dx\,dy\right].\end{aligned} \tag{2.3.13}$$

The first integral in (2.3.13) is

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\mathbb{S}^2}(1_B(x)-1_{B;\varepsilon}(x))H_2(T_\ell(x))\,dx\Big]^2 = \\
&= \int_{\mathbb{S}^2\times\mathbb{S}^2}(1_B(x)-1_{B;\varepsilon}(x))(1_B(y)-1_{B;\varepsilon}(y))\mathbb{E}[H_2(T_\ell(x))H_2(T_\ell(y))]\,dx dy \\
&\le 2\int_{\mathbb{S}^2\times\mathbb{S}^2}|1_B(x)-1_{B;\varepsilon}(x)|P_\ell^2(\langle x,y\rangle)\,dx dy \le 2\int_{\mathbb{S}^2}|1_{B;\varepsilon}(x)-1_B(x)|\cdot\int_{\mathbb{S}^2}P_\ell^2(\langle x,y\rangle)\,dy\,dx \\
&\le 2\cdot 2\pi C\varepsilon\frac{2}{2\ell+1},
\end{aligned} \tag{2.3.14}$$

where $C=2\pi$ has already been computed in Remark 2.2.1, and for the Cauchy-Schwarz inequality, the same bound holds for the third integral in (2.3.13). Then, Lemma 2.3.2 with Lemma 2.3.3 conclude the proof. □

Terms of the chaotic components for $q \geq 3$

Let us consider the chaotic components of order q for $q \geq 3$.
The variance of the third term in (2.1.5) can be bounded by its absolute value, and hence we can bound the integral on B with the one computed on $\mathbb{S}^2$ as in the following way, using (1.2.1) we have that

$$\begin{aligned}
\mathrm{Var}\Big(\int_B \frac{J_3(z)}{3!}H_3(T_\ell(x))dx\Big) &= \frac{J_3(z)^2}{3!}\int_B\int_B P_\ell(\langle x,y\rangle)^3\,dx dy \le \frac{J_3(z)^2}{3!}\int_B\int_{\mathbb{S}^2}|P_\ell(\langle x,y\rangle)|^3\,dx dy \\
&= \frac{J_3(z)^2}{3!}2\pi m(B)\int_0^{\pi/2}|P_\ell(\cos\theta)|^3\sin\theta\,d\theta = \frac{J_3(z)^2}{3!}2\pi m(B)\int_0^1|P_\ell(x)|^3\,dx;
\end{aligned} \tag{2.3.15}$$

the Cauchy-Schwartz inequality implies that (2.3.15) is

$$\le \frac{J_3(z)^2}{3!}2\pi m(B)\Big(\int_0^1 P_\ell(x)^2\,dx\Big)^{1/2}\Big(\int_0^1 P_\ell(x)^4\,dx\Big)^{1/2} \tag{2.3.16}$$

and since it has been proved in [51] and [47] that $\int_0^1 P_\ell(x)^2\,dx = O\big(\frac{1}{\ell}\big)$ and $\int_0^1 P_\ell(x)^4\,dx = O\big(\frac{\log\ell}{\ell^2}\big)$, (2.3.16) has order $O\big(\frac{\sqrt{\log\ell}}{\ell\sqrt{\ell}}\big)$, as $\ell\to\infty$.

Likewise, for the variance of the fourth chaotic projection in (2.1.5), we obtain that

$$\begin{aligned}
\mathrm{Var}\Big(\int_B \frac{J_4(z)}{4!}H_4(T_\ell(x))dx\Big) &= \frac{J_4(z)^2}{(4!)^2}\int_B\int_B P_\ell(\langle x,y\rangle)^4 dx dy \\
&\le \frac{J_4(z)^2}{(4!)^2}\int_B\int_{\mathbb{S}^2} P_\ell(\langle x,y\rangle)^4 dx dy \\
&= \frac{J_4(z)^2}{(4!)^2}m(B)2\pi\int_0^1 P_\ell(x)^4\,dx
\end{aligned}$$

which behaves as $\dfrac{\log \ell}{\ell^2}$, as $\ell \to \infty$ [51].

Eventually, for the remaining terms in (2.1.5), in the same way we get

$$\begin{aligned}
\operatorname{Var}\Big(\int_B \sum_{q=5}^{\infty} \frac{J_q(z)}{q!} H_q(T_\ell(x))\, dx \Big) &= \mathbb{E}\Big[\int_B \sum_{q=5}^{\infty} \frac{J_q(z)}{q!} H_q(T_\ell(x))\, dx \Big]^2 \\
&= \sum_{q=5}^{\infty} \frac{J_q(z)^2}{(q!)^2} \int_{B\times B} \mathbb{E}[H_q(T_\ell(x)) H_q(T_\ell(y))]\, dx dy = \sum_{q=5}^{\infty} \frac{J_q(z)^2}{(q!)^2} \int_{B\times B} q! P_\ell(\langle x, y\rangle)^q \, dx dy \\
&\le \sum_{q=5}^{\infty} \frac{J_q(z)^2}{q!} \int_{B\times B} |P_\ell(\langle x, y\rangle)|^q \, dx dy \le \sum_{q=5}^{\infty} \frac{J_q(z)^2}{q!} \int_{B\times \mathbb{S}^2} |P_\ell(\langle x, y\rangle)|^q \, dx dy \\
&\le \sum_{q=5}^{\infty} \frac{J_q(z)^2}{q!} 2\pi m(B) \int_0^{\pi/2} |P_\ell(\cos\theta)|^q \sin\theta \, d\theta = \sum_{q=5}^{\infty} \frac{J_q(z)^2}{q!} 2\pi m(B) \int_0^1 |P_\ell(x)|^q \, dx
\end{aligned}$$

and $\int_0^1 |P_\ell(x)|^q \, dx = O\big(\frac{1}{\ell^2}\big)$ ([52], Lemma 5.7 or [47], Proposition 1.1).

Quantitative Central Limit Theorem

In this subsection, we finally prove Theorem 0.0.1, assuming Proposition 2.1.7; the argument is quite similar to the one for the full sphere given in [47].

Proof of Theorem 0.0.1 assuming Proposition 2.1.7. As in [47], we denote

$$S_\ell(M) := \int_B M(T_\ell(x))\, dx,$$

with

$$M(T_\ell(x)) := 1_{((T_\ell(x))>z)}(T_\ell(x)).$$

Now, we consider the chaotic expansion

$$S'_\ell(M) := S_\ell(M) - \mathbb{E}[S_\ell(M)] = \int_B \sum_{q=1}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!}\, dx,$$

which we write as

$$\begin{aligned}
S'_\ell(M) &= J_1(M) h_{\ell;1}(B) + \frac{J_2(M)}{2} h_{\ell;2}(B) + \frac{J_3(M)}{3!} h_{\ell;3}(B) + \frac{J_4(M)}{4!} h_{\ell;4}(B) + \int_B \sum_{q=5}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!}\, dx \\
&= S_\ell(M,1) + S_\ell(M,2)
\end{aligned}$$

where

$$S_\ell(M;1) = J_1(M) h_{\ell;1}(B) + \frac{J_2(M)}{2} h_{\ell;2}(B) + \frac{J_3(M)}{3!} h_{\ell;3}(B) + \frac{J_4(M)}{4!} h_{\ell;4}(B),$$

$$S_\ell(M;2) = \int_B \sum_{q=5}^{\infty} \frac{J_q(M) H_q(T_\ell(x))}{q!}\, dx.$$

Hence, one has that

$$\begin{aligned} d_W\Big(\frac{S_\ell'(M)}{\sqrt{\text{Var}[S_\ell(M)]}}, \mathscr{N}(0,1)\Big) &\le d_W\Big(\frac{S_\ell'(M)}{\sqrt{\text{Var}[S_\ell(M)]}}, \frac{S_\ell(M;1)}{\sqrt{\text{Var}[S_\ell(M)]}}\Big)+ \\ &+ d_W\Big(\frac{S_\ell(M;1)}{\sqrt{\text{Var}[S_\ell(M)]}}, \mathscr{N}\Big(0, \frac{\text{Var}[S_\ell(M;1)]}{\text{Var}[S_\ell(M)]}\Big)\Big) + d_W\Big(\mathscr{N}\Big(0, \frac{\text{Var}[S_\ell(M;1)]}{\text{Var}[S_\ell(M)]}\Big), \mathscr{N}(0,1)\Big) \\ &\le \frac{1}{\sqrt{\text{Var}[S_\ell(M)]}}\mathbb{E}\Big[\Big(\int_B \sum_{q=5}^\infty \frac{J_q(M)H_q(T_\ell(x))}{q!}\,dx\Big)^2\Big]^{1/2}+ \\ &+ d_W\Big(\frac{S_\ell(M;1)}{\sqrt{\text{Var}[S_\ell(M)]}}, \mathscr{N}\Big(0, \frac{\text{Var}[S_\ell(M;1)]}{\text{Var}[S_\ell(M)]}\Big)\Big) + d_W\Big(\mathscr{N}\Big(0, \frac{\text{Var}[S_\ell(M;1)]}{\text{Var}[S_\ell(M)]}\Big), \mathscr{N}(0,1)\Big). \end{aligned} \tag{2.3.17}$$

We have seen that

$$\text{Var}(S_\ell(M;2)) \ll \frac{1}{\ell^2}$$

and since $\text{Var}(S_\ell(M))$ has the same asymptotic order as the second chaotic component, we have that

$$\frac{\text{Var}(S_\ell(M;2))}{\text{Var}(S_\ell(M))} \ll \frac{1}{\ell};$$

moreover, the triangular inequality gives

$$\begin{aligned} d_W\Big(\frac{S_\ell(M;1)}{\sqrt{\text{Var}(S_\ell(M))}}, \mathscr{N}\Big(0, \frac{\text{Var}(S_\ell(M;1))}{\text{Var}(S_\ell(M))}\Big)\Big) &\le d_W\Big(\frac{J_2(M)}{2\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;2}(B), \mathscr{N}\Big(0, \frac{\text{Var}(S_\ell(M;1))}{\text{Var}(S_\ell(M))}\Big)\Big)+ \\ &+ d_W\Big(\frac{J_1(M)}{\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;1}(B) + \frac{J_2(M)}{2\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;2}(B)+ \\ &+ \frac{J_3(M)}{3!\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;3}(B) + \frac{J_4(M)}{4!\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;4}(B), \mathscr{N}\Big(0, \frac{\text{Var}(S_\ell(M;1))}{\text{Var}(S_\ell(M))}\Big)\Big). \end{aligned} \tag{2.3.18}$$

For the first term in (2.3.18), we can use, again, the triangular inequality to obtain that

$$\begin{aligned} &d_W\Big(\frac{J_2(M)}{2\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;2}(B), \mathscr{N}\Big(0, \frac{\text{Var}(S_\ell(M;1))}{\text{Var}(S_\ell(M))}\Big)\Big) \\ \le &d_W\Big(\frac{J_2^*(M)}{2\sqrt{\text{Var}(S_\ell(M))}} h^*_{\ell;2}(B), \mathscr{N}\Big(0, \frac{\text{Var}(S_\ell(M;1))}{\text{Var}(S_\ell(M))}\Big)\Big) + d_W\Big(\frac{J_2(M)h_{\ell;2}(B)}{2\sqrt{\text{Var}(S_\ell(M))}}, \frac{J_2(M)^* h^*_{\ell;2}(B)}{2\sqrt{\text{Var}(S_\ell(M))}}\Big), \end{aligned} \tag{2.3.19}$$

where $J_2(M)^*$ is the coefficient of the second chaotic component of the chaos expansion of $1_{B;\varepsilon}$. In light of (2.1.19), the latter summand in (2.3.19) is an $o\Big(\frac{1}{\ell}\Big)$, thanks to the triangular inequality; whereas, in view of Proposition 2.1.7 and hence of the Fourth Moment Theorem (Corollary 1.2.4, see also [57], Theorem 5.2.7), the former is $O\Big(\frac{1}{\sqrt{\ell}}\Big)$. For the second term in (2.3.18), we have that

$$\begin{aligned} &d_W\Big(\frac{J_1(M)}{\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;1}(B) + \frac{J_3(M)}{3!\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;3}(B) + \frac{J_4(M)}{4!\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;4}(B), 0\Big) \le \\ &d_W\Big(\frac{J_1(M)}{\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;1}(B), 0\Big) + d_W\Big(\frac{J_3(M)}{3!\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;3}(B), 0\Big) + d_W\Big(\frac{J_4(M)}{4!\sqrt{\text{Var}(S_\ell(M))}} h_{\ell;4}(B), 0\Big); \end{aligned} \tag{2.3.20}$$

since

$$d_W\left(\frac{J_1(M)}{\sqrt{\mathrm{Var}(S_\ell(M))}}h_{\ell;1}(B),0\right)\le\sqrt{\mathbb{E}\left[\left(\frac{J_1(M)}{\sqrt{\mathrm{Var}(S_\ell(M))}}h_{\ell;1}(B)\right)^2\right]}=o\left(\frac{1}{\sqrt{\ell}}\right),$$

$$d_W\left(\frac{J_3(M)}{\sqrt{3!\,\mathrm{Var}(S_\ell(M))}}h_{\ell;3}(B),0\right)\le\sqrt{\mathbb{E}\left[\left(\frac{J_3(M)}{3!\sqrt{\mathrm{Var}(S_\ell(M))}}h_{\ell;3}(B)\right)^2\right]}=O\left(\sqrt{\frac{\sqrt{\log\ell}}{\ell\sqrt{\ell}}}\right)$$

and

$$d_W\left(\frac{J_4(M)}{\sqrt{4!\,\mathrm{Var}(S_\ell(M))}}h_{\ell;4}(B),0\right)\le\sqrt{\mathbb{E}\left[\left(\frac{J_4(M)}{4!\sqrt{\mathrm{Var}(S_\ell(M))}}h_{\ell;4}(B)\right)^2\right]}=O\left(\sqrt{\frac{\log\ell}{\ell^2}}\right),$$

one has that

$$d_W\left(\frac{S_\ell(M;1)}{\sqrt{\mathrm{Var}(S_\ell(M))}},\mathcal{N}\left(0,\frac{\mathrm{Var}(S_\ell(M;1))}{\mathrm{Var}(S_\ell(M))}\right)\right)=O\left(\frac{1}{\sqrt{\ell}}\right).$$

Finally, Proposition 3.6.1 in [57] leads to

$$d_W\left(\mathcal{N}\left(0,\frac{\mathrm{Var}(S_\ell(M;1))}{\mathrm{Var}(S_\ell(M))}\right),\mathcal{N}(0,1)\right)\le\sqrt{\frac{2}{\pi}}\left|\frac{\mathrm{Var}(S_\ell(M;1))}{\mathrm{Var}(S_\ell(M))}-1\right|=O\left(\frac{1}{\ell}\right)$$

and the thesis of the theorem follows. □

2.4 Technical details

In this section we give all the technical details of the proofs of the propositions and the lemmas whereby Theorem 0.0.1 has been proved.

2.4.1 Proof of Lemma 2.3.2

Proof. The aim here is to prove (2.3.11). As we have already explained, the idea is to write the integral in terms of spherical harmonics and then to exploit the properties of Clebsch-Gordan coefficients. We split the proof in these two steps in order to make the argument clearer.

Step 1. Let us consider the left hand side of (2.3.11), we can write it as

$$\begin{aligned}\mathrm{Var}\left(\int_{\mathbb{S}^2}1_{B;\varepsilon}(x)H_2(T_\ell(x))\,dx\right)&=\mathbb{E}\left[\left(\int_{\mathbb{S}^2}1_{B;\varepsilon}(x)H_2(T_\ell(x))\,dx\right)^2\right]\\&=\mathbb{E}\left[\int_{\mathbb{S}^2\times\mathbb{S}^2}1_{B;\varepsilon}(x)H_2(T_\ell(x))1_{B;\varepsilon}(y)H_2(T_\ell(y))\,dx\,dy\right].\end{aligned}\tag{2.4.1}$$

Exchanging the integral and the mean, we have that (2.4.1) is

$$=\int_{\mathbb{S}^2\times\mathbb{S}^2}1_{B;\varepsilon}(x)1_{B;\varepsilon}(y)\mathbb{E}[H_2(T_\ell(x))H_2(T_\ell(y))]dxdy\tag{2.4.2}$$

and in view of (1.2.1) (see Remark 4.10 in [45]), it follows that (2.4.2) is

$$= 2! \int_{\mathbb{S}^2\times\mathbb{S}^2} 1_{B;\varepsilon}(x)1_{B;\varepsilon}(y)\mathbb{E}[T_\ell(x)T_\ell(y)]^2\,dxdy = 2! \int_{\mathbb{S}^2\times\mathbb{S}^2} 1_{B;\varepsilon}(x)1_{B;\varepsilon}(y)P_\ell(\langle x,y\rangle)^2\,dxdy. \tag{2.4.3}$$

Along the same lines as the proof of Lemma 2.3.1, we replace $P_\ell^2(\langle x,y\rangle)$ with

$$P_\ell(\langle x,y\rangle)^2 = \left(\frac{4\pi}{2\ell+1}\right)^2 \sum_{m_1=-\ell}^{\ell}\sum_{m_2=-\ell}^{\ell} Y_{\ell m_1}(x)\overline{Y_{\ell m_1}(y)}\;\overline{Y_{\ell m_2}(x)}Y_{\ell m_2}(y) \tag{2.4.4}$$

and (2.3.6) to obtain

$$\begin{aligned}\int_{\mathbb{S}^2\times\mathbb{S}^2} & P_\ell(\langle x,y\rangle)^2 1_{B;\varepsilon}(x)1_{B;\varepsilon}(y)\,dxdy = \\ &= \int_{\mathbb{S}^2\times\mathbb{S}^2}\left(\frac{4\pi}{2\ell+1}\right)^2\sum_{m_1}\sum_{m_2} Y_{\ell m_1}(x)\overline{Y_{\ell m_2}(x)}\;Y_{\ell m_2}(y)\overline{Y_{\ell m_1}(y)}Y_{\ell m_2}(y)\times \\ &\qquad\qquad \times\sum_{\ell_1}\sum_{\ell_2} b_{\ell_1;\varepsilon}b_{\ell_2;\varepsilon}Y_{\ell_1 0}(x)Y_{\ell_2 0}(y)dxdy\end{aligned} \tag{2.4.5}$$

$$\begin{aligned}= \left(\frac{4\pi}{2\ell+1}\right)^2 \sum_{\ell_1}\sum_{\ell_2}\sum_{m_1}\sum_{m_2} b_{\ell_1;\varepsilon}b_{\ell_2;\varepsilon}\int_{\mathbb{S}^2} Y_{\ell m_1}(x)\;Y_{\ell_1 0}(x)\overline{Y_{\ell m_2}(x)}\,dx\times \\ \times\int_{\mathbb{S}^2} Y_{\ell m_2}(y)Y_{\ell_2 0}(y)\;\overline{Y_{\ell m_1}(y)}\,dy;\end{aligned} \tag{2.4.6}$$

we already justified the exchange between the series and the integral in Lemma 2.3.1, which follows from (2.1.6).
Now, (2.4.6) is known as a Gaunt integral and it is given in [45] by the following relation:

$$\int_{S^2} Y_{\ell_1 m_1}(x)Y_{\ell_2 m_2}(x)\overline{Y}_{\ell_3 m_3}(x)d\sigma(x) = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_3+1)}}C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2}C^{\ell_3 0}_{\ell_1 0 \ell_2 0}, \tag{2.4.7}$$

for all ℓ_1, ℓ_2, ℓ_3, with the convention that $C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} = 0$ for those integers ℓ_1, ℓ_2, ℓ_3 not satisfying the triangle conditions. Replacing (2.4.7) in (2.4.6), one has

$$\begin{aligned}\left(\frac{4\pi}{2\ell+1}\right)^2 \sum_{\ell_1}\sum_{\ell_2}\sum_{m_1}\sum_{m_2} b_{\ell_1;\varepsilon}\sqrt{\frac{(2\ell+1)(2\ell_1+1)}{4\pi(2\ell+1)}}C^{\ell m_2}_{\ell m_1 \ell_1 0}C^{\ell 0}_{\ell 0 \ell_1 0}\times \\ \times b_{\ell_2;\varepsilon}\sqrt{\frac{(2\ell+1)(2\ell_2+1)}{4\pi(2\ell+1)}}C^{\ell m_1}_{\ell m_2 \ell_2 0}C^{\ell 0}_{\ell 0 \ell_2 0}\end{aligned} \tag{2.4.8}$$

$$= \left(\frac{4\pi}{2\ell+1}\right)^2 \frac{1}{4\pi}\sum_{\ell_1} b_{\ell_1;\varepsilon}\sqrt{2\ell_1+1}C^{\ell 0}_{\ell 0 \ell_1 0}\sum_{\ell_2} b_{\ell_2;\varepsilon}\sqrt{2\ell_1+1}C^{\ell 0}_{\ell 0 \ell_2 0}\sum_{m_1 m_2} C^{\ell m_2}_{\ell m_1 \ell_1 0}C^{\ell m_1}_{\ell m_2 \ell_2 0} \tag{2.4.9}$$

and then, this is the value of the left hand side in (2.3.11).

Step 2. Now, we just exploit some properties of Clebsch-Gordan coefficients to simplify (2.4.9) and finally to prove that it is equal to the right hand side in (2.3.11). Hence, Recalling that the considered coefficients are related to the Wigner 3j coefficients by the identities (see [45], Section 3.5.3):

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{\ell_3+m_3} \frac{1}{\sqrt{2\ell_3+1}} C^{\ell_3 m_3}_{\ell_1 - m_1 \ell_2 - m_2} \tag{2.4.10}$$

$$C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} = (-1)^{\ell_1-\ell_2+m_3} \sqrt{2\ell_3+1} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}, \tag{2.4.11}$$

and using their permutation property of columns

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{\ell_1+\ell_2+\ell_3} \begin{pmatrix} \ell_1 & \ell_3 & \ell_2 \\ m_1 & m_3 & m_2 \end{pmatrix}, \tag{2.4.12}$$

it follows that

$$\begin{aligned} C^{\ell m_2}_{\ell m_1 \ell_1 0} &= (-1)^{\ell-\ell_1+m_2} \sqrt{2\ell+1} \begin{pmatrix} \ell & \ell_1 & \ell \\ m_1 & 0 & -m_2 \end{pmatrix} \\ &= (-1)^{\ell-\ell_1+m_2} \sqrt{2\ell+1} (-1)^{\ell+\ell_1+\ell} \begin{pmatrix} \ell & \ell & \ell_1 \\ m_1 & -m_2 & 0 \end{pmatrix} \\ &= (-1)^{\ell+m_2+2\ell} \sqrt{2\ell+1} (-1)^{\ell_1+2\ell} \frac{1}{\sqrt{2\ell_1+1}} C^{\ell_1 0}_{\ell - m_1 \ell m_2} \\ &= (-1)^{\ell+m_2+\ell_1} \frac{\sqrt{2\ell+1}}{\sqrt{2\ell_1+1}} C^{\ell_1 0}_{\ell - m_1 \ell m_2} \end{aligned} \tag{2.4.13}$$

and

$$C^{\ell m_1}_{\ell m_2 \ell_2 0} = (-1)^{\ell+m_1+\ell_2} \sqrt{2\ell+1} \frac{1}{\sqrt{2\ell_2+1}} C^{\ell_2 0}_{\ell - m_2 \ell m_1};$$

so equation (2.4.9) is equal to

$$\begin{aligned} &\left(\frac{4\pi}{2\ell+1}\right)^2 \frac{1}{4\pi} \sum_{\ell_1} b_{\ell_1;\varepsilon} \sqrt{2\ell_1+1} C^{\ell 0}_{\ell 0 \ell_1 0} \sum_{\ell_2} b_{\ell_2;\varepsilon} \sqrt{2\ell_1+1} C^{\ell 0}_{\ell 0 \ell_2 0} \times \\ &\sum_{m_1 m_2} (-1)^{m_1+m_2} (-1)^{\ell_1+\ell_2} \frac{\sqrt{2\ell+1}}{\sqrt{2\ell_1+1}} \frac{\sqrt{2\ell+1}}{\sqrt{2\ell_2+1}} C^{\ell_1 0}_{\ell - m_1 \ell m_2} C^{\ell_2 0}_{\ell - m_2 \ell m_1} = \\ &= \frac{4\pi}{2\ell+1} \sum_{\ell_1} b_{\ell_1;\varepsilon} C^{\ell 0}_{\ell 0 \ell_1 0} \sum_{\ell_2} b_{\ell_2;\varepsilon} C^{\ell 0}_{\ell 0 \ell_2 0} (-1)^{\ell_1+\ell_2} \cdot \sum_{m_1 m_2} (-1)^{m_1+m_2} C^{\ell_1 0}_{\ell - m_1 \ell m_2} C^{\ell_2 0}_{\ell - m_2 \ell m_1} \end{aligned} \tag{2.4.14}$$

and for the triangular condition

$$m_1 - m_2 = 0 \Rightarrow m_1 = m_2$$

and the unitary relation ([75]):

$$\sum_{m_1 m_2} C^{jm}_{j_1 m_1 j_2 m_2} C^{j'm'}_{j_1 m_1 j_2 m_2} = \delta^{j'}_{j} \delta^{m'}_{m}, \tag{2.4.15}$$

(2.4.14) yields

$$\frac{4\pi}{2\ell+1}\left\{\sum_{\ell_1}(-1)^{\ell_1} b_{\ell_1;\varepsilon} C^{\ell 0}_{\ell 0 \ell_1 0} \sum_{\ell_2}(-1)^{\ell_2} b_{\ell_2;\varepsilon} C^{\ell 0}_{\ell 0 \ell_2 0}\right\}\delta^{\ell_2}_{\ell_1}. \tag{2.4.16}$$

As in (2.4.13) one has

$$\begin{aligned} C^{\ell 0}_{\ell 0 \ell_1 0} &= (-1)^{\ell-\ell_1}\sqrt{2\ell+1}\begin{pmatrix} \ell & \ell_1 & \ell \\ 0 & 0 & 0 \end{pmatrix} \\ &= (-1)^{\ell-\ell_1}\sqrt{2\ell+1}\,(-1)^{2\ell+\ell_1}\begin{pmatrix} \ell & \ell & \ell_1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= (-1)^{\ell}\sqrt{2\ell+1}\,(-1)^{\ell_1+2\ell}\frac{1}{\sqrt{2\ell_1+1}} C^{\ell_1 0}_{\ell 0 \ell 0} \\ &= (-1)^{\ell+\ell_1}\frac{\sqrt{2\ell+1}}{\sqrt{2\ell_1+1}} C^{\ell_1 0}_{\ell 0 \ell 0} \end{aligned} \tag{2.4.17}$$

and then (2.4.16) is

$$\begin{aligned} &\frac{4\pi}{2\ell+1}\left\{\sum_{\ell_1}(-1)^{\ell_1} b_{\ell_1;\varepsilon}(-1)^{\ell+\ell_1}\frac{\sqrt{2\ell+1}}{\sqrt{2\ell_1+1}} C^{\ell_1 0}_{\ell 0 \ell 0}\sum_{\ell_2} b_{\ell_2;\varepsilon}(-1)^{\ell_2}(-1)^{\ell+\ell_2}\frac{\sqrt{2\ell+1}}{\sqrt{2\ell_2+1}} C^{\ell_2 0}_{\ell 0 \ell 0}\right\}\delta^{\ell_2}_{\ell_1} \\ &= 4\pi\sum_{\ell_1} b_{\ell_1;\varepsilon}\frac{1}{\sqrt{2\ell_1+1}} C^{\ell_1 0}_{\ell 0 \ell 0}\sum_{\ell_1}\frac{1}{\sqrt{2\ell_1+1}} b_{\ell_1;\varepsilon} C^{\ell_1 0}_{\ell 0 \ell 0} \\ &= 4\pi\left\{\sum_{\ell_1} b_{\ell_1;\varepsilon}\frac{1}{\sqrt{2\ell_1+1}} C^{\ell_1 0}_{\ell 0 \ell 0}\right\}^2 \\ &= 4\pi\sum_{\ell_1} b^2_{\ell_1;\varepsilon}\frac{1}{2\ell_1+1}\left(C^{\ell_1 0}_{\ell 0 \ell 0}\right)^2, \end{aligned} \tag{2.4.18}$$

where the last step is due to the previous property (2.4.15) with $m_1 = m_2 = m_3 = 0$; finally Lemma 2.3.2 is proved. □

2.4.2 Proof of Lemma 2.3.3

Proof. The variance in (2.3.11) is bounded from below by a single term of the series in the right hand side of (2.3.11), i.e.,

$$8\pi b^2_{\bar{\ell}_1;\varepsilon}\frac{1}{2\bar{\ell}_1+1}\left(C^{\bar{\ell}_1 0}_{\ell 0 \ell 0}\right)^2,$$

for a fixed $\bar{\ell}_1$ of the sum; for instance we take $\bar{\ell}_1 = 0$, then

$$\mathrm{Var}\left[\int_{S^2} 1_{B;\varepsilon}(x) H_2(T_\ell(x))\,dx\right] = 8\pi\sum_{\ell_1} b^2_{\ell_1;\varepsilon}\frac{1}{2\ell_1+1}\left(C^{\ell_1 0}_{\ell 0 \ell 0}\right)^2 \geq 8\pi b^2_{0;\varepsilon}\left(C^{00}_{\ell 0 \ell 0}\right)^2 = 8\pi b^2_{0;\varepsilon}\frac{1}{2\ell+1},$$

by the property

$$C^{00}_{\ell_1 m_1 \ell_2 m_2} = (-1)^{\ell_1 - m_1}\frac{\delta^{\ell_2}_{\ell_1}\delta^{-m_2}_{m_1}}{\sqrt{2\ell_1+1}} \tag{2.4.19}$$

(see [75]). To find an upper bound, it is sufficient to recall that for any ℓ_1, ℓ_2, ℓ_3,

$$\left|\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\right| \leq [\max\{2\ell_1+1, 2\ell_2+1, 2\ell_3+1\}]^{-1/2} \tag{2.4.20}$$

(see [45] p. 110) so that

$$|C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2}| \leq \sqrt{2\ell_3+1}[\max\{2\ell_1+1, 2\ell_2+1, 2\ell_3+1\}]^{-1/2} \tag{2.4.21}$$

and then, it is easy to see that

$$8\pi \sum_{\ell_1} b^2_{\ell_1;\varepsilon} \frac{1}{2\ell_1+1}\left(C^{\ell_1 0}_{\ell 0 \ell 0}\right)^2 \leq 8\pi \sum_{\ell_1} b^2_{\ell_1;\varepsilon} \frac{1}{2\ell_1+1} \frac{2\ell_1+1}{2\ell+1} = \frac{8\pi}{2\ell+1} \sum_{\ell_1} b^2_{\ell_1;\varepsilon}.$$

The series is finite by Remark 2.1.5. In conclusion, (2.3.11) is bounded above and below by

$$8\pi b^2_{0;\varepsilon} \frac{1}{2\ell+1} \leq 8\pi \sum_{\ell_1} b^2_{\ell_1;\varepsilon} \frac{1}{2\ell_1+1}\left(C^{\ell_1 0}_{\ell 0 \ell 0}\right)^2 \leq \frac{8\pi}{2\ell+1} \sum_{\ell_1} b^2_{\ell_1;\varepsilon} \leq \frac{8\pi}{2\ell+1} m(\mathbb{S}^2) = \frac{8\pi}{2\ell+1} 4\pi \tag{2.4.22}$$

and in light of Remark 2.2.2, the lemma is proved. □

2.4.3 Proof of Proposition 2.1.7

Proof. The purpose here is to compute the fourth cumulant of $h^*_{\ell;2}(B)$, which, in view of the Diagram Formula (see Section 4.3 [45], or [50]), is given by

$$\begin{aligned} cum_4(h^*_{\ell;2}(B)) = \int_{\mathbb{S}^2} 1_{B;\varepsilon}(x) \int_{\mathbb{S}^2} 1_{B;\varepsilon}(z) \int_{\mathbb{S}^2} P_\ell(\langle x, y\rangle) P_\ell(\langle y, z\rangle) 1_{B;\varepsilon}(y)\, dy \\ \times \int_{\mathbb{S}^2} P_\ell(\langle z, w\rangle) P_\ell(\langle w, x\rangle) 1_{B;\varepsilon}(w)\, dw\, dx\, dz; \end{aligned} \tag{2.4.23}$$

the idea is always to write the integral in terms of spherical harmonics and hence, of the Clebsch-Gordan coefficients. Then, the second step is to handle them in order to derive an expression with less parameters, namely equation (2.4.29). In the third step we split the series given in (2.4.29) to make neater some terms of the series. At this point, in step 4, we study the asymptotic behavior of all these terms proving, finally, the thesis of the lemma.

Step 1. Putting together (2.1.14) and (2.1.16) in (2.4.23), we obtain four Gaunt integrals and (2.4.7) implies that (2.4.23) is equal to

$$\begin{aligned} \left(\frac{4\pi}{2\ell+1}\right)^4 \frac{1}{(4\pi)^2} \sum_{\ell_1=-\ell}^{\ell} b_{\ell_1;\varepsilon} \sqrt{(2\ell_1+1)} C^{\ell 0}_{\ell 0 \ell_1 0} \sum_{\ell_2=0}^{\infty} b_{\ell_2;\varepsilon} \sqrt{(2\ell_2+1)} C^{\ell 0}_{\ell 0 \ell_2 0} \\ \sum_{\ell_3=-\ell}^{\ell} b_{\ell_3;\varepsilon} \sqrt{(2\ell_3+1)} C^{\ell 0}_{\ell 0 \ell_3 0} \sum_{\ell_4=-\ell}^{\ell} b_{\ell_4;\varepsilon} \sqrt{(2\ell_4+1)} C^{\ell 0}_{\ell 0 \ell_4 0} \\ \sum_{m_1=-\ell}^{\ell} \sum_{m_2=-\ell}^{\ell} \sum_{m_3=-\ell}^{\ell} \sum_{m_4=-\ell}^{\ell} C^{\ell m_2}_{\ell m_1 \ell_1 0} C^{\ell m_4}_{\ell m_3 \ell_2 0} C^{\ell m_1}_{\ell m_4 \ell_3 0} C^{\ell m_3}_{\ell m_2 \ell_4 0}. \end{aligned} \tag{2.4.24}$$

Step 2. In this step we reduce the number of parameters. Indeed, the triangular condition implies that $m_1 = m_2 = m_3 = m_4$, so that (2.4.24) is

$$= \left(\frac{4\pi}{2\ell+1}\right)^4 \frac{1}{(4\pi)^2} \sum_{\ell_1=-\ell}^{\ell} b_{\ell_1;\varepsilon} \sqrt{(2\ell_1+1)} C^{\ell 0}_{\ell 0 \ell_1 0} \sum_{\ell_2=0}^{\infty} b_{\ell_2;\varepsilon} \sqrt{(2\ell_2+1)} C^{\ell 0}_{\ell 0 \ell_2 0}$$
$$\sum_{\ell_3=-\ell}^{\ell} b_{\ell_3;\varepsilon} \sqrt{(2\ell_3+1)} C^{\ell 0}_{\ell 0 \ell_3 0} \sum_{\ell_4=-\ell}^{\ell} b_{\ell_4;\varepsilon} \sqrt{(2\ell_4+1)} C^{\ell 0}_{\ell 0 \ell_4 0} \tag{2.4.25}$$
$$\sum_{m_1=-\ell}^{\ell} C^{\ell m_1}_{\ell m_1 \ell_1 0} C^{\ell m_1}_{\ell m_1 \ell_2 0} C^{\ell m_1}_{\ell m_1 \ell_3 0} C^{\ell m_1}_{\ell m_1 \ell_4 0}.$$

Besides, for the symmetry properties (A.2.5 or [75]), one has that

$$C^{\ell m_1}_{\ell m_1 \ell_1 0} = (-1)^{\ell - m_1} \sqrt{\frac{2\ell+1}{2\ell_1+1}} C^{\ell_1 0}_{\ell m_1 \ell - m_1},$$

and then

$$\sum_{m_1=-\ell}^{\ell} C^{\ell m_1}_{\ell m_1 \ell_1 0} C^{\ell m_1}_{\ell m_1 \ell_2 0} C^{\ell m_1}_{\ell m_1 \ell_3 0} C^{\ell m_1}_{\ell m_1 \ell_4 0} =$$
$$= \sum_{m_1=-\ell}^{\ell} (-1)^{4(\ell - m_1)} \sqrt{\frac{(2\ell+1)^4}{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)(2\ell_4+1)}} C^{\ell_1 0}_{\ell m_1 \ell m_1} C^{\ell_2 0}_{\ell m_1 \ell m_1} C^{\ell_3 0}_{\ell m_1 \ell m_1} C^{\ell_4 0}_{\ell m_1 \ell m_1}. \tag{2.4.26}$$

By (A.2.16) and (A.2.24) ([75]), (2.4.26) becomes

$$= \left(\frac{4\pi}{2\ell+1}\right)^2 \sum_{\ell_1=-\ell}^{\ell} b_{\ell_1;\varepsilon} C^{\ell 0}_{\ell 0 \ell_1 0} \sum_{\ell_2=0}^{\infty} b_{\ell_2;\varepsilon} C^{\ell 0}_{\ell 0 \ell_2 0} \times$$
$$\sum_{\ell_3=-\ell}^{\ell} b_{\ell_3;\varepsilon} C^{\ell 0}_{\ell 0 \ell_3 0} \sum_{\ell_4=-\ell}^{\ell} b_{\ell_4;\varepsilon} C^{\ell 0}_{\ell 0 \ell_4 0} \times \tag{2.4.27}$$
$$\prod_{\ell_1,\ell_2,\ell_3,\ell_4} \sum_{kj} C^{kj}_{\ell_3 0 \ell_4 0} C^{kj}_{\ell_2 0 \ell_1 0} \begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell & \ell & \ell_2 \\ \ell_4 & \ell_3 & k \end{Bmatrix},$$

where the last symbol is the Wigner 9j coefficient (see [75]). Then, since the triangular condition implies $j = 0$, (2.4.27) gives

$$= \left(\frac{4\pi}{2\ell+1}\right)^2 \sum_{\ell_1=-\ell}^{\ell} b_{\ell_1;\varepsilon} C^{\ell 0}_{\ell 0 \ell_1 0} \sum_{\ell_2=0}^{\infty} b_{\ell_2;\varepsilon} C^{\ell 0}_{\ell 0 \ell_2 0} \times$$
$$\sum_{\ell_3=-\ell}^{\ell} b_{\ell_3;\varepsilon} C^{\ell 0}_{\ell 0 \ell_3 0} \sum_{\ell_4=-\ell}^{\ell} b_{\ell_4;\varepsilon} C^{\ell 0}_{\ell 0 \ell_4 0} \times \tag{2.4.28}$$
$$\prod_{\ell_1 \ell_2 \ell_3 \ell_4} \sum_{k} C^{k0}_{\ell_3 0 \ell_4 0} C^{k0}_{\ell_2 0 \ell_1 0} \begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell & \ell & \ell_2 \\ \ell_4 & \ell_3 & k \end{Bmatrix}.$$

In view of equation (A.2.5), (2.4.28) reduces to

$$= (4\pi)^2 \sum_{\ell_1=0}^{\infty} b_{\ell_1;\varepsilon} C_{\ell 0 \ell 0}^{\ell_1 0} \sum_{\ell_2=0}^{\infty} b_{\ell_2;\varepsilon} C_{\ell 0 \ell 0}^{\ell_2 0} \sum_{\ell_3=0}^{\infty} b_{\ell_3;\varepsilon} C_{\ell 0 \ell 0}^{\ell_3 0} \sum_{\ell_4=0}^{\infty} b_{\ell_4;\varepsilon} C_{\ell 0 \ell 0}^{\ell_4 0} \sum_k C_{\ell_3 0 \ell_4 0}^{k0} C_{\ell_2 0 \ell_1 0}^{k0} \begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell & \ell & \ell_2 \\ \ell_4 & \ell_3 & k \end{Bmatrix}. \tag{2.4.29}$$

Step 3. In order to simplify the notation, we define this last expression as $A_{\ell,k}(\ell_1, \ell_2, \ell_3, \ell_4)$. We split it in different cases and we study them separately; hence, we rewrite (2.4.29) as

$$\begin{aligned} =&A_{\ell,k}(\ell_1,\ell_2,\ell_3,\ell_4) + A_{\ell,k}(0,0,0,0) + A_{\ell,k}(0,\ell_2,\ell_3,\ell_4) + A_{\ell,k}(\ell_1,0,\ell_3,\ell_4) + A_{\ell,k}(\ell_1,\ell_2,0,\ell_4) + \\ &A_{\ell,k}(\ell_1,\ell_2,\ell_3,0) + 2A_{\ell,k}(0,0,\ell_3,\ell_4) + 2A_{\ell,k}(0,\ell_2,0,\ell_4) + 2A_{\ell,k}(0,\ell_2,\ell_3,0) + 2A_{\ell,k}(\ell_1,0,0,\ell_4) + \\ &2A_{\ell,k}(\ell_1,0,\ell_3,0) + 2A_{\ell,k}(\ell_1,\ell_2,0,0), \end{aligned} \tag{2.4.30}$$

where

$$A_{\ell,k}(\ell_1,\ell_2,\ell_3,\ell_4) := (4\pi)^2 \sum_{\ell_1=1}^{\infty} b_{\ell_1;\varepsilon} C_{\ell 0 \ell 0}^{\ell_1 0} \sum_{\ell_2=1}^{\infty} b_{\ell_2;\varepsilon} C_{\ell 0 \ell 0}^{\ell_2 0} \sum_{\ell_3=1}^{\infty} b_{\ell_3;\varepsilon} C_{\ell 0 \ell 0}^{\ell_3 0} \sum_{\ell_4=1}^{\infty} b_{\ell_4;\varepsilon} C_{\ell 0 \ell 0}^{\ell_4 0} \sum_k C_{\ell_3 0 \ell_4 0}^{k0} C_{\ell_2 0 \ell_1 0}^{k0} \begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell & \ell & \ell_2 \\ \ell_4 & \ell_3 & k \end{Bmatrix}, \tag{2.4.31}$$

and so that

$$A_{\ell,k}(0,0,0,0) = +(4\pi)^2 b_{0;\varepsilon}^4 (C_{\ell 0 \ell 0}^{00})^4 C_{0000}^{00} C_{0000}^{00} \begin{Bmatrix} \ell & \ell & 0 \\ \ell & \ell & 0 \\ 0 & 0 & 0 \end{Bmatrix},$$

$$A_{\ell,k}(0,\ell_2,\ell_3,\ell_4) = (4\pi)^2 b_{0;\varepsilon} C_{\ell 0 \ell 0}^{00} \sum_{\ell_2,\ell_3,\ell_4=1}^{\infty} b_{\ell_2;\varepsilon} C_{\ell 0 \ell 0}^{\ell_2 0} b_{\ell_3;\varepsilon} C_{\ell 0 \ell 0}^{\ell_3 0} b_{\ell_4;\varepsilon} C_{\ell 0 \ell 0}^{\ell_4 0} \sum_k C_{\ell_3 0 \ell_4 0}^{k0} C_{\ell_2 000}^{k0} \begin{Bmatrix} \ell & \ell & 0 \\ \ell & \ell & \ell_2 \\ \ell_4 & \ell_3 & k \end{Bmatrix}$$

(and similar expressions hold for $A_{\ell,k}(\ell_1,0,\ell_3,\ell_4)$, $A_{\ell,k}(\ell_1,\ell_2,0,\ell_4)$, $A_{\ell,k}(\ell_1,\ell_2,\ell_3,0)$),

$$\begin{aligned} A_{\ell,k}(0,0,\ell_3,\ell_4) = (4\pi)^2 (b_{0,\varepsilon} C_{\ell 0 \ell 0}^{00})^2 \sum_{\ell_3,\ell_2=1}^{\infty} b_{\ell_3;\varepsilon} C_{\ell 0 \ell 0}^{\ell_3 0} b_{\ell_2;\varepsilon} C_{\ell 0 \ell 0}^{\ell_2 0} \times \\ \times \sum_k C_{\ell_3 000}^{k0} C_{\ell_2 000}^{k0} \begin{Bmatrix} \ell & \ell & 0 \\ \ell & \ell & \ell_2 \\ 0 & \ell_3 & k \end{Bmatrix}. \end{aligned}$$

Note that all the terms with three indexes among $\ell_1, \ell_2, \ell_3, \ell_4$ equal to zero, are zero for the triangular condition, in fact, if we look at the term

$$3(4\pi)^2 (b_{0,\varepsilon} C_{\ell 0 \ell 0}^{00})^3 \sum_{\ell_3=1}^{\infty} b_{\ell_3;\varepsilon} C_{\ell 0 \ell 0}^{\ell_3 0} \sum_k C_{\ell_3 000}^{k0} C_{0000}^{k0} \begin{Bmatrix} \ell & \ell & 0 \\ \ell & \ell & 0 \\ 0 & \ell_3 & k \end{Bmatrix},$$

in the last sum, the Clebsch-Gordan coefficient $C_{\ell_3 000}^{k0}$ is different from zero only if $\ell_3 = 0$, but this value of ℓ_3 is not considered in the current series.

Now we simplify a bit these expressions.

- As far as $A_{\ell,k}(0,0,0,0)$ is concerned, for the symmetry properties of the 9j symbols [75], one has

$$\begin{Bmatrix} \ell & \ell & 0 \\ \ell & \ell & 0 \\ 0 & 0 & 0 \end{Bmatrix} = \frac{1}{2\ell+1}.$$

Therefore, remembering that $C^{00}_{0000} = 1$ (from (A.2.16)) and $C^{00}_{\ell 0 \ell 0} = \frac{(-1)^\ell}{\sqrt{2\ell+1}}$ (from (A.2.15)),

$$A_{\ell,k}(0,0,0,0) = (4\pi)^2 b^4_{0,\varepsilon} \frac{1}{(2\ell+1)^3}. \tag{2.4.32}$$

- Look at the term $A_{\ell;k}(0,\ell_2,\ell_3,\ell_4)$; for the triangular condition the only term in the sum in k which does not vanish is $k=\ell_2$ and for the symmetry properties of the Wigner 9j coefficients (A.2.22) and for (A.2.20), it follows that

$$\begin{Bmatrix} \ell & \ell & 0 \\ \ell & \ell & \ell_2 \\ \ell_4 & \ell_3 & \ell_2 \end{Bmatrix} = \begin{Bmatrix} \ell_3 & \ell_4 & \ell_2 \\ \ell & \ell & \ell_2 \\ \ell & \ell & 0 \end{Bmatrix} = \frac{(-1)^{\ell_4+\ell_2}}{[(2\ell_2+1)(2\ell+1)]^{1/2}} \begin{Bmatrix} \ell_3 & \ell_4 & \ell_2 \\ \ell & \ell & \ell \end{Bmatrix}.$$

Likewise,

$$\begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell & \ell & 0 \\ \ell_4 & \ell_3 & \ell_1 \end{Bmatrix} = \begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell_3 & \ell_4 & \ell_1 \\ \ell & \ell & 0 \end{Bmatrix} = \frac{(-1)^{\ell_3+\ell_1}}{[(2\ell_1+1)(2\ell+1)]^{1/2}} \begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell_4 & \ell_3 & \ell \end{Bmatrix} =$$

$$= \frac{(-1)^{\ell_3+\ell_1}}{[(2\ell_1+1)(2\ell+1)]^{1/2}} \begin{Bmatrix} \ell_3 & \ell_4 & \ell_1 \\ \ell & \ell & \ell \end{Bmatrix},$$

where the last equality is due to the invariance under permutation of the Wigner 6j coefficients (A.2.17). Similarly,

$$\begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell & \ell & \ell_2 \\ \ell_4 & 0 & \ell_4 \end{Bmatrix} = \begin{Bmatrix} \ell & \ell_2 & \ell \\ \ell & \ell_1 & \ell \\ \ell_4 & \ell_4 & 0 \end{Bmatrix} = \frac{(-1)^{\ell_2+\ell_4}}{[(2\ell_4+1)(2\ell+1)]^{1/2}} \begin{Bmatrix} \ell & \ell_2 & \ell \\ \ell_1 & \ell & \ell_4 \end{Bmatrix} =$$

$$= \frac{(-1)^{\ell_2+\ell_4}}{[(2\ell_4+1)(2\ell+1)]^{1/2}} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_4 \\ \ell & \ell & \ell \end{Bmatrix}$$

and

$$\begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell & \ell & \ell_2 \\ 0 & \ell_3 & \ell_3 \end{Bmatrix} = \begin{Bmatrix} \ell & \ell_2 & \ell \\ \ell & \ell_1 & \ell \\ \ell_4 & \ell_4 & 0 \end{Bmatrix} = \frac{(-1)^{\ell_1+\ell_3}}{[(2\ell_3+1)(2\ell+1)]^{1/2}} \begin{Bmatrix} \ell_2 & \ell & \ell \\ \ell & \ell_1 & \ell_3 \end{Bmatrix}$$

$$= \frac{(-1)^{\ell_1+\ell_3}}{[(2\ell_3+1)(2\ell+1)]^{1/2}} \begin{Bmatrix} \ell_2 & \ell_1 & \ell_3 \\ \ell & \ell & \ell \end{Bmatrix}.$$

Renaming the indexes of the similar terms with ℓ_2, ℓ_3, we can write

$$
\begin{aligned}
&A_{\ell,k}(0,\ell_2,\ell_3,\ell_4) = A_{\ell,k}(\ell_2,0,\ell_3,\ell_4) = A_{\ell,k}(\ell_2,\ell_3,0,\ell_4) = A_{\ell,k}(\ell_2,\ell_3,\ell_4,0) = \\
&= (4\pi)^2 b_{0,\varepsilon} \frac{(-1)^\ell}{\sqrt{2\ell+1}} \sum_{\ell_2,\ell_3,\ell_4=1}^{\infty} b_{\ell_2;\varepsilon} C^{\ell_2 0}_{\ell 0 \ell 0} b_{\ell_3;\varepsilon} C^{\ell_3 0}_{\ell 0 \ell 0} b_{\ell_4;\varepsilon} C^{\ell_4 0}_{\ell 0 \ell 0} C^{\ell_2 0}_{\ell_3 0 \ell_4 0} \times \\
&\times \frac{(-1)^{\ell_2+\ell_4}}{\sqrt{(2\ell+1)(2\ell_2+1)}} \begin{Bmatrix} \ell_3 & \ell_4 & \ell_2 \\ \ell & \ell & \ell \end{Bmatrix} = \\
&= (4\pi)^2 b_{0,\varepsilon} \frac{(-1)^\ell}{2\ell+1} \sum_{\ell_2,\ell_3,\ell_4=1}^{\infty} b_{\ell_2;\varepsilon} C^{\ell_2 0}_{\ell 0 \ell 0} b_{\ell_3;\varepsilon} C^{\ell_3 0}_{\ell 0 \ell 0} b_{\ell_4;\varepsilon} C^{\ell_4 0}_{\ell 0 \ell 0} C^{\ell_2 0}_{\ell_3 0 \ell_4 0} \frac{(-1)^{\ell_2+\ell_4}}{\sqrt{(2\ell_2+1)}} \begin{Bmatrix} \ell_3 & \ell_4 & \ell_2 \\ \ell & \ell & \ell \end{Bmatrix}
\end{aligned}
\tag{2.4.33}
$$

and since $C^{\ell' 0}_{\ell 0 \ell 0} = 0$ if ℓ' is odd, the series only run on even indexes and this implies that $(-1)^{\ell_2+\ell_4} = 1$.

- Regarding $A_{\ell,k}(0,0,\ell_3,\ell_4)$, for the triangular condition, the only term of the sum in k which is non-zero is $k=0$ and the symmetry properties of the 9j symbol (A.2.21) and the relation (A.2.23) imply

$$
\begin{Bmatrix} \ell & \ell & 0 \\ \ell & \ell & 0 \\ \ell_4 & \ell_3 & 0 \end{Bmatrix} = \begin{Bmatrix} \ell & \ell & \ell_4 \\ \ell & \ell & \ell_3 \\ 0 & 0 & 0 \end{Bmatrix} = \frac{\delta^{\ell_4}_{\ell_3}}{[(2\ell_4+1)(2\ell+1)^2]^{1/2}}. \tag{2.4.34}
$$

The same properties give

$$
\begin{Bmatrix} \ell & \ell & 0 \\ \ell & \ell & \ell_2 \\ \ell_2 & 0 & \ell_2 \end{Bmatrix} = \begin{Bmatrix} \ell & \ell_2 & \ell \\ \ell & 0 & \ell \\ \ell_2 & \ell_2 & 0 \end{Bmatrix} = \frac{1}{[(2\ell_2+1)(2\ell+1)]^{1/2}} \begin{Bmatrix} \ell & \ell_2 & \ell \\ 0 & \ell & \ell_2 \end{Bmatrix}
$$

and since if one of the argument is zero the value of the 6j symbol can be written explicitly as in (A.2.19), we have that

$$
\begin{Bmatrix} \ell & \ell & 0 \\ \ell & \ell & \ell_2 \\ \ell_2 & 0 & \ell_2 \end{Bmatrix} = \frac{1}{[(2\ell_2+1)(2\ell+1)]^{1/2}} \frac{(-1)^{\ell_2}}{\sqrt{(2\ell_2+1)(2\ell+1)}} = \frac{(-1)^{\ell_2}}{(2\ell_2+1)(2\ell+1)}.
$$

Analogously, we get

$$
\begin{Bmatrix} \ell & \ell & 0 \\ \ell & \ell & \ell_2 \\ 0 & \ell_2 & \ell_2 \end{Bmatrix} = \frac{(-1)^{\ell_2}}{[(2\ell_2+1)(2\ell+1)]^{1/2}} \begin{Bmatrix} \ell_2 & 0 & \ell_2 \\ \ell & \ell & \ell \end{Bmatrix} = \frac{1}{(2\ell_2+1)(2\ell+1)},
$$

$$
\begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell & \ell & 0 \\ \ell_1 & 0 & \ell_1 \end{Bmatrix} = \frac{(-1)^{\ell_1}}{[(2\ell_1+1)(2\ell+1)]^{1/2}} \begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell_1 & 0 & \ell \end{Bmatrix} = \frac{1}{(2\ell_1+1)(2\ell+1)},
$$

$$
\begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell & \ell & 0 \\ 0 & \ell_1 & \ell_1 \end{Bmatrix} = \frac{1}{[(2\ell_1+1)(2\ell+1)]^{1/2}} \begin{Bmatrix} \ell & \ell & \ell_1 \\ 0 & \ell_1 & \ell \end{Bmatrix} = \frac{(-1)^{\ell_1}}{(2\ell_1+1)(2\ell+1)}
$$

and

$$\begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell & \ell & \ell_1 \\ 0 & 0 & 0 \end{Bmatrix} = \frac{1}{[(2\ell_1+1)(2\ell+1)^2]^{1/2}}.$$

Same computations of (2.4.33) lead to

$$\begin{aligned} &A_{\ell,k}(0,0,\ell_3,\ell_4) = A_{\ell,k}(0,\ell_3,0,\ell_4) = A_{\ell,k}(\ell_3,0,\ell_4,0) = A_{\ell,k}(\ell_3,\ell_4,0,0) = \\ &= (4\pi)^2 \frac{b_{0,\varepsilon}^2}{2\ell+1} \sum_{\ell_3=1}^{\infty} b_{\ell_3;\varepsilon}^2 (C_{\ell 0 \ell 0}^{\ell_3 0})^2 \frac{(-1)^{\ell_3}}{(2\ell+1)(2\ell_3+1)} \\ &= (4\pi)^2 \frac{b_{0,\varepsilon}^2}{(2\ell+1)^2} \sum_{\ell_3=1}^{\infty} b_{\ell_3;\varepsilon}^2 (C_{\ell 0 \ell 0}^{\ell_3 0})^2 \frac{(-1)^{\ell_3}}{(2\ell_3+1)} \end{aligned} \tag{2.4.35}$$

and

$$A_{\ell,k}(0,\ell_2,\ell_3,0) = A_{\ell,k}(\ell_1,0,0,\ell_4) = (4\pi)^2 \frac{b_{0,\varepsilon}^2}{(2\ell+1)^2} \sum_{\ell_3=1}^{\infty} b_{\ell_3;\varepsilon}^2 (C_{\ell 0 \ell 0}^{\ell_3 0})^2 \frac{1}{(2\ell_3+1)};$$

as for the previous case, since $C_{\ell 0 \ell 0}^{\ell' 0} = 0$ if ℓ' is odd, the series only run on even indexes and then $(-1)^{\ell_3} = 1$.

Finally, equation (2.4.30) reduces to

$$cum_4(h_{\ell;2}^*) = A_{\ell,k}(\ell_1,\ell_2,\ell_3,\ell_4) + A_{\ell,k}(0,0,0,0) + 4A_{\ell,k}(0,\ell_2,\ell_3,\ell_4) + 12A_{\ell,k}(0,0,\ell_3,\ell_4). \tag{2.4.36}$$

Step 4. The aim now is to understand the asymptotic behavior of expression (2.4.36) and to prove that it is $O\left(\frac{1}{\ell^3}\right)$. As in the previous step we study the summand in (2.4.36) separately.

- First, we can note that, by the results of the second chaotic component, it is easily seen that the last summand of (2.4.36) is $O\left(\frac{1}{\ell^3}\right)$, as $\ell \to \infty$. The same holds for $A_{\ell,k}(0,0,0,0)$ directly from (2.4.32).

- Concerning the third term of (2.4.36), because of (A.2.18) (see [45]), the following upper bound holds

$$\left| \begin{Bmatrix} \ell_3 & \ell_4 & \ell_2 \\ \ell & \ell & \ell \end{Bmatrix} \right| \le \frac{1}{\sqrt{2\ell+1}} \min\left(\frac{1}{\sqrt{2\ell_2+1}}, \frac{1}{\sqrt{2\ell_3+1}}, \frac{1}{\sqrt{2\ell_4+1}} \right) \tag{2.4.37}$$

and from (A.2.12),

$$|C_{\ell 0 \ell 0}^{\ell_2 0}| \le \frac{\sqrt{2\ell_2+1}}{\sqrt{2\ell+1}}, \tag{2.4.38}$$

taking the absolute value, one has that $A_{\ell,k}(0,\ell_2,\ell_3,\ell_4)$ is bounded by

$$\leq (4\pi)^2 b_{0,\varepsilon}\frac{(-1)^\ell}{2\ell+1}\sum_{\ell_2,\ell_3,\ell_4=1}^{\infty} b_{\ell_2;\varepsilon}C_{\ell 0\ell 0}^{\ell_2 0}b_{\ell_3;\varepsilon}C_{\ell 0\ell 0}^{\ell_3 0}b_{\ell_4;\varepsilon}C_{\ell 0\ell 0}^{\ell_4 0}C_{\ell_3 0\ell_4 0}^{\ell_2 0}\frac{1}{\sqrt{(2\ell_2+1)}}\begin{Bmatrix}\ell_3 & \ell_4 & \ell_2\\ \ell & \ell & \ell\end{Bmatrix}$$
$$\leq (4\pi)^2 b_{0,\varepsilon}\frac{1}{2\ell+1}\sum_{\ell_2,\ell_3,\ell_4=1}^{\infty}|b_{\ell_2;\varepsilon}b_{\ell_3;\varepsilon}b_{\ell_4;\varepsilon}|\frac{\sqrt{2\ell_2+1}}{\sqrt{2\ell+1}}\frac{\sqrt{2\ell_3+1}}{\sqrt{2\ell+1}}\frac{\sqrt{2\ell_4+1}}{\sqrt{2\ell+1}}1\times$$
$$\frac{1}{\sqrt{(2\ell_2+1)}}\frac{1}{\sqrt{2\ell+1}}\min\left(\frac{1}{\sqrt{2\ell_2+1}},\frac{1}{\sqrt{2\ell_3+1}},\frac{1}{\sqrt{2\ell_4+1}}\right)$$
$$\leq (4\pi)^2 b_{0,\varepsilon}\frac{1}{(2\ell+1)^3}\sum_{\ell_2,\ell_3,\ell_4=1}^{\infty}|b_{\ell_2;\varepsilon}b_{\ell_3;\varepsilon}b_{\ell_4;\varepsilon}|\sqrt{2\ell_3+1}\sqrt{2\ell_4+1}\times$$
$$\times\min\left(\frac{1}{\sqrt{2\ell_2+1}},\frac{1}{\sqrt{2\ell_3+1}},\frac{1}{\sqrt{2\ell_4+1}}\right). \tag{2.4.39}$$

We have already discussed the absolute convergence of the series, which allows us to say that (2.4.39) is $O\left(\frac{1}{\ell^3}\right)$ as $\ell\to\infty$.

- It remains to study the term $A_{\ell,k}(\ell_1,\ell_2,\ell_3,\ell_4)$;. its absolute value can be bounded by

$$(4\pi)^2\sum_{\ell_1=1}^{\infty}|b_{\ell_1;\varepsilon}|\sum_{\ell_2=1}^{\infty}|b_{\ell_2;\varepsilon}|\sum_{\ell_3=1}^{\infty}|b_{\ell_3;\varepsilon}|\sum_{\ell_4=1}^{\infty}|b_{\ell_4;\varepsilon}|\frac{\sqrt{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)(2\ell_4+1)}}{(2\ell+1)^2}\times$$
$$\left|\sum_k C_{\ell_3 0\ell_4 0}^{k0}C_{\ell_2 0\ell_1 0}^{k0}\begin{Bmatrix}\ell & \ell & \ell_1\\ \ell & \ell & \ell_2\\ \ell_4 & \ell_3 & k\end{Bmatrix}\right|. \tag{2.4.40}$$

For equation (A.2.24) one has that

$$\sum_k C_{\ell_3 0\ell_4 0}^{k0}C_{\ell_2 0\ell_1 0}^{k0}\begin{Bmatrix}\ell & \ell & \ell_1\\ \ell & \ell & \ell_2\\ \ell_4 & \ell_3 & k\end{Bmatrix}=\frac{1}{\sqrt{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)(2\ell_4+1)}}(-1)^{\ell_1+\ell_2}\times$$
$$\times\sum_{s\sigma}(2s+1)\sqrt{2\ell_1+1}\sqrt{2\ell_3+1}C_{\ell_1 0 s\sigma}^{\ell_4 0}C_{\ell_3 0 s\sigma}^{\ell_2 0}\begin{Bmatrix}\ell & \ell & \ell_1\\ \ell_4 & s & \ell\end{Bmatrix}\begin{Bmatrix}\ell & \ell & \ell_3\\ \ell_2 & s & \ell\end{Bmatrix}; \tag{2.4.41}$$

the triangular condition implies $\sigma=0$, so that (2.4.41) is

$$=\frac{1}{\sqrt{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)(2\ell_4+1)}}\times$$
$$\times\sum_{s}(2s+1)\sqrt{2\ell_1+1}\sqrt{2\ell_3+1}C_{\ell_1 0 s 0}^{\ell_4 0}C_{\ell_3 0 s 0}^{\ell_2 0}\begin{Bmatrix}\ell & \ell & \ell_1\\ \ell_4 & s & \ell\end{Bmatrix}\begin{Bmatrix}\ell & \ell & \ell_3\\ \ell_2 & s & \ell\end{Bmatrix} \tag{2.4.42}$$

and thanks to the fact that

$$\left|\begin{Bmatrix}\ell & \ell & \ell_1\\ \ell_4 & s & \ell\end{Bmatrix}\begin{Bmatrix}\ell & \ell & \ell_3\\ \ell_2 & s & \ell\end{Bmatrix}\right|\leq\frac{1}{2\ell+1}\times$$

$$\times \min\left(\frac{1}{\sqrt{2\ell_1+1}}, \frac{1}{\sqrt{2s+1}}, \frac{1}{\sqrt{2\ell_4+1}}\right) \min\left(\frac{1}{\sqrt{2\ell_2+1}}, \frac{1}{\sqrt{2s+1}}, \frac{1}{\sqrt{2\ell_3+1}}\right)$$

((A.2.18), see [45]), and (2.4.38), one gets that

$$A_{\ell,k}(\ell_1,\ell_2,\ell_3,\ell_4) \leq (4\pi)^2 \frac{1}{(2\ell+1)^3} \sum_{\ell_1\ell_2\ell_3\ell_4} |b_{\ell_1;\varepsilon} b_{\ell_2;\varepsilon} b_{\ell_3;\varepsilon} b_{\ell_4;\varepsilon}| \sum_s (2s+1)\sqrt{2\ell_1+1}\sqrt{2\ell_3+1}\times$$
$$\times C^{\ell_4 0}_{\ell_1 0 s 0} C^{\ell_2 0}_{\ell_3 0 s 0} \min\left(\frac{1}{\sqrt{2\ell_1+1}}, \frac{1}{\sqrt{2s+1}}, \frac{1}{\sqrt{2\ell_4+1}}\right) \min\left(\frac{1}{\sqrt{2\ell_2+1}}, \frac{1}{\sqrt{2s+1}}, \frac{1}{\sqrt{2\ell_3+1}}\right) \tag{2.4.43}$$

and since

$$\min\left(\frac{1}{2\ell_1+1}, \frac{1}{\sqrt{2s+1}}, \frac{1}{\sqrt{2\ell_4+1}}\right) \leq 1$$

and

$$\min\left(\frac{1}{\sqrt{2\ell_2+1}}, \frac{1}{\sqrt{2s+1}}, \frac{1}{\sqrt{2\ell_3+1}}\right) \leq 1,$$

then (2.4.43) is bounded by

$$\leq (4\pi)^2 \frac{1}{(2\ell+1)^3} \sum_{\ell_1\ell_2\ell_3\ell_4} |b_{\ell_1;\varepsilon} b_{\ell_2;\varepsilon} b_{\ell_3;\varepsilon} b_{\ell_4;\varepsilon}| \left| \sum_s (2s+1)\sqrt{2\ell_1+1}\sqrt{2\ell_3+1} C^{\ell_4 0}_{\ell_1 0 s 0} C^{\ell_2 0}_{\ell_3 0 s 0} \right|. \tag{2.4.44}$$

In view of the Cauchy-Schwarz inequality, it follows that

$$\sum_s (2s+1) C^{\ell_4 0}_{\ell_1 0 s 0} C^{\ell_2 0}_{\ell_3 0 s 0} \leq \left(\sum_s (\sqrt{2s+1} C^{\ell_4 0}_{\ell_1 0 s 0})^2 \right)^{1/2} \left(\sum_s (\sqrt{2s+1} C^{\ell_2 0}_{\ell_3 0 s 0})^2 \right)^{1/2} \tag{2.4.45}$$

and since $\sum_{\ell=|\ell_2-\ell_1|}^{\ell_2+\ell_1} (C^{\ell 0}_{\ell_1 0 \ell_2 0})^2 = 1$, and the permutations properties

$$1 = \sum_\ell (C^{\ell 0}_{\ell_1 0 \ell_2 0})^2 = \sum_\ell \frac{2\ell+1}{2\ell_2+1} (C^{\ell_2 0}_{\ell_1 0 \ell 0})^2,$$

it entails that

$$\sum_\ell (2\ell+1)(C^{\ell_2 0}_{\ell_1 0 \ell 0})^2 = 2\ell_2+1$$

and then, the right hand side of (2.4.45) is equal to

$$= \sqrt{2\ell_4+1}\sqrt{2\ell_2+1}.$$

Eventually, $A_{\ell,k}(\ell_1,\ell_2,\ell_3,\ell_4)$ is smaller than

$$(4\pi)^2 \sum_{\ell_1=1}^{\infty} b_{\ell_1;\varepsilon} C^{\ell_1 0}_{\ell 0 \ell 0} \sum_{\ell_2=1}^{\infty} b_{\ell_2;\varepsilon} C^{\ell_2 0}_{\ell 0 \ell 0} \sum_{\ell_3=1}^{\ell} b_{\ell_3;\varepsilon} C^{\ell_3 0}_{\ell 0 \ell 0} \sum_{\ell_4=1}^{\infty} b_{\ell_4;\varepsilon} C^{\ell_4 0}_{\ell 0 \ell 0} \sum_k C^{k0}_{\ell_3 0 \ell_4 0} C^{k0}_{\ell_2 0 \ell_1 0} \begin{Bmatrix} \ell & \ell & \ell_1 \\ \ell & \ell & \ell_2 \\ \ell_4 & \ell_3 & k \end{Bmatrix} \tag{2.4.46}$$
$$\leq (4\pi)^2 \frac{1}{(2\ell+1)^3} \sum_{\ell_1\ell_2\ell_3\ell_4} |b_{\ell_1;\varepsilon} b_{\ell_2;\varepsilon} b_{\ell_3;\varepsilon} b_{\ell_4;\varepsilon}| \sqrt{2\ell_2+1}\sqrt{2\ell_4+1}\sqrt{2\ell_1+1}\sqrt{2\ell_3+1}$$

and since the series are absolutely convergent, the proof is completed.

□

Remark 2.4.1. *A detailed investigation for the asymptotic behavior of Clebsch-Gordan coefficients and bounds like (2.4.37) and (2.4.41) can be found also in [42] and [41].*

2.5 Further Remark on the Area of the Excursion Sets

In this section, we present some numerical evidence. We proved that the leading term of the chaotic expansion of the area of a spherical cap is the second, being $O(\frac{1}{\ell})$. Indeed, for all the chaotic components corresponding to $q \geq 3$ we derive an upper bound with a higher order (this is true also for $q = 1$ but with a different proof). Then, we make here some numerical investigations to determine these bounds explicitly. More precisely, considering the variance of (2.1.2), namely

$$h_{\ell,q}(B) = \int_B H_q(T_\ell(x))\, dx,$$

in view of the computations we did in Section 2.3 (for the chaotic component of order $q \geq 3$), it is easily seen that, for each even q, $q \geq 2$,

$$\operatorname{Var}\left(\int_B H_q(T_\ell(x))\, dx \right) \leq \frac{m(B)}{m(\mathbb{S}^2)} \operatorname{Var}\left(\int_{\mathbb{S}^2} H_q(T_\ell(x)\, dx \right).$$

Denoting with $h_{\ell,q}$ the integral on the right hand side as in (1.4.4), it is known from previous works, for example in [50], that for $q = 3$ and $q \geq 5$, one has

$$\operatorname{Var}(h_{\ell,q}) = (4\pi)^2 q! \int_0^{\pi/2} P_\ell^q(\cos\theta) \sin\theta\, d\theta \sim (4\pi)^2 q! \frac{c_q}{\ell^2}$$

with

$$c_q = \int_0^\infty \psi J_0(\psi)^q d\psi \geq 0, \quad J_0(x) = \sum_{k=0}^\infty \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} \tag{2.5.1}$$

being the J_0 Bessel function. Moreover for $q = 2, 4$, the order of magnitude of the corresponding variance is larger, namely:

$$\operatorname{Var}(h_{q;\ell}) \sim \frac{1}{\ell} \quad \text{for } q = 2$$

$$\operatorname{Var}(h_{q;\ell}) \sim \frac{\log \ell}{\ell^2} \quad \text{for } q = 4.$$

If we normalize the constants c_q and define

$$C_2 := \frac{1}{L} \int_0^L J_0(\psi)^2 \psi\, d\psi,$$

$$C_4 := \frac{1}{\log L} \int_0^L J_0(\psi)^4 \psi\, d\psi$$

and

$$C_q := \int_0^L J_0(\psi)^q \psi\, d\psi, \quad \text{for } q = 3 \text{ and } q \geq 5;$$

for the values $L = 50$, 100, 200, we find, exploiting Matlab, the following results.

C_q	$L = 50$	$L = 100$	$L = 200$
C_5	0.3286	0.3289	0.3290
C_6	0.3344	0.3352	0.3356
C_7	0.2600	0.2600	0.2600
C_8	0.2369	0.2369	0.2369
C_9	0.2085	0.2085	0.2085
C_{10}	0.1897	0.1897	0.1897
C_{11}	0.1727	0.1727	0.1727
C_{12}	0.1590	0.1590	0.1590
C_{13}	0.1472	0.1472	0.1472
C_{17}	0.1134	0.1134	0.1134
C_{18}	0.1072	0.1072	0.1072
C_{24}	0.0808	0.0808	0.0808
C_{25}	0.0776	0.0776	0.0776

In conclusion, it can be seen from Figure 2.3 and Figure 2.4 (realized for $L = 100$) that the behavior of C_q, for $q \geq 5$, is approximately

$$C_q \sim \frac{2}{q}. \tag{2.5.2}$$

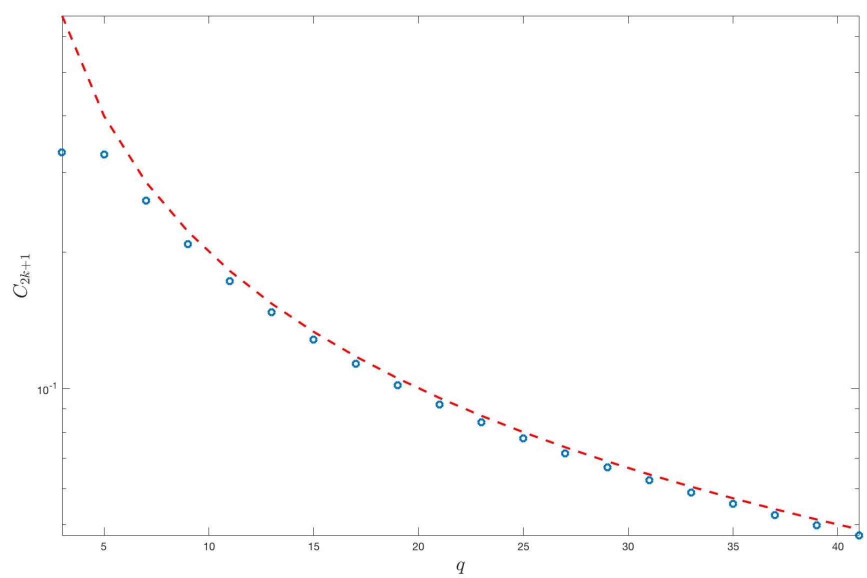


Figure 2.3: The red dashes represent the function $\frac{2}{q}$; whereas the blu circles, the coefficients C_q for odd q. The plot is realized setting $L = 100$.

In [52], the authors investigated the defect case (defined in (1.4.10)) and the variance was

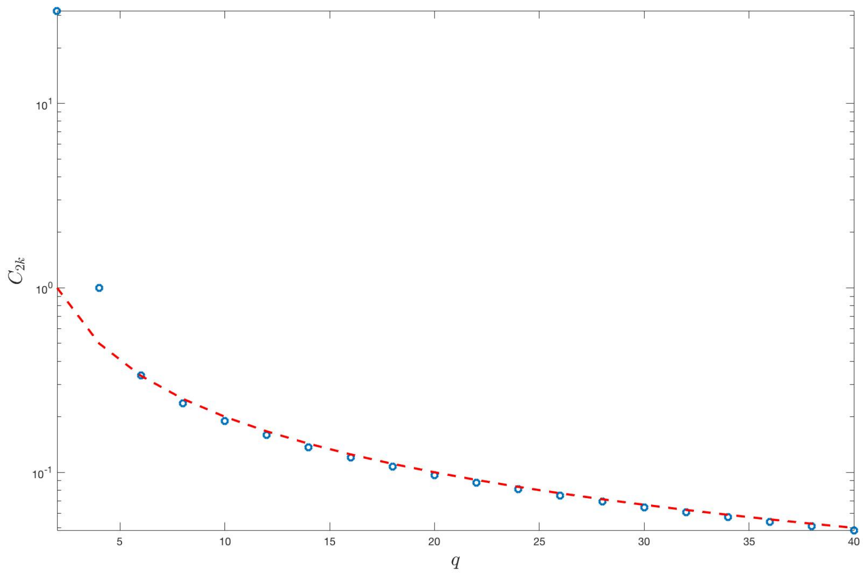


Figure 2.4: The red dashes represent the function $\frac{2}{q}$; whereas the blu circles, the coefficients C_q for even q. The plot is realized setting $L = 100$.

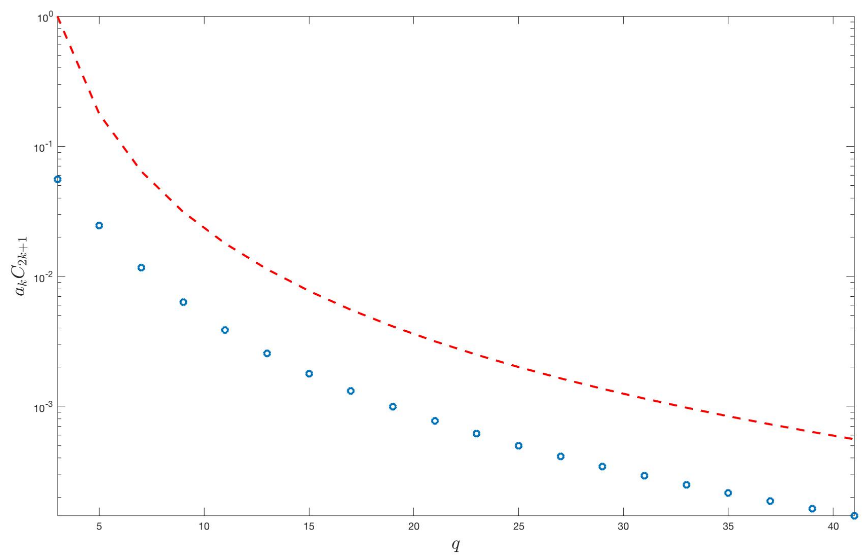


Figure 2.5: The red dashes represents the function $\dfrac{1}{q^{5/2}}$; whereas the blue circles the points $a_k C_{2k+1}$. The plot is realized setting $L = 100$.

given by (1.4.11), which we report again for simplicity sake

$$\mathrm{Var}(D_\ell) = \frac{C}{\ell^2} + o\left(\frac{1}{\ell^2}\right), \tag{2.5.3}$$

where the constant C can be computed by

$$C = 32\pi \sum_{k=1}^{\infty} a_k C_{2k+1} \quad \text{and} \quad a_k = \frac{(2k)!}{4^k (k!)^2 (2k+1)} \tag{2.5.4}$$

(formula (25), [52]). Coming across the coefficients C_q, we have noticed that it is possible to perform a numerical approximation of the constant C; hence, we report this result even if slightly off-topic, as not concerning the spherical cap. To have a precision of 1.0×10^{-4}, it is sufficient to sum the terms in (2.5.4) until $q = 20$ and we found the value

$$C = 32\pi \sum_{k=1}^{20} a_k C_{2k+1} = 32\pi \times 0.1182. \tag{2.5.5}$$

Note that for c_3 the exact value is known, computed in [52],

$$c_3 = \frac{2}{\pi\sqrt{3}} = 0.3676$$

and we used it in (2.5.5) to have a better approximation.

Remark 2.5.1. *In [52], the asymptotic behavior of the coefficients a_k is proved to be $\frac{a}{k^{3/2}}$, where a is a constant; then, in view of (2.5.2) the product $a_k C_{2k+1}$ has the same asymptotic behavior as $\frac{2a}{q^{5/2}}$. Indeed, Figure 2.5 compares the points $a_k C_{2k+1}$ with the function $\frac{1}{q^{5/2}}$.*

Remark 2.5.2. *It is easily seen that the* 50% *of the contribution of the sum in (2.5.5) comes from the first term, which is* 0.0613*. Moreover, the sum of the first and the second is* 0.0860*, which is almost the* 80%*. Thus,* 80% *of the variance for the defect is explained by the third and fifth chaoses alone.*

CHAPTER 3

Nodal Lengths in Shrinking Domains for Random Eigenfunctions on $\mathbb{S}^2$

This chapter investigates the asymptotic behavior of the lengths of the zero set defined in (0.0.5), and hence, in particular, it includes the proofs of Theorem 0.0.2, Theorem 0.0.3 and Proposition 0.0.4 (see also [74]). More precisely, we show that nodal lengths are dominated by a single term, corresponding to the fourth chaotic projection; in particular, they are asymptotically fully equivalent (in the L^2-sense) to the integral on the ball of the fourth-order Hermite polynomial (the so-called "local sample trispectrum"), and their asymptotic variance is logarithmic. As a consequence a Central Limit Theorem is also established.
Our argument is based on a re-adaptation of the proof in [77], where the key role is played by an auxiliary function (see definition 3.1.2). The difficulty here is that this map depends on the position of the point in which it is calculated. Nevertheless, thanks to the decreasing of the radius, this problem is by-passed through a tangent plane approximation.
The Central Limit Theorem is proved exploiting the Fourth Moment Theorem ([57]) applied to the local sample trispectrum. Note that in the full sphere ([50]), the CLT was established by a careful analysis of the asymptotic behavior of Clebsch-Gordan coefficients; however, in this case, we cannot use this result and hence we exploit Hilb's asymptotics approximation.
Our results can be compared with the ones obtained for the torus in [11]; the differences and similarities between nodal lengths in shrinking ball on the torus and on the sphere are stressed in the Introduction. Briefly, in the former manifold, it results a full correlation between nodal lengths in decreasing domains and in the full torus, which leads to conclude the validity of a nonCentral Limit Theorem for these subregions (since it is nonCentral in the total manifold [46]). On the contrary, full correlation does not hold on the sphere between local and global statistics, that is exactly the way to read Proposition 0.0.4; however, the limiting distribution is proved to be still Gaussian for the subdomains.
It is also notable that the "Berry's cancellation" phenomenon takes place in this framework as well, and indeed the first and the second order chaotic components are still of lower order with respect to the leading term, although not identically equal to zero as in the full spherical case.
The Chapter is organized as follows. First, in Section 3.1 we explain the basic ideas for proving Theorem 0.0.2 and Theorem 0.0.3, while the main tools for our computations are introduced in Chapter 3.2, where auxiliary functions and their properties are discussed. Chapter 3.3 is splitted in two parts; Subsection 3.3.1 contains the proof of Theorem 0.0.2, which gives the asymptotic behavior for the variance and Subsection 3.3.2 proves the Central Limit Theorem (Theorem 0.0.3). In 3.4 further result of independent interest, namely, Proposition 0.0.4, is shown and, in the end, some technical tools are collected in Section 3.5.

3.1 On the proof of Theorem 0.0.2 and Theorem 0.0.3

In this section we give the guideline of the proof of the main results of this chapter, namely Theorem 0.0.2 and Theorem 0.0.3.
Let us consider the shrinking spherical cap $B_{r_\ell} \subset \mathbb{S}^2$ of radius r_ℓ centered in N satisfying condition (0.0.4), and the nodal lengths defined in (0.0.5). From the Kac-Rice formula (Theorem 1.1.3, see also [3], [6]), it is easy to see that

$$\mathbb{E}[\mathscr{Z}_{\ell,r_\ell}] = \sqrt{\frac{\ell(\ell+1)}{2}}\frac{m(B_{r_\ell})}{2}.$$

Note that, since the area of a spherical cap B_{r_ℓ} of radius r_ℓ is given by $m(B_{r_\ell}) = 2\pi(1-\cos r_\ell)$, we have that

$$\Rightarrow \mathbb{E}[\mathscr{Z}_{\ell,r_\ell}] = \sqrt{\frac{\ell(\ell+1)}{2}}\pi(1-\cos r_\ell).$$

In the full sphere, it is possible to write the second moment as

$$\mathbb{E}[(\mathscr{Z}(T_\ell))^2] = \int_{\mathbb{S}^2\times\mathbb{S}^2} \tilde{K}_\ell(x,y)\,dx\,dy \tag{3.1.1}$$

(see [15] Theorem 2.2, [16] Theorem 4.3, [78] Proposition 3.3), where $\tilde{K}_\ell(x,y) = \tilde{K}_\ell(d(x,y))$ is the two-point correlation function (Section 3.5, see also [77]), and the symmetry of the domain implies that, changing coordinates, (3.1.1) yields

$$\mathbb{E}[(\mathscr{Z}(T_\ell))^2] = 8\pi^2\int_0^\pi \tilde{K}_\ell(\rho)\sin\rho\,d\rho,$$

which allows to handle the computations and to establish the asymptotic behavior of the variance. Focussing instead on a subdomain, the lack of this symmetry prevents this change of coordinates. However, using (1.4.17) and the same argument as in [77], proof of Theorem 1.4, it can be shown that for any function $\varphi : \mathbb{S}^2 \to \mathbb{R}$ in $C^1(\mathbb{S}^2)$, we have that

$$\mathbb{E}[(\mathscr{Z}^\varphi(T_\ell))^2] = \int_{\mathbb{S}^2\times\mathbb{S}^2} \varphi(x)\varphi(y)\tilde{K}_\ell(x,y)\,dx\,dy.$$

Now, introducing an auxiliary function $W^\varphi : [0,\pi] \to \mathbb{R}$ ([77]), defined as

$$W^\varphi(\rho) := \frac{1}{8\pi^2}\int_{d(x,y)=\rho} \varphi(x)\varphi(y)\,dx\,dy \quad x,y\in\mathbb{S}^2, \tag{3.1.2}$$

and employing Fubini, we get that

$$\mathbb{E}[(\mathscr{Z}^\varphi(T_\ell))^2] = 8\pi^2\int_0^\pi \tilde{K}_\ell(\rho)W^\varphi(\rho)\,d\rho$$

with

$$\tilde{K}_\ell(\rho) = \tilde{K}_\ell(x,y),$$

$x,y\in\mathbb{S}^2$ being any pair of points with $d(x,y)=\rho$. The crucial observation is that the case of a spherical cap can be cast in this framework, simply taking $\varphi = \varphi_\ell = 1_{B_{r_\ell}}$, which is a function in $BV(\mathbb{S}^2)\cap L^\infty(\mathbb{S}^2)$, $\forall\ell$.

More precisely, the key role in the proof of Theorem 0.0.2 will be played by a sequence of auxiliary functions, defined as in (3.1.2) ([77]), $W^{\varphi_\ell} : [0, 2r_\ell] \to \mathbb{R}$,

$$W^{\varphi_\ell}(\rho) := \frac{1}{8\pi^2} \int_{d(x,y)=\rho} \varphi_\ell(x)\varphi_\ell(y)\, dx\, dy \quad x, y \in \mathbb{S}^2; \tag{3.1.3}$$

and using a density argument and approximating $1_{B_{r_\ell}}$ with a sequence of C^1 functions $\{\varphi_\ell^i\}_i$, the second moment could be written as

$$\mathbb{E}[(\mathscr{Z}^{\varphi_\ell^i}(T_\ell))^2] = 8\pi^2 \int_0^{2r_\ell} \tilde{K}_\ell(\rho) W^{\varphi_\ell^i}(\rho)\, d\rho.$$

Note that (3.1.3) is not zero if and only if the variables x, y are inside the spherical cap B_{r_ℓ}; hence, the maximum distance allowed between two points to make (3.1.3) different from zero is $\rho = 2r_\ell$. For $\varphi_\ell = 1_{B_{r_\ell}}$ and for $x, y \in B_{r_\ell}$, (3.1.3) can be written also as

$$W^{\varphi_\ell}(\rho) = \frac{1}{8\pi^2} \int_{B_{r_\ell}} \text{len}\{y \in B_{r_\ell} : d(x, y) = \rho\}\, dx.$$

Then, if we fix x "far" from the boundary, the integrand will be given by $\text{len}\{y \in B_{r_\ell} : d(x, y) = \rho\} = 2\pi \sin\rho$; note that, however, W^{φ_ℓ} depends on the position of x. Moreover, for decreasing sequence r_ℓ, a tangent plane approximation can be shown to hold, whence, we can also define the function on the plane $\tilde{W}_{\tilde{\varphi}} : [0, 2r_\ell] \to \mathbb{R}$ as

$$\tilde{W}_{\tilde{\varphi}_\ell}(\rho) := \frac{1}{8\pi^2} \int_{d(x,y)=\rho} \tilde{\varphi}_\ell(x)\tilde{\varphi}_\ell(y) dx dy \quad x, y \in \mathbb{R}^2, \tag{3.1.4}$$

where $\tilde{\varphi}$ is given by the composition $\varphi_\ell \circ \exp$ and exp is the exponential map. Note that $\tilde{W}_{\tilde{\varphi}_\ell}$ is nonzero if $x, y \in \tilde{B}_{r_\ell}$, which is the disc contained in $\mathbb{R}^2$ of radius r_ℓ and centered in the origin of the axes. In order to scale the support of $\tilde{\varphi}_\ell$ from $\tilde{B}_{r_\ell}$ in $\tilde{B}_1$, we define also

$$\tilde{W}_1\left(\rho \frac{1}{r_\ell}\right) := \frac{1}{8\pi^2} \int_{d(x,y)=\frac{\rho}{r_\ell}} \tilde{\varphi}_\ell(r_\ell x)\tilde{\varphi}_\ell(r_\ell y)\, dx\, dy \quad x, y \in \mathbb{R}^2; \tag{3.1.5}$$

setting $W_{r_\ell}(\rho) := W^{1_{B_{r_\ell}}}(\rho)$ (hence considering $\varphi_\ell(\cdot) = 1_{B_{r_\ell}}(\cdot)$), it is easy to check the validity of the asymptotic relation below:

$$W_{r_\ell}(\rho) = r_\ell^3 \tilde{W}_1(\rho \frac{1}{r_\ell})(1 + O(\rho^2)), \tag{3.1.6}$$

as $r_\ell \to 0$ uniformly in ρ (see Lemma 3.2.2).
We will then need to show that moments computed on approximating sequences converge to those for the functions of interest. It is easy to see that, for a sequence $\{\varphi^i\}_i$ such that $\varphi^i \to \varphi$ in $L^1(\mathbb{S}^2)$, we also have, for every fixed ℓ,

$$\mathbb{E}[\mathscr{Z}^{\varphi^i}(T_\ell)] \to \mathbb{E}[\mathscr{Z}^{\varphi}(T_\ell)];$$

indeed, it follows from the expected value of a linear statistic:

$$\mathbb{E}[\mathscr{Z}^{\varphi}(T_\ell)] = \frac{\int_{\mathbb{S}^2} \varphi(x)\, dx}{2^{3/2}} \sqrt{\ell(\ell+1)}$$

([77] Proposition 1.4, starting from (121)). The analogous result holds for the variance by Proposition 3.1.7. However, before passing to the limit to obtain Theorem 0.0.2, we need the two propositions below, where $\tilde{\varphi} := \varphi_\ell^i \circ \exp$.

Proposition 3.1.1. *Under the previous notation, as $\ell \to \infty$, the variance* $\mathrm{Var}(\mathscr{Z}^{\varphi^i_\ell}(T_\ell))$ *is given by*

$$\mathrm{Var}(\mathscr{Z}^{\varphi^i_\ell}(T_\ell)) = \frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}}{256\pi} \cdot r_\ell^2 \log(r_\ell \ell) + O_{||\tilde{\varphi}^i||_\infty, V(\tilde{\varphi}^i)}(r_\ell^2). \tag{3.1.7}$$

Proposition 3.1.2. *We have that, as $\ell \to \infty$,*

$$\mathbb{E}[\mathscr{Z}^{\varphi^i_\ell}(T_\ell)^2] = O(\ell r_\ell^4 ||\tilde{\varphi}^i||_{L^1(\tilde{B}_1)} ||\tilde{\varphi}^i||_\infty + \frac{1}{\ell} r_\ell^2 ||\tilde{\varphi}^i||_\infty ||\tilde{\varphi}^i||_{L^1(B_1)}).$$

As argued above, the computations of the variance in the previous propositions will follow from the analysis of the integral of the two-point correlation function (see Section 3.5.1) and W^φ; the main contribution will actually be given from points far from the diagonal $x = y$.

The next step will be the derivation of the Central Limit Theorem. To this aim, we will start following a similar argument as in [48]; more precisely we define first as in [48], the sequence of centered random variables ("local sample trispectrum")

$$\mathscr{M}_{\ell,r_\ell} := -\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \int_{B_{r_\ell}} H_4(T_\ell(x))\, dx = -\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} h_{\ell,r_\ell;4} \tag{3.1.8}$$

where for $\ell = 1, 2, \dots$,

$$h_{\ell,r_\ell;4} := \int_{B_{r_\ell}} H_4(T_\ell(x))\, dx.$$

The key idea is to prove the asymptotic full correlation between the "local" nodal length and the "local sample trispectrum":

Proposition 3.1.3. *The correlation between $\mathscr{Z}_{\ell,r_\ell}$ and $\mathscr{M}_{\ell,r_\ell}$ is given by*

$$\mathrm{Corr}(\mathscr{Z}_{\ell,r_\ell}; \mathscr{M}_{\ell,r_\ell}) = 1 + O\left(\frac{1}{\log r_\ell \ell}\right) = 1 + o(1), \tag{3.1.9}$$

in the high energy limit $\ell \to \infty$.

This result requires the evaluation of the variance of $\mathscr{M}_{\ell,r_\ell}$.

Proposition 3.1.4. *The variance of $\mathscr{M}_{\ell,r_\ell}$ is, as $\ell \to \infty$, given by*

$$\mathrm{Var}(\mathscr{M}_{\ell,r_\ell}) = \frac{1}{256} r_\ell^2 \log r_\ell \ell + O(r_\ell^2).$$

The strategy of the proof is the same as for the variance of $\mathscr{Z}_{\ell,r_\ell}$; namely the propositions below are involved.

Proposition 3.1.5. *Defining*

$$\mathscr{M}^{\varphi^i_\ell} := -\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}} \frac{1}{4!} \int_{\mathbb{S}^2} \varphi^i_\ell(y) H_4(T_\ell(y))\, dy; \tag{3.1.10}$$

the variance of $\mathscr{M}^{\varphi^i_\ell}$, as $\ell \to \infty$, is given by

$$\mathrm{Var}(\mathscr{M}^{\varphi^i_\ell}) = \frac{||\tilde{\varphi}^i||_{L^1(\tilde{B}_1)}}{256\pi} r_\ell^2 \log r_\ell \ell + O_{||\tilde{\varphi}^i||_\infty, V(\tilde{\varphi}^i)}(r_\ell^2).$$

Proposition 3.1.6. *We have that, as $\ell \to \infty$,*

$$\mathbb{E}[\mathscr{M}^{\varphi_\ell^i}(T_\ell)^2] = O\left(r_\ell^2 \|\tilde{\varphi}^i\|_\infty \|\tilde{\varphi}^i\|_{L^1(\tilde{B}_1)} + r_\ell^2 \log(r_\ell \ell) \|\tilde{\varphi}^i\|_\infty \|\tilde{\varphi}^i\|_{L^1(\tilde{B}_1)} \right).$$

In view of the orthogonality of the projections, the result in (3.1.9) implies that the fourth chaotic component is the leading term of the chaos expansion of $\mathscr{Z}_{\ell,r_\ell}$ and hence it is enough to study its asymptotic behavior. In particular, exploiting the Stein-Malliavin approach (Section 1.2, see also [57]), it is enough to focus on the behavior of its fourth order cumulant. Here, it is important to note that our argument is quite different from the proof given by [48]; in particular, in the full sphere the behavior of the fourth-order cumulant was already established by means of Clebsch-Gordan coefficients: the latter cannot be used here due to the lack of analogous explicit results on subdomains. Hence, we derive efficient bounds by a careful exploitation of Hilb's asymptotics for powers of Legendre polynomials.

3.2 Auxiliary functions

Here and in the rest of the paper we will denote with $B_r \subset \mathbb{S}^2$ the ball of radius r, $0 < r < \pi$ centered in N, while with $\tilde{B}_r$ the disc of radius r in $\mathbb{R}^2$.

We introduce the auxiliary functions, announced in Section 3.1, involved into the proofs of the main results of this chapter.

3.2.1 Approximation with continuously differentiable functions

Let us consider the indicator function $1_{B_{r_\ell}}$; since it belongs to the space $BV(\mathbb{S}^2) \cap L^\infty(\mathbb{S}^2)$, to make some computations easier, it is more convenient to deal with continuously differentiable functions. In order to control the error term of the variance for the approximating functions (and thus pass to the limit), it is sufficient that φ_ℓ^i is uniformly bounded and with uniformly bounded variation (see [77]) and to prove that the same conditions still hold for $\tilde{\varphi}_\ell^i$, obtained through the composition with the exponential map. In [77] the existence of such a sequence was established. So, let $\{\varphi_\ell^i\}_i$ be a sequence of C^∞ functions such that

$$\begin{aligned} \varphi_\ell^i &\to 1_{B_{r_\ell}} \text{ in } L^1(\mathbb{S}^2), \\ V(\varphi_\ell^i) &\to V(1_{B_{r_\ell}}) \\ \text{and } \|\varphi_\ell^i\|_\infty &\leq \|1_{B_{r_\ell}}\|_\infty. \end{aligned} \tag{3.2.1}$$

Our goal is to check whether analogous conditions still hold for $\tilde{\varphi}_\ell^i = \varphi_\ell^i \circ \exp$, defined on $\mathbb{R}^2$. To simplify the notation we set $\tilde{\varphi}^i(x) := \tilde{\varphi}_\ell^i(r_\ell x), x \in \mathbb{R}^2$. Note that, since φ_ℓ^i has support on $\mathbb{S}^2$, which is compact, it follows that $\tilde{\varphi}^i$ has compact support in $\tilde{B}_1$. Hence, we prove the lemma below.

Lemma 3.2.1. *Let $\tilde{\varphi}^i(x) := \tilde{\varphi}_\ell(r_\ell x), x \in \mathbb{R}^2$, where $\tilde{\varphi}_\ell = \varphi_\ell^i \circ \exp$, and $\{\varphi_\ell^i\}_i$ the sequence satisfying (3.2.1). Then, $\tilde{\varphi}_\ell^i : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable functions such that, as*

$i \to \infty$,

$$\begin{aligned} &\tilde{\varphi}^i \to 1_{\tilde{B}_1} \ in\ L^1(\mathbb{R}^2) \\ &V(\tilde{\varphi}^i) \to V(1_{\tilde{B}_1}) \\ &||\tilde{\varphi}^i||_\infty \le ||1_{\tilde{B}_1}||_\infty. \end{aligned} \tag{3.2.2}$$

Proof. The first result in (3.2.2) is easily obtained, indeed

$$||\tilde{\varphi}^i - 1_{\tilde{B}_1}||_{L^1(\mathbb{R}^2)} = ||\varphi_\ell^i \circ \exp - 1_{B_{r_\ell}} \circ \exp||_{L^1(\mathbb{R}^2)} \le C||\varphi_\ell^i - 1_{B_{r_\ell}}||_{L^1(\mathbb{S}^2)}.$$

Concerning the second part of the statement, since the support of $\tilde{\varphi}$ is $\tilde{B}_1$, by the definition of total variation (see [77]), we have that

$$\begin{aligned} |V(\tilde{\varphi}^i) - V(1_{\tilde{B}_1})| &= \left| \int_{\tilde{B}_1} ||\nabla \tilde{\varphi}^i(x)||\, dx - \sup_{\tilde{g}} \int_{\tilde{B}_1} \tilde{\varphi}^i(x) \operatorname{div} \tilde{g}(x)\, dx \right| \\ &= \left| \int_{\mathbb{R}^2} ||\nabla \tilde{\varphi}^i(x)||\, dx - \sup_{\tilde{g}} \int_{\tilde{B}_1} \tilde{\varphi}^i(x) \operatorname{div} \tilde{g}(x)\, dx \right| \\ &= \left| \int_{\mathbb{R}^2} ||\nabla \tilde{\varphi}_\ell^i(\exp(x))||\, d(\exp x) - \sup_{\tilde{g}} \int_{\tilde{B}_1} \tilde{\varphi}_\ell^i(\exp(x)) \operatorname{div} \tilde{g}(\exp(x))\, d(\exp x) \right| \\ &\le \left| \int_{\mathbb{S}^2} ||\nabla \varphi_\ell^i(x)||\, dx - \sup_{g} \int_{\mathbb{S}^2} \varphi_\ell^i(x) \operatorname{div} g(x)\, dx \right| \\ &= |V(\varphi_\ell^i) - V(1_{B_{r_\ell}})| \to 0, \end{aligned} \tag{3.2.3}$$

$\forall g \in C_c^1(\tilde{B}_1, T\tilde{B}_1)$ continuously differentiable compactly supported vector fields with $|g(x)| \le 1$, for all $x \in \tilde{B}_1$. Finally,

$$\begin{aligned} ||\tilde{\varphi}^i||_\infty &= \sup_{x\in\tilde{B}_1} |\tilde{\varphi}^i(x)| = \sup_{x\in\tilde{B}_1} |\tilde{\varphi}_\ell^i(r_\ell x)| = \sup_{x\in\mathbb{R}^2} |\tilde{\varphi}_\ell^i(r_\ell x)| = \sup_{x\in\mathbb{R}^2} |\tilde{\varphi}_\ell^i(\exp(r_\ell x))| \\ &= \sup_{x\in\mathbb{S}^2} |\varphi_\ell^i(x)| \le \sup_{x\in\mathbb{S}^2} |1_{B_{r_\ell}}(x)| = 1 = \sup_{x\in\tilde{B}_1} |1_{\tilde{B}_1}(x)|. \end{aligned} \tag{3.2.4}$$

□

3.2.2 W^{φ_ℓ} and its properties

Let $\varphi_\ell : \mathbb{S}^2 \to \mathbb{R}$ be the indicator function $1_{B_{r_\ell}}$, $\forall \ell$. We denote $W_{r_\ell}(\cdot)$ the function defined in (3.1.3) with this choice of φ_ℓ and $\tilde{W}_1(\cdot)$ the one in (3.1.4). As we have already stated in Section 3.1, it is easy to establish the following asymptotic geometric relation between W_{r_ℓ} and $\tilde{W}_1$.

Lemma 3.2.2. *Let $W_{r_\ell}(\cdot)$ and $\tilde{W}_1(\cdot)$ as in (3.1.3) and (3.1.4), respectively; then,*

$$W_{r_\ell}(\rho) = r_\ell^3 \tilde{W}_1(\rho \frac{1}{r_\ell})(1 + O(\rho^2)) \tag{3.2.5}$$

as $r_\ell \to 0$ uniformly for $\rho \in [0, 2r_\ell]$.

Proof. We set $D_\rho := \{x \in B_{r_\ell} : B_\rho(x) \subset B_{r_\ell}\}$; then

$$W_{r_\ell}(\rho) = \frac{1}{8\pi^2}\int_{D_\rho} \text{len}\{y \in B_{r_\ell} : d(x,y) = \rho\}\, dx + \frac{1}{8\pi^2}\int_{B_{r_\ell}-D_\rho} \text{len}\{y \in B_{r_\ell} : d(x,y) = \rho\}\, dx;$$

we denote

$$A := \frac{1}{8\pi^2}\int_{D_\rho} \text{len}\{y \in B_{r_\ell} : d(x,y) = \rho\}\, dx$$

and

$$B := \frac{1}{8\pi^2}\int_{B_{r_\ell}-D_\rho} \text{len}\{y \in B_{r_\ell} : d(x,y) = \rho\}\, dx.$$

For each point x in D_ρ we have that

$$\begin{aligned} A &= \frac{1}{8\pi^2}\int_{D_\rho} \text{len}\{y \in B_{r_\ell} : d(x,y) = \rho\}\, dx = \frac{1}{8\pi^2} 2\pi \sin\rho |D_\rho| \\ &= \frac{1}{8\pi^2} 2\pi \sin\rho \cdot 2\pi(1-\cos(r_\ell - \rho)). \end{aligned} \tag{3.2.6}$$

Let us define also $\tilde{D}_{\rho/r_\ell} := \{x \in \tilde{B}_1 : \tilde{B}_{\rho/r_\ell}(x) \subset \tilde{B}_1\}$; likewise, we write

$$\tilde{W}_1(\rho\frac{1}{r_\ell}) = \tilde{A} + \tilde{B};$$

where

$$\tilde{A} := \frac{1}{8\pi^2}\int_{\tilde{D}_{\rho/r_\ell}} \text{len}\{y \in \tilde{B}_1 : d(x,y) = \frac{\rho}{r_\ell}\}\, dx$$

and

$$\tilde{B} := \frac{1}{8\pi^2}\int_{\tilde{B}_1-\tilde{D}_{\rho/r_\ell}} \text{len}\{y \in \tilde{B}_1 : d(x,y) = \frac{\rho}{r_\ell}\}\, dx.$$

Note that

$$\tilde{A} = \frac{1}{8\pi^2} 2\pi \frac{\rho}{r_\ell} |\tilde{D}_{\rho/r_\ell}| = \frac{1}{8\pi^2} 2\pi \frac{\rho}{r_\ell} \pi\left(1 - \frac{\rho}{r_\ell}\right)^2;$$

then, using the Taylor expansion of the sine and cosine as $r_\ell \to 0$ (and so $\rho \to 0$), we get

$$\begin{aligned} A &= \frac{1}{8\pi^2} 2\pi\rho(1+O(\rho^2))\pi \cdot \left(r_\ell - \rho\right)^2 (1+O(\rho^2)+O(r_\ell^2)) = \\ &= \frac{1}{8\pi^2} 2\pi \frac{\rho}{r_\ell} \cdot \pi\left(1 - \frac{\rho}{r_\ell}^2\right) r_\ell^3 (1+O(\rho^2))(1+O(\rho^2)+O(r_\ell^2)) = \\ &= r_\ell^3 \tilde{A}(1+O(\rho^2)+O(r_\ell^2)). \end{aligned} \tag{3.2.7}$$

Now we prove that

$$|B - \tilde{B}| \ll O(r_\ell^4 + \rho^4)$$

and thus (3.2.5) follows. So,

$$\begin{aligned} &|B-\tilde{B}| \le \\ &\le \left| \frac{1}{8\pi^2}\int_{B_{r_\ell}-D_\rho} \text{len}\{y \in B_{r_\ell} : d(x,y) = \rho\}\, dx - \frac{1}{8\pi^2}\int_{\tilde{B}_1-\tilde{D}_{\rho/r_\ell}} \text{len}\{y \in \tilde{B}_1 : d(x,y) = \frac{\rho}{r_\ell}\}\, dx \right| \\ &= \left| \frac{1}{8\pi^2}\int_{B_{r_\ell}-D_\rho} \text{len}\{y \in B_{r_\ell} : d(x,y) = \rho\}\, dx - \frac{1}{8\pi^2}\int_{\tilde{B}_{r_\ell}-\tilde{D}_\rho} \text{len}\{y \in \tilde{B}_{r_\ell} : d(x,y) = \rho\}\, dx \right|, \end{aligned} \tag{3.2.8}$$

where $\tilde{B}_{r_\ell} \subset \mathbb{R}^2$ is the disc of radius r_ℓ and $\tilde{D}_\rho := \{x \in \tilde{B}_{r_\ell} : \tilde{B}_\rho(x) \subset \tilde{B}_{r_\ell}\}$; then (3.2.8) results to be

$$\ll 2\pi(1-\cos r_\ell) - 2\pi(1-\cos(r_\ell-\rho)) - (\pi r_\ell^2 - \pi(r_\ell-\rho)^2) \ll O(r_\ell^4) + O(\rho^4).$$

□

Let us consider the sequence φ_ℓ^i satisfying (3.2.1), then relation (3.2.5) holds for $W^{\varphi_\ell^i}$ and $\tilde{W}^{\tilde{\varphi}^i}$; actually,

$$\begin{aligned}|W^{\varphi_\ell^i}(\rho) - r_\ell^3 \tilde{W}^{\tilde{\varphi}^i}(\rho\frac{1}{r_\ell})(1+O(\rho^2))| &\le |W^{\varphi_\ell^i}(\rho) - W_{r_\ell}(\rho)| + \\ &+ |W_{r_\ell}(\rho) - r_\ell^3 \tilde{W}_1(\rho\frac{1}{r_\ell})(1+O(\rho^2))| \\ &+ |r_\ell^3 \tilde{W}_1(\rho\frac{1}{r_\ell})(1+O(\rho^2)) - r_\ell^3 \tilde{W}^{\tilde{\varphi}^i}(\rho\frac{1}{r_\ell})(1+O(\rho^2))|\end{aligned} \tag{3.2.9}$$

and the former and the latter quantities of (3.2.9) go to zero for the L^1-convergence of $\varphi_\ell^i \to 1_{B_{r_\ell}}$ and $\tilde{\varphi}^i \to 1_{\tilde{B}_1}$; in fact

$$\begin{aligned}|W^{\varphi_\ell^i}(\rho) - W_{r_\ell}(\rho)| &\le \int_{\mathbb{S}^2\times\mathbb{S}^2} |\varphi_\ell^i(x)\varphi_\ell^i(y) - 1_{B_{r_\ell}}(x)1_{B_{r_\ell}}(y)|\, dx dy \\ &\le \int_{\mathbb{S}^2\times\mathbb{S}^2} |\varphi_\ell^i(x)||\varphi_\ell^i(y) - 1_{B_{r_\ell}}(y)|\, dx dy + \\ &+ \int_{\mathbb{S}^2\times\mathbb{S}^2} |1_{B_{r_\ell}}(y)||\varphi_\ell^i(x) - 1_{B_{r_\ell}}(x)|\, dx dy \to 0.\end{aligned} \tag{3.2.10}$$

So, we conclude that, as $\ell \to \infty$,

$$W^{\varphi_\ell^i}(\rho) = r_\ell^3 \tilde{W}^{\tilde{\varphi}^i}(\rho\frac{1}{r_\ell})(1+O(\rho^2)). \tag{3.2.11}$$

Furthermore, we can get further informations on $\tilde{W}^{\tilde{\varphi}^i}$, i.e., using polar coordinates centered in x, for each $x \in \mathbb{R}^2$, for y (i.e. $y = (y_1, y_2) \to (\zeta, \phi)$ with $\zeta = \rho$ and $\phi = \arctan \frac{y_2-x_2}{y_1-x_1}$), we write

$$\tilde{W}^{\tilde{\varphi}^i}(\rho) = \frac{1}{8\pi^2}\int_{d(x,y)=\rho} \tilde{\varphi}^i(x)\tilde{\varphi}^i(y)\, dx dy = \frac{\rho}{8\pi^2}\int_{\mathbb{R}^2} \tilde{\varphi}^i(x) \int_0^{2\pi} \tilde{\varphi}_x^i(\rho\cos\phi, \rho\sin\phi)\, d\phi\, dx \tag{3.2.12}$$

for a suitably defined function $\tilde{\varphi}_x : \mathbb{R}^2 \to \{0,1\}$; and then, defining

$$\tilde{W}_0^{\tilde{\varphi}^i}(\rho) := \int_{\mathbb{R}^2} \tilde{\varphi}^i(x) \int_0^{2\pi} \tilde{\varphi}_x^i(\rho\cos\phi, \rho\sin\phi)\, d\phi\, dx \tag{3.2.13}$$

we have that

$$\tilde{W}^{\tilde{\varphi}^i}(\rho) = \frac{\rho}{8\pi^2}\tilde{W}_0^{\tilde{\varphi}^i}(\rho). \tag{3.2.14}$$

Note that $\tilde{W}_0^{\tilde{\varphi}_i}(\rho)$ is bounded by

$$|\tilde{W}_0^{\tilde{\varphi}_i}(\rho)| \le 2\pi||\tilde{\varphi}^i||_\infty ||\tilde{\varphi}^i||_{L^1(B_1)} \le 2\pi^2||\tilde{\varphi}^i||_\infty^2, \tag{3.2.15}$$

moreover in zero, it is equal to

$$\tilde{W}_0^{\tilde{\varphi}^i}(0) = 2\pi ||\tilde{\varphi}^i||^2_{L^2(B_1)} \tag{3.2.16}$$

and at last, the derivative is uniformly bounded by

$$|\tilde{W}_0^{\tilde{\varphi}^i\prime}(\rho)| \le 2\pi ||\tilde{\varphi}^i||_\infty V(\tilde{\varphi}^i); \tag{3.2.17}$$

indeed, we can exchange the derivative and the integral to obtain

$$\begin{aligned} \left|\frac{\partial}{\partial\rho}\tilde{W}_0^{\tilde{\varphi}^i}(\rho)\right| &\le \int_{\mathbb{R}^2}\left|\tilde{\varphi}^i(x)\int_0^{2\pi}\frac{\partial}{\partial\rho}\tilde{\varphi}^i_x(\rho\cos\phi,\rho\sin\phi)\,d\phi\right|dx \\ &\le 2\pi\int_{B_1}|\tilde{\varphi}^i(x)|||\nabla\tilde{\varphi}^i(x)||\,dx = 2\pi||\tilde{\varphi}^i||_\infty V(\tilde{\varphi}^i). \end{aligned} \tag{3.2.18}$$

From now on $\{\varphi^i_\ell\}_i$ will denote a sequence satisfying (3.2.1) and $\{\tilde{\varphi}^i\}_i$ the one satisfying (3.2.2).

3.3 Proof of Theorem 0.0.2 and Theorem 0.0.3

3.3.1 Proof of Theorem 0.0.2 (Asymptotic for the variance)

In this subsection we prove Theorem 0.0.2, assuming the validity of Proposition 3.1.1 and Proposition 3.1.2. As we have already stated in Section 3.1, we apply an approximation argument.

Proof of Theorem 0.0.2 assuming Proposition 3.1.1 and Proposition 3.1.2 . Let $\varphi^i_\ell \in C^\infty$ be a sequence of smooth functions satisfying (3.2.1) and let $\tilde{\varphi}^i = \varphi^i_\ell \circ \exp$. Proposition 3.1.1 states that

$$\mathrm{Var}(\mathscr{Z}^{\varphi^i_\ell}(T_\ell)) = \frac{||\tilde{\varphi}^i||_{L^2(\tilde{B}_1)}}{256\pi}\cdot r_\ell^2\log(r_\ell\ell) + O_{||\tilde{\varphi}^i||_{\infty,V(\tilde{\varphi}^i)}}(r_\ell^2);$$

since $\tilde{\varphi}^i$ and $1_{\tilde{B}_1}$ are uniformly bounded, $L^1(\mathbb{R}^2)$-convergence implies $L^2(\mathbb{R}^2)$-convergence,

$$||\tilde{\varphi}^i||_{L^2(\mathbb{R}^2)} \to ||1_{\tilde{B}_1}||_{L^2(\mathbb{R}^2)} = \sqrt{\pi}$$

and it remains to prove that

$$\mathrm{Var}(\mathscr{Z}^{\varphi^i_\ell}(T_\ell)) \to \mathrm{Var}(\mathscr{Z}_{\ell,r_\ell}).$$

Applying Proposition 3.1.2 to the difference $\varphi^i_\ell - 1_{B_{r_\ell}}$, we have that

$$\begin{aligned} \mathbb{E}[(\mathscr{Z}^{\varphi^i_\ell}(T_\ell) - \mathscr{Z}_{\ell,r_\ell})^2] &= \mathbb{E}[(\mathscr{Z}^{\varphi^i_\ell - 1_{B_{r_\ell}}})^2] \\ &= O(\ell^2 r_\ell^4 ||\tilde{\varphi}^i - 1_{\tilde{B}_1}||_{L^1(\tilde{B}_1)} + r_\ell^2||\varphi^i_\ell - 1_{\tilde{B}_1}||_\infty ||\varphi^i_\ell - 1_{\tilde{B}_1}||_{L^1(\tilde{B}_1)}) \to 0, \end{aligned}$$

as $i \to \infty$, hence

$$\begin{aligned} |\mathrm{Var}(\mathscr{Z}^{\varphi^i_\ell}(T_\ell)) - \mathrm{Var}(\mathscr{Z}_{\ell,r_\ell})| &\le |\mathbb{E}[(\mathscr{Z}^{\varphi^i_\ell}(T_\ell) - \mathscr{Z}_{\ell,r_\ell})^2]| + 2|\mathbb{E}[(\mathscr{Z}^{\varphi^i_\ell}(T_\ell) - \mathscr{Z}_{\ell,r_\ell})\mathscr{Z}_{\ell,r_\ell}]| \\ &\quad + |[\mathbb{E}\mathscr{Z}^{\varphi^i_\ell}(T_\ell)]^2 - [\mathbb{E}(\mathscr{Z}_{\ell,r_\ell})]^2| \to 0. \end{aligned}$$

To conclude, taking the limit $i \to \infty$ in (3.1.7), we obtain that the variance $\mathrm{Var}(\mathscr{Z}_{\ell,r_\ell})$ is given by

$$\mathrm{Var}(\mathscr{Z}_{\ell,r_\ell}) = \frac{1}{256} \cdot r_\ell^2 \log(r_\ell \ell) + O(r_\ell^2),$$

as $\ell \to \infty$. □

Now we prove Proposition 3.1.1 and Proposition 3.1.2.

Proof of Proposition 3.1.1. In [77] in the proof of Theorem 1.4, it is shown that for functions in $C^1(\mathbb{S}^2)$, it is possible to write

$$\mathbb{E}[(\mathscr{Z}^{\varphi_\ell^i}(T_\ell))^2] = \int_{\mathbb{S}^2 \times \mathbb{S}^2} \varphi_\ell^i(x)\varphi_\ell^i(y)\tilde{K}_\ell(x,y)\,dx\,dy,$$

where $\tilde{K}_\ell(x,y) = \tilde{K}_\ell(d(x,y))$ is the two-point correlation function (see Section 3.5 and [77]). Employing Fubini, we get that

$$\mathbb{E}[(\mathscr{Z}^{\varphi_\ell^i}(T_\ell))^2] = 8\pi^2 \int_0^{2r_\ell} \tilde{K}_\ell(\rho) W^{\varphi_\ell^i}(\rho)\,d\rho;$$

with

$$\tilde{K}_\ell(\rho) = \tilde{K}_\ell(x,y),$$

$x, y \in \mathbb{S}^2$ being any pair of points with $d(x,y) = \rho$. Denoting $L := \ell + \frac{1}{2}$ and changing coordinates $\rho = \frac{\psi}{L}$, we have that

$$\mathbb{E}[(\mathscr{Z}^{\varphi_\ell^i}(T_\ell))^2] = \frac{2\pi m(\mathbb{S}^2)}{L} \int_0^{2r_\ell L} \tilde{K}_\ell\left(\frac{\psi}{L}\right) W^{\varphi_\ell^i}\left(\frac{\psi}{L}\right) d\psi;$$

setting $\tilde{K}_\ell\left(\frac{\psi}{L}\right) := \frac{\ell(\ell+1)}{2} K_\ell(\psi)$, we obtain that

$$\mathbb{E}[(\mathscr{Z}^{\varphi_\ell^i}(T_\ell))^2] = \frac{\pi m(\mathbb{S}^2)}{L} \ell(\ell+1) \int_0^{2r_\ell L} K_\ell(\psi) W^{\varphi_\ell^i}\left(\frac{\psi}{L}\right) d\psi$$

and hence the variance is given by

$$\mathrm{Var}(\mathscr{Z}^{\varphi_\ell^i}(T_\ell)) = \frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L} \int_0^{2r_\ell L} (K_\ell(\psi) - \frac{1}{4}) W^{\varphi_\ell^i}(\frac{\psi}{L})\,d\psi. \tag{3.3.1}$$

Substituting (3.2.11), i.e.

$$W^{\varphi_\ell^i}(\frac{\psi}{L}) = r_\ell^3 \tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{L r_\ell})(1 + O(\frac{\psi^2}{L^2}))$$

in (3.3.1), it follows that the variance is equal to

$$\begin{aligned} \mathrm{Var}((\mathscr{Z}^{\varphi_\ell^i}(T_\ell))) = &\frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L} r_\ell^3 \int_0^{2r_\ell L} (K_\ell(\psi) - \frac{1}{4}) \tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{r_\ell L})\,d\psi \\ &+ O\Big(\frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L} r_\ell^3 \int_0^{2r_\ell L} \tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{r_\ell L})(K_\ell(\psi) - \frac{1}{4})\frac{\psi^2}{L^2}\,d\psi\Big) \end{aligned} \tag{3.3.2}$$

and splitting the interval of the integral in $[0,C]$ and $[C,2r_\ell L]$ with $C>0$, it results that

$$\begin{aligned}\mathrm{Var}[(\mathscr{Z}^{\varphi^i_\ell}(T_\ell))^2] =& \frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L} r_\ell^3 \int_0^C (K_\ell(\psi)-\frac{1}{4})\tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{r_\ell L})\,d\psi \\ &+ \frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L} r_\ell^3 \int_C^{2r_\ell L} (K_\ell(\psi)-\frac{1}{4})\tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})\,d\psi \\ &+ O\Big(\frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L} r_\ell^3 \int_0^C (K_\ell(\psi)-\frac{1}{4})\frac{\psi^2}{L^2}\tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{r_\ell L})\,d\psi\Big) \\ &+ O\Big(\frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L} r_\ell^3 \int_C^{2r_\ell L} (K_\ell(\psi)-\frac{1}{4})\frac{\psi^2}{L^2}\tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{r_\ell L})\,d\psi\Big).\end{aligned} \tag{3.3.3}$$

For $0<\psi<C$ we may bound K_ℓ as

$$|K_\ell(\psi)| = O\Big(\frac{1}{\psi}\Big), \tag{3.3.4}$$

([77] equation (98)) and in view of (3.2.16) and (3.2.15) we get that the third term in (3.3.3) is

$$O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L} r_\ell^3 \int_0^C \Big|(K_\ell(\psi)-\frac{1}{4})\frac{\psi^3}{L^3 r_\ell}\Big|\,d\psi\Big) = O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{\ell^2}\int_0^C \Big|\frac{1}{\psi}-\frac{1}{4}\Big|\psi^3\,d\psi\Big)$$
$$= O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{\ell^2}\Big);$$

while the first integral in (3.3.3), for (3.2.14) and (3.2.16), is equal to

$$\frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L} r_\ell^3 \int_0^C (K_\ell(\psi)-\frac{1}{4})\tilde{W}^{\tilde{\varphi}^i}\Big(\frac{\psi}{Lr_\ell}\Big)\,d\psi = O\Big(\frac{\ell^2 r_\ell^3}{\ell}\int_0^C \Big|\frac{1}{\psi}-\frac{1}{4}\Big|\frac{\psi}{Lr_\ell}2\pi^2||\tilde{\varphi}^i||_\infty^2\,d\psi\Big)$$
$$= O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2).$$

To compute the contribution given by the points in $[C,2r_\ell L]$, we set

$$I_1 := \frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L} r_\ell^3 \int_C^{2r_\ell L} (K_\ell(\psi)-\frac{1}{4})\tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{r_\ell L})\,d\psi \tag{3.3.5}$$

and

$$I_2 := O\Big(\frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L} r_\ell^3 \int_C^{2r_\ell L} (K_\ell(\psi)-\frac{1}{4})\frac{\psi^2}{L^2}\tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{r_\ell L})\,d\psi\Big). \tag{3.3.6}$$

Concerning I_1, from (3.2.14) it follows that

$$I_1 := \frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{8\pi^2 L} r_\ell^3 \int_C^{2r_\ell L} (K_\ell(\psi)-\frac{1}{4})\frac{\psi}{Lr_\ell}\tilde{W}_0^{\tilde{\varphi}^i}\Big(\frac{\psi}{Lr_\ell}\Big)\,d\psi, \tag{3.3.7}$$

and Taylor expansion implies that

$$\tilde{W}_0^{\tilde{\varphi}^i}\Big(\frac{\psi}{Lr_\ell}\Big) = 2\pi||\tilde{\varphi}^i||^2_{L^2(B_1)} + O_{||\tilde{\varphi}^i||_\infty, V(\tilde{\varphi}^i)}\Big(\frac{\psi}{Lr_\ell}\Big), \tag{3.3.8}$$

thus, substituting (3.3.8) in (3.3.7), we have that

$$I_1 := \underbrace{\frac{\pi||\tilde{\varphi}^i||_{L^1(B_1)}\ell(\ell+1)}{L^2} r_\ell^2 \int_C^{2r_\ell L} (K_\ell(\psi)-\frac{1}{4})\psi\,d\psi}_{(a)} +$$

$$\underbrace{+O_{||\tilde{\varphi}^i||_\infty, V(\tilde{\varphi}^i)}\left(\frac{m(\mathbb{S}^2)\pi\ell(\ell+1)}{8\pi^2L^2}r_\ell^2\int_C^{2r_\ell L}(K_\ell(\psi)-\frac{1}{4})\frac{\psi^2}{Lr_\ell}\,d\psi\right)}_{(b)}.$$

First we compute term (a); we replace $K_\ell(\psi)$ with its asymptotic behavior given in (3.5.4) (see [77]); the second summand of (3.5.4) gives the main contribution, i.e.,

$$\frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}\pi\ell(\ell+1)}{L^2}r_\ell^2\int_C^{2r_\ell L}\frac{1}{256\pi^2\ell}\frac{1}{\sin\psi/L}\,d\psi. \tag{3.3.9}$$

Applying the Taylor expansion to the cosecant

$$\frac{1}{\sin t}=\frac{1}{t}+O(t)\ 0<|t|<\pi, \tag{3.3.10}$$

(3.3.9) is equal to

$$\begin{aligned}
&\frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}\ell(\ell+1)}{256\pi\ell L^2}r_\ell^2\int_C^{2r_\ell L}\frac{L}{\psi}+O\left(\frac{\psi}{L}\right)d\psi=\\
&=\frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}}{256\pi}r_\ell^2\log(r_\ell L)+O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2)+O_{||\tilde{\varphi}^i||_\infty}\left(\frac{r_\ell^2\ell^2}{\ell^4}\int_C^{2r_\ell L}\psi\,d\psi\right)=\\
&=\frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}}{256\pi}r_\ell^2\log(r_\ell\ell)+O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2)+O_{||\tilde{\varphi}^i||_\infty}\left(r_\ell^4+\frac{r_\ell^2}{\ell^2}\right)=\\
&=\frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}}{256\pi}r_\ell^2\log(r_\ell\ell)+O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2)+O_{||\tilde{\varphi}^i||_\infty}(r_\ell^4).
\end{aligned} \tag{3.3.11}$$

The first term of (3.5.4) leads to

$$\frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}\pi\ell(\ell+1)}{L^2}r_\ell^2\int_C^{2r_\ell L}\frac{1}{2\pi\ell}\frac{\sin(2\psi)}{\sin\psi/L}\psi\,d\psi,$$

which is, for (3.3.10),

$$O_{||\tilde{\varphi}^i||_\infty}\left(\frac{r_\ell^2}{\ell}\int_C^{2r_\ell L}(\frac{L}{\psi}+O_{||\tilde{\varphi}^i||_\infty}(\psi/L))\sin(2\psi)\psi\,d\psi\right); \tag{3.3.12}$$

since the sine is bounded, (3.3.12) is

$$\begin{aligned}
&O_{||\tilde{\varphi}^i||_\infty}\left(r_\ell^2\int_C^{2r_\ell L}\sin 2\psi\,d\psi\right)+O_{||\tilde{\varphi}^i||_\infty}\left(\frac{r_\ell^2}{\ell^2}\int_C^{2r_\ell L}\psi^2\sin(2\psi)\,d\psi\right)=\\
&=O_{||\tilde{\varphi}^i||_\infty}\left(r_\ell^2\int_C^{2r_\ell L}\sin 2\psi\,d\psi\right)+O_{||\tilde{\varphi}^i||_\infty}\left(\frac{r_\ell^2}{\ell^2}(r_\ell^3L^3-C^3)\right)=O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2).
\end{aligned} \tag{3.3.13}$$

The error term given by the third term of (3.5.4) is

$$\begin{aligned}
&\frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}\pi\ell(\ell+1)}{L^2} r_\ell^2 \int_C^{2r_\ell L} \frac{9}{32\pi\ell\psi}\frac{\cos(2\psi)}{\sin\psi/L}\psi\, d\psi \\
&= O_{||\tilde{\varphi}^i||_\infty}\left(\frac{r_\ell^2}{\ell}\int_C^{2r_\ell L}\cos(2\psi)(\frac{L}{\psi}+O(\psi/L))\,d\psi\right) \\
&= O_{||\tilde{\varphi}^i||_\infty}\left(r_\ell^2\int_C^{2r_\ell L}\frac{\sin(2\psi)}{\psi}\,d\psi\right) + O_{||\tilde{\varphi}^i||_\infty}\left(\frac{r_\ell^2}{\ell^2}\int_C^{2r_\ell L}\psi\,d\psi\right) \\
&= O_{||\tilde{\varphi}^i||_\infty}\left(r_\ell^2\left[\frac{\sin 2\psi}{\psi}\Big|_C^{2r_\ell L} + \int_C^{2r_\ell L}\frac{\sin(2\psi)}{\psi^2}\right]\right) + O_{||\tilde{\varphi}^i||_\infty}\left(\frac{r_\ell^2}{\ell^2}+r_\ell^4\right) = O_{||\tilde{\varphi}^i||_\infty}(\frac{r_\ell}{\ell}+r_\ell^2) \\
&= O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2)
\end{aligned} \tag{3.3.14}$$

and the one obtained by the fourth is

$$\begin{aligned}
&\frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}\pi\ell(\ell+1)}{L^2} r_\ell^2 \int_C^{2r_\ell L} \frac{\frac{27}{64}\sin 2\psi - \frac{75}{256}\cos 4\psi}{\pi^2\ell\psi\sin\psi/L}\psi\, d\psi \\
&= O_{||\tilde{\varphi}^i||_\infty}\left(\frac{r_\ell^2}{\ell}\int_C^{2r_\ell L}\left(\frac{27}{64}\sin(2\psi)-\frac{75}{256}\cos(4\psi)\right)(\frac{L}{\psi}+O(\psi/L))\,d\psi\right)
\end{aligned} \tag{3.3.15}$$

and with the same computations as in (3.3.14), (3.3.15) reduces to be

$$O_{||\tilde{\varphi}^i||_\infty}(\frac{r_\ell}{\ell}+r_\ell^2) = O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2).$$

Regarding the contribution of the fifth term of (3.5.4), we get

$$\begin{aligned}
&\frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}\pi\ell(\ell+1)}{L^2} r_\ell^2 \int_C^{2r_\ell L} O\left(\frac{1}{\psi^3}+\frac{1}{\ell\psi}\right)\psi\, d\psi = O_{||\tilde{\varphi}^i||_\infty}\left(r_\ell^2\left(\frac{1}{\psi}\Big|_C^{2r_\ell L} + \frac{1}{\ell}(2r_\ell L - C)\right)\right) = \\
&O_{||\tilde{\varphi}^i||_\infty}\left(r_\ell^2(\frac{1}{2r_\ell L}-\frac{1}{C}) + r_\ell^3 - \frac{r_\ell^2}{\ell}\right) = O_{||\tilde{\varphi}^i||_\infty}\left(\frac{r_\ell}{L}+r_\ell^2+r_\ell^3+\frac{r_\ell^2}{\ell}\right) = O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2).
\end{aligned} \tag{3.3.16}$$

Finally, integral (a) is given by

$$(a) = \frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}}{256} r_\ell^2 \log r_\ell L + O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2).$$

Now, look at the error term in (b); it is equal to

$$\begin{aligned}
(b) &= O_{||\tilde{\varphi}^i||_\infty, V(\tilde{\varphi}^i)}\left(\frac{m(\mathbb{S}^2)\pi\ell(\ell+1)}{8\pi^2L^2} r_\ell^2 \int_C^{2r_\ell L}(K_\ell(\psi)-\frac{1}{4})\frac{\psi^2}{Lr_\ell}\,d\psi\right) \\
&= O_{||\tilde{\varphi}^i||_\infty, V(\tilde{\varphi}^i)}\left(\frac{r_\ell^2}{\ell r_\ell}\int_C^{2r_\ell L}(K_\ell(\psi)-\frac{1}{4})\psi^2\,d\psi\right);
\end{aligned} \tag{3.3.17}$$

substituting $K_\ell(\psi)$ with (3.5.4), the main contribution comes from the second summand of the expansion, so that we can simplify (b) to be

$$\begin{aligned}(b) &= O_{\|\tilde\varphi^i\|_\infty, V(\tilde\varphi^i)}\Big(\frac{r_\ell}{\ell}\int_C^{2r_\ell L}\frac{1}{256}\frac{1}{\pi^2\ell\sin(\psi/L)\psi}\psi^2\,d\psi\Big)\\ &= O_{\|\tilde\varphi^i\|_\infty, V(\tilde\varphi^i)}\Big(\frac{r_\ell}{\ell}\frac{1}{\ell}\int_C^{2r_\ell L}\frac{\psi}{\sin(\psi/L)}\,d\psi\Big) = O_{\|\tilde\varphi^i\|_\infty, V(\tilde\varphi^i)}\Big(\frac{r_\ell}{\ell}\frac{1}{\ell}\int_C^{2r_\ell L}\psi(\frac{L}{\psi}+O(\frac{\psi}{L}))\,d\psi\Big)\\ &= O_{\|\tilde\varphi^i\|_\infty, V(\tilde\varphi^i)}\Big(\frac{r_\ell}{\ell}(r_\ell L + r_\ell^3 L + \frac{1}{\ell}+\frac{1}{\ell^2})\Big) = O_{\|\tilde\varphi^i\|_\infty, V(\tilde\varphi^i)}(r_\ell^2).\end{aligned} \tag{3.3.18}$$

Let consider I_2 in (3.3.6), thanks to (3.2.14) and (3.2.15), it is

$$= O_{\|\tilde\varphi^i\|_\infty}\Big(\frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L^3}r_\ell^3\int_C^{2r_\ell L}(K_\ell(\psi)-\frac{1}{4})\psi^2\frac{\psi}{Lr_\ell}\,d\psi\Big) \tag{3.3.19}$$

and the expansion (3.5.4) leads (3.3.19) to become

$$\begin{aligned}&O_{\|\tilde\varphi^i\|_\infty}\Big(\frac{r_\ell^2}{\ell^3}\int_C^{2r_\ell L}\frac{\psi^2}{\psi\sin(\psi/L)}\psi\,d\psi\Big) = O_{\|\tilde\varphi^i\|_\infty}\Big(\frac{r_\ell^2}{\ell^3}\int_C^{2r_\ell L}\psi^2\Big(\frac{L}{\psi}+O(\psi/L)\Big)\,d\psi\Big)\\ &= O_{\|\tilde\varphi^i\|_\infty}\Big(\frac{r_\ell^2}{L^2}\int_C^{2r_\ell L}\psi\,d\psi\Big) = O_{\|\tilde\varphi^i\|_\infty}\Big(\frac{r_\ell^2}{L}((r_\ell\ell)^2-C)\Big) = O_{\|\tilde\varphi^i\|_\infty}(r_\ell^4).\end{aligned} \tag{3.3.20}$$

In conclusion, the variance of $\mathscr{Z}^{\varphi_\ell^i}(T_\ell)$ is given by

$$\mathrm{Var}(\mathscr{Z}^{\varphi_\ell^i}(T_\ell)) = \frac{\|\tilde\varphi^i\|^2_{L^2(\tilde B_1)}}{256\pi}r_\ell^2\log(r_\ell\ell) + O_{\|\tilde\varphi^i\|_\infty, V(\tilde\varphi^i)}(r_\ell^2).$$

□

Proof of Proposition 3.1.2. As we did in the proof of Proposition 3.1.1 we write

$$\mathbb{E}[(\mathscr{Z}^{\varphi_\ell^i})^2] = \int_{\mathbb{S}^2\times\mathbb{S}^2}\tilde K_\ell(x,y)\varphi_\ell^i(x)\varphi_\ell^i(y)\,dx\,dy = \frac{\pi m(\mathbb{S}^2)\ell(\ell+1)}{L}\int_0^{2r_\ell L}K_\ell(\psi)W^{\varphi_\ell^i}(\frac{\psi}{L})\,d\psi$$

and we split the integral in two parts

$$\int_0^C K_\ell(\psi)W^{\varphi_\ell^i}(\frac{\psi}{L})\,d\psi + \int_C^{2r_\ell L}K_\ell(\psi)W^{\varphi_\ell^i}(\frac{\psi}{L})\,d\psi,$$

with $C>0$ a constant. For $C<\psi<\frac{\pi L}{2}$, $K_\ell(\psi)$ is bounded by a constant

$$|K_\ell(\psi)| = O_C(1)$$

(see [77] p.35) and for $0<\psi<C$,

$$|K_\ell(\psi)| = O_C(\frac{1}{\psi})$$

(see [77] eq. 98). Then,

$$\begin{aligned}&\Big|\int_C^{2r_\ell L}K_\ell(\psi)W^{\varphi_\ell^i}(\frac{\psi}{L})\,d\psi\Big| \ll_C \int_C^{2r_\ell L}\Big|W^{\varphi_\ell^i}(\frac{\psi}{L})\Big|\,d\psi \le \int_0^{2r_\ell L}\Big|W^{\varphi_\ell^i}(\frac{\psi}{L})\Big|\,d\psi\\ &\le L\int_0^{2r_\ell}|\tilde W^{\varphi_\ell^i}(\rho)|\,d\rho \ll_C Lr_\ell^3\int_0^{2r_\ell}\Big|\tilde W^{\tilde\varphi^i}(\frac{\rho}{r_\ell})\Big|\,d\rho \ll_C r_\ell^3\ell\int_0^{2r_\ell}\Big|\frac{\rho}{r_\ell}\tilde W_0(\frac{\rho}{r_\ell})\Big|\,d\rho\\ &\ll Lr_\ell^4\int_0^2\rho|\tilde W_0(\rho)|\,d\rho \ll_C Lr_\ell^4\|\tilde\varphi^i\|_\infty\|\tilde\varphi^i\|_{L^1(\tilde B_1)}\end{aligned} \tag{3.3.21}$$

and

$$\left|\int_0^C K_\ell(\psi) W^{\varphi_\ell^i}(\frac{\psi}{L})\, d\psi\right| \ll \int_0^C \frac{1}{\psi}\left|W^{\varphi_\ell^i}(\frac{\psi}{L})\right| d\psi \ll r_\ell^3 \int_0^C \frac{1}{\psi}\left|\tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{r_\ell L})\right| d\psi$$
$$= r_\ell^3 \int_0^{C/\ell r_\ell} \frac{1}{\rho}|\tilde{W}^{\tilde{\varphi}^i}(\rho)|\, d\rho \ll_C r_\ell^3 \int_0^{C/(\ell r_\ell)} |\tilde{W}_0(\rho)|\, d\rho \ll_C r_\ell^2 \frac{1}{L}||\tilde{\varphi}^i||_\infty ||\tilde{\varphi}^i||_{L^1(\tilde{B}_1)} \qquad (3.3.22)$$

and the thesis of the proposition follows. □

3.3.2 Proof of Theorem 0.0.3 (Central Limit Theorem)

In this subsection, we finally prove the Central Limit Theorem. We split this section in more subsections to make our argument clearer. Firstly, we show that the nodal length and the integral of $H_4(T_\ell(x))$ in the shrinking ball are fully correlated (Proposition 3.1.3); secondly, we prove that the second chaotic component has actually a smaller order than the fourth chaotic projection (Lemma 3.3.3). Moreover, we compute the fourth cumulant of the "local" sample trispectrum (Lemma 3.3.4) in order to apply Corollary 1.2.4 and to conclude the proof of Theorem 0.0.3.

Correlation between $\mathscr{L}_{\ell,r_\ell}$ and $\mathscr{M}_{\ell,r_\ell}$

Here we show the asymptotic equivalence (in the $L^2(\Omega)$-sense) of the nodal length $\mathscr{L}_{\ell,r_\ell}$ and the trispectrum $\int_{B_{r_\ell}} H_4(T_\ell(x))\, dx$. In [48], the case of the full sphere was considered and it was established that

$$\mathrm{Corr}(\mathscr{L}(T_\ell), \mathscr{M}_\ell) = 1 + O\Big(\frac{1}{\log \ell}\Big),$$

where $\mathscr{M}_\ell$ is the integral of $H_4(T_\ell(x))$ on $\mathbb{S}^2$. In the decreasing domains the full correlation still holds. Let us define the sequence of centered random variables $\mathscr{M}_{\ell,r_\ell}$ as in (3.1.8); we shall need the lemma here below and Proposition 3.1.4 to prove Proposition 3.1.3.

Lemma 3.3.1. *The covariance between $\mathscr{L}_{\ell,r_\ell}$ and $\mathscr{M}_{\ell,r_\ell}$, as $\ell \to \infty$, is given by*

$$\mathrm{Cov}(\mathscr{L}_{\ell,r_\ell}; \mathscr{M}_{\ell,r_\ell}) = \frac{1}{256} r_\ell^2 \log r_\ell \ell + O(r_\ell^2). \qquad (3.3.23)$$

Putting together this lemma, Proposition 3.1.4 and (0.0.6), Proposition 3.1.3 follows easily by definition of correlation, i.e.,

$$\mathrm{Corr}(\mathscr{L}_{\ell,r_\ell}; \mathscr{M}_{\ell,r_\ell}) = \frac{\mathrm{Cov}(\mathscr{L}_{\ell;r_\ell}; \mathscr{M}_{\ell;r_\ell})}{\sqrt{\mathrm{Var}(\mathscr{L}_{\ell;r_\ell})\mathrm{Var}(\mathscr{M}_{\ell;r_\ell})}} = 1 + O\Big(\frac{1}{\log r_\ell \ell}\Big).$$

To prove Lemma 3.3.1 we need the covariance computed in Lemma 3.3.2.

Lemma 3.3.2. *Let φ_ℓ^i a sequence of functions satisfying (3.2.1) and $\mathscr{M}^{\varphi_\ell^i}$ defined as in (3.1.10); then, the covariance between $\mathscr{L}^{\varphi_\ell^i}$ and $\mathscr{M}^{\varphi_\ell^i}$, as $\ell \to \infty$, is given by*

$$\mathrm{Cov}(\mathscr{L}^{\varphi_\ell^i}; \mathscr{M}^{\varphi_\ell^i}) = \frac{||\tilde{\varphi}^i||_{L^1(\tilde{B}_1)}}{256\pi} r_\ell^2 \log r_\ell \ell + O_{||\tilde{\varphi}^i||_\infty, V(\tilde{\varphi}^i)}(r_\ell^2). \qquad (3.3.24)$$

Proof of Lemma 3.3.1 assuming Lemma 3.3.2. The thesis follows immediately, indeed, as $i \to \infty$,

$$|\operatorname{Cov}(\mathscr{Z}^{\varphi_\ell^i}; \mathscr{M}_{\ell,r_\ell}) - \operatorname{Cov}(\mathscr{Z}_{\ell,r_\ell}; \mathscr{M}_{\ell,r_\ell})| = |\operatorname{Cov}(\mathscr{Z}^{\varphi_\ell^i} - \mathscr{Z}_{\ell,r_\ell}; \mathscr{M}_{\ell,r_\ell})| \le$$

$$\le \mathbb{E}[(\mathscr{Z}^{\varphi_\ell^i} - \mathscr{Z}_{\ell,r_\ell})\mathscr{M}_{\ell,r_\ell}] - \mathbb{E}[\mathscr{Z}^{\varphi_\ell^i} - \mathscr{Z}_{\ell,r_\ell}]\mathbb{E}[\mathscr{M}_{\ell,r_\ell}] \to 0.$$

□

Let us now turn our attention to Lemma 3.3.2.

Proof of Lemma 3.3.2. In the same line as in [48], we introduce the two point cross-correlation function

$$J_\ell(\psi;4) = \left[-\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4!}\right] \times \frac{8\pi^2}{L}\mathbb{E}[\psi_\ell(\bar{x},4)H_4(T_\ell(y(\frac{\psi}{L})))],$$

whose expansion can be found in the Appendix (3.5.2) (see [48]) and $\psi_\ell(\bar{x},4)$ is the fourth term of the $L^2(\Omega)$ expansion of the nodal length defined in (1.4.14), namely

$$\mathscr{Z}(T_\ell) - \mathbb{E}[\mathscr{Z}(T_\ell)] = \sum_{q=2}^{\infty}\int_{\mathbb{S}^2}\psi_\ell(\bar{x},q)\,dx$$

(see [48] for more details).

We define, for $\varepsilon > 0$,

$$\mathscr{Z}^{\varphi_\ell^i,\varepsilon} := \int_{\mathbb{S}^2}\varphi_\ell^i(x)||\nabla T_\ell(x)||\chi_\varepsilon(T_\ell(x))\,dx, \tag{3.3.25}$$

and, with the same notation as [48], the "approximate nodal length"

$$\Psi_\varepsilon(x) := ||\nabla T_\ell(x)||\chi_\varepsilon(T_\ell(x)), \quad \chi_\varepsilon(\cdot) = \frac{1}{2\varepsilon}1_{[-\varepsilon,\varepsilon]}(\cdot).$$

The almost-sure convergence and the L^2-convergence of $\mathscr{Z}^{\varphi_\ell^i,\varepsilon}$, as $\varepsilon \to 0$, follow with the same argument suggested in [48]. By continuity of the inner product in L^2-spaces, we need to prove that

$$\operatorname{Cov}\{\mathscr{Z}^{\varphi_\ell^i}; \mathscr{M}^{\varphi_\ell^i}\} = \lim_{\varepsilon\to 0}\operatorname{Cov}\{\mathscr{Z}^{\varphi_\ell^i,\varepsilon}; \mathscr{M}^{\varphi_\ell^i}\}.$$

Moreover,

$$\Psi_\varepsilon(x) = \mathbb{E}\Psi_\varepsilon(x) + \sum_{q=2}^{\infty}\Psi_{\ell,\varepsilon}(x,q)$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{S}^2}\varphi_\ell^i(x)\Psi_\varepsilon(x)\,dx = \int_{\mathbb{S}^2}\lim_{\varepsilon\to 0}\{||\nabla T_\ell(x)||\chi_\varepsilon(T_\ell(x))\varphi_\ell^i(x)\}\,dx = \mathscr{Z}^{\varphi_\ell^i};$$

thus

$$\begin{aligned}\mathrm{Cov}(\mathscr{Z}^{\varphi^i_\ell,\varepsilon};\mathscr{M}^{\varphi^i_\ell}) &= \mathrm{Cov}\Big(\int_{\mathbb{S}^2}\varphi^i_\ell(x)\Psi_\varepsilon(x)\,dx; -\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4!}\int_{\mathbb{S}^2}\varphi^i_\ell(y)H_4(T_\ell(y))\,dy\Big)\\ &= -\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4!}\,\mathrm{Cov}\Big(\int_{\mathbb{S}^2}\varphi^i_\ell(x)\Psi_\varepsilon(x)\,dx; \int_{\mathbb{S}^2}\varphi^i_\ell(y)H_4(T_\ell(y))\,dy\Big)\\ &= -\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4!}\mathbb{E}\Big[\int_{\mathbb{S}^2}\varphi^i_\ell(x)\Psi_\varepsilon(x)\,dx\int_{\mathbb{S}^2}\varphi^i_\ell(y)H_4(T_\ell(y))\,dy\Big]\\ &= -\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4!}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\mathbb{E}[\Psi_\varepsilon(x)H_4(T_\ell(y))]\varphi^i_\ell(x)\varphi^i_\ell(y)\,dxdy\\ &= -\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4!}\int_{\mathbb{S}^2\times\mathbb{S}^2}\mathbb{E}[\sum_{q=2}^{\infty}\Psi_{\ell;\varepsilon}(x,q)H_4(T_\ell(y))]\varphi^i_\ell(x)\varphi^i_\ell(y)\,dxdy.\end{aligned} \tag{3.3.26}$$

In addition, Fubini implies that (3.3.26) can be written as

$$= -\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4!}\times 8\pi^2\lim_{\varepsilon\to 0}\int_0^{2r_\ell}\mathbb{E}[\psi_{\ell,\varepsilon}(\bar{x},4)H_4(T_\ell(y(\rho)))]W^{\varphi^i_\ell}(\rho)d\rho \tag{3.3.27}$$

and changing coordinates $\rho = \dfrac{\psi}{L}$ with $L = \ell + \frac{1}{2}$, (3.3.27) becomes

$$= \int_0^{2r_\ell L} J_\ell(\psi,4)W_{r_\ell}(\frac{\psi}{L})d\psi.$$

We split the integral in two parts:

$$I_1 := \int_0^C J_\ell(\psi,4)W^{\varphi^i_\ell}\Big(\frac{\psi}{L}\Big)d\psi$$

and

$$I_2 := \int_C^{2r_\ell L} J_\ell(\psi,4)W^{\varphi^i_\ell}\Big(\frac{\psi}{L}\Big)d\psi;$$

as far as I_1 is concerned, thanks to (3.5.5), it is bounded by

$$\begin{aligned}I_1 &= \int_0^C J_\ell(\psi,4)W^{\varphi^i_\ell}\Big(\frac{\psi}{L}\Big)d\psi \le \ell\int_0^C W^{\varphi^i_\ell}(\frac{\psi}{L}) \le \ell r_\ell^3\int_0^C \tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})d\psi\\ &= \ell r_\ell^3\int_0^C\frac{\psi}{Lr_\ell}\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})d\psi = r_\ell^2\int_0^C\psi\tilde{W}_0(\frac{\psi}{Lr_\ell})d\psi = O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2)\end{aligned} \tag{3.3.28}$$

as $\ell\to\infty$. Regarding I_2, we substitute the expansion of $J_\ell(\psi,4)$ in (3.5.6) (see [48]), to obtain

$$\begin{aligned}I_2 = \int_C^{2r_\ell L}\Big[&\frac{1}{64}\frac{1}{\psi\sin(\psi/L)} + \frac{5}{64}\frac{\cos(4\psi)}{\psi\sin(\psi/L)} - \frac{3}{16}\frac{\sin(2\psi)}{\psi\sin(\psi/L)} + O\Big(\frac{1}{\psi^2}\frac{1}{\sin(\psi/L)}\Big)\\ &+O\Big(\frac{1}{\ell\psi}\frac{1}{\sin(\frac{\psi}{L})}\Big)\Big]W^{\varphi^i_\ell}(\frac{\psi}{L})d\psi\end{aligned}$$

and, (3.2.13) and (3.2.14) imply

$$W^{\varphi^i_\ell}\Big(\frac{\psi}{L}\Big)=r_\ell^3\tilde W^{\tilde\varphi^i}\Big(\frac{\psi}{Lr_\ell}\Big)\Big(1+O\big(\frac{\psi^2}{L^2}\big)\Big)=\frac{r_\ell^2}{8\pi^2L}\psi\tilde W_0^{\tilde\varphi^i}\big(\frac{\psi}{Lr_\ell}\big)\big(1+O\big(\frac{\psi^2}{L^2}\big)\big),\tag{3.3.29}$$

replacing (3.3.29) in I_2, we get that

$$\begin{aligned}I_2=&\int_C^{2r_\ell L}\frac{1}{64}\frac{1}{\sin(\psi/L)}\frac{r_\ell^2}{8\pi^2L}\tilde W_0^{\tilde\varphi^i}\big(\frac{\psi}{Lr_\ell}\big)d\psi+\int_C^{2r_\ell L}\frac{5}{64}\frac{\cos\psi}{\sin(\psi/L)}\frac{r_\ell^2}{8\pi^2L}\tilde W_0^{\tilde\varphi^i}\big(\frac{\psi}{Lr_\ell}\big)d\psi\\&-\int_C^{2r_\ell L}\frac{3}{16}\frac{\sin(2\psi)}{\sin(\psi/L)}\frac{r_\ell^2}{8\pi^2L}\tilde W_0^{\tilde\varphi^i}\big(\frac{\psi}{Lr_\ell}\big)d\psi+O\Big(\int_C^{2r_\ell L}\frac{r_\ell^2}{L}\frac{\psi}{\psi^2\sin(\psi/L)}\tilde W_0^{\tilde\varphi^i}\big(\frac{\psi}{Lr_\ell}\big)d\psi\Big)\\&+O\Big(\int_C^{2r_\ell L}\frac{1}{\ell\psi}\frac{r_\ell^2\psi}{\sin(\psi/L)L}\tilde W_0^{\tilde\varphi^i}\big(\frac{\psi}{Lr_\ell}\big)d\psi\Big)+O\Big(\int_C^{2r_\ell L}J_\ell(\psi,4)\frac{\psi^2}{L^2}\frac{\psi}{L}r_\ell^2\tilde W_0^{\tilde\varphi^i}\big(\frac{\psi}{Lr_\ell}\big)d\psi\Big).\end{aligned}\tag{3.3.30}$$

The first integral in (3.3.30) gives the main contribution and thanks to (3.3.10) this term is equal to

$$\begin{aligned}&\frac{r_\ell^2}{64L}\int_C^{2r_\ell L}\frac{1}{\sin(\psi/L)}\frac{1}{8\pi^2}\Big(2\pi||\tilde\varphi^i||^2_{L^2(B_1)}+O_{||\tilde\varphi^i||_\infty,V(\tilde\varphi^i)}\big(\frac{\psi}{Lr_\ell}\big)\Big)d\psi\\&=\frac{||\tilde\varphi^i||^2_{L^2(B_1)}}{256\pi}\frac{r_\ell^2}{L}\int_C^{2r_\ell L}\frac{1}{\sin(\psi/L)}d\psi+O_{||\tilde\varphi^i||_\infty,V(\tilde\varphi^i)}\Big(\frac{r_\ell}{L^2}\int_C^{2r_\ell L}\frac{1}{\sin(\psi/L)}\psi\,d\psi\Big).\end{aligned}\tag{3.3.31}$$

For the Taylor expansion of $\dfrac{1}{\sin t}$, as $t\to 0$, (3.3.31) becomes

$$\begin{aligned}&=\frac{||\tilde\varphi^i||^2_{L^2(\tilde B_1)}}{256\pi}\frac{r_\ell^2}{L}\int_C^{2r_\ell L}\Big(\frac{L}{\psi}+O\Big(\frac{\psi}{L}\Big)\Big)d\psi+O_{||\tilde\varphi^i||_\infty,V(\tilde\varphi^i)}\Big(\frac{r_\ell}{L^2}\int_C^{2r_\ell L}\Big(L+O\Big(\frac{\psi^2}{L}\Big)\Big)d\psi\Big)\\&=\frac{||\tilde\varphi^i||^2_{L^2(\tilde B_1)}}{256}r_\ell^2\log(2r_\ell L)+O\Big(\frac{r_\ell^2}{L^2}\Big(\frac{(2r_\ell L)^2}{2}-\frac{C^2}{2}\Big)\Big)+\\&+O_{||\tilde\varphi^i||_\infty,V(\tilde\varphi^i)}\Big(\frac{r_\ell}{L^2}(r_\ell L-C)+\frac{r_\ell}{L^3}((2r_\ell L)^3-C^3)\Big)\\&=\frac{||\tilde\varphi^i||^2_{L^2(\tilde B_1)}}{256}r_\ell^2\log(2r_\ell L)+O(r_\ell^4)+O(r_\ell^2/L^2)+O_{||\tilde\varphi^i||_\infty,V(\tilde\varphi^i)}(r_\ell^2)+O_{||\tilde\varphi^i||_\infty,V(\tilde\varphi^i)}(r_\ell/L)\\&+O_{||\tilde\varphi^i||_\infty,V(\tilde\varphi^i)}(r_\ell^4)+O_{||\tilde\varphi^i||_\infty,V(\tilde\varphi^i)}(r_\ell/L^3)\\&=\frac{||\tilde\varphi^i||^2_{L^2(\tilde B_1)}}{256\pi}r_\ell^2\log(2r_\ell L)+O_{||\tilde\varphi^i||_\infty,V(\tilde\varphi^i)}(r_\ell^2).\end{aligned}\tag{3.3.32}$$

The second summand of (3.3.30), since $\tilde W_0$ is bounded, can be written as

$$\begin{aligned}&\frac{5r_\ell^2}{64L}\frac{1}{8\pi^2}\int_C^{2r_\ell L}\frac{\cos\psi}{\sin(\psi/L)}\tilde W_0\big(\frac{\psi}{Lr_\ell}\big)d\psi\ll\frac{r_\ell^2}{L}\frac{2\pi^2 5||\tilde\varphi^i||_\infty}{8\pi^2\pi^2 64}\int_C^{2r_\ell L}\frac{\cos\psi}{\sin(\psi/L)}d\psi\\&=O_{||\tilde\varphi^i||_\infty}\Big(\frac{r_\ell^2 5}{64\cdot 4L}\int_C^{2r_\ell L}\cos\psi\Big(\frac{L}{\psi}+O\big(\frac{\psi}{L}\big)\Big)d\psi\Big)\\&=O_{||\tilde\varphi^i||_\infty}\Big(r_\ell^2\int_C^{2r_\ell L}\frac{\cos\psi}{\psi}d\psi+\frac{r_\ell^2}{L^2}\int_C^{2r_\ell L}\psi\,d\psi\Big);\end{aligned}\tag{3.3.33}$$

integration by parts implies that (3.3.33) is

$$= O_{||\tilde{\varphi}^i||_\infty}\Big(r_\ell^2\Big[\frac{\sin(2r_\ell L)}{2r_\ell L} - \frac{sinC}{C} + \int_C^{2r_\ell L} \frac{\sin\psi}{\psi^2}\Big]\Big) + O\Big(\frac{r_\ell^2}{L^2}(\frac{(2r_\ell L)^2}{2} - \frac{C^2}{2})\Big)$$
$$= O_{||\tilde{\varphi}^i||_\infty}(\frac{r_\ell}{L}) + O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2) + O(r_\ell^4 + \frac{r_\ell^2}{\ell^2}) = O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2). \tag{3.3.34}$$

The third term of (3.3.30) is

$$= \frac{3}{16\cdot 8\pi^2 L} r_\ell^2 \int_C^{2r_\ell L} \frac{\sin(2\psi)}{\sin(\psi/L)} \tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})\,d\psi \ll \frac{3\cdot 2\pi^2 ||\tilde{\varphi}^i||_\infty}{16\cdot 8\pi^2 L} r_\ell^2 \int_C^{2r_\ell L} \frac{\sin(2\psi)}{\sin(\psi/L)}\,d\psi$$
$$= O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{3}{16\cdot 8\pi^2 L} r_\ell^2 2\pi^2 \int_C^{2r_\ell L} \sin(2\psi)\Big(\frac{L}{\psi} + O(\frac{\psi}{L})\Big)\,d\psi\Big) \tag{3.3.35}$$

and in the same way as for the second integral of (3.3.30), we deduce that (3.3.35) is $O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2)$. The error term in the fourth summand of (3.3.30) is

$$O\Big(\int_C^{2r_\ell L} \Big|\frac{1}{\psi\sin(\psi/L)} \frac{r_\ell^2}{L}\Big| \tilde{W}_0\Big(\frac{\psi}{Lr_\ell}\Big)\,d\psi\Big) = O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{L} \int_C^{2r_\ell L} \Big|\frac{1}{\psi\sin(\psi/L)}\Big|\,d\psi\Big) =$$
$$= O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{\ell}\int_C^{2r_\ell L} \frac{L}{\psi^2} + \frac{1}{\psi} O\Big(\frac{\psi}{L}\Big)\,d\psi\Big) = O_{||\tilde{\varphi}^i||_\infty}\Big(r_\ell^2\Big(\frac{1}{2r_\ell L} - \frac{1}{C}\Big)\Big) + O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{\ell^2}(2r_\ell L - C)\Big)$$
$$= O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell}{\ell}\Big) + O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2) + O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^3}{\ell} - \frac{r_\ell^2}{\ell^2}\Big) = O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2) \tag{3.3.36}$$

and similarly, for the fifth we get that

$$= O\Big(\int_C^{2r_\ell L} \frac{r_\ell^2}{L^2} \frac{1}{\sin(\psi/L)} \tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})\,d\psi\Big) = O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{L^2}\int_C^{2r_\ell L} \frac{L}{\psi} + O\Big(\frac{\psi}{L}\Big)\Big)$$
$$= O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{\ell}\log(2r_\ell L) + \frac{r_\ell^2}{\ell} + \frac{r_\ell^2}{\ell^3}(2r_\ell^2 L^2 - C^2)\Big) = O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2 \log 2r_\ell L}{\ell} + \frac{r_\ell^2}{\ell} + \frac{r_\ell^4}{\ell} - \frac{r_\ell^2}{L^3}\Big)$$
$$= O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2). \tag{3.3.37}$$

Finally the last contribution of (3.3.30) is bounded by

$$O\Big(\int_C^{2r_\ell L} J_\ell(\psi,4)\frac{\psi^3}{L^3} r_\ell^2 \tilde{W}_0^{\tilde{\varphi}^i}(\psi/Lr_\ell)\,d\psi\Big) \le O_{||\tilde{\varphi}^i||_\infty}\Big(\int_C^{2r_\ell L} J_\ell(\psi,4) r_\ell^2 \frac{\psi^3}{L^3}\,d\psi\Big)$$
$$= O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{\ell^3}\int_C^{2r_\ell L} J_\ell(\psi,4)\psi^3\,d\psi\Big) = O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{\ell^3}\int_C^{2r_\ell L} \frac{1}{64\psi\sin(\psi/L)}\psi^3\,d\psi\Big)$$
$$= O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{\ell^3}\int_C^{2r_\ell L} \psi^2\Big(\frac{L}{\psi} + O(\frac{\psi}{L})\Big)\,d\psi\Big) \tag{3.3.38}$$
$$= O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{\ell^2}((2r_\ell L)^2 - C^2)\Big) + O_{||\tilde{\varphi}^i||_\infty}\Big(\frac{r_\ell^2}{\ell^4}((2r_\ell L)^4 - C^4)\Big)$$
$$= O_{||\tilde{\varphi}^i||_\infty}\Big(r_\ell^4 + \frac{r_\ell^2}{\ell^2}\Big) + O_{||\tilde{\varphi}^i||_\infty}\Big(r_\ell^6 + \frac{r_\ell^2}{\ell^4}\Big) = O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2);$$

then, we conclude that

$$\mathrm{Cov}(\mathscr{L}^{\varphi_\ell^i}, \mathscr{M}^{\varphi_\ell^i}) = \frac{||\tilde{\varphi}^i||^2_{L^2(\tilde{B}_1)}}{256\pi} r_\ell^2 \log(2r_\ell L) + O_{||\tilde{\varphi}^i||_\infty, V(\tilde{\varphi}^i)}(r_\ell^2)$$

□

Proposition 3.1.4 is easily seen as a corollary of Proposition 3.1.5 and Proposition 3.1.6.

Proof of Proposition 3.1.4 assuming Proposition 3.1.5 and Proposition 3.1.6 . Applying Proposition 3.1.6 to the function $\varphi_\ell^i - 1_{B_{r_\ell}}$, and since

$$\mathbb{E}[(\mathscr{M}^{\varphi_\ell^i} - \mathscr{M}_{\ell,r_\ell})^2] = \mathbb{E}[(\mathscr{M}^{\varphi_\ell^i - 1_{B_{r_\ell}}})^2],$$

we have that

$$|\mathrm{Var}(\mathscr{M}^{\varphi_\ell^i}) - \mathrm{Var}(\mathscr{M}_{\ell,r_\ell})| \le |\mathbb{E}[(\mathscr{M}^{\varphi_\ell^i - 1_{B_{r_\ell}}})^2]| + 2|\mathbb{E}[(\mathscr{M}^{\varphi_\ell^i} - \mathscr{M}_{\ell,r_\ell})\mathscr{M}_{\ell;r_\ell}]| +$$

$$|\mathbb{E}[\mathscr{M}^{\varphi_\ell^i}]^2 - \mathbb{E}[\mathscr{M}_{\ell,r_\ell}]^2|$$

goes to zero; in view of Proposition 3.1.5, passing to the limit $i \to \infty$, the thesis follows. □

Let us now prove Proposition 3.1.5 and Proposition 3.1.6.

Proof of Proposition 3.1.5. The idea of the proof is quite similar to the one in Proposition 3.1.1; actually, we write the variance of $\mathscr{M}^{\varphi_\ell^i}$ as

$$\begin{aligned}
\mathrm{Var}(\mathscr{M}^{\varphi_\ell^i}) &= \mathrm{Var}\Big[-\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4!}\int_{\mathbb{S}^2}\varphi_\ell^i(x)H_4(T_\ell(x))\,dx\Big] \\
&= \frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!^2}\mathbb{E}\Big[\int_{\mathbb{S}^2}\varphi_\ell^i(x)H_4(T_\ell(x))\,dx\int_{\mathbb{S}^2}\varphi_\ell^i(y)H_4(T_\ell(y))\,dy\Big] = \\
&= \frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!^2}\int_{\mathbb{S}^2\times\mathbb{S}^2}\mathbb{E}[H_4(T_\ell(x))H_4(T_\ell(y))]\varphi_\ell^i(x)\varphi_\ell^i(y)\,dx\,dy \\
&= \frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!^2}4!\int_{\mathbb{S}^2\times\mathbb{S}^2}P_\ell(\langle x,y\rangle)^4\varphi_\ell^i(x)\varphi_\ell^i(y)\,dxdy,
\end{aligned} \tag{3.3.39}$$

employing Fubini, this term is equal to

$$= \frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}8\pi^2\int_0^{2r_\ell}P_\ell(\cos\rho)^4 W^{\varphi_\ell^i}(\rho)\,d\rho$$

and changing the variable $\rho = \dfrac{\psi}{L}$, we have that

$$\begin{aligned}
&= \frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}8\pi^2\frac{1}{L}\int_0^{2r_\ell L}P_\ell(\cos\frac{\psi}{L})^4 W^{\varphi_\ell^i}(\frac{\psi}{L})\,d\psi \\
&= \frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}8\pi^2\frac{r_\ell^3}{L}\int_0^{2r_\ell L}P_\ell(\cos\frac{\psi}{L})^4\tilde{W}^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})(1+O(\frac{\psi^2}{L^2}))\,d\psi.
\end{aligned} \tag{3.3.40}$$

Relation (3.2.14) implies that (3.3.40) becomes

$$\begin{aligned}
&\frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}8\pi^2\frac{r_\ell^2}{L^2 8\pi^2}\int_0^{2r_\ell L} P_\ell(\cos\frac{\psi}{L})^4\psi\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})(1+O(\frac{\psi^2}{L^2}))\,d\psi\\
&=\frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}8\pi^2\frac{r_\ell^2}{L^2 8\pi^2}\int_0^{2r_\ell L} P_\ell(\cos\frac{\psi}{L})^4\psi\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})\,d\psi+\\
&O\Big(\frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}8\pi^2\frac{r_\ell^2}{L^2 8\pi^2}\int_0^{2r_\ell L} P_\ell(\cos\frac{\psi}{L})^4\frac{\psi^3}{L^2}\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})\,d\psi\Big)
\end{aligned}\tag{3.3.41}$$

and dividing as usual the integral over the two regions $[0,C]$ and $[C,2r_\ell L]$, we get four terms

$$\begin{aligned}
&=\underbrace{\frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}\frac{r_\ell^2}{L^2}\int_0^{C} P_\ell(\cos\frac{\psi}{L})^4\psi\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})\,d\psi+}_{(i)}\\
&\underbrace{O\Big(\frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}\frac{r_\ell^2}{L^2}\int_0^{C} P_\ell(\cos\frac{\psi}{L})^4\frac{\psi^3}{L^2}\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})\,d\psi\Big)}_{(ii)}\\
&+\underbrace{\frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}\frac{r_\ell^2}{L^2}\int_C^{2r_\ell L} P_\ell(\cos\frac{\psi}{L})^4\psi\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})\,d\psi}_{(iii)}\\
&+\underbrace{O\Big(\frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}\frac{r_\ell^2}{L^2}\int_C^{2r_\ell L} P_\ell(\cos\frac{\psi}{L})^4\frac{\psi^3}{L^2}\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})\,d\psi\Big)}_{(iv)}.
\end{aligned}\tag{3.3.42}$$

Let us focus on (iii), which contains the leading part. Replacing $P_\ell(\cos\frac{\psi}{L})^4$ with its expansion, given in (A.1.14) ([77]), it results that

$$\begin{aligned}
(iii)&=\frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}\frac{r_\ell^2}{L^2}\int_C^{2r_\ell L}\Big[\frac{\frac{3}{2}+2\sin(2\psi)-\frac{1}{2}\cos(4\psi)}{\pi^2\ell^2(\sin\frac{\psi}{L})^2}+O\Big(\frac{1}{\psi^3}\Big)\Big]\psi\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})\,d\psi\\
&=\frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}\frac{r_\ell^2}{L^2\ell^2\pi^2}\int_C^{2r_\ell L}\frac{\frac{3}{2}+2\sin(2\psi)-\frac{1}{2}\cos(4\psi)}{(\sin\frac{\psi}{L})^2}\psi\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})d\psi+\\
&+O\Big(r_\ell^2\int_C^{2r_\ell L}\frac{1}{\psi^3}\psi\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{Lr_\ell})\,d\psi\Big)\\
&=\underbrace{\frac{1}{16}\frac{\ell(\ell+1)}{2}\frac{1}{4!}\frac{r_\ell^2}{L^2\ell^2\pi^2}\int_C^{2r_\ell L}\frac{\frac{3}{2}+2\sin(2\psi)-\frac{1}{2}\cos(4\psi)}{(\sin\frac{\psi}{L})^2}\psi 2\pi||\tilde{\varphi}^i||^2_{L^2(B_1)}d\psi+}_{(a)}\\
&\underbrace{O_{||\tilde{\varphi}^i||_\infty,V(\tilde{\varphi}^i)}\Big(\frac{r_\ell^2}{\ell^2}\int_C^{2r_\ell L}\frac{\frac{3}{2}+2\sin(2\psi)-\frac{1}{2}\cos(4\psi)}{\sin\frac{\psi}{L}^2}\psi\frac{\psi}{Lr_\ell})\,d\psi\Big)}_{(b)}+O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2).
\end{aligned}\tag{3.3.43}$$

By Taylor expansion, integral (a) is equal to

$$\frac{||\tilde{\varphi}^i||_{L^2(\tilde{B}_1)}}{16\pi}\frac{\ell(\ell+1)}{2}\frac{1}{4!}\frac{r_\ell^2}{L^2\ell^2}\int_C^{2r_\ell L}\Big(\frac{3}{2}+2\sin(2\psi)-\frac{1}{2}\cos(4\psi)\Big)\Big(\frac{L}{\psi}+O\Big(\frac{\psi}{L}\Big)\Big)^2 2\psi\,d\psi$$

$$= \frac{||\tilde{\varphi}^i||_{L^2(\tilde{B}_1)}}{16\pi} \frac{\ell(\ell+1)}{2} \frac{1}{4!} \frac{r_\ell^2}{L^2\ell^2} \int_C^{2r_\ell L} \left(\frac{3}{2}+2\sin(2\psi)-\frac{1}{2}\cos(4\psi)\right)\left(\frac{L^2}{\psi^2}+O(1)+O\left(\frac{\psi^2}{L^2}\right)\right)2\psi\, d\psi; \tag{3.3.44}$$

the dominant summand is the first one, which is

$$\frac{||\tilde{\varphi}^i||_{L^2(\tilde{B}_1)}}{16\pi} \frac{\ell(\ell+1)}{2} \frac{1}{4!} \frac{r_\ell^2}{L^2\ell^2} \int_C^{2r_\ell L} \frac{3}{2}\frac{L^2}{\psi^2} 2\psi\, d\psi = \frac{||\tilde{\varphi}^i||_{L^2(\tilde{B}_1)}}{32\pi\cdot 4!} \frac{\ell(\ell+1)}{\ell^2} r_\ell^2 3(\log 2r_\ell L - \log C)$$
$$= \frac{||\tilde{\varphi}^i||_{L^2(\tilde{B}_1)}}{256\pi} r_\ell^2 \log(r_\ell \ell) + O(r_\ell^2). \tag{3.3.45}$$

It is easy to verify that, with similar calculations, all the other terms are $O_{||\tilde{\varphi}^i||_\infty}(r_\ell^2)$ and the thesis follows.

□

Proof of Proposition 3.1.6. In a similar way of the proof of Proposition 3.1.2,

$$\begin{aligned}\mathbb{E}[\mathscr{M}^{\varphi_\ell^i}(T_\ell)^2] &= O\left(\frac{\ell(\ell+1)}{L} \int_0^{2r_\ell L} P_\ell(\cos\frac{\psi}{L})^4 W^{\varphi_\ell^i}\left(\frac{\psi}{L}\right) d\psi\right) \\ &= O\left(\frac{\ell(\ell+1)}{L} r_\ell^3 \int_0^{2r_\ell L} P_\ell(\cos\frac{\psi}{L})^4 \tilde{W}^{\tilde{\varphi}^i}\left(\frac{\psi}{r_\ell L}\right) d\psi\right) \\ &= O\left(r_\ell^2 \int_0^{2r_\ell L} \psi P_\ell(\cos\frac{\psi}{L})^4 |\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{L r_\ell})|\, d\psi\right)\end{aligned} \tag{3.3.46}$$

and splitting the integral, we obtain that (3.3.46) is

$$O\left(r_\ell^2 \int_0^C \psi P_\ell(\cos\frac{\psi}{L})^4 |\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{L r_\ell})|\, d\psi\right) = O\left(r_\ell^2 ||\tilde{\varphi}^i||_\infty ||\tilde{\varphi}||_{L^1(\tilde{B}_1)}\right)$$

and, thanks to the Hilb's asymptotic formula (see (A.1.3) and [77] p. 40), since $\left|P_\ell(\cos\frac{\psi}{L})\right| = O\left(\frac{1}{\sqrt{\psi}}\right)$, it follows that

$$O\left(r_\ell^2 \int_C^{2r_\ell L} \psi P_\ell(\cos\frac{\psi}{L})^4 |\tilde{W}_0^{\tilde{\varphi}^i}(\frac{\psi}{L r_\ell})|\, d\psi\right) = O\left(r_\ell^2 \int_C^{2r_\ell L} \frac{1}{\psi^2} \psi ||\tilde{\varphi}^i||_\infty ||\tilde{\varphi}^i||_{L^1(\tilde{B}_1)}\right)$$
$$= O(r_\ell^2 ||\tilde{\varphi}^i||_\infty \log r_\ell \ell). \tag{3.3.47}$$

□

The second chaotic component

In the lemma below, we show that the second chaotic component of the nodal length has lower order than the fourth one.

Lemma 3.3.3. *The second component of the chaos expansion of $\mathscr{Z}_{\ell,r_\ell}$ is*

$$Proj(\mathscr{Z}_{\ell,r_\ell}|2) = O(r_\ell^2)$$

as $\ell \to \infty$.

Proof. For a general number $z \in \mathbb{R}$, if we define $\mathscr{L}_{\ell,r_\ell}(z) := \{x \in \mathbb{S}^2 \cap B_{r_\ell} : T_\ell(x) = z\}$, the projection of the length of the level curves into the second chaotic component is given by (see [64], [65], [66])

$$Proj(\mathscr{L}_{\ell,r_\ell}(z)|C_2) = \sqrt{\frac{\ell(\ell+1)}{2}}\left\{\frac{\alpha_{0,0}\beta_2(z)}{2!}\int_{B_{r_\ell}} H_2(T_\ell(x))\,dx + \frac{\alpha_{0,2}\beta_0(z)}{2}\int_{B_{r_\ell}} H_2(\tilde{\partial}_2 T_\ell(x))\,dx + \frac{\alpha_{2,0}\beta_0}{2!}\int_{B_{r_\ell}} H_2(\tilde{\partial}_1 T_\ell(x))\,dx\right\}; \tag{3.3.48}$$

where $\partial_j \tilde{T}_\ell(x) := \sqrt{\frac{2}{\ell(\ell+1)}}\frac{\partial}{\partial\theta_j}T_\ell(\theta);$

$$\alpha_{0,0} = \sqrt{\frac{\pi}{2}}, \quad \alpha_{0,2} = \alpha_{2,0} = \sqrt{\frac{\pi}{2}}\frac{1}{2}$$

and $\beta_\ell(z) := \Phi(z)H_\ell(z), \quad \beta_0 = \Phi(z), \quad \beta_2 = \Phi(z)(z^2-1).$

Evaluating $H_2(T_\ell(x))$, we get that

$$\begin{aligned}
Proj(\mathscr{L}_{\ell,r_\ell}(z)|C_2) &= \sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{2}\{\alpha_{0,0}\beta_2(z)\int_{B_{r_\ell}} T_\ell(x)^2 - 1\,dx + \alpha_{0,2}\beta_0(z)\int_{B_{r_\ell}} \tilde{\partial}_2 T_\ell(x)^2 - 1\,dx \\
&+ \alpha_{2,0}\beta_0(z)\int_{B_{r_\ell}} \tilde{\partial}_1 T_\ell(x)^2 - 1\,dx\} \\
&= \sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{2}\{\alpha_{0,0}\beta_2(z)\int_{B_{r_\ell}} T_\ell(x)^2\,dx + \frac{\alpha_{0,2}}{\beta_0(z)}\ell(\ell+1)2\int_{B_{r_\ell}} \partial_2 T_\ell(x)^2\,dx \\
&+ \frac{\alpha_{2,0}\beta_0(z)2}{\ell(\ell+1)}\int_{B_{r_\ell}} \partial_1 T_\ell(x)^2\,dx - m(B_{r_\ell})(\alpha_{0,2}\beta_0(z)2 + \alpha_{0,0}\beta_2(z))\}.
\end{aligned}$$

Moreover, Green's identity implies that

$$\int_{B_{r_\ell}} \partial_2 T_\ell(x)^2\,dx = \int_{B_{r_\ell}} \partial_2 T_\ell(x)\partial_2 T_\ell(x)\,dx = -\int_{B_{r_\ell}} T_\ell(x)\partial_2^2 T_\ell(x)\,dx + \int_{\partial B_{r_\ell}} T_\ell(x)\partial_2 T_\ell(x)\,dx,$$

then,

$$\begin{aligned}
Proj(\mathscr{L}_{\ell,r_\ell}(z)|C_2) &= \sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{2}\Big\{\alpha_{0,0}\beta_2(z)\int_{B_{r_\ell}} T_\ell(x)^2\,dx - \frac{\alpha_{0,2}\beta_0(z)}{\ell(\ell+1)}2\int_{B_{r_\ell}} T_\ell(x)\Delta T_\ell(x)\,dx \\
&+ \frac{\alpha_{0,2}\beta_0(z)}{\ell(\ell+1)}2\int_{\partial B_{r_\ell}} T_\ell(x)[\partial_2 T_\ell(x)+\partial_1 T_\ell(x)]\,dx - m(B_{r_\ell})(\alpha_{0,2}\beta_0(z)2+\alpha_{0,0}\beta_2(z)\Big\} \\
&= \sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{2}\Big\{\alpha_{0,0}\beta_2(z)\int_{B_{r_\ell}} T_\ell(x)^2\,dx - \frac{\alpha_{0,2}\beta_0(z)}{\ell(\ell+1)}2\int_{B_{r_\ell}} T_\ell(x)(-T_\ell(x)\ell(\ell+1))\,dx \\
&+ \frac{\alpha_{0,2}\beta_0(z)}{\ell(\ell+1)}2\int_{\partial B_{r_\ell}} T_\ell(x)[\partial_2 T_\ell(x)+\partial_1 T_\ell(x)]\,dx - m(B_{r_\ell})(\alpha_{0,2}\beta_0(z)2+\alpha_{0,0}\beta_2(z))\Big\} \\
&= \sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{2}\Big\{(\alpha_{0,0}\beta_2(z)+2\alpha_{0,2}\beta_0(z))\int_{B_{r_\ell}} T_\ell(x)^2\,dx \\
&+ \frac{\alpha_{0,2}\beta_0(z)}{\ell(\ell+1)}2\int_{\partial B_{r_\ell}} T_\ell(x)[\partial_2 T_\ell(x)+\partial_1 T_\ell(x)]\,dx - m(B_{r_\ell})(\alpha_{0,2}\beta_0(z)2+\alpha_{0,0}\beta_2(z))\Big\} \\
&= \sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{2}(\alpha_{0,0}\beta_2(z)+2\alpha_{0,2}\beta_0(z))\int_{B_{r_\ell}} T_\ell(x)^2-1\,dx \\
&+ \frac{1}{2}\sqrt{\frac{2}{\ell(\ell+1)}}\alpha_{0,2}\beta_0(z)\int_{\partial B_{r_\ell}} T_\ell(x)[\partial_2 T_\ell(x)+\partial_1 T_\ell(x)]\,dx.
\end{aligned} \tag{3.3.49}$$

Since in our case $z=0$, the first term vanishes and hence

$$Proj(\mathscr{L}_{\ell,r_\ell}|2) = \frac{1}{2}\sqrt{\frac{2}{\ell(\ell+1)}}\alpha_{0,2}\beta_0(z)\int_{\partial B_{r_\ell}} T_\ell(x)[\partial_2 T_\ell(x)+\partial_1 T_\ell(x)]\,dx;$$

we show that its variance is of order $O(r_\ell^2)$, indeed

$$\begin{aligned}
\mathrm{Var}(Proj(\mathscr{L}_{\ell,r_\ell})|2) &= \mathrm{Var}\left(\frac{1}{2}\sqrt{\frac{2}{\ell(\ell+1)}}\alpha_{0,2}\beta_0(z)\int_{\partial B_{r_\ell}} T_\ell(x)[\partial_2 T_\ell(x)+\partial_1 T_\ell(x)]\,dx\right) \\
&= \frac{1}{4}\frac{2}{\ell(\ell+1)}\frac{\pi}{2}\frac{1}{4}\Phi(0)\Big\{\mathrm{Var}\left(\int_{\partial B_{r_\ell}} T_\ell(x)\partial_2 T_\ell(x)\,dx\right)+\mathrm{Var}\left(\int_{\partial B_{r_\ell}} T_\ell(x)\partial_1 T_\ell(x)\,dx\right) \\
&+\mathrm{Cov}\left(\int_{\partial B_{r_\ell}} T_\ell(x)\partial_2 T_\ell(x)\,dx;\int_{\partial B_{r_\ell}} T_\ell(x)\partial_1 T_\ell(x)\,dx\right)\Big\}
\end{aligned}$$

and after proving that

$$\mathrm{Var}\left(\int_{\partial B_{r_\ell}} T_\ell(x)\partial_1 T_\ell(x)\,dx\right) = \mathrm{Var}\left(\int_{\partial B_{r_\ell}} T_\ell(x)\partial_2 T_\ell(x)\,dx\right) = O\left(\ell^2 r_\ell^2\right), \tag{3.3.50}$$

by Cauchy-Schwarz inequality the covariance is bounded by (3.3.50) and hence the thesis follows. It remains to check (3.3.50); to this aim we write

$$\mathrm{Var}\left(\int_{\partial B_{r_\ell}} T_\ell(x)\partial_2 T_\ell(x)\,dx\right) = \mathbb{E}\left[\left(\int_{\partial B_{r_\ell}} T_\ell(x)\partial_2 T_\ell(x)\,dx\right)^2\right] - \mathbb{E}\left[\int_{\partial B_{r_\ell}} T_\ell(x)\partial_2 T_\ell(x)\,dx\right]^2$$

and since $\mathbb{E}[T_\ell(x)\partial_2 T_\ell(x)] = 0$, $\mathbb{E}[T_\ell(x)\partial_1 T_\ell(x)] = 0$ (computed in [48]), the variance is

$$\mathrm{Var}\Big(\int_{\partial B_{r_\ell}} T_\ell(x)\partial_2 T_\ell(x)\,dx\Big) = \int_{\partial B_{r_\ell}\times\partial B_{r_\ell}} \mathbb{E}[T_\ell(x)\partial_2 T_\ell(x) T_\ell(y)\partial_2 T_\ell(y)]\,dxdy.$$

In view of the Cauchy-Schwarz inequality, the absolute value of this integral is bounded by

$$\begin{aligned}
&\Big|\int_{\partial B_{r_\ell}\times\partial B_{r_\ell}} \mathbb{E}[T_\ell(x)\partial_2 T_\ell(x) T_\ell(y)\partial_2 T_\ell(y)]\,dxdy\Big| \\
&\le \int_{\partial B_{r_\ell}\times\partial B_{r_\ell}} \mathbb{E}[|T_\ell(x)\partial_2 T_\ell(x) T_\ell(y)\partial_2 T_\ell(y)|]\,dxdy \\
&\le \int_{\partial B_{r_\ell}\times\partial B_{r_\ell}} (\mathbb{E}[T_\ell(x)^2\partial_2 T_\ell(x)^2]\mathbb{E}[T_\ell(y)^2\partial_2 T_\ell(y)^2])^{1/2}\,dxdy \\
&\le \int_{\partial B_{r_\ell}\times\partial B_{r_\ell}} (\mathbb{E}[T_\ell(x)^4]\mathbb{E}[\partial_2 T_\ell(x)^4]\mathbb{E}[T_\ell(y)^4]\mathbb{E}[\partial_2 T_\ell(y)^4])^{1/4}\,dxdy;
\end{aligned} \tag{3.3.51}$$

since they are random Gaussian variables, $\mathbb{E}[X^4] = 3(\mathbb{E}X^2)^2$ and then (3.3.51) is

$$\begin{aligned}
&\le \int_{\partial B_{r_\ell}\times\partial B_{r_\ell}} 3(\mathbb{E}[T_\ell(x)^2])^{1/2} 3(\mathbb{E}[\partial_2 T_\ell(x)^2])^{1/2} 3(\mathbb{E}[T_\ell(y)^2])^{1/2} 3(\mathbb{E}[\partial_2 T_\ell(y)^2])^{1/2}\,dxdy \\
&= \Big[\int_{\partial B_{r_\ell}} 9(\mathbb{E}[T_\ell(x)^2])^{1/2}(\mathbb{E}[\partial_2 T_\ell(x)^2])^{1/2}\,dx\Big]^2.
\end{aligned} \tag{3.3.52}$$

Now, since $\mathbb{E}[T_\ell(x)^2]^{1/2} = 1$, for the same computations as in [48], we have that

$$\mathbb{E}[\partial_1 T_\ell(y)^2] = \Big[P_\ell'(\cos\theta)\cos\theta - P_\ell''(\cos\theta)\sin^2\theta\Big|_{\theta=0} = P_\ell'(1) = \frac{\ell(\ell+1)}{2},$$

it follows that (3.3.52) reduces to

$$81\Big(\int_{\partial B_{r_\ell}} \sqrt{\frac{\ell(\ell+1)}{2}}\,dx\Big)^2 = \frac{\ell(\ell+1)}{2} 81(2\pi\sin r_\ell)^2.$$

The variance $\mathrm{Var}(\int_{\partial B_{r_\ell}} T_\ell(x)\partial_1 T_\ell(x)\,dx)$ is computed in the same way, with $\mathbb{E}[\partial_1 T_\ell(x)^2]^{1/2} = \Big[P_\ell'(\cos\theta)\cos\theta - P_\ell''(\cos\theta)\sin^2\theta\Big|_{\theta=0} = P_\ell'(1)$ and finally, we find that

$$\mathrm{Var}(Proj(\mathscr{Z}_{\ell,r_\ell}|C_2)) = O\Big(\frac{1}{16}\frac{\pi}{2}\Phi(0)^2\frac{2}{\ell(\ell+1)}\cdot 2\Big(2\cdot 81\ell(\ell+1)\pi^2\sin^2 r_\ell\Big)\Big) = O(r_\ell^2).$$

□

Fourth cumulant of the fourth chaotic component

In light of the orthogonality of the chaotic components, the full correlation between $\mathscr{Z}_{\ell,r_\ell}$ and $\mathscr{M}_{\ell,r_\ell}$ implies that

$$\mathrm{Corr}(\mathscr{M}_{\ell,r_\ell}; Proj(\mathscr{Z}_{\ell,r_\ell}|C_4)) = 1 + O\Big(\frac{1}{\log\ell r_\ell}\Big);$$

hence to apply the CLT, we investigate the fourth cumulant of (3.1.8), which we rewrite for simplicity

$$h_{\ell,r_\ell,4} := \int_{B_{r_\ell}} H_4(T_\ell(x))\,dx, \tag{3.3.53}$$

proving that it has a lower order than the square of its variance (to conclude the CLT by Corollary 1.2.4).

Lemma 3.3.4. *Let $h_{\ell,r_\ell,4}$ be defined as (3.3.53), for each $0<\delta<r_\ell$, as $\ell\to\infty$,*

$$cum_4(h_{\ell,r_\ell,4}) = O\Big(\frac{r_\ell^4}{\ell^4}(\log\delta\ell)^{1/4}(\log r_\ell\ell)^{7/4}\Big). \tag{3.3.54}$$

Proof. Let consider (3.3.53), with the same notation as in [50], if $\Gamma_c(4,4,4,4)$ is the set of all connected graphs, the searched cumulant is

$$cum_4(h_{\ell,r_\ell,4}) = \sum_{\gamma\in\Gamma_c(4,4,4,4)} M(\eta(\gamma))$$

with

$$M(\eta) = \int_{B^4_{r_\ell}} \prod_{i<j} P_\ell(\langle x_i,y_j\rangle)^{\eta_{i,j}}\,dx.$$

For a small parameter $\delta := \delta(\ell) > 0$ such that $\delta < r_\ell$ and $r_\ell\ell\to\infty$, we introduce the set

$$\mathscr{L}(\delta) := \{x\in(B_{r_\ell})^4 : \theta(x_i,x_j)>\delta\} \tag{3.3.55}$$

and its complementary set

$$\mathscr{L}(\delta)^c = \{x\in(B_{r_\ell})^4 : \exists(i,j) : \theta(x_i,x_j)\le\delta\}.$$

We decompose the domain of integration as following

$$(B_{r_\ell})^4 = \mathscr{L}(\delta)\cup\mathscr{L}(\delta)^c$$

and we split the set of $M(\eta)$ in

$$M(\eta) = M_{glob}(\eta;\delta) + M_{loc}(\eta;\delta),$$

where

$$M_{glob}(\eta;\delta) := \int_{\mathscr{L}(\delta)} \prod_{i<j} P_\ell(\langle x_i,y_j\rangle)^{\eta_{i,j}}\,dx$$

and

$$M_{loc}(\eta;\delta) := \int_{\mathscr{L}(\delta)^c} \prod_{i<j} P_\ell(\langle x_i,y_j\rangle)^{\eta_{i,j}}\,dx.$$

Moreover, the Diagram Formula (Section 4.3.1 [45]), implies that, for the fourth chaos, it is sufficient to evaluate

$$A_1 = \int_{B^4_{r_\ell}} P_\ell(\langle x_1,x_2\rangle)P_\ell(\langle x_1,x_3\rangle)^3 P_\ell(\langle x_3,x_4\rangle)P_\ell(\langle x_2,x_4\rangle)^3\,dx$$

and

$$A_2 = \int_{B^4_{r_\ell}} P_\ell(\langle x_1,x_2\rangle)^2 P_\ell(\langle x_1,x_3\rangle)^2 P_\ell(\langle x_3,x_4\rangle)^2 P_\ell(\langle x_2,x_4\rangle)^2\,dx.$$

We split the domain of the integral in $\mathscr{L}(\delta)$ and its complementary; we refer to $A_i(glob)$ and $A_i(loc)$ for $i=1,2$ the one computed in $\mathscr{L}(\delta)$ and the one on $\mathscr{L}(\delta)^c$, respectively (similarly in [50]). Considering the global part, on $\mathscr{L}(\delta)$, for every $i<j$ we have the uniform upper bound

$$|P_\ell(\langle x_i, x_j\rangle)| \ll \frac{1}{\sqrt{\ell\delta}};$$

hence it results that

$$\begin{aligned} A_1(glob) &= \int_{\mathscr{L}(\delta)} P_\ell(\langle x_1,x_2\rangle)P_\ell(\langle x_1,x_3\rangle)^3 P_\ell(\langle x_3,x_4\rangle)P_\ell(\langle x_2,x_4\rangle)^3\,dx \\ &\le \frac{1}{\ell\delta}\int_{\mathscr{L}(\delta)} |P_\ell(\langle x_1,x_2\rangle)|P_\ell(\langle x_1,x_3\rangle)^2|P_\ell(\langle x_3,x_4\rangle)|P_\ell(\langle x_2,x_4\rangle)^2\,dx \end{aligned} \tag{3.3.56}$$

and by Cauchy-Schwarz inequality, (3.3.56) is

$$\begin{aligned} &\le \frac{1}{\ell\delta}\Big(\int_{\mathscr{L}(\delta)} P_\ell(\langle x_1,x_2\rangle)^2 P_\ell(\langle x_3,x_4\rangle)^2\,dx\Big)^{1/2}\Big(\int_{\mathscr{L}(\delta)} P_\ell(\langle x_1,x_3\rangle)^4 P_\ell(\langle x_1,x_4\rangle)^4\,dx\Big)^{1/2} \\ &\le \frac{1}{\ell\delta}\Big(\int_{B^2_{r_\ell}} P_\ell(\langle x_1,x_2\rangle)^2\,dx_1dx_2\int_{B^2_{r_\ell}} P_\ell(\langle x_3,x_4\rangle)^2\,dx_3dx_4\Big)^{1/2}\Big(\int_{B^4_{r_\ell}} P_\ell(\langle x_1,x_3\rangle)^4 P_\ell(\langle x_2,x_4\rangle)^4\,dx\Big)^{1/2}. \end{aligned} \tag{3.3.57}$$

Since in Section 3.3.2 we saw that

$$\int_{B^2_{r_\ell}} P_\ell(\cos\langle x_1,x_2\rangle)^4\,dx = O\Big(\frac{r_\ell^2}{\ell^2}\log r_\ell\ell\Big)$$

and in Lemma 3.3.3 that

$$\int_{B^2_{r_\ell}} P_\ell(\cos\langle x_1,x_2\rangle)^2\,dx = O\Big(\frac{r_\ell^2}{\ell^2}\Big);$$

then,

$$A_1(glob) \ll \frac{1}{\ell\delta}\frac{r_\ell^2}{\ell^2}\frac{r_\ell^2\log(r_\ell\ell)}{\ell^2} = \frac{1}{\ell\delta}\frac{r_\ell^4}{\ell^4}\log(r_\ell\ell).$$

Likewise,

$$\begin{aligned} A_2(glob) &= \int_{\mathscr{L}(\delta)} P_\ell(\langle x_1,x_2\rangle)^2 P_\ell(\langle x_1,x_3\rangle)^2 P_\ell(\langle x_3,x_4\rangle)^2 P_\ell(\langle x_2,x_4\rangle)^2\,dx \\ &\le \frac{1}{\ell\delta}\int_{\mathscr{L}(\delta)} |P_\ell(\langle x_1,x_2\rangle)|P_\ell(\langle x_1,x_3\rangle)^2 P_\ell(\langle x_2,x_4\rangle)^2|P_\ell(\langle x_3,x_4\rangle)|\,dx \\ &\le \frac{1}{\ell\delta}\Big(\int_{\mathscr{L}(\delta)} P_\ell(\langle x_1,x_2\rangle)^2 P_\ell(\langle x_3,x_4\rangle)^2\,dx\Big)^{1/2}\Big(\int_{\mathscr{L}(\delta)} P_\ell(\langle x_1,x_3\rangle)^4 P_\ell(\langle x_2,x_4\rangle)^4\,dx\Big)^{1/2} \\ &\le \frac{1}{\ell\delta}\frac{r_\ell^2}{\ell^2}\frac{r_\ell^2}{\ell^2}\log r_\ell\ell = \frac{1}{\ell\delta}\frac{r_\ell^4}{\ell^4}\log r_\ell\ell. \end{aligned} \tag{3.3.58}$$

In order to investigate the local term, we may assume with no loss of generality that

$$\theta(x_1,x_2) \le \delta$$

in the domain $\mathscr{L}(\delta)^c$. The Cauchy-Schwarz inequality leads $A_2(loc)$ to be

$$\begin{aligned}
A_2(loc) &= \int_{\mathscr{L}(\delta)^c} P_\ell(\langle x_1,x_2\rangle)^2 P_\ell(\langle x_1,x_3\rangle)^2 P_\ell(\langle x_3,x_4\rangle)^2 P_\ell(\langle x_2,x_4\rangle)^2\,dx \\
&\le \Big(\int_{\mathscr{L}(\delta)^c} P_\ell(\langle x_1,x_2\rangle)^4 P_\ell(\langle x_3,x_4\rangle)^4\,dx\Big)^{1/2}\Big(\int_{\mathscr{L}(\delta)^c} P_\ell(\langle x_1,x_3\rangle)^4 P_\ell(\langle x_2,x_4\rangle)^4\,dx\Big)^{1/2} \\
&\le \Big(\int_{\{(x_1,x_2)\in B^2_{r_\ell}:\theta(x_1,x_2)<\delta\}} P_\ell(\langle x_1,x_2\rangle)^4\,dx_1\,dx_2 \int_{B^2_{r_\ell}} P_\ell(\langle x_3,x_4\rangle)^4\,dx_3\,dx_4\Big)^{1/2} \\
&\times \Big(\int_{B^2_{r_\ell}} P_\ell(\langle x_2,x_4\rangle)^4\,dx_2\,dx_4 \int_{B^2_{r_\ell}} P_\ell(\langle x_1,x_3\rangle)^4\,dx_1\,dx_3\Big)^{1/2}.
\end{aligned}$$

and hence we need to compute

$$\int_{\{(x_1,x_2)\in B^2_{r_\ell}:\theta(x_1,x_2)<\delta\}} P_\ell(\cos\langle x_1,x_2\rangle)^4 1_{B_{r_\ell}}(x_1)1_{B_{r_\ell}}(x_2)\,dx_1dx_2.$$

As we have already done, since we need to treat with continuously differentiable functions, we replace $1_{B_{r_\ell}}$ with the approximation φ^i_ℓ and we exploit the function $\tilde{W}^{\tilde{\varphi}^i}$ to solve the integral

$$\int_{\{\theta(x_1,x_2)<\delta\}} P_\ell(\cos(\langle x_1,x_2\rangle))^4 \varphi^i_\ell(x_1)\varphi^i_\ell(x_2)\,dx_1dx_2. \tag{3.3.59}$$

Using Fubini, (3.3.59) becomes

$$8\pi^2\int_0^\delta P_\ell(\cos\theta)^4 W^{\varphi^i_\ell}(\theta)d\theta, \tag{3.3.60}$$

changing variable $\theta = \frac{\psi}{L}$ and in view of (3.2.14), (3.3.60) is equal to

$$\frac{2\pi m(\mathbb{S}^2)}{L} r^3_\ell \int_0^{\delta L} P_\ell(\cos\frac{\psi}{L})^4 \frac{\psi}{Lr_\ell}\tilde{W}_0(\frac{\psi}{Lr_\ell})d\psi. \tag{3.3.61}$$

We decompose the integral in the two domains $[0,C]$ and $[C,\delta L]$, $C>0$; the leading term of the expansion of $P_\ell(\cos\frac{\psi}{L})^4$ in (A.1.14) ([77]), namely the first, and (3.2.16) imply that (3.3.61) is

$$\ll \frac{r^2_\ell}{L^2}\int_C^{\delta L} \frac{3}{2}\frac{1}{\ell^2(\sin\frac{\psi}{L})^2}\psi 2\pi||\varphi||_{L^2(\tilde{B}_1)}\,d\psi \ll \frac{r^2_\ell}{L^2}\int_C^{\delta L}\frac{1}{\psi}\,d\psi \sim \frac{r^2_\ell}{L^2}\log(\delta L).$$

Regarding the first interval, we can bound the Legendre polynomial with 1 and use (3.2.16), so that

$$\frac{r^2_\ell}{L^2}\int_0^C \psi 2\pi||\varphi||_{L^2(\tilde{B}_1)}\,d\psi \ll \frac{r^2_\ell}{\ell^2}.$$

Finally, passing to the limit, we get the value of

$$\int_{\{(x_1,x_2)\in B^2_{r_\ell}:\theta(x_1,x_2)<\delta\}} P_\ell(\cos\langle x_1,x_2\rangle)^4 1_{B_{r_\ell}}(x_1)1_{B_{r_\ell}}(x_2)\,dx_1dx_2.$$

For the local contribution, $A_2(loc)$ is easily seen to be

$$A_2(loc) \ll \Big(\frac{r_\ell^2}{\ell^2}\log r_\ell \ell \cdot \frac{r_\ell^2}{\ell^2}\log \delta\ell\Big)^{1/2} \cdot \Big(\frac{r_\ell^2}{\ell^2}\log r_\ell \ell\Big)^{2/2} = \frac{r_\ell^4}{\ell^4}\log r_\ell \ell \sqrt{\log r_\ell \ell}\sqrt{\log \delta\ell}$$

and $A_1(loc)$, by Jensen inequality is

$$\begin{aligned} A_1(loc) &= \int_{\mathscr{L}(\delta)^c} P_\ell(\langle x_1, x_2\rangle)P_\ell(\langle x_1, x_3\rangle)^3 P_\ell(\langle x_3, x_4\rangle)P_\ell(\langle x_2, x_4\rangle)^3\, dx \\ &\le \Big(\int_{\mathscr{L}(\delta)^c} P_\ell(\langle x_1, x_2\rangle)^4 P_\ell(\langle x_3, x_4\rangle)^4\, dx\Big)^{1/4}\Big(\int_{\mathscr{L}(\delta)^c} P_\ell(\langle x_1, x_3\rangle)^4 P_\ell(\langle x_2, x_4\rangle)^4\, dx\Big)^{3/4} \\ &\ll \Big(\frac{r_\ell^4}{\ell^4}\log r_\ell \ell \log \delta\ell\Big)^{1/4}\Big(\frac{r_\ell^4}{\ell^4}\log^2 r_\ell \ell\Big)^{3/4} = \frac{r_\ell^4}{\ell^4}(\log r_\ell \ell)^{7/4}(\log \delta\ell)^{1/4}. \end{aligned} \tag{3.3.62}$$

In the end, the fourth cumulant of the fourth chaotic projection is

$$\begin{aligned} cum_4\{h_{\ell,r_\ell,4}\} &= O\Big(\frac{1}{\ell\delta}\frac{r_\ell^4 \log r_\ell \ell}{\ell^4} + \frac{r_\ell^4 \log(\delta\ell)^{1/4}\log(r_\ell \ell)^{7/4}}{\ell^4} + \frac{r_\ell^4}{\ell^4}\log r_\ell \ell \sqrt{\log r_\ell \ell}\sqrt{\log \delta\ell}\Big) \\ &= O\Big(\frac{r_\ell^4}{\ell^4}(\log \delta\ell)^{1/4}(\log r_\ell \ell)^{7/4}\Big). \end{aligned} \tag{3.3.63}$$

□

Proof of Theorem 0.0.3

Denoting

$$\tilde{\mathscr{M}}_{\ell,r_\ell} = \frac{\mathscr{M}_{\ell,r_\ell}}{\mathrm{Var}(\mathscr{M}_{\ell,r_\ell})},$$

we have that

$$cum_4(\tilde{\mathscr{M}}_{\ell,r_\ell}) = O\Big(\frac{(\log(\delta\ell))^{1/8}}{(\log r_\ell \ell)^{1/8}}\Big); \tag{3.3.64}$$

hence, choosing

$$\delta := \delta(\ell) = O\big(\frac{\log r_\ell \ell}{\ell}\big), \tag{3.3.65}$$

(3.3.64) goes to zero and the CLT holds, i.e.,

$$d_W(\tilde{\mathscr{M}}_{\ell,r_\ell}, \mathscr{N}(0,1)) \le \sqrt{\frac{1}{2\pi}\{\mathbb{E}[\tilde{\mathscr{M}}_{\ell,r_\ell} - 3]\}} = O\Big(\Big(\frac{\log(\delta\ell)}{\log r_\ell \ell}\Big)^{1/8}\Big)$$

and thus

$$d_W(\tilde{\mathscr{Z}}_{\ell,r_\ell}, \mathscr{N}(0,1)) \le d_W(\tilde{\mathscr{M}}_{\ell,r_\ell}) + \sqrt{\mathbb{E}[\tilde{\mathscr{Z}}_{\ell,r_\ell} - \tilde{\mathscr{M}}_{\ell,r_\ell}]^2} = O\Big(\Big(\frac{\log(\delta\ell)}{\log r_\ell \ell}\Big)^{1/8}\Big),$$

which if $o(1)$, because of (3.3.65).

Remark 3.3.5. *Summarizing, in order to prove the CLT, δ (introduced in 3.3.55) is a sequence $\delta := \delta(\ell)$ such that*

- $\delta < r_\ell$,
- $\delta = O\big(\frac{\log r_\ell \ell}{\ell}\big)$.

3.4 Further results

3.4.1 Correlation between $\mathscr{L}_{\ell,r_\ell}$ and $\mathscr{L}(T_\ell)$

As anticipated in the Introduction, contrary to the 2-dimensional torus, the nodal length on the total sphere and the one on its subregions are not correlated; indeed we prove here Proposition 0.0.4. First, we compute the covariance in the lemma below.

Lemma 3.4.1. *The covariance between $\mathscr{L}_{\ell,r_\ell}$ and $\mathscr{L}(T_\ell)$ is given by*

$$\mathrm{Cov}(\mathscr{L}_{\ell,r_\ell},\mathscr{L}(T_\ell)) = \frac{m(B_{r_\ell})}{m(\mathbb{S}^2)}\mathrm{Var}(\mathscr{L}(T_\ell)).$$

Proof. We can write the covariance as

$$\begin{aligned}\mathbb{E}[\mathscr{L}_{\ell,r_\ell}\cdot\mathscr{L}(T_\ell)] &= E\Big[\int_{\mathbb{S}^2}||\nabla(T_\ell(x))||\delta(T_\ell(x))\,dx\int_{B_{r_\ell}}||\nabla T_\ell(y)||\delta(T_\ell(y))\Big]dy\\ &= \int_{\mathbb{S}^2\times B_{r_\ell}}\mathbb{E}[||\nabla T_\ell(x)||||\nabla T_\ell(y)||\delta(T_\ell(x))\delta(T_\ell(y))]\,dx\,dy \qquad (3.4.1)\\ &= \int_{\mathbb{S}^2\times B_{r_\ell}}\tilde{K}_\ell(x,y)\,dx\,dy = m(B_{r_\ell})\int_{\mathbb{S}^2}\tilde{K}_\ell(N,y)\,dy,\end{aligned}$$

which exploiting the isotropy of the two-point correlation function becomes

$$= 2\pi m(B_{r_\ell})\int_0^\pi \tilde{K}_\ell(N,x(\rho))\sin\rho\,d\rho.$$

Hence, we have that

$$\mathrm{Cov}(\mathscr{L}_{\ell,r_\ell};\mathscr{L}(T_\ell)) = 2\pi m(B_{r_\ell})\int_0^\pi \tilde{K}_\ell(N,x(\rho))\sin\rho\,d\rho - \frac{\ell(\ell+1)}{2}\frac{m(B_{r_\ell})}{2}2\pi$$

and since the variance of $\mathscr{L}(T_\ell)$ is

$$\mathrm{Var}(\mathscr{L}(T_\ell)) = 2\pi m(\mathbb{S}^2)\int_0^\pi \tilde{K}_\ell(\rho)\sin(\rho)d\rho - \frac{4\pi^2\ell(\ell+1)}{2},$$

([77] eq. 35) we get that

$$\mathrm{Cov}(\mathscr{L}_{\ell,r_\ell};\mathscr{L}(T_\ell)) = \frac{m(B_{r_\ell})}{m(\mathbb{S}^2)}\mathrm{Var}(\mathscr{L}(T_\ell)).$$

□

Now, we prove Proposition 0.0.4,

Proof of Proposition 0.0.4. By definition the correlation is

$$\mathrm{Corr}(\mathscr{L}_{\ell,r_\ell};\mathscr{L}(T_\ell)) = \frac{\mathrm{Cov}(\mathscr{L}_{\ell,r_\ell};\mathscr{L}(T_\ell))}{\sqrt{\mathrm{Var}(\mathscr{L}_{\ell,r_\ell})}\sqrt{\mathrm{Var}(\mathscr{L}(T_\ell))}} \qquad (3.4.2)$$

and Lemma 3.4.1 implies

$$\begin{aligned}\mathrm{Corr}(\mathscr{Z}_{\ell,r_\ell};\mathscr{Z}(T_\ell)) &= \frac{m(B_{r_\ell})}{m(\mathbb{S}^2)}\frac{\sqrt{\mathrm{Var}(\mathscr{Z}(T_\ell))}}{\sqrt{\mathrm{Var}(\mathscr{Z}_{\ell,r_\ell})}} = \frac{2\pi(1-\cos r_\ell)}{4\pi}\frac{\sqrt{\mathrm{Var}(\mathscr{Z}(T_\ell))}}{\sqrt{\mathrm{Var}(\mathscr{Z}_{\ell,r_\ell})}} \\ &= \frac{(1-\cos r_\ell)}{2}\frac{\sqrt{\mathrm{Var}(\mathscr{Z}(T_\ell))}}{\sqrt{\mathrm{Var}(\mathscr{Z}_{\ell,r_\ell})}};\end{aligned} \tag{3.4.3}$$

since the variance of $\mathscr{Z}(T_\ell)$ is

$$\mathrm{Var}(\mathscr{Z}(T_\ell)) = \frac{1}{32}\log \ell + O(1) \tag{3.4.4}$$

and in view of (3.4.4) and (0.0.6), it results that

$$\begin{aligned}\mathrm{Corr}(\mathscr{Z}_{\ell;r_\ell};\mathscr{Z}(T_\ell)) &= \frac{1-\cos r_\ell}{2}\sqrt{\frac{\frac{1}{32}\log\ell+O(1)}{\frac{r_\ell^2}{256}\log \ell r_\ell + O(r_\ell^2)}} = \frac{1-\cos r_\ell}{2r_\ell}\sqrt{\frac{\log \ell}{\log(r_\ell \ell)}+O(1)}\sqrt{8} \\ &= \frac{1-\cos r_\ell}{2r_\ell^2}\sqrt{r_\ell^2\frac{\log\ell}{\log r_\ell \ell}+O(r_\ell^2)}\sqrt{8} = O\left(\sqrt{{r_\ell}^2\frac{\log \ell}{\log r_\ell \ell}}\right),\end{aligned} \tag{3.4.5}$$

as claimed. □

Remark 3.4.2. *Note that*

$$r_\ell^2\frac{\log \ell}{\log r_\ell \ell} \to 0$$

as $\ell \to \infty$; *indeed,*

$$r_\ell^2\frac{1}{\log r_\ell \ell} < \frac{1}{\log \ell} \iff r_\ell^2 < \frac{\log r_\ell \ell}{\log \ell},$$

the second inequality holding by assumptions.

3.5 Technical tools

In this section we add some technical results proved in [77] and [48], which have been exploited in some computations.

3.5.1 2-point correlation function

Let $\tilde{K}_\ell(x,y) = \tilde{K}_\ell(d(x,y))$ be the 2-point correlation function, defined as

$$\tilde{K}_\ell(x,y) = \frac{1}{(2\pi)\sqrt{1-P_\ell(x,y)^2}}\mathbb{E}[||\nabla T_\ell(x)||\cdot||\nabla T_\ell(y)|| | T_\ell(x) = T_\ell(y) = 0].$$

We report for completeness the computations presented in [77], i.e.

$$\begin{aligned}\tilde{K}_\ell(x,y) = \frac{1}{\sqrt{1-P_\ell(x,y)^2}}\int_{\mathbb{R}^2\times\mathbb{R}^2} ||w_1||\cdot||w_2||\times \exp\left(-\frac{1}{2}(w_1,w_2)\Omega_\ell(x,y)^{-1}(w_1,w_2)^t\right) \\ \times\frac{dw_1 dw_2}{(2\pi)^3\sqrt{det\Omega_\ell(x,y)}},\end{aligned} \tag{3.5.1}$$

where $\Omega_\ell(x,y) = C - B^t A^{-1} B$,

$$A = \begin{pmatrix} 1 & P_\ell(x,y) \\ P_\ell(x,y) & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} \overrightarrow{0} & \nabla_y P_\ell(x,y) \\ \nabla_x(x,y) & \overrightarrow{0} \end{pmatrix}$$

and

$$C = \begin{pmatrix} \frac{\ell(\ell+1)}{2} I_2 & H \\ H^t & \frac{\ell(\ell+1)}{2} I_2 \end{pmatrix}$$

with $H = (h_{jk})_{j,k=1,2}$ with entries given by

$$h_{jk} = \frac{\partial}{\partial e_j^x \partial e_k^y P_\ell(x,y)}.$$

Using polar coordinates one can rewrite it as

$$\tilde{K_\ell}(\rho) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{\sqrt{1-P_\ell(x)^2}} ||w_1||\cdot||w_2|| \times \exp\left(-\frac{1}{2}(w_1,w_2)\Omega_\ell(\rho)^{-1}(w_1,w_2)^t\right) \frac{dw_1 dw_2}{(2\pi)^3\sqrt{det\Omega_\ell(\rho)}}$$

where

$$\Omega_\ell(\rho) = \begin{pmatrix} \frac{\ell(\ell+1)}{2}+\tilde{a} & 0 & \tilde{b} & 0 \\ 0 & \frac{\ell(\ell+1)}{2} & 0 & \tilde{c} \\ \tilde{b} & 0 & \frac{\ell(\ell+1)}{2}+\tilde{a} & 0 \\ 0 & \tilde{c} & 0 & \frac{\ell(\ell+1)}{2} \end{pmatrix}$$

and

$$\tilde{a} = \tilde{a}_\ell(\rho) = -\frac{1}{1-P_\ell(\cos\rho)} \cdot P_\ell'(\cos\theta)^2(\sin\theta)^2,$$
$$\tilde{b} = \tilde{b}_\ell(\rho) = P_\ell'(\cos\rho)\cos\rho - P_\ell''(\cos\rho)(\sin\rho)^2 - \frac{P_\ell(\cos\rho)}{1-P_\ell(\cos\rho)^2} \cdot P_\ell'(\cos\theta)^2(\sin\theta)^2,$$
$$\tilde{c} = \tilde{c}_\ell(\rho) = P_\ell'(\cos\rho).$$

The scaled two-point correlation function results to be

$$K_\ell(\psi) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{\sqrt{1-P_\ell(x)^2}} ||w_1||\cdot||w_2|| \times \exp\left(-\frac{1}{2}(w_1,w_2)\Delta_\ell(\psi)^{-1}(w_1,w_2)^t\right) \frac{dw_1 dw_2}{(2\pi)^3\sqrt{det\Delta_\ell(\psi)}},$$

with scaled covariance matrix

$$\Delta_\ell(\psi) = \frac{\Omega(\psi/L)}{(\ell(\ell+1))/2} = \begin{pmatrix} 1+2a & 0 & 2b & 0 \\ 0 & 1 & 0 & 2c \\ 2b & 0 & 1+2a & 0 \\ 0 & 2c & 0 & 1 \end{pmatrix}, \tag{3.5.2}$$

whose entries are explicitly given by

$$
\begin{aligned}
a = a_\ell(\psi) &= \frac{1}{\ell(\ell+1)}\tilde{a}_\ell(\psi/L) = -\frac{1}{\ell(\ell+1)}\frac{1}{1-P_\ell(\cos(\psi/L))^2}P_\ell'(\cos(\psi/L))^2\sin(\psi/L)^2,\\
b = b_\ell(\psi) &= \frac{1}{\ell(\ell+1)}\tilde{b}_\ell(\psi/L) = \frac{1}{\ell(\ell+1)}\Big[P_\ell'(\cos(\psi/L))\cos(\psi/L) - P_\ell''(\cos(\psi/L))\times\\
&\quad\times\sin(\psi/L)^2 - \frac{P_\ell(\cos(\psi/L))}{1-P_\ell(\cos(\psi/L))^2}P_\ell'(\cos(\psi/L))^2\sin(\psi/L)^2\Big]\\
c = c_\ell(\psi) &= \frac{1}{\ell(\ell+1)}\tilde{c}_\ell(\psi/L) = \frac{1}{\ell(\ell+1)}P_\ell'(\cos(\psi/L)).
\end{aligned}
\tag{3.5.3}
$$

In [77] the following proposition is proved.

Proposition 3.5.1. *For any choice of $C > 0$, as $\ell \to \infty$, one has*

$$
\begin{aligned}
K_\ell(\psi) &= \frac{1}{4} + \frac{1}{2}\frac{\sin(2\psi)}{\pi\ell\sin(\psi/L)} + \frac{1}{256}\frac{1}{\pi^2\ell\sin(\psi/L)\psi} + \frac{9}{32}\frac{\cos(2\psi)}{\pi\ell\psi\sin(\psi/L)} +\\
&\quad + \frac{\frac{27}{64}\sin(2\psi) - \frac{75}{256}\cos(4\psi)}{\pi^2\ell\psi\sin(\psi/L)} + O\left(\frac{1}{\psi^3} + \frac{1}{\ell\psi}\right)
\end{aligned}
\tag{3.5.4}
$$

uniformly for $C < \psi < \dfrac{\pi L}{2}$.

3.5.2 Expansion of the 2-point cross correlation function

Let $J_\ell(\psi, 4)$ be the 2-point cross correlation function, defined as

$$
J_\ell(\psi;4) = \left[-\frac{1}{4}\sqrt{\frac{\ell(\ell+1)}{2}}\frac{1}{4!}\right] \times \frac{8\pi^2}{L}\mathbb{E}[\psi_\ell(\bar{x},4)H_4(T_\ell(y(\frac{\psi}{L})))];
$$

the following expansion is proved in [48].

Proposition 3.5.2. *For any constant $C > 0$, uniformly over ℓ we have that, for $0 < \psi < C$,*

$$
J_\ell = O(\ell), \tag{3.5.5}
$$

and, for $C < \psi < m\frac{\pi}{2}$,

$$
J_\ell(\psi,4) = \frac{1}{64}\frac{1}{\psi\sin(\psi/L)} + \frac{5}{64}\frac{\cos 4\psi}{\psi\sin(\psi/L)} - \frac{3}{16}\frac{\sin(2\psi)}{\psi\sin(\psi/L)} + O(\frac{1}{\psi^2}\frac{1}{\sin(\psi/L)}) + O(\frac{1}{\ell\psi}\frac{1}{\sin(\psi/L)}). \tag{3.5.6}
$$

APPENDIX A

In this appendix, we report for completeness some further materials used in the thesis. More precisely, in the first part, we give explicit characterizations of Legendre polynomials and spherical harmonics, referring also to the Hilb's formula; in the second part, some properties and definitions about Clebsch-Gordan coefficients are collected. Most of the results below are taken from [72], [45] and [75].

A.1 Orthogonal Polynomials and Spherical Harmonics

A.1.1 Legendre Polynomials

We indicate with $P_\ell(x),\ x \in [-1,1]$, the orthonormal system of Legendre Polynomials, which are defined by

$$P_\ell(x) := \frac{(-1)^\ell}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(1-x^2)^\ell = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell.$$

They are solution of the *Legendre's equation*

$$(1-x^2)\frac{d^2}{dx^2}P_\ell(x) - 2x\frac{d}{dx}P_\ell(x) + \ell(\ell+1)P_\ell(x) = 0$$

and they satisfy the following recurrent relation

$$(\ell+1)P_{\ell+1}(x) = (2\ell+1)xP_\ell(x) - \ell P_{\ell-1}(x), \quad \forall \ell \in \mathbb{N}, \quad x \in [-1,1].$$

The following properties are recalled.

Remark A.1.1. $P_\ell(1) = 1$ *for all* ℓ.

Remark A.1.2. *The Legendre polynomials are orthogonal. The normalized constant is given by*

$$\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx = \frac{2}{2\ell+1}\delta_\ell^{\ell'}.$$

Proof. We follow the proof in [72]. Then, we have that

$$\int_{-1}^{1} P_\ell(x)^2\,dx = l_\ell \int_{-1}^{1} P_\ell(x)x^\ell\,dx, \tag{A.1.1}$$

where

$$l_\ell = \lim_{x\to\infty} \frac{P_\ell(x)}{x^\ell} = 2^{-\ell}\binom{2\ell}{\ell}.$$

Now integrating by parts ℓ times, (A.1.1) is equal to

$$\begin{aligned}\int_{-1}^{1} P_\ell(x)^2\,dx &= \frac{l_\ell \ell!}{2^\ell \ell!}\int_{-1}^{1}(1-x)^\ell(1+x)^\ell\,dx = \frac{l_\ell}{2^\ell}\int_{-1}^{1}(1-x)^\ell(1+x)^\ell\,dx \\ &= \frac{l_\ell}{2^{\ell-1}}\int_{0}^{1}(1-x)^\ell(1+x)^\ell\,dx = \frac{l_\ell}{2^{\ell-1}}\int_{0}^{1}\left(\sum_{k=0}^{\ell}\binom{\ell}{k}x^{\ell-k}\right)(1-x)^\ell\,dx\end{aligned} \tag{A.1.2}$$

for the Newton binomial's formula. Since

$$\int_0^1 x^{p+1}(1-x)^{q+1}\,dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

$\Gamma(\cdot)$ denoting, as usual, the Gamma function, using that $\Gamma(\ell+1)=\ell!$, we have that (A.1.2) becomes

$$\begin{aligned}&\frac{l_\ell}{2^{\ell-1}}\sum_{k=0}^{\ell}\binom{\ell}{k}\frac{\Gamma(\ell-k+1)\Gamma(\ell+1)}{\Gamma(2\ell-k+2)} = \frac{l_\ell}{2^{\ell-1}}\Gamma(\ell+1)\sum_{k=0}^{\ell}\frac{\ell!}{k!(2\ell-k+1)!} \\ =&\frac{l_\ell}{2^{\ell-1}}\ell!\frac{\ell!}{(2\ell+1)!}\sum_{k=0}^{\ell}\binom{2\ell+1}{k}\end{aligned} \tag{A.1.3}$$

substituting the value of l_ℓ, and the series with

$$\sum_{k=0}^{\ell}\binom{2\ell+1}{k} = \sum_{k=\ell+1}^{2\ell+1}\binom{2\ell+1}{k} = \frac{2^{2\ell+1}}{2},$$

it follows that (A.1.3) is equal to

$$= \frac{\ell!}{2^{\ell-1}}\frac{\ell!}{(2\ell+1)!}2^{-\ell}\binom{2\ell}{\ell}2^{2\ell} = \frac{2}{2\ell+1}$$

and the proof is completed. □

A.1.2 Spherical Harmonics and associated Legendre functions

As we have already stated in Section 1.3, the spherical harmonics are a complete set of orthonormal eigenfunctions of $\Delta_{\mathbb{S}^2}$, the spherical Laplacian, which satisfy

$$\Delta_{\mathbb{S}^2} Y_{\ell,m} = -\ell(\ell+1)Y_{\ell,m} \quad \ell=0,1,2,\ldots,\; m=-\ell,\ldots,\ell.$$

A possible suitable basis of the spherical harmonics is given by (1.3.2) ([45]), which we rewrite for sake of simplicity

$$Y_{\ell,m}(\theta,\varphi) := \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}P_{\ell,m}(\cos\theta)\exp(im\varphi), \text{ for } m\geq 0, \tag{A.1.4}$$

$$Y_{\ell,m}(\theta,\varphi) := (-1)^m \overline{Y}_{\ell,-m}(\theta,\varphi)\text{ , for } m<0, \tag{A.1.5}$$

where the associated Legendre functions $P_{\ell m}(x)$ are given by, for $m \geq 0$

$$P_{\ell,m}(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x), \tag{A.1.6}$$

$$P_{\ell,-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_{\ell m}(x). \tag{A.1.7}$$

It should be noted that $P_{\ell,m}$ is actually a polynomial if and only if m is even. Moreover they form an orthogonal (but not orthonormal) set of functions on the interval $[-1,1]$,

$$\int_{-1}^{1} P_{\ell,m}(x)P_{\ell',m}(x)\,dx \;=\; \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell}^{\ell'}$$

(see also [45]). We recall now some properties of the spherical harmonics, whose proofs are given, for instance, in [45].

- *Orthonormality*: for all ℓ,ℓ',m,m',

$$\int_0^{\pi}\int_0^{2\pi} Y_{\ell,m}(\theta,\phi)\overline{Y_{\ell',m'}}(\theta,\phi)\sin\theta\,d\phi\,d\theta = \delta_{\ell}^{\ell'}\delta_{m}^{m'}, \tag{A.1.8}$$

- *Simmetry*: for all $x\in\mathbb{S}^2$

$$\overline{Y}_{\ell,m}(x) = (-1)^m Y_{\ell,-m}(x),$$

- *Addition Formula* ([45] Chapter 3): for all $x,y\in\mathbb{S}^2$

$$\sum_{m=-\ell}^{\ell} Y_{\ell,m}(x)\overline{Y_{\ell,m}(y)} = \frac{2\ell+1}{4\pi}P_\ell(\langle x,y\rangle), \tag{A.1.9}$$

- *Central Spherical Harmonics:*

$$Y_{\ell,0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta),$$

- *Duplication property:*

$$\int_{\mathbb{S}^2}\frac{2\ell+1}{4\pi}P_\ell(\langle x,y\rangle)\frac{2\ell'+1}{4\pi}P_{\ell'}(\langle y,z\rangle)\,dy = \frac{2\ell+1}{4\pi}P_\ell(\langle x,z\rangle)\delta_{\ell}^{\ell'}. \tag{A.1.10}$$

A.1.3 Hilb's asymptotics

As far as the Legendre Polynomials are concerned, it is important to adduce the Hilb's formula, which has been largely exploited along the computations of this thesis.

Lemma A.1.3 (Hilb's Asymptotics (formula (8.21.17) on page 197 in [72])).

$$P_\ell(\cos\phi) = \left(\frac{\phi}{\sin\phi}\right)^{1/2} J_0((\ell+1/2)\phi) + \delta(\phi), \tag{A.1.11}$$

uniformly for $0\le\phi\le\pi/2$, J_0 the Bessel J function of order 0, defined in (2.5.1), and the error term is

$$\delta(\phi) \ll \begin{cases} \phi^{1/2}O(\ell^{-3/2}), & C\ell^{-1}<\phi<\pi/2 \\ \phi^2 O(1), & 0<\phi<C\ell^{-1}, \end{cases}$$

where $C>0$ is any constant and the constants involved in the "O"-notation depend on C only.

In particular, for $\theta \in [0, \pi/2]$,

$$P_\ell(\cos\theta) \ll \frac{1}{\sqrt{\ell\theta}}.$$

Actually, changing variable $\Psi = L\theta$, with $L = \ell + \frac{1}{2}$, we have that

$$P_\ell\left(\cos\left(\frac{\psi}{\ell + 1/2}\right)\right) \sim J_0(\psi)$$

and

$$J_0(\psi) = \sqrt{\frac{2}{\pi}} \frac{\cos(\psi - \pi/4)}{\sqrt{\psi}} + O\left(\frac{1}{\psi^{3/2}}\right)$$

(see also [52]). The Hilb's asymptotic has been applied in many framework, for instance in [77], where the following two lemmas have been proved.

Lemma A.1.4. *(Lemma B.3 [77]) The Legendre polynomials P_ℓ and its couple of derivatives satisfy uniformly for $\ell \geq 1, \psi > C$:*

$$\begin{aligned} P_\ell(\cos\psi/L) &= \sqrt{\frac{2}{\pi\ell\sin(\psi/L)}}\left(\sin(\psi + \frac{\pi}{4}) - \frac{1}{8}\frac{\cos(\psi + \frac{\pi}{4})}{\psi}\right) + O\left(\frac{1}{\psi^{5/2}} + \frac{1}{\sqrt{\psi}\ell}\right) \\ P_\ell''(\cos\psi/L) &= -\frac{\ell^2}{\sin(\psi/L)^2}P_\ell(\cos\psi/L) + \frac{2}{\sin(\psi/L)^2}P_\ell'(\cos\psi/L) + O\left(\frac{\ell^3}{\psi^{5/2}}\right) \end{aligned} \tag{A.1.12}$$

Lemma A.1.5. *For $\ell \geq 1, C < \psi < \pi L/2$ we have the following estimate*

$$P_\ell(\cos(\psi/L))^2 = \frac{1 + \sin(2\psi)}{\pi\ell\sin(\psi/L)} + O\left(\frac{1}{\psi^2}\right). \tag{A.1.13}$$

Finally, we report also the following expansion for $P_\ell(\cos\frac{\psi}{L})^4$, which has been used in Section 3.3.2, given in [77]: for $\ell \geq 1$ and $C < \psi < \pi L/2$,

$$P_\ell(\cos(\psi/L))^4 = \frac{\frac{3}{2} - 2\sin(2\psi) - \frac{1}{2}\cos(4\psi)}{\pi^2\ell^2\sin(\psi/L)^2} + O\left(\frac{1}{\psi^3}\right). \tag{A.1.14}$$

A.2 The Clebsch-Gordan coefficients

For a complete treatment of this topic, we refer to [45] and [75].

A.2.1 Euler Angles

The *Euler Angles* ([45]) are a tool to define an element $g \in SO(3)$. Let us consider the reference system $(x, y, z) \in \mathbb{R}^3$ and the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, whose equator plane coincides with the xy plane and the north pole is along the direction of the z axis.
A matrix $g \in SO(3)$ acts on the sphere, rotating the reference system. So, let n be the so-called *line of nodes*, that is the line obtained by the intersection of the planes xy and $x'y'$, where (x', y', z') are the new axes; we characterize such rotation with three angles (φ, θ, ψ) as follows: the angle φ is from x to n, the angle θ is from z to z', the angle ψ is from n to

x'. Hence, we associate the triple $(\varphi, \theta, \psi) \in [0, 2\pi] \times [0, \pi] \times [0, 2\pi]$ to the matrix g. These angles are uniquely determined except when the two planes xy and $x'y'$ coincide, the z axis and the z' axis having the same or opposite directions. Indeed, in the first case, $\theta = 0$ and only $(\varphi + \psi)$ is uniquely defined, and, similarly, when they have opposite direction, $\theta = \pi$ and only $(\varphi - \psi)$ is uniquely defined. Then, these angles allow us to decompose the rotation g into three easier rotations (see [45] for a proof): a rotation by an angle ψ around the z axis such that x coincides with n, a rotation by an angle θ around the *new* x' axis (that coincides with n) and a rotation by an angle φ around the *new* axis z'. Thus, we can write

$$g = R_z(\varphi) R_x(\theta) R_z(\psi)$$

according our condition of effecting rotations about the *new* axis.

A.2.2 Wigner's D matrices

A complete set of irreducible matrix representations for SO(3) is given by the so-called Wigner's D matrices [45]

$$\{D^\ell(g) : \ell = 0, 1, 2, \dots\}$$

having dimension $(2\ell + 1) \times (2\ell + 1)$ for every $\ell = 0, 1, 2, \dots$.
Wigner's matrices $D^\ell(\cdot)$ can be defined as a function of Euler angles, by

$$D^\ell(g^{-1}) = \{D^\ell_{mn}(\varphi, \theta, \psi)\}_{m,n=-\ell,\dots,\ell},$$

$$D^\ell_{mn}(\varphi, \theta, \psi) = \exp(-im\varphi) d^\ell_{m,n}(\cos\theta) \exp(-in\psi),$$

and the $d^\ell_{m,n}(\cos\theta)$ are the so-called Wigner's $d(\cdot)$ functions, which are given by [45]

$$d^\ell_{m,n}(\cos\theta) = \left[\frac{(\ell - n)!(\ell + n)!}{(\ell - m)!(\ell + m)!} \right] \left(\sin\frac{\theta}{2} \right)^{m-n} \left(\cos\frac{\theta}{2} \right)^{m+n} P^{m-n, m+n}_{\ell - m}(\cos\theta).$$

It is important to note that there is a relation between the element of these matrices and spherical harmonics, namely

$$Y_{\ell m}(\theta, \varphi) = (-1)^m \sqrt{\frac{2\ell + 1}{4\pi}} D^\ell_{-m,0}(0, \theta, \psi)$$

[45]. In words, it means that the spherical harmonics correspond to the "central column" of the Wigner's D matrices.

A.2.3 Clebsch-Gordan coefficients

Clebsch-Gordan matrices and coefficients have a central role in the evaluation of multiple integrals of spherical harmonics. A way to explain them is through the Wigner D matrices [45]. For a fixed pair (ℓ_1, ℓ_2), the tensor product representation $D^{\ell_1} \otimes D^{\ell_2}$ and the direct sum representation $\oplus_{\ell = |\ell_2 - \ell_1|}^{\ell_2 + \ell_1} D^\ell$ have dimension

$$(2\ell_1 + 1)(2\ell_2 + 1) \times (2\ell_1 + 1)(2\ell_2 + 1)$$

and they are unitarily equivalent. Thanks to the theory of group representation (Chap 2.4.2 [45]), there exists a unitary matrix $C_{\ell_1\ell_2}$, such that

$$\left\{D^{\ell_1} \times D^{\ell_2}\right\} = C_{\ell_1\ell_2}\left\{ \oplus_{\ell=|\ell_2-\ell_1|}^{\ell_2+\ell_1} D^{\ell}\right\}C^*_{\ell_1\ell_2},$$

where C^* denotes the transpose matrix of C. This unitary matrix is known as a $Clebsch-Gordan\ matrix$; it is a $\{(2\ell_1+1)(2\ell_2+1)\times(2\ell_1+1)(2\ell_2+1)\}$ block matrix, whose blocks, of dimensions $(2\ell_2+1)\times(2\ell+1)$, are usually denoted by $C^{\ell}_{\ell_1(m_1)\ell_2}, \ell = |\ell_2-\ell_1|,\dots,\ell_1+\ell_2, m_1 = -\ell_1,\dots,\ell_1$. The elements of the ℓth block are indexed by m_2 (over rows) and m (over columns). More precisely,

$$C_{\ell_1\ell_2} = \left[C^{\ell\cdot}_{\ell_1(m_1)\ell_2\cdot}\right]_{m_1=-\ell_1,\dots,\ell_1;\ell=|\ell_2-\ell_1|,\dots,\ell_2+\ell_1}$$

$$C^{\ell\cdot}_{\ell_1(m_1)\ell_2\cdot} = \left\{C^{\ell m}_{\ell_1 m_1 \ell_2 m_2}\right\}_{m_2=-\ell_2,\dots,\ell_2;m=-\ell,\dots,\ell}.$$

In the case of $SO(3)$, they are defined as the set $\{C^{\ell m}_{\ell_1 m_1 \ell_2 m_2}\}$ of the elements of the unitary matrices $C_{\ell_1\ell_2}$, with the convention of Triangle conditions: The Clebsch-Gordan coefficient vanishes unless

$$|\ell_1-\ell_2| \le \ell_3 \le \ell_1+\ell_2,$$

and the requirement

$$m_1+m_2=m_3$$

is satisfied. As a consequence of the orthonormality of row vectors and the orthonormality of columns, the following two *unitary* relations hold (see [45], [75]):

$$\sum_{m_1 m_2} C^{\ell m}_{\ell_1 m_1 \ell_2 m_2} C^{\ell' m'}_{\ell_1 m_1 \ell_2 m_2} = \delta^{\ell'}_{\ell}\delta^{m'}_{m}, \tag{A.2.1}$$

$$\sum_{\ell m} C^{\ell m}_{\ell_1 m_1 \ell_2 m_2} C^{\ell m}_{\ell_1 m_1' \ell_2 m_2'} = \delta^{m_1'}_{m_1}\delta^{m_2'}_{m_2}; \tag{A.2.2}$$

The analytic expression is known (see [45], [72]):

$$\begin{aligned} C^{\ell_3 - m_3}_{\ell_1 m_1 \ell_2 m_2} &:= (-1)^{\ell_1+m_1}\sqrt{2\ell_3+1}\left[\frac{(\ell_1+\ell_2-\ell_3)!(\ell_1-\ell_2+\ell_3)!(\ell_1-\ell_2+\ell_3)!}{(\ell_1+\ell_2+\ell_3+1)!}\right]^{1/2} \\ &\times\left[\frac{(\ell_3+m_3)!(\ell_3-m_3)!}{(\ell_1+m_1)!(\ell_1-m_1)!(\ell_2+m_2)!(\ell_2-m_2)!}\right]^{1/2} \\ &\times\sum_z \frac{(-1)^z(\ell_2+\ell_3+m_1-z)!(\ell_1-m_1+z)!}{z!(\ell_2+\ell_3-\ell_1-z)!(\ell_3+m_3-z)!(\ell_1-\ell_2-m_3+z)!}, \end{aligned} \tag{A.2.3}$$

where the summation runs over all z's such that the factorials are non-negative. This expression becomes neater for $m_1=m_2=m_3=0$, where we have

$$C^{\ell_3 0}_{\ell_1 0 \ell_2 0} = \begin{cases} 0, & \\ & \text{for } \ell_1+\ell_2+\ell_3 \text{ odd} \\ \frac{(-1)^{\frac{\ell_1+\ell_2-\ell_3}{2}}[(\ell_1+\ell_2+\ell_3)/2]!}{[(\ell_1+\ell_2-\ell_3)/2]![(\ell_1-\ell_2+\ell_3)/2]![(-\ell_1+\ell_2+\ell_3)/2]!}\left\{\frac{(\ell_1+\ell_2-\ell_3)!(\ell_1-\ell_2+\ell_3)!(-\ell_1+\ell_2+\ell_3)!}{(\ell_1+\ell_2+\ell_3+1)!}\right\}^{1/2}, & \\ & \text{for } \ell_1+\ell_2+\ell_3 \text{ even} \end{cases} \tag{A.2.4}$$

Furthermore, other important properties which are satisfied by these coefficients are given here below (see [45], [72]).

- *Symmetry* Properties

$$\begin{aligned} C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} &= (-1)^{\ell_1+\ell_2-\ell_3} C^{\ell_3 m_3}_{\ell_2 m_2 \ell_1 m_1} = (-1)^{\ell_1-m_1}\sqrt{\frac{2\ell_3+1}{2\ell_2+1}} C^{\ell_2 -m_2}_{\ell_1 m_1 \ell_3 -m_3} \\ &= (-1)^{\ell_1-m_1}\sqrt{\frac{2\ell_3+1}{2\ell_2+1}} C^{\ell_2 m_2}_{\ell_3 m_3 \ell_1 -m_1} (-1)^{\ell_2 m_2}\sqrt{\frac{2\ell_3+1}{2\ell_1+1}} C^{\ell_1 -m_1}_{\ell_3 -m_3 \ell_2 m_2} \\ &= (-1)^{\ell_2 m_2}\sqrt{\frac{2\ell_3+1}{2\ell_1+1}} C^{\ell_1 m_1}_{\ell_2 -m_2 \ell_3 m_3}, \end{aligned} \tag{A.2.5}$$

$$C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} = (-1)^{\ell_1+\ell_2-\ell_3} C^{\ell_3 -m_3}_{\ell_1 -m_1 \ell_2 -m_2}; \tag{A.2.6}$$

- *Mirror* Properties

$$C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} = C^{\ell_2 m_2}_{\bar{\ell}_1 \bar{m}_1 \ell_3 m_3}. \tag{A.2.7}$$

Finally, the integrals of multiple spherical harmonics are related to these coefficients by the following proposition.

Proposition A.2.1. *For all ℓ_1, ℓ_2, ℓ_3,*

$$\int_{S^2} Y_{\ell_1 m_1}(x) Y_{\ell_2 m_2}(x) \overline{Y}_{\ell_3 m_3}(x) d\sigma(x) = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_3+1)}} C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} C^{\ell_3 0}_{\ell_1 0 \ell_2 0}, \tag{A.2.8}$$

where we used the convention for those integers ℓ_1, ℓ_2, ℓ_3 not verifying the triangle conditions. In particular, the previous relation implies that $C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} = 0$ whenever $m_1 + m_2 \neq m_3$.

This integral is called *Gaunt integral*. For the proof of Proposition A.2.1 see [45], Section 3.5.2.

Wigner 3j coefficients

Wigner 3j coefficients (see [45], [75]) are related to Clebsch-Gordan coefficients by the identities

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{\ell_3+m_3} \frac{1}{\sqrt{2\ell_3+1}} C^{\ell_3 m_3}_{\ell_1 -m_1 \ell_2 -m_2} \tag{A.2.9}$$

$$C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} = (-1)^{\ell_1-\ell_2+m_3}\sqrt{2\ell_3+1} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}. \tag{A.2.10}$$

From [45], we have that for any ℓ_1, ℓ_2, ℓ_3, the following upper bound holds

$$\left| \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \right| \leq [\max\{2\ell_1+1, 2\ell_2+1, 2\ell_3+1\}]^{-1/2} \tag{A.2.11}$$

and this implies that

$$|C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2}| \leq \sqrt{2\ell_3+1}[\max\{2\ell_1+1, 2\ell_2+1, 2\ell_3+1\}]^{-1/2}. \tag{A.2.12}$$

The following symmetry properties of the 3j Symbols are valid:

- Permutations of columns

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{\ell_1+\ell_2+\ell_3} \begin{pmatrix} \ell_1 & \ell_3 & \ell_2 \\ m_1 & m_3 & m_2 \end{pmatrix}, \tag{A.2.13}$$

- change of signs of momentum projection

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{\ell_1+\ell_2+\ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \tag{A.2.14}$$

For special values of the arguments, namely if $\ell_3 = 0$ or $\ell_2 = 0$, one has explicit forms of this coefficients:

$$C^{00}_{\ell_1 m_1 \ell_2 m_2} = (-1)^{\ell_1 - m_1} \frac{\delta^{\ell_2}_{\ell_1} \delta^{-m_2}_{m_1}}{\sqrt{2\ell_1 + 1}}, \tag{A.2.15}$$

$$C^{\ell_3 m_3}_{\ell_1 m_1 00} = \delta^{\ell_3}_{\ell_1} \delta^{m_3}_{m_1}. \tag{A.2.16}$$

For details see [75].

Wigner 6j coefficients

The Wigner 6j coefficient ([75], Chapter 9) is a sum of this type

$$\sum_{\alpha,\beta,\gamma,\epsilon,\delta,\phi} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} a & e & f \\ \alpha & \epsilon & -\phi \end{pmatrix} \begin{pmatrix} d & b & f \\ -\delta & \beta & \gamma \end{pmatrix} \begin{pmatrix} d & e & c \\ \delta & -\epsilon & \gamma \end{pmatrix}$$

and it is denoted by the symbol

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}.$$

The 6j symbol is invariant under any permutation of its columns or under interchange of the upper and lower arguments in each of any two columns (see [75]):

$$\begin{aligned}
&\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{Bmatrix} a & c & b \\ d & f & e \end{Bmatrix} = \begin{Bmatrix} b & a & c \\ e & d & f \end{Bmatrix} = \begin{Bmatrix} b & c & a \\ e & f & d \end{Bmatrix} = \begin{Bmatrix} c & a & b \\ f & d & e \end{Bmatrix} = \begin{Bmatrix} c & b & a \\ f & e & d \end{Bmatrix} \\
&\begin{Bmatrix} a & e & f \\ d & b & c \end{Bmatrix} = \begin{Bmatrix} a & f & e \\ d & c & b \end{Bmatrix} = \begin{Bmatrix} e & a & f \\ b & d & c \end{Bmatrix} = \begin{Bmatrix} e & f & a \\ b & c & d \end{Bmatrix} = \begin{Bmatrix} f & a & e \\ c & d & b \end{Bmatrix} = \begin{Bmatrix} f & e & a \\ c & b & d \end{Bmatrix} \\
&\begin{Bmatrix} d & e & c \\ a & b & f \end{Bmatrix} = \begin{Bmatrix} d & c & e \\ a & f & b \end{Bmatrix} = \begin{Bmatrix} e & d & c \\ b & a & f \end{Bmatrix} = \begin{Bmatrix} e & c & d \\ b & f & a \end{Bmatrix} = \begin{Bmatrix} c & d & e \\ f & a & b \end{Bmatrix} = \begin{Bmatrix} c & e & d \\ f & b & a \end{Bmatrix} \\
&\begin{Bmatrix} d & b & f \\ a & e & c \end{Bmatrix} = \begin{Bmatrix} d & f & b \\ a & c & e \end{Bmatrix} = \begin{Bmatrix} b & d & f \\ e & a & c \end{Bmatrix} = \begin{Bmatrix} b & f & d \\ e & c & a \end{Bmatrix} = \begin{Bmatrix} f & d & b \\ c & a & e \end{Bmatrix} = \begin{Bmatrix} f & b & d \\ c & e & a \end{Bmatrix}.
\end{aligned} \tag{A.2.17}$$

Moreover, the following upper bound holds (see [45]):

$$\left| \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \right| \leq \min\left(\frac{1}{\sqrt{(2c+1)(2f+1)}}, \frac{1}{\sqrt{(2a+1)(2d+1)}}, \frac{1}{\sqrt{(2b+1)(2e+1)}} \right). \tag{A.2.18}$$

Finally, we recall that, if one of the arguments is equal to zero, one has

$$\begin{aligned}
\begin{Bmatrix} a & b & c \\ 0 & e & f \end{Bmatrix} &= (-1)^{a+b+e} \frac{\delta_b^f \delta_c^e}{\sqrt{(2b+1)(2c+1)}}, \\
\begin{Bmatrix} a & 0 & c \\ d & e & f \end{Bmatrix} &= (-1)^{a+d+e} \frac{\delta_a^c \delta_d^f}{\sqrt{(2a+1)(2d+1)}}, \\
\begin{Bmatrix} a & b & c \\ d & 0 & f \end{Bmatrix} &= (-1)^{a+b+d} \frac{\delta_a^f \delta_c^d}{\sqrt{(2a+1)(2c+1)}}.
\end{aligned} \tag{A.2.19}$$

Wigner 9j coefficients

For the proper definition of the Wigner 9j coefficient see [75]; it is denoted by

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix}.$$

Here we just give the relation with the 6j symbol in the easier case where one of the arguments is equal to zero, namely

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & 0 \end{Bmatrix} = \delta_c^f \delta_g^h \frac{(-1)^{b+c+d+g}}{[(2c+1)(2g+1)]^{1/2}} \begin{Bmatrix} a & b & c \\ e & d & g \end{Bmatrix} \tag{A.2.20}$$

Symmetry properties hold also in these context (see [75] for all of them), the one used in (2.4.34) is the following:

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{Bmatrix} = \begin{Bmatrix} a & d & g \\ b & e & h \\ c & f & j \end{Bmatrix}. \tag{A.2.21}$$

Then, in view of these properties, (A.2.20) and (A.2.21), we get also that

$$\begin{aligned}
\begin{Bmatrix} 0 & c & c \\ g & e & b \\ g & d & a \end{Bmatrix} &= \begin{Bmatrix} c & 0 & c \\ d & g & a \\ e & g & b \end{Bmatrix} = \begin{Bmatrix} g & g & 0 \\ e & d & c \\ b & a & c \end{Bmatrix} = \begin{Bmatrix} g & b & e \\ 0 & c & c \\ g & a & d \end{Bmatrix} = \begin{Bmatrix} a & g & d \\ c & 0 & c \\ b & g & e \end{Bmatrix} = \begin{Bmatrix} b & a & c \\ g & g & 0 \\ e & d & c \end{Bmatrix} \\
&= \begin{Bmatrix} c & e & d \\ c & b & a \\ 0 & g & g \end{Bmatrix} = \begin{Bmatrix} d & c & e \\ a & c & b \\ g & 0 & g \end{Bmatrix} = \begin{Bmatrix} a & b & c \\ d & e & c \\ g & g & 0 \end{Bmatrix} \\
&= \frac{(-1)^{b+d+c+g}}{[(2c+1)(2g+1)]^{1/2}} \begin{Bmatrix} a & b & c \\ e & d & g \end{Bmatrix}.
\end{aligned} \tag{A.2.22}$$

An explicit formula is known in the following particular case

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{Bmatrix} = \frac{\delta_a^d \delta_b^e \delta_c^f}{[(2a+1)(2b+1)(2c+1)]^{1/2}}. \tag{A.2.23}$$

Finally, the last important relation we need to collect is the one given by the sums involving the products of four Clebsch-Gordan coefficients, namely, one has that

$$\sum_{\beta\gamma\epsilon\varphi} C^{a\alpha}_{b\beta c\gamma} C^{d\delta}_{e\epsilon f\varphi} C^{g\eta}_{e\epsilon b\beta} C^{j\mu}_{f\varphi c\gamma} = (-1)^{a-b+c+d+e-f} \sum_{s\sigma} \prod_{ssag} C^{j\mu}_{a\alpha s\sigma} C^{d\delta}_{g\eta s\sigma} \begin{Bmatrix} b & c & a \\ j & s & f \end{Bmatrix} \begin{Bmatrix} b & e & g \\ d & s & f \end{Bmatrix} \tag{A.2.24}$$

$$= \prod_{adgj} \sum_{ki} C^{ki}_{g\eta j\mu} C^{ki}_{d\delta a\alpha} \begin{Bmatrix} c & b & a \\ f & e & d \\ j & g & k \end{Bmatrix}$$

(see p. 260 [75]).

Acknowledgments

In this last part I would like to thank all the people who have been part of my life in these three years and, hence, have contributed, with their help, their support and, most of all, their friendship, to the achievement of the end of my PhD.

First of all, Prof. Domenico Marinucci. There are so many reasons to say thanks. To begin, for teaching me a lot of beautiful math and for all your tentatives to explain me how to write properly. Then, for following my work and for addressing me during this research, for all the suggestions, the ideas, for the patience (most of all for the many times you explained me the Diagram Formula), for your constant presence at the university (I really appreciated to find you in the office even without warning you) and for the trust. Thank you for your general advices, not only regarding math but also life, for the book and "Hermite". In the end, for suggested me to apply to this PhD, which gave me beautiful experiences and emotions!

I sincerely thank Prof. Igor Wigman, for sharing with me ideas, suggestions and for all the useful discussions. Thank you, for giving me the opportunity to visit your nice department and for your essential help in the development of part of this research.

Thanks to Prof. Marcati, for allowing me to go at King's College of London and to visit other departments, giving me the possibility to meet other researchers and to grow up in the university academy; thank you for the trust.

Many thanks to Maurizia and Valentina for some clarifications and comments, to Claudio and Marco for some suggestions, to Alessia for the help with Python and HEALPix and Roberto for Mathematica; you all, for the friendship.

Thanks also to Fabio Durastante for Matlab notions and Susanna Risa for some Riemannian questions.

I would like to thank the department of Rome of Tor Vergata, in particular the family of the office 1225: Gianluca, Rossana, Dario, Michele, Giulia, Lorenzo, Duccio, for the friendship and the support. Specially Michele, for the help in some analytic staffs and for always replying to my messages in the middle of night when I could not sleep.

Thanks to the people of the King's College of London for the warm hospitality, to Giuseppe Tinaglia for the help with the bureaucracy, and most of all to the PhD students, for making me feel at ease right away. To Emilio and Riccardo for the advices, to Gioia and Sirio for your friendship and the musicals/movies (even if Sirio slept).

Many thanks to my London flatmates: Ale and Tony, for all the chocolate and the dinners together. In particular, Tony for teaching me some meals and Ale for pretending to support the Roma football team and watching its matches with me and pizza!

I heartily thank the two “moschettieri”: Martina and Elena; for your friendship, the musicals, and all the adventures and the amazing time we spent together in London.

I deeply thank the “Bombers in Math”: Martina, Giulia, Anna, Maurizia. For all the conversations and the laughs, for sharing with me nice moments but also the bad ones, for your help, for the sweet words and, when necessary, the strongest ones. Thank you because although the distance you were close to me.

Thanks to my “old” University colleagues. In particular, Jacopo and Giulio, for the support, the funny time and for changing the wheel of my car when I drilled; Roberto, for helping me to use Latex at the beginning; to Giovanna, Simone and the Puglia’s Girls for the great holiday.

Thanks to all the people of the GSSI, specially my PhD colleagues, for sharing these three years with me. In particular, Costanza even for splitting the office (you know what I mean), Michele, even for your kind words in a difficult moment, Aleksander, Filippo and “Prince” Ahmed. Moreover, thank you Antonio for helping me with the template of the poster and for the dinners with mozzarella di bufala!

A special acknowledgment is for Gennaro, Lars and Roberto (it is alphabetic order, do not discuss!). Thank you for your patience all the time I complained or I did not come in L’Aquila, for your help, for all the time I felt sick and you took care of me, for the lift to Amiternum, for all the laughs and the beautiful, funny and crazy time we spent together! Thank you for being always by my side. All moments with you are dear to me.

Thanks to “the best apartment”: Silvia and Myrto, for your kindness, for all the dinners and the “merendine” together, for giving me a bed when I closed myself outside in my new apartment, simply for your friendship.

Many thanks to Cug, because even if sometimes we lose contacts, we are always ready to find each other; Sivi, for the “profile”, the “gelatini” and most of all for your strength and positivity, Raffaella for letting me speak and complain, Fede, Marty, Mukky and Fru always close to me. Thanks to Elena, and to my English teacher Egy, for helped me to study for the IELTS.

Thanks to the Beach volley team for all the spare time and the competitions. To Adriano, Cindy (and Milo), Daniele and Edoardo, for the walks and the friendship, to Emanuele, for the wise suggestions and the breakfasts, to Aldo, thank you, even for the songs.

Thanks to all my family, my parents, my brother Francesco and my sisters Valentina, Nicoletta, Raffaella, to Alessandro, Giulia, Giulio and in particular to Alessio, the most important person in our life together with your brother on the way, your smile gives us joy everyday.

I like finishing thanking and remembering Prof. Di Biagio, always in my memories.

To conclude, the research of this thesis has been financially supported by the GSSI and the UMI with the Grant for Visiting student, I would like to thank these Institutions for this.

BIBLIOGRAPHY

[1] Abramowitz, M.; Stegun, I. A. (1964) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. National Bureau of Standards Applied Mathematics Series, 55 For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1964 xiv+1046 pp. (Reviewer: D. H. Lehmer) 33.00 (65.05)

[2] Adler R. J. (1981) *The Geometry of Random Fields*, Wiley and Sons, New York.

[3] Adler, R. J.; Taylor, J. E. (2007) *Random Fields and Geometry*. Springer Monographs in Mathematics. Springer, New York.

[4] Andrews, G. E; Askey, R.; Roy, R. (1999) *Special Functions Enciclopedia of Mathematics and its Applications*, 71, Cambridge University Press, Cambridge.

[5] Atkinson, K.; Han, W. (2012) *Spherical Harmonics and Approximations on the Unit Sphere: an introduction, Lecture Notes in Mathematics*, 2044, Springer.

[6] Azais, J. M.; Wschebor, M. (2009) *Level Sets and Extrema of Random Processes and Fields*, Wiley and Sons, New Jersey.

[7] Baldi, P.; Kerkyacharian, G.; Marinucci, D.; Picard, D. (2009) Asymptotics for Spherical Needlets, *Annals of Statistics*, Vol. 37, No. 3, 1150-1171.

[8] Baldi, P.; Kerkyacharian, G.; Marinucci, D.; Picard, D. (2009) Subsampling Needlet Coefficients on the Sphere, *Bernoulli*, Vol. 15, 438-463.

[9] Baldi, P.; Trapani, S. (2015) Fourier Coefficients of Invariant Random Fields on Homogeneous Spaces of Compact Lie Groups, *Ann. Inst. Henri Poincaré Probab. Stat.* 51, no. 2, 648-671.

[10] Beffara, V.; Gayet, D. (2017) Percolation of Random Nodal Lines. Publ. *Math. Inst. Hautes Etudes* Sci. 126, 131-176.

[11] Benatar, J.; Marinucci, D.; Wigman I. (2017) Planck-Scale Distribution of Nodal Length of Arithmetic Random Waves, preprint, arXiv:1710.06153.

[12] Berard, P. (1985) Volume des Ensembles Nodaux des Fonctions Propres du Laplacien. (French) [Volume of the nodal sets of eigenfunctions of the Laplacian] Bony-Sjöstrand-Meyer seminar, 1984–1985, Exp. No. 14 , 10 pp., École Polytech., Palaiseau.

[13] Berry, M. V. (2002) Statistics of Nodal Lines and Points in Chaotic Quantum Billiards: Perimeter Corrections, Fluctuations, Curvature, *J. Phys.* A 35, no. 13, 3025–3038.

[14] Berry, M. V. (1977) Regular and Irregular Semiclassical Wavefunctions. *J. Phys.* A 10, no. 12, 2083–2091. 81.58.

[15] Bleher, P.; Shiffman, B.; Zelditch, S. (2000) Universality and Scaling of Correlations between Zeros on Complex Manifolds *Invent. Math.* 142, no. 2, 351-395.

[16] Bleher, P.; Shiffman, B.; Zelditch, S. (2001). Universality and Scaling of Zeros on Symplectic Manifolds Random Matrix Models and their Applications, 31-69, *Math.Sci. Res. Inst. Publ.*, 40, Cambridge Univ. Press, Cambridge.

[17] Bourgain, J., Rudnick, Z. (2011) On the Geometry of the Nodal Lines of Eigenfunctions on the two-dimensional Torus, *Ann. Henri Poincaré* 12, no. 6, 1027-1053.

[18] Brockwell, P. J.; Davis, R. A. (1996) *Introduction to Time Series and Forecasting*, Springer-Verlag, New York, Inc.

[19] Buckley, J.; Wigman, I. (2016) On the Number of Nodal Domains of Toral Eigenfunctions. (English summary), *Ann. Henri Poincaré 17*, no. 11, 3027-3062.

[20] Cammarota, V. (2017) Nodal Area Distribution For Arithmetic Random Waves, arXiv:1708.07679v1.

[21] Cammarota, V.; Marinucci, D. (2018) A Quantitative Central Limit Theorem for the Euler-Poincaré Characteristic of Random Spherical Eigenfunctions. *Annals of Probabability* 46, no. 6, 3188–3228.

[22] Cammarota, V.; Marinucci, D., (2015) On the Limiting Behavior of Needlets Polyspectra, (English, French summary) *Ann. Inst. Henri Poincaré Probab. Stat.* 51, no. 3, 1159–1189.

[23] Cammarota, V.; Marinucci, D.; Wigman, I. (2016) Fluctuations of the Euler-Poincaré Characteristic for Random Spherical Harmonics, *Proceedings of the American Mathematical Society*, 11, 4759-4775.

[24] Cammarota, V.; Marinucci, D.; Wigman, I. (2016) On the Distribution of the Critical Values of Random Spherical Harmonics, *Journal of Geometric Analysis*, 4, 3252-3324.

[25] Chatterjee, S. (2009) Fluctuations of Eigenvalues and Second Order Poincaré Inequalities. *Probab. Theory Related Fields* 143, no. 1-2, 1–40.

[26] Cheng, S. Y. (1976) Eigenfunctions and Nodal Sets, *Comm. Math. Helv.* 51, no. 1, 43–55.

[27] Cramér, H.; Leadbetter, M. R. (1967), *Stationary and Related Stochastic Processes*, Wiley, New York.

[28] Dalmao, F.; Nourdin, I.; Peccati, G.; Rossi, M. (2016) Phase Singularities in Complex Arithmetic Random Waves. arXiv:1608.05631v3.

[29] Donnelly, H.; Fefferman, C. (1988) Nodal Sets of Eigenfunctions on Riemannian Manifolds, *Invent. Math.* 93, 161-183.

[30] Fantaye, Y.; Hansen, F.K.; Maino, D.; Marinucci, D. (2015) Cosmological Applications of the Gaussian Kinematic Formula, *Physical Review D*, Volume 91, 063501.

[31] Geller, D.; Mayeli, A. (2009) Continuous Wavelets on Manifolds, *Math. Z.*, Vol. 262, pp. 895-927.

[32] Granville, A.; Wigman, I. (2017) Planck-Scale Mass Equidistribution of Toral Laplace Eigenfunctions. *Comm. Math. Phys.*, 355, no. 2, 767–802.

[33] Krishnapur, M.; Kurlberg P.; Wigman, I. (2013) Nodal Length Fluctuations for Arithmetic Random Waves. *Annals of Mathematics* (2) 177, no. 2, 699-737.

[34] Lan, X.; Marinucci, D.; Xiao, Y. (2018) Strong Local Nondeterminism and Exact Modulus of Continuity for Spherical Gaussian Fields. *Stochastic Process. Appl.* 128, no. 4, 1294-1315.

[35] Lang, A.; Schwab, C. (2015) Isotropic Gaussian Random Fields on the Sphere: Regularity, Fast Simulation and Stochastic Partial Differential Equations. *Ann. Appl. Probab.* 25, no. 6, 3047-3094.

[36] Lester, S.; Rudnick, Z. (2017) Small Scale Equidistribution of Eigenfunctions on the Torus, *Comm. Math. Phys.*, 350 (1), 279-300.

[37] Lewis, A. (2012) The Full Squeezed CMB Bispectrum from Inflation. *Journal of Cosmology and Astroparticle Physics*, 06, 023.

[38] Logunov, A. (2018) Nodal sets of Laplace Eigenfunctions: Polynomial Upper Estimates of the Hausdorff Measure. *Ann. of Math.* (2) 187, no. 1, 221-239.

[39] Logunov, A. (2018) Nodal Sets of Laplace Eigenfunctions: Proof of Nadirashvili's Conjecture and of the Lower Bound in Yau's Conjecture. *Ann. of Math.* (2) 187, no. 1, 241-262.

[40] Logunov, A.; Malinnikova, E. (2015) On Ratios of Harmonics Functions. *Adv. Math.* 274, 241-262.

[41] Marinucci, D. (2008) A Central Limit Theorem and Higher Order Results for the Angular Bispectrum, *Probability Theory and Related Fields*, Vol. 141, N.3-4, pp. 389-409, math.pr/0509430.

[42] Marinucci, D. (2006) High Resolution Asymptotics for the Angular Bispectrum of Spherical Random Fields, *Annals of Statistics*, Vol.34, N.1, pp. 1-4.

[43] Marinucci, D. (2015) Lecture Notes on Spherical Random Fields. Lecture notes, Finnish School in Probability and Statistics, June 2015.

[44] Marinucci, D.; Peccati, G. (2013) Mean-square Continuity on Homogeneous Spaces of Compact Groups. (English summary) *Electron. Commun. Probab.* 18, no. 37, 10 pp.

[45] Marinucci, D.; Peccati, G. (2011) *Random fields on the Sphere. Representation, Limit Theorems and Cosmological Applications*. London Mathematical Society Lecture Note Series, 389. Cambridge University Press, Cambridge.

[46] Marinucci, D.; Peccati, G.; Rossi, M.; Wigman, I. (2016) Non-Universality of Nodal Length Distribution for Arithmetic Random Waves. (English summary) *Geom. Funct. Anal.* 26, no. 3, 926-960.

[47] Marinucci, D.; Rossi, M. (2015) Stein-Malliavin Approximations for Nonlinear Functionals of Random Eigenfunctions on $\mathbb{S}^d$. *J. Funct. Anal.* 268, no. 8, 2379-2420.

[48] Marinucci, D.; Rossi, M.; Wigman, I. (2017) The Asymptotic Equivalence of the Sample Trispectrum and the Nodal Length for Random Spherical Harmonics, preprint, arXiv:1705.05747.

[49] Marinucci, D.; Vadlamani, S. (2016) High-Frequency Asymptotics for Lipschitz-Killing Curvatures of Excursion Sets on the Sphere. *Ann. Appl. Probab.* 26, no. 1, 462-506.

[50] Marinucci, D.; Wigman, I. (2014) On Nonlinear Functionals of Random Spherical Eigenfunctions. *Comm. Math. Phys.* 327, no. 3, 849-872.

[51] Marinucci, D.; Wigman, I. (2011) On the Excursion Sets of Spherical Gaussian Eigenfunctions. *J. Math. Phys.* 52, no. 9, 093301, 21 pp.

[52] Marinucci, D.; Wigman, I. (2011) The Defect Variance of Random Spherical Harmonics. *Journal of Physics A: Mathematical and Theoretical*, 44, 355206.

[53] Matsubara, T. (2010) Analytic Minkowski Functionals of the Cosmic Microwave Background: Second-Order non-Gaussianity with Bispectrum and Trispectrum. *Phys. Rev. D* 81, 083505.

[54] Narcowich, F. J.; Petrushev, P.; Ward, J.D. (2006) Decomposition of Besov and Triebel-Lizorkin Spaces on the Sphere, *Journal of Functional Analysis*, 238, 2, 530-564.

[55] Narcowich, F. J.; Petrushev, P.; Ward, J.D. (2006) Localized Tight Frames on Spheres, *SIAM Journal of Mathematical Analysis*, 38, 2, 574-594.

[56] Neuheisel, J. (2000) The Asymptotic Distribution of Nodal Sets on Spheres, Johns Hopkins Ph.D. thesis.

[57] Nourdin, I.; Peccati, G. (2012) *Normal Approximations with Malliavin Calculus. From Stein's Method to Universality.* Cambridge Tracts in Mathematics, 192. Cambridge University Press, Cambridge.

[58] Nourdin, I.; Peccati, G. (2010) Stein's Method Meets Malliavin Calculus: a Short Survey with new Estimates. Recent Development in Stochastic Dynamics and Stochastic Analysis, 207–236, *Interdiscip. Math. Sci.*, 8, World Sci. Publ., Hackensack, NJ, 2010.

[59] Nourdin, I.; Peccati, G.; Rossi, M. (2017) Nodal Statistics of Planar Random Waves. arXiv:1708.02281v1.

[60] Peccati, G.; Rossi, M. (2017). Quantitative Limit Theorems for Local Functionals of Arithmetic Random Waves. *Computation and Combinatorics in Dynamics, Stochastics and Control, The Abel Symposium 2016* – Springer (in press). arXiv:1702.03765v1.

[61] Peccati, G.; Tudor, C.A. (2007) Anticipating Integrals and Martingales on the Poisson Space, *Random Oper. Stoch. Equ.* 15, no. 4, 327–352.

[62] Planck 2013 Results. XXIV. Constraints on Primordial non-Gaussianity (Planck Collaboration), *Astronomy and Astrophysics*, Volume 571, idA24, 58 pp. (2014).

[63] Planck 2013 Results. XXIII. Isotropy and Statistics of the CMB (Planck Collaboration), *Astronomy and Astrophysics*, Volume 571, idA23. (2014)

[64] Rossi, M. (2018) Random Nodal Lengths and Wiener Chaos. *Proceedings of the Workshop "Probabilistic Methods in Spectral Geometry and PDE" in Montréal, August 2016* (to appear). arXiv:1803.09716.

[65] Rossi, M. (2016) The Defect of Random Hyperspherical Harmonics, *Journal of Theoretical Probability* (in press). arXiv:1605.03491.

[66] Rossi, M. (2016) The Geometry of Spherical Random Fields - PhD thesis. arXiv:1603.07575.

[67] Rossi, M.; Wigman, I. (2018) Asymptotic Distribution of Nodal Intersections for Arithmetic Random Waves. *Nonlinearity* 31, no. 10, 4472-4516.

[68] Rudnick, Z.; Wigman I. (2016) Nodal Intersections for Random Eigenfunctions on the Torus, *American Journal of Mathematics*, 138, no. 6, 1605-1644.

[69] Rudnick, Z.; Wigman, I. (2008) On the Volume of Nodal Sets for Eigenfunctions of the Laplace on the Torus, *Annales Henri Poincaré*, Vol.9, No 1, 109-130.

[70] Rudnick, Z.; Wigman, I.; Yesha N. (2015) Nodal Intersections for Random Waves on the 3-dimensional Torus, *Annales Institut Fourier*, 66, no. 6, 2455-2484.

[71] Schoenberg, I.J. (1942) Positive Definite Functions on Spheres. *Duke Math. J.* 9, 96-108.

[72] Szego, G. (1975) Orthogonal Polynomials, Fourth edition. *American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I.*

[73] Todino, A. P. (2018) A Quantitative Central Limit Theorem for the Excursion Area of Random Spherical Harmonics over Subdomains of $\mathbb{S}^2$. arXiv:1807.06982

[74] Todino, A. P. (2018) Nodal Lengths in Shrinking Domains for Random Eigenfunctions on $\mathbb{S}^2$. arXiv:1807.11787.

[75] Varshalovich, D. A.; Moskalev, A. N.; Khersonski, V. K. (1988) *Quantum Theory of Angular Momentum. Irreducible Tensors, Spherical Harmonics, Vector Coupling Coefficients, 3nj symbols*. Translated from the Russian. World Scientific Publishing Co., Inc., Teaneck, NJ.

[76] Wigman, I. (2011) On the Nodal Lines of Random and Deterministic Laplace Eigenfunctions, *Spectral geometry*, *Proc. Sympos. Pure Math.*, vol. 84, *Amer. Math. Soc.*, *Providence, RI*, 2012, pp. 285-297.

[77] Wigman, I. (2010) Fluctuations of the Nodal Length of Random Spherical Harmonics, *Communications in Mathematical Physics, 398 no. 3 787-831*.

[78] Wigman, I. (2009) On the Distribution of the Nodal Sets of Random Spherical Harmonics. *Journal of Mathematical Physics*, 50, no. 1, 013521, 44 pp.

[79] Yau, S.T. (1993) Open Problems in Geometry. Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, CA, 1990), 1-28, *Proc. Sympos. Pure Math.*, 54, Part 1, *Amer. Math. Soc.*, *Providence, RI*.

[80] Yau, S.T. (1982) Survey on Partial Differential Equations in Differential Geometry. Seminar on Differential Geometry, pp. 3-71, *Ann. of Math. Stud.*, 102, Princeton Univ. Press, Princeton, N.J.