Contents

CHAPTER 1

Motivation

This paper studies the interplay between probability and PDE, which is divided into two major parts. The one is called first passage time problem (FPT problem) in Chapter 2 and the other is a free boundary problem arising from particle systems in Chapters 3 and 4. For the time being, let us start to discuss about some examples of the connection between probability and PDE by summarizing Einstein's argument ([21]). Considering the increment of particle positions on $\mathbb{R}$ as a random variable with a probability density function φ, let us say φ is an even function. Denote by $\rho(x,t)$ the density at location x at time t, which is the number of particles per unit volume at location x at time t. In a short time interval τ, a displacement of a particle takes place by ϵ. If we assume the conservation of mass (the total number of particles), then we have

$$\rho(x,t+\tau)=\int_{-\infty}^{\infty}\rho(x+\epsilon,t)\varphi(\epsilon)d\epsilon. \tag{1.0.1}$$

By Taylor's theorem with ignoring higher order terms, we obtain

$$\rho(x,t+\tau)\approx\rho(x,t)+\rho_t(x,t)\tau \tag{1.0.2}$$

and

$$\rho(x+\epsilon,t)\approx\rho(x,t)+\rho_x(x,t)\epsilon+\frac{1}{2}\rho_{xx}(x,t)\epsilon^2. \tag{1.0.3}$$

Putting (1.0.2) and (1.0.3) into (1.0.1), let $k=\frac{1}{2}\int_{-\infty}^{\infty}\epsilon^2\varphi(\epsilon)d\epsilon$, finally we deduce that ρ satisfies the heat equation:

$$\rho_t=k\rho_{xx}. \tag{1.0.4}$$

A mathematical rigorous description is that Brownian motion is a time-homogeneous Markov process with transition density

$$f(x,t\mid y)=\frac{1}{\sqrt{2\pi t}}e^{-(x-y)^2/2t}, \tag{1.0.5}$$

as a function of (x,t), the transition density satisfies the heat equation

$$\frac{\partial f}{\partial t}=\frac{1}{2}\frac{\partial^2 f}{\partial x^2} \tag{1.0.6}$$

with the initial condition

$$f(x,0\mid y)=\delta_y(x). \tag{1.0.7}$$

These are about well known diffusive behaviour of particles in the sense that the microscopic particle system can be described as the macroscopic point of view by using PDE.

In the other way, suppose that $u_0 : \mathbb{R} \to \mathbb{R}$ is continuous with compact support and let us define $u(x,t) := \int_{-\infty}^{\infty} u_0(y) f(x,t \mid y) dy$, then u solves the following PDE (IVP):

$$\begin{cases} u_t = \frac{1}{2} u_{xx}, \ x \in \mathbb{R}, \ t > 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}. \end{cases} \tag{1.0.8}$$

Next consider to establish stochastic representation when we are given not only the initial value but the boundary value called the initial Dirichlet boundary problem:

$$\begin{cases} u_t = \frac{1}{2} u_{xx}, \ \ x < 0, \ t > 0, \\ u(x,t) = 0, \ x = 0, \ t > 0, \\ u(x,0) = u_0(x), \ x < 0. \end{cases} \tag{1.0.9}$$

where u_0 is a sufficently regular function defined in $(-\infty, 0)$ such that $u_0(0^-) = 0$. One of the most common way to solve is to find the fundamental solution and then the solution is given by the convolution of the initial datum u_0 with it. (1.0.9) turns into the following initial Dirichlet boundary problem: for $x_0 < 0$,

$$\begin{cases} g_t = \frac{1}{2} g_{xx}, \ \ x < 0, \ t > 0 \\ g(x,t) = 0, \ x = 0, \ t > 0 \\ g(x,0) = \delta_{x_0}(x), \ x < 0. \end{cases} \tag{1.0.10}$$

As a consequence of the relation between Brownian motion and the heat equation, (1.0.10) can be interpreted as the probability density diffuses out over time; a Brownian particle starting at $x_0 < 0$ moves without touching absorbing boundary 0. Thus unexpectedly first passage time (FPT) problem is engaged in solving PDE. Actually g the probability density function satisfying 0 boundary condition is given by

$$g(x,t;x_0) = \frac{1}{\sqrt{2\pi t}} \left[\exp\left\{ -\frac{(x-x_0)^2}{2t} \right\} - \exp\left\{ -\frac{(x+x_0)^2}{2t} \right\} \right] \tag{1.0.11}$$

such that the solution of (1.0.10) is also given by

$$u(x,t) = \int_{-\infty}^{0} u_0(x_0) P_{x_0}[\tau_{x_0} > t] dx_0 = \int_{-\infty}^{0} u_0(x_0) g(x,t;x_0) dx_0, \tag{1.0.12}$$

where τ_{x_0} is the first hitting time as $\tau_{x_0} = \inf\{t \geq 0 : B_t = 0\}$ with $B_0 = x_0$.
Generally we are able to extend (1.0.10) to time varying boundaries X_t,

$$\begin{cases} g_t = \frac{1}{2} g_{xx}, \ \ x < X_t, \ t > 0 \\ g(x,t) = 0, \ x = X_t, \ t > 0 \\ g(x,0) = \delta_{x_0}(x), \ x < 0. \end{cases} \tag{1.0.13}$$

Thus it is necessary to study the regularity of X_t for the existence of the density function g. Unfortunately, this problem appears to be very difficult: an analytical explicit solution is known in only few cases (the examples shown in Chapter 2). In Chapter 2, we discuss about this and as a main result, for given Hölder continuous curve with exponent greater than 1/2, we show that there is a first passage density function through it.

As consider the particles move on the real line $\mathbb{R}$ so that each particle evolves as a brownian motion independently and duplicates a new particle via a branching process on the same position of its ancestor (Chap.3) or distributes it randomly with given probability (Chap.4). Moreover, at the same rate of creation in order to keep the total number of particles (mass conservation), the leftmost particle is eliminated. As particle dynamics is processing, the position of the leftmost changes in time where the killing procedure takes place. By a scaling limit procedure (hydrodynamic limit), one can have a macroscopic system from a microscopic evolution, which is a free boundary problem with Dirichlet boundary condition(the leftmost position corresponds the boundary).

In [32], it is stuided that under diffusive rescaling of particle system shows a hydrodynamic behaviour described by the solution of a Cauchy-Stefan problem. Let us introduce one of the simple form of the one-dimensional one-phase Stefan problem with Dirichlet boundary as follows:

$$\begin{cases} u_t = u_{xx}, \;\; 0 < x < s(t), \; t > 0, \\ u(s(t), t) = 0, \; t > 0, \\ u(x, 0) = u_0(x), \; 0 < x < s(t), \\ \dfrac{ds(t)}{dt} = -\dfrac{\partial u}{\partial x}(s(t), t), \; t > 0. \end{cases} \tag{1.0.14}$$

It is well known that under proper initial datum u_0, we can find a local in time solution pair (u, s) of (1.0.14)(see [22]) and we will study its variations.

Another example is in [8]; a standard branching Brownian motion is a continuous time Markov process in which each particle moves according to independent Brownian motions with reproduction at rate 1 and birth at rate 2. The process starts with a single particle at $0 \in \mathbb{R}$ at time t $= 0$. This particle moves according to a standard Brownian motion until time T, which is an independent exponential holding time with $P[T > t] = e^{-t}$. Then it is eliminated at time T and but simultaneously gives birth to two descendants on the same position at rate 1. Denoting by $\mathcal{N}_t$ the set of existing particles at time t in the process, for $w \in \mathcal{N}_t$, let $Y_t(w)$ be the position of the ancestor of w at time t if it was in the process at time t. The function

$$u(x, t) = P(\max_{w \in \mathcal{N}_t} Y_t(w) \geq x)$$

is a travelling wave solution of F-KPP equation $\partial_t u = \frac{1}{2}\partial_x^2 u + u - u^2$.

In addition, we introduce the following two dimensional FBP

$$\begin{cases} u_t = \dfrac{1}{2}\Delta_x u, \;\; \epsilon < |x| < R_t, \; t > 0 \\ u(x, t) = 0, \; |x| = R_t, \; t \geq 0 \\ u(x, 0) = u_0(x), \; \epsilon < |x| < R_0, \\ u_r(x, t) = -\dfrac{1}{2\pi|x|}, \;\; |x| = \epsilon, \; R_t, \; t \geq 0 \end{cases} \tag{1.0.15}$$

which is an experiment of mass transport: the system is confined in a moving annulus that only the outer boundary changes in time, we inject mass into the system from the inner boundary at rate 1, the mass diffuses in the annulus and it is removed once it reaches the outer boundary, the removal rate at the outer boundary is also 1 so that the total mass is conserved. We can show that there is a local in time solution by using a change of variables reducing to the one dimensional FBP so that we are able to adopt PDE techniques of [9](we do not show the proof in this paper!). In Chapters 3 and 4 similarly as (1.0.15), we prove local existence for classical solutions of a free boundary problem which arises in one of the biological selection models proposed by Brunet and Derrida, [5] and Durrett and Remenik, [20]. The problem we consider describes the limit evolution of branching brownian particles on the line with death of the leftmost particle at each creation time as studied in [18]. We use extensively results in [9] and [22].

CHAPTER 2

First Passage Time Densities through Hölder curves

First Passage Time(FPT) problem is a classical topic that has a number of applications that there are a bunch of articles; finance(e.g.[1],[40]), biology(e.g.[12],[13],[25]) and so on. The idea can be roughly described as we see the first time of Markov process M_t starting at 0 until it hits a moving boundary X_t with $X_0 > 0$. Precisely one can define the first passage time as

$$\tau = \inf\{t \geq 0 : M_t \geq X_t\}.$$

In this paper we focus on the case for M_t is a standard Brownian motion and X_t is a continuous curve. Then we have F the distribution induced by τ, that is,

$$F(t) = P(\tau \leq t).$$

In [35], it is proved that for any continuous curve X_t and $r < X_0$, there is a distribution F_r^X of τ_r^X which satisfies the following integral equation(called the Master Equation):

$$\Psi\left(\frac{z-r}{\sqrt{t}}\right) = \int_0^t \Psi\left(\frac{z-X_s}{\sqrt{t-s}}\right) F_r^X(ds), \tag{2.0.1}$$

where $z \geq X_t$, $t > 0$ and $\Psi(z) = \displaystyle\int_z^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$. This can be proved intuitively as follows; the left hand side of (2.0.1) is the probability that a Brownian motion starts at r at time 0 and reaches z greater or equal to X_t at time t. Then it should hit the boundary at least once which implies the right hand side of of (2.0.1). Moreover, it is showed that if X_t is C^1, then there exists a continuous density function f of F_r^X. There is an extension, in [36], to curves X which are differentiable with $\left|\dfrac{dX_t}{dt}\right| \leq Ct^{-\alpha}$ for some constant $C > 0$ and $\alpha < 1/2$, then F_r^X has a probability density function. For the case when $X_t = a + bt$, it is well known(see for instance [30]) that for $r < a$, τ_r^X has a probability density function given by $f(t) = \dfrac{a-r}{\sqrt{2\pi t^3}} \exp\left(-\dfrac{(a+bt-r)^2}{2t}\right) \mathbf{1}_{t>0}$. In addition, there is another result in [36] when $X_t = a + bt^{\frac{1}{p}}$ for $p \geq 2$, $r < a$, then τ_r^X has a probability density function.

The aim of this chapter is to find a sufficient condition for the existence of the derivative of F, which is the density function f, then we need to study a regularity of a boundary X. Hereafter we show that there is a first passage density function through a Hölder continuous curve with exponent greater than $1/2$, it seems to be plausible in terms of well known fact that Brownian motion is almost surely locally Hölder continuous with exponent α for any $\alpha < 1/2$, however it is hard to find out this result in any existing literature. For the case when Hölder continuous

with exponent less or equal to 1/2, we discuss about it at the end of this chapter.
By (2.0.1), for any continuous curve X_t and $r < X_0$, we have

$$\Psi\left(\frac{X_t - r}{\sqrt{t}}\right) = \int_0^t \Psi\left(\frac{X_t - X_s}{\sqrt{t-s}}\right) F_r^X(ds). \tag{2.0.2}$$

which can be regarded as an integral equation for $F_r^X(ds)$. In [37], it is studied the equation (2.0.2) when X is Hölder continuous with exponent greater than 1/2, it is proved that there exists a unique continuous function q such that

$$\Psi\left(\frac{X_t - r}{\sqrt{t}}\right) = \int_0^t \Psi\left(\frac{X_t - X_s}{\sqrt{t-s}}\right) q(s)ds. \tag{2.0.3}$$

To conclude that $F_r^X(ds) = q(s)ds$, one still needs that $F_r^X(ds)$ is absolutely continuous with respect to Lebesgue measure. In [2], when a moving boundary is infinitely differentiable, it is showed that the space derivative of the Green function of the heat equation at the boundary is proportional to the hitting time density function. We extend this relation between probability and PDE to our problem by using techniques in [9] extensively.

2.1 Introduction

To state the main result, we need some notations on a Brownian motion. Let us call $P_{r,s}, r \in \mathbb{R}, s \geq 0$, the law on $C([s,\infty))$ of the Brownian motion B_t, $t \geq s$, which starts from r at time s, i.e. $B_s = r$. For each $t > s$ the law of B_t is absolutely continuous with respect to the Lebesgue measure and has a density $G_{s,t}(r,\cdot)$ which is the Gaussian $G(\cdot,t;r,s) = \dfrac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\dfrac{(\cdot-r)^2}{2(t-s)}\right)$. We denote by $E_{r,s}$ the expectation under $P_{r,s}$. For a given curve $X = \{t \to X_t\}$, $s \geq 0$ and $r < X_s$, we define

$$\tau_{r,s}^X = \inf\{t \geq s : B_t \geq X_t\},\ and = \infty \text{ if the set is empty,} \tag{2.1.1}$$

where $B_s = r$ and denote by $F_{r,s}^X(dt)$ the distribution of $\tau_{r,s}^X$ induced by $P_{r,s}$. For s=0, we use abbreviated forms P_r, E_r, τ_r^X, $F_r^X(dx)$ instead of $P_{r,0}$, $E_{r,0}$, $\tau_{r,0}^X$, $F_{r,0}^X(dx)$ respectively whenever it is needed. In addition, for $t > 0$, let us call $d\mu_{r_0}(\cdot,t)$ the positive measure on $(-\infty, X_t)$ such that

$$\int_{(-\infty,X_t)} d\mu_{r_0}(x,t)f(x) = E_{r_0}[f(B_t); \tau_{r_0}^X > t] \tag{2.1.2}$$

for all $f \in C_c^\infty(\mathbb{R})$ with supp $f \Subset (-\infty, X_t)$. The main result in the paper is;

Theorem 2.1.1. *If X is Hölder continuous on any finite interval in $[0,\infty)$ with exponent $\gamma \in (1/2,1]$ and $r_0 < X_0$, then*

1. $d\mu_{r_0}(x,t) = G_{0,t}^X(r_0,x)dx$ *where for all* $x < X_t$,

$$G_{0,t}^X(r_0,x) = G_{0,t}(r_0,x) - \int_{[0,t)} F_{r_0}^X(ds) G_{s,t}(X_s,x). \tag{2.1.3}$$

2. $F^X_{r_0}(ds)$ *has a density function* p *on* $[0,\infty)$, *namely* $F^X_{r_0}(ds) = p(s)ds$.

3. $p(t) = -\frac{1}{2}\frac{\partial}{\partial x}G^X_{0,t}(r_0,x)\Big|_{x=X_t^-}$ *for all* $t > 0$.

4. $G^X_{0,t}(r_0,x)$ *solves*

$$v_t = \frac{1}{2}v_{xx}, \ \ -\infty < x < X_t, \ t > 0, \tag{2.1.4}$$

$$\lim_{(x,t)\to(X_s,s)} v(x,t) = 0, \ s > 0, \tag{2.1.5}$$

$$\lim_{(x,t)\to(y,0)} v(x,t) = \delta_{r_0}(y), \ -\infty < y < X_0. \tag{2.1.6}$$

Remarks

Item 4 of **Theorem 2.1.1** states that for any $r_0 < X_0$ the function $G^X_{0,\cdot}(r_0,\cdot)$ given by (2.1.3) is the Green function of the heat equation with Dirichlet boundary conditions at the moving boundary X. Likewise by items 2 and 3 of **Theorem 2.1.1**, the space derivative of $G^X_{0,\cdot}(r_0,\cdot)$ at the moving boundary is proportional to the hitting time density function p.

2.2 Outline of the proof of Theorem 2.1.1

Let X be a continuous curve defined on $[0,\infty)$ and let $r_0 < X_0$. Using the strong Markov property, we have, for $f \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \Subset (-\infty, X_t)$ and $0 \le s \le t$,

$$E_{r_0}[f(B_t)|\tau^X_{r_0} = s] = E_{X_s,s}[f(B_t)]. \tag{2.2.1}$$

Thus we get

$$E_{r_0}[f(B_t);\tau^X_{r_0} \le t] = \int_{[0,t]} F^X_{r_0}(ds)E_{X_s,s}[f(B_t)]. \tag{2.2.2}$$

It follows that item 1 of **Theorem 2.1.1** holds with $G^X_{0,t}(r_0,x)$ given by (2.1.3).
To show the other items of **Theorem 2.1.1**, we define D as

$$D := \{(x,t) : -\infty < x < X_t, \ t > 0\}$$

and

$$u(x,t) := \int_{-\infty}^{0} h(\xi)G^X_{0,t}(\xi,x)d\xi \tag{2.2.3}$$

for given $h \in C_c^\infty((-\infty,X_0);\mathbb{R}_+)$ and all $(x,t) \in D$.
If X is Hölder continuous with exponent $\gamma \in [1/2,1]$, then u solves the heat equation in D with the initial condition h and Dirichlet boundary condition.(See **Theorem 2.2.1**.) However, in general, u is not a unique solution of the heat equation with the initial condition h and Dirichlet boundary condition.(See **Remark 2.2.2 and 2.2.3** below)

To have the uniqueness, first of all, we restrict time variable of D in a finite interval. Thus we fix $T > 0$ and define the parabolic cylinder D_T which is a subset of D as

$$D_T := \{(x,t) : -\infty < x < X_t,\ 0 < t \le T\}.$$

Consider the following initial-boundary value problem

$$\begin{cases} v \in C(\overline{D_T}) \cap C^{2,1}(D_T), \\ v_t = \frac{1}{2} v_{xx}, \quad (x,t) \in D_T, \\ v(X_t, t) = 0,\ 0 < t \le T, \\ v(x,0) = h(x),\ -\infty < x < X_0, \\ \lim_{x \to -\infty} \sup_{0<t<T} |v(x,t)| = 0. \end{cases} \tag{2.2.4}$$

We prove the following weaker form of **Theorem 2.1.1** in **Section 2.3**.

Theorem 2.2.1. *Let X be a Hölder continuous curve on any finite interval in $[0,\infty)$ with exponent $\gamma \in [1/2, 1]$ and let $X_0 = 0$.*

1. *The function u defined in (2.2.3) is the unique solution of (2.2.4).*

2. *If $\gamma \in (1/2, 1]$, then u has the left hand derivative at the boundary $u_x(X_t^-, t)$ which is continuous on $(0,\infty)$. Moreover, $p_h(t) := -\frac{1}{2} u_x(X_t^-, t)$ satisfies*

$$p_h(t) = -\int_{-\infty}^{0} h(\xi) G_x(X_t, t; \xi, 0) d\xi + \int_0^t G_x(X_t, t; X_\tau, \tau) p_h(\tau) d\tau \quad \forall t > 0. \tag{2.2.5}$$

Remark 2.2.2. *The uniqueness for (2.2.4) is not guaranteed if we do not assume $v \in C(\overline{D_T})$. When $X_t = 0$ for all t and h is identically 0, if $v(x,t)$ is given by $\frac{1}{\sqrt{2\pi t}} \left\{-\frac{x}{t}\right\} \exp\left(-\frac{x^2}{2t}\right)$, then this satisfies the heat equation with the initial data 0 and is also 0 on the boundary, but this is not continuous at $(0,0)$.*

Remark 2.2.3. ***(Tychonoff)*** *There is another nontrivial solution when $X_t = 0$ for all $t \ge 0$, the initial data and the boundary condition are all 0, then the function*

$$v(x,t) = \sum_{n=0}^{\infty} f^{(n)}\left(\frac{t}{2}\right) \frac{x^{2n+1}}{(2n+1)!},$$

where

$$f(t) = \begin{cases} \exp\left(-\frac{1}{t^2}\right), & t > 0, \\ 0, & t \le 0. \end{cases}$$

satisfies the heat equation with the initial data 0 and is also 0 on the boundary. Furthermore, $v \in C(\overline{D_T}) \cap C^{2,1}(D_T)$. Thus we need the condition $\lim_{x \to -\infty} \sup_{0<t<T} |v(x,t)| = 0$ to have the uniqueness.

Remark 2.2.4. *It can be shown $\lim_{t \to 0} p_h(t) = 0$ by (2.2.5) and **Lemma** 2.4.2 so that $u_x(X_t^-, t)$ is continuous on $[0,\infty)$.*

By approximating the initial delta measure of **Theorem 2.1.1**, in **Section 2.4**, we prove the following proposition.

Proposition 2.2.5. *Let X be a Hölder continuous curve on any finite interval in $[0,\infty)$ with exponent $\gamma \in (1/2, 1]$ and let $X_0 = 0$. We fix $r_0 < X_0 = 0$ and choose a sequence $\{h_n\} \subset C_c^\infty(\mathbb{R};\mathbb{R}_+)$ with supp $h_n = [r_0 - \frac{1}{n}, r_0 + \frac{1}{n}] \Subset (-\infty, 0)$ and $\|h_n\|_1 = 1$. For each h_n, there exist a corresponding solution u_n with $-\frac{1}{2}\frac{\partial u_n}{\partial x}\Big|_{(X_t^-,t)} =: p_n(t)$ in the sense of **Theorem 2.2.1**. Then we have the following statements:*

1. *There is a unique $p \in C([0,\infty))$ with $p(0) = 0$ such that*

$$p(t) = -G_x(X_t, t; r_0, 0) + \int_0^t G_x(X_t, t; X_\tau, \tau)p(\tau)d\tau \; for \; all \; t > 0. \tag{2.2.6}$$

2. *p_n converges to p in $C^0_{(\eta)}((0,T])$ for all $0 < \eta < 1/2$.*

Finally, we show that p is the density function **Section 2.4** and conclude **Theorem 2.1.1** in **Section 2.5**.

2.3 Proof of of Theorem 2.2.1

Following [9], we introduce $C^0_{(\nu)}((0,T])$, $0 < \nu \le 1$, as the subspace of $C((0,T])$ that consists of those functions φ such that

$$\|\varphi\|_T^{(\nu)} = \sup_{0<t\le T} t^{1-\nu}|\varphi(t)| < \infty.$$

Then $C^0_{(\nu)}((0,T])$ is a Banach space under the norm $\|\cdot\|_T^{(\nu)}$. Moreover, we also introduce the following lemmas from [9] which plays an essential role in our analysis.

Lemma 2.3.1 (*jump relation*)**.** *For $\varphi \in C^0_{(\nu)}((0,T])$, we have*

$$\lim_{x\to X_t^\pm} \frac{\partial w_\varphi}{\partial x}(x,t) = \mp\varphi(t) + \int_0^t G_x(X_t, t; X_\tau, \tau)\varphi(\tau)d\tau, \tag{2.3.1}$$

where $w_\varphi(x,t) = \displaystyle\int_0^t G(x,t;X_\tau,\tau)\varphi(\tau)d\tau$.

Proof. See Lemma 14.2.5. of [9]. □

For two continuous curves s_1, s_2 such that $s_1(t) < s_2(t), t \in [0,T]$, let us set $E_T := \{(x,t) : s_1(t) < x < s_2(t),\ 0 < t \le T\}$ and $B_T := \{(s_i(t),t) : 0 \le t \le T,\ i \in \{1,2\}\} \cup \{(x,0) : s_1(0) < x < s_2(0)\}$.

Lemma 2.3.2 (*The Weak Maximum(Minimum) Principle*)**.** *For a solution u of $u_t = \frac{1}{2}u_{xx}$ in E_T, which is continuous in $E_T \cup B_T$,*

$$\max_{E_T\cup B_T} u = \max_{B_T} u. \quad \left(\min_{E_T\cup B_T} u = \min_{B_T} u.\right) \tag{2.3.2}$$

Before going to the proof of **Theorem 2.2.1**, we need the following proposition.

Proposition 2.3.3. *Let X be a Hölder continuous curve on any finite interval in $[0,\infty)$ with exponent $\gamma \in [1/2,1]$. If the starting point of the Brownian motion is close to X, the first hitting time converges to 0. Precisely, $\lim_{\xi \to X_0} P_\xi\left[\tau_\xi^X > s\right] = 0$ for all $s > 0$.*

Proof. Without loss of generality, we may reduce this problem as the case for Brownian motion starting at 0 and $X_0 = \epsilon > 0$ and let ϵ go to 0. For $s > 0$, let $m := \sup_{0 \le t_1 < t_2 \le s} \dfrac{|X_{t_2} - X_{t_1}|}{|t_2 - t_1|^\gamma}$. Since $\limsup_{t \downarrow 0} \dfrac{B_t}{\sqrt{t}} = \infty$ a.e., for $M > m$, we have a sequence $t_k \downarrow 0$ such that $M\sqrt{t_k} \le B_{t_k}$ a.e. and $M\sqrt{t_k} - m t_k^\gamma \downarrow 0$ for all k. Thus, for $0 < t \le s$, we deduce that a.e.

$$\sup_{0 \le l \le t}\{B_l - X_l\} \ge \sup_{0 \le l \le t}\{B_l - (\epsilon + m l^\gamma)\} \ge \sup_{t_k \le t}\left\{B_{t_k} - (\epsilon + m t_k^\gamma)\right\} \ge \sup_{t_k \le t}\left\{M\sqrt{t_k} - (\epsilon + m t_k^\gamma)\right\}.$$

Therefore,

$$P_0\left(\sup_{0 \le l \le t}\{B_l - X_l\} < 0\right) \le P_0\left(\sup_{t_k \le t}\left\{M\sqrt{t_k} - m t_k^\gamma\right\} < \epsilon\right).$$

For each sufficently small $\epsilon > 0$, there is a greatest $k(\epsilon)$ such that $M\sqrt{t_{k(\epsilon)}} - m t_{k(\epsilon)}^\gamma \ge \epsilon$. Thus we obtain that $\tau_0^X \le t_{k(\epsilon)}$ a.e. for all sufficiently small $\epsilon > 0$. Since $k(\epsilon)$ is an increasing function as ϵ decreases and $\lim_{\epsilon \to 0} k(\epsilon) = \infty$, the proposition follows. □

Proof of Theorem 2.2.1
Let us define $X'_\tau := X_{t-\tau}$ for all $0 \le \tau \le t$. Using the invariance of the law of the Brownian motion under time reversal, we have

$$u(x,t) = \int_{-\infty}^0 h(r') G_{0,t}^X(r', x)\, dr' = E_x[h(B_t); \tau_x^{\tilde{X}} > t]. \tag{2.3.3}$$

Using this equality, we also have

$$|u(x,t)| = \left| E_x[h(B_t); \tau_x^{\tilde{X}} > t] \right| \le \|h\|_\infty P_x[\tau_x^{\tilde{X}} > t]. \tag{2.3.4}$$

For $s > 0$, let us choose $0 < s^* < s$. Then for all (x,t) sufficiently close to (X_s, s), we obtain

$$P_x[\tau_x^{\tilde{X}} > t] \le P_x[\tau_x^{\tilde{X}} > s^*] \tag{2.3.5}$$

which vanishes when $(x,t) \to (X_s, s)$ by **Proposition 2.3.3** so that $\lim_{(x,t) \to (X_s, s)} u(x,t) = 0$.
In addition, we have

$$|u(x,t)| = \left| E_x[h(B_t); \tau_x^{\tilde{X}} > t] \right| \le E_x[|h(B_t)|] = \int_{-\infty}^0 |h(\xi)| G_{0,t}(x,\xi)\, d\xi \tag{2.3.6}$$

which also vanishes when $(x,t) \to (0,0)$, since the support of h is strictly away from 0. To prove that u satisfies the initial data h, we write $y_t = \min_{s \in [0,t]} X'_s$. For $x < 0$ and any positive $\lambda > 0$,

$$P_x\left[\tau_x^{\tilde{X}} \le t\right] \le P_x\left[\max_{s \in [0,t]} B_s \ge y_t\right] = P_x\left[\max_{s \in [0,t]} \exp(\lambda B_s) \ge \exp(\lambda y_t)\right]. \tag{2.3.7}$$

Since the exponential of Brownian motion is a positive submartingale, we can apply Doob's inequality, then

$$P_x\left[\max_{s\in[0,t]}\exp(\lambda B_s)\geq\exp(\lambda y_t)\right]\leq\frac{E_x[\exp(\lambda B_t)]}{\exp(\lambda y_t)}=\exp\left(\frac{1}{2}\lambda^2 t-\lambda(y_t-x)\right).\quad(2.3.8)$$

From (2.3.7) and (2.3.8), we get

$$\lim_{(x,t)\to(y,0)}P_x\left[\tau_x^{\tilde{X}}\leq t\right]\leq\exp(-\lambda(X_0-y))=\exp(\lambda y)\quad(2.3.9)$$

so that the left hand side vanishes since λ is arbitrary. Thus we obtain that

$$\lim_{(x,t)\to(y,0)}\int_{-\infty}^{0}h(\xi)G_{0,t}^{X}(\xi,x)d\xi=\lim_{(x,t)\to(y,0)}E_x[h(B_t)]=h(y).\quad(2.3.10)$$

By the properties of the Gaussian kernel, we deduce that u solves (2.2.4). We now prove uniqueness. If v_1, v_2 satisfy all the above conditions, then $v_1-v_2\in C(\overline{D_T})\cap C^{2,1}(D_T)$ satisfies the heat equation with the initial data 0 and is also 0 on the boundary. Moreover, $\lim_{x\to-\infty}\sup_{0<t<T}|v_1(x,t)-v_2(x,t)|=0$ so that by the weak maximum(minimum) principle, v_1-v_2 is all 0 in D_T. Therefore, $v_1=v_2$ so that item 1 is proved.
To prove item 2, assuming that $\gamma\in(1/2,1]$, we define, $(x,t)\in D_T$,

$$v(x,t):=\int_{-\infty}^{0}h(\xi)G(x,t;\xi,0)d\xi+\int_0^t G(x,t;X_\tau,\tau)\varphi(\tau)d\tau,\quad(2.3.11)$$

where $\varphi\in C^0_{(\gamma)}((0,T])$ satisfies

$$0=\int_{-\infty}^{0}h(\xi)G(X_t,t;\xi,0)d\xi+\int_0^t G(X_t,t;X_\tau,\tau)\varphi(\tau)d\tau.\quad(2.3.12)$$

If we apply the Abel inverse operator A defined by

$$(AF)(t)=\frac{1}{\pi}\frac{d}{dt}\int_0^t\frac{F(\eta)}{(t-\eta)^{\frac{1}{2}}}d\eta$$

on both sides of (2.3.12), then from Chapter 14 in [9] we have a equivalent Volterra integral equation of the second kind:

$$\varphi(t)=\psi(t)+\int_0^t H(t,\tau)\varphi(\tau)d\tau,\quad(2.3.13)$$

where $\psi\in C^0_{(\gamma)}((0,T])$ and $|H(t,\tau)|\leq C(t-\tau)^{2\gamma-2}$. The existence of $\varphi\in C^0_{(\gamma)}((0,T])$ which satisfies (2.3.13) can be proved similarly to the proof of **Proposition 2.2.5** that we will show later. Then v is well-defined and solves (2.2.4) so that $v=u$ since u is the unique solution of (2.2.4). By the jump relation, we have

$$u_x(X_t^-,t)=\int_{-\infty}^{0}h(\xi)G_x(X_t,t;\xi,0)d\xi+\varphi(t)+\int_0^t G_x(X_t,t;X_\tau,\tau)\varphi(\tau)d\tau\quad(2.3.14)$$

so that $u_x(X_t^-,t)\in C^0_{(\gamma)}((0,T])$. Since T is arbitrary, we deduce $u_x(X_t^-,t)$ is continuous on $(0,\infty)$.

To show (2.2.5), let us fix $(x,t) \in D_T$ and let us define $D^{(t)}_{\epsilon,r} := \{(\xi,\tau) : r < \xi < X_\tau - \epsilon,\ \epsilon < \tau < t-\epsilon\}$ for each $\epsilon > 0$ and $r \in \mathbb{R}$. By the Green's identity, we have

$$\frac{1}{2}(u_\xi G - uG_\xi)_\xi - (uG)_\tau = 0 \implies \oint_{\partial D^{(t)}_{\epsilon,r}} \frac{1}{2}(u_\xi G - uG_\xi)d\tau + (uG)d\xi = 0. \tag{2.3.15}$$

It can be also shown that $\lim_{x\to-\infty} \sup_{0<t<T} |u_x(x,t)| = 0$ by the properties of the Gaussian kernel. Hence we obtain another representation of u by letting $\epsilon \to 0$, $r \to -\infty$,

$$u(x,t) = \int_{-\infty}^{0} h(\xi)G(x,t;\xi,0)d\xi + \frac{1}{2}\int_0^t G(x,t;X_\tau,\tau)u_x(X_\tau^-,\tau)d\tau. \tag{2.3.16}$$

Differentiating both sides of (2.3.16) with respect to x and applying the jump relation, we get

$$\frac{1}{2}u_x(X_t^-,t) = \int_{-\infty}^{0} h(\xi)G_x(X_t,t;\xi,0)d\xi + \frac{1}{2}\int_0^t G_x(X_t,t;X_\tau,\tau)u_x(X_\tau^-,\tau)d\tau. \tag{2.3.17}$$

which implies (2.2.5). □

2.4 Existence of a density function

From now on, X is a Hölder continuous curve defined on $[0,\infty)$ with exponent $\gamma \in (1/2,1]$ and $X_0 = 0$. Comparing the definition of u and (2.3.16), we see the following equality:

$$\int_{[0,t)} G_{\tau,t}(X_\tau,x)\int_{-\infty}^{0} h(\xi)F_\xi^X(d\tau)d\xi = -\frac{1}{2}\int_0^t G(x,t;X_\tau,\tau)u_x(X_\tau^-,\tau)d\tau. \tag{2.4.1}$$

Set $F_h^X(d\tau) := \int_{-\infty}^{0} h(\xi)F_\xi^X(d\tau)d\xi$. To conclude $F_h^X(d\tau) = -\frac{1}{2}u_x(X_\tau^-,\tau)d\tau$, we introduce the mass lost $\Delta_I^X(u)$, $I = [t_1,t_2] \subset [0,T]$, $t_1 \le t_2$, is defined by

$$\Delta_I^X(u) = \int_{-\infty}^{X_{t_1}} u(r,t_1)dr - \int_{-\infty}^{X_{t_2}} u(r,t_2)dr. \tag{2.4.2}$$

If we see the right hand side of (2.3.16), we can extend u to $\bar{u}$ defined in $\{(x,t) : x \in \mathbb{R},\ 0 < t \le T\}$ as

$$\bar{u}(x,t) = \int_{-\infty}^{0} h(\xi)G(x,t;\xi,0)d\xi + \frac{1}{2}\int_0^t G(x,t;X_\tau,\tau)u_x(X_\tau^-,\tau)d\tau. \tag{2.4.3}$$

Then this satisfies the heat equation with $\lim_{(x,t)\to(y,0)} \bar{u}(x,t) = 0$ for all $y \ge 0$ and also satisfies $\bar{u}(X_t,t) = 0$ for all $0 < t \le T$. Moreover, by the properties of Gaussian kernel, we have

$$\lim_{x\to\infty} \sup_{0<t<T} |\bar{u}(x,t)| = 0. \tag{2.4.4}$$

It follows that $\bar{u}(x,t) = 0$ in $\{(x,t) : x \ge X_t, 0 < t \le T\}$ by the weak maximum(minimum) principle. Thus we assume that u is defined $\{(x,t) : x \in \mathbb{R},\ 0 \le t \le T\}$ such that it is 0 in

$\{(x,t) : x \geq X_t,\ 0 \leq t \leq T\}$.
Heuristically

$$\Delta_I^X(u) = -\int\int_{t_1}^{t_2} u_t(x,t)dtdx = -\int\int_{t_1}^{t_2} \frac{1}{2}u_{xx}(x,t)dxdt = -\frac{1}{2}\int_{t_1}^{t_2} u_x(X_t^-, t)dt.$$

Since we do not control u_{xx} at the moving boundary, we cannot make this argument rigorously. Thus we use a different approach.

Proposition 2.4.1. $-\frac{1}{2}\int_I u_x(X_t^-, t)dt = \Delta_I^X(u) = F_h^X(I)$.

Proof. If we integrate both sides of (2.3.16), then

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty}\int_{-\infty}^{0} h(\xi)G(x,t;\xi,0)d\xi dx + \int_{-\infty}^{\infty}\frac{1}{2}\int_0^t G(x,t;X_\tau,\tau)u_x(X_\tau^-,\tau)d\tau dx.$$

Applying Fubini's theorem, we get

$$\int_{-\infty}^{X_t} u(x,t)dx = \int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{0} h(\xi)d\xi + \frac{1}{2}\int_0^t u_x(X_\tau^-,\tau)d\tau. \tag{2.4.5}$$

Thus we get the first equality of the proposition,

$$\Delta_{[t_1,t_2]}^X(u) = -\frac{1}{2}\int_{t_1}^{t_2} u_x(X_\tau^-,\tau)d\tau. \tag{2.4.6}$$

From (2.1.2) and item 1 of **Theorem 2.1.1**, we get

$$P_\xi[\tau_\xi^X > t] = \int_{-\infty}^{X_t} G_{0,t}^X(\xi,x)dx. \tag{2.4.7}$$

For $0 = t_1 < t_2$, using Fubini's theorem again, we get

$$\begin{aligned}
\Delta_I^X(u) &= \int_{-\infty}^{0} h(\xi)d\xi - \int_{-\infty}^{X_{t_2}}\int_{-\infty}^{0} h(\xi)G_{0,t_2}^X(\xi,x)d\xi dx \\
&= \int_{-\infty}^{0} h(\xi)d\xi - \int_{-\infty}^{0} h(\xi)P_\xi[\tau_\xi^X > t_2]d\xi = \int_{-\infty}^{0} h(\xi)P_\xi[0 \leq \tau_\xi^X \leq t_2]d\xi.
\end{aligned}$$

For $0 < t_1 < t_2$, similarly,

$$\begin{aligned}
\Delta_I^X(u) &= \int_{-\infty}^{X_{t_1}}\int_{-\infty}^{0} h(\xi)G_{0,t_1}^X(\xi,x)d\xi dx - \int_{-\infty}^{X_{t_2}}\int_{-\infty}^{0} h(\xi)G_{0,t_2}^X(\xi,x)d\xi dx \\
&= \int_{-\infty}^{0} h(\xi)P_\xi[t_1 < \tau_\xi^X \leq t_2]d\xi.
\end{aligned}$$

Then for $I_\epsilon = [t_1 - \epsilon, t_1]$, we get

$$\begin{aligned}
\lim_{\epsilon\to 0}\Delta_{I_\epsilon}^X(u) &= \lim_{\epsilon\to 0} -\frac{1}{2}\int_{I_\epsilon} u_x(X_t^-,t)dt = 0 = \lim_{\epsilon\to 0}\int_{-\infty}^{0} h(\xi)P_\xi[t_1-\epsilon < \tau_\xi^X \leq t_1]d\xi \\
&= \int_{-\infty}^{0} h(\xi)P_\xi[\tau_\xi^X = t_1]d\xi.
\end{aligned}$$

Finally we conclude that

$$\Delta_I^X(u) = \int_{-\infty}^{0} h(\xi) P_\xi[t_1 \le \tau_\xi^X \le t_2] d\xi = F_h^X(I). \tag{2.4.8}$$

□

We use the following lemma extensively to prove **Proposition 2.2.5**:

Lemma 2.4.2. $\int_0^t \tau^{\alpha_1}(t-\tau)^{\alpha_2} d\tau = \frac{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}{\Gamma(2+\alpha_1+\alpha_2)} t^{1+\alpha_1+\alpha_2}$ *for* $\alpha_1, \alpha_2 > -1$, *where* Γ *is the gamma function.*

Proof of Proposition 2.2.5

Let $T_s > 0$. Since $|G_x(X_t, t; X_\tau, \tau)| = \left| \frac{1}{\sqrt{2\pi(t-\tau)}} \left\{ -\frac{X_t - X_\tau}{t-\tau} \right\} \exp\left(-\frac{(X_t - X_\tau)^2}{2(t-\tau)} \right) \right| \le \frac{C_0}{(t-\tau)^{\frac{3}{2}-\gamma}}$, we deduce that for $q_1, q_2 \in C([0, T_s])$, $0 < t \le T_s$,

$$\left| \int_0^t G_x(X_t, t; X_\tau, \tau)(q_1(\tau) - q_2(\tau)) d\tau \right| \le \int_0^t \frac{C_0 \|q_1 - q_2\|_{T_s}}{(t-\tau)^{\frac{3}{2}-\gamma}} d\tau$$
$$= C_1 t^{\gamma - \frac{1}{2}} \|q_1 - q_2\|_{T_s} \le C_1 T_s^{\gamma - \frac{1}{2}} \|q_1 - q_2\|_{T_s}.$$

We define $F : C([0, T_s]) \to C([0, T_s])$ as, for $q \in C([0, T_s])$,

$$(Fq)(t) = -G_x(X_t, t; r_0, 0) + \int_0^t G_x(X_t, t; X_\tau, \tau) q(\tau) d\tau \;\; for \;\; t > 0, \tag{2.4.9}$$

and $(Fq)(0) = 0$. Since $\lim_{t \to 0} G_x(X_t, t; r_0, 0) = 0$, it is well-defined. If we choose T_s such that $C_1 T_s^{\gamma - \frac{1}{2}} < 1$, then F is a contraction mapping so that F has a unique fixed point since $C([0, T_s])$ is a Banach space. Let us call this p_{T_s}.

Now we assume that we have p_{T_s} for some $T_s > 0$. For $T^\star > T_s$, we define $H : C([T_s, T^\star]) \to C([T_s, T^\star])$ as, for $q \in C([T_s, T^\star])$,

$$(Hq)(t) = -G_x(X_t, t; r_0, 0) + \int_0^{T_s} G_x(X_t, t; X_\tau, \tau) p_{T_s}(\tau) d\tau + \int_{T_s}^t G_x(X_t, t; X_\tau, \tau) q(\tau) d\tau.$$

Then for $q_1, q_2 \in C([T_s, T^\star])$, we have

$$\|Hq_1 - Hq_2\|_\infty \le C_2 (t - T_s)^{\gamma - \frac{1}{2}} \|q_1 - q_2\|_\infty \le C_2 (T^\star - T_s)^{\gamma - \frac{1}{2}} \|q_1 - q_2\|_\infty. \tag{2.4.10}$$

Similarly, if we choose $T^\star$ such that $C_2(T^\star - T_s)^{\gamma - \frac{1}{2}} < 1$, then H is a contraction mapping so that H has a unique fixed point since $C([T_s, T^\star])$ is a Banach space.

Therefore, if we have p defined $[0, T_s]$, p_{T_s}, then we can extend this to time $T_s + C_3$ where C_3 is an independent constant. Thus if we repeat this step inductively, we have p defined on $[0, \infty)$ which satisfies (2.2.6).

We now prove that p_n converges to p in $C^0_{(\eta)}((0, T_s])$ for all sufficiently small $T_s > 0$. By (2.2.5),

$$p_n(t) = -\int_{-\infty}^{0} h_n(\xi) G_x(X_t, t; \xi, 0) d\xi + \int_0^t G_x(X_t, t; X_\tau, \tau) p_n(\tau) d\tau. \tag{2.4.11}$$

For $0 < t \le T_s$, taking the difference between (2.2.6) and (2.4.11), we get

$$
\begin{aligned}
t^{1-\eta}|p_n(t)-p(t)| &\le t^{1-\eta}\left|\int_{-\infty}^{0} h_n(\xi)[G_x(X_t,t;\xi,0)-G_x(X_t,t;r_0,0)]d\xi\right| \\
&+ t^{1-\eta}\left|\int_0^t G_x(X_t,t;X_\tau,\tau)(p_n(\tau)-p(\tau))d\tau\right|.
\end{aligned}
$$

For the second term of the right hand side, we have

$$
\begin{aligned}
t^{1-\eta}\left|\int_0^t G_x(X_t,t;X_\tau,\tau)(p_n(\tau)-p(\tau))d\tau\right| &\le C_0 t^{1-\eta}\int_0^t \frac{\|p_n-p\|_{T_s}^{(\eta)}}{(t-\tau)^{\frac{3}{2}-\gamma}\tau^{1-\eta}}d\tau \\
&= C_1 t^{\gamma-\frac{1}{2}}\|p_n-p\|_{T_s}^{(\eta)} \le C_1 T_s^{\gamma-\frac{1}{2}}\|p_n-p\|_{T_s}^{(\eta)}.
\end{aligned}
$$

Let us choose $T_s > 0$ such that $C_1 T_s^{\gamma-\frac{1}{2}} < 1$. Then

$$
\begin{aligned}
(1-C_1T_s^{\gamma-\frac{1}{2}})\|p_n-p\|_{T_s}^{(\eta)} &\le t^{1-\eta}\left|\int_{-\infty}^{0} h_n(\xi)[G_x(X_t,t;\xi,0)-G_x(X_t,t;r_0,0)]d\xi\right|. \\
&\le \int_{-\infty}^{0} h_n(\xi) \sup_{\substack{0<t\le T_s \\ \xi\in[r_0-\frac{1}{n},r_0+\frac{1}{n}]}} t^{1-\eta}\,|G_x(X_t,t;\xi,0)-G_x(X_t,t;r_0,0)|\,d\xi.
\end{aligned}
$$

For all sufficiently large n, $0 < t \le \frac{1}{n}$ and $r_0 - \dfrac{1}{n} \le \xi \le r_0 + \dfrac{1}{n}$, there exists C_2 such that

$$
|G_x(X_t,t;\xi,0)| = \left|\frac{1}{\sqrt{2\pi}}\left\{-\frac{X_t-\xi}{t^{\frac{3}{2}}}\right\}\exp\left(-\frac{(X_t-\xi)^2}{2t}\right)\right| \le C_2,
$$

Thus

$$
\sup_{\substack{0<t\le\frac{1}{n} \\ \xi\in[r_0-\frac{1}{n},r_0+\frac{1}{n}]}} t^{1-\eta}\,|G_x(X_t,t;\xi,0)-G_x(X_t,t;r_0,0)| \le \sup_{0<t\le\frac{1}{n}} 2t^{1-\eta}C_2 \le \frac{2C_2}{n^{1-\eta}}.
$$

For all sufficiently large n, $\frac{1}{n} < t \le T_s$ and $r_0 - \dfrac{1}{n} \le \xi \le r_0 + \dfrac{1}{n}$, since $|G_{xx}(\cdot,t;\cdot,0)| \le \dfrac{C_3}{t^{\frac{3}{2}}}$, we have

$$
|G_x(X_t,t;\xi,0)-G_x(X_t,t;r_0,0)| \le \frac{C_3|\xi-r_0|}{t^{\frac{3}{2}}} \le \frac{C_3}{nt^{\frac{3}{2}}},
$$

Thus

$$
\sup_{\substack{\frac{1}{n}<t\le T_s \\ \xi\in[r_0-\frac{1}{n},r_0+\frac{1}{n}]}} t^{1-\eta}\,|G_x(X_t,t;\xi,0)-G_x(X_t,t;r_0,0)| \le \sup_{\frac{1}{n}<t\le T_s} \frac{C_3}{nt^{\frac{1}{2}+\eta}} \le \frac{C_3}{n^{\frac{1}{2}-\eta}}.
$$

Therefore, we conclude that p_n converges to p in $C^0_{(\eta)}((0,T_s])$ for all sufficiently small $T_s > 0$. To extend from T_s to $T^\star$, assuming that p_n converges to p in $C^0_{(\eta)}((0,T_s])$ for some $T_s > 0$ and writing $\|p_n - p\|_{[T_s,T^\star]} = \sup_{\tau\in[T_s,T^\star]} |p_n(\tau)-p(\tau)|$, for $T_s \le t \le T^\star$, we deduce that

$$
\begin{aligned}
|p_n(t)-p(t)| &\le \left|\int_{-\infty}^{0} h_n(\xi)[G_x(X_t,t;\xi,0)-G_x(X_t,t;r_0,0)]d\xi\right| \\
&+ \left|\int_{0}^{T_s} G_x(X_t,t;X_\tau,\tau)(p_n(\tau)-p(\tau))d\tau\right| + \left|\int_{T_s}^{t} G_x(X_t,t;X_\tau,\tau)(p_n(\tau)-p(\tau))d\tau\right| \\
&\le C_3\int_{-\infty}^{0} h_n(\xi)\frac{|\xi-r_0|}{t^{\frac{3}{2}}}d\xi + C_0\int_0^{T_s}\frac{\|p_n-p\|_{T_s}^{(\eta)}}{(t-\tau)^{\frac{3}{2}-\gamma}\tau^{1-\eta}}d\tau + C_0\int_{T_s}^{t}\frac{\|p_n-p\|_{[T_s,T^\star]}}{(t-\tau)^{\frac{3}{2}-\gamma}}d\tau \\
&\le \frac{C_3}{nt^{\frac{3}{2}}} + C_0\int_0^{T_s}\frac{\|p_n-p\|_{T_s}^{(\eta)}}{(T_s-\tau)^{\frac{3}{2}-\gamma}\tau^{1-\eta}}d\tau + C_4\|p_n-p\|_{[T_s,T^\star]}(t-T_s)^{\gamma-\frac{1}{2}} \\
&\le \frac{C_3}{nT_s^{\frac{3}{2}}} + C_1T_s^{\gamma-\frac{3}{2}+\eta}\|p_n-p\|_{T_s}^{(\eta)} + C_4\|p_n-p\|_{[T_s,T^\star]}(T^\star-T_s)^{\gamma-\frac{1}{2}}.
\end{aligned}
$$

Let us choose $T^\star > T_s$ such that $C_4(T^\star-T_s)^{\gamma-\frac{1}{2}} < 1$, then we have

$$
(1-C_4(T^\star-T_s)^{\gamma-\frac{1}{2}})\|p_n-p\|_{[T_s,T^\star]} \le \frac{C_3}{nT_s^{\frac{3}{2}}} + C_1T_s^{\gamma-\frac{3}{2}+\eta}\|p_n-p\|_{T_s}^{(\eta)}.
$$

The right term vanishes when n goes to ∞ so that p_n converges to p in $C^0_{(\eta)}((0,T_s+C_5])$ for some independent constant $C_5>0$. By repeating this argument inductively, it follows that p_n converges to p in $C^0_{(\eta)}((0,T])$. □

Hence we have

$$
\lim_{n\to\infty}\int_{-\infty}^{0} h_n(\xi)P_\xi(\tau_\xi^X\in I)d\xi = \lim_{n\to\infty}\int_I p_n(t)dt = \int_I p(t)dt.
$$

For $I=[0,t]\subset[0,T]$, $P_\xi(\tau_\xi^X\in I)$ is an increasing function of ξ, so if we choose h_n so that $\lim_{n\to\infty}\int_{r_0}^{r_0+\frac{1}{n}} h_n(\xi)d\xi = 1$, we get

$$
\lim_{\xi\to r_0^+} P_\xi(\tau_\xi^X\in I) \le \lim_{n\to\infty}\int_{r_0}^{r_0+\frac{1}{n}} h_n(\xi)P_\xi(\tau_\xi^X\in I)d\xi = \lim_{n\to\infty}\int_{-\infty}^{0} h_n(\xi)P_\xi(\tau_\xi^X\in I)d\xi. \quad (2.4.12)
$$

Similarly, if we choose h_n so that $\lim_{n\to\infty}\int_{r_0-\frac{1}{n}}^{r_0} h_n(\xi)d\xi = 1$, we get

$$
\lim_{n\to\infty}\int_{-\infty}^{0} h_n(\xi)P_\xi(\tau_\xi^X\in I)d\xi = \lim_{n\to\infty}\int_{r_0-\frac{1}{n}}^{r_0} h_n(\xi)P_\xi(\tau_\xi^X\in I)d\xi \le \lim_{\xi\to r_0^-} P_\xi(\tau_\xi^X\in I). \quad (2.4.13)
$$

Therefore, we obtain that

$$
F_{r_0}^X([0,t]) = P_{r_0}(\tau_{r_0}^X \le t) = \int_0^t p(\tau)d\tau, \quad (2.4.14)
$$

which implies that p is the density function of $F_{r_0}^X$.

2.5 Proof of Theorem 2.1.1

Items 1 and 2 of **Theorem 2.1.1** were proved in the previous sections. By **Theorem 2.2.1** and the properties of the Gaussian kernel, $G^X_{0,t}(r_0,x)$ solves (2.1.4), (2.1.5) and (2.1.6). Hence G^X is the Green function of the heat equation with Dirichlet boundary condition which implies item 4 of **Theorem 2.1.1**. Furthermore, $G^X_{0,t}(r_0,x)$ can be written as

$$G^X_{0,t}(r_0,x) = G_{0,t}(r_0,x) - \int_0^t G_{\tau,t}(X_\tau,x)p(\tau)d\tau. \tag{2.5.1}$$

Differentiating with respect to x, applying the jump relation and (2.2.6), we have

$$\begin{aligned}\frac{\partial}{\partial x}G^X_{0,t}(r_0,x)\Big|_{x=X_t^-} &= \frac{\partial}{\partial x}G_{0,t}(r_0,X_t) - p(t) - \int_0^t \frac{\partial}{\partial x}G_{\tau,t}(X_\tau,X_t)p(\tau)d\tau & (2.5.2)\\ &= -2p(t). & (2.5.3)\end{aligned}$$

Thus $p(t) = -\frac{1}{2}\frac{\partial}{\partial x}G^X_{0,t}(r_0,x)\Big|_{x=X_t^-}$, so it proves item 3 of **Theorem 2.1.1**.

2.6 Discussion

In [37] and [38], the importance of having a continuous density function through Hölder continuos with exponent greater than 1/2 is explained in the widely-studied case of a neural coding theory. It has not been founded any pure analytic proof for when exponent is less or equal to 1/2 so far, but again according to [38], it shows the difficulty of having a density function by using numerical methods.

We can try to extend what we have done to two dimensional case by formulating it as follows: let $\dot{D}$ be a subset of $\mathbb{R}^n \times \mathbb{R}_+$ such that the projection of $\dot{D}$ onto the time axis is $[0,T)$ with $0 < T \le \infty$, and that for each $0 \le t < T$, $D_t = \{x \in \mathbb{R} : (x,t) \in \dot{D}\}$ is a bounded connected open set in $\mathbb{R}^n$. Let us consider the following initial-boundary value heat equation:

$$\begin{cases} u_t = \frac{1}{2}\Delta_x u, \ \ (x,t) \in \dot{D}, \\ u(x,t) = 0, \ (x,t) \in \partial\dot{D}, \\ u(x,0) = u_0(x), \ x \in D_0, \end{cases} \tag{2.6.1}$$

where $u_0 \in C_c^\infty(D_0)$. It is known that there is a unique solution of class $C^2(D_t)\cap C(\overline{D_t})$ which satisfies (2.6.1) and then we need to approximate u_0 to the Dirac measure concentrated at some point $r_0 \in D_0$. Then

$$\begin{cases} G^D_t = \frac{1}{2}\Delta_x G^D, \ \ (x,t) \in \dot{D}, \\ G^D(x,t) = 0, \ (x,t) \in \partial\dot{D}, \\ G^D(x,0) = \delta_{r_0}(x), \ x \in D_0. \end{cases} \tag{2.6.2}$$

Similarly as before, we have to show that there is a unique solution that satisfies (2.6.2). To adopt the arguments before, we also need normal derivatives so that it is necessary to study a sufficient condition for it related to regularity of the boundary $\partial\dot{D} \cap \mathbb{R}^n \times (0,T)$.

In [39], it is studied that given a two-dimensional correlated diffusion process, we determine the joint density of the first passage times of the process to some constant boundaries which is applicable for rectangular boundaries. In [4], it is studied that Brownian motion reflected on rather than killed at the boundary of a time-dependent domain so that the analytic counterpart of the model is a heat equation with Neumann rather than Dirichlet boundary condition when $\partial \dot{D} \cap \mathbb{R}^n \times (0, T)$ is C^3.

CHAPTER 3

A free boundary problem in biological selection models

3.1 Introduction

Brunet and Derrida [5], have proposed a class of models for biological selection processes including the one we consider here. This is a system of N brownian particles on the line which branch independently at rate 1 creating a new brownian particle on the same position of the father; simultaneously the leftmost particle (which is the less fit) disappears. Thus the total number of particles does not change.
In [17] it is proved that the particles density has a limit as N diverges (under suitable assumptions on the initial datum). It is also proved that if the following FBP has a "classical solution" then this is the same as the limit density of the branching brownians, similar results have been obtained for other FBP, see [11].
Let $b \in \mathbb{R}$, say $b > 0$, and $\rho_0 \in C_c^2([b,\infty))$ such that $\rho_0(b) = 0$, $\frac{d}{dx}\rho_0(b) = 2$, $\int_b^\infty \rho_0(x)dx = 1$. The free boundary problem FBP $(*)$ that we consider is: fixed b and ρ_0 as above find a continuous curve X_t, $t \geq 0$, starting from $X_0 = b$, and $\rho(x,t)$, $x \geq X_t$; $t \geq 0$, so that

$$(*)\begin{cases} \rho_t(x,t) = \dfrac{1}{2}\rho_{xx}(x,t) + \rho(x,t), & \text{if } X_t < x, \quad t > 0, \\ \rho(X_t,t) = 0, & \text{if } t \geq 0, \\ \rho(x,0) = \rho_0(x), & \text{if } b \leq x, \\ \displaystyle\int_{X_t}^\infty \rho(x,t)dx = \int_b^\infty \rho_0(x)dx = 1, & \text{if } t > 0. \end{cases}$$

Classical solutions of the FBP $(*)$ are defined in the next section. In this paper we prove the local in time existence of a classical solution of the FBP $(*)$, observing that uniqueness follows from the results in [17]. Existence of traveling wave solutions for a large class of free boundary problems including the FBP $(*)$ is proved in [3].

3.2 Main results

By classical solutions we mean the following:

Definition. The pair (X,ρ) is a classical solution of the FBP $(*)$ in the time interval $[0,T]$, $T > 0$, if:

- $X \in C^1([0,T])$, $X_0 = b$

- $\rho \in C(\overline{D_{X,T}}) \cap C^{2,1}(D_{X,T})$, where $D_{X,T} = \{(x,t) : X_t < x,\ 0 < t < T\}$
- (X,ρ) satisfies the equations in $(*)$.

Theorem 3.2.1. *There is $T > 0$ so that the FBP $(*)$ has a classical solution in the time interval $[0,T]$ (denoted by (X,ρ)).*

We will prove Theorem 3.2.1 in the next sections. We will also prove that the derivative $\rho_x(x,t)$ has a limit when $x \to X_t$ denoted by $\rho_x(X_t,t)$ and that it vanishes as $x \to \infty$. This together with the conservation law implies $\rho_x(X_t,t) = 2$ for all $t \in [0,T]$. Furthermore if also $\rho_{xx}(X_t,t)$ exists, by differentiating $\rho(X_t,t) = 0$ we get $\dot{X}_t = -\frac{1}{4}\rho_{xx}(X_t,t)$.

Given (X,ρ) as above we define $v(x,t) := e^{-t}\rho_x(x,t)$ and observe that (X,v) satisfies the following free boundary problem FBP $(**)$:

$$(**)\begin{cases} v_t(x,t) = \frac{1}{2}v_{xx}(x,t), & \text{if } X_t < x, \quad t > 0, \\ v(X_t,t) = 2e^{-t}, & \text{if } t > 0, \\ v(x,0) = h(x), & h \in C^1_c([b,\infty)) \\ \dot{X}_t = -\frac{1}{4}e^t v_x(X_t,t), & \text{if } t > 0, \end{cases}$$

To prove Theorem 3.2.1 we will first prove the existence of a classical solution of the FBP $(**)$ in $[0,T]$:

Theorem 3.2.2. *There is $T > 0$ and a pair (X,v) which satisfies the FBP $(**)$ in $[0,T]$ with: $X \in C^1([0,T])$, $X_0 = b$, and $v \in C^{2,1}(D_{X,T})$, where $D_{X,T} = \{(x,t) : X_t < x,\ 0 < t \le T\}$.*

We did not find a proof of Theorem 3.2.2 in the existing literature, see for instance [7], [14] and references therein. Our proof exploits the one dimensionality of the problem and uses extensively the Cannon estimates, [9], following the strategy proposed by Fasano in [22]. In the next sections we will prove Theorem 3.2.2 and then, taking h equal to the space derivative of ρ_0, we will prove Theorem 3.2.1 as a corollary of Theorem 3.2.2.

3.3 Strategy of the proof of Theorem 3.2.2

The idea is to reduce the analysis of the FBP $(**)$ to a fixed point problem:

- Take a curve $X_t, t \in [0,T]$ and find v which solves from the first equation to the third one of $(**)$ in the domain $[X_t,\infty)$, $t \in [0,T]$
- Construct a new curve $K[X](t) = b - \frac{1}{4}\int_0^t e^\tau v_x(X_\tau,\tau)d\tau$ for $0 \le t \le T$
- Find X so that $K[X] = X$ and prove that the corresponding pair (X,v) solves the FBP (**)

The first task is to prove existence and smoothness of v, that we do in this section using the lemmas below, see [9]. Let $A > 0$ and

$$\Sigma(A,T) := \left\{ X \in C[0,T] : X_0 = b,\ \left| \frac{X_{t_2} - X_{t_1}}{t_2 - t_1} \right| \leq A \quad \text{for } 0 \leq t_1 < t_2 \leq T \right\}.$$

Lemma 3.3.1. *Let $X \in \Sigma(A,T)$, then there exists a unique $\varphi \in C^0_{(1)}((0,T])$ such that* $\int_b^\infty h(\xi)G(X_t,t;\xi,0)d\xi - \varphi(t) + \int_0^t G_x(X_t,t;X_\tau,\tau)\varphi(\tau)d\tau = 2e^{-t}$ *for all $t \in (0,T]$.*

Proof. Since $|G_x(X_t,t;X_\tau,\tau)| \leq \dfrac{C}{\sqrt{t-\tau}}$ and $\int_b^\infty h(\xi)G(X_t,t;\xi,0)d\xi \in C^0_{(1)}((0,T])$, we have a Volterra integral equation:

$$\varphi(t) = \psi(t) + \int_0^t G_x(X_t,t;X_\tau,\tau)\varphi(\tau)d\tau \tag{3.3.1}$$

where $\psi(t) = -2e^{-t} + \int_b^\infty h(\xi)G(X_t,t;\xi,0)d\xi \in C^0_{(1)}((0,T])$. As in [9] p.247 the right hand side of 3.3.1 defines a contraction map from $C^0_{(1)}((0,T])$ into itself for T small. Thus there is a unique $\varphi \in C^0_{(1)}((0,T])$ which satisfies (3.3.1). □

Proposition 3.3.2. *For any $X \in \Sigma(A,T)$ let φ as in Lemma 3.3.1. Define*

$$v(x,t) := \int_b^\infty h(\xi)G(x,t;\xi,0)d\xi + \int_0^t G_x(x,t;X_\tau,\tau)\varphi(\tau)d\tau \tag{3.3.2}$$

Then $v \in C^{2,1}(D_{X,T})$ satisfies from the first equation to the third one of $(**)$ and $\lim_{x\to\infty} \sup_{0\leq t\leq T} |v(x,t)| = 0$. *In addition, v has the right derivative at the boundary $v_x(X_t,t) \in C^0_{(1)}((0,T])$.*

Proof. By (3.3.2) v satisfies the heat equation with initial datum h and boundary conditions $v(X_t,t) = 2e^{-t}$, $\lim_{x\to\infty} \sup_{0\leq t\leq T} |v(x,t)| = 0$. Let us choose $R > \sup_{t\in[0,T]} X_t$. Then we have $v(R,t) \in C^0_{(1)}((0,T])$ such that by Lemma 14.4.4. and Theorem 14.4.1. of [9], we also obtain $v_x(X_t,t) \in C^0_{(1)}((0,T])$. □

3.4 Proof of Theorem 3.2.2

Theorem 3.2.2 is proved at the end of the section.

Lemma 3.4.1. *Let $X \in \Sigma(A,T)$ and v as in* (3.3.2)*, then there are positive constants C_1, C_2, C_3 such that*

$$|v_x(X_t,t)| \leq C_1 \int_0^t \frac{|v_x(X_\tau,\tau)|}{\sqrt{t-\tau}} d\tau + (C_2\sqrt{t} + C_3). \tag{3.4.1}$$

Moreover, we have $v_x(X_t,t) \in C([0,T])$.

Proof. Let us fix $(x,t) \in D_{X,T}$ and let us define $D^{(t)}_{\epsilon,R} := \{(\xi,\tau) : X_\tau + \epsilon < \xi < R,\ \epsilon < \tau < t-\epsilon\}$ for each $\epsilon > 0$ and $R \in \mathbb{R}$. By the Green's identity, we have

$$\frac{1}{2}(v_\xi G - v G_\xi)_\xi - (vG)_\tau = 0 \implies \oint_{\partial D^{(t)}_{\epsilon,R}} \frac{1}{2}(v_\xi G - v G_\xi) d\tau + (vG) d\xi = 0. \tag{3.4.2}$$

Hence we obtain another representation of v by letting $\epsilon \to 0$, $R \to \infty$,

$$\begin{aligned} v(x,t) = \int_b^\infty h(\xi) G(x,t;\xi,0) d\xi - \frac{1}{2}\int_0^t G(x,t;X_\tau,\tau) v_\xi(X_\tau,\tau) d\tau \\ + \int_0^t e^{-\tau} G_\xi(x,t;X_\tau,\tau) d\tau. \end{aligned} \tag{3.4.3}$$

Differentiating both sides of (3.4.3) with respect to x and integrating by parts, we get

$$\begin{aligned} v_x(x,t) = \int_b^\infty h_\xi(\xi) G(x,t;\xi,0) d\xi - \frac{1}{2}\int_0^t G_x(x,t;X_\tau,\tau) v_x(X_\tau,\tau) d\tau \\ +2\int_0^t e^{-\tau} G(x,t;X_\tau,\tau) d\tau, \end{aligned} \tag{3.4.4}$$

$$\begin{aligned} \frac{1}{2} v_x(X_t,t) = \int_b^\infty h_\xi(\xi) G(X_t,t;\xi,0) d\xi - \frac{1}{2}\int_0^t G_x(X_t,t;X_\tau,\tau) v_x(X_\tau,\tau) d\tau \\ +2\int_0^t e^{-\tau} G(X_t,t;X_\tau,\tau) d\tau. \end{aligned} \tag{3.4.5}$$

Since $|G_x(X_t,t;X_\tau,\tau)| \le \dfrac{C}{\sqrt{t-\tau}}$ and there are constants C_2 and C_3 such that

$$\left| 2\int_0^t e^{-\tau} G(x,t;X_\tau,\tau) d\tau \right| \le C_2\sqrt{t}, \quad \left| \int_b^\infty h_\xi(\xi) G(x,t;\xi,0) d\xi \right| \le \|h_\xi\|_\infty \le C_3, \tag{3.4.6}$$

then

$$|v_x(X_t,t)| \le (C_2\sqrt{t} + C_3) + C_1 \int_0^t \frac{|v_x(X_\tau,\tau)|}{\sqrt{t-\tau}} d\tau. \tag{3.4.7}$$

Using (3.4.5), we have

$$v_x(X_t,t) \in C([0,T]). \tag{3.4.8}$$

□

Lemma 3.4.2. *Let $A > \frac{C_3}{4}$, C_3 as in Lemma 3.4.1. Then for all sufficiently small T, $K[X](t) := b - \frac{1}{4}\int_0^t e^{\tau} v_x(X_\tau,\tau) d\tau$, $0 \le t \le T$, maps $K : \Sigma(A,T) \longrightarrow \Sigma(A,T)$.*

Proof. By Lemma 17.7.1. in [9] applied to (3.4.7), we obtain

$$|v_x(X_t,t)| \le [1+2C_1\sqrt{T}]\exp\{\pi C_1^2 T\}(C_2\sqrt{t}+C_3). \tag{3.4.9}$$

Then $\left|\frac{d}{dt}K[X](t)\right| = \frac{1}{4}\left|e^t v_x(X_t,t)\right| \le \frac{1}{4}e^T[1+2C_1\sqrt{T}]\exp\{\pi C_1^2 T\}(C_2\sqrt{T}+C_3) \le A$ for all sufficiently small T so that the map K is well defined. □

Hereafter $A > \frac{C_3}{4}$ is fixed. We will show that K is continuous, then, since $\Sigma(A,T)$ is convex and compact we can apply the Schauder fixed point theorem to conclude that K has a fixed point.

Lemma 3.4.3. *For any sufficiently small $T>0$ the map K defined in Lemma 3.4.2 is continuous on $\Sigma(A,T)$ with sup norm.*

Proof. Let $T>0$ be such that $0<b-AT$ and so small that K is well defined. For $X\in\Sigma(A,T)$, we have $0<b-AT\le \inf X \le \sup X \le b+AT$. Let v be the function determined by X via Proposition 3.3.2. Then v satisfies Green's identity as follows; for $D_{\epsilon,R}^{(t)}$ defined in the proof of Lemma 3.4.1,

$$\oint_{\partial D_{\epsilon,R}^{(t)}} [\xi v\, d\xi + \frac{1}{2}(\xi v_\xi - v)d\tau] = 0. \tag{3.4.10}$$

Let $v^{(1)}, v^{(2)}$ be the functions which correspond to $X^{(1)}, X^{(2)} \in \Sigma(A,T)$. We denote $K[X^{(i)}] = \sigma^{(i)}$, $i=1,2$, we apply (3.4.10) and let $\epsilon\to 0$, $R\to\infty$. We get

$$\int_0^t 2e^{-\tau}X_\tau^{(1)}\left[\frac{d}{d\tau}(\sigma_\tau^{(1)}-\sigma_\tau^{(2)})\right]d\tau + \int_0^t 2e^{-\tau}\left[\frac{d}{d\tau}\sigma_\tau^{(2)}\right](X_\tau^{(1)}-X_\tau^{(2)})d\tau$$
$$= \int_{X_t^{(1)}}^\infty \xi v^{(1)}(\xi,t)d\xi - \int_{X_t^{(2)}}^\infty \xi v^{(2)}(\xi,t)d\xi.$$

By integration by parts we have

$$2e^{-t}X_t^{(1)}[\sigma_t^{(1)}-\sigma_t^{(2)}] = \int_0^t (-2e^{-\tau}X_\tau^{(1)} + 2e^{-\tau}\frac{d}{d\tau}X_\tau^{(1)})(\sigma_\tau^{(1)}-\sigma_\tau^{(2)})d\tau$$

$$-\int_0^t 2e^{-\tau}\left[\frac{d}{d\tau}\sigma_\tau^{(2)}\right](X_\tau^{(1)}-X_\tau^{(2)})d\tau + \left[\int_{X_t^{(1)}}^\infty \xi v^{(1)}(\xi,t)d\xi - \int_{X_t^{(2)}}^\infty \xi v^{(2)}(\xi,t)d\xi\right]. \tag{3.4.11}$$

To control $\int_{X_t^{(1)}}^\infty \xi v^{(1)}(\xi,t)d\xi - \int_{X_t^{(2)}}^\infty \xi v^{(2)}(\xi,t)d\xi$, using (3.4.3) and Fubini's theorem, we obtain for $i=1,2$:

$$\int_{X_t^{(i)}}^\infty x v^{(i)}(x,t)dx = \int_b^\infty h(\xi)\int_{X_t^{(i)}}^\infty xG(x,t;\xi,0)dx\,d\xi - \frac{1}{2}\int_0^t v_\xi^{(i)}(X_\tau^{(i)},\tau)\int_{X_t^{(i)}}^\infty xG(x,t;X_\tau^{(i)},\tau)dx\,d\tau$$

$$+\int_0^t e^{-\tau}\int_{X_t^{(i)}}^\infty xG_\xi(x,t;X_\tau^{(i)},\tau)dx\,d\tau. \tag{3.4.12}$$

Taking the difference for the first term of (3.4.12) , we have

$$I_1 := \int_b^\infty h(\xi)t[G(X_t^{(1)},t;\xi,0)-G(X_t^{(2)},t;\xi,0)]d\xi + \int_{X_t^{(1)}}^{X_t^{(2)}} \int_b^\infty \xi h(\xi)G(x,t;\xi,0)d\xi dx.$$

so that $|I_1| \le C \sup_{0\le\tau\le T} \left|X_\tau^{(1)} - X_\tau^{(2)}\right|$.

Since

$$-\frac{1}{2}\int_0^t v_\xi^{(i)}(X_\tau^{(i)},\tau)\int_{X_t^{(i)}}^\infty xG(x,t;X_\tau^{(i)},\tau)dxd\tau = 2\int_0^t e^{-\tau}\left[\frac{d}{d\tau}\sigma_\tau^{(i)}\right]\int_{X_t^{(i)}}^\infty xG(x,t;X_\tau^{(i)},\tau)dxd\tau,$$

we also have by taking the difference for the second term of (3.4.12),

$$I_2 := 2\int_0^t e^{-\tau}\frac{d}{d\tau}(\sigma_\tau^{(1)}-\sigma_\tau^{(2)})\int_{X_t^{(1)}}^\infty xG(x,t;X_\tau^{(1)},\tau)dxd\tau$$

$$+2\int_0^t \left[\int_{X_t^{(1)}}^\infty xG(x,t;X_\tau^{(1)},\tau)dx - \int_{X_t^{(2)}}^\infty xG(x,t;X_\tau^{(2)},\tau)dx\right]e^{-\tau}\frac{d}{d\tau}\sigma_\tau^{(2)}d\tau. \quad (3.4.13)$$

By integration by parts, we get for the first term on right hand side of (3.4.13)

$$2\int_0^t e^{-\tau}\frac{d}{d\tau}(\sigma_\tau^{(1)}-\sigma_\tau^{(2)})\int_{X_t^{(1)}}^\infty xG(x,t;X_\tau^{(1)},\tau)dxd\tau$$

$$= e^{-t}[\sigma_t^{(1)}-\sigma_t^{(2)}]X_t^{(1)} - 2\int_0^t [\sigma_\tau^{(1)}-\sigma_\tau^{(2)}]\frac{\partial}{\partial\tau}\left(e^{-\tau}\int_{X_t^{(1)}}^\infty xG(x,t;X_\tau^{(1)},\tau)dx\right)d\tau.$$

so that

$$I_2 = e^{-t}[\sigma_t^{(1)}-\sigma_t^{(2)}]X_t^{(1)} - 2\int_0^t [\sigma_\tau^{(1)}-\sigma_\tau^{(2)}]\frac{\partial}{\partial\tau}\left(e^{-\tau}\int_{X_t^{(1)}}^\infty xG(x,t;X_\tau^{(1)},\tau)dx\right)d\tau$$

$$+2\int_0^t \left[\int_{X_t^{(1)}}^\infty xG(x,t;X_\tau^{(1)},\tau)dx - \int_{X_t^{(2)}}^\infty xG(x,t;X_\tau^{(2)},\tau)dx\right]e^{-\tau}\frac{d}{d\tau}\sigma_\tau^{(2)}d\tau. \quad (3.4.14)$$

To estimate the last term of (3.4.14), we use the following identity for i=1,2;

$$\int_{X_t^{(i)}}^\infty xG(x,t;X_\tau^{(i)},\tau)dx = -(t-\tau)\int_{X_t^{(i)}}^\infty G_x(x,t;X_\tau^{(i)},\tau)dx + X_\tau^{(i)}\int_{X_t^{(i)}}^\infty G(x,t;X_\tau^{(i)},\tau)dx$$

$$= (t-\tau)G(X_t^{(i)},t;X_\tau^{(i)},\tau) + X_\tau^{(i)}\int_{X_t^{(i)}}^\infty G(x,t;X_\tau^{(i)},\tau)dx = (t-\tau)G(X_t^{(i)},t;X_\tau^{(i)},\tau) + X_\tau^{(i)}\Psi\left(\frac{X_t^{(i)}-X_\tau^{(i)}}{\sqrt{t-\tau}}\right),$$

where $\Psi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{|z|^2}{2}\right\}dz$.

By applying the mean value theorem on Ψ, we have

$$\left|\Psi\left(\frac{X_t^{(1)}-X_\tau^{(1)}}{\sqrt{t-\tau}}\right)-\Psi\left(\frac{X_t^{(2)}-X_\tau^{(2)}}{\sqrt{t-\tau}}\right)\right|\le C_1\left|\frac{X_t^{(1)}-X_\tau^{(1)}-X_t^{(2)}+X_\tau^{(2)}}{\sqrt{t-\tau}}\right|\le\frac{C_1\sup_{0\le\eta\le T}\left|X_\eta^{(1)}-X_\eta^{(2)}\right|}{\sqrt{t-\tau}}$$

By applying the mean value theorem on $\exp\left\{-\frac{|\cdot|^2}{2}\right\}$, for some $0<\theta<1$,

$$\begin{aligned}&|G(X_t^{(1)},t;X_\tau^{(1)},\tau)-G(X_t^{(2)},t;X_\tau^{(2)},\tau)|\\&\le\frac{1}{\sqrt{2\pi(t-\tau)}}\left|\theta\frac{X_t^{(1)}-X_\tau^{(1)}}{\sqrt{t-\tau}}+(1-\theta)\frac{X_t^{(2)}-X_\tau^{(2)}}{\sqrt{t-\tau}}\right|\left|\frac{X_t^{(1)}-X_\tau^{(1)}-X_t^{(2)}+X_\tau^{(2)}}{\sqrt{t-\tau}}\right|\\&\le\frac{C_2\sup_{0\le\eta\le T}\left|X_\eta^{(1)}-X_\eta^{(2)}\right|}{\sqrt{t-\tau}}\end{aligned}$$

Thus we obtain

$$\left|\int_{X_t^{(1)}}^\infty xG(x,t;X_\tau^{(1)},\tau)dx-\int_{X_t^{(2)}}^\infty xG(x,t;X_\tau^{(2)},\tau)dx\right|\le C_3\sup_{0\le\eta\le T}\left|X_\eta^{(1)}-X_\eta^{(2)}\right|+C_4\frac{\sup_{0\le\eta\le T}\left|X_\eta^{(1)}-X_\eta^{(2)}\right|}{\sqrt{t-\tau}}.$$

Also we observe

$$\left|\frac{\partial}{\partial\tau}\left(e^{-\tau}\int_{X_t^{(1)}}^\infty xG(x,t;X_\tau^{(1)},\tau)dx\right)d\tau\right|$$

$$=\left|\frac{\partial}{\partial\tau}\left[e^{-\tau}\left\{(t-\tau)G(X_t^{(1)},t;X_\tau^{(1)},\tau)+X_\tau^{(1)}\Psi\left(\frac{X_t^{(1)}-X_\tau^{(1)}}{\sqrt{t-\tau}}\right)\right\}\right]\right|\le C_5+\frac{C_6}{\sqrt{t-\tau}}.$$

For the third term of (3.4.12) we have

$$\int_0^t e^{-\tau}\int_{X_t^{(i)}}^\infty xG_\xi(x,t;X_\tau^{(i)},\tau)dxd\tau=\int_0^t e^{-\tau}\left[X_t^{(i)}G(X_t^{(i)},t;X_\tau^{(i)},\tau)+\int_{X_t^{(i)}}^\infty G(x,t;X_\tau^{(i)},\tau)\right]d\tau.$$

Then we have

$$I_3:=\int_0^t e^{-\tau}\left[X_t^{(1)}G(X_t^{(1)},t;X_\tau^{(1)},\tau)+\int_{X_t^{(1)}}^\infty G(x,t;X_\tau^{(1)},\tau)-X_t^{(2)}G(X_t^{(2)},t;X_\tau^{(2)},\tau)-\int_{X_t^{(2)}}^\infty G(x,t;X_\tau^{(2)},\tau)\right]d\tau$$

so that

$$|I_3|\le C_7\sup_{0\le\eta\le T}\left|X_\eta^{(1)}-X_\eta^{(2)}\right|.$$

Combining all the results and (3.4.11) finally we obtain for all sufficiently small $T>0$,

$$\begin{aligned}\left|\sigma_t^{(1)}-\sigma_t^{(2)}\right|&\le C_8\int_0^t\left|\sigma_\tau^{(1)}-\sigma_\tau^{(2)}\right|d\tau+C_9\int_0^t\frac{\left|\sigma_\tau^{(1)}-\sigma_\tau^{(2)}\right|}{\sqrt{t-\tau}}d\tau+C_{10}\sup_{0\le\tau\le T}\left|X_\tau^{(1)}-X_\tau^{(2)}\right|\\&\le(C_8+C_9)\int_0^t\frac{\left|\sigma_\tau^{(1)}-\sigma_\tau^{(2)}\right|}{\sqrt{t-\tau}}d\tau+C_{10}\sup_{0\le\tau\le T}\left|X_\tau^{(1)}-X_\tau^{(2)}\right|.\end{aligned}$$

By Lemma 17.7.1 in [9], we get $\sup_{0\le\tau\le T}\left|\sigma_t^{(1)}-\sigma_t^{(2)}\right|\le C_{11}\sup_{0\le\tau\le T}\left|X_\tau^{(1)}-X_\tau^{(2)}\right|$ so that K is a continuous map. □

Proof of Theorem 3.2.2
Let (X, v) as in Lemma 3.4.3 then by (3.4.8) $K[X] = X$ is in $C^1([0, T])$. This completes the proof. □

3.5 Proof of Theorem 3.2.1

Suppose that $h(x) = \dfrac{d\rho_0(x)}{dx}$ and let (X, v) be as in Theorem 3.2.2 with such h as initial condition. Let $\rho(x,t) := \int_{X_t}^{x} e^t v(y,t)dy$. It can be readily shown that ρ solves from the first to the third equations in $(*)$, see Section 3.1. To prove the last equation in $(*)$ we first remark that by (3.3.2) $v \in L^1([X_t, \infty))$. We then differentiate $\int_{X_t}^{\infty} v(y,t)dy$ with respect to t:

$$\frac{d}{dt}\left(\int_{X_t}^{\infty} v(x,t)dx\right) = -\dot{X}_t v(X_t, t) + \int_{X_t}^{\infty} v_t(x,t)dx$$
$$= -2\dot{X}_t e^{-t} + \int_{X_t}^{\infty} \frac{1}{2} v_{xx}(x,t) = -2\dot{X}_t e^{-t} - \frac{1}{2} v_x(X_t, t) = 0.$$

Since $\int_b^{\infty} h(x)dx = \int_b^{\infty} \dfrac{d\rho_0(x)}{dx}dx = 0$, we obtain that $\int_{X_t}^{\infty} v(x,t)dx = 0$. By using this, (3.3.2), and Fubini's theorem, we also have

$$\rho(x,t) = e^t\left[\int_0^t G(x,t;X_\tau,\tau)\varphi(\tau)d\tau - \int_b^{\infty} h(\xi)\int_x^{\infty} G(y,t;\xi,0)dyd\xi\right] \tag{3.5.1}$$

so that $\int_{X_t}^{\infty} \rho(x,t)dx < \infty$. Similarly, if we differentiate $\int_{X_t}^{\infty} \rho(x,t)dx$ with respect to t and get $\dfrac{d}{dt}\left(\int_{X_t}^{\infty} \rho(x,t)dx\right) = \int_{X_t}^{\infty} \rho_t(x,t)dx = \int_{X_t}^{\infty}\left(\frac{1}{2}\rho_{xx}(x,t) + \rho(x,t)\right)dx = -\frac{1}{2}\rho_x(X_t,t) + \int_{X_t}^{\infty} \rho(x,t)dx = -1 + \int_{X_t}^{\infty} \rho(x,t)dx$. Since $\int_b^{\infty} \rho_0(x)dx = 1$, we obtain that $\int_{X_t}^{\infty} \rho(x,t)dx = 1$. The derivative $\rho_x(x,t)$ has a limit when $x \to X_t$ and $\rho_x(X_t,t) = 2$ and vanishes as $x \to \infty$ by Proposition 3.3.2. This completes the proof of Theorem 3.2.1; the statements below Theorem 3.2.1 also follow from what proved for $v(x,t)$.

CHAPTER 4

A free boundary problem with non local interaction

4.1 Introduction

In [5] Brunet and Derrida have proposed several models to study selection mechanisms in biological systems which give rise to very interesting questions not only in the applications to biology but also in the areas of stochastic particle systems and of PDE's with free boundaries. This paper concerns mostly the last issue but it is worth, we think, to give first a more general overview.

In the line of the Brunet-Derrida's proposal Durrett and Remenik in [20] have introduced and studied a model of particles on $\mathbb{R}$ which independently at rate 1 creates a new particle whose position is chosen randomly with probability $p(x,y)dy$, $p(x,y) = p(0,y-x)$, if x is the position of the generating particle. Instantaneously after the creation the leftmost particle is deleted so that the total number of particles is constant.

The biological interpretation is that the position of a particle is "its degree of fitness", the larger the position the higher the fitness. The removal of the leftmost (and hence less fitted) particle gives rise to an improvement of the general fitness of the population and in fact Durrett and Remenik have proved the existence of traveling fronts moving with positive velocity.

The main difficulty in the analysis of the model is the apparently simple deleting mechanism of killing the leftmost particle. In fact the notion of leftmost particle is highly non local: one needs to know the positions of all the particles to determine which is the leftmost one. This is therefore a "topological" interaction which cannot be treated with the usual methods of interacting particle systems, it is the analogue in PDE's of free boundary problems when the domain where the PDE's are defined is itself one of the unknowns, see for instance the survey by Carinci, De Masi, Giardinà and Presutti, [10], on topological interactions and their relation in the "hydrodynamic limit" with free boundary problems.

In the biological applications the size of the population is very large and therefore the main interest is in the analysis of the asymptotic behavior of the particle system in the continuum limit when N, (i.e. the total number of particles) diverges. Under suitable assumptions on the initial datum Durrett and Remenik have proved that as $N \to \infty$ a limit density exists and it satisfies:

$$\frac{\partial}{\partial t}\rho(x,t) = \int_{X_t}^{\infty} dy\, p(y,x)\rho(y,t)dy, \quad \rho(x,0) = \rho_0(x) \tag{4.1.1}$$

where $X_t = \inf\{r : \rho(r,t) > 0\}$. Notice that the domain of integration on the right hand side of (4.1.1) is also an unknown since one needs to know the whole function $\rho(x,t)$ to determine the value X_t of "the edge".

As it stands (4.1.1) does not select $\rho(x,t)$ because we can give "arbitrarily" X_t and still solve

(4.1.1). To get uniqueness we would need to know a priori X_t which should be the limit position (as $N \to \infty$) of the leftmost particle in the system. This is in itself an interesting issue but apparently very difficult to carry out. Durrett and Remenik have circumvented the difficulty by using the other information coming from the particle system, namely that the total number of particles is conserved. In the continuum limit where $N \to \infty$ the above is reflected into the condition that

$$\int_{X_t}^{\infty} dx\, \rho(x,t) = 1, \text{for all } t \geq 0 \tag{4.1.2}$$

The pair (4.1.1)–(4.1.2) is a "free boundary problem" but not in its more usual formulation where (4.1.1) is usually replaced by a parabolic diffusion equation and instead of (4.1.2) there is a condition relating the velocity of the edge to the spatial derivative of the solution at the edge. This is indeed what happens in the classical Stefan problem, see for instance the survey by Fasano, [22].

Under suitable assumptions on the initial datum ρ_0 and on the probability kernel $p(x,y)$ Durrett and Remenik have been able to prove that the pair (4.1.1)–(4.1.2) has a unique solution which is the limit density of the particles system.

An important ingredient in the proof is that X_t is monotonically non decreasing, a feature is clear at the particles level where in fact the position of the leftmost particle if it moves can only increase: it stays put when the new particle is created to its left (because then this is the one which is deleted) while, in the other case, the previous second leftmost particle becomes the leftmost one. Such a simplifying effect is not present in the next models we are going to discuss.

In [17] De Masi, Ferrari, Presutti and Soprano-Loto (in the sequel DFPS for brevity), have studied the so called N-BBM model, which is an acronym for N branching Browniam motions. The selection mechanism in the N-BBM model is similar to the Durrett-Remenik one: once a new particle is created the leftmost one is deleted. There are however two main differences: the particles move as independent Brownian motions and the new particle is created at exactly the same position of the generating one (the previous kernel $p(x,y)$ becomes a Dirac delta, $\delta(x-y)$). Biologically this means that the individual fitness changes randomly in time and the duplicating processes are exact, the fitness of the son is exactly equal to that of the father. Believing that the Durrett-Remenik arguments extend to this case one would conjecture that the limit density $\rho(x,t)$ satisfies the equation

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho(x,t) + \rho(x,t), \quad x > X_t, \quad \rho(x,0) = \rho_0(x) \tag{4.1.3}$$

(where again $X_t = \inf\{x : \rho(x,t) > 0\}$). (4.1.3) is in fact obtained from (4.1.1) by adding on the right hand side the Laplacian which takes into account the Brownian motion of the particles while the last term $\rho(x,t)$ is the right hand side of (4.1.1) when $p(x,y) = \delta(x-y)$. The free boundary problem (4.1.3)–(4.1.2) is "incomplete" because even if X_t is known yet (4.1.3) does not have a unique solution: we must also give the value of $\rho(x,t)$ at the edge X_t.

The natural choice would be to derive it as the limit particles density at the edge which is still not at all easy due to the poor control of the position of the leftmost particle. However taking into account the regularizing effect of the heat diffusion one may suppose that

$$\rho(X_t,t) = 0 \quad \text{at all times } t \geq 0 \tag{4.1.4}$$

(notice that in the Durrett-Remenik model (4.1.4) does not hold, recall however that in (4.1.1) there is no Laplacian !).
DFPS have proved (under suitable assumptions on the initial datum) that in the limit $N \to \infty$ the particle density has a limit $\rho(x,t)$ for any $t \geq 0$. It is also proved that $\rho(x,t)$ satisfies (4.1.3)–(4.1.2)–(4.1.4) if this has a "regular" solution. As far as we know there is only a "local" existence theorem under suitable assumptions on the initial datum (as discussed in the next section) which therefore coincides with the limit density of the N-BBM system. From [17] we know that $\rho(x,t)$ is well defined at all times, but it is not clear if at times larger than for local existence it is still a solution of (4.1.3)–(4.1.2)–(4.1.4) at least in a "weak sense".
Notice that uniqueness in the local existence theorem follows from [17] as DFPS have shown that any "smooth solution" is necessarily equal to the limit density of the particles system and hence unique.
The question of traveling fronts is of great interest: in [3] Berestycki, Brunet and Derrida have considered (4.1.2) complemented by conditions on the values of the solution and its derivative at the edge. They were mainly interested in the precise asymptotics of the velocity of the front underlying connections with the Fisher-KPP type fronts, see also [26] where Groisman and Jonckheere discuss front propagation and quasi-stationary distributions.
The analysis of the front before the limit $N \to \infty$ is also particularly interesting, see for instance the work of Maillard, [33] on its large fluctuations.

We are mainly interested here on the existence of solutions for a free boundary problem introduced in [18]. The particles system is an extension of the N-BBM model obtained by making the branching mechanism non local as in the case considered by Durrett and Remenik. In [18] the conjectured evolution equation is in fact

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho(x,t) + \int_{X_t}^{\infty} dy\, p(y,x)\rho(y,t)dy, \quad x > X_t, \qquad \rho(x,0) = \rho_0(x) \tag{4.1.5}$$

which is a combination of (4.1.1) (for the branching) and (4.1.3) for the Brownian diffusion. The results for the N-BBM model have been extended in [18] to this case, a limiting density $\rho(x,t)$ exists and it is uniquely defined, moreover if there is a smooth solution of the free boundary problem (4.1.5)–(4.1.2)–(4.1.4) then this is the limit particles density of the model. As mentioned the proof of local existence of smooth solutions for (4.1.5)–(4.1.2)–(4.1.4) is the main result in this paper, the precise statement is as follows.

Assumptions.

- *On the initial datum.* We suppose that: $\rho_0(x) = 0$ for $x \leq 0$, it is in C^3 for $x \geq 0$ and it has compact support. Moreover

$$\frac{d}{dx}\rho_0(x)\Big|_{x=0^+} = 2\int_0^\infty \int_0^\infty \rho_0(y)p(y,x)dydx > 0 \tag{4.1.6}$$

- *On the kernel $p(x,y)$.* We suppose that: $p(x,y) = p(0,y-x)$, $p(0,x)$ is non negative with compact support, it is in C^1 and its integral is equal to 1 (i.e. $p(x,y)$ is a transition probability kernel).

Remarks. By the first assumption $X_0 = 0$: by translation invariance there is no loss in generality by fixing the edge initially at 0. The regularity assumption on ρ_0 comes from the necessity of controlling the velocity of the edge which involves, as we will see, the second derivative of $\rho(x,t)$ with respect to x. Finally the "strange condition (4.1.6)" is required to avoid initial layer problems as discussed in the next section.

Theorem 4.1.1. *Under the above assumptions there are $T>0$, X_t, $t\in[0,T]$, and $\rho(x,t)$, $x\ge X_t$, $t\in[0,T]$, such that:*

- $X_0=0$*, X_t is differentiable and its derivative V_t is Hölder continuous with exponent* $1/2$.
- $\rho(x,t)$ *is $C^{3,1}$ (three derivatives in x and one in t) in the domain $x>X_t, t>0$.*
- *The pair $(X_t,\rho(x,t))$, $x\ge X_t, t\in[0,T]$, solves the free boundary problem* (4.1.5)–(4.1.2)–(4.1.4).

In the next section we outline the strategy of the proof and discuss what known in the literature. In Section 4.5 we give the proof of Theorem 4.1.1 and in a last section we discuss the extension to the other free boundary problems mentioned in this introduction.

4.2 Strategy of proof

In the case we are mainly interested here there is a non local term and this prevents us to use at least directly the above approach. An alternative way to study the free boundary problems is to look at the evolution from the edge. This is often done at the particles level to study the shape and structure of the traveling waves independently of their location. A derivation of the hydrodynamic equations for the N-BBM model and its non local variant is however missing. We suppose tacitly hereafter that the initial position of the edge is $X_0=0$. The advantage of studying the free boundary problem in the frame where the edge is always at the origin is that by its very definition the spatial domain is fixed, it is no longer one of unknowns. The difficulties however have not disappeared as in the evolution equations appears a drift term which depends on the velocity of the edge. The natural setting from the problem requires now that the motion of the edge is C^1 (we will also require that the derivative V_t is Hölder continuous with exponent $1/2$). More precisely we call $u(x,t)=\rho_x(x,t)$ and then change variables: $\rho(x,t)\to\rho(x-X_t,t)$, $u(x,t)\to u(x-X_t,t)$. By an abuse of notation we denote by the same symbols ρ and u the new functions and we get the following system of equations:

$$\rho_t(x,t)=\frac{1}{2}\rho_{xx}(x,t)+V_t\rho_x(x,t)+\int_0^\infty dy\; p(y,x)\rho(y,t)dy,\quad x>0 \tag{4.2.1}$$

$$\rho(x,0)=\rho_0(x),\ x\ge 0,\quad \rho(0,t)=0,\ t\ge 0 \tag{4.2.2}$$

$$u_t(x,t)=\frac{1}{2}u_{xx}(x,t)+V_tu_x(x,t)+\int_0^\infty dy\; p(y,x)u(y,t)dy,\quad x>0 \tag{4.2.3}$$

$$u(x,0)=u_0(x):=\frac{d\rho_0(x)}{dx}\ \text{ for } x\ge 0 \tag{4.2.4}$$

$$u(0,t)=2\int_0^\infty\int_0^\infty dy\,\rho(y,t)p(y,x)dx \tag{4.2.5}$$

We also have:

$$u(0,t)V_t = -\frac{1}{2}u_x(0,t) + \int_0^\infty \int_0^\infty dy\, u(y,t)p(y,x)dydx \tag{4.2.6}$$

which is obtained by differentiating (4.1.4) with respect to time.
In the way the above equations have been derived u is the spatial derivative of ρ, but we will regard the system (4.2.1) to (4.2.5) without imposing such relation. Namely we fix a function V_t which is Hölder continuous with exponent $1/2$; we then solve (4.2.1)-(4.2.2) and find $\rho(x,t)$. We then solve (4.2.3)-(4.2.5) with $\rho(x,t)$ as determined above and thus get $u(x,t)$. With such $\rho(x,t)$ and $u(x,t)$ we determine a new speed V_t via (4.2.6) and thus get an iterative scheme. We will prove that all this can be done and the iterative scheme has a fixed point. For such fixed point we re-establish the identity that u is the spatial derivative of ρ and then get a proof of Theorem 4.1.1.
The change of variables which fixes the position of the edge has been used in [24]. Our approach is similar but we have extra difficulties for the presence of the non local term. Moreover, [24] relies on the result of [31] which does not include our case since its initial and boundary conditions are stronger. As one of our analysis we give an improved estimate in Lemma 4.4.4 than (4.24) of [24].

4.3 Main results

Let us denote by G a Gaussian function and K a Green function as

$$G(x,t;\xi,\tau) = \frac{1}{\sqrt{2\pi(t-\tau)}}\exp\left\{-\frac{|x-\xi|^2}{2(t-\tau)}\right\},\ K(x,t;\xi,\tau) = G(x,t;\xi,\tau) - G(x,t;-\xi,\tau).$$

From now on, we write independent positive constants as $\{c_i\}_{i\ge 1}$ and use the following facts extensively

$$\int_{-\infty}^\infty G(x,t;\xi,\tau)dx = 1,\quad \int_{-\infty}^\infty |G_x(x,t;\xi,\tau)|dx = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{1}{\sqrt{t-\tau}}.$$

Proposition 4.3.1. *Let $X \in C^1([0,T])$ where $T > 0$ and $X_0 = 0$. There is a unique solution $\rho \in C(\overline{D_T})$ where $D_T = \{(x,t) : 0 < x,\ 0 < t \le T\}$ with $\rho_x \in C(\overline{D_T})$ and $\|\rho\|_\infty + \|\rho_x\|_\infty < \infty$ which satisfies* (4.2.1)-(4.2.2).

Proof. Similarly as Theorem 20.3.1 of [9], let us define a mapping $\mathscr{F} : \mathscr{B}_\eta \to \mathscr{B}_\eta$, where $\mathscr{B}_\eta = \{\rho(x,t) \in C([0,\infty)\times[0,\eta]) : \rho_x \in C([0,\infty)\times[0,\eta]), \|\rho\|_\infty + \|\rho_x\|_\infty < \infty\}$ as

$$\mathscr{F}\rho(x,t) := \int_0^\infty K(x,t;\xi,0)\rho_0(\xi)d\xi + \int_0^t\int_0^\infty K(x,t;\xi,\tau)\left[V_\tau \rho_\xi(\xi,\tau) + \int_0^\infty dy\rho(y,\tau)p(y,\xi)\right]d\xi d\tau. \tag{4.3.1}$$

Then $\mathscr{B}_\eta$ is a Banach space with the norm $\|\cdot\|_\infty + \|\frac{\partial}{\partial x}(\cdot)\|_\infty$ and we have

$$\|\mathscr{F}\rho_1 - \mathscr{F}\rho_2\|_\infty \le 2\eta\left[\|V\|_\infty\|\rho_{1x}-\rho_{2x}\|_\infty + \|\rho_1-\rho_2\|_\infty\right], \tag{4.3.2}$$

and

$$\left\| \frac{\partial \mathscr{F}\rho_1}{\partial x} - \frac{\partial \mathscr{F}\rho_2}{\partial x} \right\|_\infty \leq c_1\sqrt{\eta}\big[\|V\|_\infty \|\rho_{1x}-\rho_{2x}\|_\infty + \|\rho_1-\rho_2\|_\infty\big]. \tag{4.3.3}$$

Thus for all sufficiently small $\eta > 0$, we get $\mathscr{F}$ is a contraction mapping so that there is a unique fixed point ρ^η. To extend η to T, if we define $\mathscr{H}$ as

$$\begin{aligned}\mathscr{H}\rho(x,t) := &\int_0^\infty K(x,t;\xi,0)\rho_0(\xi)d\xi \\ &+\int_0^\eta\int_0^\infty K(x,t;\xi,\tau)\left[V_\tau {\rho^\eta}_\xi(\xi,\tau)+\int_0^\infty dy\rho^\eta(y,\tau)p(y,\xi)\right]d\xi d\tau \\ &+\int_\eta^t\int_0^\infty K(x,t;\xi,\tau)\left[V_\tau\rho_\xi(\xi,\tau)+\int_0^\infty dy\rho(y,\tau)p(y,\xi)\right]d\xi d\tau.\end{aligned}$$

Then we have

$$\|\mathscr{H}\rho_1-\mathscr{H}\rho_2\|_\infty \leq 2(t-\eta)\big[\|V\|_\infty\|\rho_{1x}-\rho_{2x}\|_\infty+\|\rho_1-\rho_2\|_\infty\big], \tag{4.3.4}$$

and

$$\left\| \frac{\partial \mathscr{H}\rho_1}{\partial x} - \frac{\partial \mathscr{H}\rho_2}{\partial x} \right\|_\infty \leq c_1\sqrt{t-\eta}\big[\|V\|_\infty \|\rho_{1x}-\rho_{2x}\|_\infty + \|\rho_1-\rho_2\|_\infty\big] \tag{4.3.5}$$

such that we have $\mathscr{H}$ is a contraction mapping from $\{\rho(x,t)\in C([0,\infty)\times[\eta,2\eta]) : \rho_x \in C([0,\infty)\times[\eta,2\eta]), \|\rho\|_\infty+\|\rho_x\|_\infty < \infty\}$ to itself. Thus we can extend η to 2η, inductively also to T and this completes the proof. $\square$

We will prove the existence of a classical solution of the FBP (4.2.3)-(4.2.6) in $[0,T]$ in the next sections:

Theorem 4.3.2. *There is $T > 0$ and a pair (X,u) which satisfies the FBP* (4.2.3)-(4.2.6) *in* $[0,T]$ *with: $X \in C^1([0,T])$, $X_0 = 0$, and $u \in C(\overline{D_T})\cap C^{2,1}(D_T)$, where $D_T = \{(x,t) : 0 < x,\ 0 < t \leq T\}$.*

We did not find a proof of Theorem 4.3.2 in the existing literature, see for instance [7], [14] and references therein. Our proof exploits the one dimensionality of the problem and uses extensively the Cannon estimates, [9], following the strategy proposed by Fasano in [22]. After then we will prove Theorem 4.1.1 as a corollary of Theorem 4.3.2.

4.4 Proof of Theorem 4.3.2

Theorem 4.3.2 is proved at the end of the section. The idea is to reduce the analysis of the FBP (4.2.3)-(4.2.6) to a fixed point problem:

- Take a curve V_t, $t\in[0,T]$, let $X_t = \int_0^t V_\tau d\tau$, and find u such that (X,u) solves (4.2.3)-(4.2.5).

- Construct a new curve $Q[V](t) = \dfrac{-\frac{1}{2}u_x(0,t) + \int_0^\infty \int_0^\infty u(y,t)p(y,x)dydx}{u(0,t)}$ for $0 \le t \le T$

- Find V so that $Q[V] = V$ and prove that the corresponding pair (X,u) solves the FBP (4.2.3)-(4.2.6).

The first task is to prove existence and smoothness of u, that we do in this section using the lemmas below, see [9]. Let $A > 0$ and

$$\Sigma(A,T) := \left\{ V \in C([0,T]) : V_0 = \frac{-\frac{1}{2}\rho_0''(0) + \int_0^\infty \int_0^\infty dy \rho_0'(y)p(y,x)dx}{\rho_0'(0)}, |V|_{\frac{1}{2}} \le A \right\},$$

where $|\cdot|_{\frac{1}{2}}$ is the Hölder seminorm with exponent $\dfrac{1}{2}$.

Proposition 4.4.1. *Let $X \in C^1([0,T])$ where $T > 0$ and $X_0 = 0$. There is a unique solution $u \in C(\overline{D_T})$ where $D_T = \{(x,t) : 0 < x,\ 0 < t \le T\}$ with $u_x \in C(\overline{D_T})$ and $\|u\|_\infty + \|u_x\|_\infty < \infty$ such that (X,u) satisfies* (4.2.3)-(4.2.5).

Proof. Let ρ as in Proposition 4.3.1 and let us define a mapping $\mathscr{F} : \mathscr{B}_\eta \to \mathscr{B}_\eta$ where $\mathscr{B}_\eta = \{u(x,t) \in C([0,\infty) \times [0,\eta]) : u_x \in C([0,\infty) \times [0,\eta]), \|u\|_\infty + \|u_x\|_\infty < \infty\}$ as

$$\mathscr{F}u(x,t) := -\int_0^t G_x(x,t;0,\tau)g(\tau)d\tau + \int_0^\infty K(x,t;\xi,0)\varphi(\xi)d\xi$$
$$+ \int_0^t \int_0^\infty K(x,t;\xi,\tau)\left[V_\tau u_\xi(\xi,\tau) + \int_0^\infty dy u(y,\tau)p(y,\xi) \right] d\xi d\tau,$$

where

$$u(0,t) = 2\int_0^\infty \int_0^\infty dy \rho(y,t)p(y,x)dx =: g(t),\ \varphi(\xi) := \rho_0'(\xi).$$

Then we can show that $\mathscr{F}$ is a contraction mapping for all sufficiently small $\eta > 0$ and extend η to T as same as the proof of Proposition 4.3.1 so that $\mathscr{F}$ has a unique fixed point $u \in C(\overline{D_T})$. This completes the proof. □

Thus we can write u as

$$u(x,t) = -\int_0^t G_x(x,t;0,\tau)g(\tau)d\tau + \int_0^\infty K(x,t;\xi,0)\varphi(\xi)d\xi$$
$$+ \int_0^t \int_0^\infty K(x,t;\xi,\tau)\left[V_\tau u_\xi(\xi,\tau) + \int_0^\infty dy u(y,\tau)p(y,\xi) \right] d\xi d\tau, \quad (4.4.1)$$

where

$$u(0,t) = 2\int_0^\infty \int_0^\infty dy \rho(y,t)p(y,x)dx =: g(t),\ \varphi(\xi) := \rho_0'(\xi).$$

In addition, by differentiating (4.4.1) and integration by parts, we have

$$u_x(x,t) = -2\int_0^t G(x,t;0,\tau)g'(\tau)d\tau + \int_0^\infty [G(x,t;\xi,0) + G(x,t;-\xi,0)]\varphi'(\xi)d\xi$$
$$+ \int_0^t \int_0^\infty K_x(x,t;\xi,\tau)\left[V_\tau u_\xi(\xi,\tau) + \int_0^\infty dy u(y,\tau)p(y,\xi) \right] d\xi d\tau. \quad (4.4.2)$$

We introduce the following lemma from [9] which plays an essential role in our analysis.

Lemma 4.4.2. *Let $\phi(t)$ satisfy*

$$0 \le \phi(t) \le \psi(t) + C_1 \int_0^t \frac{\phi(\tau)}{\sqrt{t-\tau}} d\tau, \ 0 \le t \le T, \tag{4.4.3}$$

where $C_1 \ge 0$ and $\psi(t)$ is nonnegative and nondecreasing. Then

$$0 \le \phi(t) \le [1 + 2C_1\sqrt{t}]\psi(t)\exp\{\pi C_1^2 t\} \le C_2\psi(t) \tag{4.4.4}$$

with

$$C_2 = [1 + 2C_1\sqrt{T}]\exp\{\pi C_1^2 T\}. \tag{4.4.5}$$

Proof. See Lemma 17.7.1 of [9]. □

Let us denote by $\mathscr{S}$ a collection of continuous functions $C : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that for each $x \in [0,\infty)$, $C(x,\cdot)$ is increasing with respect to the second variable and $C(x,0) > 0$ is independent of the first variable.

Lemma 4.4.3. *Let $V \in \Sigma(A,T)$ and let $X_t = \int_0^t V_\tau d\tau$. For (X,ρ) as in Proposition 4.3.1 and (X,u) as in Proposition 4.4.1, then $\|\rho\|_\infty \le C_1(A,T)$, $\|\rho_x\|_\infty \le C_2(A,T)$, $\sup_{0\le t\le T} \|\rho(\cdot,t)\|_{L^1} \le C_3(A,T)$, $\|u\|_\infty \le C_4(A,T)$, $\|u_x\|_\infty \le C_5(A,T)$, $\sup_{0\le t\le T} \|u(\cdot,t)\|_{L^1} \le C_6(A,T)$, where $C_i \in \mathscr{S}$ for all $1 \le i \le 6$.*

Proof. We can write ρ as

$$\begin{aligned}\rho(x,t) = &\int_0^\infty K(x,t;\xi,0)\rho_0(\xi)d\xi \\ &+ \int_0^t \int_0^\infty K(x,t;\xi,\tau)\left[V_\tau \rho_\xi(\xi,\tau) + \int_0^\infty dy\rho(y,\tau)p(y,\xi)d\xi d\tau\right] \\ = &\int_0^\infty K(x,t;\xi,0)\rho_0(\xi)d\xi - \int_0^t \int_0^\infty K_\xi(x,t;\xi,\tau)V_\tau\rho(\xi,\tau)d\xi d\tau \\ &- \int_0^t \int_0^\infty K(x,t;\xi,\tau) \int_0^\infty dy\rho(y,\tau)p(y,\xi)d\xi d\tau. \end{aligned} \tag{4.4.6}$$

Then we have

$$\begin{aligned}\sup_{x\ge0} |\rho(x,t)| \le 2\|\rho_0\|_\infty + c_1\|V\|_\infty \int_0^t \frac{\sup_{\xi\ge0}|\rho(\xi,\tau)|}{\sqrt{t-\tau}} d\tau \\ + c_2 \int_0^t \sup_{y\ge0} |\rho(y,\tau)| d\tau. \end{aligned} \tag{4.4.7}$$

Applying Lemma 4.4.2 on (4.4.7), we have

$$\sup_{x\ge0} |\rho(x,t)| \le (1 + 2c_1\|V\|_\infty \sqrt{T}) \exp\left\{\pi {c_1}^2 \|V\|_\infty^2 T\right\} \left(2\|\rho_0\|_\infty + c_2 \int_0^t \sup_{y\ge0} |\rho(y,\tau)| d\tau\right).$$

By Gronwall's lemma and $\|V\|_\infty \le |V_0| + A\sqrt{T}$, we obtain for some $C_1 \in \mathscr{S}$,

$$\|\rho\|_\infty \le C_1(A,T). \tag{4.4.8}$$

Also we can write ρ_x as

$$\begin{aligned}\rho_x(x,t) = &\int_0^\infty [G(x,t;\xi,0) + G(x,t;-\xi,0)]\varphi(\xi)d\xi \\ &+ \int_0^t \int_0^\infty K_x(x,t;\xi,\tau)\left[V_\tau \rho_\xi(\xi,\tau) + \int_0^\infty dy\rho(y,\tau)p(y,\xi)\right]d\xi d\tau.\end{aligned} \tag{4.4.9}$$

Then we have

$$\sup_{x\ge 0} |\rho_x(x,t)| \le 2\|\varphi\|_\infty + c_3\|V\|_\infty \int_0^t \frac{\sup_{\xi\ge 0}|\rho_\xi(\xi,\tau)|}{\sqrt{t-\tau}} d\tau + c_4\sqrt{T}\|\rho\|_\infty. \tag{4.4.10}$$

Using (4.4.8) and Lemma 4.4.2, we also obtain for some $C_2 \in \mathscr{S}$,

$$\|\rho_x\|_\infty \le C_2(A,T). \tag{4.4.11}$$

Taking absolute value on both sides of (4.4.6) and integrating them with respect to x, we have

$$\|\rho(\cdot,t)\|_{L^1} \le 2 + c_5\|V\|_\infty \int_0^t \frac{\|\rho(\cdot,\tau)\|_{L^1}}{\sqrt{t-\tau}} d\tau + 2\int_0^t \|\rho(\cdot,\tau)\|_{L^1} d\tau. \tag{4.4.12}$$

Similarly, we get for some $C_3 \in \mathscr{S}$,

$$\sup_{0\le t\le T} \|\rho(\cdot,t)\|_{L^1} \le C_3(A,T). \tag{4.4.13}$$

Using (4.4.1) and integration by parts, we have

$$\begin{aligned}\sup_{x\ge 0}|u(x,t)| \le &\|g\|_\infty + 2\|\varphi\|_\infty + c_6\sqrt{T}\|V\|_\infty\|g\|_\infty \\ &+ c_7\|V\|_\infty \int_0^t \frac{\sup_{\xi\ge 0}|u(\xi,\tau)|}{\sqrt{t-\tau}} d\tau + c_8\int_0^t \sup_{y\ge 0}|u(y,\tau)| d\tau.\end{aligned} \tag{4.4.14}$$

Using

$$\|g\|_\infty \le 2 \sup_{0\le t\le T}\|\rho(\cdot,t)\|_{L^1} \le 2C_3(A,T) \tag{4.4.15}$$

and we apply both Gronwall's lemma and Lemma 4.4.2 on (4.4.14), we obtain for some $C_4 \in \mathscr{S}$,

$$\|u\|_\infty \le C_4(A,T). \tag{4.4.16}$$

By integration by parts, we have

$$|g'(t)| \le c_9|\rho_x(0,t)| + (c_{10}|V_t| + c_{11})\|\rho(\cdot,t)\|_{L^1}. \tag{4.4.17}$$

Using (4.4.17), (4.4.2) and similar arguments as above, we finally get for some $C_5, C_6 \in \mathscr{S}$,

$$\|u_x\|_\infty \le C_5(A,T), \quad \sup_{0\le t\le T}\|u(\cdot,t)\|_{L^1} \le C_6(A,T). \tag{4.4.18}$$

□

Lemma 4.4.4. *Let $V \in \Sigma(A,T)$ and let $X_t = \int_0^t V_\tau d\tau$. For (X,u) as in Proposition 4.4.1, then there exists $u_{xx} \in C(\overline{D_T} \setminus \mathbf{0})$ such that $\|u_{xx}\|_\infty \le C_7(A,T)$ for some $C_7 \in \mathscr{S}$.*

Proof. By differentiating (4.4.2) with respect to spatial variable, let us define a mapping $\mathscr{F}$: $\mathscr{B}_\eta \to \mathscr{B}_\eta$ where $\mathscr{B}_\eta = \{v(x,t) \in C([0,\infty)\times[0,\eta]\setminus \mathbf{0}) : \|v\|_\infty < \infty\}$ as

- if $x \neq 0$, $t \neq 0$,

$$\begin{aligned}\mathscr{F}v(x,t) :=& -2\int_0^t G_x(x,t;0,\tau)\left[g'(\tau) - V_\tau u_x(0,\tau) - \int_0^\infty dy u(y,\tau)p(y,0)\right]d\tau \\ &+ \int_0^\infty [G_x(x,t;\xi,0) + G_x(x,t;-\xi,0)]\varphi'(\xi)dx \\ &+ \int_0^t\int_0^\infty [G_x(x,t;\xi,\tau) + G_x(x,t;-\xi,\tau)]V_\tau v(\xi,\tau)d\xi d\tau \\ &- \int_0^t\int_0^\infty [G_x(x,t;\xi,\tau) + G_x(x,t;-\xi,\tau)]\int_0^\infty dy u(y,\tau)p_y(y,\xi)d\xi d\tau, \end{aligned} \quad (4.4.19)$$

- if $x = 0$, $t \neq 0$, $\mathscr{F}v(0,t) := 2\left[g'(t) - V_t u_x(0,t) - \int_0^\infty dy u(y,t)p(y,0)\right]$,

- if $x \neq 0$, $t = 0$, $\mathscr{F}v(x,0) := \varphi''(x)$.

Then for $v_1, v_2 \in \mathscr{B}_\eta$, we have the following estimate:

$$\|\mathscr{F}v_1 - \mathscr{F}v_2\|_\infty \le c_1\sqrt{\eta}\|V\|_\infty\|v_1 - v_2\|_\infty.$$

Thus for all sufficiently small $\eta > 0$, $\mathscr{F}$ is a contraction mapping so that there is a unique fixed point v^η. We can extend η to T by a similar way of the proof of Proposition 4.3.1. Let us say a unique $v \in C(\overline{D_T} \setminus \mathbf{0})$ such that for $x \neq 0$, $t \neq 0$,

$$\begin{aligned}v(x,t) =& -2\int_0^t G_x(x,t;0,\tau)\left[g'(\tau) - V_\tau u_x(0,\tau) - \int_0^\infty dy u(y,\tau)p(y,0)\right]d\tau \\ &+ \int_0^\infty [G_x(x,t;\xi,0) + G_x(x,t;-\xi,0)]\varphi'(\xi)d\xi \\ &+ \int_0^t\int_0^\infty [G_x(x,t;\xi,\tau) + G_x(x,t;-\xi,\tau)]V_\tau v(\xi,\tau)d\xi d\tau \\ &- \int_0^t\int_0^\infty [G_x(x,t;\xi,\tau) + G_x(x,t;-\xi,\tau)]\int_0^\infty dy u(y,\tau)p_y(y,\xi)d\xi d\tau. \end{aligned} \quad (4.4.20)$$

By integrating (4.4.20) with respect to spatial variable on both sides and using integration by parts, we obtain for $x \geq 0$, $t > 0$,

$$\begin{aligned}\int_x^\infty v(y,t)dy &= 2\int_0^t G(x,t;0,\tau)\left[g'(\tau) - V_\tau u_x(0,\tau) - \int_0^\infty dy u(y,\tau)p(y,0)\right]d\tau \\ &\quad + \int_0^\infty [-G(x,t;\xi,0) - G(x,t;-\xi,0)]\varphi'(\xi)d\xi \\ &\quad + \int_0^t\int_0^\infty [-G(x,t;\xi,\tau) - G(x,t;-\xi,\tau)]V_\tau v(\xi,\tau)d\xi d\tau \\ &\quad + \int_0^t\int_0^\infty [G(x,t;\xi,\tau) + G(x,t;-\xi,\tau)]\int_0^\infty dy u(y,\tau)p_y(y,\xi)d\xi d\tau \\ &= 2\int_0^t G(x,t;0,\tau)\left[g'(\tau) - V_\tau u_x(0,\tau)\right]d\tau \\ &\quad + \int_0^\infty [-G(x,t;\xi,0) - G(x,t;-\xi,0)]\varphi'(\xi)d\xi \\ &\quad + \int_0^t\int_0^\infty K_x(x,t;\xi,\tau)V_\tau\int_\xi^\infty v(y,\tau)dy d\xi d\tau - 2\int_0^t G(x,t;0,\tau)V_\tau\int_0^\infty v(y,\tau)dy d\tau \\ &\quad - \int_0^t\int_0^\infty K_x(x,t;\xi,\tau)\int_0^\infty dy u(y,\tau)p(y,\xi)d\xi d\tau. \end{aligned} \tag{4.4.21}$$

Adding (4.4.2) to (4.4.21), we get

$$\begin{aligned}u_x(x,t) + \int_x^\infty v(y,t)dy &= 2\int_0^t G(x,t;0,\tau)V_\tau\left\{-u_x(0,\tau) - \int_0^\infty v(y,\tau)dy\right\}d\tau \\ &\quad + \int_0^t\int_0^\infty K_x(x,t;\xi,\tau)V_\tau\left\{u_\xi(\xi,\tau) + \int_\xi^\infty v(y,\tau)dy\right\}d\xi d\tau. \end{aligned} \tag{4.4.22}$$

By letting $f(x,t) := u_x(x,t) + \displaystyle\int_x^\infty v(y,t)dy \in C(\overline{D_T})$, (4.4.22) becomes

$$f(x,t) = -2\int_0^t G(x,t;0,\tau)V_\tau f(0,\tau)d\tau + \int_0^t\int_0^\infty K_x(x,t;\xi,\tau)V_\tau f(\xi,\tau)d\xi d\tau.$$

By the uniqueness of the contraction mapping argument, f should be identically 0. Thus we conclude $v = u_{xx} \in C(\overline{D_T} \setminus \mathbf{0})$ and (4.4.20) becomes

$$\begin{aligned}u_{xx}(x,t) &= -2\int_0^t G_x(x,t;0,\tau)\left[g'(\tau) - V_\tau u_x(0,\tau) - \int_0^\infty dy u(y,\tau)p(y,0)\right]d\tau \\ &\quad + \int_0^\infty K(x,t;\xi,0)\varphi''(\xi)d\xi \\ &\quad + \int_0^t\int_0^\infty [G_x(x,t;\xi,\tau) + G_x(x,t;-\xi,\tau)]V_\tau u_{xx}(\xi,\tau)d\xi d\tau \\ &\quad - \int_0^t\int_0^\infty [G_x(x,t;\xi,\tau) + G_x(x,t;-\xi,\tau)]\int_0^\infty dy u(y,\tau)p_y(y,\xi)d\xi d\tau. \end{aligned}$$

Then we have

$$\sup_{x\geq 0}|u_{xx}(x,t)| \leq 2(\|g'\|_\infty + \|V\|_\infty\|u_x\|_\infty + \|u\|_\infty) + 2\|\varphi''\|_\infty$$
$$+c_1\|V\|_\infty \int_0^t \frac{\sup_{\xi\geq 0}|u_{xx}(\xi,\tau)|}{\sqrt{t-\tau}} d\tau + c_2\sqrt{T}\|u\|_\infty. \quad (4.4.23)$$

By applying Lemma 4.4.2 and Lemma 4.4.3 on (4.4.23), we deduce that for some $C_7 \in \mathscr{S}$,

$$\|u_{xx}\|_\infty \leq C_7(A,T).$$

□

Lemma 4.4.5. *Let $V \in \Sigma(A,T)$ and let $X_t = \displaystyle\int_0^t V_\tau d\tau$. For (X,u) as in Proposition 4.4.1, then $u_x(0,t)$ is Hölder continuous with exponent $\frac{1}{2}$ with $|u_x(0,\cdot)|_{\frac{1}{2}} \leq C_8(A,T)$ for some $C_8 \in \mathscr{S}$.*

Proof.

$$u_x(0,t) = -2\int_0^t \frac{g'(\tau)}{\sqrt{2\pi(t-\tau)}} d\tau + 2\int_0^\infty \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{\xi^2}{2t}\right\} \varphi'(\xi) d\xi$$
$$+\int_0^t \int_0^\infty \frac{1}{\sqrt{2\pi(t-\tau)}} \frac{2\xi}{t-\tau} \exp\left\{-\frac{\xi^2}{2(t-\tau)}\right\} \left[V_\tau u_\xi(\xi,\tau) + \int_0^\infty dy\, u(y,\tau) p(y,\xi)\right] d\xi d\tau$$

By change of variable, $w = \frac{\xi}{\sqrt{t}}$ and $z = \frac{\xi}{\sqrt{t-\tau}}$, we have

$$u_x(0,t) = -2\int_0^t \frac{g'(\tau)}{\sqrt{2\pi(t-\tau)}} d\tau + 2\int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{w^2}{2}\right\} \varphi'(w\sqrt{t}) dw$$
$$+\int_0^t \frac{1}{\sqrt{t-\tau}} \int_0^\infty \frac{2z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} \left[V_\tau u_\xi(z\sqrt{t-\tau},\tau) + \int_0^\infty dy\, u(y,\tau) p(y, z\sqrt{t-\tau})\right] dz d\tau$$
$$= -2\int_0^t \frac{g'(\tau)}{\sqrt{2\pi(t-\tau)}} d\tau + 2\int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{w^2}{2}\right\} \varphi'(w\sqrt{t}) dw + \int_0^t \frac{H(t,\tau)}{\sqrt{t-\tau}} d\tau,$$

where $H(t,\tau) = \displaystyle\int_0^\infty \frac{2z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} \left[V_\tau u_\xi(z\sqrt{t-\tau},\tau) + \int_0^\infty dy\, u(y,\tau) p(y, z\sqrt{t-\tau})\right] dz.$

Moreover, we get for $t_1 < t_2$,

$$\left|\int_0^{t_2} \frac{H(t_2,\tau)}{\sqrt{t_2-\tau}} d\tau - \int_0^{t_1} \frac{H(t_1,\tau)}{\sqrt{t_1-\tau}} d\tau\right|$$
$$= \left|\int_0^{t_1} \frac{H(t_2,\tau) - H(t_1,\tau)}{\sqrt{t_2-\tau}} d\tau + \int_0^{t_1} H(t_1,\tau)\left[\frac{1}{\sqrt{t_2-\tau}} - \frac{1}{\sqrt{t_1-\tau}}\right] d\tau\right|$$
$$\leq c_1(\sqrt{t_2} - \sqrt{t_2-t_1}) \sup_\tau |H(t_2,\tau) - H(t_1,\tau)| + c_2\sqrt{t_2-t_1} \sup_\tau |H(t_1,\tau)|$$

Then by Lemma 4.4.3 and 4.4.4, we obtain

$$|H(t_2,\tau) - H(t_1,\tau)| \leq C(A,T)\sqrt{t_2-t_1}$$

for some $C \in \mathscr{S}$ and there is $\tilde{C} \in \mathscr{S}$ such that

$$\begin{aligned}|u_x(0,t_2)-u_x(0,t_1)| &\le c_3\|g'\|_\infty\sqrt{t_2-t_1}+c_4\sqrt{t_2-t_1}+\tilde{C}(A,T)\sqrt{t_2-t_1}\\ &\le (c_3\|g'\|_\infty+c_4+\tilde{C}(A,T))\sqrt{t_2-t_1}.\end{aligned}$$

Thus we conclude for some $C_8 \in \mathscr{S}$,

$$|u_x(0,\cdot)|_{\frac{1}{2}} \le C_8(A,T).$$

□

Lemma 4.4.6. *There is $A > 0$ such that for all sufficiently small $T > 0$,*

$$Q[V](t) = \frac{-\frac{1}{2}u_x(0,t)+\int_0^\infty\int_0^\infty u(y,t)p(y,x)dydx}{u(0,t)}, \quad 0 \le t \le T,$$

maps $Q : \Sigma(A,T) \longrightarrow \Sigma(A,T)$.

Proof. First of all, let $A > 0$ and $T > 0$ be arbitrary numbers. For $V \in \Sigma(A,T)$, let (X,u) as in Proposition 4.4.1. Since $u(0,t) = g(t)$ and $|f|_{\frac{1}{2}} \le \sqrt{T}\|f'\|_\infty$ for all $f \in C^1([0,T])$, so we have

$$\begin{aligned}&\left|\frac{-\frac{1}{2}u_x(0,t)+\int_0^\infty\int_0^\infty u(y,t)p(y,x)dydx}{u(0,t)}\right|_{\frac{1}{2}}\\ &\quad\le \frac{\|g\|_\infty\left(\frac{1}{2}|u_x(0,\cdot)|_{\frac{1}{2}}+\sqrt{T}\sup_{0\le t\le T}\left|\int_0^\infty\int_0^\infty u_t(y,t)p(y,x)dydx\right|\right)}{(\inf_{[0,T]}|g|)^2}\\ &\qquad+\frac{\sqrt{T}\|g'\|_\infty\left(\frac{1}{2}\|u_x(0,\cdot)\|_\infty+\sup_{0\le t\le T}\|u(\cdot,t)\|_{L^1}\right)}{(\inf_{[0,T]}|g|)^2}\end{aligned}$$

We also have

$$\left|\int_0^\infty\int_0^\infty u_t(y,t)p(y,x)dydx\right| \le c_1|u_x(0,t)|+(c_2|V_t|+c_3)|u(0,t)|+(c_4|V_t|+c_5)\|u(\cdot,t)\|_{L^1}$$

and

$$\inf_{[0,T]}|g| \ge |\varphi(0)|-T\|g'\|_\infty \ge |\varphi(0)|-TC(A,T)$$

for some $C \in \mathscr{S}$ by (4.4.17). Then by previous lemmas, for some $\tilde{C} \in \mathscr{S}$,

$$\left|\frac{-\frac{1}{2}u_x(0,t)+\int_0^\infty\int_0^\infty u(y,t)p(y,x)dydx}{u(0,t)}\right|_{\frac{1}{2}} \le \frac{\tilde{C}(A,T)}{(|\varphi(0)|-TC(A,T))^2}.$$

Let us choose $A > \dfrac{\tilde{C}(\cdot,0)}{|\varphi(0)|^2}$ and then for all sufficiently small $T > 0$, $Q : \Sigma(A,T) \longrightarrow \Sigma(A,T)$ is well-defined. □

Lemma 4.4.7. *The map Q defined in Lemma 4.4.6 is continuous on $\Sigma(A,T)$ with sup norm.*

Proof. Let two pairs (V, X, ρ, u), $(\tilde{V}, \tilde{X}, \tilde{\rho}, \tilde{u})$ as Proposition 4.3.1 and 4.4.1. Using abuse of notation, for each $C_i \in \mathscr{S}$, we will write simply C_i instead of $C_i(A, T)$.
Using (4.3.1) and taking a difference between ρ and $\tilde{\rho}$, we have

$$\sup_{x\geq 0}|\rho(x,t)-\tilde{\rho}(x,t)| \leq c_1\sqrt{T}\|\rho\|_\infty\|V-\tilde{V}\|_\infty + c_2\|\tilde{V}\|_\infty\int_0^t \frac{\sup_{\xi\geq 0}|\rho(\xi,\tau)-\tilde{\rho}(\xi,\tau)|}{\sqrt{t-\tau}}d\tau$$
$$+c_3\int_0^t \sup_{y\geq 0}|\rho(y,\tau)-\tilde{\rho}(y,\tau)|d\tau.$$

Then similarly as before, by applying Gronwall's lemma and Lemma 4.4.2, we obtain

$$\|\rho-\tilde{\rho}\|_\infty \leq C_1\|V-\tilde{V}\|_\infty.$$

In addition,

$$\|\rho(\cdot,t)-\tilde{\rho}(\cdot,t)\|_{L^1} \leq c_4\sqrt{T}\sup_{0\leq t\leq T}\|\rho(\cdot,t)\|_{L^1}\|V-\tilde{V}\|_\infty$$
$$+c_5\|\tilde{V}\|_\infty\int_0^t \frac{\|\rho(\cdot,\tau)-\tilde{\rho}(\cdot,\tau)\|_{L^1}}{\sqrt{t-\tau}}d\tau + c_6\int_0^t \|\rho(\cdot,\tau)-\tilde{\rho}(\cdot,\tau)\|_{L^1}d\tau$$

so that

$$\sup_{0\leq t\leq T}\|\rho(\cdot,t)-\tilde{\rho}(\cdot,t)\|_{L^1} \leq C_2\|V-\tilde{V}\|_\infty.$$

Moreover, we also get

$$\sup_{x\geq 0}|\rho_x(x,t)-\tilde{\rho}_x(x,t)| \leq c_7\sqrt{T}\|\rho_x\|_\infty\|V-\tilde{V}\|_\infty + c_8\|\tilde{V}\|_\infty\int_0^t \frac{\sup_{\xi\geq 0}|\rho_\xi(\xi,\tau)-\tilde{\rho}_\xi(\xi,\tau)|}{\sqrt{t-\tau}}d\tau$$
$$+c_9\sqrt{T}\|\rho-\tilde{\rho}\|_\infty$$

so that

$$\sup_{x\geq 0}|\rho_x(x,t)-\tilde{\rho}_x(x,t)| \leq C_3\|V-\tilde{V}\|_\infty.$$

By taking the difference of u and $\tilde{u}$ written as (4.4.1), we have

$$\sup_{x\geq 0}|u(x,t)-\tilde{u}(x,t)| \leq \|g-\tilde{g}\|_\infty + c_{10}\sqrt{T}\|\tilde{V}\|_\infty\|g-\tilde{g}\|_\infty + c_{11}\sqrt{T}\|g\|_\infty\|V-\tilde{V}\|_\infty$$
$$+c_{12}\sqrt{T}\|u\|_\infty\|V-\tilde{V}\|_\infty + c_{13}\|\tilde{V}\|_\infty\int_0^t \frac{\sup_{\xi\geq 0}|u(\xi,\tau)-\tilde{u}(\xi,\tau)|}{\sqrt{t-\tau}}d\tau$$
$$+c_{14}\int_0^t \sup_{y\geq 0}|u(y,\tau)-\tilde{u}(y,\tau)|d\tau$$

so that

$$\sup_{x\geq 0}|u(x,t)-\tilde{u}(x,t)| \leq C_4\|V-\tilde{V}\|_\infty.$$

Taking the difference between g' and $\tilde{g}'$, we also have

$$|g'(t)-\tilde{g}'(t)| \leq c_{15}|\rho_x(0,t)-\tilde{\rho}_x(0,t)| + c_{16}|V_t|\|\rho(\cdot,t)-\tilde{\rho}(\cdot,t)\|_{L^1} + c_{17}\|\rho(\cdot,t)\|_{L^1}|V_t-V_t|$$
$$+c_{18}\|\rho(\cdot,t)-\tilde{\rho}(\cdot,t)\|_{L^1}.$$

so that

$$\|g' - \tilde{g}'\|_\infty \leq C_5 \|V - \tilde{V}\|_\infty.$$

Taking the difference between u_x and $\tilde{u}_x$ and using previous results, we deduce

$$\sup_{x\geq 0} |u_x(x,t) - \tilde{u}_x(x,t)| \leq c_{19}\sqrt{T}\|g' - \tilde{g}'\|_\infty + c_{20}\sqrt{T}\|u_x\|_\infty \|V - \tilde{V}\|_\infty$$
$$+ c_{21}\|\tilde{V}\|_\infty \int_0^t \frac{\sup_{\xi\geq 0}|u_\xi(\xi,\tau) - \tilde{u}_\xi(\xi,\tau)|}{\sqrt{t-\tau}} d\tau + c_{22}\sqrt{T}\|u - \tilde{u}\|_\infty$$

so that

$$\|u_x - \tilde{u}_x\|_\infty \leq C_6 \|V - \tilde{V}\|_\infty.$$

In a similar way as before and using the estimates above, we obtain

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^1} \leq c_{23}\sqrt{T}\|g - \tilde{g}\|_\infty + c_{24}T\|g\|_\infty \|V - \tilde{V}\|_\infty + c_{25}T\|\tilde{V}\|_\infty \|g - \tilde{g}\|_\infty$$
$$+ c_{26}\sqrt{T}\|u\|_{L^1}\|V - \tilde{V}\|_\infty + c_{27}\|\tilde{V}\|_\infty \int_0^t \frac{\|u(\cdot,\tau) - \tilde{u}(\cdot,\tau)\|_{L^1}}{\sqrt{t-\tau}} d\tau$$
$$+ c_{28}\int_0^t \|u(\cdot,\tau) - \tilde{u}(\cdot,\tau)\|_{L^1} d\tau$$

so that

$$\sup_{0\leq t\leq T} \|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^1} \leq C_7 \|V - \tilde{V}\|_\infty.$$

Finally we conclude that

$$\|Q[V] - Q[\tilde{V}]\|_\infty \leq C_8 \|V - \tilde{V}\|_\infty = C_8 \|V - \tilde{V}\|_\infty.$$

□

Proofof Theorem 4.3.2
Q is continuous, then, since $\Sigma(A,T)$ is convex and compact we can apply the Schauder fixed point theorem to conclude that Q has a fixed point. This completes the proof. □

4.5 Proof of Theorem 4.1.1

Let (X,ρ,u) in Theorem 4.3.2 and let us define $\tilde{\rho}(x,t) := -\int_x^\infty u(y,t)dy$. Since u satisfies

$$\frac{d}{dt}\left(\int_0^\infty u(y,t)dy\right) = -\frac{1}{2}u_x(0,t) - V_t u(0,t) + \int_0^\infty \int_0^\infty dz u(z,t) p(z,y) dy = 0,$$

thus we have $\tilde{\rho}(0,t) = -\int_0^\infty u(y,t)dy = -\int_0^\infty \rho_0'(y)dy = 0$.
Then $\tilde{\rho}$ satisfies

$$\begin{aligned}
\tilde{\rho}_t(x,t) = -\int_x^\infty u_t(y,t)dy &= \frac{1}{2}u_x(x,t) + V_t u(x,t) - \int_x^\infty \int_0^\infty dz\tilde{\rho}_z(z,t)p(z,y)dy \\
&= \frac{1}{2}\tilde{\rho}_{xx}(x,t) + V_t\tilde{\rho}_x(x,t) + \int_x^\infty \int_0^\infty dz\tilde{\rho}(z,t)p_z(z,y)dy \\
&= \frac{1}{2}\tilde{\rho}_{xx}(x,t) + V_t\tilde{\rho}_x(x,t) - \int_0^\infty dz\tilde{\rho}(z,t)\int_x^\infty p_y(z,y)dy \\
&= \frac{1}{2}\tilde{\rho}_{xx}(x,t) + V_t\tilde{\rho}_x(x,t) + \int_0^\infty dz\tilde{\rho}(z,t)p(z,x).
\end{aligned}$$

Since $\tilde{\rho}(x,0) = \rho_0(x)$, by the uniqueness of Proposition 4.3.1, we also get $\rho = \tilde{\rho}$ such that

$$\tilde{\rho}_x(0,t) = u(0,t) = 2\int_0^\infty \int_0^\infty dy\rho(y,t)p(y,x)dx = 2\int_0^\infty \int_0^\infty dy\tilde{\rho}(y,t)p(y,x)dx. \tag{4.5.1}$$

By (4.5.1) and $\tilde{\rho}(0,t) = 0$, we have $\frac{d}{dt}\left(\int_0^\infty \tilde{\rho}(y,t)dy\right) = 0$ and finally deduce that

$$\int_0^\infty \tilde{\rho}(x,t)dx = \int_0^\infty \rho_0(x)dx = 1. \tag{4.5.2}$$

This completes the proof.

4.6 Further results

Let us try to apply our C^1-argument to the FBP of [3] as follows:

$$(\star)\begin{cases} \rho_t(x,t) = \frac{1}{2}\rho_{xx}(x,t) + \rho, & \text{if } X_t < x, \quad t > 0, \\ \rho(X_t,t) = \alpha, & \text{if } t \geq 0, \\ \rho_x(X_t,t) = \beta, & \text{if } t > 0. \\ \rho(x,0) = \rho_0(x), & \text{if } 0 \leq x, \end{cases}$$

where ρ_0 is specified later.
If $\alpha \neq 0$ and $\beta = 0$, then $\frac{d}{dt}\rho(X_t,t) = V_t\rho_x(X_t,t) + \rho_t(X_t,t) = \frac{1}{2}\rho_{xx}(X_t,t) + \alpha = 0$ so that, by change of variable $u := \rho_x$, $(\star)$ becomes

$$(\star\star)\begin{cases} u_t(x,t) = \frac{1}{2}u_{xx}(x,t) + u, & \text{if } X_t < x, \quad t > 0, \\ u(X_t,t) = 0, & \text{if } t \geq 0, \\ u_x(X_t,t) = -2\alpha, & \text{if } t > 0. \\ u(x,0) = \rho_0'(x), & \text{if } 0 \leq x, \end{cases}$$

Again by change of variable $v := -\frac{1}{2\alpha}u_x$, $(\star\star)$ becomes $(\star\star\star)$

$$(\star\star\star)\begin{cases} v_t(x,t) = \frac{1}{2}v_{xx}(x,t) + v, & \text{if } X_t < x, \quad t > 0, \\ v(X_t,t) = 1, & \text{if } t \geq 0, \\ v(x,0) = -\frac{1}{2\alpha}\rho_0''(x), & \text{if } 0 \leq x, \\ V_t = -\frac{1}{2}v_x(X_t,t), & \text{if } t > 0. \end{cases}$$

To make each step valid, it needs that the value of the initial datum at 0 is same as the boundary value which is $\rho_0(0) = \alpha$, $\rho_0'(0) = 0$, $\rho_0''(0) = -2\alpha$ and ρ_0 should be $C_c^4([0,\infty))$ to have v_{xx} of $(\star\star\star)$ as in Lemma 4.4.4.
Similarly as before, we can shift the boundary X to 0 so that we have the following equivalent FBP:

$$(\star\star\star')\begin{cases} v_t(x,t) = \frac{1}{2}v_{xx}(x,t) + V_t v_x(x,t) + v(x,t) & \text{if } 0 < x, \quad t > 0, \\ v(0,t) = 1, & \text{if } t \geq 0, \\ v(x,0) = -\frac{1}{2\alpha}\rho_0''(x), & \text{if } 0 \leq x, \\ V_t = -\frac{1}{2}v_x(0,t), & \text{if } t > 0. \end{cases}$$

By writing v as

$$\begin{aligned} v(x,t) = -\int_0^t K_x(x,t;0,\tau)d\tau + \int_0^\infty G(x,t;\xi,0)\psi(\xi)d\xi \\ + \int_0^t\int_0^\infty G(x,t;\xi,\tau)\big[V_\tau v_\xi(\xi,\tau) + v(\xi,\tau)\big]d\xi d\tau, \end{aligned} \tag{4.6.1}$$

where $\psi(\xi) = -\frac{1}{2\alpha}\rho_0''(\xi)$, we can repeat the same argument as previous sections such that there is a pair (X,v) which satisfies $(\star\star\star')$.
If $\alpha = 0$, $\beta \neq 0$, it can be done similarly as ($\alpha \neq 0$, $\beta = 0$)-case with the initial condition $\rho_0 \in C_c^3([0,\infty))$ such that $\rho_0(0) = 0$, $\rho_0'(0) = \beta$.

Bibliography

[1] L. Alili, A. E. Kyprianou: *Some remarks on first passage of Lévy processes, the American put and pasting principles*, Ann. Appl. Probab. Volume 15, Number 3 (2005), 2062-2080.

[2] L. Alili, P. Patie: *Boundary crossing identities for Brownian motion and some nonlinear ode's*, Proc. Amer. Math. Soc. 142 , 3811–3824 (2014).

[3] J. Berestycki, E. Brunet, and B. Derrida, *Exact solution and precise asymptotics of a Fisher-KPP type front*, arXiv:1705.08416v1, May 2017.

[4] Krzysyztof Burdzy, Zhen-Qing Chen, John Sylvester *The heat equation and reflected brownian motion in time-dependent domains*, The Annals of Probability 2004, Vol. 32, No. 1B, 775?804.

[5] E. Brunet, B. Derrida. (1997), *Shift in the velocity of a front due to a cutoff*, Phys. Rev. E 56:2597D2604.

[6] E. Brunet, B. Derrida, B. Mueller, S. Munier. (2007). *Effect of selection on ancestry: An exactly soluble case and its phenomenological generalization*, Phys. Rev. E (3) 76 041104, 20. MR2365627.

[7] C.M.Brauner, J.Hulshof, *A General Approach to Stability in Free Boundary Problems*, Journal of Differential Equations 164, 1648 (2000).

[8] Maury D.Bramson, *Maximal displacement of branching Brownian motion*, Comm. Pure Appl. Math. 31(1978), no. 5, 531-581. MR 0494541.

[9] J.R. Cannon: *The One-Dimensional Heat Equation*, Addison-Wesley Publishing Company (1984).

[10] G. Carinci, A. De Masi, C. Giardinà, E. Presutti, *Hydrodinamic limit in a particle system with topological interactions*, Arabian Journal of Mathematics, vol. 3, 381-417 (2014).

[11] G. Carinci, A. De Masi , C. Giardina, E. Presutti, *Free boundary problems in PDE's and particle systems*, Springer briefs in Mathematical Physics, 2016 ISBN: 978-3-319-33369-4, ISSN: 2197-1757, doi: 10.1007/978-3-319-33370-0

[12] T. Chou and M. R. D'Orsogna: *First Passage Problems in Biology. First-Passage Phenomena and Their Applications*, pp. 306-345 (2014).

[13] R.M/Capocelli, L.M.Ricciardi, *Diffusion approximation and first passage time problem for a model neuron*, Kybernetik: (1971) 8: 214. https://doi.org/10.1007/BF00288750.

[14] L.A. Caffarelli, J.L.Vazquez, *A free-boundary problem for the heat equation arising in flame propagation*, Transactions of the American Mathematical Society, Volume 347, Number 2, February (1995).

[15] A. De Masi , P.A. Ferrari (2015) *Separation versus diffusion in a two species system Brazilian Journal of Probability and Statistics*, Vol.29, no.2, 387-412 ISSN: 0103-0752, doi: 10.1214/14-BJPS276.

[16] A. De Masi, P.A. Ferrari, E. Presutti, *Symmetric simple exclusion process with free boundaries*, Probability Theory and Related fields 161, p. 155-193 (2015).

[17] A. De Masi, P.A. Ferrari, E. Presutti, N.Soprano-Loto, *Hydrodynamics of the N-BBM process*, arXiv:1707.00799, July 2017.

[18] A. De Masi, P.A. Ferrari, E. Presutti, N.Soprano-Loto, *Non local branching Brownians with annihilation and free boundary problems*, arXiv:1711.06390, Nov 2017.

[19] A. De Masi, E. Presutti, D. Tsagkarogiannis *Fourier law, phase transitions and the stationary Stefan problem*, Archive for Rational Mechanics and Analysis, Volume 201 No. 2, 681-725 ISSN: 0003-9527, doi: 10.1007/s00205-011-0423-1 (2011).

[20] R. Durrett, D. Remenik, *Brunet-Derrida particle systems, free boundary problems and Wiener-Hopf equations*, Annals of Probability 39, 2043–2078 (2011).

[21] A.Einstein, *Investigations on the Theory of the Brownian Movement (PDF). Dover Publications*, (1956) [1926] Retrieved 2013-12-25.

[22] A. Fasano, *Mathematical models of some diffusive processes with free boundaries*, SIMAI e-Lecture Notes (2008).

[23] A. Fasano, M.Primicerio, *General free-boundary problems for the heat equation*, I, J.MATH. ANAL. APPL. 57 , 694-723 (1977).

[24] A.Fasano, M.Primicerio, *Free Boundary Problems for Nonlinear Parabolic Equations with Nonlinear Free Boundary Conditions*, Journal of Mathematical Analysis and Applications 72, 247-273 (1979).

[25] Per Fauchald, Torkild Tveraa, *Using first-passage time in the analysis of area-restricted search and habitat selection*, Ecology, 84: 282?288. doi:10.1890/0012-(2003).

[26] P. Groisman and M. Jonckheere, *Front propagation and quasi-stationary distributions: the same selection principle?*, arXiv:1304.4847, April 2013.

[27] J.Lee, *First Passage Time Densities through Hölder curves*, arXiv:1607.04859, July 2016.

[28] J.Lee, *A free boundary problem in biological selection models*, arXiv:1707.01232, July 2017.

[29] J.Lee, *A free boundary problem with non local interaction*, arXiv:1801.09410, Jan. 2018.

[30] Ioannis Karatzas, Steve E.Shreve: *Brownian Motion and Stochastic Calculus*, Springer (1991).

[31] O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'ceva, *Linear and quasilinear Equations of Parabolic Type*, Amer. Math. Sot. Transl. 23 (1968).

[32] C.Landim,G.Valle, *A microscopic model for Stefan's melting and freezing problem, The Annals of Probability 2006, Vol. 34, No. 2, 779?803 DOI: 10.1214/009117905000000701*

[33] P. Maillard, Speed and fluctuations of N-particle branching Brownian motion with spatial selection, Probab. Theory Related Fields, 166(3-4):1061-1173.

[34] Bastien Mallein, Maximal displacement of d-dimensional branching Brownian motion, Electron. Commun. Probab. 20 (2015), no. 76, 1?12. DOI: 10.1214/ECP.v20-4216.

[35] G. Peskir, A. Shiryaev: Optimal Stopping and Free-Boundary Problems, Birkhäuser (2006).

[36] L. M. Ricciardi; L. Sacerdote; S. Sato: On an Integral Equation for First-Passage-Time Probability Densities, Journal of Applied Probability, Vol. 21, No. 2. 302–314 (1984).

[37] T. Taillefumier, M. Magnasco: A Transition to Sharp Timing in Stochastic Leaky Integrate-and-Fire Neurons Driven by Frozen Noisy Input, Neural Computation Vol. 26, No. 5. 819–859 (2014).

[38] T. Taillefumier, M. Magnasco: A phase transition in the first passage of a Brownian process through a fluctuating boundary:implications for neural coding, PNAS 2013 April, 110 (16) E1438-E1443. https://doi.org/10.1073/pnas.1212479110

[39] L.Sacerdote, M.Tamborrino, C.Zucca: First passage times of two-dimensional correlated processes: analytical results for the Wiener process and a numerical method for diffusion processes, Journal of Computational and Applied Mathematics, Volume 296, April 2016, Pages 275-292.

[40] Di Zhang, Roderick V. N. Melnik: First passage time for multivariate jump-diffusion processes in finance and other areas of applications, Journal Applied Stochastic Models in Business and Industry, Volume 25 Issue 5, September 2009 Pages 565-582.