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Stability and optimal control of polytopic time-inhomogeneous Markov jump linear systems

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Abstract

Linear systems subject to abrupt parameter changes due, for instance, to environmental disturbances, component failures, changes in subsystems interconnections, etc., can be modeled as a set of discrete-time linear systems with modal transition given by a discrete-time finite-state Markov chain. This family of systems is known as discrete-time Markov(ian) jump linear systems (from now on MJLSs). MJLSs represent a promising mathematical model of cyber-physical systems, the applications of which arguably have the potential to dwarf the fourth industrial revolution.

The bulk of the existing research on MJLSs is based on the fundamental assumption that parameters of the Markov chain are known and static. However, in several cyber-physical systems' applications the MJLSs' model is affected by abrupt and unpredictable perturbations on the underlying Markov chain. For instance, Markov chain models of slow fading channels are derived via measurements on real channels or via numerical reasoning, which always introduces errors. Furthermore, fading channels can partially be compensated for by adjusting the transmission power levels, with higher transmission power giving less packet errors, but increasing the energy consumption and interference with other systems. Another example can be found in the vertical take-off landing helicopter systems, where the airspeed variation is ideally modeled as homogeneous Markov process, but because of the external environment (like wind) the transition probabilities of the jumps are time-varying.

We take into account the intrinsic to the real world systems uncertainty and time-variance of the jump parameters by considering MJLSs where the underlying Markov chain is polytopic and time-inhomogeneous, i.e., its transition probability matrix is varying over time with variations that are arbitrary within a polytopic set of stochastic matrices. We show that the conditions used for time-homogeneous MJLSs are not enough to ensure the stability of the time-inhomogeneous system, and that perturbations on values of the transition probability matrix can make a stable system unstable. We present necessary and sufficient conditions for mean square stability (hereupon, MSS) of polytopic time-inhomogeneous MJLSs. We prove that deciding MSS on such systems is NP-hard and that MSS is equivalent to exponential mean square stability and to stochastic stability. We also derive necessary and sufficient conditions for robust MSS of MJLSs affected by polytopic uncertainties on transition probabilities and bounded disturbances. Then, we address and solve for this class of systems the finite horizon optimal control and filtering problems. In particular, we show that the optimal controller having only partial information on the continuous state can be obtained from two types of coupled Riccati difference equations, one associated to the control problem, and the other one associated to the filtering problem. Finally, we present and solve the finite horizon optimal control problem also for switched linear systems, where a switching signal is governed by a Markov decision process, in polytopic time-inhomogeneous setting. We call this type of systems Markov jump switched linear system. It generalizes the MJLSs' framework to optimal decision problems, with applications for example to optimal power management of wireless networked control systems.

These results construct a solid basis for the future development of novel (correct-by-design) fault and intrusion detection, isolation and reconfiguration techniques for cyber-physical systems modeled by discrete-time polytopic time-inhomogeneous Markov jump (switched) linear systems.

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Yuriy

Declaration

This thesis has been composed by myself and the presented work is my own under the guidance of my supervisors Maria Domenica Di Benedetto and Alessandro D'Innocenzo. Moreover, Chapter 3 is based on [1] and on [2], Chapters 4 and 5 are re-elaboration of [3], co-authored with Alessandro Abate, while Chapter 6 is essentially [4]. Besides, Chapter 1 uses a result from [5], which was co-authored with Ivano Malavolta, who was my tutor at Gran Sasso Science Institute. All the aforementioned scientific papers were co-authored with my supervisors. The author order indicates the relative contribution to the development of the theory as well as the paper writing for most papers.

Contents

Chapter 1

Introduction

The topic of this thesis lies within the domain of hybrid systems, i.e., dynamical systems that exhibit both continuous and discrete dynamic behavior, which are traditionally used to model several cyber-physical systems [6].

As a matter of fact, we get exposed to the notion of Markov jump linear systems (i.e., hybrid systems with stochastic switching) while conducting a systematic survey of the state of the art of security for cyber-physical systems from an automatic control perspective [5]. In the aforementioned survey, we have applied a well-established methodology from the medical and software engineering research communities, called a systematic mapping [7, 8]; we analyzed 118 studies, rigorously selected from more than a thousand potentially relevant works, and have found that, surprisingly, very few papers consider communication aspects or imperfections, and attempt to provide non-trivial mathematical models of communication protocols.

Motivated by this result, we had a deeper look into communication characteristics of cyber-physical systems [9], and how they are modeled in wireless networked control systems. We realized that discrete-time time-homogeneous Markov jump linear systems (MJLSs) are common models for networked control systems accounting for correlated (bursty) packet losses (see for instance [10, 11]).

We also noticed that, although seemingly almost all the issues on discrete-time MJLSs have been tackled [12], the obtained results are actually based on the ideal assumption that all the elements in the transition probabilities matrix are certain, completely known, and time-invariant. However, this is not true in practice: it is in general difficult to have an accurate estimation for (some of) the transition probabilities, and there are several scenarios where the transition probabilities are clearly time-varying [13].

Thus, it is more realistic to consider discrete-time Markov jump linear systems with uncertain time-varying transition probabilities. One of such models is the object of this thesis.

In the next sections of the introductory chapter we first present the general domain of our work, i.e., cyber-physical systems, hybrid systems and networked control systems; later on we focus our attention on discrete-time Markov jump linear system models. Specifically, at the outset we describe practical limitations of the customary for the most MJLSs assumption that underlying parameters of Markov chains are static and can be computed exactly. Then, we present the state of the art of discrete-time uncertain Markov jump linear systems in general, and polytopic time-inhomogeneous MJLSs in particular. Lastly, we state our contribution explicitly and outline the technical part of the thesis.

1.1 Cyber-physical systems

Cyber-physical systems (also known by their acronym, CPSs) are integrations of computation, networking, and physical processes [14, 15]. The key characteristic of cyber-physical systems is their seamless integration of both hardware and software resources for computational, communication and control purposes, all of them co-designed with the physical engineered components [16].

Cyber-physical systems are the enabling component in the fourth industrial revolution [17, 18], and their applications arguably have the potential to dwarf the 20th century information technology (IT) revolution [14].

Among the many applications of cyber-physical systems one can find high confidence medical devices and systems, assisted living, traffic control and safety, advanced automotive systems, process control, energy conservation, environmental control, avionics, instrumentation, critical infrastructure control (electric power, water resources, and communications systems for example), distributed robotics (telepresence, telemedicine), defense, manufacturing, smart structures, etc.

The dynamics of cyber-physical systems are complex, involving the stochastic nature of communication systems, discrete dynamics of computing systems, and continuous dynamics of control systems. In fact, relevant for cyber-physical systems research domains include networked control, hybrid systems, real-time computing, real-time networking, wireless sensor networks, security, and model-driven development [9]. In particular, networked control systems and hybrid systems constitute some of the theoretical foundations for design and analysis of the dynamical behavior of cyber-physical systems.

1.2 Wireless networked control systems

Wireless control networks, also known as wireless *networked control systems* (henceforward, NCSs) are distributed control systems where the communication between sensors, actuators, and computational units is supported by a wireless communication network. Wireless control networks have a wide spectrum of applications, ranging from smart grids to remote surgery, passing through industrial automation, environment monitoring, intelligent transportation, and unmanned aerial vehicles, to name few.

The use of wireless control networks in industrial automation results in flexible architectures and generally reduces installation, debugging, diagnostic and maintenance costs with respect to wired networks (see e.g. [19, 20] and references therein). However modeling, analysis and design of (wireless) networked control systems are challenging open research problems since they require to take into account the joint dynamics of physical systems, communication protocols and network infrastructures.

Recently, a huge effort has been made in scientific research on NCSs, see e.g. [10, 21–26] and references therein for a general overview. It has been widely recognized that, when dealing with networked control, it is important to take a system's perspective and develop control algorithms that can deal with communication imperfections and constraints, that can be roughly categorized in five types [27], i.e.,

(i) variable sampling / transmission intervals;

(ii) variable communication delays;

(iii) packet dropouts caused by the unreliability of the network;

(iv) communication constraints caused by the sharing of the network by multiple nodes and the fact that only one node is allowed to transmit its packet per transmission;

(v) quantization errors in the signals transmitted over the network due to the finite word length of the packets.

In particular, the *packet dropouts* have been modeled in the wireless networked control system literature either as stochastic or deterministic phenomena [22]. The proposed deterministic models specify packet losses in terms of time averages or in terms of worst case bounds on the number of consecutive dropouts (see e.g. [24]). For what concerns stochastic models, a vast amount of research assumes memoryless packet drops, so that dropouts are realizations of a Bernoulli process [10, 23, 25]. Other works consider more general correlated (bursty) packet losses and use a transition probability matrix (TPM) of a finite-state (time-homogeneous) Markov chain (see e.g. the finite-state Markov modeling of Rayleigh, Rician and Nakagami fading channels in [28] and references therein) to describe the stochastic process that rules packet dropouts (see [10, 11]). In these works networked control systems with missing packets are modeled as time-homogeneous Markov jump linear systems, which are an important family of stochastic hybrid systems.

1.3 Hybrid systems

Hybrid systems, or systems that involve the interaction of discrete and continuous dynamics, have been used as rigorous mathematical models for several important real-world technological applications, which have inspired a great deal of research from both control theory and theoretical computer science (see e.g. [29–36], and references therein for a general overview).

As described by Lygeros, Tomlin and Sastry [29], whereas researchers from computer science community have adopted “discrete state space” point of view, by approaching mainly the problems of composition, abstraction and formal verification, i.e., proving that the hybrid system satisfies certain specifications, researchers in the areas of dynamical systems and control have approached hybrid systems from a *“continuous state space and continuous/discrete time”* point of view, by extending the standard modeling and simulation techniques to capture the interaction between the continuous and discrete dynamics, and by developing new analysis and controller design techniques.

While the results presented in this thesis rely mostly on control-theoretic approaches, they introduce hybrid models, that simultaneously account for both deterministic and uncertain stochastic discrete dynamics, thus allowing to advance the topic of formally verified robust fault detection, isolation and reconfiguration in cyber-physical systems, as outlined in Chapter 7.

Typically, the discrete dynamics of hybrid systems are modeled either within a deterministic or a stochastic framework.

Among stochastic hybrid systems, a widely investigated class is given by *Markov jump linear systems* (from here on, also referred to as MJLSs), which are the first mathematical model on which we build our contribution, as detailed in Chapter 2.

MJLSs are described by a set of linear systems with commutations, or jumps, generated by a finite-state Markov chain. Due to the probabilistic description of commutations, Markov jump linear systems are well suited to model unexpected events, uncontrolled configuration changes, random faults, and other kinds of system changes ascribable to exogenous uncontrollable events, i.e., changes induced by external causes (which can be referred to as “environment”, or “nature”).

When instead the switching mechanism can be governed by a supervisor (also called a decision maker, or discrete controller), the deterministic models are more adequate. For instance, configuration changes may be decided by the supervisor in order to improve, or optimize, some performance index (see e.g., [37] as a general reference).

In practice, however, the stochastic and deterministic views of hybrid systems are not mutually exclusive.

Dual switching systems [35, 38, 39] are characterized by the simultaneous presence of a deterministic switching mechanism and a second stochastic switching signal giving rise to jumps. When the stochastic switching is governed by a Markov chain, these systems are also referred to as *switching Markov jump linear systems*. For a real world example, Bolzern, Colaneri and De Nicolao [39] consider a wind turbine connected to an energy storage device. The transition between the operating modes of the turbine (standby, power-optimization, power-limitation) governed by a deterministic switching signal whose schedule is decided by the discrete controller, while the transitions between the modes of the storage device (charging, discharging, disconnected) depend on causes exogenous to the wind generation system and are better described by a stochastic model, e.g., a Markov chain. Notice that the stochastic switching here is independent from the deterministic switching mechanism, and the transition probabilities are assumed to be time-independent and exactly known.

There are also application scenarios where the transition probabilities between the operational modes of the system are influenced by the actions of a supervisor. For instance, the wireless communication channels used to convey information between sensors, actuators, and computational units of a networked control system are frequently subject to time-varying fading and interference, where the effects of fading can partially be compensated for by adjusting the transmission power levels (see e.g., [40]), such that a higher transmission power gives less packet errors, but increases the energy consumption and interference with other systems. As discussed in Section 1.2, for any given transition power level, the stochastic process that rules packet dropouts can be modeled by the transition probability matrix of a Markov chain. Thus, it comes naturally to use a *Markov decision process* (MDP) framework to solve the optimal power management problem, minimizing for instance the transmission power and continuous control cost.

The problem of optimal power management in networked control systems has been already studied in [41], where a restricted information structure on plant inputs and transmit powers was imposed, allowing them to be designed separately; it was shown that the optimal deterministic switching policy follows a Markov decision process minimizing transmit power at the sensor and state estimation error at the controller.

Considering a Markov decision process instead of a Markov chain in a Markov jump linear system brings to light a new type of system, that we call *Markov jump switched linear system* (hereupon, MJSLS), which provides a mathematical framework for studying optimal power management in wireless networked control systems without any restrictions on continuous plant control inputs and discrete switching policies regulating the transmission power values.

Compared to switching Markov jump linear systems considered in [35, 38, 39], the MJSLS model is based on the Markov decision process framework, where the transition probabilities between operational modes of the system depend on the actions of a discrete controller, and for each discrete action there is an associated cost. Furthermore, as described in Chapter 2, our MJSLS model accounts for time-varying uncertainties in transition probabilities, where the variations are arbitrary within a polytopic set of stochastic matrices.

A possible alternative to our formulation is to consider so-called controlled Markov set-chains (see [42] and reference therein), which are anyhow related to polytopic time-inhomogeneous point of view, as can be found in [43].

1.4 Limitations of the stationary exact model

To date, quite a few fundamental control issues, such as stability and stabilization, estimation and filtering, fault detection and diagnosis, have been addressed in the literature on discrete-time Markov jump linear systems, see [12, 13] as textbooks with important results and detailed examination of the general state of the art. However, as a crucial factor governing the behaviors of Markov jump linear systems, the transition probabilities are generally considered to be time-invariant, certain, and often completely known in the majority of studies.

Still, in most real cases the transition probability matrix cannot be computed exactly and is time-varying.

For example, the Markov chain models of slow fading channels [28] are derived via *measurements on real channels or via numerical reasoning*, which always introduces errors. Indeed, it is recognized that a fundamental issue in the design of finite-state Markov channel models is how accurate and reliable the resulting system performance measures are [28].

Actually, this argument applies to any kind of practical scenario involving measurements and/or numerical analysis.

Furthermore, due to the influence of various environmental factors, the *abrupt and unpredictable time-varying perturbations of transition probabilities* are also common in real applications. For instance, in [44] it is pointed out that in the vertical take-off landing (often referred to as VTOL) helicopter system the airspeed variation are ideally modeled as homogeneous Markov process, but because of the external environment (like the wind) the transition probabilities of the jumps are time-varying; in [45] the example of failures and repairs of subsystems is considered, where the transition probabilities deeply depend on system age and working time.

We take into account these three aspects (namely, global uncertainty due to random and systematic errors of measurement and numerical computation procedures, incomplete knowledge of some transition probabilities when adequate samples of the transitions are costly or time-consuming to obtain, and time-variance due to environmental factors) simultaneously by considering polytopic time-inhomogeneous Markov jump linear systems, as described in Chapter 2.

Our choice is motivated by the fact that uncertainty and time-variance are intrinsic to the real world systems, and all measurement and numerical analysis procedures give us confidence levels (determined by accuracy and precision of the measuring instrument and/or numerical algorithm), which bound the possible values each transition probability can assume.

1.5 Literature review of the state-of-the-art

Before presenting the literature review of the research on discrete-time MJLSs with polytopic time-inhomogeneous transition probabilities, we should mention that other works typically account for either incomplete knowledge of transition probability matrix, or time-variance, as summarized in the following Table 1.1.

Specifically, the incomplete knowledge of time-invariant transition probabilities can be represented as norm-bounded [46] or polytopic uncertainties [48–52] (where the precise values are not obtained and only the bounds of transition probabilities

Topic	Time-homogeneous transition probability matrix (is)		
	norm bounded	**partially unknown**	**polytopic**
Stability	[46]	[13, 47]	[48]
Control		[13]	[48–51]
Estimation		[13]	[52]

Table 1.1: Works on MJLSs with incomplete knowledge on time-invariant TPMs

are available), or as partially unknown transition probabilities [13, 47] (where not all values are available).

The uncertain transitions were introduced first for continuous-time Markov jump linear systems in [53], where the transition rates were described by a fixed polytope and a mode-dependent state-feedback controller was designed in the sense of the mean square stability [13].

We observe that in our work we do not discuss the continuous-time MJLSs, given a large number of technical nuances which pose a great deal of difficulties to treat them satisfactorily in parallel with the discrete-time case [12], and also because the discrete-time framework is more suitable to address networked control applications on digital communication networks [39]. Anyway, the interested reader may refer to [54], and also [13], for an introductory treatment of the topic.

For what concerns *norm-bounded uncertainties*, the sufficient conditions for the admissible element-wise probabilities variations were derived in [46] using stochastic Lyapunov function approach and Kronecker product transformation techniques, so that the time-homogeneous Markov jump linear systems, in either continuous-time or discrete-time frameworks, remain stable.

There exists a considerable number of works on discrete-time MJLSs with *polytopic* but *time-invariant uncertainties* in the transition probability matrix. In particular, we report here some notable works. The quadratic optimal mode-dependent control problems with constraints on the state and control variables were considered in [50]. The convex programming approach was used in [49] to address the state-feedback $\mathcal{H}_2$ control problem (see [12, pp. 82–83] for the introduction to the topic), with or without assuming the Markov (operational) mode availability. A sufficient condition for robust stability was proposed in [48], where also state-feedback controller design problem based on linear matrix inequalities (LMIs, see e.g. [55] for a general discussion) was stated, for both the mode-dependent or mode-independent cases. The LMIs-based methods to determine $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm bounded filters were presented in [52], in which different assumptions on Markov mode availability to the filter and on system parameter uncertainties were taken into account. Finally, the $\mathcal{H}_\infty$ output feedback control problem (see [12, pp. 143–145] for additional details on the subject) under the cluster availability of the operational mode was addressed in [51], where a sufficient LMI condition that guarantees the $\mathcal{H}_\infty$ norm of the closed-loop system is below a prescribed level was also provided.

Topic	Time-inhomogeneous transition probability matrix		
	non deterministic	non deterministic / stochastic	stochastic
	polytopic	**piecewise homogeneous**	**semi-Markov**
Stability	[62, 63]	[64]	[13]
Control	[65–72]	[13, 73]	
Estimation	[45, 74, 75]	[13, 76]	
Fault detection	[44]		
Model reduction		[13]	

Table 1.2: Works on MJLSs with incomplete knowledge on time-varying TPMs

For Markov jump linear systems with stationary (also referred to as time-invariant) transition probabilities, it is possible to consider that some elements of the corresponding transition probability matrix are not available. The so-called *partially unknown model* of transition probabilities was first proposed in [47]. Several works had followed, addressing stability analysis and stabilization problems, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance analysis, control and filtering problems, fault detection and model reduction of the underlying systems, as described in detail in [13].

Nevertheless, the partially unknown model does not account for global uncertainty and time-variance. Furthermore, it is proven in [52] that the polytopic uncertainty model is more general than the partly known element one, in the sense that the latter is a particular case of the former. Moreover, the uncertainty modeled with a fixed convex polytope may also represent an uncertainty domain more precisely than the norm-bounded uncertainty and, consequently, causes no conservatism for a particular structure [56].

An alternative approach in dealing with incomplete knowledge on transition probabilities is to characterize the relative *likelihood* for uncertain *time-varying transition probabilities* to occur at a given constant. In this case, the *truncated Gaussian probability density function* can be used to quantize the uncertain information of transition probabilities, as shown in [57–60], where the problems of finite-time stabilization, and $\mathcal{H}_\infty$ control and filtering were addressed.

The uncertainties in time-inhomogeneous characteristics of transition probabilities, in general, can be determined by either non-deterministic or stochastic variations, as summarized in the following Table 1.2. The polytopic time-inhomogeneous Markov jump (switched) linear systems studied in this thesis, and MJLSs governed by piecewise homogeneous Markov chains subject to an arbitrary high-level switching signal (e.g., the signal with average dwell time approaching zero [61, Remark 2]) account for the first type of variations, while semi-Markov jump linear systems [13, Part III] and piecewise homogeneous Markov jump linear systems with transition probabilities themselves governed by a higher-level Markov chain [13, Part II] provide a rationale for stochastic variations.

Discrete-time *piecewise homogeneous* Markov jump linear systems were first introduced in [76] to account for both arbitrary variation and stochastic variation of transition probability matrices. The distinct characteristic of these systems is that their transition probabilities are varying but invariant within a time interval. Specifically, the non-deterministic variations are governed by arbitrary switchings and slow switching signals where the number of switches in a finite interval is bounded and the average time between two consecutive switchings of transition probability matrices is larger or equal to a constant [13]. In case of stochastic switching, the variations are subject to higher-level transition probability matrices.

To date, several useful results on $\mathcal{H}_\infty$ control, state estimation, and model reduction for MJLSs with piecewise homogeneous transition probabilities have been reported [13, 73, 76], where bounded $\mathcal{H}_\infty$ performance criteria (i.e., the so-called *bounded real lemmas*, often indicated by acronym BRL) were provided in therms of linear matrix inequalities.

Other interesting results closely related to piecewise homogeneous Markov jump linear system model are the following.

For MJLSs with time-varying transition probability matrix, which takes values in a finite set randomly at each time step, Zhao and Liu [77] have studied transition probability matrix estimation problem within the Bayesian framework, finding a recursive updating formula.

For Markov jump linear systems with transition probability matrices taking values in the finite set and switching governed by possibly a priori unknown sequence, Lutz and Stilwell [64] have presented necessary and sufficient conditions for uniform exponentially mean square stability (UEMSS) and uniform stochastic disturbance attenuation, expressed as a set of finite-dimensional LMIs that can be solved efficiently. This work used time-varying quadratic Lyapunov function arguments.

When the sojourn time in discrete-time MJLSs' operational modes does not follow geometric distribution, the transition probabilities are time-varying and have a "memory" property, resulting in so-called *semi-Markov jump linear systems*. See [13] for a formal introduction of such systems, and detailed treatment of stability and stabilization via semi-Markov kernel (where the probability density function of sojourn-time is dependent on both current and next system mode) and time-varying Lyapunov function approach.

Noticeably, both semi-Markov jump linear systems and MJLSs with piecewise-constant transition probabilities subject to a higher-level transition probability matrix require a prior *knowledge of time-varying behavior* of transitions between operational modes of the system, in order to describe the involved stochastic variations. Furthermore, similarly to the case of piecewise homogeneous Markov jump linear systems governed by an arbitrary average dwell time signal, the considered variations need to be in a *finite set*. This requirement implies the fundamental assumption that the transition probabilities can be computed exactly.

The polytopic time-inhomogeneous (hereupon, PTI) MJLSs model does not have such limitations. It has been used in several recent works on (robust) $\mathcal{H}_\infty$ control, filtering and fault detection [44, 45, 65, 66, 74]. These works generally provide sufficient conditions based on linear matrix inequalities and Lyapunov functional approaches for the existence of $\mathcal{H}_\infty$ controllers [65, 66], $\mathcal{H}_\infty$ filters [45, 74] and $\mathcal{H}_\infty/\mathcal{H}_-$ fault detectors [44]. Notably, Long and Yang [44] studies fault detection based on delta operator approach, and gives a new definition of stochastic $\mathcal{H}_-$ *index* [78, 79] (i.e., a measurement of sensitivity from the residuals to the faults) and provides a sufficient condition for designing a $\mathcal{H}_\infty/\mathcal{H}_-$ fault detection filter guaranteeing a stochastic stability of considered Markov jump linear systems.

The polytopic description of time-varying transition probability matrices can be found also in works on non-linear Markov jump systems. For instance, an on-line optimal model predictive controller design algorithm, which uses stochastic stability and Lyapunov function, has been presented in [80] for a noiseless non-linear system. So-called *Markov jump Lur'e systems* (which are MJLSs with sector-bounded mode-dependent memoryless nonlinearities) subject to sensor saturation (handled by a decomposition approach) have been studied in [81], where, by constructing a stochastic multiple Lyapunov function, sufficient conditions for the existence of an observer-based controller with nonlinear feedback terms were derived in terms of LMIs, such that the closed-loop systems are stochastically stable and satisfy a given ℓ_2–ℓ_∞ performance index. Observing that most of the mode-dependent methods available rely on the ideal assumption that the switches of filters are strictly synchronized with those of the system modes, Zhang et al. [82] has studied the problem of asynchronous $\mathcal{H}_\infty$ estimation for two-dimensional Markov jump systems with nonlocal sensor nonlinearity in the Roesser model. This work has considered infinite horizon setting, where mean square (asymptotic) stability [83] is required, and presented a solution by means of linear matrix inequalities. Then, Yin et al. [84] used a parameter-dependent and a mode-dependent Lyapunov function approach to present LMI-based sufficient conditions for the existence of an admissible mode-dependent $\mathcal{H}_\infty$ filter for a Takagi and Sugeno (T-S) fuzzy model of a non-linear Markov jump system. Based on this condition, it has designed a mode-dependent fuzzy filter in a way that the augmented system is stochastically stable and satisfies a prescribed $\mathcal{H}_\infty$ performance index. The same considerations and approach were used in [85] for a robust fuzzy $\mathcal{L}_2$ - $\mathcal{L}_\infty$ filter design.

For MJLSs with polytopic time-inhomogeneous transition probability matrices, the similar approach can be found in applications to several problems, such as robust $\mathcal{L}_2$ - $\mathcal{L}_\infty$ filtering in [75], constrained model predictive control (henceforward, MPC) in case of bounded disturbances (considering the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance index) in [67] and N-step off-line MPC in noiseless case in [68], mode-dependent output peak state feedback control under unit-energy disturbance in [69], robust state-feedback $\mathcal{H}_\infty$ control and observer-based $\mathcal{H}_\infty$ control respectively in [70, 71] and in [72], the last one then extended to the case of actuator saturation, modeled as a nonlinear input, in [65].

The problem of $\mathcal{H}_\infty$ filter design for singular time-delay MJLSs with polytopic time-varying transition probability matrices has been examined in Ding et al. [74]. By using the Lyapunov functional approach and reciprocally convex technique, this work has established a less conservative delay-dependent bounded real lemma and a sufficient condition for the existence of mode-dependent full order filter which guarantees the stochastic stability satisfying $\mathcal{H}_\infty$ performance index of the resulting filtering error system.

Some works have addressed also the (mean square) stability problem for Markov jump linear systems having a transition probability matrix time-varying within a polytope. Notably, in [62] a sufficient condition for stochastic stability in terms of LMI feasibility problem was provided. The approach of [62] made use of a parameter dependent stochastic Lyapunov function. The Aberkane's [62] work also presented a solution to a state-feedback control design problem for stochastic stabilization purposes in two different cases (depending on the available information level). In [63], instead, a sufficient condition for mean square stability of a Markov jump linear system with interval transition probability matrix, which in turn can be represented as a convex polytope [43], was presented in relation to spectral radius. In general, before our contribution, described in detail in the next section, only sufficient stability conditions have been derived for MJLSs with time-inhomogeneous Markov chains having transition probability matrix arbitrarily varying within a polytopic set of stochastic matrices.

Our novel results on necessary and sufficient conditions valid for different types of stability (including not only both mean square stability and stochastic stability, but also exponential mean square stability and robust mean square stability) have opened up an unexplored research line on discrete-time polytopic time-inhomogeneous Markov jump linear systems related to problems of robust linear quadratic regulation, optimal robust filtering, separation of estimation and control, etc., which is the main contribution of this thesis, and is described in detail in the next section.

1.6 Contribution and outline

In this section, we outline the contents of the thesis and the major contributions.

The main contribution of the thesis is given in four chapters, and the material is organized as follows:

- Chapter 3 — Necessary and sufficient conditions for stability of discrete-time time-inhomogeneous Markov jump linear systems affected by polytopic uncertainties on transition probabilities, with or without a bounded process noise.
- Chapter 4 — Finite-horizon optimal filtering for discrete-time polytopic time-inhomogeneous Markov jump linear systems affected by a wide sense white noise.
- Chapter 5 — Finite-horizon linear-quadratic regulation for the same discrete-time polytopic time-inhomogeneous Markov jump linear systems affected by a wide sense white noise; the principle of separation of estimation and control, and derivation of optimal finite-horizon linear-quadratic output-feedback controller.

Chapter 6 Finite-horizon optimal linear quadratic state-feedback control for discrete-time polytopic time-inhomogeneous Markov jump switched linear systems.

In more detail, the outline of the thesis is as follows.

Chapter 2: Model formulation

In Chapter 2 we formally introduce the mathematical models of discrete-time polytopic time-inhomogeneous Markov jump linear systems and Markov jump switched linear systems, which were already named in Section 1.3. We put a particular attention on the formal definition of the stochastic bases, transition probabilities, and all system variables and matrices used throughout this text.

Chapter 3: Stability issues

In Chapter 3 we provide necessary and sufficient conditions for mean square stability of discrete-time polytopic time-inhomogeneous autonomous Markov jump linear systems. Such conditions require to decide whether the joint spectral radius (see appendix's Subsection B.4 for its formal definition and properties) of a finite family of matrices is smaller than 1. While it is well known that the stability analysis problem for general switching systems (i.e., deciding whether the joint spectral radius is smaller than 1) is NP-hard [86], we prove that it is NP-hard even for the matrix structure deriving from our particular model. We also prove that mean square stability is equivalent to exponential mean square stability and to stochastic stability. Then, we extend such results deriving necessary and sufficient conditions for mean square stability robust to energy-bounded disturbances. Notably, the considered discrete-time time-inhomogeneous MJLSs are affected not only by polytopic uncertainties on transition probabilities but also by bounded disturbances. The presented conditions also require to decide whether the joint spectral radius of a finite family of matrices is smaller than 1. Finally, we show that having the spectral radius smaller than one for each matrix associated to the second moment of the state vector is not enough to ensure the stability of the time-inhomogeneous system, and that perturbations on values of transition probability matrix can make a stable system unstable.

The chapter is based on the following papers:

- Y. Zacchia Lun, A. D'Innocenzo, and M.D. Di Benedetto, "On stability of time-inhomogeneous Markov jump linear systems," in *Proceedings of the 55th IEEE Conference on Decision and Control* (CDC 2016), pp. 5527–5532, Dec. 2016.
- Y. Zacchia Lun, A. D'Innocenzo, and M.D. Di Benedetto, "Robust stability of time-inhomogeneous Markov jump linear systems," in *Proceedings of the 20th World Congress of the International Federation of Automatic Control* (IFAC 2017), pp. 3473–3478, July 2017.

Chapters 4 and 5: Optimal robust filtering and control

Following the same research line, in Chapters 4 and 5 we address and solve the finite horizon optimal control and filtering problems for polytopic time-inhomogeneous Markov jump linear systems. In particular, we cast these optimization problems as a min-max problem of optimizing robust performance, i.e., finding the minimum over the control input or filtering error of the maximum over the transition probability disturbance. We show that, as for linear-quadratic-Gaussian (LQG) control in the case with no jumps, for the finite horizon case considered in this thesis, the optimal controller can be obtained from two types of coupled Riccati difference equations, one associated to the control problem, and the other one associated to the filtering problem. When the transition probabilities between operation modes are known at each time step, our results coincide with those presented in [12], and when there is only one mode of operation, they coincide with the traditional separation principle for the LQG control of discrete-time linear systems.

The material presented in these two chapters was accepted for presentation at

- Y. Zacchia Lun, A. D'Innocenzo, A. Abate, and M.D. Di Benedetto, "Optimal robust control and a separation principle for polytopic time-inhomogeneous Markov jump linear systems," in *Proceedings of the 56th IEEE Conference on Decision and Control* (CDC 2017) [accepted], Dec. 2017.

Chapter 6: Extension to switched systems

In Chapter 6 we consider the more general model of polytopic time-inhomogeneous switched Markov jump linear systems, where discrete inputs are present and the Markov-chain turns into a (time-inhomogeneous) Markov-decision process (MDP, see appendix's Subsection B.6 for additional details), and derive the optimal solution of the finite-horizon optimal linear quadratic state-feedback problem.

The material presented in this chapter is mainly based on the publication

- Y. Zacchia Lun, A. D'Innocenzo, and M.D. Di Benedetto, "Robust LQR for time-inhomogeneous Markov jump switched linear systems," in *Proceedings of the 20th World Congress of the International Federation of Automatic Control* (IFAC 2017), pp. 2235–2240, July 2017.

Chapter 7: Conclusions and future work

In Chapter 7 we conclude the main part of the thesis.

Appendix A

In Appendix A we list all the main abbreviations and initialisms used.

Appendix B

In Appendix B we present the notational conventions applied throughout the text and the mathematical background necessary to understand and prove the results of our work. Specifically, we recall the relevant concepts used in set theory, linear algebra, and probability theory.

Appendix C

In Appendix C we list the mathematical symbols used in our presentation.

Appendix D

We already mentioned in the introduction to this thesis, that our survey [5] of cyber-physical security from an automatic control perspective has ignited our interest in Markov jump linear systems. In Appendix D we report the main findings of the aforementioned survey, since it presents a powerful comparison framework for existing and future research on the hot topic of security in cyber-physical systems, important for both industry and academia, and presents an important application domain for the results of this thesis. The contents of this final chapter of appendix are based on the following work:

- Y. Zacchia Lun, A. D'Innocenzo, I. Malavolta, and M.D. Di Benedetto, "Cyber-physical systems security: a systematic mapping study," in arXiv preprint arXiv:1605.09641, 2016.

Chapter 2

Model formulation

Linear systems subject to abrupt parameter changes due, for instance, to environmental disturbances, component failures, changes in subsystems interconnections, changes in the operation point for a non-linear plant, etc., can be modeled by a set of discrete-time linear systems with modal transition given by a discrete-time finite-state Markov chain. This family of systems is known as discrete-time Markov(ian) jump linear systems, often abbreviated as MJLSs.

A useful diagram representing a simple autonomous noiseless Markov jump linear system, i.e., a dynamical system defined as $\mathrm{x}_{k+1}=A_{\theta_k}\mathrm{x}_k$, where x_k is system's state and A_{θ_k} is the related transformation matrix, which depends on the state of the Markov chain θ, is illustrated in Figure 2.1.

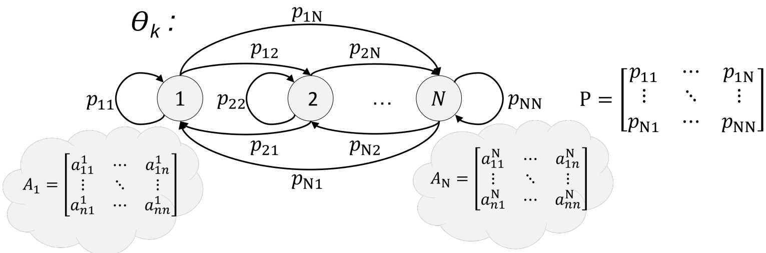

Figure 2.1: Diagram representing a simple MJLS at time step k

The parameters of Markov chain governing the jumps between N operational modes of the system are called transition probabilities and will be extensively discussed in this chapter.

The true parameters of a Markov chain are frequently time-varying and unavailable to the modeler, and a large body of research has been devoted to deal with these uncertainties (as underlined in the previous chapter) and also to the identification of the Markov chain using available observations (see [87] and references therein for an introduction to the topic of estimation of such transition probabilities, which always introduces estimation errors).

We have argued in the previous chapter that in order to account for uncertainties and time-variance inherent to real world scenarios, the time-inhomogeneous polytopic model of transition probabilities is very general and widely used.

In this chapter we present a rigorous mathematical model of MJLSs with polytopic uncertainties on transition probabilities and also the model of their natural extention, i.e., Markov jump switched linear systems (MJSLSs).

We introduce first the general model of Markov jump linear systems in Section 2.1, where we neglect some measure-theoretical details on the probability space and stochastic basis on which both MJLSs' and MJSLSs' models rely on, because these details are not essential to get a flavor of the model and are formally explained in Section 2.2. Then, in Section 2.3 we present a model of polytopic time-varying uncertainties on transition probabilities, which is used in both Markov jump linear systems and Markov jump switched linear systems, i.e., the stochastic hybrid system models with which we are dealing with in this thesis. Finally, in Section 2.4 we formally introduce the model of MJSLSs used to solve the related problem of the optimal robust linear quadratic regulation in Chapter 6.

On a side note, in Appendix B one can find a formal introduction to the relevant concepts from set theory, linear algebra, and probability theory.

2.1 Discrete-time Markov jump linear systems

Essentially, when we think about discrete-time Markov jump linear systems, we have in mind variants of the following class of dynamical systems:

$$\begin{cases} \mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k + B_{\theta_k}\mathrm{u}_k + H_{\theta_k}\mathrm{v}_k, \\ \mathrm{y}_k = F_{\theta_k}\mathrm{x}_k + G_{\theta_k}\mathrm{w}_k, \\ \mathrm{z}_k = C_{\theta_k}\mathrm{x}_k + D_{\theta_k}\mathrm{u}_k, \\ \mathrm{x}_0 = \boldsymbol{x}_0,\ \theta_0 = \vartheta_0,\ \mathrm{p}_0 = \boldsymbol{p}_0 \end{cases} \tag{2.1}$$

where $k \in \mathbb{T}$ is a discrete-time instant, $\mathbb{T}$ is a discrete-time set, $\mathbb{T} = \mathbb{Z}_0$, with $\mathbb{Z}_0$ indicating the set of all nonnegative integers, i.e., $\mathbb{Z}_0 \triangleq \{i \in \mathbb{Z} : i \geq 0\}$, $\mathbb{Z}$ being the set of integers. Then, x_k is a vector of n_{x} either real or complex *state* variables of the Markov jump linear system. Formally, we write that $n_{\mathrm{x}} \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ is a set of positive integers, i.e., $\{i \in \mathbb{Z} : i > 0\}$, and $\mathrm{x}_k \in \mathbb{F}^{n_{\mathrm{x}}}$, where $\mathbb{F}^{n_{\mathrm{x}}}$ denotes an n_{x}-dimensional linear space, with entries in $\mathbb{F}$. See Appendix B for a mathematical background on linear spaces, and other topics necessary for a formal treatment of stochastic hybrid systems. As one can expect from what was stated before, $\mathbb{F}$ indicates the set of all either real numbers (denoted by $\mathbb{R}$) or complex numbers (identified by $\mathbb{C}$). See Appendix C for the complete list and explanation of all mathematical symbols used throughout the thesis. For what concerns other system variables in the aforementioned state-space representation of an MJLS, u_k stands for a vector of n_{u} *control input* variables, $\mathrm{u}_k \in \mathbb{F}^{n_{\mathrm{u}}}$; then, $\mathrm{v}_k \in \mathbb{F}^{n_{\mathrm{v}}}$ and $\mathrm{w}_k \in \mathbb{F}^{n_{\mathrm{w}}}$ are vectors of exogenous input variables, known as *process noise* and *observation noise*, respectively; $\mathrm{y}_k \in \mathbb{F}^{n_{\mathrm{y}}}$ represent a vector of *measured state* variables available to the controller; and $\mathrm{z}_k \in \mathbb{F}^{n_{\mathrm{z}}}$ denotes a vector of measured *system output* variables. Clearly, $n_{\mathrm{u}}, n_{\mathrm{v}}, n_{\mathrm{w}}, n_{\mathrm{y}}, n_{\mathrm{z}} \in \mathbb{Z}_+$.

We should note that we define the system variables on a field of either real or complex numbers $\mathbb{F}$. In general, when studying MJLSs, it is a standard practice to work with complex fields [12], but as observed in Remark B.1 of appendix's Subsection B.4, one can consider complex operators acting on $\mathbb{C}^{m,n}$ as real (block) matrices acting on $\mathbb{R}^{2m,2n}$ [88]. Having this remark in mind, in the following we write $\mathbb{F}^{m,n}$ to denote a set of matrices with m rows, n columns, and entries in $\mathbb{F}$, or, equivalently, a set of linear maps between two linear spaces $\mathbb{F}^n$ and $\mathbb{F}^m$. See appendix's Subsections B.4 and B.4 for additional details on linear operators and transformation matrices.

Consequently, the elements of the system matrices A_{θ_k}, B_{θ_k}, etc., are also defined on a field of either real or complex numbers $\mathbb{F}$. The subscript θ_k indicates that system matrices active in a time instant k are determined by the value of the *jump* variable θ_k, which is a random variable having the set $\mathbb{M} \triangleq \{i \in \mathbb{Z}_+ : i \leq N\}$ as its *state space*, where $N \in \mathbb{Z}_+$ is the cardinality of the set, i.e., a number of its elements. We write the last statement formally as $|\mathbb{M}| = N$. The set $\mathbb{M}$ is generally referred to as the (index) set of *operational modes* of the Markov jump linear system. Before the formal introduction of the exact domain of the random variable θ_k, without loss of generality we present it here simply as the identity function of set of operational modes, i.e., $\theta_k : \mathbb{M} \to \mathbb{M}$, and $\forall i \in \mathbb{M}$, we have that $\theta_k(i) = i$. See appendix's Section B.6 for additional details on stochastic processes and related random variables, and Section 2.2 for a formal definition of the probability space and the stochastic basis of a Markov jump linear system as a whole. For every operational mode there is a correspondent system matrix, and the collection of the system matrices of each type is generally represented by a sequence of N matrices, which are not necessarily all distinct. Specifically, $\boldsymbol{A} \triangleq (A_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{x}}, n_{\mathrm{x}}}$ is a sequence of so-called *state matrices*, each of which is associated to an operational mode of the (switching) system. Noticeably, ${}_{\mathrm{N}}\mathbb{F}^{m,n}$ indicates a *linear space* made up of all N-*sequences of* $m \times n$ *matrices* with entries in $\mathbb{F}$. Similarly, $\boldsymbol{B} \triangleq (B_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{x}}, n_{\mathrm{u}}}$ is an N-sequence of *input matrices*; $\boldsymbol{C} \triangleq (C_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{z}}, n_{\mathrm{x}}}$ is a sequence of *output matrices*; $\boldsymbol{D} \triangleq (D_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{z}}, n_{\mathrm{u}}}$ is a sequence of *direct transition* (also known as feed-forward or feedthrough) *matrices*; $\boldsymbol{F} \triangleq (F_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{y}}, n_{\mathrm{x}}}$ is a sequence of *observation matrices*; $\boldsymbol{G} \triangleq (G_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{y}}, n_{\mathrm{w}}}$ is a sequence of *observation noise matrices*; and $\boldsymbol{H} \triangleq (H_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{x}}, n_{\mathrm{v}}}$ is a sequence of *process noise matrices*.

The transitions, or jumps, between operational modes of an MJLS are governed by a discrete-time Markov chain θ, which, loosely speaking, is a collection of random variables θ_t all taking values in the same state space, i.e., $\{\theta_t : t \in \mathbb{T}\}$, and satisfying the Markov property, which is formally expressed in appendix's Subsection B.6 by (B.82). See next Section 2.2 for a rigorous measure-theoretic definition of θ, and Section 2.3 for accurate definition of polytopic time-inhomogeneous model of transition probabilities.

Last but not least, the initial conditions for a Markov jump linear system consist of the initial state of the system, x_0 (which has $\boldsymbol{x}_0 \in \mathbb{F}^{n_\mathrm{x}}$ as its value), initial state of the Markov chain, θ_0 (with $\vartheta_0 \in \mathbb{M}$ being its value), and initial probability distribution of the states of the jump variable θ_0, denoted by p_0 (and having $\boldsymbol{p}_0 \in \mathbb{R}_0^N$ s.t. $\|\boldsymbol{p}_0\|_1 = 1$, as its value; clearly, $\|\cdot\|_1$ indicates the standard grid norm of a vector, formally defined in appendix's Subsection B.3, and $\mathbb{R}_0^N$ evidently represents the N-dimensional linear space with entries in the set of nonnegative real numbers, i.e., $\mathbb{R}_0 = \{i \in \mathbb{R} : i \geq 0\}$).

2.2 Probability space and stochastic basis

In this section we present in detail the probabilistic framework we shall consider throughout the rest of the thesis. The appendix's Sections B.5 and B.6 provide a theoretical background for this topic.

The most formal, measure-theoretic definition of a stochastic hybrid system, such as a Markov jump linear system in equation (2.1), requires an explicit presentation of its stochastic basis, which is determined by a product space of involved variables.

We have seen in the previous section that at each time instant k the active system matrices depend on the value of the jump variable θ_k. This discrete random variable takes values in the set $\mathbb{M} = \{i \in \mathbb{Z}_+ : i \leq N\}$ and, when considered on a generic probability space, without loss of generality is specified as an identity function, that is, $\theta_k : \mathbb{M} \to \mathbb{M}$ and $\theta_k(i) = i$. Thus, the measurable space, on which the target set (a.k.a. codomain) of θ_k is defined, is a pair $(\mathbb{M}, \mathcal{M})$, while the probability space determining the domain of θ_k is defined by a triple $(\mathbb{M}, \mathcal{M}, \Pr)$, where $\mathcal{M} \subseteq 2^{\mathbb{M}}$ is a σ-algebra of measurable events of $\mathbb{M}$, and $\Pr : \mathcal{M} \to [0,1]$ is a probability measure. Clearly, $[0,1]$ indicates a closed interval on the real line, i.e., $\{x \in \mathbb{R} : x \geq 0, x \leq 1\}$. Under a standard for MJLSs assumption that θ_k is measurable, and that the operational modes of the system are elementary, mutually exclusive (i.e., disjoint) events, we have that the probability distribution of θ_k is its probability mass function, i.e., $\forall i \in \mathbb{M}$, we have that $p_i \triangleq \Pr(\{i : \theta_k(i) = i\}) = \Pr(\theta_k = i)$. We note that the expression of p_i is intended for a random variable θ_k alone, i.e., $p_i = p_i(\theta_k)$. When θ_k is a part of a Markov chain $\{\theta_k : k \in \mathbb{T}\}$, the probability distribution of θ_k is denoted by $p_i(k)$, i.e., similarly to the random variable itself, the probability distribution is also indexed by time instant k. In that case, $\forall k \in \mathbb{T}$, the total mass of the probability distribution of the discrete random variable θ_k is written as $\sum_{i=1}^{N} p_i(k) = 1$.

Although in engineering problems the operation modes are not often available, there are enough cases where the knowledge of random changes in system structure is directly available to make these applications of great interest [12, 89]. The typical examples include a ship steering autopilot, control of pH in a chemical reactor, combustion control of a solar-powered boiler, fuel air control in a car engine and flight control systems [12]. We remark that MJLSs' model is well suited also for the wireless networked control system scenario, when a channel estimation is performed (see e.g. [90] and the references therein), so the channel state information is known at each time step. In fact, the knowledge of θ_k at each time instant k is a standard assumption, which is used in vast majority of works reviewed in Section 1.5. So, throughout this thesis we do the following assumption.

Assumption 2.1. *At every time step $k \in \mathbb{T}$, the jump variable θ_k is measurable and available to a controller.*

Depending on the considered problem, (some of) the system's vector variables x_k, u_k, y_k and z_k may also be viewed as measurable.

Notably, the set of possible values of x_k is generally a subset of $\mathbb{F}^{n_\mathrm{x}}$, i.e., a product space of n_x either real or complex variables. When it is defined as a set of real numbers, the corresponding σ-algebra is Borel-σ-algebra $\mathcal{R}^{n_\mathrm{x}}$. See Subsection B.5 of appendix for a formal definition of product Borel-σ-algebras. When the system's state vector is defined on a complex field, the relevant σ-algebra is Borel-σ-algebra $\mathcal{R}^{2n_\mathrm{x}}$. In the following, we denote a σ-algebra of x_k by $\mathcal{X}$. Thus, the measurable space of the system's state vector x_k is denoted by $(\mathbb{F}^{n_\mathrm{x}}, \mathcal{X})$, that corresponds to either $(\mathbb{R}^{n_\mathrm{x}}, \mathcal{R}^{n_\mathrm{x}})$ or $\left(\mathbb{C}^{n_\mathrm{x}}, \mathcal{R}^{2n_\mathrm{x}}\right)$.

Similarly, the set of possible values of u_k is a subset of $\mathbb{F}^{n_\mathrm{u}}$. The connected σ-algebra is either Borel-σ-algebra $\mathcal{R}^{n_\mathrm{u}}$ or $\mathcal{R}^{2n_\mathrm{u}}$, and is denoted by $\mathcal{U}$. The measurable space of the system's control input u_k is then a pair $(\mathbb{F}^{n_\mathrm{u}}, \mathcal{U})$, interpreted as either $(\mathbb{R}^{n_\mathrm{u}}, \mathcal{R}^{n_\mathrm{u}})$ or $\left(\mathbb{C}^{n_\mathrm{u}}, \mathcal{R}^{2n_\mathrm{u}}\right)$.

On the same line, the set of possible values of y_k is a subset of $\mathbb{F}^{n_\mathrm{y}}$. The related σ-algebra is either Borel-σ-algebra $\mathcal{R}^{n_\mathrm{y}}$ or $\mathcal{R}^{2n_\mathrm{y}}$. It is denoted by $\mathcal{Y}$. The correspondent measurable space of the vector y_k is a pair $(\mathbb{F}^{n_\mathrm{y}}, \mathcal{Y})$, which is understood as either $(\mathbb{R}^{n_\mathrm{y}}, \mathcal{R}^{n_\mathrm{y}})$ or $\left(\mathbb{C}^{n_\mathrm{y}}, \mathcal{R}^{2n_\mathrm{y}}\right)$.

Analogously, the set of possible values of z_k is a subset of $\mathbb{F}^{n_\mathrm{z}}$. The associated σ-algebra is either Borel-σ-algebra $\mathcal{R}^{n_\mathrm{z}}$ or $\mathcal{R}^{2n_\mathrm{z}}$, and is denoted by $\mathcal{Z}$. Consequently, the measurable space of the system's output vector z_k is a pair $(\mathbb{F}^{n_\mathrm{z}}, \mathcal{Z})$, corresponding to either $(\mathbb{R}^{n_\mathrm{z}}, \mathcal{R}^{n_\mathrm{z}})$ or $\left(\mathbb{C}^{n_\mathrm{z}}, \mathcal{R}^{2n_\mathrm{z}}\right)$.

For what concerns specific problems, from (2.1) it is evident that for any given value of current system's state x_k, control input u_k and process noise v_k, the value of the MJLS's state vector in the immediate future, i.e., x_{k+1}, depends on the latest value of the jump variable θ_k through the system matrices A_{θ_k}, B_{θ_k} and H_{θ_k}. In other words, the next future state x_{k+1} of the system is determined by the current operational mode. Thus, once the current values of the system's state x_k, control input u_k and process noise v_k are fixed, the current system's state x_k behaves as a random variable having a number equal to N of possible future values, which occurrence is determined by a discrete probability distribution of θ_k. So, the probability space of x_k is a triple $(\mathbb{F}^{n_\mathrm{x}}, \mathcal{X}, \mathrm{Pr})$, where its sample space, σ-algebra of events and probability measure are defined as before.

Scenario 1: All system's state variables are measurable

In the problems considering stability and state-feedback control, which are treated formally in Chapters 3 and 5, the system's state vector x_k together with the jump variable θ_k are regarded as measurable at each time step k.

Thus, the underlying stochastic basis is constructed as follows.

The sample space which takes into account all time values $k \in \mathbb{T}$ is defined as

$$\Omega_{\mathrm{x}} \triangleq \prod_{k=0}^{\infty} (\mathbb{M} \times \mathbb{F}^{n_{\mathrm{x}}}) \tag{2.2}$$

where $\prod$ denotes the *Cartesian product* of a sequence, and subscript x indicates that together with Markov chain θ, the sample space Ω_{x} accounts for the collection of random variables x_k, which are indexed by the discrete time set $\mathbb{T}$.

The corresponding filtration $\mathcal{G}_k$ requires the introduction of bounded discrete-time set $\mathbb{T}_k$, which is defined as $\{t \in \mathbb{T} : t \leq k\}$.

We have the following recursive definition of the filtration:

$$\mathcal{G}_k \triangleq \sigma \left\{ \prod_{t=0}^{k} (\Theta_t \times \mathbb{S}_t) \times \prod_{\tau=k+1}^{\infty} (\mathbb{M} \times \mathbb{F}^{n_{\mathrm{x}}}) : \Theta_t \in \mathcal{M},\, \mathbb{S}_t \in \mathcal{X},\, \forall t \in \mathbb{T}_k \right\} \tag{2.3}$$

so that $(\mathcal{G}_k) \nearrow$, i.e., it is a monotone non-decreasing sequence of product σ-algebras, since by construction its elements satisfy the property of being $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \cdots$

Then, the product σ-algebra $\mathcal{G}$ is defined simply by

$$\mathcal{G} \triangleq \sigma \left\{ \prod_{t=0}^{\infty} (\Theta_t \times \mathbb{S}_t) : \Theta_t \in \mathcal{M},\, \mathbb{S}_t \in \mathcal{X},\, \forall t \in \mathbb{T} \right\} \tag{2.4}$$

Clearly, $\mathcal{G}_k \subseteq \mathcal{G}$.

Hence, the corresponding *stochastic basis* (a.k.a. filtered probability space) of Markov jump linear systems considered in the problems of stability and state-feedback control is defined by the quadruple $(\Omega_{\mathrm{x}}, \mathcal{G}, (\mathcal{G}_k), \mathrm{Pr})$, where the sample space Ω_{x} is defined as in (2.2), the σ-algebra $\mathcal{G}$ is characterized by (2.4), and the filtration $(\mathcal{G}_k)$ is determined by (2.3), while the probability measure $\mathrm{Pr} : \mathcal{G} \to [0,1]$ is such that, according to the Markov property (B.82), $\forall j \in \mathbb{M}$

$$\mathrm{Pr}(\theta_{k+1} = j \mid \mathcal{G}_k) = \mathrm{Pr}(\theta_{k+1} = j \mid \theta_k) = p_{\theta_k j}(k) \tag{2.5}$$

where the random variable θ_k has the sample space Ω_{x} as its domain, i.e., the jump variable is formally defined here as $\theta_k : \Omega_{\mathrm{x}} \to \mathbb{M}$, such that $\forall \omega \in \Omega_{\mathrm{x}}$, with $\omega \triangleq \{(\phi_k, \chi_k) : k \in \mathbb{T},\, \phi_k \in \mathbb{M},\, \chi_k \in \mathbb{F}^{n_{\mathrm{x}}}\}$, one has that $\theta_k(\omega) = \phi_k$.

The transition probability between the operational modes $i, j \in \mathbb{M}$ of a Markov jump linear system is formally defined as

$$p_{ij}(k) \triangleq \mathrm{Pr}(\omega : \theta_{k+1}(\omega) = j \mid \theta_k = i) = \mathrm{Pr}(\theta_{k+1} = j \mid \theta_k = i) \geq 0 \tag{2.6}$$

Clearly, since $p_{ij}(k)$ is a probability distribution, $\forall i \in \mathbb{M}$, one has that the total mass of the distribution equals to 1, i.e.,

$$\sum_{j=1}^{N} p_{ij}(k) = 1 \tag{2.7}$$

Evidently, $\theta_x \triangleq \{\theta_k : k \in \mathbb{T}\}$, with $\theta_k : \Omega_x \to \mathbb{M}$ defined above, is a Markov chain with transition probability matrix (henceforth, TPM)

$$P(k) \triangleq [p_{ij}(k)] \in \mathbb{R}^{N,N} \tag{2.8}$$

In this thesis we assume that transition probabilities between the operational modes of a stochastic system are arbitrarily time-varying within a bounded domain. The formal model of this behavior will be presented in the next Section 2.3.

The initial probability distribution of the Markov chain θ_x is defined $\forall i \in \mathbb{M}$ by

$$p_i(0) \triangleq \Pr(\omega : \theta_0(\omega) = i) = \Pr(\theta_0 = i) \tag{2.9}$$

Then, the initial probability distribution of all the operational modes is defined as

$$\mathrm{p}_0 \triangleq \begin{bmatrix} p_1(0) \\ p_2(0) \\ \vdots \\ p_N(0) \end{bmatrix} \in \mathbb{R}^{N,1} \tag{2.10}$$

In order to present our result on mean square stability robust to energy-bounded disturbances, we need to formally introduce also the related linear spaces, defined on the presented probability space.

Following the line of reasoning of Costa et al. [12], we set $\mathbb{H}^n \triangleq \mathbb{L}^2(\Omega_x, \mathcal{G}, \Pr, \mathbb{F}^n)$ the *Hilbert space* of all $\mathbb{F}^n$-valued $\mathcal{G}$-measurable random variables with inner product given $\forall \mathrm{x}, \mathrm{y} \in \mathbb{H}^n$ by $\langle \mathrm{x}, \mathrm{y} \rangle = \mathbb{E}(\mathrm{x}^* \mathrm{y})$, and Euclidean norm denoted by $\|\cdot\|_2$. Henceforth, $\mathbb{E}(\cdot)$ denotes the expected value of a random variable. See appendix's Section B.6 for a standard definition of the expected value, and Section B.3 for additional details on Hilbert spaces and inner products.

We set $\ell_2(\mathbb{H}^n) \triangleq \bigoplus_{k \in \mathbb{T}} \mathbb{H}^n$, the direct sum of countably infinite copies of $\mathbb{H}^n$, which is a Hilbert space made up of collections of all $\mathbb{F}^n$-valued $\mathcal{G}$-measurable random variables indexed by the discrete time set $\mathbb{T}$, i.e., $\mathrm{f} = \{\mathrm{f}_k \in \mathbb{H}^n : k \in \mathbb{T}\}$, such that $\|\mathrm{f}\|_2^2 \triangleq \sum_{k=0}^{\infty} \mathbb{E}\Big(\|\mathrm{f}_k\|^2\Big) < \infty$. For any given $\mathrm{f}, \mathrm{g} \in \ell_2(\mathbb{H}^n)$, the inner product is $\langle \mathrm{f}, \mathrm{g} \rangle \triangleq \sum_{k=0}^{\infty} \mathbb{E}(\mathrm{f}_k^* \mathrm{g}_k) \leq \|\mathrm{f}\|_2 \|\mathrm{g}\|_2$. See appendix's Section B.4 for additional details on direct sum, and Section B.3 for a brief introduction to series and absolute convergence of sequences.

Then, we define $\mathcal{H}^n \subseteq \ell_2(\mathbb{H}^n)$ as follows. We say that $\mathrm{f} = \{\mathrm{f}_k \in \mathbb{H}^n : k \in \mathbb{T}\} \in \mathcal{H}^n$ if $\mathrm{f} \in \ell_2(\mathbb{H}^n)$ and $\mathrm{f}_k \in \mathbb{L}^2(\Omega_x, \mathcal{G}_k, \Pr, \mathbb{F}^n)$ $\forall k \in \mathbb{T}$, where $\mathcal{G}_k$ is defined by (2.3). We have that $\mathcal{H}^n$ is a closed linear subspace of $\ell_2(\mathbb{H}^n)$ and therefore a Hilbert space [12, p.21]. We also define $\mathcal{H}_k^n$ as formed by sequences $(\mathrm{f}_t)_{t=0}^{k}$, s.t. $\mathrm{f}_t \in \mathbb{L}^2(\Omega_x, \mathcal{G}_t, \Pr, \mathbb{F}^n)$, $\forall t \in \mathbb{T}_k$.

Finally, we denote by Θ_0 the set of all $\mathcal{G}_0$-measurable random variables taking values in $\mathbb{M}$. This permits us to state the initial conditions for a Markov jump linear system with θ_k and x_k measurable $\forall k \in \mathbb{T}$ as

$$\mathrm{x}_0 \in \mathcal{H}_0^{n_x}, \quad \theta_0 \in \Theta_0 \tag{2.11}$$

When a MJLS is defined on a stochastic basis $(\Omega_{\mathrm{x}}, \mathcal{G}, (\mathcal{G}_k), \mathrm{Pr})$, one have that θ_k and x_k are measured in each time step $k \in \mathbb{T}$, and thus $\mathrm{y}_k = \mathrm{x}_k$, i.e., $n_{\mathrm{x}} = n_{\mathrm{y}}$, meaning that all the system's state variables are available to the controller. Then, the control input u_k is a (non-random) signal the controller chooses to apply to the system at time step k. In adaptive control scenario, when u_k is defined as an (operational) mode-dependent function of x_k, one has that $\mathrm{u}_k : \Omega_{\mathrm{x}} \to \mathbb{F}^{n_{\mathrm{u}}}$ is $(\mathcal{G}_k, \mathcal{U})$-measurable. In such scenario, from (2.1) it is evident that $\mathrm{z}_k : \Omega_{\mathrm{x}} \times \mathbb{F}^{n_{\mathrm{u}}} \to \mathbb{F}^{n_{\mathrm{z}}}$ is $(\mathcal{G}_k \times \mathcal{U}, \mathcal{Z})$-measurable. See appendix's Subsection B.5 for additional details on measurable functions.

Scenario 2: Some system's state variables are measurable

When not all the state variables are available to the controller, the system's state estimation is required. The aforementioned scenario is described in Chapter 4. There, at each time step $k \in \mathbb{T}$, instead of x_k, y_k is considered to be measurable together with θ_k. Remarkably, we have that $n_{\mathrm{y}} \leq n_{\mathrm{x}}$, where n_{y} evidently denotes the number of measured state variables, and n_{x} is a total number of system's state variables. The dynamics of y_k clearly follow the dynamics of x_k and the measured system's state is regarded as a random variable. The construction of the stochastic basis is very similar to the previous one.

In particular, the sample space is defined as

$$\Omega_{\mathrm{y}} \triangleq \prod\nolimits_{k=0}^{\infty} (\mathbb{M} \times \mathbb{F}^{n_{\mathrm{y}}}) \tag{2.12}$$

where the subscript y states that the collection of random variables y_k is considered.

The corresponding filtration is defined recursively as

$$\mathcal{F}_k \triangleq \sigma \left\{ \prod\nolimits_{t=0}^{k} (\Theta_t \times \mathbb{S}_t) \times \prod\nolimits_{\tau=k+1}^{\infty} (\mathbb{M} \times \mathbb{F}^{n_{\mathrm{y}}}) : \Theta_t \in \mathcal{M},\ \mathbb{S}_t \in \mathcal{Y},\ \forall t \in \mathbb{T}_k \right\} \tag{2.13}$$

so that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_k$, i.e., $(\mathcal{F}_k) \nearrow$.

The product σ-algebra $\mathcal{F}$ is defined by

$$\mathcal{F} \triangleq \sigma \left\{ \prod\nolimits_{t=0}^{\infty} (\Theta_t \times \mathbb{S}_t) : \Theta_t \in \mathcal{M},\ \mathbb{S}_t \in \mathcal{Y},\ \forall t \in \mathbb{T} \right\} \tag{2.14}$$

Obviously, $\mathcal{F}_k \subseteq \mathcal{F}\ \forall k \in \mathbb{T}$. The corresponding stochastic basis of Markov jump linear systems considered in the problems of optimal state estimation is defined by the quadruple $(\Omega_{\mathrm{y}}, \mathcal{F}, (\mathcal{F}_k), \mathrm{Pr})$, where the probability measure $\mathrm{Pr} : \mathcal{F} \to [0, 1]$ is such that $\forall j \in \mathbb{M}$

$$\mathrm{Pr}(\theta_{k+1} = j \mid \mathcal{F}_k) = \mathrm{Pr}(\theta_{k+1} = j \mid \theta_k) = p_{\theta_k j}(k) \tag{2.15}$$

where the random variable θ_k has the sample space Ω_{y} as its domain, i.e., the jump variable is formally defined here as $\theta_k : \Omega_{\mathrm{y}} \to \mathbb{M}$, such that $\forall \omega \in \Omega_{\mathrm{y}}$, with $\omega \triangleq \{(\phi_k, \chi_k) : k \in \mathbb{T},\ \phi_k \in \mathbb{M},\ \chi_k \in \mathbb{F}^{n_{\mathrm{y}}}\}$, one has that $\theta_k(\omega) = \phi_k$.

The equations (2.6)–(2.10) remain unchanged. The transition probabilities there are assumed to be time-varying arbitrarily within a polytopic domain, as formally presented in the next section.

2.3 Polytopic time-varying transition probabilities

In the previous chapter, we have seen in Section 1.4 that in most real cases the transition probability matrix $P(k)$ introduced in (2.8) cannot be computed exactly and is time-varying, and that there exists a considerable number of works on discrete-time Markov jump systems (both linear and nonlinear) with polytopic uncertainties, which can be either time-varying or time-invariant, as was extensively discussed in Section 1.5.

In this thesis we assume that $P(k)$ is varying over time, with variations that are arbitrary within a polytopic set of stochastic matrices.

In order to express this statement formally, let $V \in \mathbb{Z}_+$ be a number of vertices of a convex polytope, and $\mathbb{V}$ be an index set of vertices of a convex polytope, i.e., $\mathbb{V} \triangleq \{i \in \mathbb{Z}_+ : i \leq V\}$. Then, the set of vertices of a convex polytope of transition probability matrices is formally defined as

$$ {}_{\mathbb{V}}\mathbb{P} \triangleq \left\{ P_l \in \mathbb{R}^{N,N} : l \in \mathbb{V} \right\} \tag{2.16}$$

Clearly, being a transition probability matrix, each vertex P_l satisfies (2.6)–(2.8).

These vertices are obtained from measurement on the real system or via numerical reasoning, taking into account accuracy and precision of the measuring instruments and/or numerical algorithms. They bound the possible values each transition probability can assume.

Then, the polytopic time-inhomogeneous assumption is stated as follows.

Assumption 2.2. *The time-varying transition probability matrix $P(k)$ is **polytopic**, that is, for all $k \in \mathbb{T}$, one has that*

$$ P(k) = \sum\nolimits_{l=1}^{V} \lambda_l(k) P_l, \quad \lambda_l(k) \geq 0, \quad \sum\nolimits_{l=1}^{V} \lambda_l(k) = 1 \tag{2.17}$$

where for each $l \in \mathbb{V}$, $P_l \in {}_{\mathbb{V}}\mathbb{P} \subset {}_{\mathbb{V}}\mathbb{R}^{N,N}$, i.e, P_l are elements of a given finite set of transition probability matrices, which are the vertices of a convex polytope; moreover, $\lambda_l(k)$ are unmeasurable.

A visual representation of the concept of arbitrarily variations within a convex hull of points is illustrated in the following Figure 2.2, where P_1, P_2 and P_3 represent the vertices, and $P(k)$ shows a possible evolution in time of an element satisfying polytopic time-varying assumption.

Assumption 2.2 plays important role also in our model of Markov jump switched linear systems, which we present in the following Section 2.4.

2.4 Markov jump switched linear systems

Switched linear systems, where a switching signal is governed by a Markov decision process (henceforward, MDP) instead of a Markov chain, is the subject of this last section of Chapter 2.

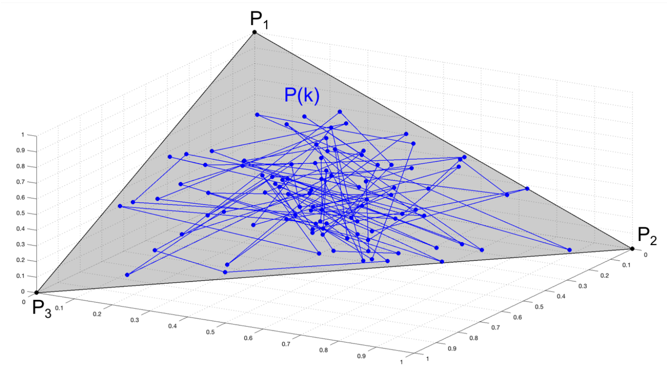

Figure 2.2: Dynamics of an element satisfying polytopic time-varying assumption

An MDP enriches the structure of a Markov chain by adding actions (which allow a choice) and costs (that give a motivation). So, as before, there is a finite, or countable, (index) set of states of a process, that we denote by $\mathbb{M}$. Again, $|\mathbb{M}| = N$. Then, on top of these discrete states of a Markov decision process, there is a a finite number $M \in \mathbb{Z}_+$ of actions among which a decision maker (a.k.a. a discrete controller or a supervisor) is able to chose. We denote the set of all these actions by $\mathbb{A}$. Typically, only a subset of $\mathbb{A}$ is available in any given state of an MDP, as illustrated in appendix's Section B.6. We take this into account by defining for each state $i \in \mathbb{M}$ the related set $\mathbb{A}_i$ of actions α available in that state. We write this statement symbolically as $\mathbb{A}_i \subseteq \mathbb{A}$, $\alpha \in \mathbb{A}_i$. Selecting an (available) action in any given state of a Markov decision process entails a (non-negative) cost, which is seen as a function $g : \mathbb{M} \times \mathbb{A} \to \mathbb{G}$, where $\mathbb{G} \subseteq \mathbb{R}_0$ is a set of immediate costs. See appendix's Section B.6 for an example from wireless transmission power management scenario.

A Markov decision process is still a stochastic process and should be defined on a proper stochastic basis. As of now, we denote by s_k a stochastic variable representing a current state of a Markov decision process. Evidently, its codomain is $\mathbb{M}$. In the same way as done before for a Markov jump variable, we make the following assumption.

Assumption 2.3. *The state s_k of the Markov decision process is measurable and available for the discrete controller at each time step $k \in \mathbb{T}$.*

Then, once more, we denote by $\mathcal{M} \subseteq 2^{\mathbb{M}}$ the σ-algebra of measurable events of $\mathbb{M}$, so that a pair $(\mathbb{M}, \mathcal{M})$ indicates the measurable space of s_k.

Hence, a Markov jump switched linear system is a dynamical system having the same form as (2.1), with the only notational difference being that the operational modes of the system are determined by the stochastic variable s_k, which will be formally defined shortly.

Specifically, a continuous state-space model of an MJSLS is the following system of recursive equations:

$$\begin{cases} \mathrm{x}_{k+1} = A_{s_k}\mathrm{x}_k + B_{s_k}\mathrm{u}_k \\ \quad \mathrm{z}_k = C_{s_k}\mathrm{x}_k + D_{s_k}\mathrm{u}_k, \\ \quad \mathrm{x}_0 = \boldsymbol{x}_0,\ s_0 = \boldsymbol{s}_0,\ \mathrm{p}_0 = \boldsymbol{p}_0 \end{cases} \tag{2.18}$$

where the system's variables and matrices are the same as in Section 2.1.

This model is used to provide a solution to a finite-horizon optimal state-feedback control problem in noiseless setting, as will be presented in Chapter 6. For this type of problems, we make the standard assumption that the system's state variables are all measurable and available to the controller. The action selected by a decision maker should also be measurable. Then, the adopted information pattern is similar to the pattern used in Subsection 2.2, i.e., $\forall k \in \mathbb{T}$, one has that $\mathcal{I}_k \triangleq \{s_k, \alpha_k, \mathrm{x}_k\}$. With a slight abuse of notation, we will indicate by $\mathcal{I}_k$ also the induced filtration, which will be defined immediately after an introduction of a proper sample space, since the addition of the actions in the structure of a stochastic process requires an adjustment in the sample space's definition. Specifically, in the rest of this section we will consider

$$\Omega_{\mathrm{s}} \triangleq \prod_{k=0}^{\infty} (\mathbb{M} \times \mathbb{A} \times \mathbb{F}^{n_{\mathrm{x}}}) \tag{2.19}$$

where the subscript s indicates that this sample space accounts for the collection of random variables s_k.

At this point, we have the following recursive definition of the filtration

$$\mathcal{I}_k \triangleq \sigma\left\{\prod_{t=0}^{k}(\Theta_t \times \Upsilon_t \times \mathbb{S}_t) \times \prod_{\tau=k+1}^{\infty}(\mathbb{M} \times \mathbb{A} \times \mathbb{F}^{n_{\mathrm{x}}}) : \Theta_t \in \mathcal{M},\ \Upsilon_t \in \mathcal{A},\ \mathbb{S}_t \in \mathcal{X}, \forall t \in \mathbb{T}_k\right\}$$

where $\mathcal{A} \subseteq 2^{\mathbb{A}}$ is the σ-algebra of measurable events of $\mathbb{A}$. Noticeably, we have that $(\mathcal{I}_k)\nearrow$, meaning that $\mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_k$.

The product σ-algebra $\mathcal{I}$ is then written as

$$\mathcal{I} \triangleq \sigma\left\{\prod_{t=0}^{\infty}(\Theta_t \times \Upsilon_t \times \mathbb{S}_t) : \Theta_t \in \mathcal{M},\ \Upsilon_t \in \mathcal{A},\ \mathbb{S}_t \in \mathcal{X},\ \forall t \in \mathbb{T}\right\} \tag{2.20}$$

Clearly, $\mathcal{I}_k \subseteq \mathcal{I}\ \forall k \in \mathbb{T}$.

The corresponding stochastic basis of a MJSLS represented by (2.18) is defined by the quadruple $(\Omega_{\mathrm{s}}, \mathcal{I}, (\mathcal{I}_k), \Pr)$, where the probability measure $\Pr : \mathcal{I} \to [0,1]$ is such that $\forall j \in \mathbb{M}$

$$\Pr(s_{k+1} = j \mid \mathcal{I}_k) = \Pr(s_{k+1} = j \mid s_k = i, \alpha_k = \alpha) = p_{ij}^{\alpha}(k) \tag{2.21}$$

where the random variable s_k has the sample space Ω_{s} as its domain, and the action chosen by discrete controller is measurable (on $\mathcal{I}_k$).

Then, the random variable representing the current operational mode of a Markov jump switched linear system is formally defined as $s_k : \Omega_\mathrm{s} \to \mathbb{M}$, such that $\forall\omega\in\Omega_\mathrm{s}$, with $\omega \triangleq \{(\phi_k, \mu_k, \chi_k) : k\in\mathbb{T},\ \phi_k\in\mathbb{M},\ \mu_k\in\mathbb{A},\ \chi_k\in\mathbb{F}^{n_\mathrm{x}}\}$, one has that $s_k(\omega)=\phi_k$.

Obviously, when considered together with immediate cost function $g : \mathbb{M}\times\mathbb{A}\to\mathbb{G}$, the stochastic process $s\triangleq\{s_k : k\in\mathbb{T}\}$, with $s_k : \Omega_\mathrm{s}\to\mathbb{M}$ defined above, is a Markov decision process. Its transition probabilities $p_{ij}^{\alpha}\geq 0$ are s.t. $\sum_{j=1}^{N} p_{ij}^{\alpha}(k)=1\ \forall\alpha\in\mathbb{A}_i$.

Since $\mathbb{A}_i\subseteq\mathbb{A}$, not all the actions may be available in an operational mode $s_k=i$. In other words, there may be some actions that are not available in all operational modes. For those actions we do not have a correspondent transition probability matrix (which by definition requires every row to have elements with nonnegative values such that their total sum equals to 1; when an action is not available in an operational mode, the associated transition probabilities are intuitively all equal to 0 or not defined). As a result, when dealing with such MJSLSs, we may not have transition probability matrices, but only a collection of transition probability row vectors $\mathrm{p}_{i\bullet}^{\alpha}(k)$, each of which is associated to an operational mode $s_k=i$ where $\alpha\in\mathbb{A}_i$ is available. So, Assumption 2.2 is modified from the notational point of view as follows.

Assumption 2.4. *The time-varying transition probability row vector* $\mathrm{p}_{i\bullet}^{\alpha}(k)$ *is* ***polytopic****, that is, for all* $k\in\mathbb{T}$*,* $i\in\mathbb{M}$*, and* $\alpha\in\mathbb{A}_i$ *one has that*

$$\mathrm{p}_{i\bullet}^{\alpha}(k) = \sum\nolimits_{l=1}^{V_\alpha} \lambda_l(k)\mathrm{p}_{i\bullet l}^{\alpha}, \quad \lambda_l(k)\geq 0, \quad \sum\nolimits_{l=1}^{V_\alpha} \lambda_l(k) = 1 \tag{2.22}$$

where for each $l\in\mathbb{V}_\alpha$*,* $\mathrm{p}_{i\bullet l}^{\alpha}\in {}_{\mathbb{V}_\alpha}\mathbb{P}\subset {}_{\mathbb{V}_\alpha}\mathbb{R}^{1,N}$*, i.e,* $\mathrm{p}_{i\bullet l}^{\alpha}$ *are elements of a given finite set of transition probability row vectors, which are the vertices of a convex polytope; moreover,* $\lambda_l(k)$ *are unmeasurable.*

We stress the fact that if an action is available for all operational modes, all N associated transition probability (TP) row vectors form a transition probability matrix related to that action.

Loosely speaking, instead of having only one polytope of transition probability matrices, a MJSLS has several. See appendix's Subsection B.4 for additional thoughts on sets of matrices and convex polytopes.

In Chapter 6 we will use the presented model to solve the problem of optimal quadratic regulation which is robust to the polytopic time-varying uncertainties on transition probabilities between operational modes of the system, associated to each action.

In the next chapter, instead, we will examine the problem of stability of discrete-time polytopic time-inhomogeneous Markov jump linear systems illustrated in the previous sections of this chapter.

Chapter 3

Stability issues

Among the requirements in any control system design problem, stability is certainly a mandatory one [12]. Various types of stability may be discussed for the solutions of difference equations describing discrete-time dynamical systems.

For the systems without switching behavior, the most important type of stability is that concerning the stability of solutions near to a point of equilibrium, i.e., the stability in sense of Lyapunov. Although Markov jump linear systems seem to be a natural extension of the class of linear dynamical systems without switching, their subtleties are such that the standard linear theory cannot be directly applied, as well described in the reference book on discrete-time MJLSs [12].

The notion of stability of Markov jump linear systems that parallels the ideas of Lyapunov stability theory is so-called mean square stability (often abbreviated to MSS). The necessary and sufficient conditions for this type of stability derived for Markov jump linear systems with time-varying uncertain transition probabilities is the main result of this section.

This result was first derived for the noiseless setting in [1] and presented at the 55th IEEE Conference on Decision and Control (CDC 2016) held in Las Vegas, Nevada, USA, in December 2016. An extension considering also a bounded process noise was obtained in [2], which was presented at the 20th World Congress of the International Federation of Automatic Control in July 2017.

In order to render this chapter straightforward even for readers not familiar with the mathematical stability theory, we first recall in Section 3.1 the basic concepts of asymptotic stability for dynamical systems without switching, and mean square stability, exponential mean square stability and stochastic stability for Markov jump linear systems.

In Section 3.1, we also provide a motivational example showing that the conditions guaranteeing the stability of MJLSs with known time-invariant transition probabilities between the operational modes are not enough to ensure stability in case the transition probability matrices are uncertain and time-varying, evincing a reader that perturbations on values of transition probabilities can make a stable

Markov jump linear system unstable.

Then, in Section 3.2 we present necessary and sufficient conditions for the mean square stability of autonomous noiseless MJLSs with polytopic time-inhomogeneous transition probabilities. These conditions are based on the notion of joint spectral radius of a set of matrices. We then show that the presented necessary and sufficient conditions ensure also the exponential mean square stability and stochastic stability.

In Section 3.3 we extend such results by deriving necessary and sufficient conditions for robust mean square stability of a discrete-time time-inhomogeneous MJLSs affected not only by polytopic time-inhomogeneous uncertainties on transition probabilities but also by bounded disturbances. In addition, we provide an illustrative example that shows that the presence of the bounded process noise does not change the considerations on stability in relation to the joint spectral radius.

Finally, in Section 3.4 we introduce the definitions of mean square stabilizability and mean square detectability, together with respective conditions to test them, which extend the notions of stability to controlled systems, where possibly not all the state variables may be available to the controller.

3.1 Autonomous systems and stability

In order to recollect the concept of asymptotic stability, let us consider the following difference equations

$$\mathrm{x}_{k+1} = f(\mathrm{x}_k) \tag{3.1}$$

$$\mathrm{x}_{k+1} = A\mathrm{x}_k \tag{3.2}$$

with $k \in \mathbb{T}$, $\mathrm{x}_k \in \mathbb{F}^{n_\mathrm{x}}$, $f: \mathbb{F}^{n_\mathrm{x}} \to \mathbb{F}^{n_\mathrm{x}}$, and $A \in \mathbb{F}^{n_\mathrm{x}, n_\mathrm{x}}$.

The sequence (x_k) generated according to one of the previous difference equations is called a *trajectory* of the dynamical system. It describes the evolution of a dynamical system, as time passes by.

The second equation is a particular case of the first one and is of greater interest to us, since it represents a particular case of a Markov jump linear system, when there is only one operational mode. Equation (3.2) defines a so-called *discrete-time linear time-invariant system*. Notably, this system is also *autonomous*, since there is no control input, and *noiseless*, because there is no process noise representing the discrepancies between the model and the real process, due to unmodeled dynamics or disturbances, for instance. For more information on dynamical systems with only one operational mode, an interested reader may refer for eaxample to [91–93].

A point $\boldsymbol{x}_e \in \mathbb{F}^{n_\mathrm{x}}$ is called an *equilibrium point* of a dynamical system described by (3.1) if $f(\boldsymbol{x}_e) = \boldsymbol{x}_e$. Notably, $\boldsymbol{x}_e = 0$ is an equilibrium point of a dynamical system represented by (3.2). As usual, we denote by $\mathbb{R}_+$ the set of positive real numbers, i.e., $\{i \in \mathbb{R} : i > 0\}$, by $\|\cdot\|$ any norm in $\mathbb{F}^{n_\mathrm{x}}$, and by $\boldsymbol{x}_0$ the initial state of the system, that is, $\mathrm{x}_0 = \boldsymbol{x}_0$. Then, *Lyapunov stability* of a dynamical system as in (3.1) or in (3.2) is defined as follows.

Definition 3.1. *An equilibrium point $\boldsymbol{x}_e$ is said to be* ***stable in the sense of Lyapunov*** *if for each $\epsilon \in \mathbb{R}_+$ there exists $\delta_\epsilon \in \mathbb{R}_+$ such that $\|\mathrm{x}_k - \boldsymbol{x}_e\| \leq \epsilon$ for all $k \in \mathbb{T}$ whenever $\|\boldsymbol{x}_0 - \boldsymbol{x}_e\| \leq \delta_\epsilon$.*

A stronger version of Lyapunov stability is the (global) *asymptotic stability*, which is formally defined as follows.

Definition 3.2. *An equilibrium point $\boldsymbol{x}_e$ is said to be* ***asymptotically stable*** *if it is stable in the sense of Lyapunov and there exists $\delta \in \mathbb{R}_+$ such that whenever $\|\boldsymbol{x}_0 - \boldsymbol{x}_e\| \leq \delta$ we have that $\mathrm{x}_k \to \boldsymbol{x}_e$ as k increases. It is* ***globally asymptotically stable*** *if it is asymptotically stable and $\mathrm{x}_k \to \boldsymbol{x}_e$ as k increases for any $\boldsymbol{x}_0$ in the state space.*

The definition above simply states that the equilibrium point is stable if, given any spherical region surrounding the equilibrium point, we can find another spherical region surrounding the equilibrium point such that trajectories starting inside this second region do not leave the first one. Besides, if the trajectories also converge to this equilibrium point, then it is asymptotically stable [12].

A classical result in Lyapunov stability theory states that $\boldsymbol{x}_e = 0$ *is the only globally asymptotically stable equilibrium point for a discrete-time linear time-invariant system* (3.2) and *there exists a globally asymptotically stable equilibrium point for the system* (3.2) *if and only if* $\rho(A) < 1$, where $\rho(\cdot)$ denotes a spectral radius of a square matrix. A proof of this statement can be found for instance in [91]. For a definition and some properties of a spectral radius, see appendix's Section B.4.

The above result was extended to Markov jump linear systems through the notion of mean square stability, as presented in book written by Costa et al. [12].

In the aforementioned book, the authors first of all show on a simple numerical example [12, pp. 4–6] that in case of stochastic switching between two operational modes, one of which is stable and other not, some trajectories may be unstable while others tend to zero as k increases. They also show that, depending on transition probabilities between operational modes, a system having all operational modes stable, may be unstable, and that a system with unstable operational modes may be stable [12, pp. 37–41], underlining a connection between the mean square stability (introduced next) and the probability of visits to the unstable modes.

Let us consider an autonomous discrete-time Markov jump linear system described by the following state-space model defined on a stochastic basis introduced in Subsection 2.2, i.e., $(\Omega_\mathrm{x}, \mathcal{G}, (\mathcal{G}_k), \mathrm{Pr})$:

$$\begin{cases} \mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k + H_{\theta_k}\mathrm{v}_k, \\ \quad \mathrm{x}_0 = \boldsymbol{x}_0,\ \theta_0 = \vartheta_0 \end{cases} \tag{3.3}$$

where, as presented in Section 2.1, x_k is a column vector of n_x either real or complex *state* variables, $\mathrm{v}_k \in \mathbb{F}^{n_\mathrm{v}}$ is a vector of *process noise* variables, $\boldsymbol{A} \triangleq (A_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_\mathrm{x}, n_\mathrm{x}}$ is a sequence of *state matrices*, and $\boldsymbol{H} \triangleq (H_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_\mathrm{x}, n_\mathrm{v}}$ is a sequence of *process noise matrices*, each of which is associated to an operational mode of the (switching) system; the values of $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ and $\theta_0 \in \Theta_0$, i.e., $\boldsymbol{x}_0 \in \mathbb{F}^{n_\mathrm{x}}$ and $\vartheta_0 \in \mathbb{M}$, respectively, represent the initial conditions.

As in the previous chapter, let us denote by $\mathbb{E}(\cdot)$ the expected value of a random variable, and by $\|\cdot\|$ either any vector norm or any matrix norm. See appendix's Sections B.3 and B.4 for additional details on equivalent norms. Then, the *mean square stability* of a system (3.3) is defined as follows [12, p. 36–37].

Definition 3.3. *A Markov jump linear system* (3.3) *is* ***mean square stable*** *if for any initial condition* $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ *and* $\theta_0 \in \Theta_0$ *there exist* $\boldsymbol{x}_e \in \mathbb{F}^{n_\mathrm{x}}$ *and* $Q_e \in \mathbb{F}_+^{n_\mathrm{x}, n_\mathrm{x}}$ *(independent from initial conditions* x_0 *and* θ_0*), such that*

$$\lim_{k\to\infty} \|\mathbb{E}(\mathrm{x}_k) - \boldsymbol{x}_e\| = 0, \tag{3.4a}$$

$$\lim_{k\to\infty} \|\mathbb{E}(\mathrm{x}_k \mathrm{x}_k^*) - Q_e\| = 0 \tag{3.4b}$$

Remark 3.1. It is worth mentioning [12, p. 37, Remark 3.10] that in noiseless case, i.e., when $\mathrm{v}_k = 0$ in (3.3), the conditions (3.4) defining mean square stability become

$$\lim_{k\to\infty} \mathbb{E}(\mathrm{x}_k) = 0, \qquad \lim_{k\to\infty} \mathbb{E}(\mathrm{x}_k \mathrm{x}_k^*) = 0 \tag{3.5}$$

There exist also other forms of stability for Markov jump linear systems without process noise, notably *exponential mean square stability* (as of now, EMSS) and *stochastic stability* (sometimes abbreviated to SS), that we define as follows.

Definition 3.4. *An MJLS* (3.3) *is* ***exponentially mean square stable*** *if for some reals* $\beta \geq 1$, $0 < \zeta < 1$, *we have for all initial conditions* $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ *and* $\theta_0 \in \Theta_0$ *that, for every* $k \in \mathbb{T}$, *if* $\mathrm{v}_k = 0$, *then*

$$\mathbb{E}\Big(\|\mathrm{x}_k\|^2\Big) \leq \beta \zeta^k \|\mathrm{x}_0\|_2^2 \tag{3.6}$$

We observe that $\|\cdot\|_2$ denotes the Euclidean norm, also known as $\mathbb{L}^2$-norm or simply 2-norm. See appendix's Section B.3 for additional details.

Definition 3.5. *A Markov jump linear system* (3.3) *is* ***stochastically stable*** *if for all initial conditions* $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ *and* $\theta_0 \in \Theta_0$, *we have that, if* $\mathrm{v}_k = 0$ *for every* $k \in \mathbb{T}$, *then*

$$\sum_{k=0}^{\infty} \mathbb{E}\Big(\|\mathrm{x}_k\|^2\Big) \leq \infty \tag{3.7}$$

In *time-homogeneous* case, i.e., when the transition probability matrix defined by (2.6)–(2.8), is such that $P(k) = P$ for all $k \in \mathbb{T}$, there is a condition based on a value of a spectral radius of a matrix associated to the second moment of x_k that is necessary and sufficient for the mean square stability of the system (3.3); furthermore, in noiseless setting, MSS, EMSS and SS are equivalent [12, pp. 36–44].

Specifically, the matrix related to the second moment of x_k that we have mentioned above is

$$\Lambda \triangleq \left(P^T \otimes I_{n_\mathrm{x}^2}\right) \left(\bigoplus_{i=1}^{N} \left(\bar{A}_i \otimes A_i\right)\right) \tag{3.8}$$

where $\otimes$ denotes the Kronecker product, $I_{n_\mathrm{x}^2}$ is the identity matrix of size n_x^2, and the direct sum $\oplus$ of the manipulated elements of a sequence of state matrices $\boldsymbol{A}$ produces a block diagonal matrix, having the matrices $\left(\bar{A}_i \otimes A_i\right)$ on the main diagonal blocks. See appendix's Sections B.4 and B.4 for additional information.

The necessary and sufficient condition for the mean square stability of time-homogeneous Markov jump linear systems we have hinted at before is

$$\rho(\Lambda) < 1 \tag{3.9}$$

This condition for mean square stability does not hold in time-inhomogeneous case, as we show in the following example.

Motivational example

To motivate the necessity of studying the characteristics of Markov jump linear systems with dynamic perturbations on values in transition probability matrix, let us consider an autonomous noiseless system satisfying Assumption 2.2. In the following example we show that having the spectral radius smaller than one for each matrix Λ associated to the second moment of x_k (computed for each vertex of polytope bounding the variations in transition probabilities between operational modes) is not enough to ensure the (mean square) stability of the time-inhomogeneous system.

Let us consider a noiseless autonomous MJLS with $N = 3$ operational modes, where the state matrices associated with the operational modes are

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1.2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1.13 & 0 \\ 0.16 & 0.48 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0.3 & 0.13 \\ 0.16 & 1.14 \end{bmatrix}$$

and the time-varying transition probability matrix $P(k)$ is uncertain and belongs to a polytope with $V = 2$ vertices

$$P_1 = \begin{bmatrix} 0 & 0.35 & 0.65 \\ 0.6 & 0.4 & 0 \\ 0.4 & 0.6 & 0 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0.25 & 0.75 & 0 \\ 0 & 0.6 & 0.4 \\ 0 & 0.4 & 0.6 \end{bmatrix}$$

Then, any transition probability matrix within a polytope is represented by

$$P(k) = \lambda(k) P_1 + (1 - \lambda(k)) P_2, \qquad 0 \leq \lambda(k) \leq 1$$

Let us consider, for instance, also the matrix

$$P' = 0.5\ P_1 + 0.5\ P_2 = \begin{bmatrix} 0.125 & 0.55 & 0.325 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$$

The spectral radii ρ of the matrices Λ are:

$$\rho(\Lambda_1) = 0.901601, \quad \rho(\Lambda_2) = 0.905686, \quad \rho(\Lambda') = 0.937965$$

Thus, the time-homogeneous Markov jump linear systems with transition probability matrices P_1, P_2 and P' are mean square stable.

In order to present this result visually, we report one possible dynamical behavior of the system, obtained for $\mathrm{x}_0 = [100; 85]$ and the initial probability distribution $\mathrm{p}_0 = [0.33, 0.34, 0.33]^T$.

Figure 3.1 shows us a trajectory of the system state vector when the time-homogeneous transition probability matrix is P_1.

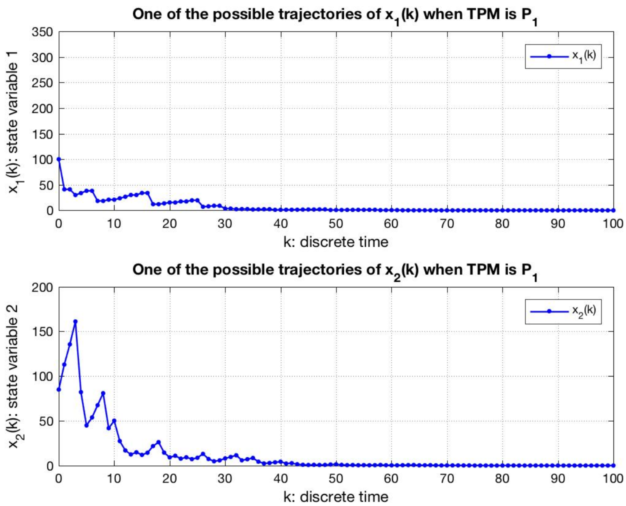

Figure 3.1: One of the possible trajectories of x_k when TPM is P_1

Repeating the trial under the same conditions, except that the considered transition probability matrix being P_2, one obtains a system's state trajectory presented in Figure 3.2.

However, repeating the experiment again under the same conditions, except that the transition probability matrix being allowed to switch arbitrarily between P_1 and P_2 at each time step, brings to light a trajectory shown in Figure 3.3. The considered Markov jump linear system is clearly unstable.

In Section 3.2 we present a condition, proved to be necessary and sufficient, for the mean square stability of polytopic time-inhomogeneous Markov jump linear systems. The condition is based on the generalization of the notion of spectral radius to sets of matrices. This generalization is known as joint spectral radius (or JSR). Our condition requires the JSR, denoted by $\hat{\rho}(\cdot)$, of all matrices Λ_l, $l \in \mathbb{V}$,

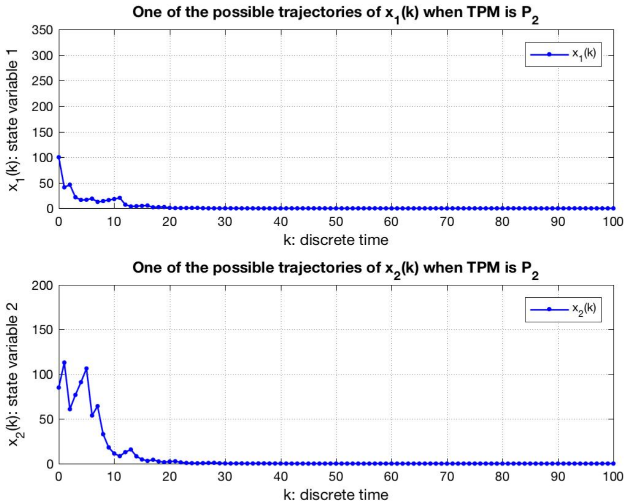

Figure 3.2: One of the possible trajectories of x_k when TPM is P_2

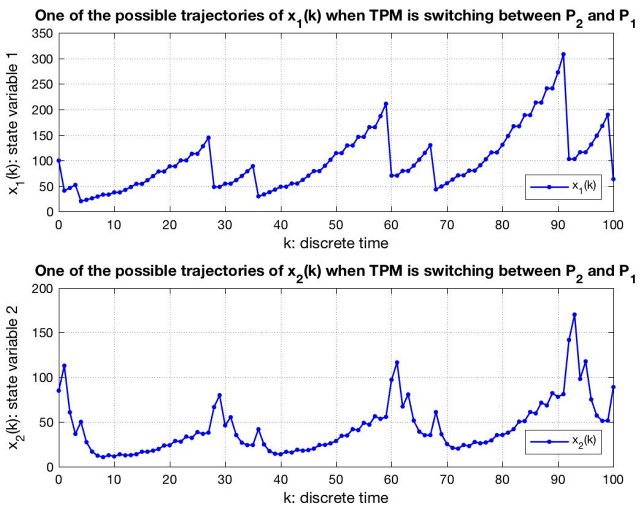

Figure 3.3: Trajectory of x_k when TPM is switching between P_1 and P_2

associated to the second moment of state vector x_k, to be smaller than one. For the set of matrices in example we have instead that

$$\hat{\rho}(\{\Lambda_l : l \in \mathbb{V}\}) = \hat{\rho}(\{\Lambda_1, \Lambda_2\}) \geq 1.024442$$

The value of the joint spectral radius has been computed via JSR toolbox [94].

Standard notational conventions

We have seen in Section 3.1 that different definitions of stability of MJLSs are all based on some properties of the system's state vector x_k. However, it is easy to see from the state-space model of the system (3.3) that the stochastic process $\{\mathrm{x}_k : k \in \mathbb{T}\}$ considered alone does not satisfy the Markov property (B.82). The stochastic process that satisfies the aforementioned property is instead a joint process $\{(\theta_k, \mathrm{x}_k) : k \in \mathbb{T}\}$. Thus, it is a common practice in analysis of Markov jump linear systems to use the indicator function (B.1) defined on a relevant σ-algebra (i.e., $\mathcal{G}$ in the considered case) to take advantage of the Markov property also for system's state x_k. The values $i \in \mathbb{M}$ of the jump variable θ_k are all measurable elementary events on $\mathcal{G}$. As a consequence, the indicator function $\mathbf{1}_{\{\theta_k = i\}}$ is such that, for any $\omega \in \Omega_{\mathrm{x}}$, one has

$$\mathbf{1}_{\{\theta_k = i\}}(\omega) = \begin{cases} 1 & \text{if } \theta_k(\omega) = i \\ 0 & \text{otherwise} \end{cases} \tag{3.10}$$

The indicator function $\mathbf{1}_{\{\theta_k = i\}}$ allows us to obtain recursive difference equations for the first and second moments of the system's state, which are fundamental in deriving our result on stability.

Let us denote by $\mathbb{F}_0^{n,n}$ a set of all positive semi-definite matrices of order n with entries in $\mathbb{F}$, and by ${}_{\mathrm{N}}\mathbb{F}_0^{n,n}$ the set of all N-sequences of square matrices in $\mathbb{F}_0^{n,n}$.

Then, following the standard workflow for Markov jump linear systems [12, p. 31], we use the subsequent notation, where $k \in \mathbb{T}$ and $i \in \mathbb{M}$.

$$\mathrm{q}_i(k) \triangleq \mathbb{E}\big(\mathrm{x}_k \mathbf{1}_{\{\theta_k = i\}}\big) \in \mathbb{F}^{n_{\mathrm{x}}} \tag{3.11}$$

$$\mathrm{q}(k) \triangleq \begin{bmatrix} \mathrm{q}_1(k) \\ \vdots \\ \mathrm{q}_N(k) \end{bmatrix} \in \mathbb{F}^{N n_{\mathrm{x}}}$$

$$\mathrm{r}_i(k) \triangleq \mathbb{E}\big(\mathrm{v}_k \mathbf{1}_{\{\theta_k = i\}}\big) \in \mathbb{F}^{n_{\mathrm{v}}} \tag{3.12}$$

$$\mathrm{r}(k) \triangleq \begin{bmatrix} \mathrm{r}_1(k) \\ \vdots \\ \mathrm{r}_N(k) \end{bmatrix} \in \mathbb{F}^{N n_{\mathrm{v}}}$$

$$Q_i(k) \triangleq \mathbb{E}\big(\mathrm{x}_k \mathrm{x}_k^* \mathbf{1}_{\{\theta_k = i\}}\big) \in \mathbb{F}_0^{n_{\mathrm{x}}, n_{\mathrm{x}}} \tag{3.13}$$

$$\boldsymbol{Q}(k) \triangleq (Q_i(k))_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}_0^{n_{\mathrm{x}},n_{\mathrm{x}}} \tag{3.14}$$

$$R_i(k) \triangleq \mathbb{E}\big(\mathrm{v}_k\mathrm{v}_k^*\mathbf{1}_{\{\theta_k=i\}}\big) \in \mathbb{F}_0^{n_{\mathrm{v}},n_{\mathrm{v}}} \tag{3.15}$$

$$\boldsymbol{R}(k) \triangleq (R_i(k))_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}_0^{n_{\mathrm{v}},n_{\mathrm{v}}} \tag{3.16}$$

$$\boldsymbol{H}\boldsymbol{R}(k)\boldsymbol{H}^* \triangleq (H_iR_i(k)H_i^*)_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}_0^{n_{\mathrm{x}},n_{\mathrm{x}}} \tag{3.17}$$

$$W_i(k) \triangleq \mathbb{E}\big(\mathrm{x}_k\mathrm{v}_k^*\mathbf{1}_{\{\theta_k=i\}}\big) \in \mathbb{F}^{n_{\mathrm{v}},n_{\mathrm{x}}}, \tag{3.18}$$

$$\boldsymbol{W}(k) \triangleq (W_i(k)) \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{v}},n_{\mathrm{x}}} \tag{3.19}$$

$$\boldsymbol{A}\boldsymbol{W}(k)\boldsymbol{H}^* \triangleq (A_iW_i(k)H_i^*) \in {}_{\mathrm{N}}\mathbb{F}_0^{n_{\mathrm{x}},n_{\mathrm{x}}} \tag{3.20}$$

This permits us to define the *expected value* of x_k as

$$\mathbb{E}(\mathrm{x}_k) = \sum\nolimits_{i=1}^{N} \mathrm{q}_i(k) \in \mathbb{F}^{n_{\mathrm{x}}} \tag{3.21}$$

and the *second moment* of x_k as

$$\mathbb{E}(\mathrm{x}_k\mathrm{x}_k^*) = \sum\nolimits_{i=1}^{N} Q_i(k) \in \mathbb{F}_0^{n_{\mathrm{x}},n_{\mathrm{x}}} \tag{3.22}$$

The expressions of the first and second moment of x_k above can be easily derived from the definition of the expected value (B.78) and from (3.10).

This notation is used throughout the rest of the chapter to derive the necessary and sufficient conditions for mean square stability of polytopic time-inhomogeneous MJLSs as in (3.3).

3.2 Stability conditions in noiseless setting

The results of this section are based on a noiseless version of (3.3), i.e., when $\mathrm{v}_k = 0$ for every $k \in \mathbb{T}$. They are based on our first work on MJLSs [1].

Let us consider a noiseless autonomous discrete-time Markov jump linear system defined on a stochastic basis $(\Omega_{\mathrm{x}}, \mathcal{G}, (\mathcal{G}_k), \Pr)$ and described by the following system of difference equations

$$\begin{cases} \mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k, \\ \mathrm{x}_0 = \boldsymbol{x}_0,\ \theta_0 = \vartheta_0 \end{cases} \tag{3.23}$$

where, as before, $\mathrm{x}_k \in \mathbb{F}^{n_{\mathrm{x}}}$ is a system's state vector, $\boldsymbol{A} \triangleq (A_i)_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{x}},n_{\mathrm{x}}}$ is a sequence of *state matrices*, each of which is associated to an operational mode; while $\mathrm{x}_0 \in \mathcal{H}_0^{n_{\mathrm{x}}}$ and $\theta_0 \in \Theta_0$ are initial conditions.

Let the transition probability matrix $P(k) = [p_{ij}(k)]$ of the system (3.23) be polytopic time-inhomogeneous, i.e., satisfying Assumption 2.2. Then, we can easily see that the recursive equations for $\mathrm{q}_i(k)$ and $Q_i(k)$ defined by (3.11) and by (3.13), respectively, have the same structure as their counterpart in the time-homogeneous case with known probability matrix [12, p. 32]. The extension to this more general time-varying case is straightforward, but the formal proof requires an additional lemma on inequality between trace of any positive semi-definite matrix and any matrix norm.

Lemma 3.1. *For any $Q \in \mathbb{F}_0^{n,n}$ we have the following inequality*

$$\operatorname{tr}(Q) \leq n \|Q\| \tag{3.24}$$

Proof. The proof is based on the relationship between the trace, eigenvalues and the spectral radius $\rho(Q)$ of positive semi-definite matrices. Since Q is positive semi-definite, all its eigenvalues are nonnegative real numbers. Thus, from the definition (B.2) of the absolute value for real numbers, the property (B.39) of the trace of being the sum of all the eigenvalues of a square matrix, and definition (B.40) of the spectral radius as the largest absolute value of the eigenvalues, we have that

$$\operatorname{tr}(Q) \leq n\rho(Q)$$

Then, let $\mathrm{v} \in \mathbb{F}^n$ be the eigenvector associated to the maximal eigenvalue $\nu_{\max}$ of Q, which for both real and complex-valued positive semi-definite matrices equals to $\rho(Q)$. By definition of the eigenvalue provided in (B.36), we have that

$$Q\mathrm{v} = \nu_{\max}\mathrm{v}$$

By absolute homogeneity (B.4) of any vector norm and triangle inequality (B.43) of any matrix norm, we have for any $\nu_{\max} \in \mathbb{R}_0$ that

$$\nu_{\max} \|\mathrm{v}\| = |\nu_{\max}| \, \|\mathrm{v}\| = \|\nu_{\max}\mathrm{v}\| = \|Q\mathrm{v}\| \leq \|Q\| \, \|\mathrm{v}\|$$

Thus, $\rho(Q) = |\nu_{\max}| \leq \|Q\|$. Together with the first equation in the proof, this implies the thesis, and the lemma is proved. □

Now we can present the aforementioned recursive equations for $\mathrm{q}_i(k)$ and $Q_i(k)$.

Proposition 3.2. *Consider the system* (3.23). *For all $k \in \mathbb{T}$, $j \in \mathbb{M}$, we have that*

$$\mathrm{q}_j(k+1) = \sum\nolimits_{i=1}^{N} p_{ij}(k) A_i \mathrm{q}_i(k) \tag{3.25}$$

$$Q_j(k+1) = \sum\nolimits_{i=1}^{N} p_{ij}(k) A_i Q_i(k) A_i^* \tag{3.26}$$

$$\mathbb{E}\Big(\|\mathrm{x}_k\|_2^2\Big) = \mathbb{E}\left(\left\| \left(\prod\nolimits_{i=0}^{k-1} A_{\theta(i)}^*\right)^* \mathrm{x}_0 \right\|_2^2\right) \leq n_{\mathrm{x}} \|\boldsymbol{Q}(k)\|_1 \tag{3.27}$$

Proof. Regarding the first statement, we have that

$$
\begin{aligned}
\mathrm{q}_j(k+1) &= \mathbb{E}\big(\mathrm{x}_{k+1}\mathbf{1}_{\{\theta_{k+1}=j\}}\big) && \text{by (3.11)}\\
&= \mathbb{E}\big(A_{\theta_k}\mathrm{x}_k\mathbf{1}_{\{\theta_{k+1}=j\}}\big) && \text{by (3.23)}\\
&= \sum\nolimits_{i=1}^{N} \mathbb{E}\big(A_i\mathrm{x}_k\mathbf{1}_{\{\theta_k=i\}}\mathbf{1}_{\{\theta_{k+1}=j\}}\big) && \text{by (B.78) and (3.10)}\\
&= \sum\nolimits_{i=1}^{N} A_i\mathbb{E}\big(\mathrm{x}_k\mathbf{1}_{\{\theta_k=i\}}\mathbf{1}_{\{\theta_{k+1}=j\}}\big) && \text{by linearity of expected value}\\
&= \sum\nolimits_{i=1}^{N} A_i\mathbb{E}\big(\mathrm{x}_k\mathbf{1}_{\{\theta_k=i\}}\Pr\{\theta_{k+1}=j \mid \mathcal{G}_k\}\big) && \text{by (2.5) and (3.10)}\\
&= \sum\nolimits_{i=1}^{N} A_i\mathbb{E}\big(\mathrm{x}_k\mathbf{1}_{\{\theta_k=i\}}\big)\, p_{ij}(k) && \text{by (B.78) and (2.6)}\\
&= \sum\nolimits_{i=1}^{N} A_i\mathrm{q}_i(k)p_{ij}(k) && \text{by (3.11)}
\end{aligned}
$$

which proves the first result.

Regarding the second statement, we have from the definition (3.13) of the matrix $Q_i(k)$ and exactly the same considerations made before, that

$$
Q_j(k+1) = \sum\nolimits_{i=1}^{N} \mathbb{E}\big(A_i\mathrm{x}_k\,(A_i\mathrm{x}_k)^*\,\mathbf{1}_{\{\theta_k=i\}}\mathbf{1}_{\{\theta_{k+1}=j\}}\big) = \sum\nolimits_{i=1}^{N} A_iQ_i(k)A_i^* p_{ij}(k)
$$

By commutative property of scalar multiplication on either real or complex linear spaces, this implies the thesis.

For what concerns the third statement, the equality, stated in the compact form (B.26) of the matrix product in reverse order, comes from the repeated applications of the recursive equation (3.23) describing the evolution of the system's state x_k, while the inequality is derived as follows.

$$
\begin{aligned}
\mathbb{E}\Big(\|\mathrm{x}_k\|_2^2\Big) &= \sum\nolimits_{i=1}^{N} \mathbb{E}\Big(\|\mathrm{x}_k\|_2^2\,\mathbf{1}_{\{\theta_k=i\}}\Big) && \text{from (B.78) and (3.10)}\\
&= \sum\nolimits_{i=1}^{N} \mathbb{E}\big(\mathrm{tr}(\mathrm{x}_k\mathrm{x}_k^*)\,\mathbf{1}_{\{\theta_k=i\}}\big) && \text{from (B.33) and (B.10)}\\
&= \sum\nolimits_{i=1}^{N} \mathrm{tr}\big(\mathbb{E}\big(\mathrm{x}_k\mathrm{x}_k^*\mathbf{1}_{\{\theta_k=i\}}\big)\big) && \text{from the linearity of } \mathrm{tr}(\cdot) \text{ and } \mathbb{E}(\cdot)\\
&= \sum\nolimits_{i=1}^{N} \mathrm{tr}(Q_i(k)) && \text{from (3.13)}\\
&= \mathrm{tr}\Big(\sum\nolimits_{i=1}^{N} Q_i(k)\Big) && \text{by linearity of the trace}\\
&\le n_{\mathrm{x}}\,\Big\|\sum\nolimits_{i=1}^{N} Q_i(k)\Big\| && \text{from (3.24)}\\
&\le n_{\mathrm{x}}\sum\nolimits_{i=1}^{N} \|Q_i(k)\| && \text{by triangle inequality (B.43)}\\
&= n_{\mathrm{x}}\,\|\boldsymbol{Q}(k)\|_1 && \text{by definition of 1-norm in (B.54)}
\end{aligned}
$$

The proposition is proved. □

Similarly to the time-homogeneous case [12, pp. 33-35], also here, via application of Proposition 3.2 describing through (3.26) the dynamics of the matrices $Q_i(k)$, the definition (B.55) of the linear operator $\mathrm{vec}^2(\cdot)$, and the related definition (B.29) of the linear mapping $\mathrm{vec}(\cdot)$, the properties (B.30) and (B.31) of the Kronecker product (described in appendix's Subsection B.4) to $\boldsymbol{Q}(k)$, defined by (3.14), we have that

$$\mathrm{vec}^2(\boldsymbol{Q}(k+1)) = \Lambda(k)\mathrm{vec}^2(\boldsymbol{Q}(k)) \tag{3.28}$$

where $\Lambda(k)$ is a time-varying version of (3.8), i.e.,

$$\Lambda(k) \triangleq \left(P^T(k) \otimes I_{n_\mathrm{x}^2}\right) \left(\bigoplus_{i=1}^{N} \left(\bar{A}_i \otimes A_i\right)\right) \tag{3.29}$$

Proposition 3.3. *The matrix $\Lambda(k)$ is polytopic, i.e., $\forall k \in \mathbb{T}$*

$$\Lambda(k) = \sum_{l=1}^{V} \lambda_l(k)\Lambda_l, \ \lambda_l(k) \geq 0, \quad \sum_{l=1}^{V} \lambda_l(k) = 1 \tag{3.30a}$$

$$\Lambda_l \triangleq \left(P_l^T \otimes I_{n_\mathrm{x}^2}\right) \left(\bigoplus_{i=1}^{N} \left(\bar{A}_i \otimes A_i\right)\right) \tag{3.30b}$$

where for each $l \in \mathbb{V}$, $P_l \in {}_{\mathbb{V}}\mathbb{P} \subset {}_{\mathbb{V}}\mathbb{R}^{N,N}$, i.e, P_l are elements of a given finite set of transition probability matrices, which are the vertices of a convex polytope.

Proof. The result follows from Assumption 2.2 on time-varying unmeasurable transition probability matrix $P(k)$ of being polytopic, by direct application of the related equation (2.17) and (bi-)linearity (B.31) of the Kronecker product, to the definition (3.29) of the matrix $\Lambda(k)$. □

Similarly to ${}_{\mathbb{V}}\mathbb{P}$ defined by (2.16), let us indicate by ${}_{\mathbb{V}}\boldsymbol{\Lambda}$ the set of vertices of the convex polytope of the matrices $\Lambda(k)$ related to the second moment of x_k through $\boldsymbol{Q}(k)$. Formally, we write that

$${}_{\mathbb{V}}\boldsymbol{\Lambda} \triangleq \left\{\Lambda_l \in \mathbb{F}^{Nn_\mathrm{x}^2, Nn_\mathrm{x}^2} : l \in \mathbb{V}\right\} \tag{3.31}$$

Recalling the definition (B.62) of a convex hull, we can write that for each $k \in \mathbb{T}$

$$\Lambda(k) \in \mathrm{conv}_{\mathbb{V}}\boldsymbol{\Lambda} \tag{3.32}$$

It worth noting that the set of possible values of $\Lambda(k)$ is bounded, but uncountable. Also, by Assumption 2.2, the values of $\Lambda(k)$ are unmeasurable.

Anyway, the repeated applications of (3.29) gives us the following recursion:

$$\mathrm{vec}^2(\boldsymbol{Q}(k)) = \left(\prod_{t=0}^{k-1} \Lambda^*(t)\right)^* \mathrm{vec}^2(\boldsymbol{Q}(0)) \tag{3.33}$$

The previous equation will be used in the proof of our first main result, presented in the next subsection.

Conditions for the mean square stability

It is well known, that the maximal rate of growth among all products of matrices from a bounded set is given by its joint spectral radius, which is the generalization of the notion of spectral radius to sets of matrices. See appendix's Subsection B.4 for the definition and some relevant properties of the joint spectral radius, that will be used in the following theorem, which presents necessary and sufficient conditions for the mean square stability of polytopic time-inhomogeneous MJLSs.

Theorem 3.4. *The discrete-time Markov jump linear system* (3.23) *with unknown and time-varying transition probability matrix* $P(k) \in \operatorname{conv}_{\mathbb{V}}\mathbb{P}$ *is mean square stable if and only if* $\hat{\rho}({}_{\mathbb{V}}\mathbf{\Lambda}) < 1$.

Proof. We first prove the *necessity* of the presented condition for the mean square stability, i.e., MSS $\Rightarrow$ $\hat{\rho}({}_{\mathbb{V}}\mathbf{\Lambda}) < 1$.

By hypothesis (3.5), for any $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ and $\theta_0 \in \Theta_0$,

$$\lim_{k\to\infty} \mathbb{E}(\mathrm{x}_k \mathrm{x}_k^*) = 0$$

First of all, we observe that the elements of the main diagonal of the positive semi-definite matrix $\mathrm{x}_k \mathrm{x}_k^*$ are all real and nonnegative.

Formally, if we write x_k as a column vector

$$\mathrm{x}_k = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_\mathrm{x}} \end{bmatrix} \in \mathbb{F}^{n_\mathrm{x},1}$$

where we do not state explicitly the time-dependence of the values only for the sake of conciseness, then, by the definition of the conjugate transposition and the matrix multiplication, the i-th element of the main diagonal of $\mathrm{x}_k \mathrm{x}_k^*$ is $x_i \bar{x}_i$, which has always a nonnegative real value, as reminded in appendix's Subsection B.2.

In order to be able to recall this fact in what follows, we write that

$$(x_i \bar{x}_i)_{i=1}^{n_\mathrm{x}} \in \mathbb{R}_+^{n_\mathrm{x}} \tag{3.34}$$

where $\mathbb{R}_+^{n_\mathrm{x}}$ indicates an n_x-dimensional linear space, with entries in $\mathbb{R}_+$.

From the definition (3.22) of the second moment of x_k and the definition (3.13) of $Q_i(k)$, we have that

$$\lim_{k\to\infty} \sum\nolimits_{i=1}^{N} Q_i(k) = 0$$

It may be useful to write the same expression explicitly as

$$\lim_{k\to\infty} \sum\nolimits_{i=1}^{N} \mathbb{E}\big(\mathrm{x}_k \mathrm{x}_k^* \mathbf{1}_{\{\theta_k = i\}}\big) = 0$$

Since limits of sequences behave well with respect to the usual arithmetic operations, we have that

$$\sum_{i=1}^{N} \lim_{k\to\infty} \mathbb{E}\big(\mathrm{x}_k \mathrm{x}_k^* \mathbf{1}_{\{\theta_k=i\}}\big) = 0$$

From the definition (3.10) of the indicator function in a set of operational modes $\mathbb{M}$, considered together with (3.34), one has that, for each $i \in \mathbb{M}$

$$\lim_{k\to\infty} \mathbb{E}\big(\mathrm{x}_k \mathrm{x}_k^* \mathbf{1}_{\{\theta_k=i\}}\big) = \lim_{k\to\infty} Q_i(k) = 0$$

Thus, from the definition (3.14) of $\boldsymbol{Q}(k)$ follows that

$$\lim_{k\to\infty} \boldsymbol{Q}(k) = 0 \tag{3.35}$$

As explained in appendix's Subsection B.4, the linear mapping $\mathrm{vec}^2(\cdot)$ is uniform homeomorphic. As a consequence, the convergent behavior of $\boldsymbol{Q}(k)$ is preserved by $\mathrm{vec}^2(\boldsymbol{Q}(k))$, i.e.,

$$\lim_{k\to\infty} \mathrm{vec}^2(\boldsymbol{Q}(k)) = 0$$

Applying the expression (3.33) for the recursion of $\mathrm{vec}^2(\boldsymbol{Q}(k))$, we obtain that

$$\lim_{k\to\infty} \left(\prod_{t=0}^{k-1} \Lambda^*(t)\right)^* \mathrm{vec}^2(\boldsymbol{Q}(0)) = 0 \tag{3.36}$$

From Proposition 3.3, for each $k \in \mathbb{T}$

$$\Lambda(k) \in \mathrm{conv}_{\mathbb{V}} \boldsymbol{\Lambda}$$

Thus, from the proposition on the convergence of matrix products, reported in appendix's Subsection B.4 as Proposition B.2, we have that (3.36) holds for any $\boldsymbol{Q}(0)$ if and only if $\hat{\rho}(\mathrm{conv}_{\mathbb{V}} \boldsymbol{\Lambda}) < 1$.

Thus, from the proposition on the value of the joint spectral radius of a convex hull of a set of matrices, stated in appendix's Subsection B.4 as Proposition B.1, follows the thesis.

Now, let us prove that the presented condition is indeed sufficient, by showing that the mean square stability of system (3.23) is implied by $\hat{\rho}(_{\mathbb{V}} \boldsymbol{\Lambda}) < 1$.

As the definition of mean square stability (3.5) provides two requirements, one for the expected value, and other for the second moment of the system's state x_k, for k approaching infinity, the proof of sufficiency is divided in two parts.

The first part of the proof follows the inverse pattern of the proof of the necessity.

We start with the expression (3.33) for the recursion of $\mathrm{vec}^2(\boldsymbol{Q}(k))$, i.e.,

$$\mathrm{vec}^2(\boldsymbol{Q}(k)) = \left(\prod_{t=0}^{k-1} \Lambda^*(t)\right)^* \mathrm{vec}^2(\boldsymbol{Q}(0))$$

By its definition, provided by (3.14) and (3.13), $\boldsymbol{Q}(0)$ accounts for all possible initial operational modes $\theta_0 \in \mathbb{M}$; it depends only on the initial state x_0 of the system, and the initial probability distribution p_0 of all the operational modes. Thus, there always exists a $\boldsymbol{Q}(0) \in {}_{\mathrm{N}}\mathbb{F}_0^{n_\mathrm{x},n_\mathrm{x}}$ for any initial condition, represented by the values of $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ and $\theta_0 \in \Theta_0$. Besides, the matrix $\boldsymbol{Q}(0)$ accounts also for any initial probability distribution p_0. Since, by Proposition 3.3, $\Lambda(k) \in \mathrm{conv}_{\mathbb{V}}\boldsymbol{\Lambda}$ for each $k \in \mathbb{T}$, we have that

$$\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Lambda}) < 1 \Rightarrow \lim_{k\to\infty} \mathbb{E}(\mathrm{x}_k\mathrm{x}_k^*) = 0, \qquad \forall \mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}},\ \theta_0 \in \Theta_0$$

by Propositions B.1 and B.2, uniform homeomorphism between the spaces ${}_{\mathrm{N}}\mathbb{F}_0^{n_\mathrm{x},n_\mathrm{x}}$ and $\mathbb{F}_0^{Nn_\mathrm{x}^2}$ through the mapping $\mathrm{vec}^2(\cdot)$, together with the application of the definitions of matrices $\boldsymbol{Q}(k)$, $Q_i(k)$ and of the second moment of x_k, i.e., (3.14), (3.13) and (3.22), respectively.

To complete the proof, in this second part we need to show that

$$\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Lambda}) < 1 \Rightarrow \lim_{k\to\infty} \mathbb{E}(\mathrm{x}_k) = 0, \qquad \forall \mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}},\ \theta_0 \in \Theta_0$$

From the first part of the proof, we already have (3.35), i.e., that the matrix $\boldsymbol{Q}(k)$ converges to the zero matrix as $k \in \mathbb{T}$ approaches infinity.

Then, the equation (3.27) from Proposition 3.2 tells us that the value of the expected value of $\|\mathrm{x}_k\|_2^2$ is bounded by $\|\boldsymbol{Q}(k)\|_1$.

Thus, we obtain that

$$\lim_{k\to\infty} \mathbb{E}\Big(\|\mathrm{x}_k\|_2^2\Big) = 0$$

Since limits of sequences behave well with respect to the usual arithmetic operations, including multiplication, and thus, exponentiation, we have that

$$\lim_{k\to\infty} \mathbb{E}(\|\mathrm{x}_k\|_2) = 0$$

which implies the thesis and concludes the proof. □

The presented condition is very useful from theoretical point of view, but it is computationally demanding, as will be shown in the next subsection. For additional details on the topic of computational complexity in general and NP-hardness in particular, see for instance [95].

Computational complexity of the stability analysis

While it is well known that the stability analysis problem for general switching systems (that is, deciding whether the joint spectral radius is smaller than 1) is NP-hard [86], we prove in the following theorem that it is NP-hard even in our particular model.

Theorem 3.5. *Given a discrete-time Markov jump linear system* (3.23) *with unknown time-varying transition probability matrix* $P(k) \in \mathrm{conv}_{\mathbb{V}}\mathbb{P}$, *unless* $P = NP$, *there is no polynomial-time algorithm that decides whether it is mean square stable.*

Proof. Let us denote by $\mathbb{Q}$ the set of all rational numbers, and by $\mathbb{Q}_0$ the set of all nonnegative rationals, i.e., $\{i \in \mathbb{Q} : i \geq 0\}$. Let us indicate by $\mathbb{Q}^{n,n}$ the set of all square matrices of order n with entries in $\mathbb{Q}$.

Our proof works by reduction from the *matrix semigroup stability*, which is well known to be NP-hard [88, Theorem 2.4 and Theorem 2.6]. In this problem, one is given a set of two matrices $\mathbb{S}_{\mathrm{M}} = \{\mathrm{M}, \mathrm{M}'\} \subset \mathbb{Q}^{n,n}$ such that $\mathrm{M} = [m_{ij}]$, $\mathrm{M}' = [m'_{ij}]$, and for any $i, j \in \mathbb{Z}_+$, $m_{ij} \in \mathbb{Q}_0$ and $m'_{ij} \in \mathbb{Q}_0$, i.e., the entries of the two square matrices of order n are all nonnegative rational numbers. Then, one is asked whether the product of length k of any sequence of matrices M, M' converges to the zero matrix when $k \to \infty$.

Let us consider a particular instance of the matrix semigroup stability problem. With a slight abuse of notation, this instance of the problem is also indicated by

$$\mathbb{S}_{\mathrm{M}} = \{\mathrm{M} = [m_{ij}],\ \mathrm{M}' = [m'_{ij}] : \forall i, j, n \in \mathbb{Z}_+,\ i \leq n,\ j \leq n,\ m_{ij} \in \mathbb{Q}_0,\ m'_{ij} \in \mathbb{Q}_0\}$$

We will build a discrete-time autonomous noiseless MJLS (3.23) with a set of (scalar) state matrices $\{a_i \in \mathbb{R}_0 : i, n, N \in \mathbb{Z}_+,\ N = n+1,\ i \leq N\}$, with unknown and time-varying transition probability matrices $P(k) \in \mathrm{conv}\, {}_{\mathbb{V}}\mathbb{P}$, ${}_{\mathbb{V}}\mathbb{P} = \{P, P'\} \subset \mathbb{R}^{N,N}$, where $P = [p_{ij}]$ and $P' = [p'_{ij}]$ are stochastic matrices (i.e., for any $i, j \in \mathbb{Z}_+$, $p_{ij} \in \mathbb{R}_0$, $p'_{ij} \in \mathbb{R}_0$, and such that any row of these two matrices is a distribution, thus it satisfies (B.84)) and prove that the constructed Markov jump linear system (3.23) is mean square stable if and only if the set $\mathbb{S}_{\mathrm{M}}$ is stable.

By (3.30) it follows that

$$\Lambda = P^T \left(\bigoplus\nolimits_{i=1}^{N} a_i^2 \right), \qquad \Lambda' = (P')^T \left(\bigoplus\nolimits_{i=1}^{N} a_i^2 \right)$$

Explicitly,

$$\Lambda = \begin{bmatrix} p_{11} & \cdots & p_{n1} & p_{N1} \\ \vdots & \ddots & \vdots & \vdots \\ p_{1n} & \cdots & p_{nn} & p_{Nn} \\ p_{1N} & \cdots & p_{nN} & p_{NN} \end{bmatrix} \begin{bmatrix} a_1^2 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_n^2 & 0 \\ 0 & \cdots & 0 & a_N^2 \end{bmatrix}$$

Which shows us that

$$\Lambda = \begin{bmatrix} p_{11}a_1^2 & \cdots & p_{n1}a_n^2 & p_{N1}a_N^2 \\ \vdots & \ddots & \vdots & \vdots \\ p_{1n}a_1^2 & \cdots & p_{nn}a_n^2 & p_{Nn}a_N^2 \\ p_{1N}a_1^2 & \cdots & p_{nN}a_r^2 & p_{NN}a_N^2 \end{bmatrix}$$

Similarly,

$$\Lambda' = \begin{bmatrix} p'_{11}a_1^2 & \cdots & p'_{n1}a_r^2 & p'_{N1}a_N^2 \\ \vdots & \ddots & \vdots & \vdots \\ p'_{1n}a_1^2 & \cdots & p'_{nn}a_r^2 & p'_{Nn}a_N^2 \\ p'_{1N}a_1^2 & \cdots & p'_{nN}a_r^2 & p'_{NN}a_N^2 \end{bmatrix}$$

Our construction is as follows.
Assign arbitrarily for each $j \in \mathbb{Z}_+$, s.t. $j \leq n$,

$$a_j^2 \in \mathbb{Q}_0 : \quad a_j^2 \geq n \max_i \{m_{ij}, m'_{ij} : i \in \mathbb{Z}_+, i \leq n\}$$

Assign for all $i, j \in \mathbb{Z}_+$ such that $i \leq n$, $j \leq n$

$$p_{ij} \triangleq \frac{m_{ji}}{a_i^2}, \qquad p'_{ij} \triangleq \frac{m'_{ji}}{a_i^2}$$

Obviously, $p_{ij}, p'_{ij} \in \mathbb{Q}_0$, $p_{ij} \leq \frac{1}{n}$, and $p'_{ij} \leq \frac{1}{n}$.
Then, assign for every $i \in \mathbb{Z}_+$ s.t. $i \leq N$

$$p_{iN} \triangleq 1 - \sum\nolimits_{j=1}^{n} p_{ij}, \quad p'_{iN} \triangleq 1 - \sum\nolimits_{j=1}^{n} p'_{ij}$$

Clearly, by construction $p_{iN}, p'_{iN} \in \mathbb{Q}_0$, and $p_{iN} \leq 1$, $p'_{iN} \leq 1$.
As a next step, assign

$$a_N \triangleq 0$$

Finally, for each $j \in \mathbb{Z}_+$, $j \leq N$ assign

$$p_{Nj} = p'_{Nj} \triangleq \frac{1}{N}$$

As a consequence of the above assignments, it follows that P, P' are stochastic matrices and that

$$\Lambda = \begin{bmatrix} \mathrm{M} & 0 \\ \mathrm{R} & 0 \end{bmatrix}, \qquad \Lambda' = \begin{bmatrix} \mathrm{M}' & 0 \\ \mathrm{R}' & 0 \end{bmatrix}$$

with $\mathrm{R}, \mathrm{R}' \in \mathbb{Q}^{1,n}$, having nonegative elements. By Theorem 3.4, (3.23) is mean square stable if and only if the joint spectral radius of the set $\{\Lambda, \Lambda'\}$ is smaller than 1. From this, it is straightforward to see that (3.23) is mean square stable if and only if $\mathbb{S}_{\mathrm{M}}$ is stable. This concludes the proof. □

Remark 3.2. It is not known (to the best of our knowledge) whether the matrix semigroup stability problem is Turing decidable (say, for matrices with rational nonnegative entries). Thus, the above proof does not allow us to conclude that mean square stability is undecidable for Markov jump linear systems with polytopic unknown and time-varying transition probability matrices. This is why we only claim that the stability problem is NP-hard.

In the next subsection we present a theorem that links mean square stability to exponential mean square stability and to stochastic stability.

Stability equivalence

Our last but not least important result on stability of autonomous noiseless Markov jump linear systems as in (3.23) having polytopic time-inhomogeneous transition probabilities is presented in the following theorem.

Theorem 3.6. *The following assertions are equivalent.*

1. *The system* (3.23) *is mean square stable (MSS);*
2. *The system* (3.23) *is exponentially mean square stable (EMSS);*
3. *The system* (3.23) *is stochastically stable (SS).*

As explained in appendix's Subsections B.3, B.4 and B.4, we are working on a finite-dimensional linear spaces, for which all norms are *equivalent* from a topological point of view, as they induce the same topology. Thus, in the following proof we will make use of the Euclidean (B.10) and the grid (B.11) norms for vectors, ℓ_1-norm (B.49) and Frobenius norm (B.48) for matrices, and 1-norm (B.54) for sequences of matrices. The proof shown in this text is an extended version of the proof presented at the 55th IEEE Conference on Decision and Control [1]. It states explicitly all the norms we apply in different steps. It slightly differs from the proof introduced in the conference article [1] in that the proof we describing here uses ℓ_1-norm in conjunction with Frobenius norm. This permits us to simplify the presentation and improve the formal rigor without going into presentation of the operations on the matrices at the level of elements. We invite an interested reader to see [96, Section 5.6, pp. 340–370] for a detailed presentation of the topic of matrix norms, especially as a reference for the exact constants used in the inequalities (B.9) involving the equivalent norms.

Proof. It is trivially verified that the second assertion in the statement of the theorem implies the third one, i.e., EMSS $\Rightarrow$ SS. The result follows directly from the definitions of exponential mean square stability (3.6) and stochastic stability (3.7).

Thus, let us show that the third statement implies the first, that is, SS $\Rightarrow$ MSS.

We have already seen in the proof of (3.27) in Proposition 3.2 that from the definition of the Euclidean norm for vectors (B.10) and the definition of trace (B.33) as a linear mapping (together with the definition of the matrix product (B.22) and linearity of the expected value described by (B.78)), one obtains that

$$\mathbb{E}\Big(\|\mathrm{x}_k\|_2^2\Big) = \mathbb{E}(\mathrm{tr}(\mathrm{x}_k\mathrm{x}_k^*)) = \mathrm{tr}(\mathbb{E}(\mathrm{x}_k\mathrm{x}_k^*)) = \mathbb{E}(\mathrm{x}_k^*\mathrm{x}_k) \geq 0 \tag{3.37}$$

In appendix's Subsection B.3 it is explained that the absolute convergence of a series in a normed linear space implies the convergence of a series in the same space, and thus (B.7). So, from the definition of the stochastic stability (3.7) and (3.37), one has that, for all initial conditions $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ and $\theta_0 \in \Theta_0$

$$\lim_{k\to\infty} \mathrm{tr}(\mathbb{E}(\mathrm{x}_k\mathrm{x}_k^*)) = \lim_{k\to\infty} \mathbb{E}(\mathrm{x}_k^*\mathrm{x}_k) = 0$$

As a consequence (since $\mathrm{x}_k\mathrm{x}_k^*$ defines a positive semi-definite matrix, for which, by [96, Corollary 7.1.5, p. 430], $\mathrm{tr}(\mathrm{x}_k\mathrm{x}_k^*)=0$ if and only if $\mathrm{x}_k\mathrm{x}_k^*=0$), this implies

$$\lim_{k\to\infty}\mathbb{E}(\mathrm{x}_k\mathrm{x}_k^*)=0$$

We have already seen in the proof of the sufficiency of Theorem 3.4 that this last statement implies mean square stability of the system (3.23). Hence, this part of the proof is concluded.

Moreover, as a result, we have also that EMSS $\Rightarrow$ MSS.

Now, let us show that the opposite is true as well, that is, MSS $\Rightarrow$ EMSS.

From Theorem 3.4 we know that if the system (3.23) is mean square stable, then $\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Lambda})<1$. Since from the definition of the joint spectral radius

$$\lim_{k\to\infty}\left\|\left(\prod_{t=0}^{k-1}\Lambda^*(t)\right)^*\right\|^{\frac{1}{k}}\leq\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Lambda})$$

by the *radical test* for infinite series (a.k.a., *Cauchy root test*) we state that

$$\left\|\left(\prod_{t=0}^{k-1}\Lambda^*(t)\right)^*\right\|<\zeta^k,\quad\forall k\geq k',\quad\forall\zeta\in\mathbb{R}_+:\zeta\in(\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Lambda}),1)$$

for some integer $k'\geq 0$. With

$$\beta'=\zeta^{-k'}\sup_{\mathrm{P}\in\mathcal{P}_j(_{\mathbb{V}}\boldsymbol{\Lambda}),\ 0\leq j\leq k'}\|\mathrm{P}\|,\quad\beta'\geq 1$$

where $\mathcal{P}_j(_{\mathbb{V}}\boldsymbol{\Lambda})$ indicates the set of all possible products of length j whose factors are elements of $_{\mathbb{V}}\boldsymbol{\Lambda}$, as formally defined by (B.63). So, we obtain that

$$\left\|\left(\prod_{t=0}^{k-1}\Lambda^*(t)\right)^*\right\|\leq\beta'\zeta^k,\quad\forall k\in\mathbb{T}\tag{3.38}$$

Now, in the proof of (3.27) in Proposition 3.2 we have seen that

$$\mathbb{E}\Big(\|\mathrm{x}_k\|_2^2\Big)\leq n_{\mathrm{x}}\sum_{i=1}^{N}\|Q_i(k)\|\tag{3.39}$$

To proceed with our proof, we use the ℓ_1-norm (B.49), denoted by $\|\cdot\|_{\mathbf{1}}$, as the particular norm for $Q_i(k)$, which we are going to examine next.

Following the notation presented in appendix's Subsection B.4, we indicate by

$$\Lambda_{[(i-1)n_{\mathrm{x}}^2+1,\,in_{\mathrm{x}}^2]\bullet}$$

a matrix obtained by taking n_{x} consecutive rows (starting from the $((i-1)n_{\mathrm{x}}^2+1)$-th row, with $i\in\mathbb{Z}_+$, $i\leq N$) of Λ.

From the recursion (3.33) for $\boldsymbol{Q}(k)$, by using the linear mapping $\mathrm{vec}^2(\cdot)$ described in (B.30), the definitions of the ℓ_1-norm (B.49), the matrix product (B.22), the sub-multiplicative property (B.43), and triangle inequality (B.43), we have that

$$\begin{aligned}\|Q_i(k)\|_{\mathbf{1}} = \|\mathrm{vec}(Q_i(k))\|_1 &= \left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t)\right)^*_{[(i-1)n_{\mathrm{x}}^2+1, in_{\mathrm{x}}^2]\bullet} \mathrm{vec}^2(\boldsymbol{Q}(0)) \right\|_1 \\ &\leq \left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t)\right)^*_{[(i-1)n_{\mathrm{x}}^2+1, in_{\mathrm{x}}^2]\bullet} \right\|_{\mathbf{1}} \left\|\mathrm{vec}^2(\boldsymbol{Q}(0))\right\|_1 \\ &\leq \left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t)\right)^* \right\|_{\mathbf{1}} \left\|\mathrm{vec}^2(\boldsymbol{Q}(0))\right\|_1 \end{aligned} \tag{3.40}$$

where the last inequality is justified by the fact that the matrix norm of a submatrix is always less than or equal to a norm of the whole matrix.

Since (3.40) is valid for each $i \in \mathbb{M}$, we rewrite (3.39) as

$$\mathbb{E}\left(\|\mathrm{x}_k\|_2^2\right) \leq n_{\mathrm{x}} N \left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t)\right)^* \right\|_{\mathbf{1}} \left\|\mathrm{vec}^2(\boldsymbol{Q}(0))\right\|_1 \tag{3.41}$$

From the definition (B.49) of the ℓ_1-norm, it is trivial to verify that

$$\left\|\mathrm{vec}^2(\boldsymbol{Q}(0))\right\|_1 = \sum_{i=1}^{N} \|\mathrm{vec}(Q_i(0))\|_1 = \sum_{i=1}^{N} \|Q_i(0)\|_{\mathbf{1}} \tag{3.42}$$

At this point, we recall that the ℓ_1-norm is related to the Frobenius norm by the inequality (B.50), i.e.,

$$\|Q_i(0)\|_{\mathbf{1}} \leq n_{\mathrm{x}} \|Q_i(0)\|_F \tag{3.43}$$

Since by its definition (3.13), $Q_i(0) \in \mathbb{F}^{n_{\mathrm{x}}, n_{\mathrm{x}}}$ for each $i \in \mathbb{M}$, we apply the property (B.53) of the trace of a positive semi-definite matrix, obtaining that

$$\|Q_i(0)\|_F \leq \mathrm{tr}(Q_i(0)) \tag{3.44}$$

After combining (3.41) with (3.42), (3.43), and (3.44), we obtain that

$$\mathbb{E}\left(\|\mathrm{x}_k\|_2^2\right) \leq n_{\mathrm{x}}^2 N \left\| \left(\prod_{t=0}^{k-1} \Lambda^*(t)\right)^* \right\|_{\mathbf{1}} \sum_{i=1}^{N} \mathrm{tr}(Q_i(0)) \tag{3.45}$$

Within the proof of (3.27) in Proposition 3.2, we have seen that (with $k=0$)

$$\sum_{i=1}^{N} \mathrm{tr}(Q_i(0)) = \mathbb{E}\left(\|\mathrm{x}_0\|_2^2\right) = \|\mathrm{x}_0\|_2^2 \tag{3.46}$$

Then, by putting together (3.45), (3.46), and (3.38), we obtain

$$\mathbb{E}\left(\|\mathrm{x}_k\|_2^2\right) \leq n_{\mathrm{x}}^2 N \beta' \zeta^k \|\mathrm{x}_0\|_2^2 = \beta \zeta^k \|\mathrm{x}_0\|_2^2$$

This proves the assertion that mean square stability implies exponential mean square stability also for the autonomous noiseless Markov jump linear systems (3.23) with time-varying uncertain transition probabilities. All the remaining implications follow from the already proved ones. Thus, the proof is concluded. □

The results presented in this section, including several steps of the related proofs, are the basis for what is coming in the next section, where on top of time-varying perturbations in uncertain transition probability matrices, we consider also the presence of a bounded process noise.

3.3 Stability conditions with bounded process noise

In order to underline how the time-varying disturbances in uncertain transition probability matrices affect the stability of discrete-time Markov jump linear systems, until now we have focused on state-space models without noise, control input, or any type of uncertainties in system parameters. Obviously, these parts of the model are not immune to the disturbances. Notably, the discrepancies between the modeled system states and the real process are often represented by an additive process noise, which in this section is described by an ℓ_2-stochastic signal (see Subsection 2.2 for its definition). Such problem setup is particularly useful for the $\mathcal{H}_\infty$-control problems, as described by Costa et al. [12, Chapter 7, pp. 143–166] for the Markov jump linear systems with time-invariant and exactly known transition probabilities between the operational modes of the system.

So, let us consider again an autonomous discrete-time Markov jump linear system described as in (3.3), i.e.,

$$\begin{cases} \mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k + H_{\theta_k}\mathrm{v}_k, \\ \mathrm{x}_0 = \boldsymbol{x}_0,\ \theta_0 = \vartheta_0 \end{cases}$$

where $\mathrm{x}_k \in \mathbb{F}^{n_\mathrm{x}}$ is a system's state, $\mathrm{v}_k \in \mathbb{F}^{n_\mathrm{v}}$ an exogenous input representing a process noise, while $\boldsymbol{A} \triangleq (A_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_\mathrm{x},n_\mathrm{x}}$ and $\boldsymbol{H} \triangleq (H_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_\mathrm{x},n_\mathrm{v}}$ are the respective transformation matrices. As before, the transition probability matrices are time-varying, with variations that are arbitrary within a polytopic set, as formally stated by Assumption 2.2. The initial conditions are $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ and $\theta_0 \in \Theta_0$.

It is easy to see by repeated applications of the recursion for x_k that the system state evolves as

$$\mathrm{x}_k = \left(\prod\nolimits_{t=0}^{k-1} A_{\theta_t}^*\right)^* \mathrm{x}_0 + \sum\nolimits_{t=0}^{k-1} \left(\prod\nolimits_{j=t+1}^{k-1} A_{\theta_j}^*\right)^* H_{\theta_t}\mathrm{v}_t \tag{3.47}$$

Let us indicate by $\mathrm{Re}[\cdot]$ either the real part of a complex number or, when applied to matrices, the operation of taking the real part of each entry of a complex matrix.

As in the noiseless case, we find that the recursive equations for $q_i(k)$ and $Q_i(k)$ for polytopic time-inhomogeneous MJLSs as in (3.3) again have the same structure as their counterpart with time-homogeneous exactly known transition probability matrices [12, p. 50–52]. We show this in the following proposition.

Proposition 3.7. *Consider the system* (3.3). *For all $k \in \mathbb{T}$, $j \in \mathbb{M}$, we have that*

$$\mathrm{q}_j(k+1) = \sum\nolimits_{i=1}^N p_{ij}(k) A_i \mathrm{q}_i(k) + \sum\nolimits_{i=1}^N p_{ij}(k) H_i \mathrm{r}_i(k) \tag{3.48}$$

$$Q_j(k+1) = \sum_{i=1}^{N} p_{ij}(k) A_i Q_i(k) A_i^* + \sum_{i=1}^{N} p_{ij}(k) H_i R_i(k) H_i^* + \\ 2\,\mathrm{Re}\left(\sum_{i=1}^{N} p_{ij}(k) A_i W_i(k) H_i^*\right) \tag{3.49}$$

Proof. The proof is very similar to the proof of Proposition 3.2, so we only outline the procedure.

Regarding the first statement, from the definition (3.11) of the vector $\mathrm{q}_i(k)$ of expected values of the system state variables in correspondence of the i-th operational mode, the recursive equation (3.3) describing the evolution of the system's state x_k, the definition (3.12) of the vector $\mathrm{r}_i(k)$ of expected values of the process noise related to the i-th operational mode, with $i \in \mathbb{M}$, by linearity of the expected value, we have that

$$\begin{aligned} \mathrm{q}_j(k+1) &= \sum_{i=1}^{N} \mathbb{E}\big((A_i \mathrm{x}_k + H_i \mathrm{v}_k)\, \mathbf{1}_{\{\theta_k = i\}} \mathbf{1}_{\{\theta_{k+1} = j\}}\big) \\ &= \sum_{i=1}^{N} p_{ij}(k) A_i \mathrm{q}_i(k) + \sum_{i=1}^{N} p_{ij}(k) H_i \mathrm{r}_i(k) \end{aligned}$$

The second statement can be proven in the same manner, i.e., by linearity of the expected value, from the definition (3.13) of the matrix $Q_i(k)$, the state-space representation (3.3) of the Markov jump linear system, the definition (3.15) of the matrix $R_i(k)$, and the definition (3.18) of the matrix $W_i(k)$, after remembering the properties of complex conjugation reported in appendix's Subsection B.4, and the fact that the sum of a complex number with its complex conjugate gives us two times the real part of the complex number. □

Following the same line of the previous section, we rewrite the recursive equation (3.49) for $Q_i(k)$ in a matrix form.

In particular, the recursive equation of $\boldsymbol{Q}(k)$ for Markov jump linear systems that accounts for a process noise is obtained by applying the equation (3.49) describing the dynamics of $Q_i(k)$ (from Proposition 3.7), together with the definition (B.55) of the linear transformation $\mathrm{vec}^2(\cdot)$, the correspondent definition (B.29) of the linear map $\mathrm{vec}(\cdot)$, and the relevant properties (B.30) and (B.31) of the Kronecker product, to $\boldsymbol{Q}(k)$, defined by (3.14). Notably,

$$\mathrm{vec}^2(\boldsymbol{Q}(k+1)) = \Lambda(k)\mathrm{vec}^2(\boldsymbol{Q}(k)) + \Gamma(k)\mathrm{vec}^2(\boldsymbol{R}(k)) + 2\,\mathrm{Re}\big(\Xi(k)\mathrm{vec}^2(\boldsymbol{W}(k))\big) \tag{3.50}$$

where $\Lambda(k)$ is defined as in (3.29), and $\Gamma(k)$, $\Xi(k)$ are defined similarly, i.e.,

$$\Gamma(k) \triangleq \big(P^T(k) \otimes I_{n_\mathrm{x}^2}\big) \left(\bigoplus_{i=1}^{N} \big(\bar{H}_i \otimes H_i\big)\right) \tag{3.51}$$

$$\Xi(k) \triangleq \big(P^T(k) \otimes I_{n_\mathrm{x}^2}\big) \left(\bigoplus_{i=1}^{N} \big(\bar{H}_i \otimes A_i\big)\right) \tag{3.52}$$

We recall that $\boldsymbol{R}(k)$ is defined by (3.16), and $\boldsymbol{W}(k)$ is shown in (3.19).

From the repeated applications of (3.50), we obtain the following relation

$$\begin{aligned}\operatorname{vec}^2(\boldsymbol{Q}(k)) = &\left(\prod_{t=0}^{k-1} \Lambda^*(t)\right)^* \operatorname{vec}^2(\boldsymbol{Q}(0)) + \\ &\sum_{t=0}^{k-1} \left(\prod_{j=t+1}^{k-1} \Lambda^*(j)\right)^* \Gamma(t)\operatorname{vec}^2(\boldsymbol{R}(t)) + \\ &2\operatorname{Re}\left(\sum_{t=0}^{k-1} \left(\prod_{j=t+1}^{k-1} \Lambda^*(j)\right)^* \Xi(t)\operatorname{vec}^2(\boldsymbol{W}(t))\right)\end{aligned} \tag{3.53}$$

Now we are ready to state the main result of this section, which will be presented in the following dedicated subsection.

Mean square stability with process noise

In the next theorem we will show that the mean square stability for the system (3.3) is equivalent to the discrete-time Markov jump linear system being a bounded linear operator that maps ℓ_2-stochastic exogenous input signals into ℓ_2-stochastic output signals. As before, this result represents a useful generalization of the notion already known for time-homogeneous Markov jump linear systems. In fact, when there is only one time-invariant transition probability matrix, joint spectral radius corresponds to a spectral radius.

Theorem 3.8. *Given a discrete-time Markov jump linear system* (3.3) *with unknown time-varying transition probability matrix* $P(k) \in \operatorname{conv}_{\mathbb{V}}\mathbb{P}$*, then* $\hat{\rho}({}_{\mathbb{V}}\boldsymbol{\Lambda}) < 1$ *if and only if* $\mathrm{x} = \{\mathrm{x}_k : k \in \mathbb{T}\} \in \mathcal{H}^{n_\mathrm{x}}$ *for every* $\mathrm{v} = \{\mathrm{v}_k : k \in \mathbb{T}\} \in \mathcal{H}^{n_\mathrm{v}}$*, and any initial condition* $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$ *and* $\theta_0 \in \Theta_0$*.*

Proof. To prove the **necessity** (that is, $\hat{\rho}({}_{\mathbb{V}}\boldsymbol{\Lambda}) < 1 \Rightarrow \mathrm{x} \in \mathcal{H}^{n_\mathrm{x}}\ \forall\, \mathrm{v} \in \mathcal{H}^{n_\mathrm{v}}$, $\mathrm{x}_0 \in \mathcal{H}_0^{n_\mathrm{x}}$, $\theta_0 \in \Theta_0$), all we have to show is that $\|\mathrm{x}\|_2 < \infty$ since on the considered stochastic basis $(\Omega_\mathrm{x}, \mathcal{G}, (\mathcal{G}_k), \Pr)$ described in detail in Subsection 2.2 we clearly have that $(\mathrm{x}_t)_{t=0}^k \in \mathcal{H}_k^{n_\mathrm{x}}$ for each $k \in \mathbb{T}$.

While looking at the equation (3.47) describing x_k as a function of x_0, i.e.,

$$\mathrm{x}_k = \left(\prod_{t=0}^{k-1} A_{\theta_t}^*\right)^* \mathrm{x}_0 + \sum_{t=0}^{k-1} \left(\prod_{j=t+1}^{k-1} A_{\theta_j}^*\right)^* H_{\theta_t}\mathrm{v}_t = \check{\mathrm{x}}_k + \sum_{t=0}^{k-1} \check{\mathrm{v}}_t$$

we notice that the first addend $\check{\mathrm{x}}_k$ on the right-hand side of the equality is clearly related to the noiseless version of system (3.3), that is, when $\mathrm{v}_k = 0$ for each $k \in \mathbb{T}$. The other addend describes the contribution of the noise.

So, the proof of necessity is divided into three parts: the first one is related to the noiseless part of the evolution of the system's state $\check{\mathrm{x}}_k$, the second part is connected to the partial dynamics due to the noise, i.e., $\check{\mathrm{v}}_t$, while the last part corresponds to the combination of the previous two.

By hypothesis $\hat{\rho}({}_{\mathbb{V}}\mathbf{\Lambda}) < 1$, and for the first part we apply the same steps of the proof of the fact that, for Markov jump linear systems without process noise, exponential mean square stability implies mean square stability.

This procedure is illustrated in the proof of Theorem 3.6, where it is shown that for some $k' \in \mathbb{T}$, there always exists a $k \in \mathbb{T}$, $k \geq k'$, such that for any $N, n_{\mathrm{x}} \in \mathbb{Z}_+$,

$$\|\check{\mathrm{x}}_k\|_2^2 = \mathbb{E}\Big(\|\check{\mathrm{x}}_k\|_2^2\Big) \leq \beta\zeta^k \|\mathrm{x}_0\|_2^2 = N n_{\mathrm{x}}^2 \beta' \zeta^k \|\mathrm{x}_0\|_2^2 \tag{3.54}$$

with $\zeta \in \mathbb{R}_+$ such that

$$\zeta \in (\hat{\rho}({}_{\mathbb{V}}\mathbf{\Lambda}), 1) \tag{3.55}$$

and $\beta' \in \mathbb{R}_+$, $\beta' \geq 1$, being defined as

$$\beta' = \zeta^{-k'} \sup_{\mathrm{P} \in \mathcal{P}_j({}_{\mathbb{V}}\mathbf{\Lambda}),\ 0 \leq j \leq k'} \|\mathrm{P}\| \tag{3.56}$$

where $\mathcal{P}_j({}_{\mathbb{V}}\mathbf{\Lambda})$ indicates the set of all possible products of length j whose factors are elements of ${}_{\mathbb{V}}\mathbf{\Lambda}$, as formally defined by (B.63).

Regarding the second part, which is related to the evolution in time of the partial contribution of the noise, the procedural steps are similar.

For each $t \in \mathbb{T}_{k-1}$ we consider

$$\check{\mathrm{v}}_t = \left(\prod_{j=t+1}^{k-1} A_{\theta_j}^*\right)^* H_{\theta_t}\mathrm{v}_t \tag{3.57}$$

It is clear from the expression (3.57) that $\check{\mathrm{v}}_t$ behaves as an autonomous noiseless Markov jump linear system with the initial condition given by $H_{\theta_t}\mathrm{v}_t$.

The second moment for this initial condition is obtained as

$$\mathbb{E}\big(H_{\theta_t}\mathrm{v}_t\,(H_{\theta_t}\mathrm{v}_t)^*\big) = \sum_{i=1}^{N} \mathbb{E}\big(H_{\theta_t}\,(\mathrm{v}_t\mathrm{v}_t\mathbf{1}_{\{\theta_t=i\}})\,H_{\theta_t}^*\big) = \sum_{i=1}^{N} H_i R_i(t) H_i^* p_i(t)$$

So, it is trivial to verify that the second moment of $\check{\mathrm{v}}_t$ is already computed in matrix form in the equation (3.53) describing the evolution of the second moment of the autonomous system (3.3) with a process noise. It is expressed by

$$\begin{aligned}\mathrm{vec}^2\Big(\check{\boldsymbol{R}}(t)\Big) &= \left(\prod_{j=t+1}^{k-1} \Lambda^*(j)\right)^* \Gamma(t)\mathrm{vec}^2(\boldsymbol{R}(t)) \\ &= \left(\prod_{j=t+1}^{k-1} \Lambda^*(j)\right)^* \mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)\end{aligned} \tag{3.58}$$

where $\Gamma(t)$ is given by (3.51), $\boldsymbol{R}(t)$ and $R_i(t)$ are expressed via (3.16) and (3.15), respectively, $\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*$ is represented in (3.17), while $\check{\boldsymbol{R}}(t)$ and $\check{R}_i(k)$ are defined as follows:

$$\check{R}_i(t) \triangleq \mathbb{E}\big(\check{\mathrm{v}}_t\check{\mathrm{v}}_t^*\mathbf{1}_{\{\theta_t=i\}}\big) \in \mathbb{F}_0^{n_{\mathrm{x}},n_{\mathrm{x}}} \tag{3.59}$$

$$\check{\boldsymbol{R}}(t) \triangleq \Big(\check{R}_i(t)\Big)_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}_0^{n_{\mathrm{x}},n_{\mathrm{x}}} \tag{3.60}$$

From here on, we follow the line of the proof of Theorem 3.6. Thus, we only outline the main points, without explaining every passage. First of all, we make the same considerations used in the proof of the third statement (3.27) in Proposition 3.2, especially the inequality (3.24) between the trace of a matrix and any norm of the same matrix, proved in Lemma 3.1, obtaining that $\forall t \in \mathbb{T}_{k-1}$

$$\|\check{\mathrm{v}}_t\|_2^2 = \mathbb{E}\Big(\|\check{\mathrm{v}}_t\|_2^2\Big) = \sum_{i=1}^{N} \mathrm{tr}\big(\mathbb{E}\big(\check{\mathrm{v}}_t \check{\mathrm{v}}_t^* \mathbf{1}_{\{\theta_t = i\}}\big)\big) = \sum_{i=1}^{N} \mathrm{tr}\Big(\check{R}_i(k)\Big) \le n_{\mathrm{x}} \sum_{i=1}^{N} \Big\|\check{R}_i(k)\Big\|$$

which holds for any equivalent matrix norm, including ℓ_1-norm.

We apply ℓ_1-norm to $\check{R}_i(k)$, obtaining that

$$\begin{aligned} \Big\|\check{R}_i(k)\Big\|_{\mathbf{1}} &= \Big\|\mathrm{vec}\Big(\check{R}_i(k)\Big)\Big\|_1 = \left\|\Big(\prod_{j=t+1}^{k-1} \Lambda^*(j)\Big)^*_{[(i-1)n_{\mathrm{x}}^2+1,\, i n_{\mathrm{x}}^2]\bullet} \mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)\right\|_1 \\ &\le \left\|\Big(\prod_{j=t+1}^{k-1} \Lambda^*(j)\Big)\right\|_{\mathbf{1}} \big\|\mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)\big\|_1 \end{aligned}$$

Since the previous inequality is valid for each $i \in \mathbb{M}$, we write that

$$\|\check{\mathrm{v}}_t\|_2^2 \le n_{\mathrm{x}} N \left\|\Big(\prod_{j=t+1}^{k-1} \Lambda^*(j)\Big)\right\|_{\mathbf{1}} \big\|\mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)\big\|_1 \tag{3.61}$$

From the definition (B.49) of the ℓ_1-norm, it follows that

$$\big\|\mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)\big\|_1 = \sum_{i=1}^{N} \|\mathrm{vec}(H_i R_i(t) H_i^*)\|_1 = \sum_{i=1}^{N} \|H_i R_i(t) H_i^*\|_{\mathbf{1}}$$

Recollecting that the ℓ_1-norm is related to the Frobenius norm by the inequality (B.50), we write that

$$\|H_i R_i(t) H_i^*\|_{\mathbf{1}} \le n_{\mathrm{x}} \|H_i R_i(t) H_i\|_F$$

As before, by construction $H_i R_i(t) H_i^* \in \mathbb{F}_0^{n_{\mathrm{x}}, n_{\mathrm{x}}}$ for each $i \in \mathbb{M}$. Thus, we apply the property (B.53) of the trace of a positive semi-definite matrix, obtaining that

$$\|H_i R_i(t) H_i^*\|_F \le \mathrm{tr}(H_i R_i(t) H_i^*) = \mathrm{tr}\big(H_i \mathbb{E}\big(\mathrm{v}_t \mathrm{v}_t^* \mathbf{1}_{\{\theta_t = i\}}\big)\, H_i^*\big)$$

where we used the definition (3.15) of $R_i(t)$.

Consequently, from the previous three equations, by linearity of the trace and definition (B.54c) of the max-norm on the linear space made up of all N-sequences of either real or complex matrices, we obtain that

$$\big\|\mathrm{vec}^2(\boldsymbol{H}\boldsymbol{R}(t)\boldsymbol{H}^*)\big\|_1 \le n_{\mathrm{x}} \sum_{i=1}^{N} H_i \mathrm{tr}\big(\mathbb{E}\big(\mathrm{v}_t \mathrm{v}_t^* \mathbf{1}_{\{\theta_t = i\}}\big)\big)\, H_i^* \le n_{\mathrm{x}} \|\boldsymbol{H}\|_{\max}^2 \|\mathrm{v}_t\|_2^2$$

Then, by the radical test for infinite series, valid for all equivalent matrix norms, we also have that

$$\left\|\Big(\prod_{j=t+1}^{k-1} \Lambda^*(j)\Big)\right\|_{\mathbf{1}} \le \beta' \zeta^{k-t-1}$$

where ζ and β' are those defined by (3.55) and (3.56), respectively.

Putting together both parts of (3.61), we obtain that

$$\|\check{\mathrm{v}}_t\|_2^2 \leq N n_{\mathrm{x}}^2 \beta' \zeta^{k-i-1} \|\boldsymbol{H}\|_{\max}^2 \|\mathrm{v}_t\|_2^2 \tag{3.62}$$

which holds for each $t \in \mathbb{T}_{k-1}$.

So, we have that

$$\mathrm{x}_k = \left(\prod_{t=0}^{k-1} A_{\theta_t}^*\right)^* \mathrm{x}_0 + \sum_{t=0}^{k-1} \left(\prod_{j=t+1}^{k-1} A_{\theta_j}^*\right)^* H_{\theta_t} \mathrm{v}_t = \check{\mathrm{x}}_k + \sum_{t=0}^{k-1} \check{\mathrm{v}}_t$$

together with bounds on $\|\check{\mathrm{x}}_k\|_2^2$ expressed by (3.54) and on $\|\check{\mathrm{v}}_t\|_2^2$, given by (3.62).

By triangle inequality (B.3), we have that

$$\|\mathrm{x}_k\|_2 \leq \|\check{\mathrm{x}}_k\|_2 + \sum_{t=0}^{k-1} \|\check{\mathrm{v}}_t\|_2$$

We still need to show that $\|\mathrm{x}_k\|_2 < \infty$.

From now on, in this last part of the proof of the necessity, we follow the steps of the proof provided in [12, Theorem 3.34, pp. 55-57] for time-homogeneous MJLSs with bounded process noise.

Applying the bounds obtained for $\|\check{\mathrm{x}}_k\|_2^2$ in (3.54) and for $\|\check{\mathrm{v}}_t\|_2^2$ in (3.62), and also considering the expressions (3.55), (3.56) of respectively ζ and β', we can state that there exist $\zeta \in (\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Lambda}), 1)$ and $\beta' \geq 1$ such that

$$\|\mathrm{x}_k\|_2 \leq \sum_{t=0}^{k} \zeta_{k-t} \beta_t$$

where

$$\zeta_{k-t} \triangleq \sqrt{\zeta^{k-t}}$$

$$\beta_0 \triangleq n_{\mathrm{x}} \sqrt{N\beta'} \|\mathrm{x}_0\|_2$$

$$\beta_t \triangleq n_{\mathrm{x}} \sqrt{N\beta'} \|\boldsymbol{H}\|_{\max} \|\mathrm{v}_{t-1}\|_2 , \quad t \geq 1$$

We set $a \triangleq (\zeta_i)_{i=0}^k$ and $b \triangleq (\beta_i)_{i=0}^k$. Since $a \in \ell_1$ (i.e., $\sum_{i=0}^{\infty} |\zeta_i| < \infty$) and $b \in \ell_2$ (that is, $\sum_{i=0}^{\infty} |\beta_i|^2 < \infty$), it follows that the convolution $c \triangleq a * b = (c_i)_{i=0}^k$, $c_i \triangleq \sum_{t=0}^{i} \zeta_{i-t}\beta_t$, lies itself in ℓ_2 with $\|c\|_2 \leq \|a\|_1 \|b\|_2$ [12, p. 56]. Hence,

$$\|\mathrm{x}\|_2 = \sqrt{\sum_{k=0}^{\infty} \mathbb{E}\left(\|\mathrm{x}_k\|_2^2\right)} \leq \sqrt{\sum_{i=0}^{\infty} c_i^2} = \|c\|_2 < \infty$$

This concludes the proof of necessity.

To prove the **sufficiency** (i.e., $\mathrm{x} \in \mathcal{H}^{n_{\mathrm{x}}}\ \forall\, \mathrm{v} \in \mathcal{H}^{n_{\mathrm{v}}}$, $\mathrm{x}_0 \in \mathcal{H}_0^{n_{\mathrm{x}}}$, $\theta_0 \in \Theta_0 \Rightarrow \hat{\rho}(_{\mathbb{V}}\boldsymbol{\Lambda}) < 1$) we observe that, by hypothesis

$$\|\mathrm{x}\|_2^2 = \sum_{k=0}^{\infty} \mathbb{E}\left(\|\mathrm{x}_k\|_2^2\right) < \infty$$

for all $\mathrm{v}\in\mathcal{H}^{n_\mathrm{v}}$, $\mathrm{x}_0\in\mathcal{H}_0^{n_\mathrm{x}}$, and $\theta_0\in\Theta_0$.

Since, as reported in appendix's Subsection B.3, the absolute convergence of a series in a normed linear space implies the convergence of a series in that space, by (B.7) we have that

$$\lim_{k\to\infty}\mathbb{E}(\mathrm{x}_k\mathrm{x}_k^*)=0$$

for all $\mathrm{x}_0\in\mathcal{H}_0^{n_\mathrm{x}}$, $\theta_0\in\Theta_0$, and for any $\mathrm{v}\in\mathcal{H}^{n_\mathrm{v}}$.

Since this last statement holds for every $\mathrm{v}\in\mathcal{H}^{n_\mathrm{v}}$, we can make $\mathrm{v}_k=0$ for all $k\in\mathbb{T}$ in the state-space representation of the autonomous Markov jump linear system (3.3), obtaining the noiseless model (3.23). Thus, we have exactly the same conditions found at the beginning of the proof of necessity of Theorem 3.4. Application of the procedure illustrated there brings us to the thesis and concludes our proof of sufficiency. □

Illustrative example with bounded noise

This subsection is a follow-up of the motivational example presented in Subsection 3.1, where we have seen that the perturbations on values of transition probabilities can make a stable Markov jump linear system unstable.

First, we show that the addition of the bounded process noise does not change the considerations on stability, i.e., time-homogeneous systems remain stable, while the perturbed MJLSs remain unstable. Then, we show that by shrinking the polytope defining the uncertainty of the transition probability matrix to a new one, with joint spectral radius of the set of vertices smaller then one, we obtain that both noiseless autonomous discrete-time Markov jump linear system as in (3.23), and its counterpart perturbed by a bounded process noise (3.3), become stable.

So, we consider the autonomous Markov jump linear system with $N=3$ operational modes, having the same as in example of Subsection 3.1 state matrices associated with the operational modes, and the same set of vertices defining the polytope bounding the values of time-varying transition probability matrix $P(k)$:

$$A_1=\begin{bmatrix}1&0\\0&1.2\end{bmatrix},\quad A_2=\begin{bmatrix}1.13&0\\0.16&0.48\end{bmatrix},\quad A_3=\begin{bmatrix}0.3&0.13\\0.16&1.14\end{bmatrix}$$

$$P_1=\begin{bmatrix}0&0.35&0.65\\0.6&0.4&0\\0.4&0.6&0\end{bmatrix},\quad P_2=\begin{bmatrix}0.25&0.75&0\\0&0.6&0.4\\0&0.4&0.6\end{bmatrix}$$

The spectral radii ρ of the matrices Λ are:

$$\rho(\Lambda_1)=0.901601,\quad \rho(\Lambda_2)=0.905686$$

and the joint spectral radius (computed via JSR toolbox [94]) is

$$\hat{\rho}(\{\Lambda_1,\Lambda_2\})\geq 1.024442$$

Here, we also take into account a process noise $\mathrm{v}_k \in [-1,1]\times[-1,1] \subset \mathbb{R}^2$, which perturbs the system's state variables according to the following process noise transformation matrices

$$H_1 = 2I_2, \quad H_2 = 1.5I_2, \quad H_3 = I_2$$

We report one possible dynamical behavior of the system, obtained, as before, for $\mathrm{x}_0 = [100; 85]$ and the initial probability distribution $\mathrm{p}_0 = [0.33, 0.34, 0.33]^T$. The obtained system trajectories for the system perturbed by the described process noise are illustrated in the following figures.

Figure 3.4 shows us a trajectory of the system state vector x_k perturbed by the process noise v_k, when the time-homogeneous transition probability matrix is P_1.

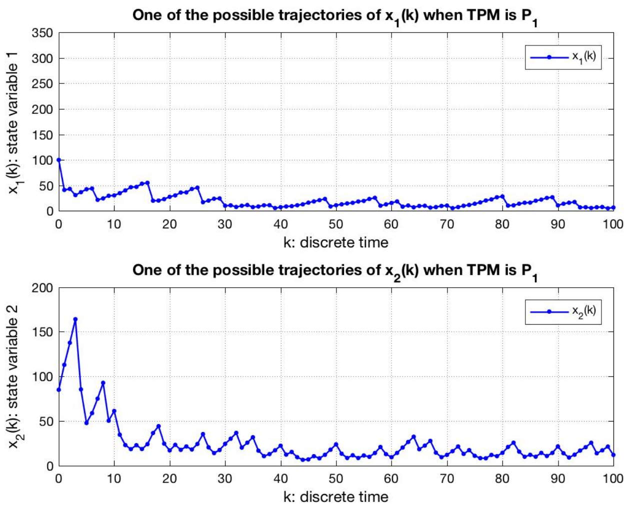

Figure 3.4: A possible trajectory of perturbed x_k when TPM is P_1

Again, after repeating the trial under the same conditions, except that the considered transition probability matrix being P_2, one obtains a system's state trajectory (which takes into account the process noise v_k) presented in Figure 3.5.

Since the joint spectral radius (of the set of the vertices defining the bounds on variations of values of the transition probability matrix) is greater than one, it comes without surprise that, repeating the experiment again under the same conditions, except that the transition probability matrix being allowed to switch arbitrarily between P_1 and P_2 at each time step, brings to light a trajectory shown in Figure 3.6. The considered Markov jump linear system is clearly unstable.

However, by shrinking the polytope defining the uncertainty of the transition probability matrix to e.g. the new vertices

$$\bar{P}_1 = 0.8P_1 + 0.2P_2, \quad \bar{P}_2 = 0.2P_1 + 0.8P_2$$

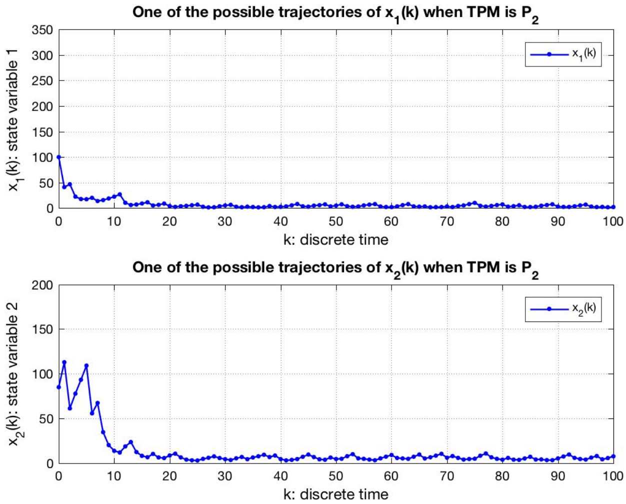

Figure 3.5: A possible trajectory of perturbed x_k when TPM is P_2

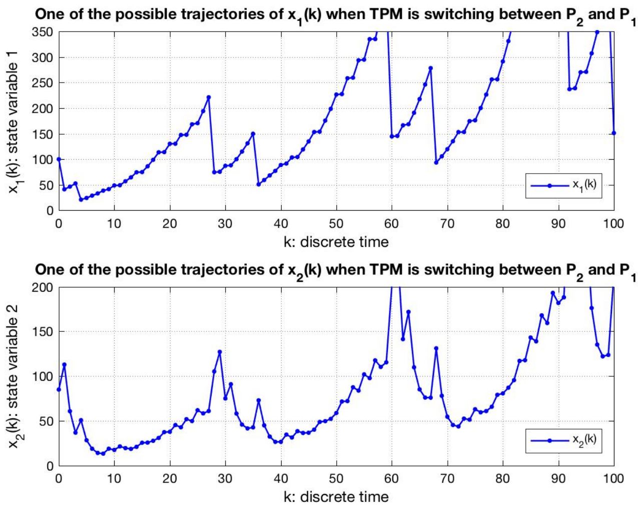

Figure 3.6: Trajectory of perturbed x_k when TPM is switching between P_1 and P_2

we obtain that the corresponding (whether perturbed by a process noise or not) time-inhomogeneous system is robustly (mean square) stable, because the joint spectral radius is

$$\hat{\rho}(\{\bar{\Lambda}_1, \bar{\Lambda}_2\}) \leq 0.972553$$

Figures 3.7 and 3.8 show us trajectories of the time-inhomogeneous Markov jump linear system where the transition probability matrix is switching arbitrarily between $\bar{P}_1$ and $\bar{P}_2$ at each time step, in, respectively, the noiseless setting and in case the system's state is perturbed by a bounded process noise described before.

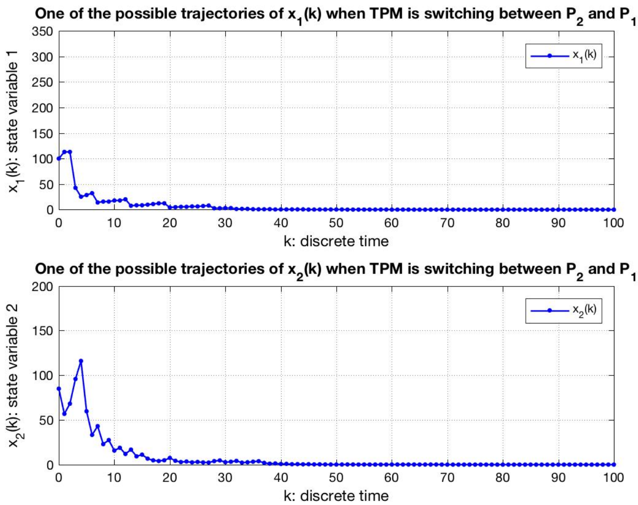

Figure 3.7: Trajectory of noiseless x_k when TPM is switching between $\bar{P}_1$ and $\bar{P}_2$

In conclusion, the presented conditions, based on the notion of the joint spectral radius of the set of vertices of the polytope of matrices characterizing the second moment of Markov jump linear system's state for all operational modes, permit us to check whether the autonomous system is stable, regardless the presence of bounded perturbations on the system's state itself.

In the next section we extend the same concepts to the controlled systems, introducing the basis of the rest of our works.

3.4 Structural properties of feedback control systems

The various "abilities" such as controllability, observability, reachability, reconstructibility, stabilizability, and detectability are basic to the study of linear control and system theory [97, pp. 1–15].

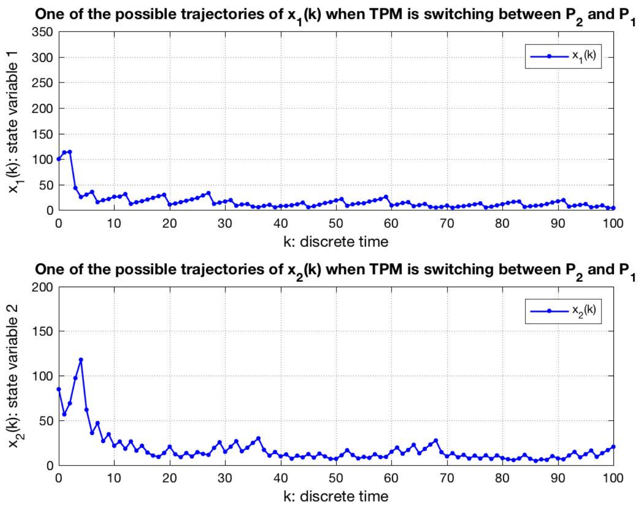

Figure 3.8: Trajectory of perturbed x_k when TPM is switching between $\bar{P}_1$ and $\bar{P}_2$

This section deals with the concepts of mean square detectability and mean square stabilizability, which are the basic notions for the study of the feedback control for (time-inhomogeneous polytopic) Markov jump linear systems.

The well-known definitions and characterization of controllability, stabilizability, observability and detectability for linear time-invariant systems, which are, as we recall, the special case of MJLSs, can be found in any textbook on the topic, also for instance in [12, pp. 24–27], and thus will be omitted.

In the next two subsections we report directly our definitions of mean square stabilizability and mean square detectability, which are generalized from the similar concepts presented in [12, pp. 57–63] for Markov jump linear systems with time-invariant transition probabilities between the operational modes.

Mean square stabilizability

Let us consider a discrete-time Markov jump linear system as in (2.1) defined on a stochastic basis introduced in Subsection 2.2, i.e., $(\Omega_{\mathrm{x}}, \mathcal{G}, (\mathcal{G}_k), \mathrm{Pr})$. Since on such filtered probability space all the system's state variables, together with all the jump variables, are (assumed to be) measurable, the state space model representation is the following:

$$\begin{cases} \mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k + B_{\theta_k}\mathrm{u}_k + H_{\theta_k}\mathrm{v}_k, \\ \quad \mathrm{x}_0 = \boldsymbol{x}_0,\ \theta_0 = \vartheta_0,\ \mathrm{p}_0 = \boldsymbol{p}_0 \end{cases}$$

The state vector x_k, the controlled input u_k and the exogenous input v_k, together with the related (sequences of) transformation matrices and initial conditions, are the same as before. They are presented in detail in Section 2.1. We have already seen in Section 3.3 that the condition for the mean square stability of a Markov jump linear system with bounded process noise represented by ℓ_2-stochastic exogenous input signal is the same as for the noiseless version of the system. Thus, as may be expected, the definition of the mean square stabilizability is related to the noiseless setting, i.e., considering v_k to be a vector of all zeros for each $k \in \mathbb{T}$, as represented by the following dynamical system

$$\begin{cases} \mathrm{x}_{k+1} = A_{\theta_k} \mathrm{x}_k + B_{\theta_k} \mathrm{u}_k \\ \quad \mathrm{x}_0 = \boldsymbol{x}_0, \ \theta_0 = \vartheta_0, \ \mathrm{p}_0 = \boldsymbol{p}_0 \end{cases} \tag{3.63}$$

The aforementioned definition of the mean square stabilizability is the following.

Definition 3.6. *The pair $(\boldsymbol{A}, \boldsymbol{B})$ of N-sequences of state and control input matrices $\boldsymbol{A} = (A_i)_{i=1}^N$ and $\boldsymbol{B} = (B_i)_{i=1}^N$ related to all N operational modes of the system* (3.63)*, which has unknown and time-varying transition probability matrix $P(k) \in \mathrm{conv}_{\mathbb{V}}\mathbb{P}$, is* ***mean square stabilizable*** *if there exists an N-sequences of control matrices $\boldsymbol{K} = (K_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{u}}, n_{\mathrm{x}}}$ such that the system* (3.63) *with synchronous state feedback controller $\mathrm{u}_k = K_{\theta_k} \mathrm{x}_k$ is mean square stable. In this case, $\boldsymbol{K}$ is said to stabilize the pair $(\boldsymbol{A}, \boldsymbol{B})$.*

Since the controller $\mathrm{u}_k = K_{\theta_k} \mathrm{x}_k$ gives to the system (3.63) a form of a noiseless autonomous system as in (3.23), that is

$$\mathrm{x}_{k+1} = (A_{\theta_k} + B_{\theta_k} K_{\theta_k})\, \mathrm{x}_k$$

we can apply the results of the corresponding Section 3.2.

Specifically, let us write

$$A'_{\theta_k} = A_{\theta_k} + B_{\theta_k} K_{\theta_k} \tag{3.64}$$

The matrix (3.29) associated to the second moment of x_k becomes

$$\Lambda'(k) \triangleq \left(P^T(k) \otimes I_{n_{\mathrm{x}}^2}\right) \left(\bigoplus_{i=1}^{N} \left(\bar{A}'_i \otimes A'_i\right) \right)$$

which is still polytopic, that is, $\forall k \in \mathbb{T}$

$$\Lambda'(k) = \sum_{l=1}^{V} \lambda_l(k) \Lambda'_l, \ \lambda_l(k) \geq 0, \quad \sum_{l=1}^{V} \lambda_l(k) = 1$$

with

$$\Lambda'_l \triangleq \left(P_l^T \otimes I_{n_{\mathrm{x}}^2}\right) \left(\bigoplus_{i=1}^{N} \left(\bar{A}'_i \otimes A'_i\right) \right) \tag{3.65}$$

where, again, for each $l \in \mathbb{V}$, $P_l \in {}_{\mathbb{V}}\mathbb{P} \subset {}_{\mathbb{V}}\mathbb{R}^{N,N}$, i.e, P_l are elements of a given finite set of transition probability matrices, which are the vertices of a convex polytope.

Similarly to what has been done before, we indicate by ${}_{\mathbb{V}}\boldsymbol{\Lambda}'$ the set of vertices of the convex polytope of the matrices $\Lambda'(k)$ related to the second moment of controlled system's state x_k. Formally, we write that

$$ {}_{\mathbb{V}}\boldsymbol{\Lambda}' \triangleq \left\{ \Lambda'_l \in \mathbb{F}^{Nn_{\mathrm{x}}^2, Nn_{\mathrm{x}}^2} : l \in \mathbb{V} \right\} \tag{3.66} $$

Then, the condition for verifying the mean square stabilizability of a MJLS (3.63) is given by the following proposition.

Proposition 3.9. *The discrete-time Markov jump linear system* (3.63) *with unknown and time-varying transition probability matrix $P(k) \in \mathrm{conv}_{\mathbb{V}}\mathbb{P}$ is mean square stabilizable if and only if $\hat{\rho}({}_{\mathbb{V}}\boldsymbol{\Lambda}') < 1$.*

Proof. From (3.64), this result is a direct consequence of Theorem 3.4. □

It is also obvious that the results of Theorems 3.6 and 3.8 are still valid and $\hat{\rho}({}_{\mathbb{V}}\boldsymbol{\Lambda}') < 1$ ensures exponential mean square stabilizability and stochastic stabilizability in the noiseless setting, and robust mean square stabilizability of polytopic time-inhomogeneous Markov jump linear systems subject to a process noise in form of an ℓ_2-stochastic exogenous input signal.

Mean square detectability

It is well-known fact in the control theory that the necessity of the state observer arises when not all the system's state variables are available to the controller. We remind that in a such scenario the stochastic basis to consider is that of the Subsection 2.2, i.e., $(\Omega_{\mathrm{y}}, \mathcal{F}, (\mathcal{F}_k), \mathrm{Pr})$. As in the case of linear time-invariant systems [98, p. 63], the (mean square) detectability ensures that a full order (synchronous state-dependent) steady-state observer exists.

In order to present the formal definition of the mean square detectability, let us consider the following noiseless Markov jump linear system:

$$ \begin{cases} \mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k + B_{\theta_k}\mathrm{u}_k, \\ \quad \mathrm{y}_k = F_{\theta_k}\mathrm{x}_k \end{cases} \tag{3.67} $$

together with a full-order Markov jump filter having a structure similar to the structure of the Luenberger observer [98, p. 63], i.e.,

$$ \begin{cases} \tilde{\mathrm{x}}_{k+1} = A_{\theta_k}\tilde{\mathrm{x}}_k + B_{\theta_k}\mathrm{u}_k - L_{\theta_k}(\mathrm{y}_k - \tilde{\mathrm{y}}_k), \\ \quad \tilde{\mathrm{y}}_k = F_{\theta_k}\tilde{\mathrm{x}}_k \end{cases} \tag{3.68} $$

with $\boldsymbol{L} \triangleq (L_i)_{i=1}^N$ being a vector of filter gain matrices, each of which related to an operational mode, and $\tilde{\mathrm{x}}_k$ being the estimated state, also known as the state of the observer, while $\tilde{\mathrm{y}}_k$ is the output of the filter.

Then, we define the observation (or, equivalently, estimation) error as

$$\tilde{\mathrm{e}}_k = \mathrm{x}_k - \tilde{\mathrm{x}}_k \tag{3.69}$$

From the state-space representations of the Markov jump linear system (3.67) and the related full-order Markov jump filter (3.68), we see that the aforementioned observation error has the following dynamical behavior:

$$\begin{aligned}
\tilde{\mathrm{e}}_{k+1} &= \mathrm{x}_{k+1} - \tilde{\mathrm{x}}_{k+1} = A_{\theta_k}(\mathrm{x}_k - \tilde{\mathrm{x}}_k) + L_{\theta_k}(\mathrm{y}_k - \tilde{\mathrm{y}}_k) \\
&= A_{\theta_k}\tilde{\mathrm{e}}_k + L_{\theta_k}(F_{\theta_k}\mathrm{x}_k - F_{\theta_k}\tilde{\mathrm{x}}_k) \\
&= (A_{\theta_k} + L_{\theta_k}F_{\theta_k})\,\tilde{\mathrm{e}}_k
\end{aligned} \tag{3.70}$$

We observe that the dynamics of the estimation error are those of the noiseless autonomous MJLS seen in Sections 3.2 and 3.2. Thus, similarly to how it was done for the stabilizability, we define the mean square detectability for Markov jump linear systems with polytopic time-inhomogeneous transition probabilities as follows.

Definition 3.7. *The pair* $(\boldsymbol{F}, \boldsymbol{A})$ *of* N*-sequences of observation and state matrices* $\boldsymbol{F} = (F_i)_{i=1}^N$ *and* $\boldsymbol{A} = (A_i)_{i=1}^N$ *related to all* N *operational modes of the system* (3.67) *with unknown and time-varying transition probability matrix* $P(k) \in \mathrm{conv}_{\mathbb{V}}\mathbb{P}$*, is* ***mean square detectable*** *if there exists an* N*-sequences of filter gain matrices* $\boldsymbol{L} \triangleq (L_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{x}, n_\mathrm{y}}$ *such that the evolution* (3.70) *of the observation error for the system* (3.67) *with the synchronous full-order Markov jump filter* (3.68) *is mean square stable.*

With exactly the same procedure seen in the previous subsection, we define

$$A''_{\theta_k} = A_{\theta_k} + L_{\theta_k}F_{\theta_k} \tag{3.71}$$

Then, the matrix associated to the second moment of e_k is again polytopic, with vertices given by

$$\Lambda''_l \triangleq \left(P_l^T \otimes I_{n_\mathrm{x}^2}\right)\left(\bigoplus_{i=1}^N \left(\bar{A}''_i \otimes A''_i\right)\right) \tag{3.72}$$

We indicate by ${}_{\mathbb{V}}\boldsymbol{\Lambda}''$ the set of all such vertices Λ''_l.

So, the condition for verifying the mean square detectability of a MJLS (3.67) is given by the following proposition.

Proposition 3.10. *The discrete-time Markov jump linear system* (3.67) *with unknown and time-varying transition probability matrix* $P(k) \in \mathrm{conv}_{\mathbb{V}}\mathbb{P}$ *is mean square detectable if and only if* $\hat{\rho}({}_{\mathbb{V}}\boldsymbol{\Lambda}'') < 1$.

Proof. From (3.71), this result is a direct consequence of Theorem 3.4. □

Chapter 4

Optimal robust filtering

As it was pointed out in the seminal work [99] of Rudolph Emil Kalman, which has introduced the renowned Kalman filter, the problem of estimation of the unobservable value of the signal at the present time step is called filtering.

Filtering problems are of great interest not only because of their wide number of applications to, e.g., problems of tracking of satellites, signal detection, stochastic control, aerospace engineering [100], but also for being the main step in studying control problems with partial observations on the state variable [12].

In fact, in our most recent work [3] (which was accepted for presentation at the 56th IEEE Conference on Decision and Control (CDC), to be held in December 2017 in Melbourne, Australia), we have showed that, as for linear-quadratic-Gaussian (LQG) control in the case with no jumps, also for the Markov jump linear systems with polytopic time-inhomogeneous transition probabilities the optimal controller having only partial information on the continuous state can be obtained from two types of coupled Riccati difference equations, one associated to the control problem, and the other one associated to the filtering problem. When the transition probabilities between operation modes are known at each time step, our results coincide with those presented in [12], and when there is only one mode of operation, they coincide with the traditional separation principle for the LQG control of discrete-time linear systems.

This chapter is based on the said work [3] and is dedicated to the study of the finite horizon optimal filtering problem for the MJLSs with time-varying transition probabilities, which are bounded, but unmeasurable, as specified in Assumption 2.2.

As one can expect, we use the same standard considerations made for the Markov jump linear systems with known and time-invariant transition probabilities, i.e., that together with the jump variable θ_k, only some of the system's state variables are measured and available to the controller. These state variables are represented by the vector y_k. So, as pointed out in the previous chapter, the appropriate stochastic basis is that of the Subsection 2.2, i.e., $(\Omega_\mathrm{y}, \mathcal{F}, (\mathcal{F}_k), \mathrm{Pr})$, and the asymptotic observer exists, if and only if the system is detectable.

We should stress the fact that, as by standard practice in the MJLSs' theory, we are restricting our attention to the family of linear Markov jump filters, since otherwise the optimal linear mean square filter would be obtained from a sample path Kalman filter [101]. The reason for choosing this family of filters is that their state depends only on the present value of the Markov parameter (rather than on the entire past history of modes, $(\theta_t)_{t=0}^{k}$), so that the closed loop system is again a Markov jump linear system. Besides, in comparison with the Kalman filter, optimal linear Markov jump filter requires much smaller number of precomputed filter's gains, as explained in [12, Remark 5.2, p. 104].

The formal definition of the problem of the optimal filtering, which is robust to the bounded dynamic perturbations of the transition probability matrices of Markov jump linear systems, is presented in the next Section 4.1, while in Section 4.2 the optimal solution to the aforementioned problem is illustrated.

4.1 Problem definition

Let us consider the discrete-time polytopic time-inhomogeneous Markov jump linear system as in (2.1) defined on the stochastic basis $(\Omega_{\mathrm{y}}, \mathcal{F}, (\mathcal{F}_k), \mathrm{Pr})$, and described by the following dynamical system

$$\begin{cases} \mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k + B_{\theta_k}\mathrm{u}_k + H_{\theta_k}\mathrm{w}_k, \\ \mathrm{y}_k = F_{\theta_k}\mathrm{x}_k + G_{\theta_k}\mathrm{w}_k, \\ \mathrm{x}_0 = \boldsymbol{x}_0,\ \theta_0 = \vartheta_0,\ \mathrm{p}_0 = \boldsymbol{p}_0 \end{cases} \tag{4.1}$$

where the system's variables, matrices, and initial states are those defined in Section 2.1. Specifically, $\mathrm{x}_k \in \mathbb{F}^{n_{\mathrm{x}}}$ is a vector of system's state variables, $\mathrm{u}_k \in \mathbb{F}^{n_{\mathrm{u}}}$ is control vector, which gathers the control actions applied to the process, $\mathrm{y}_k \in \mathbb{F}^{n_{\mathrm{y}}}$ is the vector of measured system's states (that are available for feedback, as we will see in the next chapter), with $n_{\mathrm{y}} < n_{\mathrm{x}}$, while the exogenous input $\mathrm{w}_k \in \mathbb{F}^{n_{\mathrm{w}}}$ is a wide sense white noise, which represents disturbances. Notably, far each $k, t \in \mathbb{T}$

$$\mathbb{E}(\mathrm{w}_k) = 0, \qquad \mathbb{E}(\mathrm{w}_k\mathrm{w}_k^*) = I_{n_{\mathrm{w}}}, \qquad \mathbb{E}(\mathrm{w}_k\mathrm{w}_t^*) = 0 \tag{4.2}$$

We observe that, compared to the complete model (2.1) of a Markov jump linear system, the model (4.1) used in our presentation of the filtering problem has the same wide sense white noise as both the observation noise and the process noise. This is justified by the fact that in the filtering problem considered here, y_k represents the measured part of x_k, where the control input u_k is known to the controller and thus can be accounted for in the proposed structure of the Markov jump filter, in the same manner as it was done in Subsection 3.10. So, from the filter's point of view, the noise w_k contributes to the evolution of the system's state through the process noise transformation matrix H_{θ_k}, and it is observed by the Markov jump filter thanks to the observation noise matrix G_{θ_k}. That said, the closed loop MJLS considered in the problem of the optimal control with partial observation of the system's state x_k, which will be shown in the next chapter, will use some other noise sequence v_k, which is different from w_k, as a process noise (in order to derive the separation principle described in Section 5.3). See also Section 5.1 for the additional details.

The relevant for the filtering problem system matrices are $\boldsymbol{A} \triangleq (A_i)_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{x},n_\mathrm{x}}$, $\boldsymbol{B} \triangleq (B_i)_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{x},n_\mathrm{u}}$, $\boldsymbol{H} \triangleq (H_i)_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{x},n_\mathrm{w}}$, together with $\boldsymbol{F} \triangleq (F_i)_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{y},n_\mathrm{x}}$, and $\boldsymbol{G} \triangleq (G_i)_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{y},n_\mathrm{w}}$; they are the N-sequences of state, input, process noise, observation, and observation noise matrices, respectively, where each matrix in the related sequence is associated to an operational mode.

As usual, the initial conditions for a Markov jump linear system consist of the initial state of the system, x_0 (with $\boldsymbol{x}_0 \in \mathbb{F}^{n_\mathrm{x}}$ being its value), the initial state of the Markov chain θ_y, i.e., θ_0 (which has $\vartheta_0 \in \mathbb{M}$ as its value), and the initial probability distribution of the states of the jump variable θ_0, denoted by p_0 (and having $\boldsymbol{p}_0 \in \mathbb{R}_0^N$ as its value).

For the sake of completeness, we recall that being polytopic time-inhomogeneous for a Markov jump linear system means that it satisfies the Assumption 2.2.

So, we cast a finite-horizon robust optimization problem as a min-max *problem of optimizing robust performance, i.e., finding the minimum over the filtering error of the maximum over the transition probability disturbance.*

This problem can be presented also from the game-theoretic point of view, where at each time step $k \in \mathbb{T}$ the perturbation-player (environment and/or malicious adversary) tries to maximize the cost while the filter tries to minimize the cost.

Such formulation requires to make explicit the following assumption on the information structure for the observer and the adversary.

Assumption 4.1. *The perturbation-player has no information on the choice of the filter and vice versa.*

Following the usual notational conventions of this thesis, in particular those shown in appendix's Subsections B.4 and B.3, we denote by $P_{i\bullet}(k)$ the i-th row of the transition probability matrix $P(k)$ and by $P_{\bullet j}(k)$ the j-th column of the transition probability matrix $P(k)$. It is obvious from the expression (2.17) in Assumption 2.2 that $P_{i\bullet}(k)$ and $P_{\bullet j}(k)$ are polytopic sets of stochastic vectors. We also indicate by $P_{\bullet\theta_\mathrm{y}} \triangleq \left(P_{\bullet\theta_{t+1}}(t)\right)_{t=0}^{T-1}$ the sequence of the length $T \in \mathbb{T}$ of column vectors of the transition probability matrices $P(k)$, with $k \in \mathbb{T}_{T-1}$, where the elements of $P_{\bullet\theta_\mathrm{y}}$ clearly depend on the realizations of the Markov chain θ_y described in Subsection 2.2.

We assume without loss of generality [12, Remark 5.1, pp.103–104] that $\forall i \in \mathbb{M}$

$$H_i G_i^* = 0, \qquad G_i G_i^* \succ 0 \tag{4.3}$$

that is, each $H_i G_i^*$ is a matrix with all entries equal to zero, while each $G_i G_i^*$ is a positive definite matrix.

As in [12, p. 133], we assume independence of the noise sequence from the Markov chain and the initial conditions:

Assumption 4.2. *The disturbance* $\{\mathrm{w}_k : k \in \mathbb{T}\}$ *and the Markov chain* $\{\theta_k : k \in \mathbb{T}\}$ *are independent stochastic processes, and the initial conditions* (x_0, θ_0) *are independent random variables, with*

$$\mathbb{E}(\mathrm{x}_0) = \psi_0, \qquad \mathbb{E}(\mathrm{x}_0 \mathrm{x}_0^*) = \Psi_0 \tag{4.4}$$

We examine the problem of designing the optimal robust mode-dependent synchronous dynamic full-order Markov jump filter, described by the following family of systems of stochastic equations:

$$\begin{cases} \hat{\mathrm{x}}_{k+1} = \hat{A}_{\theta_k}(k)\hat{\mathrm{x}}_k + \hat{B}_{\theta_k}(k)\mathrm{y}_k, \\ \hat{\mathrm{x}}_0 = \hat{\boldsymbol{x}}_0 \end{cases} \tag{4.5}$$

with $\hat{\mathrm{x}}_k$ being the state of the filter, and the initial state $\hat{\mathrm{x}}_0$ being deterministic, chosen in a way to minimize quadratic functional cost of filtering, which will be formally defined shortly.

We define the *observation error* introduced by any Markov jump filter as

$$\hat{\mathrm{e}}_k = \mathrm{x}_k - \hat{\mathrm{x}}_k \tag{4.6}$$

and denote the *sequence of estimation errors* over a finite time horizon $T \in \mathbb{T}$ as

$$\hat{\mathrm{e}} \triangleq (\hat{\mathrm{e}}_k)_{k=1}^{T}$$

Then, the *estimation cost* is expressed by

$$\hat{\mathcal{J}}\left(\hat{\mathrm{e}}, P_{\bullet\theta_\mathrm{y}}\right) \triangleq \sum\nolimits_{k=1}^{T} \mathbb{E}\Big(\|\hat{\mathrm{e}}_k\|_2^2\Big) = \sum\nolimits_{k=1}^{T} \mathrm{tr}(\hat{\mathrm{e}}_k \hat{\mathrm{e}}_k^*) \tag{4.7}$$

Consequently, the *cost of robust filtering* is described by

$$\hat{\mathcal{J}}(\hat{\mathrm{e}}) \triangleq \max_{P_{\bullet\theta_\mathrm{y}}} \hat{\mathcal{J}}\left(\hat{\mathrm{e}}, P_{\bullet\theta_\mathrm{y}}\right) \tag{4.8}$$

and the *optimal cost of robust filtering* is defined for any sequence $\hat{\mathrm{e}}$ as

$$\hat{\mathcal{J}} \triangleq \min_{\hat{\mathrm{e}}} \hat{\mathcal{J}}(\hat{\mathrm{e}}) = \min_{\hat{\mathrm{e}}} \max_{P_{\bullet\theta_\mathrm{y}}} \sum\nolimits_{k=1}^{T} \mathbb{E}\Big(\|\hat{\mathrm{e}}_k\|_2^2\Big) \tag{4.9}$$

So, in summary, the finite-horizon robust optimization problem we study in this chapter is formally defined as follows.

Problem 4.1. *Given a discrete-time Markov jump linear system* (4.1) *with unknown and time-varying transition probability matrix* $P(k) \in \mathrm{conv}_{\mathbb{V}}\mathbb{P}$ *and satisfying Assumption 4.2, find* $\hat{\boldsymbol{A}}(k) \triangleq (\hat{A}_i)_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{x}, n_\mathrm{x}}$, $\hat{\boldsymbol{B}}(k) \triangleq (\hat{B}_i)_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{x}, n_\mathrm{y}}$, *and* $\hat{\mathrm{x}}_0$ *in* (4.5), *such that the optimal cost* $\hat{\mathcal{J}}$ *of robust filtering* (4.9) *is achieved.*

We underline that the setting of Problem 4.1 is in finite time-horizon and does not requires the assumption on detectability. We will see in in the next chapter that for what concerns the control problem, the stabilizability will be required even in the finite time horizon.

In the next section we present the solution to the stated problem of the optimal filtering, which is robust to the dynamic perturbations (that are bounded in the polytope) of the transition probability matrices of Markov jump linear systems.

4.2 Solution to the filtering problem

Let us consider a full-order synchronous Markov jump filter having a structure similar to the structure of Luenberger observer, i.e.,

$$\begin{cases} \tilde{\mathrm{x}}_{k+1} = A_{\theta_k}\tilde{\mathrm{x}}_k + B_{\theta_k}\mathrm{u}_k - L_{\theta_k}(k)(\mathrm{y}_k - \tilde{\mathrm{y}}_k)\,, \\ \quad \tilde{\mathrm{y}}_k = F_{\theta_k}\tilde{\mathrm{x}}_k, \\ \quad \tilde{\mathrm{x}}_0 \triangleq \mathbb{E}(\mathrm{x}_0) = \psi_0 \end{cases} \tag{4.10}$$

with $\boldsymbol{L}(k) \triangleq (L_i(k))_{i=1}^N$ being a vector of filter gain matrices, each of which associated to an operational mode, and $\tilde{\mathrm{x}}_k$ being the state of the observer, while $\tilde{\mathrm{y}}_k$ is the output of the Markov jump filter.

The associated filtering error for this particular structure of Markov jump filter is denoted by

$$\tilde{\mathrm{e}}_k = \mathrm{x}_k - \tilde{\mathrm{x}}_k \tag{4.11}$$

As before, the sequence of the filtering errors over a finite time horizon $T \in \mathbb{T}$ is indicated by

$$\tilde{\mathrm{e}} \triangleq (\tilde{\mathrm{e}}_k)_{k=1}^T$$

In this section we show that the considered filter (4.10) is indeed optimal, i.e., a filter achieving the optimal cost of robust filtering, that is $\hat{\mathcal{J}}$ defined by (4.9).

From the definitions (4.11) of the observation error, Markov jump linear system (4.1) under consideration, and the proposed full-order synchronous Markov jump filter (4.10), it is immediate to see that the estimation error has the following dynamics:

$$\tilde{\mathrm{e}}_{k+1} = (A_{\theta_k} + L_{\theta_k}F_{\theta_k})\,\tilde{\mathrm{e}}_k + (H_{\theta_k} + L_{\theta_k}G_{\theta_k})\,\mathrm{w}_k \tag{4.12}$$

with the initial error and the related expectation value given respectively by

$$\tilde{\mathrm{e}}_0 = \mathrm{x}_0 - \psi_0, \qquad \mathbb{E}(\tilde{\mathrm{e}}_0) = 0 \tag{4.13}$$

We have seen in the previous section, that the estimation cost (4.7) is related to the trace of the second moment of the estimation error $\tilde{\mathrm{e}}_k$.

So, we define for each $k \in \mathbb{T}$ the N-sequence of the second moments of the estimation errors associated to each operational mode as

$$\boldsymbol{Y}(k) = (Y_i(k))_{i=1}^N$$

where $\boldsymbol{Y}(k) \in {}_{\mathrm{N}}\mathbb{F}_0^{n_{\mathrm{x}},n_{\mathrm{x}}}$, and

$$Y_i(k) \triangleq \mathbb{E}\big(\tilde{\mathrm{e}}_k \tilde{\mathrm{e}}_k^* \mathbf{1}_{\{\theta_k = i\}}\big) \tag{4.14}$$

Clearly,

$$\hat{\mathcal{J}}\big(\tilde{\mathrm{e}}, P_{\bullet\theta_{\mathrm{y}}}\big) = \sum_{k=1}^{T} \sum_{i=1}^{N} \operatorname{tr}(Y_i(k))$$

After defining

$$\hat{\mathcal{J}}(\tilde{\mathrm{e}}_k, P_{\bullet i}(k-1)) \triangleq \operatorname{tr}(Y_i(k)) \tag{4.15}$$

we have that

$$\hat{\mathcal{J}}(\tilde{\mathrm{e}}, P_{\bullet\theta_{\mathrm{y}}}) = \sum_{k=1}^{T} \sum_{i=1}^{N} \hat{\mathcal{J}}(\tilde{\mathrm{e}}_k, P_{\bullet i}(k-1)) \tag{4.16}$$

From the definition of the probability distribution (B.79) of a random variable θ_k, it is immediate to see that such a probability distribution evolves according to the transition probabilities, i.e.,

$$p_j(k+1) = \sum_{i=1}^{N} p_i(k) p_{ij}(k) \tag{4.17}$$

From the definition (4.14) of the second moment of the observation error $Y_i(k)$ associated to an operational mode $i \in \mathbb{M}$, the definition (4.13) of the initial estimation error, Assumption 4.2, the definition (B.79) of the probability distribution of a random variable θ_k and the definition (B.86) its initial probability distribution, which value is provided as initial condition in (4.1), and the definition (4.10) of the Markov jump filter, it follows that $\tilde{\mathrm{x}}_0 = \psi_0$ is deterministic, and

$$Y_i(0) = p_i(0)(\Psi_0 - \psi_0 \psi_0^*) \tag{4.18}$$

To present the recursion of the second moment of the observation error $Y_i(k)$, it is useful to find first the values of the expected values of the cross-products between the noise and the system's state, the filter's state, and the observation error. These values are provided by the following lemma.

Lemma 4.1. *The following statements hold for each $k \in \mathbb{T}$, and for all $i \in \mathbb{M}$:*

$$\mathbb{E}\big(\mathrm{w}_k \mathrm{x}_k^* \mathbf{1}_{\{\theta_k = i\}}\big) = 0 \tag{4.19}$$

$$\mathbb{E}\big(\mathrm{w}_k \tilde{\mathrm{x}}_k^* \mathbf{1}_{\{\theta_k = i\}}\big) = 0 \tag{4.20}$$

$$\mathbb{E}\big(\mathrm{w}_k \tilde{\mathrm{e}}_k^* \mathbf{1}_{\{\theta_k = i\}}\big) = 0 \tag{4.21}$$

Proof. We construct our proof by induction on $k \in \mathbb{T}$.

The first statement, (4.19), is proved as follows.

For $k = 0$, from Assumption 4.2 and the definition (4.2) of the wide sense white noise, we have that w_0, θ_0 and x_0 are independent, and $\mathbb{E}(\mathrm{w}_0) = 0$. Hence,

$$\mathbb{E}\big(\mathrm{w}_0 \mathrm{x}_0^* \mathbf{1}_{\{\theta_0 = i\}}\big) = \mathbb{E}(\mathrm{w}_0)\, \mathbb{E}(\mathrm{x}_0^*)\, \mathbb{E}\big(\mathbf{1}_{\{\theta_k = i\}}\big) = 0$$

So, we suppose that $\mathbb{E}\big(\mathrm{w}_k \mathrm{x}_k^* \mathbf{1}_{\{\theta_k=i\}}\big)=0$.

From the definition (4.1) of the dynamical system representing the Markov jump linear system under consideration, it follows that

$$\mathrm{x}_{k+1} = \Big(\prod_{i=0}^{k} A_{\theta_i}^*\Big)^* \mathrm{x}_0 + \sum_{i=0}^{k} \Big(\prod_{j=i+1}^{k} A_{\theta_j}^*\Big)^* (B_{\theta_i}\mathrm{u}_i + H_{\theta_i}\mathrm{w}_i) \tag{4.22}$$

Since for each $i,k\in\mathbb{T}$, $i\le k$, A_{θ_i}, B_{θ_i} and H_{θ_i} are constant matrices, u_i is a deterministic input, from the previous equation, (4.22), combined with Assumption 4.2 and the definition (4.2) of the wide sense white noise, we obtain the desired result

$$\mathbb{E}\big(\mathrm{w}_{k+1}\mathrm{x}_{k+1}^*\mathbf{1}_{\{\theta_{k+1}=i\}}\big)=0$$

The proof of the second statement, i.e., (4.20), follows the same line of reasoning.

For $k=0$, from Assumption 4.2 and the dynamics (4.10) of the full-order synchronous Markov jump filter under consideration, we have that w_0 is independent from θ_0, and $\tilde{\mathrm{x}}_0=\psi_0$ is deterministic. Thence, from (4.2), we have that

$$\mathbb{E}\big(\mathrm{w}_0\tilde{\mathrm{x}}_0^*\mathbf{1}_{\{\theta_0=i\}}\big) = \mathbb{E}(\mathrm{w}_0)\,\psi_0^*\,p_i(0) = 0$$

Consequently, we suppose that $\mathbb{E}\big(\mathrm{w}_k\tilde{\mathrm{x}}_k^*\mathbf{1}_{\{\theta_k=i\}}\big)=0$.

From the definition (4.10) of the proposed Markov jump filter, we have that

$$\begin{aligned}\tilde{\mathrm{x}}_{k+1} = {} & \Big(\prod_{i=0}^{k} (A_{\theta_i}+L_{\theta_i}(i)F_{\theta_i})^*\Big)^*\tilde{\mathrm{x}}_0 + \\ & \sum_{i=0}^{k}\Big(\prod_{j=i+1}^{k}\big(A_{\theta_j}+L_{\theta_j}(j)F_{\theta_j}\big)^*\Big)^*\big(B_{\theta_i}\mathrm{u}_i + G_{\theta_i}\mathrm{w}_i + L_{\theta_j}(j)F_{\theta_j}\mathrm{x}_i\big)\end{aligned} \tag{4.23}$$

where for each $i,j,k\in\mathbb{T}$, $i\le k$, x_i can be expressed through (4.22), A_{θ_i}, B_{θ_i} G_{θ_i}, F_{θ_i} (and obviously, also A_{θ_j}, and F_{θ_j}) are constant matrices, u_i is a deterministic input, $L_{\theta_i}(i)$ is a gain matrix of our choice, from the previous expression (4.23) combined with (4.22), Assumption 4.2 and the definition (4.2) of the wide sense white noise, we obtain the desired result, that is,

$$\mathbb{E}\big(\mathrm{w}_{k+1}\tilde{\mathrm{x}}_{k+1}^*\mathbf{1}_{\{\theta_{k+1}=i\}}\big)=0$$

Finally, from the definition (4.11) of the filtering error for the proposed Markov jump filter and linearity of the expected value, it is clear that (4.21) is a direct consequence of (4.19) and (4.20). Thus, the lemma is proved. $\square$

Now we are ready to present the expression of the recursion for the second moment of the observation error $Y_i(k)$. From the definition (4.14) of $Y_i(k)$, the dynamics (4.12) of the estimation error, Assumption 4.2, the evolution (4.17) of the probability distribution p_i of the jump variable θ_k, and the expected value

(4.21) of the cross-product between the wide sense white noise and the observation error $\tilde{e}_k$ for each operational mode $i \in \mathbb{M}$, we have that

$$Y_j(k+1) = \sum_{i=1}^{N} p_{ij}(k)[(A_i + L_i(k)F_i)Y_i(k)(A_i^* + F_i^* L_i^*(k)) + p_i(k)(H_i H_i^* + L_i(k)G_i G_i^* L_i^*(k))] \tag{4.24}$$

This results permits us to state the following lemma.

Lemma 4.2. *At any time step $k \in \mathbb{T}$ and for any operational mode $j \in \mathbb{M}$, the maximum in transition probabilities of the estimation cost function $\hat{\mathcal{J}}(\tilde{e}_k, P_{\bullet j}(k{-}1))$ is attained on a vertex of the convex polytope of the column-vectors that define the j-th column of the polytopic transition probability matrix* (2.17)*, i.e.,*

$$\hat{\mathcal{J}}_j(\tilde{e}_k) = \max_{P_{\bullet j}(k-1)} \hat{\mathcal{J}}(\tilde{e}_k, P_{\bullet j}(k-1)) = \max_{P^l_{\bullet j}} \hat{\mathcal{J}}(\tilde{e}_k, P^l_{\bullet j}) \tag{4.25}$$

Proof. We construct our proof by induction on $k \in \mathbb{T}$.

From the definition (4.8) of the cost of robust filtering, the expression (4.16) of the total estimation cost as a sum of partial costs defined by (4.15), we know that

$$\hat{\mathcal{J}}_j(\tilde{e}_k) = \max_{P_{\bullet j}(k)} \operatorname{tr}(Y_j(k))$$

Since $\hat{\mathcal{J}}_j(\tilde{e}_k)$ is defined on a finite time horizon $(k)_{k=1}^{T}$, we start from the recursion (4.24) of the second moment of the observation error $Y_i(k)$ by examining its first step, $Y_j(1)$, filtered through $\mathcal{G}_0$ as a function of $p_{ij}(0)$.

From the model (4.1) of the dynamical system representing the MJLS under consideration, we see that, for each operational mode $i \in \mathbb{M}$, its initial probability distribution $p_i(0)$ is available as a deterministic quantity, A_i, F_i, G_i, H_i are constant matrices, $L_i(0)$ is a filtering gain matrix, which we are free to choose (in a way that minimizes the filtering error), and $Y_i(0)$ is computed from (4.18).

For any $j \in \mathbb{M}$, the value of j is fixed for $Y_j(1)$, and we have that $Y_j(1)$ is a function of only $P_{\bullet j}(0)$, which from Assumption 2.2 is defined as a convex hull of a polytopic set of transition probability matrices. Since the trace operator (B.33) is linear, it is immediate to verify that *Jensen's inequality* [102, p. 25, Theorem 4.3] holds for $\hat{\mathcal{J}}(\tilde{e}(1), P_{\bullet j}(0))$ considered as a function of only $P_{\bullet j}(0)$, that is, for all $\lambda_l(0)$ defined in (2.17)

$$\hat{\mathcal{J}}\left(\tilde{e}(1), \sum_{l=1}^{V} \lambda_l(0) P^l_{\bullet j}\right) = \sum_{l=1}^{V} \lambda_l(0) \hat{\mathcal{J}}\left(\tilde{e}(1), P^l_{\bullet j}\right)$$

Hence, $\hat{\mathcal{J}}\left(\tilde{e}(1), P_{\bullet j}(0)\right)$ is a *convex function* in the variable $P_{\bullet j}(0)$, which belongs to a *polytopic set*. From [102, p. 343, Theorem 32.2], this means that

$$\hat{\mathcal{J}}_j(\tilde{e}(1)) = \max_{P_{\bullet j}(0)} \hat{\mathcal{J}}(\tilde{e}(1), P_{\bullet j}(0)) = \max_{P^l_{\bullet j}} \hat{\mathcal{J}}(\tilde{e}(1), P^l_{\bullet j})$$

proving the statement for the base case.

Let us denote by

$$P_{\bullet j}^{\upsilon_0} \triangleq \arg \max_{P_{\bullet j}(0)} \hat{\mathcal{J}}_j(\tilde{\mathrm{e}}(1)) \tag{4.26}$$

the vertex of the convex polytope of the column-vectors (which define the j-th columns of the polytopic transition probability matrix) that achieves the maximal filtering cost at time step $k=1$ for the j-th operational mode. The vector $P_{\bullet j}^{\upsilon_0}$ is obtained during the computation of the robust filtering cost function $\hat{\mathcal{J}}_j(\tilde{\mathrm{e}}(1))$. It defines the value of $p_j(1)$ via (4.17) and the value of $Y_j(1)$ via (4.24).

Bearing in mind this fact, we are ready to prove the inductive step.

So, we suppose that hypothesis (4.25) holds. Then, $p_i(k-1)$ is available $\forall i \in \mathbb{M}$; $p_j(k)$ can be computed through (4.17) as

$$p_j(k) = \mathrm{p}_{k-1}^T \arg \max_{P_{\bullet j}(k-1)} \hat{\mathcal{J}}_j(\tilde{\mathrm{e}}_k), \tag{4.27}$$

where p_{k-1} is the probability distribution of all the operational modes at time step $k-1$, defined similarly to p_0 in (2.10) as

$$\mathrm{p}_{k-1} \triangleq \begin{bmatrix} p_1(k-1) \\ p_2(k-1) \\ \vdots \\ p_N(k-1) \end{bmatrix} \in \mathbb{R}^{N,1} \tag{4.28}$$

and for any choice of $\boldsymbol{L}(k-1)$, we have that $\boldsymbol{Y}(k)$ is uniquely determined by the corresponding $P_{\bullet j}^{\upsilon_{k-1}}$.

Thus, in the expression (4.24) of the recursion for the second moment of the observation error we have that all the parameters, apart from $p_{ij}(k)$, are deterministic. The remaining considerations are the same as for the base case. They are based on Assumption 2.2, Jensen's inequality, and the fact that the maximum of any convex function defined on a variable, which belongs to a polytopic set, is attained on a vertex of the polytope, that bounds the domain of the named variable. This leads us to the desired result and concludes the proof. $\square$

Let us denote by

$$P_{\bullet j}^{\upsilon_{k-1}} \triangleq \arg \max_{P_{\bullet j}(k-1)} \hat{\mathcal{J}}_j(\tilde{\mathrm{e}}_k) \tag{4.29}$$

the vertex of the convex polytope of the column-vectors (which define the j-th columns of the polytopic transition probability matrix) that achieves the maximal filtering cost at time step k for the j-th operational mode.

The vector $P_{\bullet j}^{\upsilon_{k-1}}$ is obtained during the computation of the robust filtering cost function $\hat{\mathcal{J}}_j(\tilde{\mathrm{e}}_k)$. It defines the value of $p_j(k)$ via (4.17) and the value of $Y_j(k)$ via (4.24), allowing the recursion.

The question of choosing $\boldsymbol{L}(k)$, in a way to minimize the filtering error, remains open, and it is tackled in the remaining of this section.

Let us compute the value of $\mathbb{E}\big(\tilde{\mathrm{e}}_k \tilde{\mathrm{x}}_k^* \mathbf{1}_{\{\theta_k=i\}}\big)$.

For $k=0$, from the definition (4.11) of the observation error $\tilde{\mathrm{e}}_k$, its initial value and the related expectation (4.13), first and second moments (4.4) of x_0 found in Assumption 4.2, and the linearity of the expected value, we have that for every $i \in \mathbb{M}$

$$\mathbb{E}\big(\tilde{\mathrm{e}}_0 \tilde{\mathrm{x}}_0^* \mathbf{1}_{\{\theta_0=i\}}\big) = 0 \tag{4.30}$$

Following the mathematical induction technique, we assume that $\forall k \in (k)_{k=1}^{T}$, $T \in \mathbb{T}$, the structure of $\boldsymbol{L}(k-1)$ is such that for each $i \in \mathbb{M}$, $\mathbb{E}\big(\tilde{\mathrm{e}}_k \tilde{\mathrm{x}}_k^* \mathbf{1}_{\{\theta_k=i\}}\big) = 0$, and proceed to find $\boldsymbol{L}(k)$ such that $\mathbb{E}\big(\tilde{\mathrm{e}}_{k+1} \tilde{\mathrm{x}}_{k+1}^* \mathbf{1}_{\{\theta_{k+1}=j\}}\big) = 0$, for all $j \in \mathbb{M}$.

From the definition (4.11) of $\tilde{\mathrm{e}}_k$ and the related recursion (4.12), the definition of the Markov jump filter (4.10), the state-space representation (4.1) of the MJLS, linearity of the expected value, the fact that, $\forall i \in \mathbb{M}$, the matrices A_i, B_i, F_i, G_i, H_i are constant, $L_i(k)$ is determined by our choice, u_k is deterministic, independence between w_k, θ_k, and $x(0)$, given by Assumption 4.2, the characteristics (4.2) of the wide sense white noise, (4.21) from Lemma 4.1, the induction hypothesis, the definition (4.14) of the second moment of the estimation error for an operational mode, and the properties (4.3) of $H_i G_i^*$ and $G_i G_i^*$, we have that

$$\begin{aligned}\mathbb{E}\left(\tilde{\mathrm{e}}_{k+1} \tilde{\mathrm{x}}_{k+1}^* \mathbf{1}_{\{\theta_{k+1}=j\}}\right) = -\sum\nolimits_{i=1}^{N} p_{ij}(k) [(A_i + L_i(k) F_i)\, Y_i(k)\, (A_i^* + F_i^* L_i^*(k)) + \\ p_i(k) L_i(k) G_i G_i^* L_i^*(k)]\end{aligned}$$

which is equal to zero if (and only if, for every $p_{ij}(k) \neq 0$)

$$L_i(k) = \begin{cases} 0 \quad \text{if } p_i(k) = 0 \\ -A_i Y_i(k) F_i^* \left(F_i Y_i(k) F_i^* + p_i(k) G_i G_i^*\right)^{-1} \quad \text{otherwise} \end{cases} \tag{4.31}$$

which is obtained by observing that $p_i(k) = 0$ implies that $Y_i(k) = 0$ (see (4.17) and (4.14)), and thus also $L_i(k) = 0$.

For the notational convenience, we define

$$\mathbb{P}_k \triangleq \{i \in \mathbb{M} : p_i(k) \neq 0\} \tag{4.32}$$

From (4.31), (4.32), and the definition (4.29) of $P_{\bullet j}^{\upsilon_k}$, we have that

$$Y_j(k+1) = \sum\nolimits_{i \in \mathbb{P}_k} p_{ij}^{\upsilon_k} (A_i Y_i(k) A_i^* + p_i(k) H_i H_i^* + L_i(k) F_i Y_i(k) A_i^*) \tag{4.33}$$

where $p_{ij}^{\upsilon_k}$ are the elements of $P_{\bullet j}^{\upsilon_k}$.

The following two lemmas are useful within the final proof that the observer (4.10), having a filtering gain (4.31) computed interactively from (4.18), (4.33), and (4.29) is optimal.

Lemma 4.3. *The following statement holds for each $i\in\mathbb{M}$, $k\in\mathbb{T}$:*

$$\mathbb{E}\big(\tilde{\mathrm{e}}_k\hat{\mathrm{x}}_k^*\mathbf{1}_{\{\theta_k=i\}}\big)=0 \tag{4.34}$$

Proof. We construct our proof by induction on $k\in\mathbb{T}$.

For $k=0$, from (4.13), Assumption 4.2 on the initial conditions, and the fact that $\hat{\mathrm{x}}_0$ is deterministic, as stated in (4.5), we have that $\mathbb{E}\big(\tilde{\mathrm{e}}(0)\hat{\mathrm{x}}^*(0)\mathbf{1}_{\{\theta(0)=i\}}\big)=0$.

Then, we can suppose that $\mathbb{E}\big(\tilde{\mathrm{e}}_k\hat{\mathrm{x}}_k^*\mathbf{1}_{\{\theta_k=i\}}\big)=0$.

From the definition (4.11) of the estimation error $\tilde{\mathrm{e}}_k$ of the proposed filter (4.10) and the related recursion (4.12) of the estimation error, the state-space representation (4.1) of the Markov jump linear system, the definition (4.5) of a generic Markov jump filter, the fact that the matrices A_i, F_i, G_i are constant, $\hat{A}_i(k)$, $\hat{B}_i(k)$, $L_i(k)$ are deterministic, $\mathbb{E}\big(\tilde{\mathrm{e}}_k\tilde{\mathrm{x}}_k^*\mathbf{1}_{\{\theta_k=i\}}\big)=0$ by construction of $L_i(k)$, induction hypothesis, (4.21) and (4.20) from Lemma 4.1, the definition (4.2) of a wide sense white noise, Assumption 4.2, the properties (4.3) of $H_iG_i^*$ and $G_iG_i^*$, and the same argument used to prove (4.20) in stating that $\forall i\in\mathbb{M}$ and $\forall k\in\mathbb{T}$ (where $k\geq 1$ and $k\leq T$), $\mathbb{E}\big(\mathrm{w}_k\hat{\mathrm{x}}_k^*\mathbf{1}_{\{\theta_k=i\}}\big)=0$, we obtain that

$$\mathbb{E}\big(\tilde{\mathrm{e}}_{k+1}\tilde{\mathrm{x}}_{k+1}^*\mathbf{1}_{\{\theta_{k+1}=i\}}\big)=\sum_{i=1}^{N} p_{ij}(k)[A_iY_i(k)F_i^*\hat{B}_i^*(k)+L_i(k)G_iG_i^*\hat{B}_i^*(k)p_i(k)]$$

which, after substitution of $L_i(k)$ with its explicit expression from (4.31), equals to zero. Thus, the lemma is proved. □

Lemma 4.4. *Let $\hat{\mathrm{e}}_k$ be the error introduced by any Markov jump filter, as described by* (4.6)*, and $\mathbf{Y}(k)$ be the solution of the system of coupled Riccati difference equations associated to robust filtering problem at time step k, obtained from* (4.33), (4.31)*, and* (4.29)*. Then, $\forall k\in\mathbb{T}$, $\mathbf{E}\big(\|\hat{\mathrm{e}}_k\|_2^2\big)\geq\sum_{i=1}^{N}\mathrm{tr}\,(Y_i(k))$.*

Proof. From the definition (4.6) of the estimation error $\hat{\mathrm{e}}_k$ introduced by any Markov jump filter, and the definition (4.11) of the observation error $\tilde{\mathrm{e}}_k$ of the proposed Markov jump filter, the definition (4.14) of the second moment $Y_i(k)$ of $\tilde{\mathrm{e}}_k$, the definition (4.7) of the estimation cost, linearity of the trace operator and linearity of the expected value, follows that

$$\mathbb{E}\big(\|\hat{\mathrm{e}}_k\|_2^2\big)=\sum_{i=1}^{N}\mathrm{tr}\big(\mathbb{E}\big((\tilde{\mathrm{e}}_k+\tilde{\mathrm{x}}_k-\hat{\mathrm{x}}_k)(\tilde{\mathrm{e}}_k+\tilde{\mathrm{x}}_k-\hat{\mathrm{x}}_k)^*\mathbf{1}_{\{\theta_k=i\}}\big)\big)$$

From (4.34) in Lemma 4.3 and since $\mathbb{E}\big(\tilde{\mathrm{e}}_k\tilde{\mathrm{x}}_k^*\mathbf{1}_{\{\theta_k=i\}}\big)=0$ by construction of $L_i(k)$, the previous expression becomes

$$\mathbb{E}\big(\|\hat{\mathrm{e}}_k\|_2^2\big)=\sum_{i=1}^{N}\mathrm{tr}\,(Y_i(k))+\mathbf{E}\big(\|\tilde{\mathrm{x}}_k-\hat{\mathrm{x}}_k\|^2\big)\geq\sum_{i=1}^{N}\mathrm{tr}(Y_i(k))=\mathbb{E}\big(\|\tilde{\mathrm{e}}_k\|_2^2\big)$$

and the lemma is proved. □

Then, the main result of this chapter is straightforward from Lemma 4.4, and the state-space representation of a generic Markov jump filter (4.5) and the proposed filter (4.10).

Theorem 4.5. *An optimal solution for the robust filtering problem posed above is:*

$$\hat{A}_i(k) = A_i + L_i(k)F_i, \qquad \hat{B}_i(k) = -L_i(k) \tag{4.35}$$

with $L_i(k)$ *as in* (4.31)*, obtained from* (4.33) *and* (4.29)*, and the optimal robust cost defined in* (4.9) *is*

$$\hat{\mathcal{J}} = \sum_{k=1}^{T} \sum_{i=1}^{N} \operatorname{tr}(Y_i(k)) \tag{4.36}$$

Proof. Direct consequence of Lemma 4.4, considering the definitions of a generic Markov jump filter (4.5) and the proposed filter (4.10). □

This result plays an important role in the derivation of the optimal dynamic Markov jump controller having only partial information on the continuous state of a Markov jump linear system with polytopic time-inhomogeneous transition probabilities, as will be shown in the next chapter, where we present both the optimal controller for the case where all the state variables are available to the controller and the principle of separation of estimation and control for the case of partial observations of the system's states.

Chapter 5

Optimal robust control

An extensively studied and classical control problem is that of operating a dynamical system at minimum cost. The idea of minimizing a cost function is to drive the state of the system to the origin without much strain from the control variable which is, in general, a desirable behavior for control systems [12].

When the system dynamics are represented by a set of linear difference equations and the cost is described by a quadratic function, the problem is referred to as linear-quadratic regulator (from now on, LQR) problem. Its solution (see for instance the original paper, [103]) is an important part of the solution to the linear-quadratic-Gaussian (or LQG) control problem too.

In this chapter we first and foremost formally define and solve analytically via a dynamic programming approach [104] the Markov jump version in polytopic time-inhomogeneous setting of the finite-horizon robust linear quadratic regulation problem. The provided solution consists of a finite set of recursive coupled Riccati difference equations. This result is an extension of state-of-the-art which is non-trivial from the technical point of view, since in the proof we need to show that due to the time-varying nature of perturbations, at generic time step k the vertex that attains the maximum is unknown and state dependent: with respect to previous works on MJLSs having exactly known transition probabilities, we need to define and address the explosion of the number of coupled Riccati difference equations. Then, in the last section of the chapter, we present the separation principle, which allows us to solve the generalization of the classical LQG problem also for Markov jump linear systems with time-varying and unmeasurable perturbations on transition probabilities.

This presentation is based on our recent work [3], accepted by the 56th IEEE Conference on Decision and Control, as was already mentioned in the previous chapter, dedicated to the optimal robust filtering problem.

In the first section we give a formal definition of the optimal linear quadratic state-feedback control problem for Markov jump linear systems with bounded perturbations of the transition probability matrices. Then, in Section 5.2 we present the related solution, while in the last section, 5.3, we illustrate the principle of separation of estimation and control for the polytopic time-inhomogeneous Markov jump linear systems.

5.1 Problem statement

Let us consider the discrete-time polytopic time-inhomogeneous Markov jump linear system as in (2.1) defined on the stochastic basis $(\Omega_{\mathrm{x}}, \mathcal{G}, (\mathcal{G}_k), \mathrm{Pr})$ described in Subsection 2.2, and represented by the following dynamical system

$$\begin{cases} \mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k + B_{\theta_k}\mathrm{u}_k + \Phi_{\theta_k}\mathrm{v}_k, \\ \quad \mathrm{z}_k = C_{\theta_k}\mathrm{x}_k + D_{\theta_k}\mathrm{u}_k, \\ \quad \mathrm{x}_0 = \boldsymbol{x}_0,\ \theta_0 = \vartheta_0,\ \mathrm{p}_0 = \boldsymbol{p}_0 \end{cases} \tag{5.1}$$

where the system's variables, initial states and most of the matrices are those defined in Section 2.1. In particular, $\mathrm{x}_k \in \mathbb{F}^{n_{\mathrm{x}}}$ is a vector of system's state variables, $\mathrm{u}_k \in \mathbb{F}^{n_{\mathrm{u}}}$ is control vector, $\mathrm{z}_k \in \mathbb{F}^{n_{\mathrm{z}}}$ is a vector of system output variables, while the exogenous input $\mathrm{v}_k \in \mathbb{F}^{n_{\mathrm{v}}}$ is a process noise representing the discrepancies between the model and the real process, due to unmodeled dynamics or disturbances.

In order to be able to apply a separation principle (described in Section 5.3), the process noise considered here is more general than the wide sense white noise w_k in the previous chapter, being required to satisfy only the following assumption, borrowed from [12, p. 73].

Assumption 5.1. *The stochastic process* $\{\mathrm{v}_k : k \in \mathbb{T}\}$ *describing the process noise satisfies for each* $i \in \mathbb{M}$ *the following equations:*

$$\mathbb{E}\big(\mathrm{v}_k\mathrm{v}_k^*\mathbf{1}_{\{\theta_k=i\}}\big) = R_i(k) \in \mathbb{F}_0^{n_{\mathrm{v}},n_{\mathrm{v}}}, \qquad \mathbb{E}\big(\mathrm{v}_0\mathrm{x}_0^*\mathbf{1}_{\{\theta_k=i\}}\big) = 0 \tag{5.2}$$

Furthermore, for all measurable functions $f(\cdot)$ *and* $g(\cdot)$ *we have that*

$$\mathbb{E}(f(\mathrm{v}_k)g(\theta_{k+1}) \mid \mathcal{G}_k) = \mathbb{E}(f(\mathrm{v}_k) \mid \mathcal{G}_k) \sum\nolimits_{j=1}^{N} p_{\theta_k j}(k) g(j) \tag{5.3}$$

In Section 5.3 we will show that this assumption is verified for the controlled system with partial information on continuous state.

The set of admissible controllers, denoted by $\mathbb{U}_{\mathrm{T}}$, is given by the sequence of control laws

$$\mathrm{u} \triangleq (\mathrm{u}_k)_0^{T-1}$$

such that for each $k \in \mathbb{T}_{T-1}$, we have that u_k is $\mathcal{G}_k$-measurable, and both the following two expressions hold.

$$\mathbf{E}[\mathrm{v}_k\mathrm{x}_k^*\mathbf{1}_{\{\theta_k=i\}}] = 0 \tag{5.4}$$

$$\mathbf{E}[\mathrm{v}_k\mathrm{u}_k^*\mathbf{1}_{\{\theta_k=i\}}] = 0 \tag{5.5}$$

The relevant for the control problem system matrices are $\boldsymbol{A} \triangleq (A_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{x}, n_\mathrm{x}}$, $\boldsymbol{B} \triangleq (B_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{x}, n_\mathrm{u}}$, $\boldsymbol{C} \triangleq (H_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{z}, n_\mathrm{x}}$, together with $\boldsymbol{D} \triangleq (F_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{z}, n_\mathrm{u}}$, and $\boldsymbol{\Phi} \triangleq (\varPhi_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_\mathrm{x}, n_\mathrm{v}}$; they are the N-sequences of state, input, output, direct transition, and process noise matrices, respectively, where each matrix in the related sequence is associated to an operational mode.

As before, the initial conditions for a Markov jump linear system consist of the initial system's state x_0, and the initial state of the jump variable θ_0, considered together with the initial probability distribution of its states, p_0. The respective values are $\boldsymbol{x}_0$, ϑ_0 and $\boldsymbol{p}_0$.

We assume without loss of generality [12, Remark 4.1, p. 74] that $\forall i \in \mathbb{M}$

$$C_i^* D_i = 0, \qquad D_i^* D_i \succ 0 \tag{5.6}$$

that is, each $C_i^* D_i$ is a matrix with all entries equal to zero, while each $D_i^* D_i$ is assumed to be a positive definite matrix.

For the sake of completeness, we remind again that a Markov jump linear system considered also in this chapter is polytopic time-inhomogeneous, that is, it satisfies Assumption 2.2.

As in the previous chapter, we follow the usual notational conventions of this thesis, specifically those of appendix's Subsections B.4 and B.3, and denote by $P_{i\bullet}(k)$ the i-th row of the transition probability matrix $P(k)$ and by $P_{\bullet j}(k)$ its j-th column. It is clear from (2.17) in Assumption 2.2 that $P_{i\bullet}(k)$ and $P_{\bullet j}(k)$ are polytopic sets of stochastic vectors. We indicate by $P_{\theta_\mathrm{x}\bullet} \triangleq (P_{\theta_t\bullet}(t))_{t=0}^{T-1}$ the sequence of the length $T \in \mathbb{T}$ of row vectors of the transition probability matrices $P(k)$, with $k \in \mathbb{T}_{T-1}$, where the elements of $P_{\theta_\mathrm{x}\bullet}$ obviously depend on the realizations of the Markov chain θ_x described in Subsection 2.2.

Then, the problem of designing the optimal *mode-dependent* state-feedback Markov jump controller, which is robust to all possible bounded perturbations of transition probabilities, is formally defined as follows.

Problem 5.1. *Given a discrete-time Markov jump linear system* (5.1) *with unknown and time-varying transition probability matrix* $P(k) \in \mathrm{conv}_{\mathbb{V}}\mathbb{P}$ *and satisfying Assumption 5.1, find the optimal sequence* $\mathrm{u} = (\mathrm{u}_k)_0^{T-1} \in \mathbb{U}_\mathrm{T}$ *of state-feedback mode-dependent control inputs, such that the following optimal cost of robust control is achieved.*

$$\mathscr{J}(\theta_0, \mathrm{x}_0) \triangleq \min_{\mathrm{u}} \max_{P_{\theta_\mathrm{x}\bullet}} \sum_{k=0}^{T-1} \mathbb{E}\big(\|\mathrm{z}_k\|_2^2\big) + \mathbb{E}(\mathrm{x}_T^* Z_{\theta_T} \mathrm{x}_T) \tag{5.7}$$

with $\boldsymbol{Z} \triangleq (Z_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}_0^{n_\mathrm{x}, n_\mathrm{x}}$ *being a sequence of the terminal cost weighting matrices.*

We remark that, like for the optimal robust filtering problem, we cast a finite-horizon robust control optimization problem as a min-max *problem of optimizing robust performance*, that is, *finding the minimum over the control input of the maximum over the transition probability disturbance.*

Thus, this problem too can be presented from the game-theoretic point of view, where at each time step $k \in \mathbb{T}$ the perturbation-player (environment and/or malicious adversary) tries to maximize the cost while the controller tries to minimize the cost. As it was done for the robust filter, the game-theoretic formulation of the optimal robust control problem requires to make explicit the following assumption on the information structure for the controller and the adversary.

Assumption 5.2. *The perturbation-player has no information on the choice of the controller and vice versa.*

5.2 Solution to the state-feedback problem

Our solution to the problem 5.1 is based on a *dynamic programming* approach in *Bellman's optimization formulation* [104]. It is obtained by backward induction.

Since x_T is $\mathcal{G}_T$-measurable on the stochastic basis $(\Omega_{\mathrm{x}}, \mathcal{G}, (\mathcal{G}_k), \Pr)$, by linearity of the expected value, it is clear from the definition (5.7) of the cost of robust control, that the terminal cost is given by

$$\mathcal{J}(\theta_T, \mathrm{x}_T) = \mathbb{E}(\mathrm{x}_T^* X_{\theta_T}(T)\mathrm{x}_T \mid \mathcal{G}_T) = \mathrm{x}_T^* \mathbb{E}(X_{\theta_T}(T) \mid \mathcal{G}_T)\,\mathrm{x}_T = \mathrm{x}_T^* Z_{\theta_T} \mathrm{x}_T \tag{5.8}$$

where $X_{\theta_T}(T) \triangleq Z_{\theta_T}$ is a solution to the coupled Riccati difference equation (or, CRDE, for short) for the robust control at the terminal time step.

We are interested in the explicit form of $\boldsymbol{X}(k) = (X_i(k))_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}_0^{n_{\mathrm{x}}, n_{\mathrm{x}}}$.

A generic cost at time step $k \in \mathbb{T}_{T-1}$ is

$$\mathcal{J}(\theta_k, \mathrm{x}_k, \mathrm{u}_k, P_{\theta_k \bullet}(k)) = \mathbb{E}\Big(\|\mathrm{z}_k\|_2^2 + \mathcal{J}(\theta_{k+1}, x_{k+1}) \mid \mathcal{G}_k\Big) \tag{5.9}$$

The cost *cost-to-go function* is defined as

$$\mathcal{J}(\theta_k, \mathrm{x}_k) = \min_{\mathrm{u}_k} \max_{P_{\theta_k \bullet}(k)} \mathcal{J}(\theta_k, \mathrm{x}_k, \mathrm{u}_k, P_{\theta_k \bullet}(k)) \tag{5.10}$$

From the definition of the expected value (B.78), the cost-to-go function (5.10) is equal to

$$\mathcal{J}(\theta_k, \mathrm{x}_k) = \min_{\mathrm{u}_k} \max_{P_{[1,N]\bullet}(k)} \sum_{i=1}^N p_i(k) \mathcal{J}(i, \mathrm{x}_k, \mathrm{u}_k, P_{i\bullet}(k)) \tag{5.11}$$

We start by examining the cost-to-go function (5.10) at time step $k = T-1$.

To improve the readability of the following steps, let us write $\boldsymbol{\vartheta}$ as a *short notation for* θ_k, which for $k = T-1$ *corresponds to* θ_{T-1}.

First, we observe that, from the definitions of the expected value (B.78), the indicator function (3.10), and of the transition probability (2.5) between the operational modes, for $\boldsymbol{\vartheta} = i$, we have that

$$\mathbb{E}(X_{\theta_T}(T) \mid \mathcal{G}_{T-1}) = \sum_{j=1}^N p_{ij}(T-1) X_j(T) \tag{5.12}$$

Then, the generic cost for each $i \in \mathbb{M}$ in (5.10) at time step $k=T-1$ is obtained from the expressions of z_{T-1} and x_T in state-space representation (5.1) of the Markov jump linear system and from the equality (5.6) for $C_i^* D_i$.

Since x_{T-1}, θ_{T-1}, and any admissible control input u_{T-1} are $\mathcal{G}_{T-1}$-measurable, the state matrices A_i, B_i, C_i, D_i and Φ_i are constant matrices for all $i \in \mathbb{M}$, $\boldsymbol{X}(T) \in {}_{\mathrm{N}}\mathbb{F}_0^{n_\mathrm{x},n_\mathrm{x}}$, by linearity of the expected value, from Assumption 5.1 on process noise, the expressions (5.4) and (5.5) characterizing all the admissible control inputs in the sequence $\mathrm{u} \in \mathbb{U}_\mathrm{T}$, the fact (B.35) that the trace is invariant under cyclic permutations of matrix product, and just obtained expression (5.12), we have, for each $i \in \mathbb{M}$, in case $\boldsymbol{\vartheta}=i$, that

$$\begin{aligned} &\mathcal{J}(i, \mathrm{x}_{T-1}, \mathrm{u}_{T-1}, P_{i\bullet}(T-1)) = \\ &\quad \mathrm{x}_{T-1}^* \left(C_i^* C_i + A_i^* \sum\nolimits_{j=1}^{N} p_{ij}(T-1) X_j(T) A_i \right) \mathrm{x}_{T-1} + \\ &\quad 2\mathrm{x}_{T-1}^* A_i^* \sum\nolimits_{j=1}^{N} p_{ij}(T-1) X_j(T) B_i \mathrm{u}_{T-1} + \\ &\quad \mathrm{u}_{T-1}^* \left(D_i^* D_i + B_i^* \sum\nolimits_{j=1}^{N} p_{ij}(T-1) X_j(T) B_i \right) \mathrm{u}_{T-1} + \\ &\quad \sum\nolimits_{i=1}^{N} \mathrm{tr}\left(\Phi_i R_i(T-1) \Phi_i^* \sum\nolimits_{j=1}^{N} p_{ij}(T-1) X_j(T) \right) \end{aligned} \tag{5.13}$$

So, a generic cost from (5.10) at time step $k = T-1$ is expressed as follows, where the left-hand side of (5.13) may be substituted with its right-hand side.

$$\mathcal{J}(\boldsymbol{\vartheta}, \mathrm{x}_{T-1}, \mathrm{u}_{T-1}, P_{\boldsymbol{\vartheta}\bullet}(T-1)) = \sum\nolimits_{i=1}^{N} p_i(T-1) \mathcal{J}(i, \mathrm{x}_k, \mathrm{u}_{T-1}, P_{i\bullet}(T-1)) \tag{5.14}$$

We notice that the probability distribution $p_i(T-1)$ (which evolves according to (4.17)) at time step $k=T-1$ is independent from the transition probability values in the same time step. So, we consider the previous equation (5.14) as a function of only transition probabilities $P_{i\bullet}(T-1)$, examined for all $i \in \mathbb{M}$, i.e., as a function of the entire transition probability matrix $\left[P_{[1,N]\bullet}(T-1)\right]$.

From Assumption 2.2, the transition probability matrix $P(T-1)$ is polytopic. It is straightforward verifying that *Jensen's inequality* [102, p. 25, Theorem 4.3] holds for the generic cost function (5.14), with the addends in the right-hand side defined by (5.13). That is, for all $\lambda_l(T-1)$ from (2.17) in Assumption 2.2, we have that, by linearity and convexity of the considered expression

$$\begin{aligned} &\sum\nolimits_{i=1}^{N} p_i(T-1) \mathcal{J}\left(i, \mathrm{x}_{T-1}, \mathrm{u}_{T-1}, \sum\nolimits_{l=1}^{V} \lambda_l(T-1) P_{i\bullet}^l \right) = \\ &\qquad\qquad \sum\nolimits_{l=1}^{V} \lambda_l(T-1) \sum\nolimits_{i=1}^{N} p_i(T-1) \mathcal{J}\left(i, \mathrm{x}_k, \mathrm{u}_{T-1}, P_{i\bullet}^l\right) \end{aligned}$$

Hence, this generic quadratic cost is a *convex function* in a variable, that belongs to a *polytopic set.* From [102, p. 343, Theorem 32.2] this means that the maximum in transition probabilities of the quadratic cost function is attained on a vertex of the convex polytope of transition probabilities. This vertex should be the same for all operational modes at the considered time step $k = T-1$. Formally, we write that

$$\max_{P_{\boldsymbol{\vartheta}\bullet}(T-1)} \mathcal{J}(\boldsymbol{\vartheta}, \mathrm{x}_{T-1}, \mathrm{u}_{T-1}, P_{\boldsymbol{\vartheta}\bullet}(T-1)) = \max_{P_l \in \mathbb{VP}} \mathcal{J}\big(\boldsymbol{\vartheta}, \mathrm{x}_{T-1}, \mathrm{u}_{T-1}, P^l_{\boldsymbol{\vartheta}\bullet}\big)$$

Then, the expression (5.11) for the cost-to-go function at $k = T-1$ becomes

$$\mathcal{J}(\boldsymbol{\vartheta}, \mathrm{x}_{T-1}) = \min_{\mathrm{u}_{T-1}} \max_{P_l \in \mathbb{VP}} \sum\nolimits_{i=1}^{N} p_i(k) \mathcal{J}\big(i, \mathrm{x}_{T-1}, \mathrm{u}_{T-1}, P^l_{i\bullet}\big) \tag{5.15}$$

So, to find the maximum in transition probability disturbances, we need to evaluate the expectation of the generic cost $\mathcal{J}\big(\boldsymbol{\vartheta}, \mathrm{x}_{T-1}, \mathrm{u}_{T-1}, P^l_{\boldsymbol{\vartheta}\bullet}\big)$ in each vertex $P_l \in \mathbb{VP}$ of the polytopic transition probability matrix.

Since in Problem 5.1 under consideration we are interested in mode-dependent control input, we compute the minimum in u_{T-1} directly on $\mathcal{J}\big(\boldsymbol{\vartheta}, \mathrm{x}_{T-1}, \mathrm{u}_{T-1}, P^l_{\boldsymbol{\vartheta}\bullet}\big)$ by equaling to zero its derivative with respect to u_{T-1}, obtaining that

$$2\mathrm{u}^*_{T-1}\left(D^*_{\boldsymbol{\vartheta}} D_{\boldsymbol{\vartheta}} + B^*_{\boldsymbol{\vartheta}} \sum\nolimits_{j=1}^{N} p^l_{\boldsymbol{\vartheta}j} X_j(T) B_{\boldsymbol{\vartheta}}\right) + 2\mathrm{x}^*_{T-1} A^*_{\boldsymbol{\vartheta}} \sum\nolimits_{j=1}^{N} p^l_{\boldsymbol{\vartheta}j} X_j(T) B_{\boldsymbol{\vartheta}} = 0$$

Thence, it follows immediately that

$$\mathrm{u}_{T-1} = -\left(D^*_{\boldsymbol{\vartheta}} D_{\boldsymbol{\vartheta}} + B^*_{\boldsymbol{\vartheta}} \sum\nolimits_{j=1}^{N} p^l_{\boldsymbol{\vartheta}j} X_j(T) B_{\boldsymbol{\vartheta}}\right)^{-1} B^*_{\boldsymbol{\vartheta}} \sum\nolimits_{j=1}^{N} p^l_{\boldsymbol{\vartheta}j} X_j(T) A_{\boldsymbol{\vartheta}}\, \mathrm{x}_{T-1}$$

So, we see that the optimal gain at time $k = T-1$, considering the transition probability vector $P^l_{\boldsymbol{\vartheta}\bullet}$, $l \in \mathbb{V}$, $\boldsymbol{\vartheta} \in \mathbb{M}$, is given by

$$K^l_{\boldsymbol{\vartheta}}(T-1) = -\left(D^*_{\boldsymbol{\vartheta}} D_{\boldsymbol{\vartheta}} + B^*_{\boldsymbol{\vartheta}} \sum\nolimits_{j=1}^{N} p^l_{\boldsymbol{\vartheta}j} X_j(T) B_{\boldsymbol{\vartheta}}\right)^{-1} B^*_{\boldsymbol{\vartheta}} \sum\nolimits_{j=1}^{N} p^l_{\boldsymbol{\vartheta}j} X_j(T) A_{\boldsymbol{\vartheta}} \tag{5.16}$$

Then, the state-feedback control input at time $k = T-1$ for i-th operational mode and the same transition probability vector as before is obviously

$$\mathrm{u}_{T-1} = K^l_i(T-1)\, \mathrm{x}_{T-1} \tag{5.17}$$

We observe that among V vertices of convex polytope of transition probability matrices P_l, there is one, indicated by $P_{\bar{v}}$, for which $\mathcal{J}(\boldsymbol{\vartheta}, \mathrm{x}_{T-1})$ of (5.10) is attained. We denote by $K^{\bar{v}}_{\boldsymbol{\vartheta}}(T-1)$ the corresponding optimal gain.

When for each operational mode $i \in \mathbb{M}$ the corresponding optimal control input $\mathrm{u}_{T-1} = K^{\bar{v}}_i(T-1)\, \mathrm{x}_{T-1}$ is applied, we have that

$$P_{\bar{v}} \triangleq \arg \max_{P_l \in \mathbb{VP}} \mathcal{J}(\boldsymbol{\vartheta}, \mathrm{x}_{T-1}) \tag{5.18}$$

Thus, from the expression (5.14) of a generic cost at time step $k\!=\!T\!-\!1$, for each vertex $P_l \in_{\mathbb{V}}\mathbb{P}$, the last equality (5.18) implies that

$$\sum_{i=1}^{N} p_i(T-1)\mathcal{J}\big(i, \mathrm{x}_k, K_i^{\bar{\upsilon}}(T-1)\,\mathrm{x}_{T-1}, P_{i\bullet}^{\bar{\upsilon}}\big) \geq \\ \sum_{i=1}^{N} p_i(T-1)\mathcal{J}\big(i, \mathrm{x}_k, K_i^{\bar{\upsilon}}(T-1)\,\mathrm{x}_{T-1}, P_{i\bullet}^{l}\big) \qquad (5.19)$$

with equality valid when the transition probability is the one of the vertex with index $\bar{\upsilon}$. This ensures us that with Assumption 5.2 on the information structure for the controller and the adversary satisfied in game-theoretic setting, the proposed control input is indeed optimal for both controller and adversary, since the perturbation with any value of transition probabilities different from $P_{\bar{\upsilon}}$ produces a minor cost for the perturbation player. Consequently, in the following we define

$$\mathcal{J}(i, \mathrm{x}_{T-1}) \triangleq \mathcal{J}\big(i, \mathrm{x}_k, K_i^{\bar{\upsilon}}(T-1)\,\mathrm{x}_{T-1}, P_{i\bullet}^{\bar{\upsilon}}\big) \qquad (5.20)$$

so that

$$\mathcal{J}(\boldsymbol{\vartheta}, \mathrm{x}_{T-1}) = \sum_{i=1}^{N} p_i(T-1)\mathcal{J}(i, \mathrm{x}_{T-1})$$

This leads us to the following explicit expression for the cost contribution of the i-th operational mode to cost-to-go:

$$\mathcal{J}(i, \mathrm{x}_{T-1}) = \mathrm{x}_{T-1}^* X_i^{\bar{\upsilon}}(T-1)\mathrm{x}_{T-1} + \sum_{i=1}^{N} \mathrm{tr}\left(\Phi_i R_i(T-1)\,\Phi_i^* \sum_{j=1}^{N} p_{ij}^{\bar{\upsilon}} X_j(T)\right) \quad (5.21)$$

where the solution of the corresponding coupled Riccati difference equation is

$$X_i^{\bar{\upsilon}}(T-1) \triangleq C_i^* C_i + A_i^* \sum_{j=1}^{N} p_{ij}^{\bar{\upsilon}} X_j(T) A_i + A_i^* \sum_{j=1}^{N} p_{ij}^{\bar{\upsilon}} X_j(T) B_i K_i^{\bar{\upsilon}}(T-1) \quad (5.22)$$

Since the total mass of the probability distribution of the discrete random variable θ_k equals to one for any $k \in \mathbb{T}$ (see e.g., appendix's Subsection B.6), by linearity of the considered expressions we write the cost-to-go as

$$\mathcal{J}(\boldsymbol{\vartheta}, \mathrm{x}_{T-1}) = \mathrm{x}_{T-1}^* \left(\sum_{i=1}^{N} p_i(T-1) X_i^{\bar{\upsilon}}(T-1)\right) \mathrm{x}_{T-1} + \\ \sum_{i=1}^{N} p_i(T-1)\mathrm{tr}\left(\Phi_i R_i(T-1)\,\Phi_i^* \sum_{j=1}^{N} p_{ij}^{\bar{\upsilon}} X_j(T)\right) \qquad (5.23)$$

In order to emphasize the structure of the cost-to-go function, we write the same expression as

$$\mathcal{J}(\boldsymbol{\vartheta}, \mathrm{x}_{T-1}) = \mathrm{x}_{T-1}^* \left(X_{\boldsymbol{\vartheta}}^{\bar{\upsilon}}(T-1)\right) \mathrm{x}_{T-1} + \delta_{\bar{\upsilon}}(T-1) \qquad (5.24)$$

where the expressions for $X_{\boldsymbol{\vartheta}}^{\bar{\upsilon}}(T-1)$ and for $\delta_{\bar{\upsilon}}(T-1)$ are obvious from (5.23).

We underline that the cost-to-go $\mathcal{J}(\boldsymbol{\vartheta}, \mathrm{x}_{T-1})$ clearly depends on x_{T-1}, so the value of the index $\bar{\upsilon}$ can be different for distinct values of the system's state at time step $k=T-1$. Thence, it is easy to see that we write $\bar{\upsilon}$ as a compact notation for index $\bar{\upsilon}(\mathrm{x}_{T-1})$. Without knowing a priori in which state system will be at time step $T-1$, we need to consider *all the vertices as possible candidates.* So, we can have up to V optimal gains for each operational mode $i\in\mathbb{M}$ at time step $k=T-1$, since we need to consider all the solutions of coupled Riccati difference equations *that can achieve maximum in transition probabilities.* In Subsection 5.2 we will explain the way to reduce the number of equations to deal with, obtaining a smaller set of solutions that we will call *parsimonious set*, or the set of non-redundant solutions.

The indices of the vertices of polytopic transition probability matrix for which the solutions of CRDEs are parsimonious at a given time step k will be indicated by υ. The cardinality of a parsimonious set of solutions, i.e., the number of its elements, will be indicated by $n_\upsilon(k)$.

Considering one parsimonious solution $X_i^\upsilon(T-1)$ for each operational mode i gives us an N-sequence of what can be seen as terminal costs for the preceding time step, i.e., $k=T-2$. Thus, we can repeat exactly the same procedure of the time step $k=T-1$, keeping in mind that instead of one sequence of terminal costs, now we need to examine all $n_\upsilon(k)$ non-redundant solutions of coupled Riccati difference equations corresponding to each N-sequence with the same index υ. Let us index these N-sequences by ξ. Then, the total number of sequences to consider at time step $k=T-2$ is simply

$$n_\xi(T-2)=Vn_\upsilon(T-1)$$

which without the elimination of redundant solutions (performed by following the procedure presented in the next Subsection 5.2) would be

$$n_{\bar{\xi}}(T-2)=V^2$$

So, by iterating this procedure for a genetic $k=T-t$, $t\in\{i\in\mathbb{T}: i\geq 2, i\leq T\}$, the number of sequences to consider is

$$n_\xi(T-t)=Vn_\upsilon(T-t+1), \qquad n_{\bar{\xi}}(T-t)=V^t \tag{5.25}$$

and we obtain the main result of this chapter, enunciated in the following theorem.

Theorem 5.1. *An optimal solution for the robust control Problem 5.1 is given by a sequence of control inputs $(\mathrm{u}_k)_{k=0}^{T-1}$ such that for each $k\in\mathbb{T}$, $k\leq T$, $i\in\mathbb{M}$ and for all $\xi\in(i)_{i=1}^{n_\xi(k)}$, where $n_\xi(k)=Vn_\upsilon(k+1)$, one has that for any given x_k,*

$$\mathcal{J}(\theta_k,\mathrm{x}_k)=\max_\xi \mathcal{J}_\xi(\theta_k,\mathrm{x}_k) \tag{5.26}$$

$$\mathcal{J}_\xi(\theta_k,\mathrm{x}_k)=\mathrm{x}_k^*\left(\sum_{i=1}^N p_i(k)X_i^{\upsilon_\xi}(k)\right)\mathrm{x}_k+\delta_\xi(k) \tag{5.27}$$

$$p_j(k+1)=\sum_{i=1}^N p_i(k)p_{ij}^{\upsilon_\xi} \tag{5.28}$$

$$X_i^{\upsilon_\xi}(k)\triangleq C_i^*C_i+A_i^*\sum_{j=1}^N p_{ij}^{\upsilon_\xi}X_j(k+1)A_i+A_i^*\sum_{j=1}^N p_{ij}^{\upsilon_\xi}X_j(k+1)B_iK_i^{\upsilon_\xi}(k) \tag{5.29}$$

$$K_i^{\upsilon_\xi}(k) = -\left(D_i^* D_i + B_i^* \sum\nolimits_{j=1}^{N} p_{ij}^{\upsilon_\xi} X_j(k+1) B_i\right)^{-1} B_i^* \sum\nolimits_{j=1}^{N} p_{ij}^{\upsilon_\xi} X_j(k+1) A_i \quad (5.30)$$

$$\delta_\xi(k) = \sum\nolimits_{i=1}^{N} p_i(k) \operatorname{tr}\left(\varPhi_i R_i(k) \varPhi_i^* \sum\nolimits_{j=1}^{N} p_{ij}^{\upsilon_\xi} X_j(k+1)\right) + \sum\nolimits_{j=1}^{N} p_{ij}^{\upsilon_\xi} \delta(k+1) \quad (5.31)$$

$$\delta_\xi(T) \triangleq 0 \quad (5.32)$$

$$\upsilon = \arg\max_{\upsilon_\xi} \mathcal{J}(\theta_k, \mathrm{x}_k)$$
$$\mathrm{u}_k = K_{\theta_k}^{\upsilon}(k)\, \mathrm{x}_k \quad (5.33)$$

Proof. By backward induction, it follows the procedure presented in the preceding part of this section. □

In what comes next, we describe a procedure that permits to discard the redundant solutions to coupled Riccati difference equations, reducing the number of equations to deal with at each iteration of the recursive algorithm.

Parsimonious set and stabilizability

We have seen from (5.25) that the number of possible solutions to coupled Riccati difference equations to consider in the design of the optimal robust controller presented in Theorem 5.1 grows exponentially with the length of the time horizon.

However, not all of those $n_{\bar{\xi}}(T-t)$ sequences of N solutions $\boldsymbol{X}^{\bar{\upsilon}_\xi}(k)$ to CRDEs can ever let a related cost achieve the maximum in transition probabilities. The subset of a set of all $n_{\bar{\xi}}(T-t)$ sequences of N solutions to coupled Riccati difference equations that includes as members only those sequences of N solutions $\boldsymbol{X}^{\upsilon_\xi}(k)$ that let the cost $\mathcal{J}(\theta_t, \mathrm{x}_t, K_{\theta_t}^{\upsilon_\xi}(t)\mathrm{x}_t, P_{\theta_t \bullet}^{\upsilon_\xi})$ of optimal control $K_{\theta_t}^{\upsilon_\xi}(t)\mathrm{x}_t$ achieve the maximum in transition probabilities for some state x_k is sometimes called *parsimonious* (see [105]).

Let us consider two costs $\mathcal{J}(\theta_t, \mathrm{x}_t, K_{\theta_t}^{\upsilon_\xi}(t)\mathrm{x}_t, P_{\theta_t \bullet}^{\upsilon_\xi})$ and $\mathcal{J}(\theta_t, \mathrm{x}_t, K_{\theta_t}^{\bar{\upsilon}_\xi}(t)\mathrm{x}_t, P_{\theta_t \bullet}^{\bar{\upsilon}_\xi})$, both defined by (5.27). It is clear that if

$$\mathcal{J}(\theta_t, \mathrm{x}_t, K_{\theta_t}^{\upsilon_\xi}(t)\mathrm{x}_t, P_{\theta_t \bullet}^{\upsilon_\xi}) \geq \mathcal{J}(\theta_t, \mathrm{x}_t, K_{\theta_t}^{\bar{\upsilon}_\xi}(t)\mathrm{x}_t, P_{\theta_t \bullet}^{\bar{\upsilon}_\xi}) \quad \forall \mathrm{x}_t \in \mathbb{F}^{n_\mathrm{x}}$$

then the sequence $\boldsymbol{X}^{\bar{\upsilon}_\xi}(k)$ of the solutions of coupled Riccati difference equations that produces cost $\mathcal{J}(\theta_t, \mathrm{x}_t, K_{\theta_t}^{\bar{\upsilon}_\xi}(t)\mathrm{x}_t, P_{\theta_t \bullet}^{\bar{\upsilon}_\xi})$ is redundant. From the definition (5.27), the previous expression can be rewritten explicitly as

$$\sum\nolimits_{i=1}^{N} p_i(k) \mathrm{x}_k^* \left(X_i^{\upsilon_\xi}(k) - X_i^{\bar{\upsilon}_\xi}(k)\right) \mathrm{x}_k + \delta_\xi(k) - \delta_{\bar{\xi}}(k) \geq 0 \quad (5.34)$$

We can see from its recursive definition (5.28) that the value of $p_i(k)$ depends on the initial probability distribution p_0 and the sequence of the transition probability matrices describing the evolution in time of this initial probability distribution.

Thus, the probability of an operational mode $i \in \mathbb{M}$ is a priori unknown. Thence, the previous equation should hold for all $\mathrm{x}_k \in \mathbb{F}^{n_\mathrm{x}}$ and for an arbitrary $p_i(k)$.

From the definition of the positive semi-definite matrices provided in appendix's Subsection B.4, we have that

$$\sum\nolimits_{i=1}^{N} p_i(k)\mathrm{x}_k^* \left(X_i^{\upsilon_\xi}(k) - X_i^{\bar{\upsilon}_\xi}(k) \right) \mathrm{x}_k \geq 0$$

for all x_k and for any p_k if and only if

$$X_i^{\upsilon_\xi}(k) \succeq X_i^{\bar{\upsilon}_\xi}(k) \quad \forall i \in \mathbb{M}$$

We need to require that also $\delta_\xi(k) - \delta_{\bar{\xi}}(k) \geq 0$, to be able to ensure that (5.34) always holds. So, we say that if

$$\forall i \in \mathbb{M} \quad X_i^{\upsilon_\xi}(k) \succeq X_i^{\bar{\upsilon}_\xi}(k) \quad \wedge \quad \delta_\xi(k) \geq \delta_{\bar{\xi}}(k) \tag{5.35}$$

then the N-sequence $\boldsymbol{X}^{\bar{\xi}}(k)$ of solutions to CRDEs is redundant and should be discarded. If we examine all pairs of elements of a set of N-sequence of solutions to coupled Riccati difference equations and discard all the redundant elements, we obtain the parsimonious set of solutions.

As a final note, we observe that when the Markov jump linear system is stabilizable according to Definition 3.6, after a transitory period, the finite-horizon optimal state feedback robust solution becomes unique. From proposition (3.9) and Theorem 3.6 on equivalence between mean square stability and exponential mean square stability, the length of transitory period depend on how much the joint spectral radius, associated to the second moment of the state vector through vertices of the polytopic transition probability matrix, is smaller than 1. The exponential mean square stability of the closed loop system ensures that the state space reachable from the previous state becomes smaller and smaller, with fewer number of possible CDREs. When the MJLS is non-stabilizable, the behavior is opposite.

5.3 Separation principle

Following the procedure of [12, pp. 132–136], it is immediate to establish the principle of separation between estimation and control for the optimal solutions to the robust filtering and control problems presented in Chapters 4 and 5. For the sake of completeness, we briefly illustrate this result.

Let us consider the discrete-time polytopic time-inhomogeneous Markov jump linear system as in (2.1) defined on the stochastic basis $(\Omega_\mathrm{y}, \mathcal{F}, (\mathcal{F}_k), \Pr)$, and described by the following dynamical system

$$\begin{cases} \mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k + B_{\theta_k}\mathrm{u}_k + H_{\theta_k}\mathrm{w}_k, \\ \mathrm{y}_k = F_{\theta_k}\mathrm{x}_k + G_{\theta_k}\mathrm{w}_k, \\ \mathrm{z}_k = C_{\theta_k}\mathrm{x}_k + D_{\theta_k}\mathrm{u}_k, \\ \mathrm{x}_0 = \boldsymbol{x}_0,\ \theta_0 = \vartheta_0,\ \mathrm{p}_0 = \boldsymbol{p}_0 \end{cases}$$

where the system's variables and matrices are those defined in Section 2.1, and the transition probability matrix is polytopic, as stated in Assumption 2.2.

The problem of the optimal mode-dependent dynamic output feedback control for an MJLS as in (2.1) is that of designing a controller described by the following stochastic equations:

$$\begin{cases} \hat{\mathrm{x}}_{k+1} = \hat{A}_{\theta_k}(k)\hat{\mathrm{x}}_k + \hat{B}_{\theta_k}(k)\mathrm{y}_k, \\ \mathrm{u}_k = \hat{C}_{\theta_k}(k)\hat{\mathrm{x}}_k, \\ \hat{\mathrm{x}}_0 = \hat{\boldsymbol{x}}_0 \end{cases} \tag{5.36}$$

where $\hat{\mathrm{x}}_k$ is the state of the controller, with the initial state $\hat{\mathrm{x}}_0$ being deterministic, and such that the cost (5.7) is attained, i.e.,

$$\mathcal{J}(\theta_0, \mathrm{x}_0) \triangleq \min_{\mathrm{u}} \max_{P_{\theta_{\mathrm{y}\bullet}}} \sum_{k=0}^{T-1} \mathbb{E}\big(\|\mathrm{z}_k\|_2^2\big) + \mathbb{E}(\mathrm{x}_T^* Z_{\theta_T} \mathrm{x}_T)$$

Combining the results of Sections 4.2 and 5.2, we obtain the optimal solution

$$\begin{cases} \hat{A}_i(k) = A_i + L_i(k)F_i + B_i K_i^v(k), \\ \hat{B}_i(k) = -L_i(k), \\ \hat{C}_i(k) = K_i^v(k) \end{cases} \tag{5.37}$$

with $L_i(k)$ provided by (4.31) in Theorem 4.5 and $K_i^v(k)$ from (5.33) in Theorem 5.1.

This result is proved in two steps. We provide a sketch of the proof as follows.

First, we observe that from the definition (4.11) of the filtering error, one has that

$$\mathrm{x}_k = \hat{\mathrm{x}}_k + \hat{\mathrm{e}}_k$$

From the structure (5.36) of the output feedback controller, the state space representation (2.1) of the Markov jump linear system, the definition (4.14) of $Y_i(k)$, and (4.34) from Lemma 4.3, we obtain that

$$\mathbb{E}\big(\|\mathrm{z}_k\|_2^2\big) = \mathbb{E}\big(\|\hat{\mathrm{z}}_k\|_2^2\big) + \sum_{i=1}^{N} \mathrm{tr}(C_i Y_i(k) C_i^*)$$

with

$$\hat{\mathrm{z}}_k \triangleq C_{\theta_k}\hat{\mathrm{x}}_k + D_{\theta_k}\mathrm{u}_k$$

Similarly, using also the fact (B.35) that the trace is invariant under cyclic permutations of matrix product, one obtains that

$$\mathbb{E}(\mathrm{x}_T^* Z_{\theta_T} \mathrm{x}_T) = \mathbb{E}(\hat{\mathrm{x}}_T^* Z_{\theta_T} \hat{\mathrm{x}}_T) + \sum_{i=1}^{N} \mathrm{tr}(Z_i Y_i(k))$$

Clearly, the terms with $Y_i(k)$ do not depend on the control input u_k. Therefore, solving (5.7) is equivalent to finding

$$\hat{\mathcal{J}}(\theta_0, \hat{\mathrm{x}}_0) \triangleq \min_{\mathrm{u}} \max_{P_{\theta_{\mathrm{y}\bullet}}} \sum_{k=0}^{T-1} \mathbb{E}\big(\|\hat{\mathrm{z}}_k\|_2^2\big) + \mathbb{E}(\hat{\mathrm{x}}_T^* Z_{\theta_T} \hat{\mathrm{x}}_T)$$

subject to

$$\begin{cases} \hat{\mathrm{x}}_{k+1} = A_{\theta_k}\hat{\mathrm{x}}_k + B_{\theta_k}\mathrm{u}_k - \Phi_{\theta_k}\mathrm{v}_k, \\ \quad \hat{\mathrm{z}}_k = C_{\theta_k}\hat{\mathrm{x}}_k + D_{\theta_k}\mathrm{u}_k, \\ \quad \hat{\mathrm{x}}_0 = \mathbb{E}(\mathrm{x}_0) = \psi_0 \end{cases}$$

where

$$\mathrm{v}_k = \mathrm{y}_k - L_{\theta_k}(k)\hat{\mathrm{x}}_k = L_{\theta_k}(k)\hat{\mathrm{e}}_k + G_{\theta_k}(k)\mathrm{w}_k$$

Then, it is immediate to verify that Assumption 5.1 verified, and

$$R_i(k) = L_i(k)Y_i(k)L_i^*(k) + G_iG_i^*$$

This concludes our recap of the derivation of the separation principle through the procedure from [12, pp. 132–136].

In the next chapter we present a result on optimal robust state-feedback control for discrete-time polytopic time-inhomogeneous Markov jump switched linear systems, i.e., switched linear systems, where a switching signal is governed by a Markov decision process instead of a Markov chain, as introduced in Section 2.4. In chronological order, the aforementioned result was obtained before our work on output feedback control illustrated in Chapters 4 and 5. Thus, it shares several ideas of this chapter, and, in fact, it has helped us to develop the result in noisy setting presented here. For this reason, we will focus on pointing out the similarities and differences, rather than writing down the same passages for the second time.

Chapter 6

Extension to switched systems

Markov jump linear systems studied in detail in the previous chapters use probabilistic description of commutations between operational modes and are well suited to model exogenous uncontrollable events induced by external causes. Their model however does not take into account the fact that a system may have an automated mechanism or a human supervisor capable to (partially) compensate for the effects of external disturbances.

When the actions of a supervisor are required to be optimized, it comes naturally to use a Markov decision process framework. Considering a Markov decision process instead of a Markov chain in a Markov jump linear system brings to light a new type of system, that we call Markov jump switched linear system (or MJSLS).

This chapter follows the line of investigation on optimal finite time horizon state-feedback problems in polytopic time-inhomogeneous setting presented in the previous chapter, considering the issue of joint minimization of costs of continuous and discrete control inputs for the worst possible disturbance in transition probabilities. It is based on [4], which was presented at the 20th World Congress of the International Federation of Automatic Control in July 2017.

In the next section, 6.1, we formally define the optimal control problem, while in the following section, 6.2, we present an analytic solution to the aforementioned problem. See also the appendix's Subsection B.6 for some additional details on Markov decision processes.

6.1 Optimal robust control problem

Let us consider the discrete-time Markov jump switched linear system (2.18) defined on the stochastic basis $(\Omega_{\mathrm{s}}, \mathcal{I}, (\mathcal{I}_k), \mathrm{Pr})$ described in Subsection 2.4, with the switching between operational modes of the system being governed by a Markov decision process described as in appendix's Subsection B.6 by a quintuple $(\mathbb{M}, \mathbb{A}, \mathrm{Pr}, g, \gamma)$. Its transition probabilities associated to each action available in an operational mode are polytopic time-inhomogeneous, as by Assumption 2.4. Also, all the operational modes of the system are considered to be measurable (see Assumption 2.3).

In order to facilitate the readability of this section, we recall that the state space representation of the system (2.18) under consideration is

$$\begin{cases} \mathrm{x}_{k+1} = A_{s_k}\mathrm{x}_k + B_{s_k}\mathrm{u}_k \\ \quad \mathrm{z}_k = C_{s_k}\mathrm{x}_k + D_{s_k}\mathrm{u}_k, \\ \quad \mathrm{x}_0 = \boldsymbol{x}_0,\ s_0 = \boldsymbol{s}_0,\ \mathrm{p}_0 = \boldsymbol{p}_0 \end{cases}$$

where, as before, the system variables and matrices are those of Section 2.1.

As in the previous chapter, without loss of generality [12, Remark 4.1, p. 74], we assume that for each $i \in \mathbb{M}$ the following equation (5.6) holds, i.e.,

$$C_i^* D_i = 0, \qquad D_i^* D_i \succ 0$$

The set of all admissible $\mathcal{I}_k$-measurable controllers is denoted in the same way as before, that is, $\mathbb{U}_\mathrm{T}$, and is given by the sequence of (continuous) control laws

$$\mathrm{u} \triangleq (\mathrm{u}_k)_0^{T-1}$$

In addition, for each $k \in \mathbb{T}$, we denote by π_k the *hybrid control pair* (α_k, u_k), where $\alpha_k \in \mathbb{A}_i$ is a (discrete) action at time instant k.

Then, the sequence π of hybrid control pairs $(\pi_k)_{k=0}^{T-1}$ is called *hybrid control sequence.* At each time step (or decision epoch, in MDP terminology) k, a *particular choice* $\boldsymbol{u}_k$ of u_k is called the continuous control law; similarly, $\boldsymbol{\alpha}_k$ is denominated discrete switching control law. The pair $(\boldsymbol{\alpha}_k, \boldsymbol{u}_k)$ forms the hybrid control *law* $\boldsymbol{\pi}_k$, and the sequence of hybrid control laws over the horizon T constitutes a finite horizon *feedback policy*,

$$\boldsymbol{\pi} \triangleq (\boldsymbol{\pi}_k)_{k=0}^{T-1} \triangleq (\boldsymbol{\alpha}_k, \boldsymbol{u}_k)_{k=0}^{T-1}$$

We also indicate by

$$\mathrm{p}_{s\bullet}^{\alpha} \triangleq \left(\mathrm{p}_{s_t\bullet}^{\alpha}(t)\right)_{t=0}^{T-1}$$

the sequence of the length $T \in \mathbb{T}$ of the transition probability row vectors $\mathrm{p}_{i\bullet}^{\alpha}(k)$, with $k \in \mathbb{T}_{T-1}$, where the elements of $\mathrm{p}_{s\bullet}^{\alpha}$ obviously depend on the realizations of the Markov decision process s described in Subsection 2.4.

So, following the same line of reasoning of the previous chapter, we cast an optimal linear quadratic state-feedback control problem for Markov jump switched linear systems with bounded perturbations of the transition probabilities as a min-max problem of optimizing robust performance, i.e., finding the minimum over the finite horizon feedback policy of the maximum over the transition probability disturbance (obtained in correspondence of the chosen feedback policy).

Thence, the problem of designing the optimal mode-dependent state-feedback Markov jump controller, which is robust to all possible polytopic perturbations in transition probabilities, is formally defined as follows.

Problem 6.1. *Given a discrete-time Markov jump switched linear system* (2.18) *with unknown and time-varying transition probability row vectors* $\mathrm{p}^{\alpha}_{i\bullet}(k) \in {}_{\mathbb{V}_\alpha}\mathbb{P}$ *and satisfying Assumption 2.4, find the mode-dependent state feedback policy* $\boldsymbol{\pi}$ *that achieves the following optimal cost of robust control.*

$$\mathcal{J}(\mathrm{s}_0, \mathrm{x}_0) \triangleq \min_{\pi} \max_{\mathrm{p}^{\alpha}_{s\bullet}} \sum_{k=0}^{T-1} \mathbb{E}\big(\|\mathrm{z}_k\|_2^2 + g(s_k, \alpha_k) \mid \mathcal{I}_0\big) + \mathbb{E}(\mathrm{x}_T^* Z_{s_T} \mathrm{x}_T \mid \mathcal{I}_T) \tag{6.1}$$

with $\boldsymbol{Z} \triangleq (Z_i)_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}_0^{n_\mathrm{x}, n_\mathrm{x}}$ *being a sequence of the terminal cost weighting matrices.*

We remark that even if the cost $g(s_k, \alpha_k)$ of performing a discrete action α_k in an operational mode s_k here is treated as time-invariant, the result will obviously remain the same in the case of the time-varying cost $g(s_k, \alpha_k, k)$, as long as the current value of the cost is known by the decision maker.

In the next section we provide the analytic solution to Problem 6.1, underlining the similarities and differences with the solution to the Problem 5.1.

6.2 Analytic solution

Similarly to the case of the optimal robust control of Markov jump linear systems described in Section 5.2, the solution to the above Problem 6.1 is based on a *dynamic programming* approach in *Bellman's optimization formulation* [104], and is obtained by backward induction.

The relevant stochastic basis is $(\Omega_\mathrm{s}, \mathcal{I}, (\mathcal{I}_k), \mathrm{Pr})$, so x_T is $\mathcal{I}_T$-measurable. From the definition (6.1) of the cost of robust control and linearity of the expected value, the terminal cost is

$$\mathcal{J}(s_T, \mathrm{x}_T) = \mathbb{E}(\mathrm{x}_T^* X_{s_T}(T) \mathrm{x}_T \mid \mathcal{I}_T) = \mathrm{x}_T^* \mathbb{E}(X_{s_T}(T) \mid \mathcal{I}_T)\, \mathrm{x}_T = \mathrm{x}_T^* Z_{s_T} \mathrm{x}_T \tag{6.2}$$

with $X_{s_T}(T) \triangleq Z_{s_T}$ a solution to the coupled Riccati difference equation for the robust control at the terminal time step.

As before, we are interested in the explicit form of $\boldsymbol{X}(k) = (X_i(k))_{i=1}^{N} \in {}_{\mathrm{N}}\mathbb{F}_0^{n_\mathrm{x}, n_\mathrm{x}}$.

A generic cost at time step $k \in \mathbb{T}_{T-1}$ is

$$\mathcal{J}\big(s_k, \mathrm{x}_k, \pi_k, \mathrm{p}^{\alpha}_{s_k\bullet}(k)\big) = \mathbb{E}\Big(\|\mathrm{z}_k\|_2^2 + g(s_k, \alpha_k) + \mathcal{J}(s_{k+1}, x_{k+1}) \mid \mathcal{I}_k\Big) \tag{6.3}$$

while the cost cost-to-go function is defined as

$$\mathcal{J}(s_k, \mathrm{x}_k) = \min_{\pi_k} \max_{\mathrm{p}^{\alpha}_{s_k\bullet}(k)} \mathcal{J}\big(s_k, \mathrm{x}_k, \pi_k, \mathrm{p}^{\alpha}_{s_k\bullet}(k)\big) \tag{6.4}$$

To improve the readability, we write $\boldsymbol{s}$ as a *short notation for* s_k.

We start by examining the cost-to-go function (6.4) at time step $k = T - 1$. Clearly, at this time step $\boldsymbol{s}$ indicates s_{T-1}.

With the same considerations made in Section 5.2, we obtain that

$$\mathbb{E}(X_{s_T}(T) \mid \mathcal{I}_{T-1}) = \sum\nolimits_{j=1}^{N} p_{\boldsymbol{s}j}^{\alpha_{T-1}}(T-1)X_j(T) \tag{6.5}$$

$$\mathcal{J}\big(\boldsymbol{s}, \mathrm{x}_{T-1}, \pi_{T-1}, \mathrm{p}_{\boldsymbol{s}\bullet}^{\alpha_{T-1}}(T{-}1)\big) = \sum\nolimits_{i=1}^{N} p_i(T{-}1)\mathcal{J}\big(i, \mathrm{x}_{T-1}, \pi_{T-1}, \mathrm{p}_{i\bullet}^{\alpha_{T-1}}(T{-}1)\big) \tag{6.6}$$

$$\begin{aligned}\mathcal{J}\big(i, \mathrm{x}_{T-1}, \pi_{T-1}, \mathrm{p}_{i\bullet}^{\alpha_{T-1}}(T{-}1)\big) = &\\ \mathrm{x}_{T-1}^{*}\left(C_i^{*}C_i + A_i^{*}\sum\nolimits_{j=1}^{N} p_{ij}^{\alpha_{T-1}}(T-1)X_j(T)A_i\right)\mathrm{x}_{T-1} + &\\ 2\mathrm{x}_{T-1}^{*}A_i^{*}\sum\nolimits_{j=1}^{N} p_{ij}^{\alpha_{T-1}}(T-1)X_j(T)B_i\mathrm{u}_{T-1} + &\\ \mathrm{u}_{T-1}^{*}\left(D_i^{*}D_i + B_i^{*}\sum\nolimits_{j=1}^{N} p_{ij}^{\alpha_{T-1}}(T-1)X_j(T)B_i\right)\mathrm{u}_{T-1} + &\\ g(i, \alpha_{T-1})&\end{aligned} \tag{6.7}$$

So, we can verify that $\mathcal{J}\big(\boldsymbol{s}, \mathrm{x}_{T-1}, \pi_{T-1}, \mathrm{p}_{\boldsymbol{s}\bullet}^{\alpha_{T-1}}(T{-}1)\big)$ is a convex function in a variable $\mathrm{p}_{\boldsymbol{s}\bullet}^{\alpha_{T-1}}(T{-}1)$ defined on a convex set. Thus,

$$\max_{\mathrm{p}_{\boldsymbol{s}\bullet}^{\alpha_{T-1}}(T-1)} \mathcal{J}\big(\boldsymbol{s}, \mathrm{x}_{T-1}, \pi_{T-1}, \mathrm{p}_{\boldsymbol{s}\bullet}^{\alpha_{T-1}}(T-1)\big) = \max_{\mathrm{p}_{[1,N]\bullet l}^{\alpha_{T-1}} \in \mathbb{V}_{\alpha}\mathbb{P}} \mathcal{J}\big(\boldsymbol{s}, \mathrm{x}_{T-1}, \pi_{T-1}, \mathrm{p}_{\boldsymbol{s}\bullet l}^{\alpha_{T-1}}\big)$$

This implies that the maximum in transition probabilities of the generic cost (6.6) associated to a discrete control action α_{T-1} is attained on a vertex of the convex polytope of transition probabilities, all related to the same operational mode $\boldsymbol{s}$.

Hence, to find the maximum in transition probability disturbances of the minimum in available discrete actions, we need to evaluate the expectation of the generic cost (6.6) in each vertex $\mathrm{p}_{[1,N]\bullet l}^{\alpha_{T-1}} \in \mathbb{V}_{\alpha}\mathbb{P}$, for every available action $\alpha_{T-1} \in \mathbb{A}_{\boldsymbol{s}}$.

For each of those actions and vertices we compute the optimal continuous control input as in Section 5.2, finding that

$$\mathrm{u}_{T-1} = K_{\boldsymbol{s}l}^{\alpha_{T-1}}(T-1)\,\mathrm{x}_{T-1} \tag{6.8}$$

$$K_{\boldsymbol{s}l}^{\alpha_{T-1}}(T{-}1) = -\left(D_{\boldsymbol{s}}^{*}D_{\boldsymbol{s}} + B_{\boldsymbol{s}}^{*}\sum\nolimits_{j=1}^{N} p_{\boldsymbol{s}jl}^{\alpha_{T-1}}X_j(T)B_{\boldsymbol{s}}\right)^{-1} B_{\boldsymbol{s}}^{*}\sum\nolimits_{j=1}^{N} p_{\boldsymbol{s}jl}^{\alpha_{T-1}}X_j(T)A_{\boldsymbol{s}} \tag{6.9}$$

At this point, following the same line of reasoning of Section 5.2, for each discrete action $\alpha \in \mathbb{A}_{\boldsymbol{s}}$, we denote the vertex, which maximizes in transition probabilities the generic cost (6.6), as

$$\mathrm{p}_{[1,N]\bullet\bar{v}}^{\alpha_{T-1}} \triangleq \arg \max_{\mathrm{p}_{[1,N]\bullet l}^{\alpha_{T-1}}} \mathcal{J}(\boldsymbol{s}, \mathrm{x}_{T-1}, \alpha_{T-1}) \tag{6.10}$$

Then, we have that for each $l \in \mathbb{V}_\alpha$

$$\sum_{i=1}^{N} p_i(T-1)\mathcal{J}\big(i, \mathrm{x}_{T-1}, \alpha_{T-1}, K_{i\bar{\upsilon}}^{\alpha_{T-1}}(T-1)\mathrm{x}_{T-1}, \mathrm{p}_{i\bullet\bar{\upsilon}}^{\alpha_{T-1}}\big) \geq \sum_{i=1}^{N} p_i(T-1)\mathcal{J}\big(i, \mathrm{x}_{T-1}, \alpha_{T-1}, K_{i\bar{\upsilon}}^{\alpha_{T-1}}(T-1)\mathrm{x}_{T-1}, \mathrm{p}_{i\bullet l}^{\alpha_{T-1}}\big) \quad (6.11)$$

with equality valid when the transition probability is the one of the vertex with index $\bar{\upsilon}$. Consequently, in the following we define

$$\mathcal{J}(i, \mathrm{x}_{T-1}, \alpha_{T-1}) \triangleq \mathcal{J}\big(i, \mathrm{x}_k, \alpha_{T-1}, K_{i\bar{\upsilon}}^{\alpha_{T-1}}(T-1)\mathrm{x}_{T-1}, \mathrm{p}_{i\bullet l}^{\alpha_{T-1}}\big) \quad (6.12)$$

$$\mathcal{J}(\boldsymbol{s}, \mathrm{x}_{T-1}, \alpha_{T-1}) = \sum_{i=1}^{N} p_i(T-1)\mathcal{J}(i, \mathrm{x}_{T-1}, \alpha_{T-1})$$

Thence, we obtain that

$$\mathcal{J}(i, \mathrm{x}_{T-1}, \alpha_{T-1}) = \mathrm{x}_{T-1}^{*} X_{i\bar{\upsilon}}^{\alpha_{T-1}}(T-1)\mathrm{x}_{T-1} + g(i, \alpha_{T-1}) \quad (6.13)$$

$$X_{i\bar{\upsilon}}^{\alpha_{T-1}}(T-1) \triangleq C_i^{*}C_i + A_i^{*}\sum_{j=1}^{N} p_{ij\bar{\upsilon}}^{\alpha_{T-1}} X_j(T)A_i + A_i^{*}\sum_{j=1}^{N} p_{ij\bar{\upsilon}}^{\alpha_{T-1}} X_j(T)B_i K_{i\bar{\upsilon}}^{\alpha_{T-1}}(T-1) \quad (6.14)$$

So that

$$\mathcal{J}(\boldsymbol{s}, \mathrm{x}_{T-1}, \alpha_{T-1}) = \mathrm{x}_{T-1}^{*}\left(\sum_{i=1}^{N} p_i(T-1)X_{i\bar{\upsilon}}^{\alpha_{T-1}}(T-1)\right)\mathrm{x}_{T-1} + \sum_{i=1}^{N} p_i(T-1)g(i, \alpha_{T-1}) \quad (6.15)$$

Then, the cost-to-go is

$$\mathcal{J}(\boldsymbol{s}, \mathrm{x}_{T-1}) = \min_{\alpha_{T-1} \in \mathbb{A}_{\boldsymbol{s}}} \mathcal{J}(\boldsymbol{s}, \mathrm{x}_{T-1}, \alpha_{T-1}) \quad (6.16)$$

Clearly, due to the dependence on system's state, all parsimonious solutions should be considered and stored in memory. For each discrete action, a parsimonious set of vertices is obtained as in Subsection 5.2. Obviously, if all the parsimonious solutions of coupled Riccati equations related to an action produce a cost that is larger than a cost of each non redundant solution for another action, the first action is not optimal and can be removed from the pool of possible solutions at the considered time step.

Let us indicate by n_α the number of parsimonious actions, i.e., those actions that may produce a minimum cost for a certain partition of state space, and by $n_{\upsilon_a}(k)$ the cardinality of a parsimonious set of solutions of coupled Riccati equations associated to a parsimonious action. As before, we index all these solutions by ξ.

Then, by iterating the presented procedure for a genetic $k = T - t$, where $t \in \{i \in \mathbb{T} : i \geq 2, i \leq T\}$, we obtain that the total number of N-sequences of solutions to coupled Riccati difference equations (for a total of M possible discrete actions) to consider is

$$n_\xi(T-t)=MV\sum_{i=1}^{n_\alpha} n_{\upsilon_i}(T-t+1), \qquad n_{\bar{\xi}}(T-t)=(MV)^t \tag{6.17}$$

(where $n_{\bar{\xi}}(T-t)$ is a number of sequences to deal with, when the pruning of redundant solutions is not performed); hence, the number of N-sequences of solutions to coupled Riccati difference equations to deal with for each discrete action α_k is

$$n_\xi(T-t,\alpha_k)=V\sum_{i=1}^{n_\alpha} n_{\upsilon_i}(T-t+1) \tag{6.18}$$

and we obtain the main result of this chapter, stated in the following theorem.

Theorem 6.1. *An optimal solution for the robust control Problem 6.1 is given by the mode-dependent hybrid control sequence* $(\alpha_k, \mathrm{u}_k)_{k=0}^{T-1}$ *such that for each* $k \in \mathbb{T}$, $k \le T$, $i \in \mathbb{M}$ *and for all* $\xi \in (i)_{i=1}^{n_\xi(k,\alpha_k)}$, *where* $n_\xi(k,\alpha_k)$ *is obtained from* (6.18), *one has that for any given* x_k,

$$\mathcal{J}(s_k,\mathrm{x}_k) = \min_{\alpha_k \in \mathbb{A}_{s_k}} \mathcal{J}(s_k,\mathrm{x}_k,\alpha_k) \tag{6.19}$$

$$\mathcal{J}(s_k,\mathrm{x}_k,\alpha_k) = \max_{\xi} \mathcal{J}_\xi\,(s_k,\mathrm{x}_k,\alpha_k) \tag{6.20}$$

$$\mathcal{J}_\xi\,(s_k,\mathrm{x}_k,\alpha_k) = \mathrm{x}_k^*\left(\sum_{i=1}^{N} p_i(k) X_{i\upsilon_\xi}^{\alpha_k}(k)\right)\mathrm{x}_k + g_\xi(k) \tag{6.21}$$

$$p_j(k+1) = \sum_{i=1}^{N} p_i(k) p_{ij\upsilon_\xi}^{\alpha_k} \tag{6.22}$$

$$X_{i\upsilon_\xi}^{\alpha_k}(k) \triangleq C_i^* C_i + A_i^* \sum_{j=1}^{N} p_{ij\upsilon_\xi}^{\alpha_k} X_j(k+1) A_i + \tag{6.23}$$

$$A_i^* \sum_{j=1}^{N} p_{ij\upsilon_\xi}^{\alpha_k} X_j(k+1) B_i K_{i\upsilon_\xi}^{\alpha_k}(k)$$

$$K_{i\upsilon_\xi}^{\alpha_k}(k) = -\left(D_i^* D_i + B_i^* \sum_{j=1}^{N} p_{ij\upsilon_\xi}^{\alpha_k} X_j(k+1) B_i\right)^{-1} B_i^* \sum_{j=1}^{N} p_{ij\upsilon_\xi}^{\alpha_k} X_j(k+1) A_i \tag{6.24}$$

$$g_\xi(k) = \sum_{i=1}^{N} p_i(k) g(i,\alpha_k) + \sum_{j=1}^{N} p_{ij\upsilon_\xi}^{\alpha_k} g(k+1) \tag{6.25}$$

$$g_\xi(T) \triangleq 0 \tag{6.26}$$

$$\upsilon = \arg\max_{\upsilon_\xi} \mathcal{J}(s_k,\mathrm{x}_k)$$

$$\mathrm{u}_k = K_{s_k\upsilon}^{\alpha}(k)\,\mathrm{x}_k \tag{6.27}$$

Proof. By backward induction, it follows the procedure presented in the preceding part of this section. □

Chapter 7

Conclusions and future work

This thesis presents our original contributions to stability theory and optimal control of a class of stochastic hybrid systems, known as discrete-time Markov jump linear systems, putting a specific focus on dealing with abrupt and unpredictable dynamic perturbations of transition probabilities between the operational modes of such systems.

Our interest in this particular systems is inspired by their applications as possible models for wireless networked control systems and cyber-physical systems, especially in the view of ongoing efforts made by academia and industry in developing a fifth generation of mobile technology (5G), which also uses models based on Markov chains and is expected to meet the requirements of ultra-reliable low-latency communications for factory automation and safety-critical internet of things.

To motivate the necessity of studying the characteristics of Markov jump linear systems with dynamic perturbations on values in transition probability matrix we show in Chapter 3 that a stable system assumed to have time-invariant transition probability matrix may become unstable, if subject to bounded dynamic perturbations. This happens even in case when the Markov jump linear system is robust to static uncertainties in transition probabilities.

In order to account for uncertainties and time-variance inherent to real world scenarios, we use the time-inhomogeneous polytopic model of transition probabilities, which is very general and widely used.

Then, as a technical contribution to stability theory, we derive the necessary and sufficient conditions for the mean square stability of discrete-time polytopic time-inhomogeneous Markov jump linear systems, and prove that deciding mean square stability on such systems is NP-hard and that mean square stability is equivalent to exponential mean square stability and to stochastic stability. We also obtain the necessary and sufficient conditions for the robust mean square stability of Markov jump linear systems affected by both dynamic polytopic uncertainties on transition probabilities, and bounded disturbances of system states. Our conditions are based on the notion of the joint spectral radius, which is applied to a family of matrices associated to the second moment of the state vector.

For what regards our contribution to the optimal control theory, it is the following. We formally define and solve the problem of the optimal filtering, which is robust to the bounded dynamic perturbations of the transition probability matrices of Markov jump linear systems. Then we also properly define and solve the problem of the optimal linear quadratic state-feedback control that is robust to the polytopic time-inhomogeneous uncertainties on the values of the transition probability matrices. Everything is done in a way that a principle of separation of estimation and control holds true, and the optimal output feedback controller may be obtained consequently.

Lastly, we extend the optimal linear quadratic state-feedback control result to so-called Markov jump switched linear systems, in which the switching between operational modes is based on the Markov decision process framework, where the (polytopic time-inhomogeneous) transition probabilities depend on the actions of a discrete controller, and for each action there is an associated cost.

7.1 Future work

There are several research directions on both theoretical foundations and practical application of Markov jump (switched) linear systems with dynamic bounded uncertainties in transition probabilities.

First of all, in a typical Markov jump linear systems framework, the operational modes are assumed to be measurable and immediately available to the controller. In the wireless control scenario this assumption may not hold true, since the operational modes may become available to the controller with some time delay. Also, the jump variable could be obtained via an estimation procedure. In this case, the estimation error should be taken into account.

Then, it is natural to go beyond the issues of the stability and optimal control, and we are particularly interested in the topics of fault detection, isolation and reconfiguration, and in formal verification and synthesis, in relation to the Markov jump (switched) linear system models.

Notably, in the domain of cyber-physical systems, the reactive security mechanisms, i.e., intrusion detection, automatic response and recovery [5], can exploit fault detection, isolation and reconfiguration techniques [106], by relaxing some assumptions on fault signals. In general, this type of methods utilize the concept of redundancy, which can be either a hardware redundancy or analytic one. This often poses a challenge of controlling, coordinating and synchronizing the operation of several interacting sub-modules within a system. Reconfiguration task, and not just it, may have objectives expressed in temporal logic, which provides a formal specification mechanism allowing one to quantitatively define the desired behavior of a systems by prescribing the interaction between sub-modules. In fact, formal methods [107] are increasingly being used for control and verification of dynamic systems against complex specifications [108–110]. Here, abstraction and compositional verification are key to handling complexity in verification [36]. In particular, a probabilistic computation tree logic [111] (PCTL) is widely adopted to write specifications of Markov chains and Markov decision processes with possibly time-varying uncertainties on transition probabilities, with structures that often can be reconducted to polytopes.

So, one important research direction is on correct-by-design fault detection, isolation and reconfiguration solutions for polytopic time-inhomojeneous Markov jump switched linear system, satisfying PCTL specifications (possibly on approximate with explicit bound bisimilar abstractions preserving the properties of interest).

Appendix A

Abbreviations and initialisms

IN this section we explicit all the main abbreviations and initialisms used throughout the text.

A.1 Abbreviations

For the sake of conciseness, through the text of the thesis, we use a number of abbreviations reported in the following Table A.1.

Abbrev.	**Meaning**
abbrev.	abbreviation
a.k.a.	also known as
commun.	communications
comput.	computer
conf.	conference
dist.	distribution
DoS	denial-of-service
etc.	*et cetera* (Latin phrase); it means "and other similar things", "and so forth".
e.g.	*exemplī grātiā* (Latin phrase); it signifies "for example"
i.e.	*id est* (Latin phrase); it stands for "that is", "in other words"
inf.	information
int.	international
iff	if and only if
NP	non-deterministic polynomial-time
resp.	respectively
s.t.	such that
trans.	transactions

Table A.1: Main abbreviations used in the text

A.2 Initialisms

In order to improve the readability of the thesis, we use a number of initialisms (i.e., abbreviations consisting of initial letters pronounced separately) listed and described in the following Table A.2.

Acronym	Meaning
BDD	bad data detection
BRL	bounded real lemma
CPS	cyber-physical system
CRDE	coupled Riccati difference equation
EMSS	exponential mean square stability
IEEE	Institute of Electrical and Electronics Engineers
IT	information technology
JSR	joint spectral radius
KL	Kullback–Leibler
LHS	left-hand side (of the mathematical expression)
LMI	linear matrix inequality
LQG	linear-quadratic-Gaussian
LQR	linear-quadratic regulator
LTI	linear time invariant
MJLS	Markov(ian) jump linear system
MJSLS	Markov(ian) jump switched linear system
MDP	Markov decision process
MPC	model predictive control
MSS	mean square stability
NCS	networked control system
PCA	principal component analysis
PCTL	probabilistic computation tree logic
PID	proportional-integral-derivative
PTI	polytopic time-inhomogeneous
RHS	right-hand side (of the mathematical expression)
RQ	research question
SCADA	supervisory control and data acquisition
SE	state estimation
SIAM	Society for Industrial and Applied Mathematics
SMT	satisfiability modulo theory
SS	stochastic stability
TP	transition probability
TPM	transition probability matrix
WLS	weighted least squares

Table A.2: Main acronyms used in the thesis

Appendix B

Mathematical background

THE results presented in this thesis rely on several notions from different branches of mathematics, including set theory, linear algebra, and probability theory. In order to render the text self-sufficient and accessible to a broad audience, in this section we recall all the concepts and properties necessary to understand and prove the results of our work. Our treatment is theoretically oriented, with the focus on the notation.

B.1 Notational Conventions

A number of special notational conventions are used throughout this text.

As a general rule, sets and spaces are denoted by blackboard uppercase characters (such as $\mathbb{Z}$, $\mathbb{R}$). Uppercase Greek and Roman letters are used for cardinality of finite sets (i.e., a number of elements of a finite set), as well as for matrix variables and functions, while lowercase Greek and Roman letters are used for both scalar and vector variables and functions. Boldface letters generally indicate sequences of N matrices or vectors, where N is the number of operational modes of the system.

Sometimes it is not possible or convenient to adhere completely to this rules, but the exceptions should be clearly perceived based on their specific context.

B.2 Sets of numbers

Numbers are mathematical objects used to count, measure and label [112] concrete or abstract things. Different types of numbers have many different uses, e.g. counting numbers are useful as labels, while complex numbers play a central role in quantum mechanics [112]. Depending on the intended use, numbers can be classified into sets, which we present first. Then, we introduce a set-builder notation used through this thesis, followed by recurrent sets derived from the standard sets of numbers, and some additional notation peculiar to the real and complex numbers.

Standard sets of numbers

A *set* is a well-defined collection of distinct objects, concrete or abstract, considered as an object in its own right. The objects that make up a set are called *elements* or *members*. The number of members (possibly infinite) of a set is known as *cardinality* and is denoted by $|\cdot|$. A set is *finite* if it has a finite cardinality, and it is *countable* if the number of elements is countable, that is, if one can label them by the positive integers in such a manner that no element remains unlabeled. An example of an infinite countable set is a set of all integers, denoted by $\mathbb{Z}$, which is one of the standard sets of numbers. Standard sets of numbers used in this thesis are reported in the following Table B.1.

Symbol	Meaning
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers

Table B.1: Standard sets of numbers used in this thesis

All sets derived from the standard sets of numbers, and other more abstract sets of mathematical objects are defined through a set-builder notation.

Set-builder notation

We use a *set-builder notation* to define sets. For the sake of conciseness and clarity, we define sets via a predicate, rather than by explicitly enumerating the set's elements. In this form, set-builder notation has three parts: a variable with its domain of appurtenance, a colon, and a logical predicate. These three parts are contained in curly brackets. As an example, a set of non-negative integers can be declared as

$$\mathbb{Z}_0 \triangleq \{i \in \mathbb{Z} : i \geq 0\}$$

where the symbol $\in$ denotes the *set membership*, indicating that an element belongs to a set. See Appendix C for additional details on mathematical symbols used throughout the thesis.

In accordance with the set-builder notation, the (unique) set having no elements is denoted by $\{\}$ and is called the *empty set*; its cardinality is zero.

Set operations and terminology

Just as real (or complex) numbers can be added or multiplied, there exist *operations* on sets [113, pp. 3–4]. Let Ω, $\mathbb{S}_i$ (with $i \in \mathbb{Z}$, $i > 0$) be (generic) sets.

The *union* of two sets $\mathbb{S}_1$ and $\mathbb{S}_2$ is the set of elements which are in $\mathbb{S}_1$, in $\mathbb{S}_2$, or in both $\mathbb{S}_1$ and $\mathbb{S}_2$. Formally, for any domain of appurtenance of a variable x,

$$\mathbb{S}_1 \cup \mathbb{S}_2 \triangleq \{x : x \in \mathbb{S}_1 \vee x \in \mathbb{S}_2\}$$

Clearly, the symbol $\vee$ indicates a *logical disjunction*. The statement $\mathcal{S}_1 \vee \mathcal{S}_2$ is true if $\mathcal{S}_1$ or $\mathcal{S}_2$ (or both) are true; if both are false, the statement is false.

The *intersection* of two sets $\mathbb{S}_1$ and $\mathbb{S}_2$ is the set that contains all elements of $\mathbb{S}_1$ that also belong to $\mathbb{S}_2$, but no other elements. In symbols, for any domain of appurtenance of a variable x,

$$\mathbb{S}_1 \cap \mathbb{S}_2 \triangleq \{x : x \in \mathbb{S}_1, x \in \mathbb{S}_2\} = \{x : x \in \mathbb{S}_1 \wedge x \in \mathbb{S}_2\}$$

Obviously, the comma in this context indicates a *logical conjunction*, which can also be denoted by $\wedge$. The statement $\mathcal{S}_1 \wedge \mathcal{S}_2$ is true if $\mathcal{S}_1$ and $\mathcal{S}_2$ are both true; else it is false.

We also use standard notations such as $\bigcup_{i=1}^{n} \mathbb{S}_i$ and $\bigcap_{i=1}^{\infty} \mathbb{S}_i$ for unions and intersections of finitely or countably many sets [113, p. 4].

The *complement* of a set Ω is denoted by Ω^{C} and refers to elements not in Ω. For any domain of appurtenance of a variable x, we express the complement of a set Ω in set-builder notation as

$$\Omega^{\mathrm{C}} \triangleq \{x : x \notin \Omega\}$$

The *difference of sets* $\mathbb{S}_1$ and $\mathbb{S}_2$, written $\mathbb{S}_1 \backslash \mathbb{S}_2$, also termed the *relative complement* of a set $\mathbb{S}_2$ with respect to a set $\mathbb{S}_1$, is the set of elements in $\mathbb{S}_1$ but not in $\mathbb{S}_2$. Formally, for any domain of appurtenance of a variable x,

$$\mathbb{S}_1 \backslash \mathbb{S}_2 \triangleq \mathbb{S}_1 \cap \mathbb{S}_2^{\mathrm{C}} = \{x : x \in \mathbb{S}_1 \wedge x \notin \mathbb{S}_2\}$$

The *symmetric difference*, also known as the *disjunctive union*, of two sets is the union of both relative complements. The symmetric difference of the sets $\mathbb{S}_1$ and $\mathbb{S}_2$ is commonly denoted by $\mathbb{S}_1 \triangle \mathbb{S}_2$. In symbols,

$$\mathbb{S}_1 \triangle \mathbb{S}_2 \triangleq (\mathbb{S}_1 \backslash \mathbb{S}_2) \cup (\mathbb{S}_2 \backslash \mathbb{S}_1)$$

Through this thesis we use some set-theoretic terms, which are introduced in the rest of this subsection.

A set $\mathbb{S}$ is a *subset* of a set Ω, denoted by $\mathbb{S} \subseteq \Omega$, or equivalently Ω is a *superset* of $\mathbb{S}$, written as $\Omega \supseteq \mathbb{S}$, if $\mathbb{S}$ is "contained" inside Ω, i.e., all elements of $\mathbb{S}$ are also elements of Ω. Sets $\mathbb{S}$ and Ω may coincide. The *relationship* of one set being a subset of another is called *inclusion* or sometimes *containment*. Formally,

$$\mathbb{S} \subseteq \Omega \quad \text{if} \quad x \in \mathbb{S} \Rightarrow x \in \Omega$$

The symbol $\Rightarrow$ denotes the *material implication*. The statement $\mathcal{S}_1 \Rightarrow \mathcal{S}_2$ means that if $\mathcal{S}_1$ is true then $\mathcal{S}_2$ is also true; if $\mathcal{S}_1$ is false then nothing is said about $\mathcal{S}_2$.

The subset relation defines a (non-strict) *partial order* on sets, i.e., the inclusion is reflexive, antisymmetric, and transitive. Formally, for all (generic) sets Ω, $\mathbb{S}_i$ (with $i \in \mathbb{Z}$, $i > 0$), it satisfy the following properties.

- *Reflexivity*: every set is related to itself, i.e.,

$$\Omega \subseteq \Omega$$

- *Antisymmetry*: two distinct sets cannot be related in both directions, that is,

$$\mathbb{S}_1 \subseteq \mathbb{S}_2 \wedge \mathbb{S}_2 \subseteq \mathbb{S}_1 \Rightarrow \mathbb{S}_1 = \mathbb{S}_2$$

- *Transitivity*: if a first set is related to a second set, and, in turn, that set is related to a third set, then the first set is related to the third set. Formally,

$$\mathbb{S}_1 \subseteq \mathbb{S}_2 \wedge \mathbb{S}_2 \subseteq \mathbb{S}_3 \Rightarrow \mathbb{S}_1 \subseteq \mathbb{S}_3$$

Two sets are said to be *disjoint* if they have no element in common. Equivalently, disjoint sets are sets whose intersection is the empty set. Formally, we write that sets $\mathbb{S}_1$ and $\mathbb{S}_2$ are disjoint if $\mathbb{S}_1 \cap \mathbb{S}_2 = \{\}$.

The *power set* of any set Ω is the set of all subsets of Ω, including the empty set and Ω itself. We denote the power set of a set Ω by the set of all functions from Ω to a given set of two elements, 2^Ω. This notation is motivated by the fact that for a finite set Ω with cardinality $|\Omega|$, the cardinality of the power set is $2^{|\Omega|}$. This statement can be easily proved through the application of the indicator function (also known as a characteristic function) of the subset to each element of the set Ω.

Formally, for any set $\mathbb{S} \subseteq \Omega$, we define the *indicator function* $\mathbf{1}_\mathbb{S} : \mathbb{S} \to \{0, 1\}$ in the usual way [12, p. 31], that is, $\forall \omega \in \mathbb{S}$

$$\mathbf{1}_\mathbb{S}(\omega) \triangleq \begin{cases} 1 & \text{if } \omega \in \mathbb{S} \\ 0 & \text{if } \omega \notin \mathbb{S} \end{cases} \tag{B.1}$$

If we order the elements of Ω in any manner, we can write any subset $\mathbb{S}$ of Ω in the format $\left\{\mathbf{1}_\mathbb{S}(\omega_1), \mathbf{1}_\mathbb{S}(\omega_2), \ldots, \mathbf{1}_\mathbb{S}\left(\omega_{|\Omega|}\right)\right\}$. Clearly, the number of distinct subsets that can be constructed in this way is $2^{|\Omega|}$ as $\mathbf{1}_\mathbb{S}(\omega_i) \in \{0, 1\}$, $\forall i \in \mathbb{Z}$, $i \geq 1$, $i \leq |\Omega|$.

The power set of a set Ω is formally defined as

$$2^\Omega \triangleq \{\mathbb{S} : \mathbb{S} \subseteq \Omega\}$$

See Section B.5 for the related notions of sequences, limits and collections of sets, together with the fields of sets.

Sets derived from the standard sets of numbers

In order to render our notation more succinct and simple, we define recurrent subsets of standard sets of numbers as follows.

Symbol	Meaning
$\mathbb{Z}_0$	Set of nonnegative integers, i.e., $\{i\in\mathbb{Z} : i\geq 0\}$
$\mathbb{Z}_+$	Set of positive integers, i.e., $\{i\in\mathbb{Z} : i>0\}$
$\mathbb{R}_0$	Set of nonnegative real numbers, i.e., $\{i\in\mathbb{R} : i\geq 0\}$
$\mathbb{R}_+$	Set of positive real numbers, i.e., $\{i\in\mathbb{R} : i>0\}$
$\mathbb{F}$	Set of either real or complex numbers

Table B.2: Sets derived from the standard sets of numbers

Notation peculiar to the real and complex numbers

Any *complex number* can be expressed in form $x = a + \imath b$, where $a, b \in \mathbb{R}$, and $\imath$ indicates the imaginary unit, which satisfies $\imath^2 = -1$. The real number a is called the *real part* of a complex number x, and is denoted by $\mathrm{Re}(x)$. Similarly, the real number b is referred to as the *imaginary part* of a complex number x, and is denoted by $\mathrm{Im}(x)$. The *complex conjugate* of $x\in\mathbb{C}$, i.e., a complex number $a–\imath b$, is indicated by $\bar{x}$ (read "x-bar", from overbar drawn above the text).

Both real and complex numbers can be used in different types of measurements, all of which are closely related to the idea of distance. The distance from a number to the origin is given by the absolute value of that number (along the real number line, for real numbers, or in the complex plane, for complex numbers). More generally, the absolute value of the difference of two real or complex numbers is the distance between them.

The *absolute value* (sometimes referred to as modulus) of either real or complex number is denoted by $|\cdot|$.

For any real number x, the absolute value $|\cdot| : \mathbb{R}\to\mathbb{R}_0$ is defined as

$$|x| \triangleq \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases} \tag{B.2}$$

For any complex number $x=a+\imath b$, the absolute value $|\cdot| : \mathbb{C}\to\mathbb{R}_0$ is defined by

$$|x| \triangleq \sqrt{[\mathrm{Re}(x)]^2 + [\mathrm{Im}(x)]^2} = \sqrt{a^2 + b^2} = \sqrt{x\bar{x}}$$

Clearly, the absolute value is a norm on a 1-dimensional vector space $\mathbb{F}$.

B.3 Complete normed linear spaces

As a rule, we assume the most important notions and results of *linear algebra* to be known. See for instance [114], [115], [116] as textbooks on the topic. In the following, we summarize only some essential concepts, with the focus on our notation.

Linear spaces

A *linear space* (also called a *vector space*, denoted for instance by $\mathbb{X}$ or by $\mathbb{Y}$) is a collection of objects called *vectors*, which may be added together and multiplied ("scaled") by numbers, called *scalars* in this context. Linear spaces are the subject of linear algebra and are well characterized by their *dimension*, which, roughly speaking, specifies the number of *independent directions* in the space. One widely known example of a linear space is three-dimensional Euclidean space defined on Cartesian coordinate system, which is denoted by $\mathbb{R}^3$.

The basic operations (in infix notation) used to perform calculations on linear spaces are reported in Table B.3.

Symbol	Meaning
$\times$	*Cartesian product*; it returns a product set of all ordered pairs; it is used in definition of linear spaces, which are domains of our functions, e.g. $\mathbb{R}^3 = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, and $\mathbb{X}\times\mathbb{Y} = \{(\mathrm{x},\mathrm{y}) : \mathrm{x}\in\mathbb{X}, \mathrm{y}\in\mathbb{Y}\}$
$+$	*addition* (uses plus symbol), e.g $\mathrm{x}+\mathrm{y}$, $\forall \mathrm{x},\mathrm{y}\in\mathbb{X}$
$-$	*subtraction* (uses minus sign), e.g. $\mathrm{x}-\mathrm{y}$, $\forall \mathrm{x},\mathrm{y}\in\mathbb{X}$
	scalar multiplication (uses juxtaposition), e.g. $a\mathrm{x}$, $\forall \mathrm{x}\in\mathbb{X}, a\in\mathbb{F}$

Table B.3: Basic operations for (definition of, and) calculations on linear spaces

Normed linear spaces

We are interested in normed linear spaces, i.e., vector spaces on which a norm is defined. *Norm* is a (continuous) function that assigns a strictly positive length or size to each vector in a linear space (save for the zero vector, which is assigned a length of zero):

$$\|\cdot\| : \mathbb{X} \to \mathbb{R}_0$$

For any given norm, the *triangle inequality* holds, i.e., taking norms as distances, the shortest distance between any two points is a straight line. Formally,

$$\|\mathrm{x}+\mathrm{y}\| \leq \|\mathrm{x}\| + \|\mathrm{y}\|, \ \forall \mathrm{x},\mathrm{y}\in\mathbb{X} \tag{B.3}$$

This property is sometimes referred to as being subadditive.

Another property defining any given norm is *absolute homogeneity* (a.k.a. absolute scalability), stating that $\forall c\in\mathbb{F}, \mathrm{x}\in\mathbb{X}$

$$\|c\mathrm{x}\| = |c|\,\|\mathrm{x}\| \tag{B.4}$$

We remind that any norm $\|\cdot\|$ induces a *metric* (a notion of distance) and therefore a *topology* on a linear space $\mathbb{X}$. This metric is defined in the natural way: the *distance* is a function $d_{\mathbb{X}} : \mathbb{X}\times\mathbb{X} \to \mathbb{R}_0$ between any two vectors x and y in $\mathbb{X}$, which is given by $\|\mathrm{x}-\mathrm{y}\|$. This topology is precisely the weakest topology which makes $\|\cdot\|$ continuous and which is compatible with the linear structure of $\mathbb{X}$. Notably, the topology is defined so as to capture a particular notion of convergence of sequences of objects.

Sequences defined by position and by recursion

It is worth recalling that a *sequence* is an enumerated collection of objects in which repetitions are allowed. Like a set, it contains *members* (also called *elements*, or *terms*). The number of elements (possibly infinite) is called the *length* of the sequence. Unlike a set, order matters, and exactly the same elements can appear multiple times at different positions in the sequence.

In this thesis, we are dealing with discrete-time signals (and their linear transformations), which are time series consisting of sequences of quantities (vectors, and linear maps between vectors). Formally, such a sequence can be defined as a function whose domain is either the discrete-time set $\mathbb{T}$ (corresponding to the set of the natural numbers, for infinite sequences) or the bounded discrete-time set $\mathbb{T}_k$ (which basically is the set of the first k natural numbers, for a sequence of finite length $k+1$). The variable $k\in\mathbb{Z}_0$ is called discrete-time *instant* (which is an *index* of the sequence, and the set of values that it can take is generally called the *index set*). Formally,

$$\mathbb{T} \triangleq \mathbb{Z}_0 \tag{B.5}$$

$$\mathbb{T}_k \triangleq \{i\in\mathbb{T} : i\leq k\} \tag{B.6}$$

When the elements of a sequence are defined as a function of their position, the sequence is denoted by the indexed element of the sequence in round brackets, e.g.

$$(\mathrm{x}_k),\ \mathrm{x}_k\in\mathbb{X},\ k\in\mathbb{T}$$

This convention permits us to denote the *empty sequence* as (), and to state the first and the last elements of the sequence explicitly as

$$(\mathrm{x}_k)_{k=0}^{\infty}$$

For sequences whose elements are related to the previous elements in a straightforward way, we also use an alternative notation based on recursion, e.g.

$$\mathrm{x}_k = f(\mathrm{x}_{k-1}),\ f:\mathbb{R}^2\rightarrow\mathbb{R}^2,\ \mathrm{x}_0=(a,b),\ a,b\in\mathbb{R}$$

It is worth noting that in computer science, finite sequences are sometimes called strings, words or lists, the different names commonly corresponding to different ways to represent them in computer memory; infinite sequences are called streams.

Convergent sequences, Cauchy sequences, Banach spaces

An important property of a sequence is *convergence*. If a sequence converges, it "tends to" a particular finite value known as the *limit*, i.e., elements of the sequence become closer and closer to the limit value. If such a limit exists, the sequence is called *convergent*. A sequence that does not converge is *divergent*.

Formally, we define a vector x of a normed linear space $(\mathbb{X}, \|\cdot\|)$ as the limit of the sequence (x_k), if

$$\forall \epsilon \in \mathbb{R}_+ \ \exists L \in \mathbb{R}_0 : \forall k \in \mathbb{T},\ k > L,\ \|\mathrm{x}_k - \mathrm{x}\| < \epsilon$$

We denote the limit value as

$$\mathrm{x} \triangleq \lim_{k\to\infty} \mathrm{x}_k$$

Of particular interest for our work are *Cauchy sequences*, whose elements become arbitrarily close to each other as the sequence progresses. More formally,

$$\forall \epsilon \in \mathbb{R}_+ \ \exists L \in \mathbb{Z}_+ : \forall k, t \in \mathbb{T},\ k, t > L,\ \|\mathrm{x}_k - \mathrm{x}_t\| < \epsilon$$

Cauchy sequences are necessary for the definition of complete metric spaces (a.k.a. Cauchy spaces). Specifically, a normed linear space $\mathbb{X}$ is called *complete* if every Cauchy sequence of vectors in $\mathbb{X}$ has a limit that is also within the space $\mathbb{X}$. Such complete normed linear space is also called the *Banach space*.

The vector space structure of a Banach space $\mathbb{X}$ allows us to relate the behavior of Cauchy sequences (x_k) in $\mathbb{X}$ to that of (absolutely) converging series of vectors.

Series, summation, absolute convergence of sequences

We recall that the *series* generated by (x_k) is the sequence

$$\left(\sum_{k=0}^{t} \mathrm{x}_k\right)_{t=0}^{\infty}$$

As usual, the *summation* symbol $\sum$ indicates the addition of a sequence of terms,

$$\sum_{k=0}^{t} \mathrm{x}_k = \mathrm{x}_0 + \mathrm{x}_1 + \cdots + \mathrm{x}_{t-1} + \mathrm{x}_t$$

The addends of summation are added sequentially from left to right. Any intermediate result is called a *partial sum* (or *prefix sum*, or *running total*). We recall that summation notation supports two special cases, to sum up less than two addends. Specifically, if the summation has only one addend x, then the evaluated sum is x. If the summation has no addends, then the evaluated sum is zero (vector), because zero is the identity for addition. This is known as the empty sum.

For *convergent* series, we have that there exists a limit of corresponding sequence of partial sums. It is given by

$$\lim_{t\to\infty} \sum_{k=0}^{t} \mathrm{x}_k = \sum_{k=0}^{\infty} \mathrm{x}_k < \infty$$

It is well known (see [117, Proposition 1.3.7]) that

$$\text{if } \sum_{k=0}^{\infty} \mathrm{x}_k \text{ converges in } \mathbb{X}, \text{ then } \lim_{k\to\infty} \mathrm{x}_k = 0 \tag{B.7}$$

This result is obtained immediately from the fact that if $\sum_{k=0}^{\infty} \mathrm{x}_k$ is a convergent series in $\mathbb{X}$, then each x_k besides the first is the difference of two consecutive terms of the Cauchy sequence of partial sums of the series. Of course, the converse of (B.7) is not in general true since the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Another useful result reported in [117, Proposition 1.3.7] follows from the repeated applications of the triangle inequality (B.8) and from the continuity of the norm function. It states that for $t \leq \infty$

$$\text{if } \sum_{k=0}^{t} \mathrm{x}_k \text{ in } \mathbb{X}, \text{ then } \|\sum_{k=0}^{t} \mathrm{x}_k\| \leq \sum_{k=0}^{t} \|\mathrm{x}_k\| \tag{B.8}$$

It worth noting that the sum $\sum_{k=0}^{\infty} \|\mathrm{x}_k\|$ in (B.8) does not have to be finite when $\sum_{k=0}^{\infty} \mathrm{x}_k$ is a convergent infinite sum.

If $\sum_{k=0}^{\infty} \|\mathrm{x}_k\|$ converges, then the series $\sum_{k=0}^{\infty} \mathrm{x}_k$ is called *absolutely convergent*.

The absolute convergence of a series in a normed linear space $\mathbb{X}$ implies the convergence of a series in $\mathbb{X}$. Besides, a normed linear space $\mathbb{X}$ is a Banach space if and only if each absolutely convergent series in $\mathbb{X}$ converges.

Equivalent vector norms

In this text, we are dealing with finite-dimensional linear spaces, for which all norms are *equivalent* [118, Theorem 4.27] from a topological viewpoint, as they induce the same topology (although the resulting metric spaces need not be the same). Formally, any two norms $\|\cdot\|_1$, $\|\cdot\|_2$ in a vector space $\mathbb{X}$ are equivalent, if

$$\exists c_1 \in \mathbb{R}_+, \exists c_2 \in \mathbb{R}_+ : \|\mathrm{x}\|_1 \leq c_2 \|\mathrm{x}\|_2\,,\ \|\mathrm{x}\|_2 \leq c_1 \|\mathrm{x}\|_1\,,\ \forall \mathrm{x} \in \mathbb{X} \tag{B.9}$$

The notable (equivalent) vector norms we are dealing with in this thesis are all variants of *p-norms* (a.k.a. $\mathbb{L}^p$-norms) for finite-dimensional linear spaces.

For all $\mathrm{x} \in \mathbb{F}^n$, i.e., $\mathrm{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$, $x_i \in \mathbb{F}$, and for each $p \in \mathbb{R}_+$, $p \geq 1$, a p-norm of x [116, p. 274] is defined as

$$\|\mathrm{x}\|_p \triangleq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

The *Euclidean norm* in this context is a p-norm, with $p=2$. Notably, $\forall \mathrm{x} \in \mathbb{F}^n$

$$\|\mathrm{x}\|_2 \triangleq \sqrt{\sum_{i=1}^{n} |x_i|^2} \tag{B.10}$$

Similarly, the *grid norm* (a.k.a. *Manhattan* or *taxicab* norm) is a p-norm, with $p=1$. Curiously, the name relates to the distance a taxi has to drive in a rectangular street grid to get from the origin to the point x. For all $\mathrm{x} \in \mathbb{F}^n$, the 1-norm is defined as

$$\|\mathrm{x}\|_1 \triangleq \sum_{i=1}^{n} |x_i| \tag{B.11}$$

Finally, as p tends to infinity the p-norm approaches the *maximum norm*, i.e., $\forall \mathrm{x} \in \mathbb{F}^n$, $i, n \in \mathbb{Z}_+$, $i \leq n$,

$$\|\mathrm{x}\|_\infty = \|\mathrm{x}\|_{\max} \triangleq \max_i |x_i| \tag{B.12}$$

The finite-dimensional linear spaces defined over $\mathbb{F}^n$, with equivalent $\mathbb{L}^p$ norms defined as above, are sometimes called *Lebesgue spaces*.

We observe that completeness of a normed linear space is preserved if the given norm is replaced by an equivalent one. Furthermore, every finite-dimensional normed space over $\mathbb{F}^n$ is a Banach space [117, Corollary 1.4.19].

Hilbert spaces and inner products

A *Hilbert space* is a (real or complex) Banach space, possessing the structure of an inner product, and having the distance function induced by this inner product. For any linear space $\mathbb{X}$, the *inner product*, is a function $\langle \cdot , \cdot \rangle : \mathbb{X} \times \mathbb{X} \to \mathbb{F}$ that satisfies the following three axioms for all vectors $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathbb{X}$ and all scalars $a \in \mathbb{F}$:

1. *Conjugate symmetry*, i.e.,

$$\langle \mathrm{x} , \mathrm{y} \rangle = \overline{\langle \mathrm{y} , \mathrm{x} \rangle} \tag{B.13}$$

2. Linearity in the first argument:

$$\langle a\mathrm{x} , \mathrm{y} \rangle = a \langle \mathrm{x} , \mathrm{y} \rangle \tag{B.14a}$$

$$\langle \mathrm{x}+\mathrm{y} , \mathrm{z} \rangle = \langle \mathrm{x} , \mathrm{z} \rangle + \langle \mathrm{y} , \mathrm{z} \rangle \tag{B.14b}$$

3. Positive-definiteness:

$$\langle \mathrm{x} , \mathrm{x} \rangle \geq 0 \tag{B.15a}$$

$$\langle \mathrm{x} , \mathrm{x} \rangle = 0 \iff \mathrm{x} = 0 \tag{B.15b}$$

where, with a slight abuse of notation, we denote by 0 either a scalar number zero in $\mathbb{F}$, or a vector of all zeros of appropriate size.

Notably, the space $\mathbb{L}^2$, i.e., a finite-dimensional Banach space defined on $\mathbb{F}^n$, equipped with the Euclidean norm as in (B.10), is a Hilbert space.

B.4 Linear maps, matrices and related operations

For $\mathbb{X}$ and $\mathbb{Y}$ either real or complex Banach spaces, we set $\mathcal{B}(\mathbb{X}, \mathbb{Y})$ the (Banach) space of all (bounded) linear operators of $\mathbb{X}$ into $\mathbb{Y}$, with the uniform induced norm represented by $\|\cdot\|$. In this section we show that linear operators can be represented by matrices, and recall a number of useful operations on matrices.

Linear operators

We recall that linear *operators* (also called linear *transformations*, linear *maps* or linear *mappings*) are continuous functions $\mathcal{L}:\mathbb{X}\to\mathbb{Y}$ between two linear spaces that preserves the operations of addition and scalar multiplication, i.e., $\forall \mathrm{x}, \mathrm{x}_1, \mathrm{x}_2 \in \mathbb{X}$ and $a \in \mathbb{F}$ the following two conditions are satisfied:

$$\mathcal{L}(\mathrm{x}_1 + \mathrm{x}_2) = \mathcal{L}(\mathrm{x}_1) + \mathcal{L}(\mathrm{x}_2) \tag{B.16a}$$

$$\mathcal{L}(a\mathrm{x}) = a\mathcal{L}(\mathrm{x}) \tag{B.16b}$$

For any normed linear spaces $\mathbb{X}$ and $\mathbb{Y}$, and any linear operator $\mathcal{L}(\cdot):\mathbb{X}\to\mathbb{Y}$, the following statements are equivalent [117, Theorem 1.4.2]:

1. The operator $\mathcal{L}(\cdot)$ is continuous.
2. The operator $\mathcal{L}(\cdot)$ is continuous at 0, and uniformly continuous on $\mathbb{X}$.
3. The operator $\mathcal{L}(\cdot)$ is bounded.
4. For some neighborhood $\mathbb{U}$ of 0 in $\mathbb{X}$, the set $\mathcal{L}(\mathbb{U})$ is bounded in $\mathbb{Y}$.
5. There is $L \in \mathbb{R}_0$ s.t. $\|\mathcal{L}(\mathrm{x})\| \leq L\|\mathrm{x}\|$ for each x in $\mathbb{X}$.
6. The quantity $\sup\{\|\mathcal{L}(\mathrm{x})\| : \mathrm{x}\in\mathbb{X}, \|\mathrm{x}\|\leq 1\}$ is finite.

Let $\mathbb{X}$ and $\mathbb{Y}$ be normed linear spaces. For each linear transformation $\mathcal{L}(\cdot)$ in $\mathcal{B}(\mathbb{X},\mathbb{Y})$, the *operator norm* is a function $\|\cdot\| : \mathcal{B}(\mathbb{X},\mathbb{Y})\to\mathbb{R}_0$, defined by

$$\|\mathcal{L}(\cdot)\| = \sup\{\|\mathcal{L}(\mathrm{x})\| : \mathrm{x}\in\mathbb{X}, \|\mathrm{x}\|\leq 1\} \tag{B.17}$$

Let $\mathbb{X}$, $\mathbb{Y}$ be normed vector spaces. Then $\mathcal{B}(\mathbb{X},\mathbb{Y})$ is a normed linear space under the operator norm. If $\mathbb{Y}$ is a Banach space, then so is $\mathcal{B}(\mathbb{X},\mathbb{Y})$ [117, Theorem 1.4.8].

Transformation matrices

In linear algebra, linear operators can be represented by matrices.

A *matrix* (plural matrices) is a rectangular array of numbers, symbols, or mathematical expressions arranged in rows and columns. A matrix with m rows and n columns is said to be an $m\times n$ (read "m-by-n") matrix. Clearly, $m, n \in \mathbb{Z}_+$. The individual items in an $m\times n$ matrix A are denoted by a_{ij} where $\max i = m$ and $\max j = n$, are called its *elements* or entries. In this thesis, we use *square brackets* to contain constituent elements of a matrix, i.e.,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where, as usual, $\cdots$, $\vdots$ and $\ddots$ indicate omitted values from a pattern.

In what follows, it will be useful to be able to indicate concisely specific rows, columns and submatrices of any given matrix.

We denote by $A_{i\bullet}$ the i-th row of a matrix $A=[a_{ij}]\in\mathbb{F}^{m,n}$. Specifically, for all $i,m\in\mathbb{Z}_+$, $i\leq m$,

$$A_{i\bullet} = (a_{ij})_{j=1}^{n} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \tag{B.18}$$

This notation is readily extended to denote submatrices obtained from the original matrix by taking a number of consecutive rows.

We indicate by $A_{[i,i+d]\bullet}$ the submatrix of $A=[a_{ij}]\in\mathbb{F}^{m,n}$ containing $d+1$ consecutive rows, starting from $A_{i\bullet}$. Explicitly, $\forall i,d,m\in\mathbb{Z}_+$, $i+d\leq m$,

$$A_{[i,i+d]\bullet} = \Big((a_{sj})_{j=1}^{n}\Big)_{s=i}^{i+d} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(i+d)1} & a_{(i+d)2} & \cdots & a_{(i+d)n} \end{bmatrix} \tag{B.19}$$

Clearly, this notation allows us to denote the matrix $A\in\mathbb{F}^{m,n}$ as $A=[A_{[1,m]\bullet}]$, underlining the fact that in this case the matrix A is interpreted row by row.

Then, we indicate by $A_{\bullet j}$ the j-th column of a matrix $A=[a_{ij}]\in\mathbb{F}^{m,n}$. Formally, for all $j,n\in\mathbb{Z}_+$, $j\leq n$,

$$A_{\bullet j} = (a_{ij})_{i=1}^{m} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \tag{B.20}$$

The extension to denote submatrices obtained from the original matrix by taking a number of consecutive columns is straightforward.

We denote by $A_{\bullet[j,j+d]}$ the submatrix of $A=[a_{ij}]\in\mathbb{F}^{m,n}$ consisting of $d+1$ consecutive columns, starting from $A_{\bullet j}$. Formally, $\forall j,d,n\in\mathbb{Z}_+$, $j+d\leq n$,

$$A_{\bullet[j,j+d]} = ((a_{is})_{i=1}^{m})_{s=j}^{j+d} = \begin{bmatrix} a_{1j} & a_{1(j+1)} & \cdots & a_{1(j+d)} \\ a_{2j} & a_{2(j+1)} & \cdots & a_{2(j+d)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{mj} & a_{m(j+1)} & \cdots & a_{m(j+d)} \end{bmatrix} \tag{B.21}$$

Obviously, this notation allows us to write the matrix $A\in\mathbb{F}^{m,n}$ as $A=[A_{\bullet[1,n]}]$, emphasizing the interpretation of the matrix A column by column.

Evidently, when $m=1$, we have n-dimensional row-vectors, and when $n=1$, we deal with m-dimensional column-vectors. When both m, and n are equal to 1, the matrix is actually a (scalar) number. Thus, we can use matrices to represent both elements of (finite-dimensional) normed linear spaces and linear transformations between such normed linear spaces.

Notably, if $\mathcal{L} : \mathbb{F}^n \to \mathbb{F}^m$ is a linear operator, and $\mathrm{x} \in \mathbb{F}^n$ is a column vector, then $\mathcal{L}(\mathrm{x}) = A\mathrm{x}$, for some $m \times n$ matrix A, called the *transformation matrix* of $\mathcal{L}(\cdot)$. When a linear mapping $\mathcal{L}(\mathrm{x})$ is given in functional form, it is easy to determine the transformation matrix A by transforming each of the vectors of the standard basis by $\mathcal{L}(\cdot)$, then inserting the result into the columns of a matrix. We recall that the *standard basis* (also called *natural basis*) consists of the elements of the vector space such that all coefficients but one are 0 and the non-zero one is 1. Denoting the i-th vector of the natural basis by e_i, we have that

$$A = \begin{bmatrix} \mathcal{L}(\mathrm{e}_1) & \mathcal{L}(\mathrm{e}_2) & \cdots & \mathcal{L}(\mathrm{e}_n) \end{bmatrix}$$

Henceforth, we denote by $\mathbb{F}^{m,n}$ both the set of matrices with m rows, n columns, and entries in $\mathbb{F}$, and the set of linear maps between two linear spaces $\mathbb{F}^n$ and $\mathbb{F}^m$.

Diagonal and square matrices

Two important type of matrices are diagonal and square matrices.

The *main diagonal* of a generic matrix $A = [a_{ij}] \in \mathbb{F}^{m,n}$ is the collection of entries a_{ij} where $i = j$, that is a sequence $(a_{ii})_{i=1}^{\min\{m,n\}}$.

The *diagonal matrix* is a matrix in which the entries outside the main diagonal are all zero. In more detail, a matrix $A \in \mathbb{F}^{m,n}$, $A = [a_{ij}]$, is diagonal, if

$$\forall i, j : i \neq j,\ a_{ij} = 0$$

The *square matrix* is a matrix with the same number of rows and columns. Thus, we denote the set of all $n \times n$ (square) matrices as $\mathbb{F}^{n,n}$.

To construct a (square) diagonal matrix in straightforward way, there is a function diag : $\mathbb{F}^n \to \mathbb{F}^{n,n}$, which puts the elements of the vector $\mathrm{x} \in \mathbb{F}^n$ on the main diagonal. More formally, $\forall \mathrm{x} \in \mathbb{F}^n, A \in \mathbb{F}^{n,n}$, $A = [a_{ij}]$,

$$\operatorname{diag}(\mathrm{x}) = [a_{ij}] : \begin{cases} \forall j \neq i,\ a_{ij} = 0 \\ \forall x_i \in \mathrm{x}, a_{ii} = x_i \end{cases}$$

The *identity matrix* of size n, denoted by I_n, is the $n \times n$ square diagonal matrix with ones on the main diagonal. The identity matrix takes its name from the fact, that it is the identity element of the matrix multiplication.

Matrix product

When two linear operators are represented by matrices, then the *matrix product* (a.k.a. *matrix multiplication*) represents the composition of the two operators.

The matrix product is a binary operation that produces a matrix from two matrices. Specifically, if $A \in \mathbb{F}^{m,n}$ and $B \in \mathbb{F}^{n,p}$, their matrix product $AB \in \mathbb{F}^{m,p}$, in which the n entries across a row of A are multiplied with the n entries down a columns of B and summed to produce an entry of AB. More formally, for $A=[a_{ij}]$, $B=[b_{ij}]$, we have that $AB=[(ab)_{ij}]$, with

$$(ab)_{ij} = \sum\nolimits_{k=1}^{n} a_{ik}b_{kj} \tag{B.22}$$

Clearly, the matrix product is not commutative, i.e., in general $AB \neq BA$.

Other algebraic operations (on matrices) which carry the term product in their name are the outer product (of vectors in matrix form) and Kronecker product.

Before introducing them, we first need to recall some additional definitions.

Transpose and conjugate transpose

The operation of *transposition* is denoted by a superscript ^{T}. It produces a *transpose* matrix, obtained from the original one by flipping a matrix over its main diagonal, i.e., the transposition switches the row and column indices of the matrix. Specifically, $\forall A=[a_{ij}]\in\mathbb{F}^{m,n}$, $A^T=[a_{ji}]\in\mathbb{F}^{n,m}$.

The operation of *complex conjugation*, defined also for (complex) matrices, is performed entrywise, that is $\forall A=[a_{ij}]\in\mathbb{C}^{m,n}$, $\bar{A}=[\bar{a}_{ij}]\in\mathbb{C}^{m,n}$.

The *conjugate transpose* (a.k.a. *Hermitian transpose*) of a (complex) matrix is obtained by taking the transpose of the original matrix and then taking the complex conjugate of each entry. Formally, denoting the *conjugate transpose* by superscript *, we have that $\forall A = [a_{ij}] \in \mathbb{C}^{m,n}$, $A^* = [\bar{a}_{ji}] \in \mathbb{C}^{n,m}$. Clearly, for a set of real matrices, the transpose and conjugate transpose are the same.

The operation of taking the (conjugate) transpose is an involution (self-inverse), i.e.,

$$\left(A^T\right)^T = A, \quad (A^*)^* = A \tag{B.23}$$

Another important property of the (conjugate) transpose regards the matrix product. For two matrices of the appropriate size, we have that

$$(AB)^T = B^TA^T, \qquad (AB)^* = B^*A^*$$

By induction, this result extends to the general case of multiple matrices: $\forall k\in\mathbb{Z}_+$,

$$(A_1A_2\cdots A_{k-1}A_k)^T = A_k^TA_{k-1}^T\cdots A_2^TA_1^T \tag{B.24a}$$

$$(A_1A_2\cdots A_{k-1}A_k)^* = A_k^*A_{k-1}^*\cdots A_2^*A_1^* \tag{B.24b}$$

Compact notation for the matrix product of a sequence

The matrix product in the left-hand side (LHS) of the previous equations (B.24) can be written in the compact form as

$$A_1A_2\cdots A_{k-1}A_k = \prod\nolimits_{i=1}^{k} A_i \tag{B.25}$$

The symbol $\prod$ indicates the *product* of a sequence of matrices of appropriate size. Similarly to the summation notation, the subscript gives the symbol for a dummy variable (i in this case), called the "index" of multiplication together with its lower bound (here 1), whereas the superscript (in this instance k) gives its upper bound. The lower and upper bound are expressions denoting integers. The factors of the product are obtained by taking the expression following the product operator, with successive integer values substituted for the index of multiplication, starting from the lower bound and incremented by 1 up to and including the upper bound.

This notation supports two special cases, to multiply less than two factors. In particular if the product has only one factor A_i, then the evaluated product is A_i. If the product has no factors, then the evaluated result is identity matrix of appropriate size.

Even though the matrix product is not commutative, for a sequence of matrices of appropriate size we can write the matrix product in reverse order in compact form, by combining the properties (B.23) and (B.24), i.e.,

$$A_k A_{k-1} \cdots A_2 A_1 = \left(\prod_{i=1}^{k} A_i^* \right)^* \tag{B.26}$$

These compact notations for the matrix product of sequences of matrices will be widely used in the rest of this thesis.

Block diagonal matrices and direct sum

A *block matrix* is a matrix that is interpreted as having been broken into sections called blocks (a.k.a. submatrices). Intuitively, a matrix interpreted as a block matrix can be visualized as the original matrix with a collection of horizontal and vertical lines, which break it up, or partition it, into a collection of smaller matrices. Any matrix may be interpreted as a block matrix in one or more ways, with each interpretation defined by how its rows and columns are partitioned. From here on, we denote an interpretation of the partitioned matrix A by $[A_{ij}]$.

Of particular interest for our thesis are *block diagonal matrices*, which are block matrices that are square matrices, having square matrices on main diagonal blocks, and such that the off-diagonal blocks are zero matrices.

Block diagonal matrices can be obtained from the application of the *direct sum* operation, denoted by $\oplus$, to square matrices. For instance, $\forall A = [a_{ij}] \in \mathbb{F}^{n,n}$, $B = [b_{ij}] \in \mathbb{F}^{m,m}$, $(A \oplus B) \in \mathbb{F}^{n+m,n+m}$ is defined by

$$A \oplus B = \left[\begin{array}{c:c} A & 0 \\ \hdashline 0 & B \end{array} \right] = \left[\begin{array}{ccc:ccc} a_{11} & \cdots & a_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 0 \\ \hdashline 0 & \cdots & 0 & b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{m1} & \cdots & b_{mm} \end{array} \right]$$

where, with a slight abuse of notation, we denote by 0 either a scalar number zero in $\mathbb{F}$, or a square matrix of all zeros of appropriate size.

The direct sum of a sequence of $N \in \mathbb{Z}_+$ square (either real or complex) matrices $\boldsymbol{A} = (A_i)_{i=1}^N$ produces a block diagonal matrix, having the elements of $\boldsymbol{A}$ on the main diagonal blocks, i.e.,

$$\bigoplus_{i=1}^N A_i = A_1 \oplus A_2 \oplus \cdots \oplus A_N = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_N \end{bmatrix} = \mathrm{diag}(\boldsymbol{A}) \tag{B.27}$$

where 0 denotes square matrices of all zeros of appropriate size, and the function $\mathrm{diag}(\cdot)$ accepts sequence (or vector) of square matrices as input.

Kronecker product, vectorization and outer product

The *Kronecker product*, denoted by $\otimes$, is an operation on two matrices of arbitrary size resulting in a block matrix. Specifically, for any $A = [a_{ij}] \in \mathbb{F}^{m,n}$, $B = [b_{ij}] \in \mathbb{F}^{r,p}$, with $m, n, p, r \in \mathbb{Z}_+$, the Kronecker product is a function $\otimes\colon \mathbb{F}^{m,n} \times \mathbb{F}^{r,p} \to \mathbb{F}^{m+r,n+p}$, such that

$$A \otimes B \triangleq \left[\begin{array}{c|c|c} a_{11}B & \cdots & a_{1n}B \\ \hline \vdots & \ddots & \vdots \\ \hline a_{m1}B & \cdots & a_{mn}B \end{array}\right] \tag{B.28}$$

The resulting Kronecker product matrix in more explicit form is

$$A \otimes B = \left[\begin{array}{ccc|cc|ccc} a_{11}b_{11} & \cdots & a_{11}b_{1p} & \cdots & \cdots & a_{1n}b_{11} & \cdots & a_{1n}b_{1p} \\ \vdots & \ddots & \vdots & & & \vdots & \ddots & \vdots \\ a_{11}b_{r1} & \cdots & a_{11}b_{rp} & \cdots & \cdots & a_{1n}b_{r1} & \cdots & a_{1n}b_{rp} \\ \hline \vdots & & \vdots & \ddots & & \vdots & & \vdots \\ \vdots & & \vdots & & \ddots & \vdots & & \vdots \\ \hline a_{m1}b_{11} & \cdots & a_{m1}b_{1p} & \cdots & \cdots & a_{mn}b_{11} & \cdots & a_{mn}b_{1p} \\ \vdots & \ddots & \vdots & & & \vdots & \ddots & \vdots \\ a_{m1}b_{r1} & \cdots & a_{m1}b_{rp} & \cdots & \cdots & a_{mn}b_{r1} & \cdots & a_{mn}b_{rp} \end{array}\right]$$

The Kronecker product has a long list of properties presented and proved in [119].

Some of these properties rely on the *vectorization* transformation defined by Neudecker in [120], which converts the matrix into a column vector. Specifically, any given matrix $A \in \mathbb{F}^{m,n}$ is interpreted as a block matrix of all n consecutive columns, i.e., $A=[A_{\bullet[1,n]}]$; the vectorization is a linear map vec : $\mathbb{F}^{m,n}\to\mathbb{F}^{mn}$, defined as

$$\operatorname{vec}(A)\triangleq\begin{bmatrix}A_{\bullet 1}\\ A_{\bullet 2}\\ \vdots\\ A_{\bullet n}\end{bmatrix} \tag{B.29}$$

The vectorization is frequently used together with the Kronecker product to express matrix multiplication as a linear transformation on matrices.

In particular, for any A,B,C,D given either real or complex matrices of appropriate size, the following property is satisfied:

$$\operatorname{vec}(ABC)=(C^T\otimes A)\operatorname{vec}(B) \tag{B.30}$$

Another property of the Kronecker product used in this thesis is (bi-)linearity:

$$(A+B)\otimes(C+D)=A\otimes C+B\otimes C+A\otimes D+B\otimes D \tag{B.31}$$

Further, the transposition and conjugate transposition are distributive over the Kronecker product:

$$(A\otimes B)^T=A^T\otimes B^T,\qquad (A\otimes B)^*=A^*\otimes B^* \tag{B.32}$$

We also recall that the *outer product* is a special case of the Kronecker product of matrices. It is denoted by $\otimes$ and applied to (column-)vectors. Specifically, for arbitrary $\mathrm{x}\in\mathbb{F}^{m,1}$, $\mathrm{y},\mathrm{z}\in\mathbb{F}^{n,1}$, we have that $\otimes:\mathbb{F}^m\times\mathbb{F}^n\to\mathbb{F}^{m,n}$ is defined by

$$\mathrm{x}\otimes\mathrm{y}\triangleq\mathrm{x}\mathrm{y}^*=\begin{bmatrix}x_1\bar{y}_1 & \cdots & x_1\bar{y}_n\\ \vdots & \ddots & \vdots\\ x_m\bar{y}_1 & \cdots & x_m\bar{y}_n\end{bmatrix}=[x_i\bar{y}_j]$$

Moreover, in this context, the outer product is related to inner product through matrix multiplication, i.e.,

$$(\mathrm{x}\otimes\mathrm{y})\mathrm{z}=(\mathrm{x}\mathrm{y}^*)\mathrm{z}=\mathrm{x}(\mathrm{y}^*\mathrm{z})=\mathrm{x}\langle\mathrm{z},\mathrm{y}\rangle=\mathrm{x}\sum_{i=1}^{n}z_i\bar{y}_i$$

Clearly, the *inner product* of two vectors is defined only for the vectors of the same linear space. Notoriously, it can be calculated from the outer product matrix through the trace operator.

Trace, eigenvalues and determinant

The *trace* is defined (only for square matrices) by the sum of the elements on the main diagonal. Formally, the trace is linear mapping tr : $\mathbb{F}^{n,n}\to\mathbb{F}$ defined for all square matrices $A=[a_{ij}]\in\mathbb{F}^{n,n}$ by

$$\operatorname{tr}(A)=\sum\nolimits_{i=1}^{n}a_{ii} \tag{B.33}$$

The trace of product of matrices has a number of notable properties.

First of all, the trace of a matrix product can be rewritten as the sum of entry-wise products of elements, i.e., $\forall A, B, C \in \mathbb{F}^{n,n}$, $A = [a_{ij}]$, and $B = [b_{ij}]$

$$\mathrm{tr}(A^T B) = \mathrm{tr}(AB^T) = \mathrm{tr}(B^T A) = \mathrm{tr}(BA^T) = \sum\nolimits_{i=1}^{n} \sum\nolimits_{j=1}^{n} a_{ij} b_{ij} \tag{B.34}$$

The trace is invariant under cyclic permutations of matrix product

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA) \tag{B.35}$$

We note that arbitrary permutations are not allowed. In general,

$$\mathrm{tr}(ABC) \neq \mathrm{tr}(ACB)$$

The trace of the matrix product is not the product of traces:

$$\mathrm{tr}(AB) \neq \mathrm{tr}(A)\mathrm{tr}(B)$$

It is the trace of the Kronecker product of two matrices that is the product of their traces:

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$$

The trace has an important relationship with eigenvalues.

The prefix *eigen* comes from the German adjective meaning "to own", and is synonymous in English with the word characteristic. Each square matrix has its own eigen- or characteristic equation, with corresponding eigen- or characteristic values and vectors [115, p. 443].

We recall that an eigenvector of a linear map is a non-zero vector whose direction does not change when that linear map is applied to it. Formally, for any scalar $\nu \in \mathbb{F}$, and linear mapping $\mathcal{L} : \mathbb{F}^n \to \mathbb{F}^n$, represented by a transformation matrix $A \in \mathbb{F}^{n,n}$, the *eigenvector* is a non-zero (column-) vector $\mathrm{x} \in \mathbb{F}^n$, s.t.

$$\mathcal{L}(\mathrm{x}) = \nu \mathrm{x}, \quad A\mathrm{x} = \nu \mathrm{x} \tag{B.36}$$

The scalar ν is called *eigenvalue.*

Geometrically an eigenvector, corresponding to a real nonzero eigenvalue, points in a direction that is stretched (or shrunk) by the mapping, and the eigenvalue is the scale factor by which it is stretched. If the eigenvalue is negative, the direction is reversed [115, p. 444].

Equation (B.36) can be stated equivalently as

$$(A - \nu I_n)\,\mathrm{x} = 0$$

which has a non-zero solution x iff the determinant of the matrix $(A - \nu I_n)$ is zero.

The *determinant* can be viewed as the scaling factor of the mapping described by the transformation matrix. It is a function $\det : \mathbb{F}^{n,n} \to \mathbb{F}$ defined as follows.

- If $A = [a] \in \mathbb{F}$, i.e., a 1×1 matrix, then $\det(A) = a$.
- If $A \in \mathbb{F}^{n,n}$, $n \in \mathbb{Z}_+$, $n > 1$, then the *minor* M_{ij} is the determinant of the $(n-1)\times(n-1)$ submatrix of A, obtained by deleting the i-th row and j-th column of the matrix A.
- The *cofactor* associated with M_{ij} is defined as $(-1)^{i+j}\,\mathrm{M}_{ij}$.
- The determinant of $A = [a_{ij}] \in \mathbb{F}^{n,n}$, $n \in \mathbb{Z}_+$, $n > 1$ is given $\forall i = (j)_{j=1}^{n}$, and $\forall j = (i)_{i=1}^{n}$ by

$$\det(A) = \sum\nolimits_{j=1}^{n} (-1)^{i+j}\,\mathrm{M}_{ij} = \sum\nolimits_{i=1}^{n} (-1)^{i+j}\,\mathrm{M}_{ij} \tag{B.37}$$

The eigenvalues of $A = [a_{ij}] \in \mathbb{F}^{n,n}$ are values of the scalar variable ν that satisfy the equation

$$\det(A - \nu I_n) = 0 \tag{B.38}$$

From (B.37), the LHS of (B.38) is a polynomial function of the variable ν, and the degree of this polynomial is n, the order of the matrix A. Its coefficients depend on the entries of A, except that its term of degree n is always $(-1)^n\,\nu^n$. This polynomial is called the *characteristic polynomial* of A.

The fundamental theorem of algebra implies that the characteristic polynomial of $A \in \mathbb{F}^{n,n}$, being a polynomial of degree n, can be factored into the product of n linear terms, i.e.,

$$\det(A - \nu I_n) = \prod\nolimits_{i=1}^{n} (\nu_i - \nu)$$

where the scalar numbers $\nu_i \in \mathbb{C}$ (with $i \in \mathbb{Z}_+$, $i \leq n$), which may not all have distinct values, are roots of the polynomial and are the eigenvalues of A.

$$\det(A) = \prod\nolimits_{i=1}^{n} \nu_i$$

It is a well known fact that the eigenvalues of the matrix $A \in \mathbb{F}^{n,n}$ are related also to its trace [114, p. 288] as

$$\mathrm{tr}(A) = \sum\nolimits_{i=1}^{n} \nu_i \tag{B.39}$$

Lastly, the set of eigenvalues of a square matrix A is known as its *spectrum*. We denote it by $\mathfrak{S}_A$

Spectral radius and matrix norms

The *spectral radius* of a square matrix (or of a bounded linear operator) is the supremum among the absolute values of the elements in its spectrum. More formally, the spectral radius is a function $\rho : \mathbb{F}^{n,n} \to \mathbb{R}_0$ defined $\forall A \in \mathbb{F}^{n,n}$ as

$$\rho(A) = \max_{\nu_i \in \mathfrak{S}_A} |\nu_i| \tag{B.40}$$

The spectral radius is closely related to the rate of the convergence of the power sequence of a matrix [114, p. 525], i.e., $\forall A \in \mathbb{F}^{n,n}$,

$$\lim_{k\to\infty} A^k = 0 \text{ iff } \rho(A) < 1 \tag{B.41}$$

The spectral radius is linked to matrix norms, notably through Gelfand's theorem [121, p. 195], which proves that $\forall A \in \mathbb{F}^{n,n}$

$$\rho(A) = \lim_{k\to\infty} \left\| A^k \right\|^{\frac{1}{k}} \tag{B.42}$$

for any matrix norm $\|\cdot\|$.

A matrix norm is a natural extension of the notion of a vector norm to matrices.

A *matrix norm* is a norm on the vector space $\mathbb{F}^{m,n}$, i.e., a function $\|\cdot\| : \mathbb{F}^{m,n} \to \mathbb{R}_0$ satisfying the following properties (see e.g. [115, p. 438], [116, p. 280]) $\forall c \in \mathbb{R}$ and $\forall A \in \mathbb{F}^{m,n}$:

- *positive-valued*:
$$\|A\| \geq 0 \tag{B.43a}$$
- *definite*:
$$\|A\| = 0 \text{ iff } A = 0 \tag{B.43b}$$
- *absolutely homogeneous*:
$$\|cA\| = |c| \, \|A\| \tag{B.43c}$$
- *sub-additive*, i.e., satisfying the *triangle inequality* ($\forall A, B \in \mathbb{F}^{m,n}$):
$$\|A + B\| \leq \|A\| + \|B\| \tag{B.43d}$$
- *sub-multiplicative*, i.e., ($\forall A \in \mathbb{F}^{m,p}, B \in \mathbb{F}^{p,n}$):
$$\|AB\| \leq \|A\| \, \|B\| \tag{B.43e}$$

Some matrix norms are *induced* by vector norms [115, p. 438]. When a vector norm defined on a linear space $\mathbb{F}^n$ is given, any matrix $A \in \mathbb{F}^{m,n}$ is regarded as a linear operator from $\mathbb{F}^n$ to $\mathbb{F}^m$, and the corresponding matrix norm $\|\cdot\| : \mathbb{F}^{m,n} \to \mathbb{R}_0$ induced by the vector norm $\|\cdot\| : \mathbb{F}^n \to \mathbb{R}_0$ is defined as an operator norm (B.17):

$$\|A\| = \sup\{\|A\mathrm{x}\| : \mathrm{x} \in \mathbb{F}^n, \|\mathrm{x}\| = 1\} = \sup\left\{\frac{\|A\mathrm{x}\|}{\|\mathrm{x}\|} : \mathrm{x} \in \mathbb{F}^n, \mathrm{x} \neq 0\right\} \tag{B.44}$$

In particular, if the p-norm for vectors ($p \geq 1$) is used for both linear spaces $\mathbb{F}^m$ and $\mathbb{F}^n$, then the corresponding induced operator norm is:

$$\|A\|_p = \sup_{\mathrm{x} \neq 0} \frac{\|A\mathrm{x}\|_p}{\|\mathrm{x}\|_p}$$

In the special cases of $p = 1, 2, \infty$, $\forall A = [a_{ij}] \in \mathbb{F}^{m,n}$, the induced matrix norms can be computed explicitly (see e.g. [116, p. 281, p. 283]) as

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \tag{B.45}$$

which is the maximum absolute column sum of the matrix;

$$\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}| \tag{B.46}$$

which is the maximum absolute row sum of the matrix;

$$\|A\|_2 = \sigma_{\max}(A) = \sqrt{\max_{\nu_i \in \mathbb{S}_{(A^*A)}} \nu_i} \tag{B.47}$$

which is the largest *singular value* of A, i.e., the square root of the largest eigenvalue of the matrix A^*A.

The measure given to a matrix under a induced norm describes how the matrix stretches (or shrinks) unit vectors relative to that norm. The maximum stretch is the norm of the matrix [115, p. 439].

Another useful matrix norm is the Frobenius norm, also known as Schur norm, Hilbert-Schmidt norm, or ℓ_2-norm [96, p. 341] which is a special case of entry-wise matrix norms, i.e., matrix norms that treat an $m \times n$ matrix as a vector of size mn, and use one of the familiar vector norms (the Euclidean vector norm in this case).

Specifically, the *Frobenius norm* $\|\cdot\|_F : \mathbb{F}^{m,n} \to \mathbb{R}_0$ is defined $\forall A = [a_{ij}] \in \mathbb{F}^{m,n}$ as

$$\|A\|_F \triangleq \|\mathrm{vec}(A)\|_2 = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\mathrm{tr}(A^*A)} \tag{B.48}$$

Last but not least important is entry-wise norm ℓ_1-norm [96, p. 341], defined as $\|\cdot\|_{\mathbf{1}} : \mathbb{F}^{m,n} \to \mathbb{R}_0$, such that, $\forall A = [a_{ij}] \in \mathbb{F}^{m,n}$

$$\|A\|_{\mathbf{1}} \triangleq \|\mathrm{vec}(A)\|_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| \tag{B.49}$$

For any $A \in \mathbb{F}^{m,n}$, all the aforementioned matrix norms induce the same topology on the linear space $\mathbb{F}^{m,n}$, and are thus *equivalent*, that is they satisfy (B.9). This statement is true because the vector space $\mathbb{F}^{m,n}$ has the finite dimension $m \times n$, and for finite-dimensional linear spaces all norms are equivalent [118, Theorem 4.27].

It is useful to recall that for any $A \in \mathbb{F}^{n,n}$, the ℓ_1-norm and the Frobenius norm satisfy the following inequality [96, p. 365]:

$$\|A\|_{\mathbf{1}} \leq n \, \|A\|_F \tag{B.50}$$

We observe that $\forall A \in \mathbb{F}^{m,n}$ the matrix $A^*A \in \mathbb{F}^{n,n}$ is a special square matrix called positive semi-definite. In what follows next, we present this and other special types of square matrices.

Special types of square matrices

In the previous subsections B.4, B.4, B.4 and B.4 we have seen that square matrices play very important role in linear algebra. The notions of identity matrix, trace, determinant, eigenvalues and derived concepts of spectrum, spectral radius and singular values are all peculiar to square matrices. In this subsection we keep on the topic by presenting some relevant types of square matrices.

We already seen that *identity matrix* is the identity element of the matrix multiplication, i.e., $\forall A \in \mathbb{F}^{m,n}$

$$I_m A = A I_n = A$$

The identity matrix is used in the definition of the inverse element of matrix multiplication. A square matrix $A \in \mathbb{F}^{n,n}$ is called *invertible* (also nonsingular or nondegenerate) if $\exists B \in \mathbb{F}^{n,n}$, s.t.

$$AB = BA = I_n$$

If this is the case, then the matrix B is uniquely determined by A and is called the *inverse* of A, denoted by A^{-1}. A square matrix A is invertible iff $\det(A) \neq 0$.

The following properties hold for all invertible matrices $A, B \in \mathbb{F}^{n,n}$:

$$\left(A^{-1}\right)^{-1} = A \tag{B.51a}$$

$$(cA)^{-1} = c^{-1}A^{-1}, \quad \forall c \in \mathbb{F},\ c \neq 0 \tag{B.51b}$$

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T \tag{B.51c}$$

$$(AB)^{-1} = B^{-1}A^{-1} \tag{B.51d}$$

$$\det\left(A^{-1}\right) = (\det(A))^{-1} \tag{B.51e}$$

A square matrix that is equal to its transpose is said to be *symmetric matrix*. Formally, $A = [a_{ij}] \in \mathbb{F}^{n,n}$ is symmetric if $A = A^T$, or equivalently, if $a_{ij} = a_{ji}$, $\forall i, j$. We indicate the set of all symmetric matrices of order n by $\mathbb{F}^{n,n}_{\mathrm{T}}$, i.e.,

$$\mathbb{F}^{n,n}_{\mathrm{T}} \triangleq \left\{A \in \mathbb{F}^{n,n} : A = A^T\right\}$$

A *skew-symmetric* (or *antisymmetric* or antimetric) matrix is a square matrix $A = [a_{ij}] \in \mathbb{F}^{n,n}$ whose transpose is its negation, that is it satisfies the condition $-A = A^T$, or equivalently the condition $a_{ij} = -a_{ji}$, $\forall i, j$.

Remarkably, any complex number $a + \imath b$ can be identified with a skew-symmetric 2×2 matrix [122, p. 24] with equal diagonal entries

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \tag{B.52}$$

In other words, complex numbers $\mathbb{C}$ are two-dimensional vector spaces over the real numbers, so each complex number can be identified with points in the Euclidean plane $\mathbb{R}^2$. This facts justify the following remark.

Remark B.1. One can consider complex operators acting on $\mathbb{C}^{m,n}$ as real (block) matrices acting on $\mathbb{R}^{2m,2n}$ [88].

A *Hermitian* (or *self-adjoint*) matrix is a complex square matrix $A=[a_{ij}]\in\mathbb{C}^{n,n}$ that is equal to its own conjugate transpose, i.e., $A=A^*$, or, equivalently $a_{ij}=\bar{a}_{ji}$, $\forall i,j$. We denote the set of all Hermitian matrices of order n by $\mathbb{C}^{n,n}_*$. Explicitly,

$$\mathbb{C}^{n,n}_* \triangleq \{A\in\mathbb{C}^{n,n} : A=A^*\}$$

Hermitian matrices are named after Charles Hermite, who demonstrated in 1855 that matrices of this form share a property with real symmetric matrices of always having only real eigenvalues.

Since for a set of real matrices, the transpose and conjugate transpose are the same, we indicate the *set of either real symmetric or (complex) Hermitian matrices* of order n by

$$\mathbb{F}^{n,n}_* \triangleq \{A\in\mathbb{F}^{n,n} : A=A^*\}$$

A square matrix with complex entries is said to be *skew-Hermitian* (a.k.a. *antihermitian*) if its conjugate transpose is equal to its negative. Formally, a matrix $A=[a_{ij}]\in\mathbb{C}^{n,n}$ is skew-Hermitian if $A=-A^*$, i.e., $a_{ij}=-\bar{a}_{ji}$, $\forall i,j$.

A symmetric real matrix $A\in\mathbb{R}^{n,n}_{\mathrm{T}}$ is called *positive definite* if the scalar $\mathrm{x}^T A\mathrm{x}$ is positive for every non-zero column vector $\mathrm{x}\in\mathbb{R}^n$.

A concept of positive definiteness is generalized to complex matrices as follows. A Hermitian matrix $A\in\mathbb{C}^{n,n}_*$ is said to be positive definite if the scalar $\mathrm{x}^*A\mathrm{x}$ is real and positive for all non-zero column vectors $\mathrm{x}\in\mathbb{C}^n$.

We denote the set of all (either real or complex) positive definite matrices of order n by $\mathbb{F}^{n,n}_+$. Formally,

$$\mathbb{F}^{n,n}_+ \triangleq \left\{A\in\mathbb{F}^{n,n}_* : \forall \mathrm{x}\in\mathbb{F}^{n,1},\ \mathrm{x}\neq 0,\ \mathrm{x}^*A\mathrm{x}\in\mathbb{R}_+\right\}$$

All the matrices within this set have only real positive eigenvalues. Thus, every matrix $A\in\mathbb{F}^{n,n}_+$ is invertible.

Similarly, either real symmetric or complex Hermitian matrix $A\in\mathbb{F}^{n,n}_*$ is called *positive semi-definite* if the scalar $\mathrm{x}^*A\mathrm{x}$ is real and non-negative for all non-zero (either real or complex) column vectors $\mathrm{x}\in\mathbb{F}^n$.

We denote the set of all positive semi-definite matrices as $\mathbb{F}^{n,n}_0$. Explicitly,

$$\mathbb{F}^{n,n}_0 \triangleq \left\{A\in\mathbb{F}^{n,n}_* : \forall \mathrm{x}\in\mathbb{F}^{n,1},\ \mathrm{x}\neq 0,\ \mathrm{x}^*A\mathrm{x}\in\mathbb{R}_0\right\}$$

A real symmetric or complex Hermitian matrix is positive semi-definite if and only if all of its eigenvalues are non-negative.

Noticeably, for any matrix A, the matrix A^*A is positive semi-definite.

As alternative notation to $A\in\mathbb{F}_0^{n,n}$ (respectively to $A\in\mathbb{F}_+^{n,n}$) we write $A\succeq 0$ (respectively $A\succ 0$).

In conclusion of this subsection we recall that for positive semi-definite matrices, the trace dominates the Frobenius norm [123, p. 233], i.e., $\forall A\in\mathbb{F}_0^{n,n}$

$$\operatorname{tr}(A) \geq \|A\|_F \tag{B.53}$$

Sequences of matrices and their norms

As it can be seen through all the thesis, in order to analyze Markov jump (possibly switched) linear systems, we widely use the indicator function on the jump parameter to "Markovianize" the state. This, in turn, decomposes the matrices associated to the second moment and control problems into N matrices, with N being the number of the operational modes of the system. Therefore it comes up naturally that a convenient space to be used is the one we define as ${}_{\mathrm{N}}\mathbb{F}^{m,n}$ which is the linear space made up of all N-sequences of either real or complex $m{\times}n$ matrices. Formally,

$${}_{\mathrm{N}}\mathbb{F}^{m,n} \triangleq \left\{\boldsymbol{A}\triangleq(A_i)_{i=1}^N : A_i\in\mathbb{F}^{m,n}\right\}$$

Henceforth, we indicate by $N\in\mathbb{Z}_+$ the number of operational modes (also known as discrete states) of the system, and by $\mathbb{M}$ the set of operational modes, i.e.,

$$\mathbb{M}\triangleq\{i\in\mathbb{Z}_+ : i\leq N\}$$

For every sequence $\boldsymbol{A}=(A_i)_{i=1}^N\in{}_{\mathrm{N}}\mathbb{F}^{m,n}$ and any matrix norm $\|\cdot\|$, we define the following *equivalent norms* in the finite dimensional linear space ${}_{\mathrm{N}}\mathbb{F}^{m,n}$:

$$\|\boldsymbol{A}\|_1\triangleq\sum_{i=1}^N\|A_i\| \tag{B.54a}$$

$$\|\boldsymbol{A}\|_2\triangleq\sqrt{\sum_{i=1}^N\operatorname{tr}(A_i^*A_i)} \tag{B.54b}$$

$$\|\boldsymbol{A}\|_{\max}\triangleq\max_{i\in\mathbb{M}}\{\|A_i\|\} \tag{B.54c}$$

We shall omit the subscripts 1, 2, max whenever the definition of a specific norm does not affect the result being considered.

It is easy to verify that ${}_{\mathrm{N}}\mathbb{F}^{m,n}$ equipped with any of the above norms is a Banach space. To show this fact explicitly, we need to recall some additional notions related to normed linear spaces.

Let us denote by $(\mathbb{X},d_{\mathbb{X}})$ and $(\mathbb{Y},d_{\mathbb{Y}})$ two normed linear spaces, where $d_{\mathbb{X}}$ and $d_{\mathbb{Y}}$ indicate the distances induced by the respective norms.

A mapping $f:(\mathbb{X},d_{\mathbb{X}})\to(\mathbb{Y},d_{\mathbb{Y}})$ is said to be *homeomorphism* if f is invertible and both f and f^{-1} are continuous [124, p. 93]. We stress that f is homeomorphism if and only if f^{-1} is a homeomorphism. A homeomorphism f is *uniform homeomorphism* if f and f^{-1} are uniformly continuous.

Two normed linear spaces $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$ are uniformly homeomorphic (to one another) if there exists a uniform homeomorphism mapping one of them onto the other [124, p. 117].

Uniform homeomorphisms are very important in the theory of Banach spaces, since this family of mappings preserve completeness. In particular, let the normed linear spaces $(\mathbb{X}, d_{\mathbb{X}})$, $(\mathbb{Y}, d_{\mathbb{Y}})$ be uniformly homeomorphic. Then $(\mathbb{X}, d_{\mathbb{X}})$ is complete if and only if $(\mathbb{Y}, d_{\mathbb{Y}})$ is complete [124, Theorem 3.13.9].

The finite-dimensional linear spaces $\mathbb{F}^{m,n}$ and $\mathbb{F}^{mn}$ (equipped respectively with any of aforementioned equivalent matrix and vector norms) are uniformly homeomorphic through the vectorization (B.29).

Since every finite-dimensional normed space over $\mathbb{F}^n$ is a Banach space [117, Corollary 1.4.19], then also any finite-dimensional linear space $\mathbb{F}^{m,n}$ is a Banach space. As alternative argument we can use the fact mentioned in section B.4, stating that if $\mathbb{Y}$ is a Banach space, then so is $\mathcal{B}(\mathbb{X}, \mathbb{Y})$ [117, Theorem 1.4.8].

We can show that ${}_{\mathrm{N}}\mathbb{F}^{m,n}$ equipped with any of the norms (B.54) is a Banach space as follows. We denote by $\mathrm{vec}^2 : {}_{\mathrm{N}}\mathbb{F}^{m,n} \to \mathbb{F}^{Nmn}$, a linear operator defined $\forall \boldsymbol{A} = (A_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{m,n}$ as

$$\mathrm{vec}^2(\boldsymbol{A}) \triangleq \begin{bmatrix} \mathrm{vec}(A_1) \\ \mathrm{vec}(A_2) \\ \vdots \\ \mathrm{vec}(A_N) \end{bmatrix} \tag{B.55}$$

The linear spaces ${}_{\mathrm{N}}\mathbb{F}^{m,n}$ equipped with any of the norms (B.54) are uniformly homeomorphic to a finite-dimensional Banach space $\mathbb{F}^{Nmn}$ through the mapping $\mathrm{vec}^2(\cdot)$ [12, p. 17]. Thus, all these normed linear spaces are complete, that is they are Banach spaces.

Furthermore, $({}_{\mathrm{N}}\mathbb{F}^{m,n}, \|\cdot\|_2)$ is a Hilbert space [12, p. 16], with the *inner product* given, for $\boldsymbol{A} = (A_i)_{i=1}^N$, $\boldsymbol{B} = (B_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{m,n}$ by

$$\langle\, \boldsymbol{A}\,,\, \boldsymbol{B}\,\rangle \triangleq \sum\nolimits_{i=1}^{N} \mathrm{tr}(A_i^* B_i) \tag{B.56}$$

We extend the notion of being symmetric, Hermitian and positive (semi-)definite for the sequences of square matrices as follows.

For $\boldsymbol{A} = (A_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{m,n}$ we write $\boldsymbol{A}^* = (A_i^*)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n,m}$, and say that the sequence $\boldsymbol{A} = (A_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n,n}$ is either real symmetric or (complex) Hermitian if $\boldsymbol{A} = \boldsymbol{A}^*$. The related set is defined as

$${}_{\mathrm{N}}\mathbb{F}^{n,n}_{*} \triangleq \left\{ \boldsymbol{A} = (A_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n,n} : A_i \in \mathbb{F}^{n,n}_{*} \right\} \tag{B.57}$$

The sets of all N-sequences of positive (semi-)definite matrices in $\mathbb{F}^{n,n}$ are defined in similar fashion, i.e.,

$${}_{\mathrm{N}}\mathbb{F}^{n,n}_{+} \triangleq \left\{ \boldsymbol{A} = (A_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n,n} : A_i \in \mathbb{F}^{n,n}_{+} \right\} \tag{B.58}$$

$$ {}_{\mathrm{N}}\mathbb{F}_0^{n,n} \triangleq \left\{ \boldsymbol{A} = (A_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n,n} : A_i \in \mathbb{F}_0^{n,n} \right\} \tag{B.59} $$

For $\boldsymbol{A} = (A_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n,n}$ and $\boldsymbol{B} = (B_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n,n}$ we write that $\boldsymbol{A} \succ \boldsymbol{B}$ if $\boldsymbol{A}-\boldsymbol{B} = (A_i - B_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}_+^{n,n}$, and that $\boldsymbol{A} \succeq \boldsymbol{B}$ if $\boldsymbol{A}-\boldsymbol{B} = (A_i - B_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}_0^{n,n}$.

Sets of matrices and convex polytopes

The last two subsections focused on the topics of linear algebra are dealing with convex sets of matrices in $\mathbb{F}^{N,N}$. We will see in the next chapter that these sets of matrices play an important role in the treatment of (time-varying) uncertainties on the system parameters.

Let us denote by $\mathbb{P}$ any finite set (of cardinality V) of square matrices acting on the Banach space $\mathbb{F}^{N,N}$. We indicate by $\mathbb{V}$ the correspondent *index set*, that is a set whose members label (or index) elements of another set, in this instance $\mathbb{P}$:

$$ \mathbb{V} \triangleq \{ l \in \mathbb{Z}_+ : l \leq V \} \tag{B.60} $$

We define a set of all either real or complex square $N \times N$ matrices of cardinality V as follows:

$$ {}_{\mathbb{V}}\mathbb{F}^{N,N} \triangleq \left\{ P_l \in \mathbb{F}^{N,N} : l \in \mathbb{V} \right\} \tag{B.61} $$

Any set of $V \in \mathbb{Z}_+$ either real or complex square $N \times N$ matrices is then represented as $\mathbb{P} \subseteq {}_{\mathbb{V}}\mathbb{F}^{N,N}$.

We remind that any complex matrix acting on $\mathbb{C}^{m,n}$ can be seen as a real matrix acting on $\mathbb{R}^{2m,2n}$ (See Remark B.1), which in turn can be transformed in a vector acting on $\mathbb{R}^{4mn}$ through uniform homeomorphism $\mathrm{vec}(\cdot)$. Thus, without loss of generality, we recall the notions of convexity for finite linear spaces over the real numbers. Clearly, all the definitions and properties will still apply to the corresponding sets of either real or complex matrices.

A set $\mathbb{S} \subset \mathbb{R}^n$ is *convex* if (and only if) for each pair of distinct points $\mathrm{x}, \mathrm{y} \in \mathbb{S}$ the closed segment with endpoints x and y is contained in $\mathbb{S}$ [125, p. 8]. In other words, a set $\mathbb{S} \subset \mathbb{R}^n$ is said to be convex if (and only if) $\forall \mathrm{x}, \mathrm{y} \in \mathbb{S}$ and all $\{\lambda \in \mathbb{R}_0 : \lambda \leq 1\}$, the element $(1-\lambda)\,\mathrm{x} + \lambda \mathrm{y} \in \mathbb{S}$.

Let $\mathbb{S} \subset \mathbb{R}^n$ be a convex set. An element (also known as a *point*) $\mathrm{x} \in \mathbb{S}$ is an *extreme point* of $\mathbb{S}$ provided $\mathrm{y}, \mathrm{z} \in \mathbb{S}$, $\lambda \in \mathbb{R}_+$, $\lambda < 1$, and $\mathrm{x} = \lambda \mathrm{y} + (1-\lambda)\,\mathrm{z}$ imply $\mathrm{x} = \mathrm{y} = \mathrm{z}$. In other words, x is an extreme point of $\mathbb{S}$ if it does not belong to the relative interior of any segment contained in $\mathbb{S}$ [125, p. 17].

The *convex hull* of a set $\mathbb{S} \subset \mathbb{R}^n$ is the intersection of all the convex sets in $\mathbb{R}^n$ which contain $\mathbb{S}$ [125, p. 14]. It is denoted by $\mathrm{conv}\,\mathbb{S}$.

The convex hull of a nonempty set $\mathbb{S} \subset \mathbb{R}^n$ is the set of all points which may be represented as *convex combinations* of points of $\mathbb{S}$ [125, p. 14]; that is, it is a set of all points which can be written as

$$ \mathrm{conv}\,\mathbb{S} = \left\{ \sum\nolimits_{l=1}^{V} \lambda_l \mathrm{x}_l \in \mathbb{S} : \quad \mathrm{x}_l \in \mathbb{S}, \quad \lambda_l \in \mathbb{R}_0, \quad \sum\nolimits_{l=1}^{V} \lambda_l = 1, \quad V \in \mathbb{Z}_+ \right\} \tag{B.62} $$

A convex hull of a finite set of points defines a *convex polytope* (in its $\mathcal{V}$-description), i.e., a compact convex set $\mathbb{S} \subset \mathbb{R}^n$ that has a finite number of extreme points [125, p. 31, p.52a]. In this context, *compactness* is a property of a set being *closed* (i.e., containing all its limit points) and *bounded* (that is, having all its points lie within some fixed distance of each other).

Formally, a set $\mathbb{S} \subset \mathbb{R}^n$ is *open* if every $\mathrm{x} \in \mathbb{S}$ is the center of some open ball which is contained in $\mathbb{S}$. We remind that $\forall \mathrm{x} \in \mathbb{S}, \delta \in \mathbb{R}_+$, an *open ball* with center x and radius δ is a set $\{\mathrm{y} \in \mathbb{S} : d_{\mathbb{S}}(\mathrm{y}, \mathrm{x}) < \delta\}$, where $d_{\mathbb{S}}(\mathrm{y}, \mathrm{x})$ is a distance between two vectors x and y in $\mathbb{S}$.

A set $\mathbb{S} \subset \mathbb{R}^n$ is *closed* if its *complement* $\{\mathrm{x} \in \mathbb{R}^n : \mathrm{x} \notin \mathbb{S}\}$ is open. For instance, all finite sets of points, $\{\}$ and $\mathbb{R}^n$ are closed. Clearly, for any $\mathrm{x} \in \mathbb{S} \subset \mathbb{R}^n$ and $\delta \in \mathbb{R}_+$, a set $\{\mathrm{y} \in \mathbb{S} : d_{\mathbb{S}}(\mathrm{y}, \mathrm{x}) \leq \delta\}$ is closed. In fact, such a set is called a *closed ball* (with center x and radius δ) [125, p. 5].

Finally, a set $\mathbb{S} \subset \mathbb{R}^n$ is *bounded* if there exists $\delta \in \mathbb{R}_+$ and $\mathrm{x} \in \mathbb{S}$ such that $d_{\mathbb{S}}(\mathrm{y}, \mathrm{x}) < \delta$ for all $\mathrm{y} \in \mathbb{S}$.

For the sake of clarity, we provide alternative definitions of closed and compact sets as follows. A set $\mathbb{S} \subset \mathbb{R}^n$ is *closed* if the limit of every convergent sequence of points of $\mathbb{S}$ belongs to $\mathbb{S}$. From this point of view, a set $\mathbb{S} \subset \mathbb{R}^n$ is *compact* if every infinite sequence of points of $\mathbb{S}$ contains a subsequence which has a point of $\mathbb{S}$ as limit [125, p. 6].

We recall that for a polytope, it is customary to call its extreme points *vertices* [125, p. 31]. A polytope is called *full-dimensional* if it is an n-dimensional object in $\mathbb{R}^n$. This is a type of polytopes we are dealing with in this thesis. In the case of a full-dimensional polytope, the minimal $\mathcal{V}$-description is unique; it is given by the convex hull of all the vertices.

On a side note, we remind that any polytope can be equivalently represented by a bounded intersection of finitely many closed half-spaces (i.e., by an $\mathcal{H}$-description), which is also unique for full-dimensional polytopes. In fact, there are several algorithms for the conversion between the two representations of (full-dimensional) polytopes [125, p. 52a].

Joint spectral radius

The *joint spectral radius* (from now on, JSR, [126]) is a generalization of the classical notion of spectral radius of a matrix, to sets of matrices. In the last decades JSR has been subject of intense research due to its role in the study of wavelets, switching systems, approximation algorithms, and many other topics [88].

In order to define JSR formally, for each $k \in \mathbb{Z}_+$, $\mathbb{P} \subseteq {}_{\mathbb{V}}\mathbb{F}^{N,N}$ let us consider the set $\mathcal{P}_k(\mathbb{P})$ of all possible products of length k whose factors are elements of $\mathbb{P}$, i.e.,

$$\mathcal{P}_k(\mathbb{P}) \triangleq \left\{ \left(\prod_{i=1}^{k} P_{l_i}^* \right)^* \in \mathbb{F}^{N,N} : P_{l_i} \in \mathbb{P} \subseteq {}_{\mathbb{V}}\mathbb{F}^{N,N} \right\} \tag{B.63}$$

For any matrix norm $\|\cdot\|$ on $\mathbb{F}^{N,N}$, consider the supremum among the normalized norms of all products in $\mathcal{P}_k(\mathbb{P})$, $k \in \mathbb{Z}_+$ i.e.,

$$\hat{\rho}_k(\mathbb{P}) \triangleq \sup_{P \in \mathcal{P}_k(\mathbb{P})} \|P\|^{\frac{1}{k}} \tag{B.64}$$

The *joint spectral radius* of $\mathbb{P} \subseteq {}_{\mathbb{V}}\mathbb{F}^{N,N}$ is defined as

$$\hat{\rho}(\mathbb{P}) = \lim_{k \to \infty} \hat{\rho}_k(\mathbb{P}) \tag{B.65}$$

The joint spectral radius of a bounded set of matrices has some interesting properties.

Proposition B.1 (Convex hull)**.** *The convex hull of a set has the same joint spectral radius as the original set, i.e.,*

$$\hat{\rho}(\operatorname{conv} \mathbb{P}) = \hat{\rho}(\mathbb{P}) \tag{B.66}$$

Proof. This result was first obtained by Barabanov [127]. See [128], [129] for further details. □

Proposition B.2 (Convergence of matrix products)**.** *For any bounded set of matrices $\mathbb{P} \subseteq {}_{\mathbb{V}}\mathbb{F}^{N,N}$ and for any $k \in \mathbb{Z}_+$, all matrix products $P \in \mathcal{P}_k(\mathbb{P})$ converge to zero matrix as $k \to \infty$, if and only if $\hat{\rho}(\mathbb{P}) < 1$.*

Proof. See the seminal work of Berger and Whang [130, Theorem I (b)]. □

These concepts are at the basis of our main result on (robust) stability of polytopic time-inhomogeneous MJLSs, presented in Chapter 3.

B.5 Fields of sets

In order to define the concept of algebra over a set, which is necessary for a definition of a probability space (it will be presented in the next section), we need to introduce first the notions of sequences of sets, their limits, and collections of sets, together with the related set relations. The material presented here comes mainly from Allan Gut's book on probability [113, pp. 4–6].

Sequences of sets and their limits

As any sequence presented in Subsection B.3, a *sequence of sets* can be denoted by the indexed element of the sequence in round brackets, e.g., $(\mathbb{S}_k)$, with $k \in \mathbb{T}$.

A sequence of sets $(\mathbb{S}_k)$ is said to be *monotone non-decreasing*, and denoted by $(\mathbb{S}_k)\nearrow$, if $\mathbb{S}_0 \subseteq \mathbb{S}_1 \subseteq \mathbb{S}_2 \subseteq \cdots$.

Similarly, a sequence of sets $(\mathbb{S}_k)$ is said to be *monotone non-increasing*, and denoted by $(\mathbb{S}_k)\searrow$, if $\mathbb{S}_0 \supseteq \mathbb{S}_1 \supseteq \mathbb{S}_2 \supseteq \cdots$.

While working on sequences of sets, the following (de Morgan's) formulas can be useful

$$\left(\bigcup_{k=0}^{t} \mathbb{S}_k\right)^{\mathrm{C}} = \bigcap_{k=0}^{t} \mathbb{S}_k^{\mathrm{C}} \quad \text{and} \quad \left(\bigcap_{k=0}^{t} \mathbb{S}_k\right)^{\mathrm{C}} = \bigcup_{k=0}^{t} \mathbb{S}_k^{\mathrm{C}} \tag{B.67}$$

It is also possible to define limits of sets. However, not every sequence of sets has a limit. We define a limit of a sequence $(\mathbb{S}_k)$ of subsets of a set Ω in the following manner.

$$\liminf_{k\to\infty} (\mathbb{S}_k) \triangleq \bigcup_{k=0}^{\infty} \bigcap_{t=k}^{\infty} \mathbb{S}_t \quad \text{and} \quad \limsup_{k\to\infty} (\mathbb{S}_k) \triangleq \bigcap_{k=0}^{\infty} \bigcup_{t=k}^{\infty} \mathbb{S}_t$$

If these sets agree, then

$$\mathbb{S} = \liminf_{k\to\infty} (\mathbb{S}_k) = \limsup_{k\to\infty} (\mathbb{S}_k) = \lim_{k\to\infty} (\mathbb{S}_k)$$

One instance when a limit exists is when the sequence of sets is monotone.

Specifically, let $(\mathbb{S}_k)$ be a sequence of subsets of a set Ω. We have that

$$(\mathbb{S}_k)\nearrow \;\Rightarrow\; \lim_{k\to\infty} (\mathbb{S}_k) = \bigcup_{k=0}^{\infty} \mathbb{S}_k \tag{B.68}$$

Similarly,

$$(\mathbb{S}_k)\searrow \;\Rightarrow\; \lim_{k\to\infty} (\mathbb{S}_k) = \bigcap_{k=0}^{\infty} \mathbb{S}_k \tag{B.69}$$

Collections of sets

Collections of sets are defined according to a setup of rules. Different rules yield different collections. In what follows we provide definitions of some important collections of sets, together with the connections between them.

Let $\mathcal{F}$ be a non-empty collection of subsets of Ω, and consider the following set relations:

$$\mathbb{S} \in \mathcal{F} \;\Rightarrow\; \mathbb{S}^{\mathrm{C}} \in \mathcal{F} \tag{B.70a}$$

$$\mathbb{S}_1, \mathbb{S}_2 \in \mathcal{F} \;\Rightarrow\; \mathbb{S}_1 \cup \mathbb{S}_2 \in \mathcal{F} \tag{B.70b}$$

$$\mathbb{S}_1, \mathbb{S}_2 \in \mathcal{F} \;\Rightarrow\; \mathbb{S}_1 \cap \mathbb{S}_2 \in \mathcal{F} \tag{B.70c}$$

$$\mathbb{S}_1, \mathbb{S}_2 \in \mathcal{F},\ \mathbb{S}_2 \subseteq \mathbb{S}_1 \;\Rightarrow\; \mathbb{S}_1 \setminus \mathbb{S}_2 \in \mathcal{F} \tag{B.70d}$$

$$\mathbb{S}_k \in \mathcal{F}\ \ \forall k \in \mathbb{T} \;\Rightarrow\; \bigcup_{k=0}^{\infty} \mathbb{S}_k \in \mathcal{F} \tag{B.70e}$$

$$\mathbb{S}_k \in \mathcal{F}\ \ \forall k \in \mathbb{T} \ \wedge\ \mathbb{S}_i \cap \mathbb{S}_j = \{\}\ \ \forall i, j \in \mathbb{T} :\ i \neq j \;\Rightarrow\; \bigcup_{k=0}^{\infty} \mathbb{S}_k \in \mathcal{F} \tag{B.70f}$$

$$\mathbb{S}_k \in \mathcal{F}\ \ \forall k \in \mathbb{T} \;\Rightarrow\; \bigcap_{k=0}^{\infty} \mathbb{S}_k \in \mathcal{F} \tag{B.70g}$$

$$\mathbb{S}_k \in \mathcal{F}\ \ \forall k \in \mathbb{T} \ \wedge\ (\mathbb{S}_k)\nearrow \;\Rightarrow\; \bigcup_{k=0}^{\infty} \mathbb{S}_k \in \mathcal{F} \tag{B.70h}$$

$$\mathbb{S}_k \in \mathcal{F} \;\; \forall k \in \mathbb{T} \;\wedge\; (\mathbb{S}_k)\searrow \Rightarrow \bigcap_{k=0}^{\infty} \mathbb{S}_k \in \mathcal{F} \tag{B.70i}$$

Clearly, a number of relations among these rules and extensions of them can be established. For instance (B.70a) and one of (B.70b) and (B.70c), together with the de Morgan formulas (B.67), yield the other; (B.70a) and one of (B.70e) and (B.70g), together with the de Morgan formulas (B.67), yield the other; (B.70b) and induction shows that (B.70b) can be extended to any finite union of sets; (B.70c) and induction shows that (B.70c) can be extended to any finite intersection of sets, and so on.

We recall definitions of some notorious collections of sets in what follows.

Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then,

- $\mathcal{F}$ is a *field* or an *algebra* if $\Omega \in \mathcal{F}$ and properties (B.70a) and (B.70b) hold;
- $\mathcal{F}$ is a *σ-field* or a *σ-algebra* if $\Omega \in \mathcal{F}$ and properties (B.70a) and (B.70e) hold;
- $\mathcal{F}$ is a *monotone class* if properties (B.70h) and (B.70i) hold;
- $\mathcal{F}$ is a *π-system* if property (B.70c) holds;
- $\mathcal{F}$ is a *Dynkin system* if $\Omega \in \mathcal{F}$ and properties (B.70d) and (B.70h) hold.

The definition of a Dynkin system (also known as *λ-system*) varies. One alternative, in addition to the assumption that $\Omega \in \mathcal{F}$, is that (B.70a) and (B.70f) hold.

The definitions of the different collections of sets are obviously based on minimal requirements. By manipulating the different properties (B.70), for example together with the de Morgan formulas (B.67), other properties can be derived. The following relations between different collections of sets are obtained by such manipulations.

The following connections hold:

1. Every algebra is a π-system.
2. Every σ-algebra is an algebra.
3. An algebra is a σ-algebra if and only if it is a monotone class.
4. Every σ-algebra is a Dynkin system.
5. A Dynkin system is a σ-algebra if and only if it is π-system.
6. Every Dynkin system is a monotone class.
7. Every σ-algebra is a monotone class.
8. The power set of any subset of Ω is a σ-algebra on that subset.
9. The intersection of any number of σ-algebras, countable or uncountable, is, again, a σ-algebra.

10. The countable union of a non-decreasing sequence of σ-algebras is an algebra, but not necessarily a σ-algebra.

11. If $\mathcal{F}$ is a σ-algebra, and $\mathbb{S}_1 \subseteq \Omega$, then $\mathbb{S}_1 \cap \mathcal{F} = \{\mathbb{S}_1 \cap \mathbb{S}_2 : \mathbb{S}_2 \in \mathcal{F}\}$ is a σ-algebra on $\mathbb{S}_1$.

12. If Ω and Ω' are sets, $\mathcal{F}'$ a σ-algebra on Ω' and $\mathcal{L} : \Omega \to \Omega'$ a mapping, then the inverse $\mathcal{L}^{-1}(\mathcal{F}') = \left\{\mathcal{L}^{-1}(\mathbb{S}') : \mathbb{S}' \in \mathcal{F}'\right\}$ is a σ-algebra on Ω.

Measurable spaces, filtration and Borel sets

A set Ω together with an associated σ-algebra, $\mathcal{F}$, i.e., the pair $(\Omega, \mathcal{F})$, is called a *measurable space*; and the members of $\mathcal{F}$ are called *measurable sets*.

For two given measurable spaces $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$, a *function* $f : \Omega \to \Omega'$ is said to be $(\mathcal{F}, \mathcal{F}')$-*measurable* if for every Ω'-measurable set $\mathbb{S} \in \mathcal{F}'$, the inverse image is Ω-measurable, i.e., $f^{-1}(\mathbb{S}) = \{\omega \in \Omega : f(\omega) \in \mathbb{S}\} \in \mathcal{F}$.

A non-decreasing sequence of σ-algebras on a measurable space is called filtration. Formally, given a measurable space $(\Omega, \mathcal{F})$, a *filtration* is a sequence of σ-algebras $(\mathcal{F}_t)$, where $t \in \mathbb{T}_k$ and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}_k \subseteq \mathcal{F}$, i.e., $t_1 \leq t_2 \Rightarrow \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$.

In the rest of this subsection we characterize $\mathcal{R}^n$, the σ-algebra of Borel sets, which are subsets of the set $\Omega = \mathbb{R}^n$. This topic is closely related to Subsection B.4, where we provided definitions of open sets on $\mathbb{R}^n$.

We remind from Subsection B.3 that a linear space $\mathbb{R}^n$ is a product space of all ordered *n-tuples*, which are finite ordered lists of n elements, each of which is defined on $\mathbb{R}$. The Euclidean space $\mathbb{R}^n$ is obtained via Cartesian product of n real lines. We recall that the *real line* is another name for the set $\mathbb{R}$ of all real numbers. We present first the σ-algebra of Borel sets on $\mathbb{R}$, i.e., a special case of $\mathbb{R}^n$ with $n = 1$, and then extend the results to an arbitrary $n \in \mathbb{Z}_+$.

So, the sets in $\mathbb{R}$ are called Borel sets, and the measurable space $(\mathbb{R}, \mathcal{R})$ is called the Borel space. The σ-algebra of Borel sets, or the Borel-σ-algebra, is defined as the σ-algebra generated by the open subsets of $\mathbb{R}$, i.e.,

$$\mathcal{R} \triangleq \sigma\{\mathbb{S} \subset \mathbb{R} : \mathbb{S} \text{ is open}\}$$

See Subsections B.4 and B.2, B.3, B.3, for additional details on open sets, open balls, and distances for real numbers, normed linear spaces, and specific equivalent vector norms.

An important fact is that the Borel sets can, equivalently, be generated by intervals. We remind that (real) *intervals* are subsets of real line with the property that any real number that lies between two real numbers in an interval is also included in that interval. The interval of real numbers between a and b, including a and b, is denoted $[a, b]$, and called *closed interval*. Formally, $\forall a, b \in \mathbb{R}$, $a \leq b$,

$$[a, b] \triangleq \{c \in \mathbb{R} : a \leq c,\ c \leq b\}$$

These two numbers a and b are called the *endpoints* of the interval.

An *open interval* does not include its endpoints, and is indicated with parentheses. In symbols, $\forall a, b \in \mathbb{R},\ a < b$,

$$(a, b) \triangleq \{c \in \mathbb{R} : a < c,\ c < b\}$$

A *half-open interval* includes only one of its endpoints, and is denoted by mixing the notations for open and closed intervals. Specifically, $\forall a, b \in \mathbb{R},\ a < b$,

$$(a, b] \triangleq \{c \in \mathbb{R} : a < c,\ c \leq b\}\,, \quad \text{and} \quad [a, b) \triangleq \{c \in \mathbb{R} : a \leq c,\ c < b\}$$

Obviously, the usage of parentheses and square brackets in the presented concise notation for intervals causes the slight abuse of notation. However, the domain of mathematical objects contained in parentheses and square brackets makes clear which objects are represented in the specific context.

We have that Borel-σ-algebras on $\mathbb{R}$ can be equivalently defined [113, p. 15] by

$$\mathcal{R} = \sigma\{(a, b]\,,\ -\infty \leq a < b < \infty\} \tag{B.71a}$$
$$= \sigma\{[a, b)\,,\ -\infty < a < b \leq \infty\} \tag{B.71b}$$
$$= \sigma\{(a, b)\,,\ -\infty \leq a < b \leq \infty\} \tag{B.71c}$$
$$= \sigma\{[a, b]\,,\ -\infty < a \leq b < \infty\} \tag{B.71d}$$
$$= \sigma\{(-\infty, b]\,,\ -\infty < b < \infty\} \tag{B.71e}$$

We can interpret intervals as one-dimensional rectangles and extend (B.71) for Borel-σ-algebras on $\mathbb{R}^n$ by considering higher-dimensional rectangles.

In particular, let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. The σ-algebra for the corresponding product space $\Omega_1 \times \Omega_2$ is called the *product σ-algebra* and is defined by

$$\mathcal{F}_1 \times \mathcal{F}_2 \triangleq \sigma\{\mathbb{S}_1 \times \mathbb{S}_2 : \mathbb{S}_1 \in \mathcal{F}_1,\ \mathbb{S}_2 \in \mathcal{F}_2\} \tag{B.72}$$

Clearly,

$$\mathcal{R}^n = \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n \text{ times}} = \prod\nolimits_{i=1}^{n} \mathcal{R} \tag{B.73}$$

where $\prod$ denotes a Cartesian product of a sequence, $\mathcal{R}$ is defined as in (B.71), e.g.,

$$\mathcal{R}^n = \sigma\{(-\infty, b_1] \times (-\infty, b_2] \times \cdots \times (-\infty, b_n] : b_i \in \mathbb{R}\ \forall i \in \mathbb{Z}_+,\ i \leq n\} \tag{B.74}$$

B.6 Probability space and Markov processes

Probability is the extent to which an event is likely to occur, measured by the ratio of the favorable cases to the whole number of cases possible [131, p. 1414]. In probability theory, an *experiment* (or *trial*) is any procedure that can be infinitely repeated and has a well-defined set of possible outcomes, known as the sample space. Probability models aim at describing random experiments, that is, experiments that can be repeated (indefinitely) and where future outcomes (i.e., results of an experiment) cannot be exactly predicted - due to randomness - even if the experimental situation can be fully controlled [113, p. 1].

Probability space

The basis of the probability theory is the *probability space*, i.e., a triple $(\Omega, \mathcal{F}, \Pr)$, where

- Ω is the *sample space*, i.e., some (possibly abstract) set; its elements $\omega \in \Omega$ are all the possible outcomes of an experiment;
- $\mathcal{F} \subseteq 2^{\Omega}$ is the σ-algebra of sets (called *events*), which are the measurable subsets of Ω; their atoms $\omega \in \Omega$ are named *elementary events*;
- $\Pr : \mathcal{F} \to [0, 1]$ is the *probability measure*, which is a function satisfying the following

Kolmogorov's axioms:

1. For any $\mathbb{S} \in \mathcal{F}$, $\exists$ a real number $\Pr(\mathbb{S}) \geq 0$; it is called the *probability* of $\mathbb{S}$.
2. $\Pr(\Omega) = 1$.
3. Let $\{\mathbb{S}_n \subseteq \Omega : n \in \mathbb{Z}_+\}$, be a collection of disjoint subsets of Ω. Then we have a *countable additivity*, i.e.,

$$\Pr\left(\bigcup_{n=1}^{\infty} \mathbb{S}_n\right) = \sum_{n=1}^{\infty} \Pr(\mathbb{S}_n)$$

From the Kolmogorov's axioms only, we can derive various relations between probabilities of unions, subsets, complements and so on [113, p. 11].

Let $\mathbb{S}$, $\mathbb{S}_i$, with $i, n \in \mathbb{Z}_+$, be measurable sets. Then,

$$\Pr\left(\mathbb{S}^{\mathrm{C}}\right) = 1 - \Pr(\mathbb{S}) \tag{B.75a}$$

$$\Pr(\{\}) = 0 \tag{B.75b}$$

$$\Pr(\mathbb{S}_1 \cup \mathbb{S}_2) \leq \Pr(\mathbb{S}_1) + \Pr(\mathbb{S}_2) \tag{B.75c}$$

$$\mathbb{S}_1 \subseteq \mathbb{S}_2 \Rightarrow \Pr(\mathbb{S}_1) \leq \Pr(\mathbb{S}_2) \tag{B.75d}$$

$$\Pr\left(\bigcup_{i=1}^{n} \mathbb{S}_i\right) + \Pr\left(\bigcap_{i=1}^{n} \mathbb{S}_i^{\mathrm{C}}\right) = 1 \tag{B.75e}$$

Events' independence and conditional probabilities

One of the most central concepts of probability theory is *independence*, which means that successive experiments do not influence each other, that the future does not depend on the past, that knowledge of the outcomes so far does not provide any information about future experiments [113, p. 17].

We say that the events $\mathbb{S}_1, \mathbb{S}_2 \in \mathcal{F}$ are *independent* if and only if

$$\Pr(\mathbb{S}_1 \cap \mathbb{S}_2) = \Pr(\mathbb{S}_1)\Pr(\mathbb{S}_2)$$

Clearly, the product between two probability measures is indicated by *juxtaposition*.

A suggestive way to illustrate independence is to introduce the concept of conditional probability, i.e., a measure of the probability of an event given (by assumption or evidence) that another event has occurred. Formally, let $\mathbb{S}_1, \mathbb{S}_2 \in \mathcal{F}$ be two events, and suppose that $\Pr(\mathbb{S}_1) > 0$. The *conditional probability* of $\mathbb{S}_2$ given $\mathbb{S}_1$ is defined as

$$\Pr(\mathbb{S}_2 \mid \mathbb{S}_1) = \frac{\Pr(\mathbb{S}_1 \cap \mathbb{S}_2)}{\Pr(\mathbb{S}_1)} \tag{B.76}$$

Obviously, if events $\mathbb{S}_1, \mathbb{S}_2 \in \mathcal{F}$ are independent,

$$\Pr(\mathbb{S}_2 \mid \mathbb{S}_1) = \frac{\Pr(\mathbb{S}_1)\Pr(\mathbb{S}_2)}{\Pr(\mathbb{S}_1)} = \Pr(\mathbb{S}_2)$$

Lastly, the famous *Bayes' rule* (also known as Bayes-Price rule, or Bayes' theorem, Bayes' law) shows the relationship between $\Pr(\mathbb{S}_2 \mid \mathbb{S}_1)$ and $\Pr(\mathbb{S}_1 \mid \mathbb{S}_2)$. It states that

$$\Pr(\mathbb{S}_2 \mid \mathbb{S}_1) = \frac{\Pr(\mathbb{S}_1 \mid \mathbb{S}_2)\Pr(\mathbb{S}_2)}{\Pr(\mathbb{S}_1)} \tag{B.77}$$

Stochastic processes

To define a concept of the random variable, essential in the presentation of stochastic processes in general, and Markov chains in particular, let us consider a probability space $(\Omega, \mathcal{F}, \Pr)$ and a measurable space $(\mathbb{M}, \mathcal{M})$.

Here, $\mathbb{M}$ indicates any set, and $\mathcal{M} \subseteq 2^{\mathbb{M}}$ the associated σ-algebra.

A *random variable* (also known as a random quantity, aleatory variable, or stochastic variable) is a function $f : \Omega \to \mathbb{M}$ which is $(\mathcal{F}, \mathcal{M})$-measurable. It maps outcomes $\omega \in \Omega$ of an experiment to numerical quantities (labels) $f_i \in \mathbb{M}$. The set $\mathbb{M}$ is called the *state space* of the random variable. For any $\mathbb{S} \subseteq \mathbb{M}$, the associated probability measure $\Pr(\{\omega \in \Omega : f(\omega) \in \mathbb{S}\})$ is usually shortened to $\Pr(f \in \mathbb{S})$ and is called *probability distribution*.

A random variable is said to be *discrete*, if its state space $\mathbb{M}$ is discrete. One typical example of discrete state space is given by values associated to outcomes of the roll of a dice.

A discrete random variable has a discrete probability distribution, which can be encoded by a discrete list of the probabilities of the outcomes, known as a *probability mass function*, i.e., a function that gives the probability that a discrete random variable is exactly equal to some value. We write probability mass function as $\Pr(\{\omega \in \Omega : f(\omega) = f_i \in \mathbb{M}\})$, or simply $\Pr(f = f_i)$.

An important parameter of a (discrete) random variable is its *expected value* (also known as (mathematical) expectation, mean value, or first moment), that is, probability-weighted average of all possible values of random variable. In other words, each possible value the random variable can assume is multiplied by its probability of occurring, and the resulting products are summed to produce the expected value. We denote the expected value by $\mathbb{E}(\cdot)$. For $\mathbb{M} \subseteq \mathbb{Z}_+$, we have that

$$\mathbb{E}(f) \triangleq \sum\nolimits_{i=1}^{|\mathbb{M}|} f_i \Pr(f = f_i) \tag{B.78}$$

A *stochastic process* is a collection of random variables, indexed by (some) set $\mathbb{T}$, i.e., $t \in \mathbb{T}$, all defined on a common probability space $(\Omega, \mathcal{F}, \Pr)$ and taking values in the same measurable space $(\mathbb{M}, \mathcal{M})$. Thus, it is common to denote a stochastic process by using a set-builder notation, that is, as $\{\theta_t : t \in \mathbb{T}\}$, with $\theta_t : \Omega \to \mathbb{M}$.

Since any random variable within a stochastic process is actually a function of two variables, $t \in \mathbb{T}$ and $\omega \in \Omega$, with a state space $\mathbb{M}$, sometimes it is useful to denote this fact explicitly by writing that $\theta_t : \mathbb{T} \times \Omega \to \mathbb{M}$.

Since all random variables in a stochastic process are indexed by a set $\mathbb{T}$, which in this text is exactly the discrete-time set (B.5), to study the properties of a stochastic process it is useful to consider a stochastic basis, i.e., a probability space equipped with the filtration $(\mathcal{F}_t)$, $t \in \mathbb{T}_k$, of its σ-algebra. Thus, a *stochastic basis* (also known as *filtered probability space*) is simply a quadruple $(\Omega, \mathcal{F}, (\mathcal{F}_t), \Pr)$.

Recalling that a filtration on $\mathbb{T}_k$ is defined as a non-decreasing sequence of σ-algebras $\mathcal{F}_t \subseteq \mathcal{F}$, with $t \in \mathbb{T}_k$, we say that a stochastic process $\theta \triangleq \{\theta_t : t \in \mathbb{T}\}$ is *adapted* to $(\mathcal{F}_t)$, if the random variable $\theta_t : \Omega \to \mathbb{M}$ is $(\mathcal{F}_t, \mathcal{M})$-measurable for every $t \in \mathbb{T}_k$ [132, p. 97].

The filtration can be interpreted as representing all historical but not future information available about the stochastic process, with the algebraic object $\mathcal{F}_t$ gaining in complexity with time. Hence, a stochastic process that is adapted to a filtration $(\mathcal{F}_t)$, is also called non-anticipating, i.e., one that cannot see into the future [133, p. 491].

Markov processes

Stochastic processes describe random phenomena evolving in time.

If a process retains no memory of where it has been in the past, i.e., if only the current state of a stochastic process can influence where it goes next, then such a process is called a *Markov process*.

In other words, a Markov process is a stochastic process that satisfies the *Markov property* (also known as *memorylessness*), i.e., the conditional probability distribution of future states of the process (conditional on both past and present states) depends only on the present state, not on the sequence of events that preceded it.

More formally, consider a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathrm{Pr})$ and a measurable space $(\mathbb{M}, \mathcal{M})$. A Markov process θ defined as $\{\theta_t : t \in \mathbb{T}\}$, where $\theta_t : \Omega \to \mathbb{M}$, is referred to as a *Markov chain*, if it can assume only a finite or countable set of states [134, p. xiii], i.e., if $\mathbb{M}$ is finite or countable, e.g., $\mathbb{M} \subseteq \mathbb{Z}_+$.

When $\mathbb{T}$ is discrete index set, such as discrete-time set (B.5), a Markov chain is obviously called *discrete-time* Markov chain. In this thesis, discrete-time Markov chains are fundamental mathematical model used to define Markov jump (switched) linear systems, as explained in Chapter 2.

For any $k \in \mathbb{T}$ and $i \in \mathbb{M}$, the probability distribution of a random variable $\theta_k : \Omega \to \mathbb{M}$ is its probability mass function p_i, i.e.,

$$p_i \triangleq \mathrm{Pr}(\theta_k = i) = \mathrm{Pr}(\{\omega : \theta_k(\omega) = i\}) \tag{B.79}$$

Evidently, from the second Kolmogorov's axiom presented in Subsection B.6, the total mass of the distribution equals to 1, that is,

$$\sum\nolimits_{i \in \mathbb{M}} p_i = 1$$

where the summation symbol indicates the sum of the probabilities of all the states $i \in \mathbb{M}$ of the random variable θ_k.

It is a common practice to give a progressive index as a value associated to each possible outcome of an experiment in a sample space Ω. For instance, to each face of a dice is usually associated a positive integer from 1 to 6, which may be seen as a reward obtained when the rolling of a dice produces the correspondent face. In such a case, the state space of the discrete random variable is written as

$$\mathbb{M} = \{i \in \mathbb{Z}_+ : i \leq N\} \tag{B.80}$$

with $N \triangleq |\mathbb{M}|$ the cardinality of the state space.

This view of a state space $\mathbb{M}$ is useful to represent the (index) set of *operational modes* of Markov jump linear systems.

The total mass of the probability distribution of the discrete random variable θ_k with the state space $\mathbb{M}$ defined by (B.80) is written as

$$\sum\nolimits_{i=1}^{N} p_i = 1 \tag{B.81}$$

For a discrete-time Markov chain $\theta : \mathbb{T} \times \Omega \to \mathbb{M}$ with such a state space $\mathbb{M}$, the *Markov property* can be formally stated for any sequence (i_k), $i_k \in \mathbb{M}$, $k \in \mathbb{T}$, as

$$\mathrm{Pr}(\theta_k = i_k \mid \theta_{k-1} = i_{k-1}, \theta_{k-2} = i_{k-2}, \ldots, \theta_0 = i_0) = \mathrm{Pr}(\theta_k = i_k \mid \theta_{k-1} = i_{k-1}) \tag{B.82}$$

The right-hand side of (B.82) is the probability of a Markov chain being in a state i_k, conditioned to the fact that at the previous time-step it was in a state i_{k-1}. This probability is typically denoted $\forall i, j \in \mathbb{M}$ as

$$p_{ij} = \mathrm{Pr}(\theta_k = j \mid \theta_{k-1} = i) \tag{B.83}$$

All possible combinations of p_{ij} can be written in a stochastic matrix $[p_{ij}]\in\mathbb{R}^{N,N}$, i.e., a matrix P, where any row $P_{i\bullet}$ is a distribution. In other words, $\forall i,j\in\mathbb{M}$,

$$\sum\nolimits_{j=1}^{N} p_{ij} = 1 \tag{B.84}$$

The stochastic matrix $P \triangleq [p_{ij}]$ is usually called the *transition probability matrix*.

The probabilities of transition between the states of a Markov chain in (B.83) and (B.84) are implicitly independent from $k\in\mathbb{T}$. When that is the case, a Markov chain is said to be *time-homogeneous* (or *stationary*). Formally, for any $k\in\mathbb{T}$, a time-homogeneous Markov chain satisfies the following equivalence

$$\Pr(\theta_{k+1}=i \mid \theta_k=j) = \Pr(\theta_k=i \mid \theta_{k-1}=j) \tag{B.85}$$

Conversely, when (B.85) does not hold for every $k\in\mathbb{T}$, the Markov chain is called *time-inhomogeneous*, or non homogeneous in time.

Both stationary and time-inhomogeneous Markov chains are characterized by two parameters.

The first one is an *initial probability distribution*, which $\forall i\in\mathbb{M}$ is denoted by

$$p_i(0)\triangleq\Pr(\theta_0=i) \tag{B.86}$$

For convenience we list initial probability distributions of all states of a Markov chain in a column vector p_0, i.e.,

$$\mathrm{p}_0 \triangleq \begin{bmatrix} p_1(0) \\ p_2(0) \\ \vdots \\ p_N(0) \end{bmatrix} \in\mathbb{R}^{N,1}$$

The second parameter is a *transition probability matrix* $P(k)\triangleq[p_{ij}(k)]$, with

$$p_{ij}\,(k) = \Pr(\theta_{k+1}=j \mid \theta_k=i) \tag{B.87}$$

which for stationary Markov chains is a single, time-independent stochastic matrix.

Markov decision processes

One useful extension of (discrete-time) Markov chains are *Markov decision processes* (hereafter, MDPs). The difference is the addition of actions (allowing choice) and costs (giving motivation). Conversely, if only one choice exists for each state and all costs are the same (e.g. "zero"), an MDP reduces to a Markov chain.

Specifically, a Markov decision process is a quintuple $(\mathbb{M},\mathbb{A},\Pr,g,\gamma)$, where

- $\mathbb{M}$ is a finite, or countable, (index) set of states – see (B.80);
- $\mathbb{A}$ is a finite, or countable, (index) set of actions, i.e., $\mathbb{A}\triangleq\{i\in\mathbb{Z}_+ : i\leq M\}$;

- Pr is state and action dependent transition probability distribution;
- g is a state and action dependent immediate cost;
- γ is a discounting factor, which represents the difference in importance between future costs and present costs; $\gamma \in \mathbb{R}_0, \gamma \leq 1$.

Since taking the discount factor into account does not effect any theoretical results or algorithms in the finite-horizon case (but might effect the decision maker's preference for policies) [135, p. 79], we do not consider a discounting factor here.

Typically, there is a finite number $M \in \mathbb{Z}_+$ of actions that a controller is able to perform. Generally in any given state $i \in \mathbb{M}$ of a Markov decision process, not all the actions are available. For instance, in a decision problem of the optimal transmission power management in a wireless communication, the possible actions available to a controller may be those of increasing or decreasing of a transmission power; in a finite set of transmission power levels, it is impossible to increase a power from a maximum level or decrease it from a minimum level. Thus, we represent this fact by considering for each state $i \in \mathbb{M}$ the related set $\mathbb{A}_i$ of actions α available in that state. Obviously, $\mathbb{A}_i \subseteq \mathbb{A}$, and $\alpha \in \mathbb{A}_i$.

The transition probabilities between states of an MDP depend not only upon a current state of a process, but also on the action taken by a controller. To define them formally, we consider again a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \Pr)$ and a measurable space $(\mathbb{M}, \mathcal{M})$. A Markov decision process is then defined by $\{s_t : t \in \mathbb{T}\}$, with $s_t : \Omega \times \mathbb{A} \to \mathbb{M}$. For any $k \in \mathbb{T}$, $i, j \in \mathbb{M}$, $\alpha \in \mathbb{A}_i$, the future transition probability distribution, conditioned on the present state s_k of the MDP and the action α_k to be taken from that state, is denoted by

$$p_{ij}^{\alpha}(k) \triangleq \Pr\{s_{k+1} = j \mid s_k = i,\ \alpha_k = \alpha\} \tag{B.88}$$

Being a probability distribution, $p_{ij}^{\alpha}(k) \in \mathbb{R}_0$ and satisfies $\forall k \in \mathbb{T}$, $i, j \in \mathbb{M}$, and $\alpha \in \mathbb{A}_i$

$$\sum\nolimits_{j=1}^{N} p_{ij}^{\alpha}(k) = 1 \tag{B.89}$$

For any $\alpha \notin \mathbb{A}_i$, the action is not available in a given state of the MDP. Hence, $\forall j \in \mathbb{M}$

$$p_{ij}^{\alpha}(k) \triangleq 0 \tag{B.90}$$

Choosing an (available) action in any given state entails a (non-negative) cost, which is seen as a function $g : \mathbb{M} \times \mathbb{A} \to \mathbb{G}$, where $\mathbb{G} \subseteq \mathbb{R}_0$ is a set of immediate costs. For example, increase of the transmission power gives less packet errors, but implies higher energy consumption and interference with other systems. Thus it is more expensive, in terms of energy and interference, than transmission with the same or lower power level.

Appendix C

List of the mathematical symbols

THE mathematical symbols used in this thesis are listed in the following tables, where their meaning is also reported.

C.1 Basic mathematical symbols

First we recall the basic mathematical symbols used throughout the text, reporting them in Table C.1.

Symbol	Meaning
$\cdot$	*placeholder* indicates the functional nature of an expression without assigning a specific symbol for its argument
$+$	*addition* is denoted by plus symbol
$\sum$	*summation* of a sequence of elements is presented in Sigma notation
$-$	*subtraction* is indicated by minus sign
	multiplication is declared by juxtaposition; so, the product of two mathematical objects a and b is written as ab.
$\prod$	*product* of a sequence of elements is presented in capital Pi notation
$\frac{\cdot}{\cdot}$	*division* is specified in fractional notation
$\sqrt{\cdot}$	*square root* is denoted by radical symbol, a.k.a. radix
$\cdots$	*intentional omission of values from a pattern* is declared by ellipsis, i.e., by a series of dots.
$\vdots$	*intentional omission of values from a pattern*
$\ddots$	*intentional omission of values from a pattern*
$\square$	*end of proof*, or QED, i.e., an alphabetism of the Latin phrase *quod erat demonstrandum*

Table C.1: Basic mathematical symbols used in the thesis

C.2 Symbols based on equality

This section reports mathematical symbols we use to indicate equality, definition and inequality, all listed in Table C.2.

Symbol	Meaning
$=$	*equality* asserts that the quantities have the same value, or that the expressions represent the same mathematical object
$\triangleq$	*definition* is read as "is equal by definition to", "is defined as"
$\neq$	*inequality* asserts that the mathematical objects are different
$>$	*greater than* is a strict inequality between the values of elements in an ordered set
$\geq$	*greater than or equal to* is a relation between the values of elements in an ordered set
$<$	*less than* is a strict inequality between the values of elements in an ordered set
$\leq$	*less than or equal to* is a relation between the values of elements in an ordered set

Table C.2: Mathematical symbols based on equality

C.3 Logic symbols

There are several logic symbols used in this thesis. We report them in Table C.3.

Symbol	Meaning
$\mathcal{S}$	*statement* is a declarative sentence that is either true or false
$\exists$	*existential quantification* is interpreted as "there is at least one", "there exists", or "for some"
$\forall$	*universal quantification* is interpreted as "given any" or "for all"
$\vee$	*logical disjunction*, see Subsection B.2
$\wedge$	*logical conjunction*, see Subsection B.2
$,$	*comma* has the same meaning of $\wedge$
$\Rightarrow$	*material implication*, see Subsection B.2
$:$	*such that* can be read also as "with the property that"; it is used in declaration of functions and in the set-builder notation

Table C.3: Logical symbols used throughout the text

C.4 Symbols from the set theory

In this section we recall the symbols used in definition of sets. We list them in Table C.4. See also Subsection B.2 for additional details.

Symbol	**Meaning**
$\mathbb{S}$	generic *set*
$\{\}$	*empty set*; see Subsection B.2
$()$	*empty sequence*; see Subsection B.3
$\in$	*set membership* indicates that an element belongs to a set
$\notin$	*set membership* indicates that an element does not belong to a set
$\subseteq$	*subset* expresses that LHS set is contained inside RHS set, i.e., all elements of LHS set are elements of RHS set
$\subset$	*proper* (or strict) *subset* means that LHS set is a subset of RHS set, but $\exists$ element of RHS set which is not an element of LHS set
$\supseteq$	*superset* states that RHS set is contained inside LHS set, i.e., all elements of RHS set are elements of LHS set
$\cup$	*union* of sets
$\cap$	*intersection* of sets
$\mathbb{S}^{\mathrm{C}}$	*complement* of a set $\mathbb{S}$
$2^{\mathbb{S}}$	*power set* of a set $\mathbb{S}$
$\lvert\cdot\rvert$	*cardinality* of a set
$\setminus$	*difference* of sets
$\triangle$	*symmetric difference* of sets, a.k.a. the disjunctive union
$\mathbf{1}_{\mathbb{S}}$	*indicator function*, a.k.a. characteristic function, of a set $\mathbb{S}$, is a convex function that indicates the membership (or non-membership) of a given element in that set
$\times$	*Cartesian product* returns a product set of all ordered pairs; see Subsection B.3 for additional details
$\prod$	*Cartesian product* of a sequence returns a product set of all ordered tuples; see e.g. Subsection B.5
$(\mathbb{S}_k)\nearrow$	*monotone non-decreasing sequence of sets*, i.e., $\mathbb{S}_0 \subseteq \mathbb{S}_1 \subseteq \mathbb{S}_2 \subseteq \cdots$
$(\mathbb{S}_k)\searrow$	*monotone non-increasing sequence of sets*, i.e., $\mathbb{S}_0 \supseteq \mathbb{S}_1 \supseteq \mathbb{S}_2 \supseteq \cdots$
$\operatorname{conv} \mathbb{S}$	*convex hull* of a nonempty set $\mathbb{S}$; see Subsection B.4
max	*maximum* is a greatest element of a (totally ordered) set, or a largest value of a function
sup	*supremum* of a subset $\mathbb{S}$ of a partially ordered set Ω is the least element in Ω that is greater than or equal to all elements of $\mathbb{S}$, if such an element exists, i.e., it is least upper bound of $\mathbb{S}$
min	*minimum* is a least element of a (totally ordered) set, or a smallest value of a function
inf	*infimum* of a subset $\mathbb{S}$ of a partially ordered set Ω is the greatest element in Ω that is less than or equal to all elements of $\mathbb{S}$, if such an element exists, i.e., it is greatest lower bound of $\mathbb{S}$

Table C.4: Mathematical symbols from set theory

C.5 Symbols specific to real and complex numbers

For the sake of completeness, we recall the symbols used to deal with real- and complex-valued variables in Table C.5. See Subsection B.2 for additional details.

Symbol	Meaning
$\mathbb{R}$	set of *real* numbers
$\mathbb{R}_0$	set of nonnegative real numbers, i.e., $\{i \in \mathbb{R} : i \geq 0\}$
$\mathbb{R}_+$	set of positive real numbers, i.e., $\{i \in \mathbb{R} : i > 0\}$
$\mathbb{C}$	set of *complex* numbers
$\mathbb{F}$	set of either real or complex numbers
$\lvert \cdot \rvert$	*absolute value* (a.k.a. modulus) of either real or complex number
$\imath$	*imaginary unit*, i.e., a number that satisfies the relation $\imath^2 = -1$
$\mathrm{Re}(\cdot)$	*real part* of a complex number
$\mathrm{Im}(\cdot)$	*imaginary part* of a complex number
$\bar{x}$	*complex conjugate* of a complex number x

Table C.5: Symbols specific to real and complex variables

C.6 Symbols from linear algebra & functional analysis

We list symbols specific to linear algebra and functional analysis in Tables C.6–C.8.

Symbol	Meaning
$\mathbb{F}^n$	*n-dimensional linear space*, with entries in $\mathbb{F}$
$\mathbb{F}^{m,n}$	set of *matrices* with m rows, n columns, and entries in $\mathbb{F}$, or a set of *linear maps* between two linear spaces $\mathbb{F}^n$ and $\mathbb{F}^m$
$A_{i\bullet}$	*i-th row* of a matrix $A \in \mathbb{F}^{m,n}$; see Subsection B.4
$A_{\bullet j}$	*j-th column* of a matrix $A \in \mathbb{F}^{m,n}$; see Subsection B.4
I_n	*identity matrix* of size n
A^T	*transpose* of a matrix $A \in \mathbb{F}^{m,n}$; see Subsection B.4
$\bar{A}$	*complex conjugate* of a matrix $A \in \mathbb{C}^{m,n}$; see Subsection B.4
A^*	*conjugate transpose* of a matrix $A \in \mathbb{C}^{m,n}$; see Subsection B.4
A^{-1}	*inverse* of a square matrix $A \in \mathbb{F}^{n,n}$; see Subsection B.4
$\oplus$	*direct sum* of matrices; see Subsection B.4
$\otimes$	*Kronecker product* of matrices, or *outer product* of (column-)vectors; see Subsection B.4
$\langle \cdot, \cdot \rangle$	*inner product*; see Subsections B.3; B.4, and B.4

Table C.6: Symbols used in linear algebra

For the additional details of symbols listed in the following Table C.7, see Subsections B.4–B.4.

Symbol	**Meaning**
$\mathbb{F}_{\mathrm{T}}^{n,n}$	set of all *symmetric matrices* of order n, with entries in $\mathbb{F}$
$\mathbb{C}_{*}^{n,n}$	set of all *Hermitian matrices* of order n
$\mathbb{F}_{*}^{n,n}$	set of all *either real symmetric or (complex) Hermitian matrices*
$\mathbb{F}_{+}^{n,n}$	set of all *positive definite matrices* with entries in $\mathbb{F}$
$\mathbb{F}_{0}^{n,n}$	set of all *positive semi-definite matrices* with entries in $\mathbb{F}$
${}_{\mathrm{N}}\mathbb{F}^{m,n}$	*linear space* made up of all *N-sequences of $m\times n$ matrices* with entries in $\mathbb{F}$
${}_{\mathrm{N}}\mathbb{F}_{+}^{n,n}$	set of all *N-sequences of positive definite matrices* of order n
${}_{\mathrm{N}}\mathbb{F}_{0}^{n,n}$	set of all *N-sequences of positive semi-definite matrices* of order n
${}_{\mathbb{V}}\mathbb{F}^{N,N}$	*set* of cardinality V of all square $N\times N$ *matrices* with entries in $\mathbb{F}$
$\succeq$	*positive semi-definite*, i.e., $A\succeq 0$ means $A\in\mathbb{F}_{0}^{n,n}$, and $\boldsymbol{A}\succeq 0$ stands for $\boldsymbol{A}\in{}_{\mathrm{N}}\mathbb{F}_{0}^{n,n}$
$\succ$	*positive definite*, i.e., $A\succ 0$ means $A\in\mathbb{F}_{+}^{n,n}$, and $\boldsymbol{A}\succ 0$ stands for $\boldsymbol{A}\in{}_{\mathrm{N}}\mathbb{F}_{+}^{n,n}$
ν	*eigenvalue* of a square matrix
$\mathfrak{S}_{A}$	*spectrum* of a square matrix A
$\sigma_{\max}(\cdot)$	*largest singular value* of a matrix
$\mathrm{vec}(\cdot)$	*vectorization* of a matrix
$\rho(\cdot)$	*spectral radius* of a square matrix
$\mathrm{tr}(\cdot)$	*trace* of a square matrix
$\det(\cdot)$	*determinant* of a square matrix
$\mathrm{vec}^2(\cdot)$	*vectorization* of a sequence of matrices
$\mathrm{diag}(\cdot)$	diagonal (resp. block diagonal) matrix obtained by putting the elements of a vector (resp. a sequence of square matrices) on the main diagonal
$\hat{\rho}(\cdot)$	*joint spectral radius* of a set of matrices
lim	*limit* is a value to which all elements of a sequence converge to
$\to$	*tends to* symbol is used to declare a limit of a sequence
$\to$	*function arrow* is used to declare a function (for instance, f) by stating its domain and codomain, e.g. $f:\mathbb{R}\to\mathbb{R}$
arg	*argument* of a function is an independent variable (defined on a function's domain), which represents input or cause, i.e., potential reason for variation of the output

Table C.7: Notation from linear algebra and functional analysis

The notation used to indicate different norms is reported in Table C.8. See also Subsections B.3, B.4 and B.4, for additional discussion on topic.

Symbol	Meaning
$\|\cdot\|$	*any norm* in $\mathbb{F}^n$, in ${}_{\mathrm{N}}\mathbb{F}^{m,n}$, or uniform induced norm in $\mathbb{F}^{m,n}$
$\|\cdot\|_p$	*p-norm* (a.k.a. $\mathbb{L}^p$-norm) or induced p-norm
$\|\cdot\|_1$	1-*norm*, or induced 1-norm
$\|\cdot\|_2$	*Euclidean norm*, or induced Euclidean norm
$\|\cdot\|_{\max}$	max-*norm*, or induced max-norm
$\|\cdot\|_{\mathbf{1}}$	ℓ_1-*norm*
$\|\cdot\|_F$	*Frobenius norm*

Table C.8: Symbols used to denote different norms

C.7 Symbols for notable discrete-valued numerical sets

The discrete-valued (i.e., countable) numerical sets (and their notable elements) used in this thesis are listed and described in the following Table C.9.

Symbol	Meaning
$\mathbb{Z}$	set of *integers*
$\mathbb{Z}_0$	set of *nonnegative integers*, i.e., $\{i\in\mathbb{Z} : i\geq 0\}$
$\mathbb{Z}_+$	set of *positive integers*, i.e., $\{i\in\mathbb{Z} : i>0\}$
$\mathbb{Q}$	set of *rational numbers*
$\mathbb{T}$	*discrete-time* set, $\mathbb{T}=\mathbb{Z}_0$
k	discrete-time *instant*, $k\in\mathbb{T}$
$\mathbb{T}_k$	*bounded discrete-time* set, i.e., $\{i\in\mathbb{T} : i\leq k\}$
T	*discrete-time horizon*, $T\in\mathbb{T}$
N	*number of operational modes* (a.k.a. discrete states), $N\in\mathbb{Z}_+$
$\mathbb{M}$	set of *operational modes*, $\mathbb{M}\triangleq\{i\in\mathbb{Z}_+ : i\leq N\}$
M	*number of* (discrete) *actions*, $M\in\mathbb{Z}_+$
$\mathbb{A}$	set of (discrete) *actions*, $\mathbb{A}\triangleq\{i\in\mathbb{Z}_+ : i\leq M\}$
$\mathbb{A}_i$	set of *actions available* in an operational mode $i\in\mathbb{M}$, $\mathbb{A}_i\subseteq\mathbb{A}$
α	*discrete action*, $\alpha\in\mathbb{A}_i$
α_k	*action* (to be) *taken* at time instant $k\in\mathbb{T}$, $\alpha_k\in\mathbb{A}_i$
V	*number of vertices* of a convex polytope, $V\in\mathbb{Z}_+$
V_α	*number of vertices* of a polytope *for an action* $\alpha\in\mathbb{A}_i$, $V_\alpha\in\mathbb{Z}_+$
$\mathbb{V}$	*index set of vertices* of a convex polytope, $\mathbb{V}\triangleq\{i\in\mathbb{Z}_+ : i\leq V\}$
$\mathbb{V}_{\alpha_k}$	*index set of vertices* of a polytope *for an* α_k, i.e., $\{i\in\mathbb{Z}_+ : i\leq V_{\alpha_k}\}$
l	*index of a vertex* of a convex polytope, $l\in\mathbb{V}$, or $l\in\mathbb{V}_{\alpha_k}$

Table C.9: Notation for discrete sets and their elements used in the text

C.8 Symbols from probability theory & measure theory

In the following, we list the mathematical symbols from probability and measure theories in Tables C.10 and C.11. See Subsections B.5 and B.6, and Sections 2.2, 2.3 and 2.4 for additional details.

Symbol	Meaning
Ω	*sample space* is a set of all the possible outcomes of an experiment
Ω_x	sample space defined by (2.2)
Ω_y	sample space defined by (2.12)
Ω_s	sample space defined by (2.19)
ω	*element of a sample space*
$\mathcal{F}$	*σ-algebra of events*, which are the measurable subsets of Ω_y
$\mathcal{G}$	*σ-algebra of events*, which are the measurable subsets of Ω_x
$\mathcal{I}$	*σ-algebra of events*, which are the measurable subsets of Ω_s
$\mathcal{M}$	*σ-algebra of events*, which are the measurable subsets of $\mathbb{M}$
$\mathcal{R}$	*Borel-σ-algebra* is the σ-algebra generated by the open subsets of $\mathbb{R}$
$\mathcal{X}$	product *Borel-σ-algebra*, see Section 2.2
$\mathcal{U}$	product *Borel-σ-algebra*, see Section 2.2
$\mathcal{Y}$	product *Borel-σ-algebra*, see Section 2.2
$\mathcal{Z}$	product *Borel-σ-algebra*, see Section 2.2
$(\mathcal{F}_k)$	*filtration* is a monotone non-decreasing sequence of σ-algebras
$(\mathcal{G}_k)$	another *filtration*
$(\mathcal{I}_k)$	yet another *filtration*
$\mathbb{H}^n$	*Hilbert space* of all $\mathbb{F}^n$-valued $\mathcal{G}$-measurable random variables
$\ell_2(\mathbb{H}^n)$	direct sum of countably infinite copies of $\mathbb{H}^n$, it is a *Hilbert space*
$\mathcal{H}^n$	closed linear subspace of $\ell_2(\mathbb{H}^n)$; it is a Hilbert space
$\mathcal{H}^n_k$	family of sequences $(\mathrm{f}_t)_{t=0}^k$, s.t. $\mathrm{f}_t \in \mathbb{L}^2(\Omega_\mathrm{x}, \mathcal{G}_t, \Pr, \mathbb{F}^n)$, $\forall t \in \mathbb{T}_k$
$\Pr(\cdot)$	*probability measure*
$\vert$	*vertical bar* is a separator in *conditional probability* measure notation; it is read as "given that"
θ	*discrete-time Markov chain*
θ_x	*discrete-time Markov chain*
θ_y	*discrete-time Markov chain*
θ_k	*random variable*

Table C.10: Symbols from probability and measure theories, first part

Symbol	**Meaning**
$\boldsymbol{\vartheta}$	compact notation for θ_k, $\forall k \in \mathbb{T}$, useful to write e.g., θ_{T-1}
$\mathbb{E}(\cdot)$	*expected value* of a random variable
$p_i(k)$	*probability mass function* of θ_k, i.e., $\Pr(\theta_k = i) = \Pr(\{\omega : \theta_k(\omega) = i\})$
$p_{ij}(k)$	probability of θ being next in a state j, conditioned to the fact that at the current time-step k it is in a state i
P	*transition probability matrix* (TPM), i.e., $[p_{ij}]$
$_{\mathbb{V}}\mathbb{P}$	set of vertices of a convex polytope of TPMs
$_{\mathbb{V}_\alpha}\mathbb{P}$	set of vertices of a convex polytope of row TP vectors for an action
s	*discrete-time Markov decision process* (MDP)
p_0	vector with *initial probability distributions* of all states of θ, or s
$\boldsymbol{p}_0$	numerical value of p_0
s_k	*random variable*
$\boldsymbol{s}$	compact notation for s_k, $\forall k \in \mathbb{T}$, e.g., useful to write e.g., s_{T-1}
g	*immediate cost* dependent on state and action in an MDP
$\mathbb{G}$	*set of immediate costs* that depend on state and action in an MDP
$p_{ij}^{\alpha}(k)$	future *transition probability distribution* of an MDP, conditioned on the present state s_k and the action α_k to be taken from that state, i.e., $\Pr\{s_{k+1} = j \mid s_k = i,\, \alpha_k = \alpha\}$

Table C.11: Symbols from probability and measure theories, second part

C.9 Symbols from state-space representation

In this final section, we list the symbols used in a state-space model of dynamical systems studied in the thesis, namely Markov jump linear systems and Markov jump switched linear systems, in Tables C.12, C.13 and C.14.

Symbol	**Meaning**
n_{x}	number of continuous state variables of a system
n_{u}	number of continuous control input variables of a system
n_{v}	number of process noise variables
n_{y}	number of measured continuous state variables, $n_{\mathrm{y}} \leq n_{\mathrm{x}}$
n_{w}	number of observation noise variables
n_{z}	number of (measured) system output variables
$n_{\hat{\mathrm{x}}}$	number of state variables of dynamic controller or filter

Table C.12: Symbols for a state-space representation of dynamical systems, part 1

Symbol	Meaning
$n_{\hat{z}}$	number of output variables of dynamic output feedback controller
$n_{\tilde{x}}$	number of state variables of optimal dynamic Markov jump filter
$n_{\tilde{y}}$	number of output variables of optimal dynamic Markov jump filter
x_k	vector of *continuous state* variables of a system, $\mathrm{x}_k \in \mathbb{F}^{n_x}\ \forall k \in \mathbb{T}$
u_k	vector of *continuous control* input variables, $\mathrm{u}_k \in \mathbb{F}^{n_u}\ \forall k \in \mathbb{T}$
v_k	vector of *process noise* variables, $\mathrm{v}_k \in \mathbb{F}^{n_v}\ \forall k \in \mathbb{T}$
y_k	vector of *measured continuous state* variables, $\mathrm{y}_k \in \mathbb{F}^{n_y}\ \forall k \in \mathbb{T}$
w_k	vector of *observation noise* variables, $\mathrm{w}_k \in \mathbb{F}^{n_w}\ \forall k \in \mathbb{T}$
z_k	vector of measured *system output* variables, $\mathrm{z}_k \in \mathbb{F}^{n_z}\ \forall k \in \mathbb{T}$
$\hat{\mathrm{x}}_k$	vector of state variables of dynamic controller or filter, $\hat{\mathrm{x}}_k \in \mathbb{F}^{n_{\hat{x}}}$
$\hat{\mathrm{z}}_k$	vector of output variables of dynamic output feedback controller, $\hat{\mathrm{z}}_k \in \mathbb{F}^{n_{\hat{z}}}\ \forall k \in \mathbb{T}$
$\hat{\mathrm{e}}_k$	observation (or estimation) error of a generic dynamic filter
$\tilde{\mathrm{x}}_k$	vector of state variables of the optimal dynamic Markov jump filter, $\tilde{\mathrm{x}}_k \in \mathbb{F}^{n_{\tilde{x}}}\ \forall k \in \mathbb{T}$
$\tilde{\mathrm{y}}_k$	vector of output variables of the optimal dynamic Markov jump filter, $\tilde{\mathrm{y}}_k \in \mathbb{F}^{n_{\tilde{y}}}\ \forall k \in \mathbb{T}$
$\tilde{\mathrm{e}}_k$	observation (or estimation) error of the optimal dynamic filter
$\boldsymbol{x}_k$	value of x_k, $\forall k \in \mathbb{T}$
$\boldsymbol{x}_e$	equilibrium point of a dynamical system
ϑ_k	value of the jump variable θ_k
$\boldsymbol{u}_k$	value of u_k; it is called continuous control law
$\mathbb{U}_\mathrm{T}$	set of all admissible and measurable controllers
u	sequence of (continuous) control laws, i.e., $\mathrm{u} \triangleq (\mathrm{u}_k)_0^{T-1} \in \mathbb{U}_\mathrm{T}$
$\boldsymbol{\alpha}_k$	value of α_k; it is called *discrete switching control law*
π_k	*hybrid control pair* (α_k, u_k), $k \in \mathbb{T}$
π	*hybrid control sequence*, i.e., $(\pi_k)_{k=0}^{T-1}$
$\boldsymbol{\pi}_k$	*hybrid control law*, i.e., a pair $(\boldsymbol{\alpha}_k, \boldsymbol{u}_k)$
$\boldsymbol{\pi}$	*hybrid control policy*, i.e., a sequence $(\boldsymbol{\pi}_k)_{k=0}^{T-1} \triangleq (\boldsymbol{\alpha}_k, \boldsymbol{u}_k)_{k=0}^{T-1}$
$\boldsymbol{A}$	sequence of *state matrices*, i.e., $(A_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_x,n_x}$, each of which is associated to an operational mode of the (switching) system
$\boldsymbol{B}$	sequence of *input matrices*, i.e., $(B_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_x,n_u}$
$\boldsymbol{C}$	sequence of *output matrices*, i.e., $(C_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_z,n_x}$
$\boldsymbol{D}$	sequence of *direct transition* (a.k.a. feed-forward or feedthrough) *matrices*, i.e., $(D_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_z,n_u}$
$\boldsymbol{F}$	sequence of *observation matrices*, i.e., $(F_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_y,n_x}$
$\boldsymbol{G}$	sequence of *observation noise matrices*, i.e., $(G_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_y,n_w}$
$\boldsymbol{H}$	sequence of *process noise matrices*, i.e., $(H_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_x,n_v}$
$\boldsymbol{\Phi}$	sequence of *process noise matrices*, i.e., $(\Phi_i)_{i=1}^N \in {}_\mathrm{N}\mathbb{F}^{n_x,n_v}$

Table C.13: Symbols for a state-space representation of dynamical systems, part 2

Symbol	**Meaning**
$\boldsymbol{K}(k)$	sequence of mode-dependent continuous *control gain matrices*, i.e., $(K_i(k))_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{u}},n_{\mathrm{x}}}$
$\boldsymbol{L}(k)$	sequence of mode-dependent *filter gain matrices*, i.e., $(L_i(k))_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{y}},n_{\hat{\mathrm{x}}}}$
$\hat{\boldsymbol{A}}$	sequence of dynamic controller or filter *state matrices*, i.e., $(\hat{A}_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\hat{\mathrm{x}}},n_{\hat{\mathrm{x}}}}$, each one associated to an operational mode
$\hat{\boldsymbol{B}}$	sequence of *input matrices* of a dynamic controller, i.e., $(\hat{B}_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\hat{\mathrm{x}}},n_{\mathrm{y}}}$
$\hat{\boldsymbol{C}}$	sequence of *output feedback control matrices*, i.e., $(\hat{C}_i)_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{u}},n_{\hat{\mathrm{x}}}}$
$\mathrm{q}_i(k)$	expected value of x_k for the operational mode $\theta_k = i$ only, see (3.11)
$\mathrm{r}_i(k)$	expected value of v_k for the operational mode $\theta_k = i$ only, see (3.12)
$Q_i(k)$	second moment of x_k for $\theta_k = i$ only, see (3.13)
$\boldsymbol{Q}(k)$	sequence of matrices of the second moments of x_k associated to each operational mode, i.e., $(Q_i(k))_{i=1}^N \in {}_{\mathrm{N}}\mathbb{F}_0^{n_{\mathrm{x}},n_{\mathrm{x}}}$, see (3.14)
$R_i(k)$	second moment of v_k for $\theta_k = i$ only, see (3.15)
$\boldsymbol{R}(k)$	sequence of matrices of the second moments of v_k associated to each operational mode, i.e., $(R_i(k))_{i=1}^N \in \mathbb{F}_0^{n_{\mathrm{v}},n_{\mathrm{v}}}$, see (3.16)
$W_i(k)$	expected value of $\mathrm{x}_k \mathrm{v}_k^*$ for $\theta_k = i$ only, see (3.18)
$\boldsymbol{W}(k)$	sequence $(W_i(k)) \in {}_{\mathrm{N}}\mathbb{F}^{n_{\mathrm{v}},n_{\mathrm{x}}}$
ψ_0	expected value of x_0 in optimal robust filtering problem, see (4.4)
Ψ_0	second moment of x_0 in optimal robust filtering problem, see (4.4)
Λ	matrix associated to the second moment of x_k in stability problem, see (3.8)
${}_{\mathbb{V}}\mathbb{P}$	set of vertices of a convex polytope of transition probability matrices, see (2.16)
${}_{\mathbb{V}}\boldsymbol{\Lambda}$	set of vertices of the convex polytope of the matrices Λ, see (3.31)
$\mathcal{P}_j({}_{\mathbb{V}}\boldsymbol{\Lambda})$	set of all possible products of length j whose factors are elements of ${}_{\mathbb{V}}\boldsymbol{\Lambda}$, see (B.63)
$\mathcal{J}(\cdot)$	quadratic cost associated to an optimization problem
ξ	index of the solutions of coupled Riccati difference equations
υ	vertex of the transition probability matrix for which the cost is maximum in transition probabilities, see for instance Theorem 5.1

Table C.14: Symbols for a state-space representation of dynamical systems, part 3

Appendix D

State of the art of CPSs' security: an automatic control perspective

We have mentioned in both introductory and final chapters of this thesis that our exposure to and interest in Markov jump linear systems is strongly linked to our work on security of cyber-physical systems. In this final appendix we report the main findings of our systematic survey [5] on security in cyber-physical domain.

D.1 A short introduction

Due to the tight cyber-physical coupling and to the potentially disrupting consequences of failures, security is one of the primary concerns in cyber-physical systems. Our systematic mapping study reported here sheds light on how security is actually addressed when dealing with cyber-physical systems from an automatic control perspective. We provide a map of 118 selected studies which was defined empirically and is based on, for instance, application fields, various system components, related algorithms and models, attacks characteristics and defense strategies. It presents a powerful comparison framework for existing and future research on this hot topic, important for both industry and academia.

Motivations behind the choice of perspective

Cyber-physical systems (CPSs) security is attracting several research efforts from different and independent areas (e.g., secure control, intrusion detection in SCADA systems, etc.), each of them with specific peculiarities, features, and capabilities. However, if on one side having many research efforts from different and independent areas on CPSs security confirms its importance from a scientific point of view, on the other side it is very difficult to have a holistic view on this important domain. Under this perspective, even if the progress of research on cyber-physical systems has started more than ten years ago and the various research communities are very active, *the trends, characteristics, and the validation strategies of existing research on CPS security are still unclear.* The aim of this survey is to fill this gap. CPS security is presently investigated in a number of scientific (e.g. in embedded systems and wireless sensor networks) communities from different points of view. In this report we focus on research on CPS security from the point of view of the automatic control scientific community. A first motivation for our choice is that most application domains where CPS security is an issue consist of/include distributed feedback-based automation systems. In addition to this, a peculiar characteristic of the automatic control research is the attempt to combine in a unifying mathematical framework physical components (e.g. electrical/electronic devices, vehicles, and industrial automation machineries) and cyber components (e.g. SCADA systems, communication protocols, and real-time software) of the CPS, as well as to define rigorous performance and robustness/resilience metrics on security properties based on such unifying mathematical framework.

Methodology

Goal of our survey is to identify, classify, and analyze existing research on CPS security from an automatic control perspective, in order to better understand how security is actually addressed when dealing with cyber-physical systems. To tackle this goal, we have used a well-established methodology from the medical and software engineering research communities called *systematic mapping* [7, 8] (see Section D.2), applying it on the peer reviewed papers which propose and validate a method or technique for CPS security enforcing or breaching. Through our systematic mapping process, we selected 118 primary studies among more than a thousand entries fitting at best three research questions we identified (see Subsection D.2). Then, we defined a classification framework composed of more than 40 different parameters for comparing state-of-the-art approaches, and we applied it to all selected studies. Finally, we analyzed and discussed the obtained data for extracting emergent research challenges and implications for future research on CPS security.

Contributions

The main **contributions** of this study are:

- a reusable *comparison framework* for understanding, classifying, and comparing methods or techniques for cyber-physical systems security from an automatic control perspective;

- a *systematic review* of current methods or techniques for automatic control for CPS security, useful for both researchers and practitioners;

- a discussion of *emerging research challenges and implications* for future research.

To the best of our knowledge, this work presents the first systematic investigation into the state of the art of research on CPS security from an automatic control perspective. The results of this study provide a complete, comprehensive and replicable picture of the state of the art of research on CPS security, helping researchers and practitioners in finding trends, characteristics, and validation strategies of current research on security-aware cyber-physical (co-)design, intrusion detection, forecast and response, its future potential and applicability.

Main findings

The main findings produced by our analysis are discussed below.

Publication trends: even if the need for methods and techniques for CPS security has emerged only in 2008, in the last years there is an increasing need and scientific interest on methods and techniques for CPS security. Also, CPS security is turning more and more into a mature field, with more foundational and comprehensive studies published in the recent years. Cyber-physical systems security has a very multidisciplinary nature and it has been broadly considered by researchers with different research interests, such as smart grid, automatic control, communications, networked systems, parallel and distributed systems, etc.

Characteristics and focus: the bulk of the works on CPS security is focused on power grids, while somehow surprisingly, we have not found any work on the cyber-physical security of medical CPS, and only a small part of selected papers is within the application field of secure control of (unmanned) ground vehicles and aerial systems, and of heating, ventilation, and air-conditioning in large functional buildings. All the works considered in this mapping study deal with attacks, in order to either implement or to counteract them: putting together all this studies gives us the possibility to categorize the existing (cyber-physical) attack models. The defense strategies are presented in most of the studies, occupying the central spot of the research efforts on CPS security. More than 90% of the works are concerned with system integrity, threatened by various types of deception attacks. Regarding the considered system components, the approaches considering attacks on sensors and their protection completely dominate the scene; in fact the resilient state estimation under measurement attacks is a very active research topic within the area of CPS security. Somehow unexpectedly, very few papers consider communication aspects or imperfections and attempt to provide non-trivial mathematical models of the communication; the centralized schemes dominate both attack and defense solutions.

Validation strategies: most advanced and realistic validation methods have been exploited in the power networks application domain, but even there a benchmark is still missing. Even if the repeatability process, capturing how a third party may reproduce the validation results of the method or technique, is recognized as a good scientific practice, we found no studies providing a replication package. So, we put a particular attention on analysis and description of standard test systems and experimental testbeds used by researchers studying various aspects of CPS security.

By presenting and discussing the above mentioned results we are the first to provide an overview of the state of the art of research in CPS security from an automatic control perspective, thus our work can certainly be useful for both researchers (either young or experienced ones) and practitioners in the field of CPS security. Finally, we use the results of this study for discussing potential implications for future research on automatic control for CPS security.

Outline of the rest of appendix

Appendix D is organized as follows. Section D.2 describes in details our research methodology in designing, conducting, and documenting the study (and readers principally interested in the results of our survey and future research directions may directly jump to subsequent sections and come back to this section after the first read of the material), followed by a discussion of the obtained results in Sections D.3, D.4 and D.5. We discuss the implications for future research on cyber-physical security in Section D.6, and related work in Section D.7. Section D.8 closes the main part of Appendix D. Then, the list of the selected primary studies is provided in supplementary Section D.9, while the additional details on our search strategy can be found in Section D.10. In Section D.11 we discuss some additional characteristics of our primary studies, which are not related to CPS security per se, but are still useful to better understand this scientific area. Ultimately, Section D.12 describes limitations and threats to validity of our results.

D.2 Systematic mapping study methodology

A *systematic mapping study* (or *scoping study*) is a research methodology particularly intended to provide an **unbiased, objective and systematic instrument** to answer a set of research questions by finding all of the relevant research outcomes in a specific research area (CPS security in our paper) [7]. Research questions of mapping studies are designed to provide an overview of a research area by classifying and counting research contributions in relation to a set of well-defined categories such as publication type, forum, frequency, assumptions made, followed research method, etc. [8, 136]. The mapping process involves searching and analyzing the literature in order to identify, classify, and understand existing research on a specific topic of interest.

In the recent years many researchers are conducting systematic mapping studies on a number of areas and using different guidelines or methods (e.g., on technical debt [137], search-base software engineering [138], model-driven engineering for wireless sensor networks [139]). In a recent study [7] it emerged that at least ten different guidelines have been proposed for designing the systematic mapping process. We conducted our study by considering the two most commonly accepted and followed guidelines according to [7], specifically: the ones proposed by Kitchenham and Charters [8] and Petersen et al. [136], respectively. Also, we refined our mapping process according to the results of a consolidating update on how to conduct systematic mapping studies proposed by Petersen et al. in [7]. Finally, due to the various specificities of existing research on CPS (e.g., the presence of many different definitions of CPS, the intrinsic multidisciplinarity of existing research on CPS, etc.), we found it appropriate to tailor the method and classification schemes proposed in the guidelines according to our topic. The method we followed in our systematic mapping study is detailed in the rest of this section.

In order to establish the need for performing a mapping study on security for cyber-physical systems, we searched (on January 5, 2015) a set of electronic data sources (i.e., those listed in Subsection D.2), for systematic studies on security-aware cyber-physical co-design, self-protection and related security mechanisms specific to CPS without any success. None of the retrieved publications was related to any of our research questions detailed in Subsection D.2. So, we can claim that our research complements the related works described in Section D.7 to investigate the state-of-research about cyber-physical systems security.

The process we followed for carrying on our study can be divided into three main phases, which are the well-accepted ones for performing a systematic study [8, 140]: planning, conducting, and documenting.

In order to mitigate potential threats to validity, some produced artifacts in each phase have been circulated to external experts for independent review. More specifically, we identified two classes of external experts: systematic mapping studies experts who focused on the overall design of the study and domain experts focusing more on aspects related to security for cyber-physical systems. One systematic mapping study expert and two experts of CPS security reviewed our review protocol and final report independently and we refined them according to their feedback. In the following we will go through each phase of the process, highlighting its main activities and produced artifacts.

Planning

In this phase we identified the main research questions (see Subsection D.2) and we produced a well-defined review protocol describing in details the various steps of our study.The final version of the review protocol is publicly available as part of the replication package of this study at the following hyperlink:

`http://cs.gssi.infn.it/CPSSecurity`.

Conducting

In this phase we set the previously defined protocol into practice. More specifically, we performed the following activities:

- **Studies search**: we performed a combination of techniques for identifying the comprehensive set of candidate entries on automatic control for CPS security (see Subsection D.2).

- **Studies selection**: we filtered candidate entries in order to obtain the final list of primary studies to be considered in later activities of the review (see Subsection D.2).

- **Comparison framework definition**: we defined the set of parameters for comparing the primary studies. The main outcome of this activity is a document explaining the possible values and the meaning of each parameter (see Subsection D.2).

- **Data extraction**: we went into the details of each primary study and extracted data according to the comparison framework defined in the previous activity (see Subsection D.2).

- **Data synthesis**: we elaborated on the extracted data in order to address each research question of our study. This activity involved both quantitative and qualitative analysis of the extracted data (see Subsection D.2).

Documenting

The main activities performed in this phase are:

- a thorough elaboration on the data extracted in the previous phase with the main aim at setting the obtained results in their context,

- the analysis of possible threats to validity, and

- the writing of a set of reports describing the performed mapping study to different audiences.

Produced reports have been evaluated by experts on systematic mapping studies and cyber-physical systems (this appendix itself is an instance of produced final report).

Research questions

It is fundamental to clearly define the research questions (abbreviated to RQ) of a systematic literature study [141]. The research questions of this study are:

- **RQ1**: *What are the publication trends of research studies on automatic control for cyber-physical systems security?*

  Objective: to classify primary studies in order to assess interest, relevant venues, and contribution types.

- **RQ2**: *What are the characteristics and focus of existing research on automatic control for CPS security?*

Objective: to analyze and classify all the existing approaches for automatic control for CPS security with respect to the specific concerns they want to address (e.g., cyber and physical security, secure control, model-based intrusion detection, or any combination of them).

- **RQ3**: *What are the validation strategies of existing approaches for automatic control for cyber-physical systems security?*

  Objective: to analyze and classify all the existing approaches for automatic control for CPS security with respect to the strategies used for assessing their validity (e.g., controlled experiment, industrial application, prototype-based experiment, test bed, simple examples, formal proofs).

Answer to RQ1 gives a detailed overview about publication trends, venues, and research groups active on the topic. The classification resulting from our investigation on RQ2 and RQ3 provides a solid foundation for a thorough comparison of existing and future solutions for CPS security via automatic control. These contributions are especially useful for researchers willing to further contribute this research area with new approaches to CPS security or willing to better understand or refine existing ones.

Search strategy

In order to achieve maximal coverage, our search strategy consisted of three complementary methods: an automatic search, manual search, and snowballing. Figure D.1 shows the details about our search strategy.

Automatic search

Automatic search refers to the execution of a search query on a set of electronic databases and indexing systems [142]. As shown in Figure D.1, our automatic search is performed on the largest and most complete scientific databases and indexing systems available in computer science. The applied search string is the following:

```
(((("cyber physical" OR "cyber-physical" OR cyberphysical OR
"networked control") AND system*) OR CPS OR NCS) AND (attack* OR
secur* OR protect*)).
```

In the spirit of Zhang, Ali Babar and Tell [143], we established a *quasi-gold standard* for creating a good search string for the automatic search (see Section D.10). Our automatic search resulted in 1559 potentially relevant studies.

Manual search

By following the quasi-gold standard procedure defined in [143], we

- identified a subset of important venues for the domain of automatic control for cyber-physical systems security (they are provided in Section D.10), and

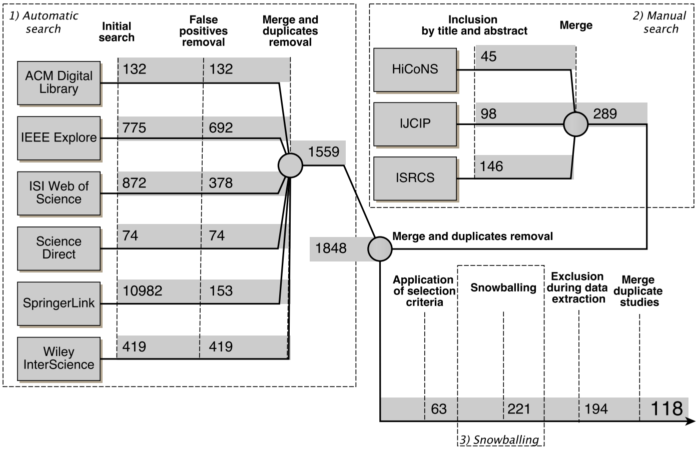

Figure D.1: Overview of the search and selection process

- performed a *manual search* of relevant publications in those venues.

By referring to Figure D.1, we manually searched and selected 289 potentially relevant studies and, after merging all the studies and removing duplicates we obtained 1848 potentially relevant studies.

In order to further restrict the number of studies to be considered during the snowballing activity, we applied the selection process depicted in Subsection D.2 to the current set of studies, thus obtaining 63 potentially relevant works.

Snowballing

We applied (backward and forward) snowballing on the 63 studies for identifying additional sources published in other journals or venues [144] which may not have been considered during the automatic or manual searches.

For the sake of replicability, we provide all the details, data, and results of our search strategy in the publicly available replication package of this study.

Selection strategy

We considered all the collected studies and filtered them according to a set of well-defined inclusion and exclusion criteria. In the following we provide the inclusion (I) and exclusion (E) criteria of our study:

- **I1**: Studies focusing on security of cyber-physical systems.

- **I2**: Studies proposing an method or technique for CPS security enforcing or breaching based on automatic control.

- **I3**: Studies providing some kind of validation of the proposed method or technique (e.g., via formal analysis, controlled experiment, exploitation in industry, example usage).
- **E1**: Studies not subject to peer review [140] (e.g., journal papers are considered, whereas white papers are discarded).
- **E2**: Studies written in any language other than English.
- **E3**: Studies focusing on security method or technique not specific to CPS (e.g studies focusing on either the physical or cyber part only of the system under consideration).
- **E4**: Studies published before 2006 (because the CPS discipline has emerged in 2006).
- **E5**: Secondary or tertiary studies (e.g., systematic literature reviews, surveys, etc.).
- **E6**: Studies in the form of tutorial papers, short papers, poster papers, editorials, because they do not provide enough information.

A study was selected as a primary study if it satisfied *all* inclusion criteria, and it was discarded if it met *any* exclusion criterion. In order to reduce bias, the selection criteria of this study have been decided during the review protocol definition (thus they have been checked by three external reviewers). By following the approach proposed in [145], two researchers classified each potentially relevant study either as *relevant*, *uncertain*, or *irrelevant*; studies classified as *irrelevant* have been excluded, whereas all the other approaches have been discussed with the help of a third researcher.

When reading a primary study in details for extracting its information, researchers could agree that the currently analyzed study was semantically out of the scope of our research, and so it has been excluded (see the *Exclusion during data extraction* stage in Figure D.1), resulting in 194 potentially primary studies.

As suggested in [140], if a primary study was published in more then one paper (e.g., if a conference paper has been extended to a journal version) then we considered only one reference paper as primary study; in those cases we considered all the related papers during the data extraction activity in order to obtain all the necessary data [8]. The final set of primary studies is composed of 118 entries after a duplicates merging step.

Data extraction

Data extraction refers to the recording of all the relevant information from the primary studies required to answer the research questions [140]. Before analysing each primary study, we defined a *comparison framework* for classifying research studies on cyber-physical systems security from an automatic control perspective.

To help the definition of a sound and complete comparison framework, we selected and adapted suitable dimensions and properties found in existing surveys and taxonomies related to CPS security, such as those proposed in [146–148]. In addition, we defined several parameters for classifying methods and techniques for CPS security; we grouped those parameters into three main dimensions: method or technique's positioning, characterization, and validation.

The **positioning** dimension characterizes the objectives and intent of existing research on CPS security (the "**what**" aspect of each method or technique).

The **characterization** dimension concerns the classification of studies based on "**how**" CPS security is addressed in research on automatic control.

Finally, the **validation** dimension concerns the strategies researchers apply for providing evidence about the validity of proposed methods or techniques.

All the dimensions and parameters of our comparison framework have been encoded in a dedicated *data extraction form*, which can be seen as the implementation of a *comparison framework*. For the sake of brevity we do not provide the description of all the parameters of our data extraction form, we will briefly elaborate on each of them while discussing the results of this study. As suggested in [140], the data extraction form (and thus also the classification framework) has been independently piloted on a sample of primary studies by two researchers, and iteratively refined accordingly. Then, the data extraction activity has been conducted by two researchers.

Data synthesis

The main goal of our data synthesis activity is to understand, analyze, and classify current research on automatic control for CPS security [8, § 6.5]. Depending on the parameters of the classification framework (see Subsection D.2), in this research we applied both quantitative and qualitative synthesis methods. We applied standard descriptive statistics for analysing quantitative data, whereas we applied the line of argument synthesis for qualitative data [140].

In the following sections we present the results of our analysis of the extracted data. In total 118 publications have been selected and analyzed as the subjects of our study. For the sake of clarity we organized the results of the analysis according to our research questions (see Subsection D.2).

D.3 Results - Publication trends (RQ1)

In order to assess the publication trends about security for cyber-physical systems we identified a set of variables focusing on the publication and bibliographic data of each primary study. In the following we describe the main facts emerging from our analysis.

Publication timeline

Figure D.2 presents the distribution of the selected publications[1] on security for cyber-physical systems over the time period from 2006 to 2015.

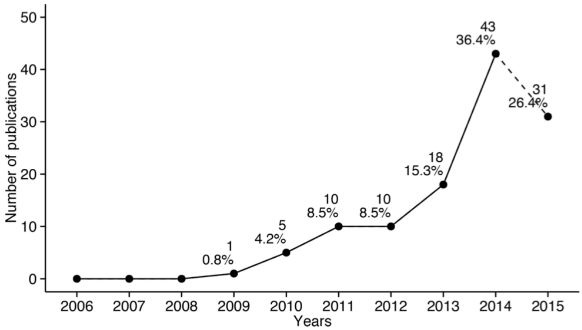

Figure D.2: Distribution by year (partial data for 2015)

The first interesting result of our study is the growth of the number of those publications in the last years. Indeed, we can observe that there was a relatively low number of publications on this topic over the time period from 2006 (zero publications) to 2010 (5 publications). Starting from 2011, we see a continuous growing trend over the years, culminating in the 2014 and 2015 years, which together amount for the 61.8% of the selected studies. From the collected data, we can offer the following observations:

- there are no selected studies until 2009; this may be because the main concepts and research interest on CPS emerged only around 2006 [14], and the need for methods and techniques for CPS security has emerged only recently;
- there is a sharp increase in the number of selected studies between 2012 and 2014; we can trace this observation to the fact that (i) in the last years methods and techniques for CPS security are gaining increasing interest and attention from a scientific point of view and (ii) methods and techniques for CPS security are getting urgently needed to produce industry-ready systems with the required levels of security and reliability;
- our study covers the studies published before April 2015; nevertheless, in this year 31 studies have been already published on CPS security, representing the 26.4% of the whole set of primary studies of our research; this result further confirms the growing attention and need of research on cyber-physical systems security; we expect that this growing trend will continue;

[1] See Subsection D.2 for details on selection strategy, which, of course, determined the results presented here.

- finally, we can notice that 117 (99.2%) out of the 118 selected studies were published during the last five years; this can be seen as an indication that CPS security is a relatively new area, which is gaining more and more traction from a scientific point of view; this observation is further strengthened by the fact that the highest slope is between 2013 and 2014, where the number of publications has more than doubled, going from 18 (15.3%) to 43 (36.4%).

Figure D.3 shows the distribution of targeted types of venues over the years (considering partial data for 2015). The most common publication types are journal and conference, with 59 (50.01%) and 50 (42.37%) of the primary studies, respectively. Book chapter and workshop are the least popular publication types, with only 6 (5.08%) and 3 (2.54%) studies falling into their categories, respectively. Such a high number of journal and conference papers on CPS security may indicate that cyber-physical systems security is becoming more and more a mature research theme, despite its relative young age (the first publication on CPS security was in 2009). Additional considerations on publication venues and on research institutions are provided in Section D.11.

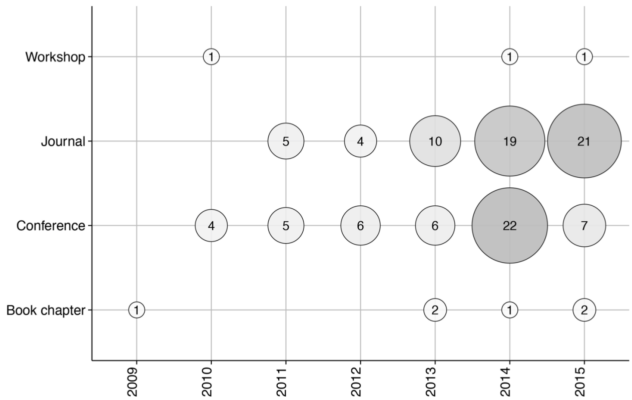

Figure D.3: Types of publications over the years

D.4 Results - Characteristics & focus of research (RQ2)

As already introduced in Subsection D.2, we identified a set of variables describing positioning and characterization of methods and techniques for cyber-physical systems security breaching and/or enforcing. With the purpose of evaluating what aspects of system are attacked or protected by an approach, in the following we indicate which application fields, points of view, security attributes, system components, plant models, state estimation and anomaly detection algorithms, controllers, communication aspects and network-induced imperfections are considered by each primary study. Furthermore, we give an account of attacks and their characteristics, attack and defense schemes, plant models used by an attacker and defense strategies, in order to understand how these methods and techniques are characterized.

CPS application field

As we can see from Figure D.4, 65 out of 118 primary studies are focused exclusively on power grids, which corresponds to the 55.08% of all selected studies. Among those, as shown in Figure D.5, 45 papers (i.e., 38.14% of all the selected studies) deal exclusively with power transmission, 8 studies address the security aspects of the electricity market ([S061-S068]), 3 studies are focused on power distribution ([S018, S032, S056]), 2 studies on power generation ([S005, S024]), and the remaining 7 on any combination of the previous ones ([S002, S013, S028, S030, S049, S050, S059]).

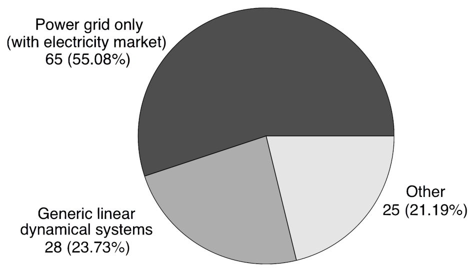

Figure D.4: Distribution of studies by application area

The second largest group of publications in Figure D.4 counts 28 works, i.e., 23.73% of the whole set of primary studies of our research. All these papers study the security of generic linear dynamical systems. The proposed approaches can be used in any suitable application. However, these works do not provide examples of a particular application.

The last group of the remaining 25 studies is detailed in Figure D.6. These works are almost uniformly distributed among the following applications: (unmanned) ground vehicles (UGV) accounting for 6 of primary studies ([S084, S097, S099, S106, S111, S115]); (unmanned) aerial systems (e.g. unmanned aerial vehicles, air traffic management systems) and hydro-systems relying on automatic control, both considered in 5 papers ([S082, S090, S093, S108, S114] and [S010, S072, S078, S081, S083], respectively); generic (linear and non linear) dynamical systems and linear dynamical systems with applications to power grids, both found in 4 studies ([S035, S080, S096, S113] and [S048, S079, S100, S118], respectively). It is worth noting that UGV-based systems deal with the navigation and control of teleoperated and autonomous ground vehicles, together with their supervisory control and vehicle

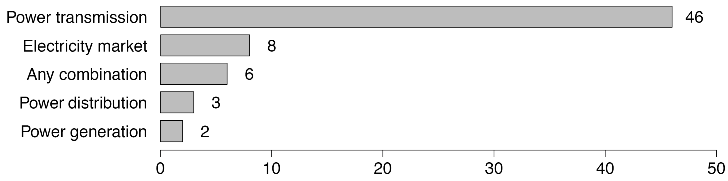

Figure D.5: Distribution of primary studies applied in power grids

platooning. Finally, the security of building automation applications is investigated in one primary study ([S088]).

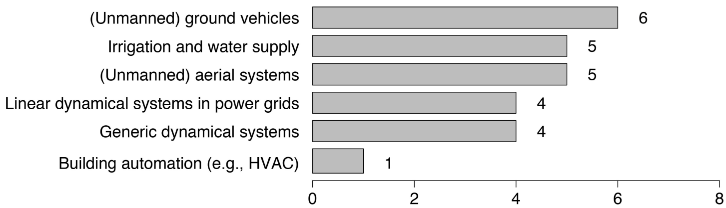

Figure D.6: Number of studies in other application fields

From the collected data, we can offer the following observations:

- the bulk of the selected works on security for cyber-physical systems is focused on power grids; this is not surprising, and may be due to the fact that smart grids are recognized as a driver for sustained economic prosperity, quality of life, and global competitiveness of a nation, attracting big research efforts to this area as a whole; also, the models used in this domain are well-known and the famous false data injection attack [S001] has been introduced in the context of power networks, giving traction to this kind of research applications. Moreover, the impressive market growth in renewable energy devices posed novel challenging problems in the design and management of power grids: as a consequence, the interest of energy providers on novel methods and technologies for optimizing network management with guaranteed performance, safety, and security provided a tremendous boost to academic research on these topics;

- only a small part of the selected papers presents the applications to the secure control of (unmanned) ground vehicles and aerial systems, and of heating, ventilation, and air-conditioning (HVAC), as well as lighting and shading, in large functional buildings; this application fields are relatively new for the approaches to the cyber-physical security, with the first studies appearing only in 2012; this result can be seen as indication of a potentially interesting direction for future research on CPS security;

- somehow surprisingly, we have not found any work focused on the cyber-physical security of medical CPS [149]. We suppose that the topics of physiological close-loop control and patient modeling are seen as not mature enough to consider the security aspects specific to this important application field from the control-theoretic point of view. In any case, we expect that these topics will be considered and addressed in the near future.

Point of view

As reported in Figure D.7[2], we distinguish primary studies based on whether they treat approaches for security breaching (i.e., *attack*) or enforcing via some kind of countermeasures (i.e., *defense*), or both. From our analysis it emerged that 62 studies over 118 focus exclusively on the various countermeasures that a CPS may put in place in response to an attack, whereas 28 studies (i.e., 22.88% of the total) focus exclusively on vulnerability analysis by proposing or improving an attack scheme using an adversary's point of view. They do not study the topic of the risk treatment, which is peculiar to the designer's or operator's perspective. The remaining 28 works treat both attack and defense strategies.

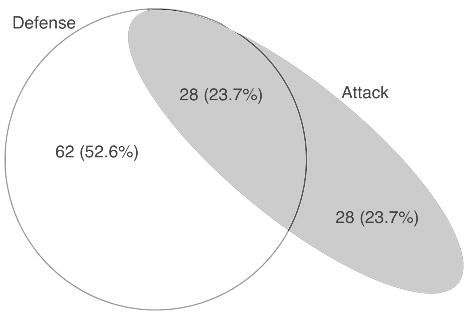

Figure D.7: Distribution by the adopted point of view

From this result we can observe that the defense strategies are presented in most (76.27%) of the selected studies, occupying the central spot of the research efforts on CPS security. A more detailed discussion of the various defense strategies proposed in research is provided in Subsection D.4.

[2]In this work we use area-proportional Euler diagrams for visualizing the distribution over parameters with multiple values in which the discussion of their intersections is relevant for this study.

Considered security attributes

Security can be seen as a composition of three main attributes, namely: confidentiality, integrity and availability [147]. Accordingly, we identified the security attributes considered by each primary study in order to understand how those attributes have been investigated by researchers on CPS security. Figure D.8 shows the distribution of the primary studies across confidentiality, integrity, and availability.

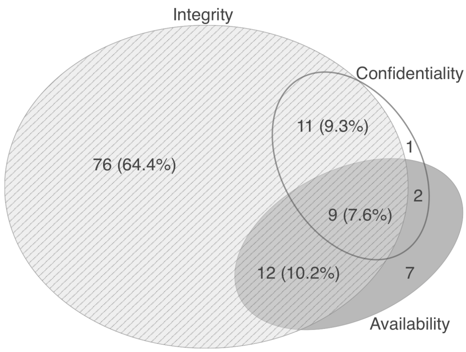

Figure D.8: Distribution by security attributes

The first thing that strikes the eye is that more than 90% of the works are concerned with CPS *integrity*, threatened by various types of deception attacks. Some of these works consider also the availability and/or confidentiality, together with integrity. On the contrary, only two studies ([S068, S105]) focus on the combination of solely *availability and confidentiality*; those papers apply game theory to the design of countermeasures to intelligent jamming attacks, which have been published between the fall 2014 and 2015. For further discussion of security attributes, see Subsection D.4.

System components

Each approach to security breaching or enforcing considers a particular set of system components to be compromised or protected. In our analysis we identified five main categories for describing the main system components to be compromised or protected, that are: sensors, actuators, network, controllers, plant. As an example, false data injection mainly targets a set of *sensors*, while load altering can attack a set of *actuators*. As for all deception and some disruption attacks, we should *"note that from a practical point of view, an attack on a sensor could either be interpreted as an attack on the node itself (making it transmit an incorrect signal), or it could also be interpreted as an attack on the communication link between the sensor and* *the receiver device; similarly an attack on an actuator could either be interpreted as an attack on the actuator itself, or on the communication link from the controller to the actuator"* [S079]. Thus, we say that an approach considers a *network* either when it does it implicitly by considering a denial-of-service (DoS) attack on communication links, or explicitly, by exploiting transmission scheduling, routing or some network-induced imperfections. Following the same line of reasoning, we say that the work takes into account a *controller* when it proposes a novel one, whereas the *plant* category comes into play with attacks at the physical layer and with eavesdropping.

Figure D.9 presents how system components have been considered among the primary studies. Sensors were taken into account 100 (84.75% of) times, 62 (52.54% of) times alone and 27 (22.88% of) times together with actuators. The actuators themselves were considered 33 (27.97% of) times, while network was taken into account in 29 (24.58% of) studies. This data suggests that the approaches considering attacks on sensors and their protection completely dominate the scene. All the other system components have received much less attention, with a slight predominance of actuators and network.

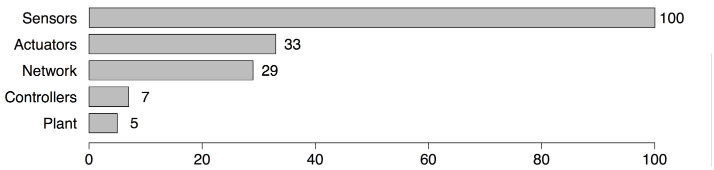

Figure D.9: Distribution of studies by system components

Plant model

We have seen in Subsection D.4 that the application domain of research on cyber-physical systems security is mainly divided between power grids and all the others. This result is reflected also in the choice of the mathematical models used to describe the physical domain.

In particular, power transmission is traditionally studied via a power flow model, which is a set of equations that depict the energy flow on each transmission line of a power grid. An AC power flow model considers both real and reactive power and is formulated by nonlinear equations, where the state variables are voltage magnitudes and phase angles of the buses [150, 151]. However, state estimation using an AC power flow model can be computationally expensive and does not always converge to a solution. Thus, power system engineers sometimes use a linearized power flow model, DC power flow model, to approximate the AC power flow model [S001]. In DC model the reactive power is completely neglected and state variables only consist of voltage phase angles of the buses. As of power generation, the model based on equations describing the electromechanical swing dynamics of the synchronous generators [152] is usually applied. In other application domains more general linear time invariant (LTI) or nonlinear dynamical models are used.

Figure D.10 shows how the above mentioned models have been used within the set of primary studies. The *DC approximation of power flow* has been used in 53 works (44.92% of whole set), while the more complicated and realistic *AC power flow model* (which is capable to capture more subtleties) has been studied 16 (13.56% of) times. In 6 studies both the AC power flow model and its linear DC approximation have been used ([S023, S028, S030, S051, S056, S057]). Other *LTI models* were applied in 51 (43.22% of) primary studies. *Nonlinear dynamic* and *swing-equation based models* were applied 13 (11.02%) and 7 (5.93 % of) times, respectively.

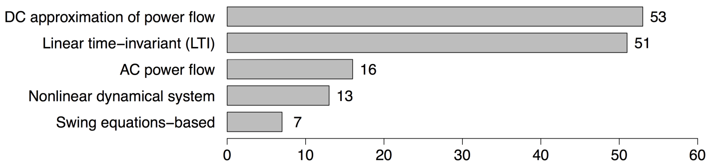

Figure D.10: Distribution of studies by plant model

Process noise

To capture any deviation in the plant model from the real dynamics of the controlled physical system, the process noise is used; from the primary studies it emerged that it can be categorized into 3 main classes: *Gaussian*, *bounded (non-stochastic)*, and *noiseless.*

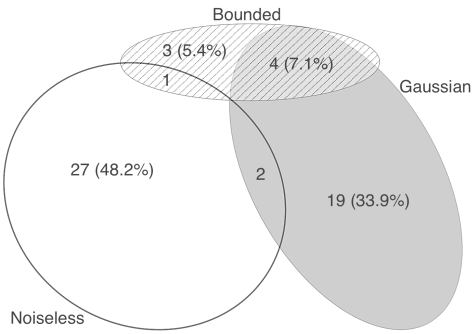

Figure D.11: Distribution of studies by process noise

The distribution of primary studies by process noise is reported in Figure D.11, where the studies considering the measurement model only (62, accounting for

52.54% of the whole set of selected papers) were not included, since for them the facet of process noise is not applicable. We can see that the noiseless and Gaussian process noise models are the most used ones (accounted 30 and 25 times, respectively). As shown in Figure D.12, the bounded non-stochastic model (used 8 times) is starting to receive a growing attention in the very last years.

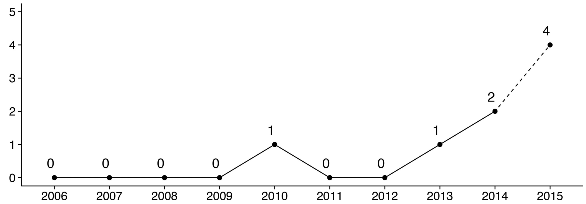

Figure D.12: Number of studies with bounded process noise year by year (partial data for 2015)

Measurement noise

Depending on the assumptions on the noise, sensor measurement models can be broadly categorized into three classes: *Gaussian*, *bounded (non-stochastic)* and *noiseless* [S116]. As shown in Figure D.13, the majority of primary studies (78, i.e., 66.10%) uses Gaussian measurement noise model; while 38 (32.20% of all) works assume noiseless measurements. Only 8 works have used bounded (non-stochastic) assumptions. Similarly for the bounded process noise, the bounded measurement noise has started to gain attention only recently in the CPS security domain, as we can see from Figure D.14.

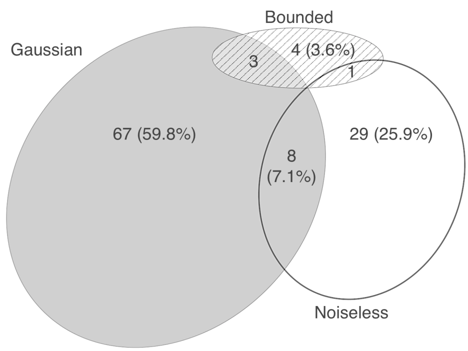

Figure D.13: Distribution by measurement noise

If a study does not consider the measurement model (e.g., when the work is not related to the secure state estimation against sensor attacks), we say that the

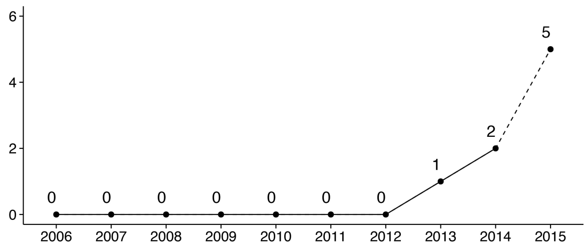

Figure D.14: Number of studies with bounded measurement noise year by year (partial data for 2015)

measurement noise is not applicable. Among the selected primary studies there were 6 such studies.

State estimation

For many situations, it may be unrealistic or unfeasible to assume that all the states of the system are measured. In fact, 89 studies were using some kind of state estimation (SE), which corresponds to 75.42% of all the primary studies (see Figure D.15). The most used SE method is *weighted least squares* (WLS), found in 54 (45.76% of all) works (interestingly, all 54 studies were related to power grids). The WLS method for power system SE is optimal under Gaussian measurement noise [S057] and, in case of DC approximation of power flow, leads to an estimator identical to the one obtained with maximum likelihood or with minimum variance methods [S001]. The (extended) *Kalman filter* was used in 21 studies (17.80% of all primary studies), while the (extended) *Luenberger observer* was used in 10 studies (8.47%), the H_∞ *filter* in 2 studies ([S087, S093]) and the *least trimmed squares* estimator in only one study ([S057]). Novel solutions for the SE were proposed in 17 (14.41%) studies.

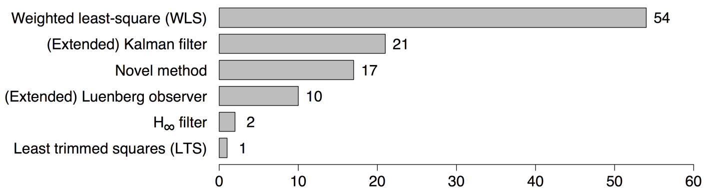

Figure D.15: Distribution of primary studies by state estimation (SE)

Novel methods range from application-specific solutions [S024, S072], distributed state estimation techniques for power networks [S011, S014, S025], to generic attack-resilient solutions inspired by Kalman filter [S091, S106, S116].

Within the domain of power grids, Giani et al. [S015] proposes SE based countermeasures to coordinated sparse attacks on power meter readings, that take advantage of graph-theoretic construct of *observable islands*, which are disjoint subsets of buses sharing the same perceived change of state [voltage phase] under the attack. As a countermeasure to leverage point attacks against WLS SE in smart grid, Tan et al. [S049] introduces a modified robust Schweppe-Huber Generalized-M estimator. The WLS estimation method for power networks has been extended by Liu et al. [S054] by merging cyber impact factor matrix into the state estimation as a reasonable adjustment of the weight values, in order to create the abnormal traffic-indexed SE.

Regarding generic CPS, to estimate the state of the plant despite attacks on sensors and actuators, Fawzi et al. [S079] propose an efficient state reconstructor inspired from techniques used in compressed sensing and error correction over the real numbers. Pajic et al. [S099] show that implementation issues such as jitter, latency and synchronization errors can be mapped into parameters of the SE procedure that describe modeling errors, and provides a bound on the SE error caused by modeling errors. Mo and Sinopoli [S096] constructs an optimal estimator of a scalar state that minimizes the "worst-case" expected cost against all possible manipulations of measurements by the attacker, while Weimer et al. [S102] introduces a minimum mean-squared error resilient (MMSE-R) estimator for stochastic systems, whose conditional mean squared error from the state remains finitely bounded and is independent of additive measurement attacks.

Lastly, for linear dynamical systems under sensor attacks, Shoukry and Tabuada [S111] present an efficient event-triggered projected Luenberger observer for systems under sparse attacks, and Shoukry et al. [S117] develop an efficient algorithm that uses a satisfiability modulo theory (SMT) approach to isolate the compromised sensors and estimate the system state despite the presence of the attack.

Together, these results are an indication that the resilient state estimation under measurement attacks is a very active research topic within the area of CPS security, making us reasonably confident about its future development and potential.

Anomaly detector

Current state estimation algorithms use bad data detection (BDD) schemes to detect random outliers in the measurement data [S006]. Two of the most used BDD hypothesis tests are the *performance index test* (also known in power system's community as *$J(\hat{x})$-test* or *χ^2-test*) and the *largest normalized residual test* (often referred as *r^N_{max}-test*) [150]. As shown in Figure D.16, among our primary studies there are 58 approaches considering performance index test, 22 approaches dealing with normalized residual test, and 13 considering both aforementioned hypothesis tests.

Also, two studies consider an arbitrary anomaly detector implemented by the controller and deployed to detect possible deviations from the nominal behavior

[S081, S101], while 36 (30.51%) primary studies do not deal at all with anomaly detection.

In an effort to minimize the detection delay, the change detection can be formulated as a quickest detection problem. Page's cumulative sum (CUSUM) algorithm is the best-known technique to tackle this type of problem. There are 5 selected primary studies, that propose or use a CUSUM-based attack detection schemes [S007, S016, S035, S060, S075]. There are also 26 (22.03%) studies, that propose other novel anomaly detection approaches, either considering them together with the performance index test or normalized residual test.

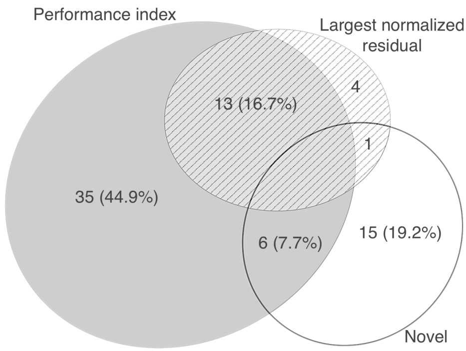

Figure D.16: Distribution of primary studies by anomaly detection

The novel solutions for bad data detection cover the topics of distributed monitoring [S010, S011, S014, S029] and application-specific anomaly detection for multi-agent distributed flocking formation control [S024], automated cascade canal irrigation systems [S072], wireless control networks, *"where the network itself acts as the controller, instead of having a specially designated node performing this task"* [S074], multi-hop control networks, *"where the communication between sensors, actuators and computational units is supported by a (wireless) multi-hop communication network and data flow is performed using scheduling, routing and network coding of sensing and actuation data"* [S088], and air transportation systems [S108].

In the power system domain, Kosut et al. [S002] proposes a generalized likelihood ratio detector, that incorporates historical data and does not compute explicitly the residue error, while Gu et al. [S058] introduces a new method to detect false data injection attacks against AC state estimation by tracking the dynamics of measurement variations: the Kullback–Leibler distance (KL divergence, known also as relative entropy) is used to calculate the distance between two probability distributions derived from measurement variations.

The KL divergence is adopted also by Mo et al. [S070, S112] in designing the optimal watermark signal in the class of stationary Gaussian processes, which is used to derive the optimal Neyman–Pearson detector of reply and covert attacks, respectively.

Valenzuela et al. [S031] use principal component analysis (PCA) to separate power flow variability into regular and irregular subspaces, with the analysis of the information in the irregular subspace determining whether the power system data has been compromised. Also Liu et al. [S033] views false data detection as matrix separation problem and, differently from the case of the PCA, proposes algorithms that exploit *"the low rank structure of the anomaly-free measurement matrix, and the fact that malicious attacks are quite sparse."*

Tiwari et al. [S097] propose an approach inspired by PCA, that uses an invariant *" — an over-approximation of the reachable states — of the system under normal conditions as the classifier";* this set is called the safety envelope. An alarm is raised whenever the system state falls outside the safety envelope.

Security-oriented cyber-physical state estimation (SCPSE) for power grid, proposed in Zonouz et al. [S026], uses stochastic information fusion algorithms on *"information provided by alerts from intrusion detection systems that monitor the cyber infrastructure for malicious or abnormal activity, in conjunction with knowledge about the communication network topology and the output of a traditional state estimator",* in order to detect intrusions and malicious data, and to assess the cyber-physical system state.

Other novel anomaly detection methods in power grid comprise a detector implementing the Euclidean distance metric [S048], and a cosine similarity matching based approach [S055]. It is worth noting that the second one requires the usage of the Kalman filter as a source of estimated/expected data.

To contrast false data injection attacks, Sedghi and Jonckheere [S034] present a decentralized detection and isolation scheme based on the Markov graph of the bus phase angles, obtained via conditional mutual information threshold (CMIT) test, while Sou et al. [S020] introduces a scheme, that considers potentially compromised information from both the active and the reactive power measurements on transmission lines. In this second scheme, based on the novel reactive power measurement residual, *"the component of the proposed residual on any particular line depends only locally on the component of the data attack on the same line".* Li and Wang [S040] presents the state summation detection using state variables' distributions, which tests hypothesis on true measurement square sum S_x (assumed to follow normal distribution, given a large number of state variables) together with test on $J(\hat{x})$. Finally, Sanandaji et al. [S041] presents a heuristic for detecting abrupt changes in the system outputs based on the singular value decomposition of a history matrix built from system observations.

For dissipative or passive CPS, Eyisi and Koutsoukos [S098] propose energy-based attack detection monitor.

To contrast stochastic cyber-attacks, Li et al. [S107] presents an algebraic detection scheme based on the frequency-domain transformation technique and linear algebra theory, together with sufficient and necessary conditions guaranteeing the detectability of such attacks.

Pasqualetti et al. [S010] characterizes fundamental monitoring limitations of descriptor systems from system-theoretic and graph-theoretic perspectives, and designs centralized and distributed monitors, which are complete, in the sense that they detect and identify every (detectable and identifiable) attack.

Finally, Jones et al. [S113] presents an automated anomaly detection mechanism based on inference via formal methods to develop an unsupervised learning algorithm, which constructs from data a signal temporal logic (STL) formula that describes normal system behavior. Trajectories that do not satisfy the learned formula are flagged as anomalous.

As a general comment, the literature described in this section appears quite fragmented, and a systematic high level view is still missing even within a specific application domain. The different results and methodologies are very difficult to relate each other and validate since both a comparison metric and a benchmark, neither academic nor industrial, have not been agreed and defined yet.

Controller

Considering the used controller, the first fact emerging from our analysis is that studies focusing on state estimation usually do not examine at all the controller. In fact, in 82 (69.49% of 118 selected) studies the controller is not available. In the remainder of this section we will focus on the remaining 36 studies, some of which consider more than one controller at once.

As shown in Figure D.17, the most considered controllers are generic state feedback or output feedback controllers with a control law restricted to be linear time invariant, found in 13 studies, together with linear quadratic regulators (LQR) and H_∞ (minimax) controllers, each of which is seen in 12 works. The variations of proportional-integral-derivative (PID) controller are considered in 7 works, while the event-triggered and self-triggered controllers can be found in 3 studies [S085, S103, S111], and sliding mode controllers in 2 studies [S013, S115].

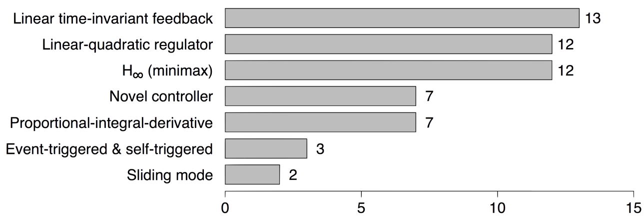

Figure D.17: Distribution of studies by controller

Interestingly, seven primary studies ([S024, S069, S073, S076, S077, S086, S093]) propose novel controllers. More specifically, inspired by the analogy to flocking behavior, Wei and Kundur [S024] developed distributed hierarchical *"control methodologies that leverage cooperation between distributed energy resources and traditional synchronous machines to maintain transient stability in the face of severe disturbances".* For a class of denial-of-service (DoS) attack models, Amin et al. [S069] presents an optimal minimax causal feedback control law, subject to the power, safety and security constraints. Gupta et al. [S073] studies a similar problem of optimal minimax control in the presence of an intelligent jammer with limited actions as dynamic zero-sum game between the jammer and the controller. Befekadu et al. [S076] introduces instead the *"measure transformation technique under which the observation and state variables become mutually independent along the sample-path (or path-estimation) of the DoS attack sequences in the system"*, thanks to which it derives the optimal control policy for the risk-sensitive control problem, under a Markov modulated DoS attack model. Zhu and Martínez [S077] proposes a variation of the receding-horizon control law to deal with the replay attacks, while Zhu et al. [S086] provides a set of coupled Riccati differential equations characterizing feedback Nash equilibrium as the solution concept for the distributed control in the multi-agent system environment subject to cyber attacks and malicious behaviors of physical agents. Finally, Kwon and Hwang [S093] proposes *"a hybrid robust control scheme that considers multiple sub-controllers, each matched to a specific type of cyber attacks"*, together with a method for designing the corresponding secure switching logic.

As a general comment, the literature described in this section derives interesting theoretical results, but there is still a lot of work to do for addressing the practical challenges in CPS security.

Communication aspects and network-induced imperfections

The introduction of the communication network in a control loop modifies the external signals of the plant and the controller due to the network-induced imperfections [97], which in turn depend on some communication aspects, such as transmission scheduling and routing.

When analyzing the primary studies on the basis of this facet we got a surprise: 100 out of 118 studies (i.e., 84.75%) do not explicitly consider any communication aspect or imperfection, while only 6 studies (i.e., 5.08%) address more than one aspect. The total number of times each communication aspect was addressed within the set of the primary studies is shown in Figure D.18.

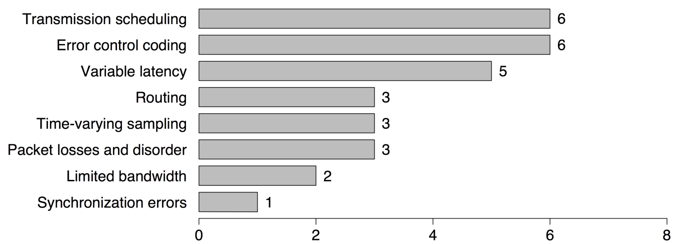

Figure D.18: Distribution by communication aspects and network-induced imperfections

Synchronization errors are considered only by Pajic et al. [S099], where also variable latency and time-varying sampling are mapped into parameters of the state estimation procedure that describe modeling errors. Time-varying sampling is taken into account also by Yilmaz and Wang [S060] and, together with transmission scheduling, by De Persis and Tesi [S103]. Limited bandwidth is considered together with error control coding by Gupta et al. [153] (which is related to [S073]), and by Sundaram et al. [S074], in which *"nodes in a network transmit linear combinations of incoming packets rather than simply routing them"*. Packet losses and disorder alone is taken into consideration in two works ([S091, S118]) and together with variable latency and transmission scheduling in another one ([S087]). Routing by itself is examined by Vuković et al. [S022], and together with error control coding, transmission scheduling and variable latency, by D'Innocenzo et al. [S088]. Only variable latency is considered by Miao and Zhu [S094] and by Jones et al. [S113]. Both error control coding and transmission scheduling by themselves are taken into account in 3 works ([S079, S109, S110] and [S085, S095, S104], respectively).

Surprisingly, very few papers (attempt to) provide non-trivial mathematical models of the communication protocol, which indeed is a fundamental actor of almost any CPS. In particular, only in D'Innocenzo et al. [S088] a specific standard for communication, i.e., WirelessHART and ISA-100, is explicitly considered in the CPS mathematical model.

Attacks and their characteristics

Regardless of the adopted point of view (see Subsection D.4), every study on CPS security deals with attacks in order to either implement or to counteract them. Each attack threats one or more primary security attributes (see Subsection D.4). More specifically, the best known attack on availability is the *denial of service* (DoS) attack, that renders inaccessible some or all the components of a control system by preventing transmissions of sensor or/and control data over the network. *"To launch a DoS an adversary can jam the communication channels, compromise devices and prevent them from sending data, attack the routing protocols, flood with network traffic some devices, etc." [S069].* Attacks on data integrity are known as *deception* attacks and represent the largest class of attacks on cyber-physical systems, including false data injection attacks. The attacks on confidentiality alone are often referred to as *disclosure* attacks, i.e., *eavesdropping*, which is discussed only in two studies [S081, S084].

Figure D.19 shows the distribution of attacks within the set of our primary studies. The false data injection, together with generic deception and DoS, with 57, 33 and 20 occurrences respectively, accounts for 74.8% of all considered attacks, while the variable structure switching, the packet scheduling, and the bias injection attacks are considered only once.

Characterization of the attacks. Generally speaking, an attack on control systems can be characterized by the amount of available resources and knowledge [S081]. The resources of an adversary can be split in *disclosure* resources, which

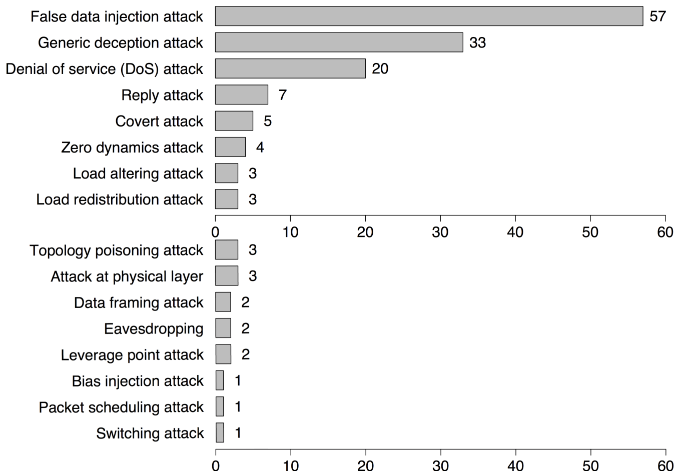

Figure D.19: Distribution of attacks considered by primary studies

enable her to obtain sensitive information about the system during the attack by violating data confidentiality, and *disruption* resources, that affect the system operation by compromising the integrity and/or availability. The amount of a priori *knowledge* regarding the control system is another core component of the adversary model, as it may be used, for instance, to render the attack undetectable. In the rest of Subsection D.4 we describe the characteristics of each type of attack individually.

In the **bias injection** attack, considered only by Teixeira et al. [S081], the adversary's goal is to inject a constant bias in the system without being detected. *No disclosure* capabilities are required for this attack, since the attack policy is open-loop. The data corruptions may be added to both the actuator and sensor data, and the amount of disruption resources should be *above the threshold of undetectability*[3]. Furthermore, the open-loop attack policy requires an extensive knowledge of the parameters of considered closed-loop system and anomaly detector.

In the coordinated **variable structure switching** attack and its extension to multi-switch attack considered in the work of Liu et al. [S013], an opponent controls multiple circuit breakers within a power system, and employs a local model of the system and local state information (i.e., some knowledge of the target generator states, which are rotor angle and frequency) to design a state-dependent breaker switching sequence, that destabilizes target synchronous generators.

[3]In other words, the attacker should have enough resources to construct an unobservable attack; a good example of the amount of disruption resources above the threshold of undetectability in the context of power transmission networks is given by the security index [154], defined as minimum number of measurements an attacker needs to compromise, in order to attack measurement k without being detected.

The attack on the **scheduling** algorithm influences the temporal characteristics of the network, as *"it results in time-varying delays and data packets possibly received out-of-order"* [S087]. To remain stealthy, the attacker is not able to delay the packets beyond a maximum allowable delay consistent with the network protocol in place. On the system level, this attack does not require any a priori knowledge of the system model, nor any disclosure resources.

The **false data injection** is a specific deception attack on state estimation, introduced in the context of electric power grids by Liu et al. [S001]. This attack on cyber-physical systems is the most studied one. To perform it, an adversary with some knowledge of the system topological information manipulates sensor measurements in order to change the state variables, while bypassing existing bad data detection schemes. This attack is based on the open-loop policy and does not require any disclosure resources. To construct the attack vectors, a common assumption in most works on false data injection attacks on power system state estimation is that the attacker has complete knowledge about the power grid topology and transmission-line admittances. This information is abstracted in the Jacobian matrix $\mathbf{H}$ [150, 155], known also as measurement or (power network) topology matrix. By contrast, Teixeira et al. [S006] assumes the attacker only possesses a perturbed model of the power system, *"such a model may correspond to a partial model of the true system, or even an out-dated model"*[S006]. In this way it quantifies a trade-off between the accuracy of the model known by adversary and possible attack impact for different BDD schemes, showing that *"the more accurate model the attacker has access to, the larger deception attack he can perform undetected"* [S006]. Similarly, Rahman and Mohsenian-Rad [S027] argues that *"a realistic false data injection attack is essentially an attack with incomplete information due to the attackers lack of real-time knowledge with respect to various grid parameters and attributes such as the position of circuit breaker switches and transformer tap changers and also because of the attacker's limited physical access to most grid facilities",* and presents a vulnerability measure for topologies of power grids subject to attacks based on incomplete information. On the same line, Bi and Zhang [S017] derives a necessary and sufficient condition to perform undetectable false data injection attack with partial topological information and develops a min-cut method to design the optimal attack, which requires the minimum knowledge of system topology.

Finally, the problem of constructing a blind false data injection attacks without explicit prior knowledge of the power grid topology is studied by Esmalifalak et al. [S012], Kim et al. [S051], and Yu and Chin [S052]. In Esmalifalak et al. [S012] attackers try to make inferences through phasor observations applying linear independent component analysis (ICA) technique. However, such technique requires that loads are statistically independent and non-Gaussian, and the technique need full sensor observations [S051]. Kim et al. [S051] instead proposes subspace methods, which requires no system parameter information. In this case the attack can be launched with only partial sensor observations. Yu and Chin [S052] proposes to use principal component analysis (PCA) approximation method without the assumption regarding the distribution of state variables, to perform the same task of making inferences from the correlations of the line measurements, in order to construct the blind false data injection attack.

Differently from the works on undetectable false data injection attacks on power grids summarized up to here, Qin et al. [S036] presents an *unidentifiable* version of this attack, in which the control center can detect that there are bad or malicious measurements, but it cannot identify which meters have been compromised.

A special type of false data injection attack on electric power grid is the **load redistribution** attack, in which only load bus power injection and line power flow measurements are attackable [S008]. It consists in increasing load at some buses and reducing loads at other buses, while maintaining the total load unchanged, in order to hide the attack from bad data detection. The construction of load redistribution attack relies on topological information of the network, that can be derived from the Jacobin matrix $\mathbf{H}$. Considering the practical issue that an attacker can only obtain the parameter information of a limited number of lines, Liu et al. [S043] presents a strategy to determine optimal local attacking region, that requires the minimum network parameter information. The undetectability is obtained by *"making sure that the variations of phase angles of all boundary buses connected to the same island of the nonattacking region are the same"* [S043].

The **data framing** attack is a deception attack on power system state estimation that exploits current bad data detection and removal mechanisms. It purposely triggers the bad data detection mechanism and frames some normally operating meters as sources of bad data such that their data will be removed. After such data removal, although the remaining data appear to be consistent with the system model, the resulting state estimate may have an arbitrarily large error [S037]. Also this attack does not require any disclosure resources, since the attack policy is open-loop. By applying the subspace methods presented in 2015 by Kim et al. [S051] to learn the system operating subspace from measurements, the data framing can be performed without knowledge of the Jacobian matrix $\mathbf{H}$. A limited a priori knowledge required consists of a basis matrix $\mathbf{U}$ of a subspace of all possible noiseless measurements $\mathcal{R}$ of $\mathbf{H}$.

The **leverage point** attack is a deception attack which creates leverage points within the factor space of the (power system) state estimation regression model [S049]. The residual of the measurement corresponded with the leverage point is very small even when it is contaminated with a very large error. Thus the adversary can freely introduce arbitrary errors into the meter measurements without being detected. This attack is based on an open-loop policy and thus does not require disclosure resources. However, to be fully effective, it requires a complete knowledge of the Jacobian matrix $\mathbf{H}$ and amount of disruption resources above the threshold of undetectability [S057].

The **load altering** attack against power grid's demand response and demand side management programs can bring down the grid or cause significant damage to the power transmission and user equipment. It consists in an attempt to control and change (usually increase) certain load types in order to damage the grid through circuit overflow or disturbing the balance between power supply and demand [S018]. The *static* load altering is mainly concerned in changing the volume of the load. Here the attacker without any prior knowledge of the plant model uses some historical data to impose a pre-programmed trajectory to the victim load (an open-loop policy). In the more advanced *dynamic* load altering attack, presented in 2015 by Amini et al. [S050], the adversary *"constantly monitors the grid conditions through the attacker's installed sensors so that it can adjust the attack trajectory based on the current conditions in the power grid"* [S050]. With this closed-loop policy, the attacker having a complete knowledge of the plant's model controls the victim load based on a feedback from the power system frequency and can make the power system unstable, without the need for increasing the scope or volume of the attack, compared to a static scenario.

The attacks at **physical layer** range from attacks that affect both the physical infrastructure and the control network (of power grids) [S053] to attacks through physical layer interactions, such as an attack on vehicle platoon traveling at a constant speed, presented by Dadras et al. [S115]. The attack studied by Soltan et al. [S053] physically disconnects some power lines within the attacked zone (which is defined as a set of buses, power lines, phasor measurement units (PMUs) and an associated phasor data concentrator (PDC) [155]) and disallows the information from the PMUs within the zone to reach the control center. This attack does not require any knowledge of the plant model, nor disclosure resources. The attack on vehicle platoons [S115] is carried out by a maliciously controlled vehicle, who attempts to destabilize or take control of the platoon by combining changes to the gains of the associated law with the appropriate vehicle movements. This closed-loop attack *"bears some resemblance to an insider version of the replay attack of [S010], in that the attacker is part of the CPS and is therefore able inject control inputs legitimately"*.

In **topology poisoning** attack an adversary covertly alters data from certain meters, network switches and line breakers to mislead the control center with an incorrect network topology. Kim and Tong [S028] shows that under certain conditions even in a local information regime, where the attacker has only local information from those meters it has gained control, undetectable topology poisoning attacks exist and can be implemented easily based on simple heuristics. Deka et al. [S039] proves that grids completely protected by secure measurements are also vulnerable to hidden topology poisoning attacks, if the adversary armed only with generic information regarding the grid structure can corrupt the breaker statuses on transmission lines and jam the communication of flow measurements on the attacked lines.

The **zero dynamics** attack, first considered in [156, 157], is one in which an adversary constructs an open-loop policy such that the attack signal produces no output. In other words, *"these attacks are decoupled from the plant output y_k, thus being stealthy with respect to arbitrary anomaly detectors"* [S081]. For an attacker with limited disruption resources, zero dynamics attacks are based on the perfect (local) knowledge of the plant dynamics. In this setting, Teixeira et al. [S083] shows that zero-dynamics attacks may not be completely stealthy since they require the system to be at a non-zero initial condition; however for the subset of attacks exciting unstable zero-dynamics, the effect of initial condition mismatch in terms of the resulting increase in the output energy can be made arbitrarily small while still affecting the system performance. We should notice that an adversary capable of changing all the measurements can, of course, force the system's output to zero without any knowledge of the model, initial state and nominal input. Furthermore, for a linear not left-invertible system, the knowledge of the initial state is not required, because an attacker can exploit the kernel of the transfer matrix and the linearity of the system.

With the covert attack, also known as a **covert misappropriation** of the plant [S078], an adversary can gain control of the plant in a manner that cannot be detected by the controller. This attack requires high levels of system knowledge and the ability of attacker to both read and replace communicated signals within the control loop, indeed *"the covert agent is assumed to have the resources to read and add to both the control actuation commands and the output measurements. In practice, this could also be accomplished by augmenting the physical actuators or modifying the sensors. Examples of such modifications include installing a controlled-flow bypass around a sluice gate in an irrigation system and connecting a controlled voltage source between a voltage measuring device and its intended connection point in an electrical network. Another potential mode of attack would involve corrupting the PLCs used by the nominal controller to implement the control and sensing operations"* [S078]. Pasqualetti et al. [S010] observe that the covert attack can be seen as a feedback version of the replay attack, while Smith [S078] examines also the effects of lower levels of system knowledge and nonlinear plants on the ability to detect a covert misappropriation of the plant.

The **replay** attack is a deception attack (possibly combined with a physical attack), in which an adversary first gathers sequences of measurement and/or control data, and then replays the recorded data while injecting an exogenous signal into the system [S081]. The adversary requires no knowledge of the system model to generate stealthy outputs. However, the attacker needs to have *"enough knowledge of the system model to design an input that may achieve its malicious objective, such as physically damaging the plant"* [S070]. The model of this attack is inspired by the Stuxnet [158].

A **generic deception** attack is an attack on data integrity, where an adversary sends false information from (one or more) sensors or/and controllers in order to deceive a compromised system's component into believing that a received false data is valid or true [S071]. Usually it is modeled as an arbitrary additive signal injected to override the original data. Since generic deception attacks can be used to represent also other, more specialized deception attacks, they are considered mostly in the studies adopting the defender's point of view, presented in Subsection D.4.

There are 23 (19.49% of all) studies using a generic deception attack model only to develop some defense strategy. The remaining 10 primary studies present (generic) deception attacks, that are different from any other attack considered above. Vrakopoulou et al. [S005] deals with a cyber-attack on the automatic generation control (AGC) signal in multi-area power system as a controller synthesis problem, where the objective is to drive the system outside the safety margins. It investigates two cases according to whether the attacker has perfect model knowledge or not, and provides different alternatives for attack synthesis, ranging from *"open loop approaches, based on Markov chain Monte Carlo (MCMC) optimization, to close loop schemes based on feedback linearization and gain scheduling"* [S005].

Always within power grids' application domain, Vuković and Dán [S029] consider a sophisticated adversary, that knows the system model and aims to disable the state-of-the-art distributed state estimation by preventing it from converging. To this end, he or she compromises the communication infrastructure of a single control center in an interconnected power system, in order to manipulate the exchanged data (i.e., state variables) used as an input to the state estimator. The stealthy cyber attacks that maximize the error in unmanned aerial systems' state estimation are studied in Kwon et al. [S082]. To consider the worst-case security problem, this study assumes the attacker has the perfect knowledge on the system model and can compromise sensors and/or actuators.

The attacks on both sensors and actuators by the adversary with a perfect knowledge of the static parameters of a CPS (modeled as a discrete LTI system equipped with a Kalman filter, LQG controller and χ^2 failure detector) are considered also by Mo and Sinopoli [S071], where the adversary's strategy is formulated as a constrained control problem. Djouadi et al. [S100] instead present optimal sensor signal attacks for the observer-based finite and infinite horizon linear quadratic (LQ) control in terms of maximizing the corresponding cost functions. Also this study assumes full-information, i.e., the system parameters are known to the adversary.

Zhang et al. [S104] studies stealthy deception attacks on remote state estimation with communication rate constraints. Here the deception attacker intrudes the sensor, learns its online transmission strategy and then modifies the event-based sensor transmission schedule, in order to degrade the estimation quality. For the domain of electricity market, Jia et al. [S062] studies the average relative perturbation of the real-time locational marginal price as an optimization problem; the adversary is assumed to have not only the perfect knowledge of the system model, but also the possibility to access the measurement values in real-time, in order to inject bad data that is state independent, partially adaptive, or even fully adaptive.

A stealthy deception scheme capable of compromising the performance of the automated cascade canal irrigation systems is presented by Amin et al. [S072]. This attack scheme is based on approximate knowledge of canal hydrodynamics and is implemented via switching the linearized shallow water partial differential equation parameters and proportional boundary control actions, to withdraw water from the pools through offtakes. Similarly, the stealthy deception attacks on process control systems performed by a very powerful adversary with knowledge of the exact linear model of the plant, the parameters of anomaly detector and control command signals, are presented by Cárdenas et al. [S075]. In the most sophisticated attack considered in this study, adversaries *"try to shift the behavior of the system very* *discretely at the beginning of the attack and then maximize the damage after the system has been moved to a more vulnerable state"* [S075].

Finally, for a single-input single-output plant, Bai et al. [S101] analytically characterizes an optimal stealthy attack strategy, that maximizes the estimation error of the Kalman filter by tampering with the control input, as a function of the system parameters, noise statistics and information available to the attacker.

From such literature a systematic characterization of "types" of attack is emerging, even if the "generic deception attack" and "false data injection attack" have been primarily addressed.

Attack scheme

In this section we distinguish the selected studies based on whether they consider centralized, distributed or local attack strategies. The distribution of studies based on this facet is shown in Figure D.20.

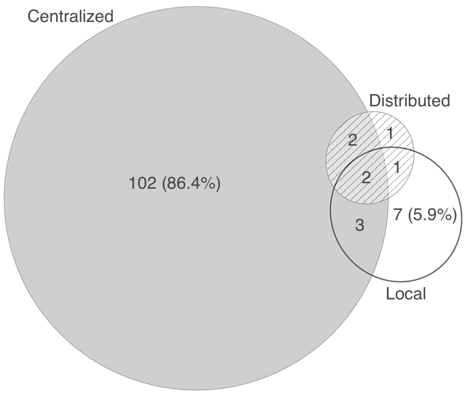

Figure D.20: Distribution of studies by attack scheme

The overwhelming majority of primary studies (102, 86.44%) considers only near omniscient adversary, capable of compromising several system components in a centralized fashion, while there are only 6 (5.17%) studies that study distributed attacks ([S010, S013, S014, S024, S025, S086]), and 13 (11.02%) studies dealing with local attacks ([S005, S013, S019, S025, S028, S029, S043, S044, S074, S084, S086, S104, S115]).

It is clear from this data that distributed and local solutions require more attention.

Plant model used by the attacker

This facet characterizes a modeling framework used by an adversary to design an attack on a CPS. Since attacker's knowledge of the control system and plant model can be limited or absent, an adversary may rely on a model of plant that is different from the actual model used by a system operator. Here our focus is on such cases, Figure D.21 shows the distribution of the primary studies by plant model used by an attacker.

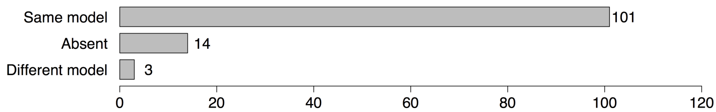

Figure D.21: Distribution of primary studies by plant model used by an attacker

In 101 studies (85.59%) it is assumed that the attacker uses the same model of the plant as the system operator, while in 14 studies (11.86%) the adversary does not use any model of plant. In the remaining 3 studies (2.54%) the attacker uses a model of plant that is simpler than the one used by operator. In particular, in the works of Kim et al. [S037, S051] data framing attacks on power transmission system are designed using a linearized system. It is shown that such attacks can successfully perturb a nonlinear *"state estimate, and the attacker is able to control the degree of perturbation as desired"* [S037]. This is an answer on the question on *"whether attacks constructed from a linear model is effective in a nonlinear system"* [S051]. Liang et al. [S044] studies both DC and AC attack models to construct the false data injection in AC state estimation, showing that the DC attack is detectable when the injected values are too large, while the AC attack model permits to *"hide the attack completely"* [S044].

Defense scheme

Similarly to attack schemes, we differentiate the studies also based on whether the proposed approach to defend a CPS focuses on the local or global scale of the system. In case of the global scale, this dimension also specifies whether a defense mechanism uses centralized or distributed coordination model.

We recall from Subsection D.4 that there are 28 primary studies adopting only an adversary's point of view and not concerned with countermeasures against attacks. We say that for them the defense schemes are not available. The distribution of remaining (90, i.e., 76.27% of all) primary studies by defense scheme is shown in Figure D.22.

Most of the studies (74) on defense mechanisms uses only centralized scheme, while the local scale is considered only in 4 works ([159] and [160], related to [S010] and [S013], respectively, together with [S020], where also the centralized scheme is taken into account, and [S105]). Distributed approaches are examined in

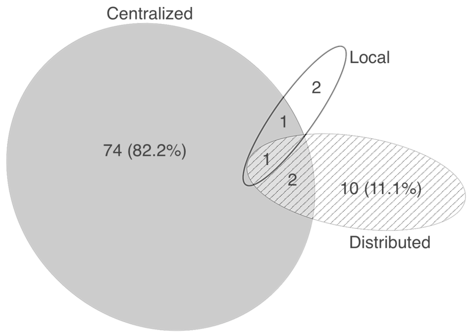

Figure D.22: Distribution of studies by defense scheme

13 works (alone in [S011, S014, S024, S029, S034, S084, S086, S100, S108, S110] and together with centralized ones in [S010, S025, S060]). We must point out that according to our selection strategy we do not consider the studies focused on the typical distributed problem of reaching consensus in the presence of malicious agents [156, 157]; this is because in these works the dynamics is part of the consensus algorithm and can be specifically designed, rather than being given as in a physical system [S058].

This data suggests that distributed and local defense solutions require more attention.

Defense strategy

We have already anticipated in Subsection D.4 that countermeasures against attacks, i.e., actions minimizing the risk of threats, are presented in more than three-fourth of primary studies, and occupy the central spot of the research efforts. The defense strategies can be classified as prevention, detection, and mitigation [161]; following the line of the fault diagnosis literature [106], we advocate isolation as a further defense strategy extending detection approaches.

Prevention aims at decreasing the likelihood of attacks by reducing the vulnerability of the system [161]. It brings together all the actions performed **offline**, before the system is perturbed or attacked. There are 43 studies (36.44%) studying prevention mechanisms. These studies range from security metrics for the vulnerability analysis of systems or their critical components to design and analysis of resilient state estimators and controllers capable to withstand some attacks, and protection-based approaches aiming to identify and secure some strategic distributed components. Figure D.23 shows the distribution of the primary studies focussing on prevention.

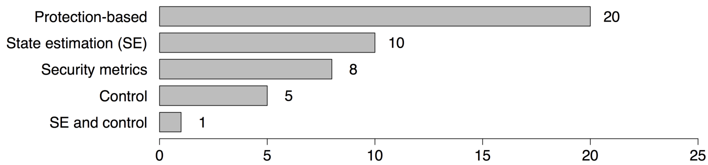

Figure D.23: Distribution of primary studies by prevention approach

Twenty studies present **protection-based** approaches. Among them, 6 studies discuss the *secure sensor allocation* against undetectable false data injection attacks in power transmission networks. More specifically, Bobba et al. [S003] show that it is necessary and sufficient to protect a set of basic measurements (in number equal to number of all the unknown state variables in the state estimation problem) to ensure that no such attack can be launched, while Giani et al. [S015] proof that placing $p + 1$ secure phasor measurement units (PMUs) at carefully chosen buses are sufficient to neutralize any collection of p sparse attacks, and Kim and Tong [S028] present a so-called cover-up protection that identifies the set of meters that need to be secured so an undetectable attack does not exist for any target topology.

Also Yang et al. [S016] identify the critical meters to protect and observes that the meters measuring bus injection powers play a more important role than the ones measuring the transmission line power flows, since they are essential in determining a specific state variable, while the measurements of line power flows are redundant to improve the accuracy of state estimation.

As finding the minimum number of protected sensors such that an adversary cannot inject false data without being detected is NP-hard[4] [S003], Kim and Poor [S009] and Deka et al. [S038] present greedy algorithms to select a subset of measurements to be protected.

To validate the correctness of customers' energy usage by detecting anomaly activities at the consumption level in the power distribution network, Lo and Ansari [S032] present *"a hybrid anomaly intrusion detection system framework, which incorporates power information and sensor placement along with grid-placed sensor algorithms using graph theory to provide network observability."*

To reveal zero-dynamics attacks, Teixeira et al. [S083] provide necessary and sufficient conditions on modifications of the CPS's structure and presents an algorithm to deploy additional measurements to this end, while Bopardikar and Speranzon [S089] develop design strategies that can prevent or make stealth attacks difficult to be carried out; the proposed modifications of the legacy control system include optimal allocation of countermeasures and design of augmented system using a Moore-Penrose pseudo-inverse.

Mohsenian-Rad and Leon-Garcia [S018] discuss the defense mechanisms against static load altering attacks and presents a cost-efficient load protection design prob-

[4] since this problem is reducible to the *hitting set problem*

lem minimizing the cost of protection while ensuring that the remaining unprotected load cannot cause circuit overflow or any other major harm to the electric grid.

For electricity market domain, Esmalifalak et al. [S065] use a two-person zero-sum game model to obtain an equilibrium solution in protecting different measurements against false data injection attacks impacting locational marginal price (LMP). Within the same domain, Ma et al. [S068] consider a multiact dynamic game where the attacker can jam a reduced number of signal channels carrying measurement information in order to manipulate the LMP creating an opportunity for gaining profit, and the defender is able to guarantee a limited number of channels in information delivery.

Other protection-based approaches include, for instance *"intentionally switch on/off one of the selected transmission lines by turns, and therefore change the system topology"* [S042]; dynamically change the set of measurements considered in state estimation and the admittances of a set of lines in the topology in a controlled fashion [S047], that is an application of a moving target defense (MTD) paradigm; use covert topological information by keeping the exact reactance of a set of transmission lines secret, possibly jointly with securing some meter measurements [S017]; use an algebric criterion to reconfigure and partition a Jacobian matrix $\mathbf{H}$ into two sub-matrices, on each of which to perform a corresponding residual test [S021]; use graph partition algorithms to decompose a power system into several subsystems, where false data do not have enough space to hide behind normal measurement errors [S030]; or even use voltage stability index [162] to identify nodes in power distribution networks with similar levels of vulnerabilities to false data injection attacks via a hybrid clustering algorithm [S056]; *"employ a coding matrix to the original sensor outputs to increase the estimation residues, such that the alarm will be triggered by the detector even under intelligent data injection attacks"* [S109], under the assumption that the attacker does not know the coding matrix yet.

Finally, in order to detect and isolate the disconnected lines and recover the phase angles, in front of the joint cyber and physical attack [S053] outlined in Section D.4, Soltan et al. [S053] present an algorithm that partitions the power grid into the minimum number of attack-resilient zones, ensuring the proposed online methods are guaranteed to succeed.

Then, the four over five **resilient controllers** [S069, S073, S076, S077] and nine over ten **state estimators** [S015, S049, S054, S091, S096, S099, S102, S111, S116] presented in the primary studies were already described in the end of Subsections D.4 and D.4, respectively.

The only works not discussed there are Bezzo et al. [S114] and Mishra et al. [S110]. The first one builds an algorithm that leverages the theory of Markov decision processes to determine the optimal policy to plan the motion of unmanned vehicles and avoid unsafe regions of a state space despite the attacks on sensor measurements, when *"the system is fully observable and at least one measurement (however unknown) returns a correct estimate of a state"* [S114], while in the second study the state estimation is performed in a private and secure manner across multiple computing nodes (observers) with an approach inspired by techniques in cryptography, i.e., decoding Reed-Solomon codes, and results from estimation theory, such as Cramer-Rao lower bound, as a guarantee on the secrecy of the plant's state against corrupting observers [S110].

Finally, Shoukry et al. [S087] present a minimax **state estimator and controller** design as a defense against packet scheduling attacks.

There are 8 works presenting **security metrics**, such as *security indices* defined in the context of power networks as a minimum number of meters to perform an unobservable attack whether including [S004] or not [S002] a given meter, and ϵ-*stealthiness*, which is a notion that quantifies the difficulty to detect an attack when an arbitrary detection algorithm is implemented by the controller [S101].

A vulnerability measure for topologies of power grids subject to false data injection attacks based on incomplete information is presented by Rahman and Mohsenian-Rad [S027], while the vulnerability of the power system state estimator to attacks performed against the communication infrastructure is analyzed by Vuković et al. [S022] via security metrics that quantify the importance of individual substations and the cost of attacking individual measurements in terms of number of substations that have to be attacked. For the domain of electricity market, Jia et al. [S062] introduces the *average relative price perturbation* as a measure of a system-wide price perturbation resulting from a deception attack described in Subsection D.4.

In the context of a canonical double-integrator-network (DIN) model of autonomous vehicle networks, to reflect the quality of the adversary's estimate of the desired nonrandom statistics Xue et al. [S084] defines *"the error covariance for a minimum-variance-unbiased estimate of the initial-condition vector as the security level matrix"* and considers its scalar measures as security levels characterizing the confidentiality of network's state.

Finally, Kwon and Hwang [S090] consider the *dynamic behavior cost* and *estimation error costs* to analytically test the behavior of unmanned aerial systems under various deception attacks and quantify their severity accordingly.

The distribution of primary studies between offline and online defense strategies is shown in Figure D.24, while the distribution of studies by online defense strategy is reported in Figure D.25.

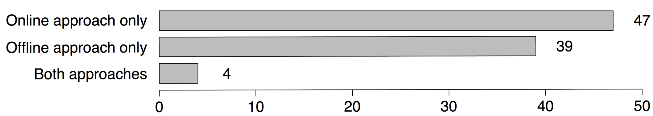

Figure D.24: Distribution between defense strategies

The **online** approaches come into play after adversarial events happen [S080]. *Detection* is an online approach in which the system is continuously monitored for anomalies caused by adversary actions [161], in order to decide whether an attack has occurred. Attack *isolation* is one step beyond attack detection, since

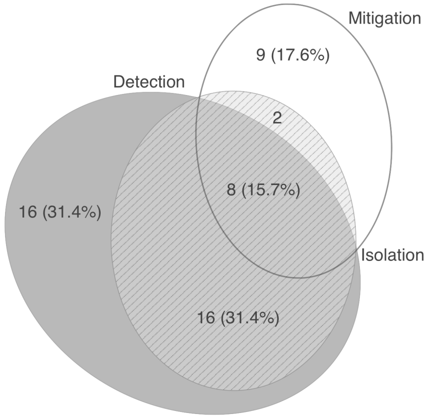

Figure D.25: Distribution by online defense strategy

it distinguishes between different types of attacks [106], and requires also that the exact location(s) of the compromised components(s) be identified [S020]. Once an anomaly or attack is detected (and isolated), *mitigation* actions may be taken to disrupt and neutralize the attack, thus reducing its impact [161]. Among the 51 studies concerned with online defenses, 16 are focused on detection only, other 16 on detection and isolation, while 8 on detection, isolation and mitigation. There are 9 works studying mitigation only, and two works on isolation and mitigation [S036, S085].

To contrast unidentifiable false data injection, Qin et al. [S036] present an algorithm to enumerate all feasible cases and proposes a mitigation strategy to minimize the average damage to the system. Another work on **isolation and mitigation** is Foroush and Martínez [S085], which introduces joint identification and control strategy, that renders the system asymptotically stable in front of unknown periodic DoS in form of pulse-width modulated jamming attacks.

Three of the works focused on **mitigation** were already described in previous Subsections (i.e., [S079] in D.4, [S086] and [S093] in D.4). Here we spend some words on the remaining 5 studies. Liu et al. [S013] recalls their study of strategies to be *"employed by a power system operator in the face of a switching attack to steer the system to a stable equilibrium through persistent co-switching and by leveraging the existence of a stable sliding mode"* [160]. Zhu and Başar [S080] presents a cross-layer, hybrid dynamic game-theoretic model that captures the coupling between the cyber and the physical layers of the system dynamics, extending the control and defense strategy designs *"to incorporate post-event system states, where resilient control and cyber strategies are developed to deal with uncertainties and events that are not taken into account in pre-event robustness and security designs"* [S080].

The overall optimal design of the cyber-physical system is characterized here by a Hamilton-Jacobi-Isaacs equation, together with a Shapley optimality criterion. Yuan et al. [S118] uses this model to construct a hierarchical Stackelberg game, in order to design a control strategy resilient to DoS launched by the intelligent attacker, which adjusts its strategy according to the knowledge of the defender's security profile. Also Barreto et al. [S092] studies a game-theory problem (via differential games and heuristic stability games) where the actions of the players are the control signals each of them has access to. It focuses on reactive security mechanisms, which change the control actions in response to attacks.

Another game-theoretic study is Liu et al. [S105], in which the objective of the defender is to guarantee the dynamic performance of the networked control system (NCS) by transmitting signals with higher power levels than that of jammer's noisy signals. The cost function of the proposed two-player zero-sum stochastic game includes *"not only the resource costs used to conduct cyber-layer defense or attack actions, but also the dynamic performance (indexed by quadratic state errors) of the NCS"* [S105].

To contrast the DoS attacks characterized by their frequency and duration, De Persis and Tesi [S103] determines suitable scheduling of the transmission times achieving input-to-state stability (ISS) of the closed-loop system. It considers periodic, event-based and self-triggering implementation of sampling logics, all of which adapt the sampling rate to the occurrence of DoS and, sometimes, to the closed-loop behavior.

Regarding **detection** mechanisms, most of all related works were already described in Section D.4. Here we introduce the remaining ones.

In order to detect a zero dynamics attack, Keller et al. [S091] proposes to destroy the stealthy strategy of the attacker by triggering data losses on the control signals corrupted by the attack and to use the (augmented state version of) intermittent unknown input Kalman filter. For a system equipped with multiple controllers/estimators/detectors, such that each combination of these components constitute a subsystem, Miao and Zhu [094] presents a moving-horizon approach to solve a zero-sum hybrid stochastic game and obtain a saddle-point equilibrium policy for balancing the system's security overhead and control cost, since each subsystem has a probability to detect specific types of attacks with different control and detection costs.

In the power systems domain, Hao et al. [S046] takes advantage of the sparse and low rank properties of the block measurements for a time interval to make use of robust PCA with element-wise constraints to improve both the error tolerance and the capability of detecting false data with partial observations.

The **detection and identification** of false data injection attacks on power transmission systems is considered by Davis et al. [S019], which outlines an "observe and perturb methodology" to compare the expected results of a control action with the observed response of the system, while Ozay et al. [S025] use a modified version of normalized residual test coupled with proposed state vector estimation methods against sparse attacks. Assuming the attack signal enters through the electro-mechanical swing dynamics of the synchronous generators in the grid as an unknown additive disturbance, Nudell et al. [S059] divide the grid into coherent areas via *"phasor-based model reduction algorithm by which a dynamic equivalent of the clustered network can be identified in real-time"*, and localizes which area the attack may have entered using relevant information extracted from the phasor measurement data.

D.5 Results - Validation strategies (RQ3)

We determined the research type and related research methods of each primary study, simulation models, simulation test systems and experimental testbeds used, repeatability and availability of replication package. In the following we describe the main facts emerging from the collected data.

Research type and related research methods

Following the guidelines of systematic mapping studies, we reuse the classification of research approaches proposed by Wieringa et al. [163], applying the research type classification presented in Petersen et al. [7]. It is worth noting that our selection strategy (see Section D.2) focuses on studies *proposing a method or technique* for CPS security, so the *philosophical papers*, *opinion papers* and *experience papers* are not considered in our study. The distribution of primary studies by research type is presented in Figure D.26.

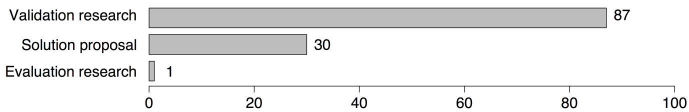

Figure D.26: Distribution of studies by research type

Validation research is applied in 87 studies (73.73%), where the techniques investigated are novel and have not yet been implemented in practice; the research methods used are formal mathematical proofs, case studies and lab experiments, together with simulations as a means for conducting an empirical study.

In particular, formal *mathematical proofs* are used in 63 studies (53.39%), in 5 of which as the only validation method adopted. There are 18 primary studies providing both mathematical proofs and illustrative numerical examples, and 14 works illustrating formal mathematical proofs and examples applied to simulation test systems.

Case studies via simulation, understood as empirical inquiries that draw on multiple sources of evidence to investigate contemporary phenomena in their real-life context, especially when the boundary between phenomenon and context cannot be clearly specified [140], are employed in 4 studies, twice as validation of a good line of argumentation [S026, S114], and twice as a follow up of formal mathematical reasoning [S082, S088]. It is worth noting that in Bezzo et al. [S114] also a hardware evaluation on a remotely controlled flying quadricopter is performed, while the case study of [S088] is extracted from its previous work cited therein [164].

Another validation research approach, considered in 46 primary studies, consists of an *experiment*, that is a formal, rigorous and controlled empirical investigation, where one factor or variable of the studied setting is manipulated, while all the other parameters are regulated at fixed levels [140].

Most of these experiments are performed in simulation: in fact, the *experimental testbeds* are employed only in 7 of these 46 works. As shown in Figure D.27, the *quadruple-tank process* [165], that is a multivariable laboratory process consisting of four interconnected water tanks, is used in 3 primary studies [S081, S083, S107]. *LandShark*[5] robot, i.e., a fully electric unmanned ground vehicle developed by Black I Robotics, is used in other 3 works [S097, S099, S106]. Finally, micro grid experimental testbed consisting of three Siemens SENTRON PAC4200 smart meters connected into the network with YanHua Industry control machine, which is used to monitor all traffic of lab network and read the data from all meters, is used only in one primary study [S054].

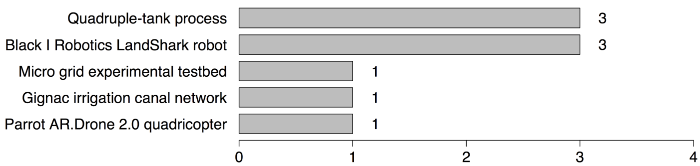

Figure D.27: Distribution of experimental testbeds

The remaining 39 works that use experiments as a validation method are employing different simulation test systems, described in Subsection D.5. Notably, simulation experiments follow a good line of argumentation of the rest of the paper in 21 primary studies, while in the remaining 18 works the experiments are coupled with formal mathematical proofs.

Then, in 30 (i.e., 25.42% of all) studies *solution proposals* for specific problems are given, where the potential benefits and the applicability of a solution is simply shown through a small example or a line of argumentation; those solutions are either novel or a significant extension of existing ones. We want to point out that often this category corresponds to the results of theoretical research. There are 2 primary studies that use only a good line of argumentation [S014, S069], while sound argument is followed by an illustrative numerical example in 6 primary studies ([S050, S092, S096, S100, S104, S109]), or by an example applied to simulation test system in 22 works. The different simulation test systems found in our primary studies are described in Subsection D.5.

[5] `http://www.blackirobotics.com/LandShark_UGV_UCOM.html`

Finally, *evaluation research*, where the techniques are implemented in practice with identification of problems in industry, is done only in one study [S072], in which the Gignac irrigation canal network is used to demonstrate the feasibility of stealthy deception attacks on water SCADA systems.

Simulation model

As in the case of plant models used by attackers, also the plant models adopted for simulation purposes can be different from the plant models used in the analysis. As we can see from Figure D.28, an overwhelming majority of primary studies uses the same model of plant for both the analysis and simulation, while only in 6 studies (5.08%) these models are different [S028, S030, S051, S057, S062, S067]. Those six studies are within the power transmission or electricity market application domains and use nonlinear AC model for simulation, while consider a DC model (sometimes together with AC model) for analysis purposes. It is worth to mention that in 32 primary studies there are no simulations. Those works account for those solution proposals and validation research papers already described in Subsection D.5 that use only good line of argumentation, formal mathematical proofs and illustrative numerical examples as the research methods. The only exception is Tiwari et al. [S097], which uses LandShark robot as the experimental testbed, without relying on simulations.

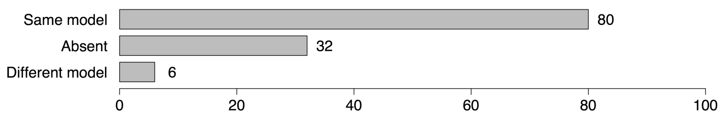

Figure D.28: Distribution of primary studies by simulation model

Simulation test system

As it was anticipated in the previous section, 85 primary studies (72,03%) use simulation test systems to validate the presented results. Within the power systems application domains, the simulation tool used in all but one primary study is *MatPower*. The distribution of its test cases is shown in Figure D.5.

The works studying applications to electricity market use a modified 5-bus PJM example (MatPower case5) [166], which is employed in 3 primary studies, and IEEE 14-bus (case14), IEEE 30-bus (case30), IEEE 118-bus (case118) test systems.

Generally speaking, IEEE 14-bus test system is the most used one, found in 38 works, treating mostly power transmission (in 34 studies), but also power generation (in 2 studies) and electricity market (in 8 studies). IEEE 30-bus test system is used in 17 primary studies, 16 of which are focused on power transmission only, and the remaining one on electricity market. IEEE 118-bus test system is second

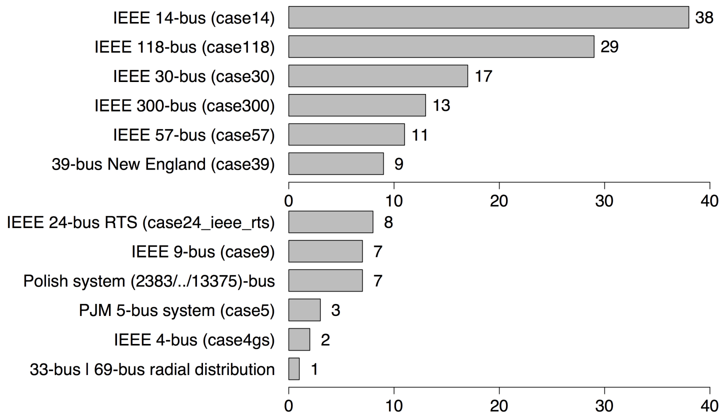

Figure D.29: Distribution of power grid test cases

most adopted one, found in 29 primary studies, dealing power transmission (in 27 studies), power generation (in 2 studies), and electricity market (in 2 studies).

Studies on power distribution use 33-bus [167] and 69-bus [162] radial distribution test systems in one primary study [S056], and IEEE 24-bus reliability test system (MatPower case24_ieee_rts) in another one [S018]. We recall that IEEE 24-bus RTS is based on IEEE RTS-79 [168] and is used in 8 primary studies, all 8 focused on power transmission, 2 of which are dealing also with power generation.

39-bus New England test system (MatPower case39), obtained from Bills [169], is used in 9 studies, 3 of which are about power generation and 8 are about power transmission.

The remaining test systems are all about power transmission. IEEE 4-bus test system (MatPower case4gs) is used in 2 studies; IEEE 9-bus (case9) is found in 9 studies; IEEE 57-bus (case57) is adopted by 11 and IEEE 300-bus (case300) by 13 studies, while MatPower cases representing the Polish 400, 220 and 110 kV networks during either peak or off-peak conditions are used in 7 studies.

Power generation is also studied on two-area Kundur system test case [152], which parameters can be found in the Matlab Power System Toolbox, in two studies ([S005, S059]); and on multi-area load frequency control schemes installed with proportional-integral controllers, as described by Jiang et al. [170], in one study [S118].

The other used test cases are summarized in Figure D.30. Irrigation system consisting of a cascade of a number of canal pools, as presented in Amin et al. [171], is used in two primary studies [S072, S078]. Also an unstable batch reactor system presented by Walsh et al. [172], which is a fourth order unstable linear system with two inputs, is employed in two works [S087, S094]. Tennessee Eastman process control system model and associated multi-loop proportional-integral control law, as proposed by Ricker [173], is adopted in three studies [S070, S075, S094]. PHANToM

Premium 1.5A [174], that is a haptic device from SensAble Technologies, is used once in a simulation setup [S105]. Finally, a rotorcraft in a cruise flight [175] is simulated in two studies [S090, S093].

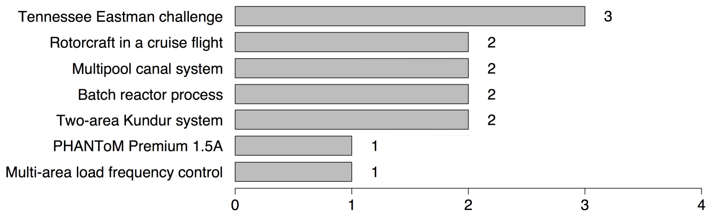

Figure D.30: Distribution of other used test cases

There are also 8 primary studies, which use ad hoc simulation test systems to validate their results. Specifically, Kwon et al. [S082] use Monte Carlo simulation with 1000 runs on an unmanned aerial system navigation system integrating the inertial navigation system and the global positioning system implemented in Matlab. D'Innocenzo et al. [S088] perform Matlab/Simulink simulations on the multi-hop wireless network deployed in a room to connect the temperature sensor to the variable-air-volume box, which is positioned nearby the room. Also Eyisi and Koutsoukos [S098] perform Matlab/Simulink simulations on a single-input single-output (SISO) system; it deals with a velocity control of a single joint robotic arm over a communication network. Bezzo et al. [S106] use robot operating system[6] (ROS) based simulator emulating electromechanical and dynamical behavior of the real robot. In Park et al. [S108] simulations are carried out using a simple model of air traffic operations. Shoukry and Tabuada [S111] use an UGV model implemented in Matlab. Jones et al. [S113] simulate a train, which uses an electronically-controlled pneumatic braking system modeled as a classical hybrid automaton. Finally, Shoukry et al. developed a *"theory solver in Matlab and interfaced it with the pseudo-Boolean SAT solver SAT4J"* [S117], where the simulations are performed on linear dynamical systems with a variable number of sensors and system states.

It is not surprising that most advanced and realistic validation methods have been exploited in the power networks application domain. Despite research on CPS Security in this domain appears quite mature, a benchmark is still missing.

Repeatability and availability of replication package

The possibility of reproducing the evaluation or validation results provided by the authors is called repeatability, while the possibility of exploring changes to experiment parameters is known as workability. The repeatability process is a good

[6] `http://www.ros.org`

scientific practice [176]. The so called Artifact Evaluation Process[7] is used in a number of conferences in computer science, and a similar concept of repeatability evaluation of computational elements has been introduced in cyber-physical systems domain in 2014 ACM Hybrid Systems Computation and Control (HSCC) conference[8]. However, such practice is rather new to several research communities working on CPS: we found no primary study with a replication package. Thus, we have isolated the information concerning the availability of a replication package and extended the simple dimension provided in Yuan et al. [146] in a way that *repeatability* is considered *high* when the authors provide enough details about

- the steps performed for evaluating or validating the study,
- the developed or used software,
- the used or simulated testbed, if any, and
- any other additional resource,

in a way that interested third parties can be able to repeat the evaluation or validation of the study. Otherwise, we have *low* repeatability.

Such high-level definition of repeatability values has ensured that the primary studies using standard test systems from Subsection D.5 and well known experimental testbeds have received high values of repeatability, where steps performed in their experiments, case studies and/or simulation examples have been described with enough details. On the other hand, the usage of some ad hoc simulation test system has caused some low values of repeatability assigned. As shown in Figure D.31, 82 studies (69.49%) have a high repeatability value, and 5 studies (4.24%) have a low repeatability score. As a note, we did not have the possibility to evaluate the repeatability of 31 studies (26.27%) since they do not present any experiment, case study or simulation example.

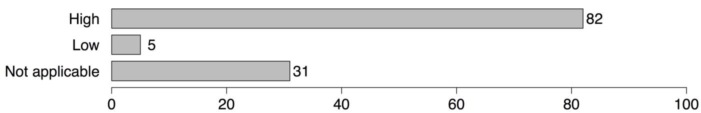

Figure D.31: Distribution of primary studies by repeatability

Overall, we advocate the improving of repeatability and workability of computational results of the papers by adopting the best practices of repeatability process and creating related replication packages, because we strongly believe in the usefulness of repeatability to empower others to build on top of the contributions of a paper[9] and thus accelerate scientific and technological progress.

[7] http://www.artifact-eval.org

[8] http://www.cs.ox.ac.uk/conferences/hscc2016/re.html

[9] http://evaluate.inf.usi.ch/artifacts/aea

D.6 Implications for future research

We discussed potential future research trends and challenges for CPS security throughout this paper in the context of the various discussions of obtained results (Sections D.3, D.4, and D.5); in the following we discuss more general observations about implications for future research on CPS security.

CPS security is a relatively young research domain that is experiencing a strong **academic and industrial interest** in the very few years, and both European Commission and NSF are very oriented in financing research in this area. From the data obtained in this systematic mapping study it can be inferred that the potential of the developed results and methodologies in addressing realistic emerging problems in several application domains (first of all, power systems) is very promising. As a consequence it is predictable that CPS security will be a “hot topic” for the forthcoming years. Our investigation, based on the current state of the art, sheds some light on challenges that will possibly represent the next steps of research in CPS security.

From a **modeling** point of view this study shows that, as usual in the control theory community, most of the research is based on the model-based paradigm. However, as experience demonstrates, e.g., in the context of energy efficient control of building automation systems, in many CPS application domains the cost of modeling is much larger than the improvement margin in terms of efficiency/-cost/performance. As a consequence, we expect that part of future research will be based on the data-based paradigm. This approach, based on “learning” techniques and thus strictly connected with the computer science research community, together with the recent large availability of (big!) data deriving from CPS infrastructures, can also be of help towards a more realistic and systematic modeling/mapping of attack/defense models/strategies/architectures.

From a **validation** point of view, selected papers, as illustrated in the previous sections, address a wide range of application domains, system architectures, problem formulations and theoretical foundations: this makes it very difficult to compare different solutions to similar problems, and we believe that time is mature for the development of academic or industrial benchmarks, test-beds and demonstrators. This could also help in disseminating how research on CPS security can make the difference in each application domain.

From the point of view of the **societal and industrial impact**, it is easy to infer from the selected papers that, even thought realistic applications are almost always the main motivation for research, a strong synergy between real industrial/societal problems and theoretical investigation and results is still not apparent from the scientific literature. It is also true that our research questions did not include analysis of relevant projects related to CPS security, however most of the selected papers do not directly relate to or derive from direct collaboration between industry and academia: we hope and expect that this will happen in the near future. Also, we were unable to find research devoted to formal certification with reference to international standards, whose satisfaction is often the biggest barrier for testing and applying novel methods and technologies. Finally, we observe the lack of workshops or symposia with the explicit target of catalysing collaboration between industry and academia on specific applications.

D.7 Related work

Cyber-physical systems security within the smart grid domain has been reviewed by Mo et al. [177] and by Sridhar et al. [178].

The work from Mo et al. [177] is a good starting point to face the area of CPS security since it gives a broad overview on cyber and system-theoretic approaches to security and shows how a combination of both of them together can provide better security level than traditional methods. The provided example describes defense against replay attack following secure control [179] method.

The article from Sridhar et al. [178] is more domain-specific. Since power system is functionally divided into generation, transmission, and distribution, the survey considers cyber vulnerabilities and security solutions for each of the underlying fields. Notably, it deals with a wide range of (sophisticated) attacks, some bad data detection techniques and mentions attack resilient control. This work provides also an overview on supporting infrastructure security, with a look on secure communication, device security, security management and awareness, cyber security evaluation, and intrusion tolerance. All in all, the paper identifies the importance of combining both power application security and supporting infrastructure security into the risk assessment process and provides a methodology for impact evaluation. Conclusively, it lists a number of emerging research challenges in risk modeling and mitigation, pointing out the importance of attack resilient control, domain-specific anomaly detection and intrusion tolerance.

Both of previous surveys [177, 178] are focused on smart grid domain-specific security. Moreover, based on the guidelines for performing systematic literature reviews from Kitchenham and Charters [8], these studies cannot be considered as a systematic literature reviews but as *informal literature surveys*.

The intrusion detection techniques for different CPS applications were surveyed by Mitchell and Chen [180]. For each presented intrusion detection system (IDS) design it was analyzed which, if any, distinguishing characteristics of CPS intrusion detection were considered. The unique characteristics of cyber-physical intrusion detection listed in this study are physical process monitoring, closed control loops, attack sophistication and legacy technologies. The conclusion was that there is a lack of IDS techniques that specifically consider most or all distinguishing aspects of CPS. Other notable remark was that behavior-specification-based detection, which formally define legitimate behavior and detects an intrusion when the system departs from this model, has a potential to be the most effective one and deserves more research attention. A similar inference was made by Zhu and Sastry [181] in their survey of SCADA-specific IDS. Although the works on intrusion detection are relevant for our study, our goal is to give a much broader holistic view on cyber-physical security, and not only on a particular family of mechanisms.

D.8 Conclusions and future work

The main goal of this survey was to analyze the publication trends, characteristics, and validation strategies of existing methods and techniques for automatic control for CPS security from a researcher's point of view. In order to achieve this goal we designed and conducted an empirical study that provides a detailed overview of publication trends, venues, and research groups active on automatic control for CPS security, and a thorough classification providing an empirically validated foundation for evaluating existing solutions. The main contribution of this research is to provide a systematic map of research on automatic control for CPS security; the map has been carried out methodologically in order to warrant the quality of the analysis and results. Additionally, another main contribution of our research is the definition of a sound and complete comparison framework for both existing and future research on automatic control for CPS security. These contributions will benefit researchers proposing new approaches for CPS security, or willing to better understand or refine existing ones.

We selected a total of 118 primary studies as a result of the systematic mapping process, each of them belonging to different research areas, such as automatic control, networked systems, smart grid, security for information systems. The main findings emerging from our study are summarized in Subsection D.1 and explained in details in Sections D.3, D.4, and D.5. The resulting implications for the future research are presented in Section D.6.

Based on the learning of this work, our future scientific research will be oriented to address CPS security problems providing mathematical models of the interaction between physical systems and non-idealities due to communication protocols, in particular regarding wireless sensor and actuator networks.

D.9 Selected primary studies

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D.10 Additional details on our search strategy

As it was introduced in Subsection D.2, in order to achieve maximal coverage, our search strategy consists of three complementary methods: an automatic search, a manual search, and the snowballing.

Automatic search

It refers to the execution of a search query on a set of electronic databases and indexing systems, in the literature it is the dominant method for identifying potentially relevant papers [142]. The applied search string is the following:

```
(((("cyber physical" OR "cyber-physical" OR cyberphysical OR
"networked control") AND system*) OR CPS OR NCS) AND (attack* OR
secur* OR protect*))
```

In the spirit of Zhang, Ali Babar and Tell [143], we established a *quasi-gold standard* (QGS) for creating a good search string for the automatic search. This procedure requires a manual search in a small number of venues (see Table D.2) and the results of these manual searches have been treated as a QGS by cross-checking the results obtained from the automatic search. So, we iteratively defined and refined the search string and conducted automatic searches on the electronic data sources until the quasi-sensitivity was above the established threshold of 80%. When the *quasi-sensitivity* became greater than 80%, the search performance was considered acceptable and the results from the automated search have been merged with the QGS. The details of the above mentioned process are provided in the replication package of this study.

In this stage it was fundamental to select papers objectively so, following the suggestions from Wohlin et al. [140], two researchers assessed a random sample of the studies and the inter-researcher agreement has been measured using the Cohen Kappa statistic [182]. Each disagreement has been discussed and resolved, with the intervention of the team administrator, if necessary, until the Cohen Kappa statistic reached a result above or equal to 0.80.

Our automatic search is performed on the largest and most complete scientific databases and indexing systems available in computer science (see Table D.1). The selection of these electronic databases and indexing systems is guided also by their high accessibility and their ability to export search results to well-defined formats.

Library	Website
ACM Digital Library	`http://dl.acm.org`
IEEE Explore	`http://ieeexplore.ieee.org`
ISI Web of Science	`http://apps.webofknowledge.com`
ScienceDirect	`http://www.sciencedirect.com`
SpringerLink	`http://link.springer.com`
Wiley InterScience	`http://onlinelibrary.wiley.com/+`

Table D.1: Electronic data sources targeted with search strings

Among the results of the automatic searches we removed a set of *false positives* in order to work on a polished set of potentially relevant studies (see Figure D.1). Examples of false positives include proceedings of conferences or workshops, tables of contents, maps, lists of program committee members, keynotes, tutorial or invited talks, and messages from (co-)chairs. As shown in Figure D.1, our automatic search resulted in 1559 potentially relevant studies.

Manual search

By following the quasi-gold standard procedure defined in [143], we

- identified a subset of important venues for the domain of cyber-physical systems security (they are shown in Table D.2), and
- performed a *manual search* of relevant publications in those venues.

By referring to Figure D.1, we manually searched and selected 289 potentially relevant studies.

Venue	Publisher
Int. Conf. on High Confidence Networked Systems (HiCoNS)	ACM
Int. Journal of Critical Infrastructure Protection (IJCIP)	Elsevier
Int. Symposium on Resilient Control Systems (ISRCS)	IEEE

Table D.2: Selected venues for manual search

After merging all the studies and removing duplicates we obtained 1848 potentially relevant studies. In order to further restrict the number of studies to be considered during the snowballing activity, we applied the selection process depicted in Section D.2 to the current set of studies, thus obtaining 63 potentially relevant studies. In order to handle studies selection in a cost effective way we used the adaptive reading depth, as the full-text reading of clearly excluded approaches is unnecessary. So, we considered *title*, *keywords* and *abstract* of each potentially relevant study and, if selection decision could not be made, other information (like *conclusion* or even *full-text*) have been exploited [143].

Snowballing

We applied the *snowballing* technique for identifying additional sources published in other journals or venues [183], which may not have been considered during the automatic and manual searches. So, as recommended in [184], we applied (backward and forward) snowballing on the primary studies selected by the automatic and manual searches. More specifically, we considered all the studies selected by the automatic and manual searches and we automatically searched all the papers referring them (i.e., forward snowballing [144]); then, we scrutinized also the references of each selected study to identify important studies that might have been missed during the initial search (i.e., backward snowballing [144]).

D.11 Additional results

This section of appendix provides the results of analysis of some additional characteristics of our primary studies, which are not related to CPS security per se, but are still useful to better understand this scientific area. In the following we first describe the venues where the primary studies were published, then the involved research institutions, and finally the theoretical foundations and time-scale models found in these selected works.

Publication venues

In accordance with our selection strategy, we selected publications which have been subject to peer review. Indeed, each primary study was published either as a journal paper, conference paper, workshop paper, or book chapter. Figure D.32 shows the distribution of primary studies over their publication types. The most common publication types are journal and conference, with 59 (50.01%) and 50 (42.37%) of the primary studies, respectively. Book chapter and workshop are the least popular publication types, with only 6 (5.08%) and 3 (2.54%) studies falling into their categories, respectively. Such a high number of journal and conference papers on CPS security may indicate that CPS security is becoming more and more a mature research theme, despite its relative young age.

Moreover, the very low number of workshop papers may be an indication of two facts: on one side researchers on CPS security are valuing more other types of publications (e.g., journal papers), given the high effort and skills required to contribute in this research area; on the other side, it may be an indication that actually the research community on CPS security still does not have a clearly defined identity, and a symptom of this situation may be the lack of a workshop or conference fully dedicated to CPS security. We will detail more on this aspect when analyzing the targeted publication venues (see Table D.3, where we recall that the HiCoNS conference has been merged into the International Conference on Cyber-Physical Systems (ICCPS) since 2015.).

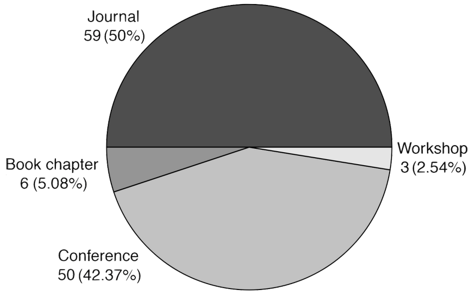

Figure D.32: Distribution by type of publication

Publication venue	**Type**	**#Studies**
IEEE Trans. on Smart Grid	Journal	19 (16.10%)
IEEE Conf. on Decision and Control (CDC)	Conf.	11 (9.32%)
IEEE Trans. on Automatic Control	Journal	9 (7.62%)
American Control Conf. (ACC)	Conf.	6 (5.08%)
IEEE Journal on Selected Areas in Communications	Journal	6 (5.08%)
IEEE Conf. on Smart Grid Communications (SmartGridComm)	Conf.	6 (5.08%)
Int. Conf. on High Confidence Networked Systems (HiCoNS)	Conf.	4(3.38%)
IEEE Control Systems	Journal	3 (2.54%)
Global Communications Conf. (GLOBECOM)	Conf.	3 (2.54%)
IEEE Trans. on Parallel and Distributed Systems	Journal	3 (2.54%)
IEEE Trans. on Power Systems	Journal	3 (2.54%)
Automatica	Journal	2 (1.69%)
ACM Symposium on Inf., Comput. and Commun. Security (ASIACCS)	Conf.	2 (1.69%)
Cyber Physical Systems Approach to Smart Electric Power Grid	Book	2 (1.69%)
International Journal of Systems Science	Journal	2 (1.69%)
TOTAL	-	**81 (68.64%)**

Table D.3: Publication venues with more than one selected study

For what concerns the evolution of publication types of the years, Figure D.3 shows that there is a growing trend in the publications in journals and conference proceedings, as 84 out of 118 studies are journal and conference papers published between 2013 and 2015. Also, almost all book chapters have been published between 2013 and 2015 (5 out of 6 book chapters). Again, this may be a further confirmation that CPS security is turning more and more into a mature field, with more foundational and comprehensive studies published in the recent years.

By looking at the specific targeted publications venues we can notice that research on CPS security is published across a number of venues spanning different research areas, such as automatic control, networked systems, smart grid, security for information systems. Indeed, the 118 selected papers of our study were published at 53 different venues. Table D.3 shows the publication venues with more than one selected study.

Those publication venues can be considered as the de facto leading venues for publishing studies on cyber-physical systems security.

From the collected data we can observe that most targeted venues are heterogeneous and pertain to different research areas, such as smart grid, automatic control, communications, networked systems, parallel and distributed systems, etc.; this is a clear indication of the very multidisciplinary nature of cyber-physical systems, even in a specific sub-area like CPS security. Moreover, according to well-acknowledged international rankings the most targeted venues for CPS security are all top-level and very reputable in their research area. Indeed, all journals are ranked in the first quartile according to the SCImago Journal Rank (SJR) indicator [185], and all conferences are ranked either as A or B according to the computer science conference rankings (CORE) [186] (depending on data availability); Finally, there is a whole book in the set of most targeted venues and it is the sole publication venue specifically targeted to research on CPS. The book has been published in 2015 and can be a useful reading for current and future researchers in the area of CPS security, with a special emphasis on power grids.

Theoretical foundation

Because of the intrinsic multidisciplinary nature of cyber-physical systems, we payed attention also on the theoretical background on which primary studies are built upon. Since the control systems are at the heart of CPS, it isn't a surprise that control theory is used in every study considered in our mapping study. The distribution of other theoretical backgrounds considered by primary studies is presented in Figure D.33.

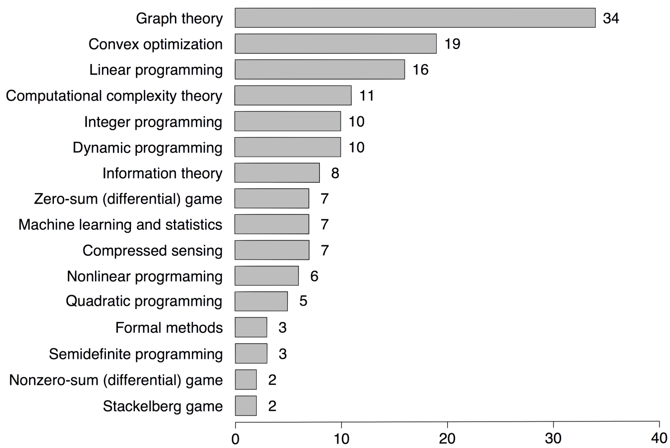

Figure D.33: Distribution of theoretical backgrounds

The study of graphs is the most used theoretical foundation, found in 34 studies (28.81%), that are [S002, S004, S009-S011, S014-S017, S020, S022-S024, S026-S030, S032, S034, S037-S039, S042, S045, S051, S053, S059, S062, S074, S084, S086, S088, S100]. *Graph theory* is well suited to represent any kind of networks, and, in fact, it was used in 26 studies on security of power transmission networks.

To asymptotically analyze the intrinsic difficulty of problems and algorithms and to decide which of these are likely to be tractable, *computational complexity theory* is employed in 11 works, all within the field of power transmission ([S001-S004, S010, S015, S016, S032, S038, S039, S053]).

Information theory is used in 8 works ([153], related to [S073], and [S018, S024, S074, S079, S101, S110, S116]), most of which treating the security of generic linear dynamical systems.

The methods of dimensionality reduction (such as principal component analysis) and of latent variable separation (e.g. independent component analysis) from *machine learning* and *statistics* provide a way to understand and visualize the structure of complex data sets and are used in 7 works ([S012, S031, S033, S052, S056, S097, S113]). Their application domain is power grids and generic dynamical systems.

Other methods of linear dimensionality reduction are used for simultaneous sensing and compression of finite-dimensional vectors. Providing means for recovering sparse high-dimensional signals from highly incomplete measurements by using efficient algorithms, *compressed sensing* is applied in 7 works on power grids and linear dynamical systems ([S004, S009, S025, S033, S046, S079, S111]).

Starting from 2014, typical *formal methods* concepts of signal temporal logic (STL, which is a rigorous formalism for specifying desired behaviors of continuous signals) and satisfiability modulo theories (SMT) have found their way in 3 studies on CPS security ([S113] and [S047, S117], respectively), with applications to anomaly detection and resilient state estimation in generic cyber-physical systems and power grids.

The mathematical *optimization* is used in several studies and application areas. The sub-fields of optimization found in primary studies include *convex optimization* (19 studies), *linear programming* (16 studies), *dynamic programming* and *integer programming* (both appeared in 10 studies), *nonlinear programming* (6 studies), *quadratic programming* (adopted in 5 works) and *semidefinite programming* (3 studies).

The most used sub-field of *game theory*, found in 7 primary studies, is zero-sum game, which do not allow for any cooperation between the players, since what one player gains incurs a loss to the other player ([S065, S068, S073, S080, S087, S094, S105]). Both non-zero sum games and Stackelberg games are formulated in 2 works ([S086, S092] and [187], related to [S077], together with [S118], respectively). As expected, all these games belong to a class of continuous-time infinite dynamic games, also known as *differential games*, wherein the evolution of the state is described by a differential equation and the players act throughout a time interval.

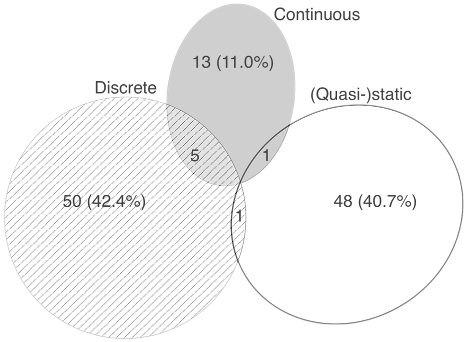

Figure D.34: Distribution of primary studies by time-scale model

Time-scale model

The dynamic system behavior can be modeled via different time-scale models, such as continuous, discrete and hybrid. In the case of the (quasi-)steady state assumption, the system is treated as (quasi-)static, and the time-scale model is named accordingly.

As shown in Figure D.34, the quasi-static model is used in 48 studies (40.68%), all of them concerned with power systems state estimation, while there are 13 studies (11.02%) considering continuous time, 50 (42.37%) discrete time, and only 5 considering both continuous and discrete time ([S080, S083, S086, S103, S113], only 3 of which actually using hybrid time [S080, S086, S113]). There is also one work with both continuous time and quasi-static model ([S015]), and one with both discrete time and quasi-static model ([S016]).

In particular, quasi-static analysis is mostly chosen for addressing control architectures like SCADA, which provide steady-state set-points to inner control loops.

D.12 Threats to validity

We assessed the level of quality of our study by applying the quality checklist proposed by Petersen et al. in [7]. The goal of Petersen's quality checklist is to assess an objective quality rating for systematic mapping studies. According to the metrics defined in Petersen's quality checklist, we achieve an outstanding score of 54%, defined as the ratio of the number of actions taken in comparison to the total number of actions reported in the quality checklist. *The quality score of our study is far beyond the scores obtained by existing systematic mapping studies in the literature*, which have a distribution with a median of 33% and 48% as absolute maximum value.

Overall, the high quality of our study has being ensured by producing a detailed research protocol document in which all of its steps have been subject to three external reviews by independent researchers (see Section D.2) and by conducting our study by following the well-accepted and updated guidelines of systematic review/mapping study [7, 8]. In the following we detail the main threats to validity of our study and how we alleviated them.

Conclusion validity

Conclusion validity refers to the relationship between the extracted data, the produced map, and the resulting findings [140].

In order to mitigate possible conclusion validities, first of all we defined the search terms systematically and we document procedures in our research protocol, so that our research can be replicated by other researchers interested in the topic. Moreover, we documented and used a rigorously defined data extraction form, so that we could reduce possible biases that may happen during the data extraction process; also, in so doing we had the guarantee that the data extraction process has been consistent to our research questions.

On the same line, the classification scheme could have been another source of threats to the conclusion validity of our study; indeed, other researchers may identify classification schemes with different facets and attributes. In this context, we mitigated this bias by

- performing an external evaluation by independent researchers who were not involved in our research, and
- having the data extraction process conducted by the principle researcher and validated by the secondary researcher.

Internal validity

Internal validity is concerned with the degree of control of our study design with respect to potential extraneous variables influencing the study itself.

In this case, having a rigorously defined protocol with a rigorous data extraction form has surely helped in mitigating biases related to the internal validity of our research. Also, for what concerns the data analysis validity, the threats have been minimal since we employed well-assessed descriptive statistics when dealing with quantitative data. When considering qualitative data, the sensitivity analysis performed on all extracted data has helped in having good internal validity.

Construct validity

It concerns the validity of extracted data with respect to our research questions. Construct validity concerns the selection of the primary studies with respect to how they really represent the population in light of what is investigated.

Firstly, as described in Subsection D.2, the automatic search has been performed on multiple electronic databases to get relevant studies independently of publishers' policies and business concerns. Moreover, we are reasonably confident about the construction of the search string used in our automatic search since the used terms have been identified by rigorously applying a systematic procedure (i.e., the quasi-gold standard systematic procedure as defined in [143]). Moreover, the automatic search is complemented by the snowballing activity performed during the search and selection activity of our review process (see Figure D.1), thus making us reasonably confident about our search strategy. Since our automated search strategy actually relies on search engines quality and on how researchers write their abstracts, the set of primary selected studies have been extended by means of the backward and forward snowballing procedure.

After having collected all relevant studies from the automatic search, we rigorously screened them according to well-documented inclusion and exclusion criteria (see Subsection D.2); this selection stage has been performed by the principle researcher, under the supervision of the secondary researcher. Also, in order to assess the quality of the selection process, both principle and secondary researchers assessed a random sample of studies, and inter-researcher agreement has been statistically measured with very good results (i.e., we obtained a Cohen-Kappa coefficient of inter-rater agreement of more than 0.80).

External validity

It concerns the generizability of the produced map and of the discovered findings [140].

In our research, the most severe threat related to external validity consists in having a set of primary studies that is not representative of the whole research on security for cyber-physical systems. In order to mitigate this possible threat, we employed a search strategy consisting of both automatic search and backward-forward snowballing of selected studies. Using these two search strategies in combination empowered us in mitigating this threat to validity. Also, having a set of well-defined inclusion and exclusion criteria contributed to reinforcing the external validity of our study.

A potential source of issues regarding the external validity of our study can be the fact that only studies published in the English language have been selected in our search. This decision may result in a possible threat to validity because potentially important primary studies published in other languages may have not been selected in our research. However, the English language is the most widely used language for scientific papers, so this bias can be reasonably considered as minimal.

Similarly, grey literature (e.g., white papers, not-peer-reviewed scientific publications, etc.) is not included in our research; this potential bias is intrinsic to our study design, since we want to focus exclusively on the state of the art presented in high-quality scientific papers, and thus undergoing a rigorous peer-reviewed publication process is a well-established requirement for this kind of scientific works.

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