Gran Sasso Science Institute

PHD PROGRAMME IN MATHEMATICS IN NATURAL, SOCIAL AND LIFE SCIENCES: XXX CYCLE

June 5, 2018

L^p-L^q decay estimates for dissipative linear hyperbolic systems

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Thinh Tien Nguyen

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La Sapienza, Roma

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Introduction

A first glance

In the last few decades, *relaxation systems* are favored by many mathematicians due to their enormous applications varying from the kinetic theory of gases [78] to almost all numerical schemes approximating conservation laws [34]. They are systems of first-order partial differential equations with zero-order terms. The zero-order terms are called *source terms* and the remain parts of the systems are called *hyperbolic parts*. Particularly, the systems are also endowed with dissipative mechanisms in the presence of the source terms.

In general, a relaxation system is not totally affected by the source term due to the fact that the *equilibrium set* of the system is nonzero and nonempty, where the set consists of every zero of the source term. This fact may give rise to some critical situations. Indeed, since the source term has a strong influence on the system outside the equilibrium set, any smooth solution to the system will be driven toward an equilibrium as time increases due to dissipative effects of the source term. However, since the equilibria are merely solutions to a homogeneous first-order hyperbolic system, they are well known to break down after a finite time due to the formation of singularities. Therefore, smooth solutions to the relaxation system may be overcontrolled after a finite time when they approach the equilibria.

The above observation yields an interesting fact. If a relaxation system has a sufficiently "good" equilibrium i.e. this equilibrium is defined for all time, then a global (in time) smooth solution to the system can be completely exploited. In fact, what we need is only a stability structure in order for the equilibrium to be stable under the perturbation of the system. If we can propose such a stability structure, then at least for initial data near the initial datum of the equilibrium, if a local smooth solution to the system exists, it will be attracted by the equilibrium as time increases. Hence, since the equilibrium is defined for all time, the local smooth solution will continue to exist for all time close to it. To prove the stability, it is sufficient to construct an appropriate Lyapunov function by the energy estimate method in the energy space. Moreover, we can also consider the strategy via linearization and the Duhamel formula.

As a matter of fact, it is obvious that constant equilibria of a relaxation system are always very good choices for the global existence of a smooth solution to the system. For their stability, we can consider the stability structures in [30,77], which are based on the coupling hypotheses: the *entropy dissipation condition* and the *Shizuta–Kawashima condition*. The (strictly) entropy dissipation condition is in general too weak to prevent singularities from developing after a finite time. However, together with the Shizuta–Kawashima condition, it can produce complete smoothing effects on local smooth solutions to the relaxation system, provided the initial data are close to the constant equilibria and the source term is *nondegenerate*. The smoothing effects yield that the global existence of smooth solutions to the system is guaranteed. Particularly, once existed, the global smooth solutions approach the constant equilibria for large time. In fact, the global existence is already showed in [30, 77] based on the energy estimate method and the commutator estimates. On the other hand, the stability is acquired in [5] through linearization and the Duhamel formula. Especially, large-time approximations of the solutions and explicit rates of the convergence of the solutions to the constant equilibria are also established in [5] by using the Chapman–Enskog expansion. Despite that, they are still restrictive.

In this thesis, we will also study the stability of constant equilibria of relaxation systems in order to improve the results in [5]. However, we focus only on the linear stability. It is then sufficient to examine linear systems arising in the linearization of the nonlinear systems about the constant equilibria. By imposing some reasonable stability structures on the linear systems based on the ones in [5, 30, 77], we study large-time behaviors of solutions to the systems as time tends to infinity. On the other hand, large-time approximations of the solutions are also sharply constructed. Here, we follow the approaches in [5, 78] with some improvements. At this linear level, the results obtained in this thesis can be seen as generalizations of [5, 49, 67, 78] with optimal rates of the convergence of the solutions to zero (after changing of variables) and the relaxation of assumptions about the initial data.

Relaxation systems

In this thesis, *relaxation systems* are systems fitting in the class

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} f_j(u) + g(u) = 0. \tag{0.1}$$

In (0.1), d is a positive integer, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $t \in [0, +\infty)$. The unknown $u = u(x, t)$ belongs to an open subset $\Omega \subseteq \mathbb{R}^n$ for a certain positive integer n. The *flux* f_j for $j \in \{1, \dots, d\}$ and the *source term* g are sufficiently smooth functions from Ω to $\mathbb{R}^n$. For simplicity, we briefly call the relaxation systems by the class (0.1) hereafter.

Based on the natural existence of nonzero equilibria in many physical systems, we thus assume that the *equilibrium set* of (0.1) is nonzero, where the set is defined by $\mathcal{M} := \{u \in \Omega : g(u) = 0\}$.

The class (0.1) is *entropy dissipative* if it admits an *entropy* $\mathcal{E} = \mathcal{E}(u) \in \mathbb{R}$ satisfying that

$$(\mathcal{E}'(u) - \mathcal{E}'(u_*)) \cdot g(u) \geqslant 0 \qquad \forall u \in \Omega,\ \forall u_* \in \mathcal{M},$$

where $\cdot$ denotes the scalar product on $\mathbb{R}^n$ and $\mathcal{E}'$ is the Jacobian of $\mathcal{E}$ with respect to u. Moreover, the entropy dissipation condition holds *strictly* if

$$(\mathcal{E}'(u) - \mathcal{E}'(u_*)) \cdot g(u) \geqslant c|g(u)|^2 \qquad \forall u \in \Omega,\ \forall u_* \in \mathcal{M}.$$

Here, $\mathcal{E}$ is a strictly convex smooth function from Ω to $\mathbb{R}$ such that $\mathcal{E}''(u)f'_j(u)$ is symmetric for all $j \in \{1, \dots, d\}$ and $u \in \Omega$, where $\mathcal{E}''$ is the Hessian of $\mathcal{E}$ and f'_j is the Jacobian of f_j with respect to u. Moreover, there are sufficiently smooth functions $\mathcal{P}_j$ for $j \in \{1, \dots, d\}$ and $\mathcal{Q}$ from Ω to $\mathbb{R}$ such that

$$\partial_t \mathcal{E}(u) + \sum_{j=1}^{d} \partial_{x_j} \mathcal{P}_j(u) + \mathcal{Q}(u) = 0$$

for every smooth solution u to (0.1), where $\mathcal{P}'_j = (f'_j)^t \mathcal{E}'$, t denotes the transpose and $\mathcal{Q}' = \mathcal{E}' \cdot g$.

Assume that $g = (0, q)$, where 0 is the zero vector in $\mathbb{R}^m$ and q is a sufficiently smooth function from $\Omega \subseteq \mathbb{R}^m \times \mathbb{R}^{n-m}$ to $\mathbb{R}^{n-m}$ for a certain integer $m \in [1, n)$. Then, (0.1) is *nondegenerate* if the Jacobian q_w of $q = q(v, w)$ with respect to $w \in \mathbb{R}^{n-m}$ is a nonsingular $(n-m) \times (n-m)$ matrix whenever it is computed at $u_* = (v_*, w_*) \in \mathcal{M}$.

Let $u_* \in \mathcal{M}$ be constant, the linearization of (0.1) about u_* is given by

$$\partial_t \tilde{u} + \sum_{j=1}^{d} A_j \partial_{x_j} \tilde{u} + B\tilde{u} = 0, \tag{0.2}$$

where $\tilde{u} := u - u_* \in \mathbb{R}^n$, $A_j := f'_j(u_*)$ for $j \in \{1, \dots, d\}$ and $B := g'(u_*)$ are $n \times n$ matrices with real constant entries. Here, g' is the Jacobian of g with respect to u. The class (0.1) satisfies the *Shizuta–Kawashima condition* if the following holds

(SK). *for* $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \backslash \{0\}$, *if* z *is an eigenvector of* $A(\xi) := \sum_{j=1}^{d} \xi_j A_j$, *then* $z \notin \ker B$.

It then follows from [30,77] that if the above assumptions hold, then (0.1) has a unique global smooth solution for initial data in $H^s(\mathbb{R}^d)$ close to u_* where $s > d/2+1$ is an integer and $u_* \in \mathcal{M}$. Although the Shizuta–Kawashima condition, which is firstly introduced in [38,64] for hyperbolic-parabolic systems, is very crucial to the proofs in [30,77], it is in general not necessary for the global existence of smooth solutions to (0.1). Some examples of this fact are the case of gas dynamics in thermal nonequilibrium [78] and the case of the Kerr–Debye model for electromagnetic waves in nonlinear Kerr medium [12–14]. Indeed, a relaxation system can be completely decomposed into a pair of a linearly degenerated subsystem and a totally dissipative subsystem in some cases, a global smooth solution to the system thus can exist without assuming that the Shizuta–Kawashima condition holds. More general discussions about this fact are provided in [51].

Contrarily, we will see in Chapter 1 that (0.2) is globally well-posed for initial data in $H^s(\mathbb{R}^d)$ with $s \in \mathbb{R}$ if and only if the matrix-valued exponential operator $e^{iA(\xi)}$ is bounded uniformly for all $\xi \in \mathbb{R}^d$.

Striking examples

Discrete-velocity models

In the kinetic theory of gases, the Boltzmann equation is given by

$$\partial_t f + \xi \cdot \nabla_x f = Q(f), \tag{0.3}$$

where $\cdot$ denotes the scalar product on $\mathbb{R}^3$, $f = f(x, t, \xi) \geqslant 0$ is the density of particles at $(x, t, \xi) \in \mathbb{R}^3 \times [0, +\infty) \times \mathbb{R}^3$ and the nonlocal operator Q describes the interaction of particles. The equation (0.3) is in fact in the relaxation form (0.1). In 1989, DiPerna and Lions in [18] showed that the Cauchy problem for (0.3) is globally well-posed. However, the same result has not been well answered in the discrete-velocity case where ξ does not vary continuously in $\mathbb{R}^3$ but is restricted to a set of a finite number of speeds. The point is that the proof in [18] is based on a regularization obtained by an average procedure which holds only if ξ is continuous. Discrete-velocity models for (0.3) are thus not only interesting in their own difficulties but also very important for approximating (0.3) via numerical methods.

The equation (0.3) can be rewritten as

$$\partial_t f + u\partial_x f + v\partial_y f + w\partial_z f = (\partial_t f)_{\text{col}} = G - L.$$

The vector $(u, v, w) \in \mathbb{R}^3$ is the velocity of the density f at the position $(x, y, z) \in \mathbb{R}^3$ and at the time $t \in [0, +\infty)$. The rate of change $(\partial_t f)_{\text{col}}$ is the difference between the gain G and the loss L of particles of the density f after each collision. We now assume that the velocity set $\{(u, v, w)\}$ is finite. Then, between collisions, there is a finite set of particles f_i located at each velocity (u_i, v_i, w_i) in the velocity space for $i = 1, 2, \ldots, r$ and $r \geqslant 1$. With only six velocities for simplicity i.e. $r = 6$, Broadwell in [10] derived a system of 6 equations by computing the differences $G_i - L_i$ for $i = 1, \ldots, 6$. In the one-dimensional space, due to symmetry properties, the system is reduced to the relaxation 3×3 system

$$\begin{cases} \partial_t f_+ + v\partial_x f_+ = f_0^2 - f_+ f_-, \\ \partial_t f_0 = -\dfrac{1}{2}(f_0^2 - f_+ f_-), \\ \partial_t f_- - v\partial_x f_- = f_0^2 - f_+ f_-, \end{cases} \tag{0.4}$$

where $(x, t) \in \mathbb{R} \times [0, +\infty)$, f_-, f_0 and f_+ are the densities of particles moving with velocities constrained in the set $\{-v, 0, v\}$ for $v > 0$.

The global existence of solutions to (0.4) is studied in [6, 17]. Nonetheless, the proofs require some restrictions on the initial data either to be positive or to be small. Even though exact solutions to (0.4) are constructed in [17], they are nonnegative only if $t \geqslant t_*$ for a certain $t_* > 0$. Thus, for $t \in [0, t_*)$, these solutions are not relevant in the physical sense that the densities are nonnegative. In parallel to the global existence, large-time behaviors of solutions to (0.4) are studied in [33]. The solutions are showed to decay to the absolute Maxwellians (positive equilibria) in $H^1(\mathbb{R})$ at the heat-decay rate $t^{-\frac{1}{4}}$ as $t \to +\infty$, provided the initial data are in $H^1(\mathbb{R}) \cap L^1(\mathbb{R})$. On the other hand, traveling-wave solutions to the Broadwell systems and their stability are also studied in [11, 37].

More general forms of discrete-velocity models for the Boltzmann equation can be found in [16, 26] and large-time behaviors of their solutions can be found in [2].

Velocity-jump processes

Similarly to the discrete-velocity models, relaxation systems arising in correlated random walks can be also derived with a finite number of speeds.

The first prior formulation of this type of relaxation systems is known as the Goldstein–Kac model [29, 35].

Assume that there are two densities $f_i \geqslant 0$ moving with two velocities $\nu_i \in \mathbb{R}$ for $i = 1, 2$; one goes to the left and one goes to the right of the x-axis where $x \in \mathbb{R}$. On the other hand, the transition rates from ν_i to ν_j are $\mu_{ii} \leqslant 0$ and $\mu_{ij} \geqslant 0$ ($i \neq j$) for $i, j \in \{1, 2\}$. Moreover, $\mu_{11} + \mu_{12} = \mu_{21} + \mu_{22} = 0$. Hence, if $|\nu_1| = |\nu_2|$ and $\mu_{12} = \mu_{21}$, the simplest Goldstein–Kac 2×2 system in the one-dimensional space is given by

$$\begin{cases} \partial_t f_1 - \nu \partial_x f_1 = -\mu f_1 + \mu f_2, \\ \partial_t f_2 + \nu \partial_x f_2 = \mu f_1 - \mu f_2 \end{cases} \tag{0.5}$$

for some $\mu \geqslant 0$ and $\nu \geqslant 0$, where $(x, t) \in \mathbb{R} \times [0, +\infty)$.

Imitating the above Goldstein–Kac 2×2 system, we can consider more general systems in the d-dimensional space with n densities $f_i \geqslant 0$ moving with n velocities $\boldsymbol{\nu}_i := \{\nu_i^1, \dots, \nu_i^d\} \in \mathbb{R}^d$ for $i \in \{1, \dots, n\}$. Let the transition rates from $\boldsymbol{\nu}_i$ to $\boldsymbol{\nu}_j$ be $\mu_{ii} \leqslant 0$ and $\mu_{ij} \geqslant 0$ ($i \neq j$) for $i, j \in \{1, \dots, n\}$. Furthermore, $\sum_{j=1}^n \mu_{ij} = 0$ for all $i \in \{1, \dots, n\}$. Then, it follows from [49] that the i-th component equation of a generalized Goldstein–Kac $n \times n$ system in the d-dimensional space is given by

$$\partial_t f_i + \boldsymbol{\nu}_i \cdot \nabla_x f_i + \sum_{j \neq i} \mu_{ij} (f_i - f_j) = 0, \qquad i = 1, 2, \dots, n, \tag{0.6}$$

where $(x, t) \in \mathbb{R}^d \times [0, +\infty)$ and $\cdot$ denotes the scalar product on $\mathbb{R}^d$.

As a pattern of transport mechanisms, other models which are similar to the Goldstein–Kac systems have been also derived for variously different phenomena, for instance, the reaction-hyperbolic equations for the transport of neurofilaments in axons [15, 22, 23, 74].

Since the Goldstein–Kac systems are in the linear form (0.2) and the associated matrix-valued operator $e^{iA(\xi)}$ where $A(\xi) = \sum_{j=1}^d \xi_j \mathrm{diag}\,(\nu_1^j, \dots, \nu_n^j)$ is uniformly bounded for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, the systems are in general well-posed for initial data in $H^s(\mathbb{R}^d)$ and $s \in \mathbb{R}$ (see Chapter 1).

Moreover, for the system (0.5), the diffusive decay of the solution to zero as $t \to +\infty$ can be observed easily due to the fact that (0.5) is equivalent to the telegraph equation

$$\partial_{tt} u + 2\mu \partial_t u - \nu^2 \partial_{xx} u = 0, \tag{0.7}$$

where $(x, t) \in \mathbb{R} \times [0, +\infty)$, $u := f_1 \pm f_2$, $\mu \geqslant 0$ and $\nu \geqslant 0$. If $\mu > 0$ and $\nu > 0$, it follows from [48] that

$$\left\| u - \phi - e^{-\mu t} \frac{u_0(\cdot + \nu t) + u_0(\cdot - \nu t)}{2} \right\|_{L^p} \leqslant C t^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p}) - 1} \|(u_0, u_1)\|_{L^q} \tag{0.8}$$

for a constant $C > 0$, $1 \leqslant q \leqslant p \leqslant \infty$ and $t \geqslant 1$, where $(u_0, u_1) \in L^q(\mathbb{R}) \times L^q(\mathbb{R})$ is an initial datum of (0.7). Moreover, ϕ is a solution to

$$\partial_t \phi - \frac{\nu^2}{2\mu} \partial_{xx} \phi = 0$$

with the initial datum $\phi|_{t=0} = u_0 + u_1$. Furthermore, ϕ decays to zero in $L^p(\mathbb{R})$ at the rate $t^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p})}$ as $t \to +\infty$ for initial data in $L^q(\mathbb{R})$ and $1 \leqslant q \leqslant p \leqslant \infty$ (see [27]).

The decay estimate (0.8) is optimal in the sense that p and q are general and the assumption that $(u_0, u_1) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ can be relaxed. Indeed, the exponentially decaying singular part of u, which may contain singularities coming from the initial datum, is already subtracted. Moreover, this part is obtained by $e^{-\mu t}\psi$, where ψ is a solution to the linear wave equation

$$\partial_{tt}\psi - \nu^2\partial_{xx}\psi = 0$$

with the initial datum $(\phi, \partial_t\psi)|_{t=0} = (0, u_0)$.

For more general cases, large-time behaviors of solutions to (0.6) with a symmetric matrix $B := (\mu_{ij})_{i,j\in\{1,\dots,n\}}$ are accomplished in [49]. The solutions decay to zero in $L^2(\mathbb{R}^d)$ at the rate $t^{-\frac{d}{4}}$ as $t \to +\infty$ for initial data in $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Furthermore, they approach solutions to parabolic systems in $L^2(\mathbb{R}^d)$ at the rate $t^{-\frac{d}{4}-\frac{1}{2}}$ as $t \to +\infty$. The parabolic systems are obtained by using the Kirchhoff's matrix-tree theorem from the graph theory. Despite that, singular terms as the one in (0.8) are not studied.

Relaxation numerical schemes

The relaxation numerical schemes are originally motivated by Jin and Xin in [34] in order to approximate the one-dimensional scalar conservation laws

$$\partial_t u + \partial_x f(u) = 0, \tag{0.9}$$

where $(x, t) \in \mathbb{R} \times [0, +\infty)$. The approximation systems are given by

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \alpha^2\partial_x u = \dfrac{1}{\varepsilon}(f(u) - v), \end{cases} \tag{0.10}$$

where $(u, v) \in \mathbb{R}\times\mathbb{R}$, f is a smooth function from $\mathbb{R}$ to $\mathbb{R}$, $\alpha > 0$ and $\varepsilon \in (0, 1]$.

The idea here is to consider the Chapman–Enskog expansion

$$v = f(u) + \varepsilon v_1 + o(\varepsilon), \qquad \varepsilon \to 0^+. \tag{0.11}$$

Inserting (0.11) into (0.10), it is easy to see that

$$v_1 = -(\alpha^2 - (f'(u))^2)\partial_x u. \tag{0.12}$$

Thus, from (0.10) - (0.12), u satisfies the approximation equation

$$\partial_t u + \partial_x f(u) - \varepsilon(\alpha^2 - (f'(u))^2)\partial_{xx} u = 0.$$

Therefore, if the sub-characteristic stability condition

$$|f'(u)| < \alpha$$

holds (see [47, 72]), then u in fact converges to a solution to (0.9) as $\varepsilon \to 0^+$. This result is especially significant due to the regularity of u.

The systems (0.10) can be generalized to systems of $n(d+1)$ equations in the d-dimensional space, namely

$$\begin{cases} \partial_t u + \displaystyle\sum_{j=1}^{d} \partial_{x_j} v_j = 0, \\ \partial_t v_j + A_j\partial_{x_j} u = \dfrac{1}{\varepsilon}(F_j(u) - v_j), \qquad j = 1, \dots, d. \end{cases} \tag{0.13}$$

In (0.13), $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $t \in [0, +\infty)$. Moreover, $(u, v_j) \in \mathbb{R}^n \times \mathbb{R}^n$, A_j is a real $n \times n$ matrix and F_j is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$ for $j \in \{1, \dots, d\}$.

The systems (0.13) are well studied in [28, 32, 34, 44–46, 50, 61]. Most of these works are related to the traveling-wave solutions. For the stability of the constant equilibria, one can see in [5, 59, 60] but with more general forms. At least for some specific classes of (0.13) with initial data in $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, the solutions converge to the constant equilibria in $L^p(\mathbb{R}^d)$ at the rate $t^{-\frac{d}{2}(1-\frac{1}{p})}$ as $t \to +\infty$ for $p \geqslant \min\{d, 2\}$ by applying the results in [5]. Particularly, they approach solutions to linear parabolic systems in $L^p(\mathbb{R}^d)$ at the rate $t^{-\frac{d}{2}(1-\frac{1}{p})-\mu}$ as $t \to +\infty$ for $p \geqslant 2$ and $d \geqslant 2$. Here, $\mu = 1/2$ and the parabolic systems are obtained by using the Chapman–Enskog expansion. In the one-dimensional space where $d = 1$, the parabolic systems include also the quadratic terms since the convolution of these terms and the Green kernels gives the decay rate $t^{-\frac{1}{2}(1-\frac{1}{p})}$ in $L^p(\mathbb{R})$ for $p \geqslant 1$ as $t \to +\infty$ once it was integrated in time. Moreover, $\mu \in [0, 1/2)$ and $p \geqslant 1$ in this case, provided the initial data are small enough.

The isentropic compressible Euler equations with damping

The isentropic compressible Euler equations with damping for perfect fluid flows have the form

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p(\rho) = -\alpha \rho u. \end{cases} \tag{0.14}$$

Here, $(x, t) \in \mathbb{R}^d \times [0, +\infty)$, $\cdot$ denotes the scalar product on $\mathbb{R}^d$, $\otimes$ denotes the tensor product of two d-dimensional vectors and the damping coefficient $\alpha > 0$. The fluid density $\rho = \rho(x, t) \in \mathbb{R}$ and the fluid velocity $u = u(x, t) \in \mathbb{R}^d$. The pressure p depending on ρ in the sense that $p(\rho) = \frac{1}{\gamma}\rho^\gamma$ where the adiabatic exponent $\gamma > 1$. The systems (0.14) describe ideal compressible fluids passing through porous media that generate friction forces.

The systems (0.14) indeed can be put in the class (0.1) as proved in [73]. Without loss of generality, we can assume $\alpha = 1$. It is then sufficient to consider the new variable

$$v = (v_1, v_2) := \begin{pmatrix} -\dfrac{|m|^2}{2\rho^2} + h'(\rho) & \dfrac{m}{\rho} \end{pmatrix}^t,$$

where t denotes the transpose, $m := \rho u$ and $h(\rho) := \displaystyle\int_1^\rho \frac{p'(s)}{s}\,ds$. In the variable v, the systems (0.14) are written as

$$\partial_t v + \sum_{j=1}^d A_0^{-1}(v) A_j(v) \partial_{x_j} v + A_0^{-1}(v) B(v) = 0. \tag{0.15}$$

In (0.15), the positive-definite symmetric matrix

$$A_0(v) := \begin{pmatrix} 1 & v_2^t \\ v_2 & v_2 \otimes v_2 + p'(\rho) I_d \end{pmatrix},$$

where I_d is the identity matrix in $\mathbb{R}^{d\times d}$. The matrix $A_j(v)$ is given by

$$A_j(v) := \begin{pmatrix} v_2^j & v_2^t v_2^j + p'(\rho)e_j^t \\ v_2 v_2^j + p'(\rho)e_j & v_2^j(v_2 \otimes v_2 + p'(\rho)I_d) + p'(\rho)(v_2 \otimes e_j + e_j \otimes v_2) \end{pmatrix},$$

where v_2^j is the j-th component of v_2 and e_j is the j-th unit vector of the standard basis of $\mathbb{R}^d$ for $j \in \{1,\dots,d\}$. On the other hand, the vector $B(v)$ satisfies

$$B(v) := \begin{pmatrix} 0 & -p'(\rho)v_2 \end{pmatrix}^t.$$

The global existence and uniqueness of smooth solutions to (0.14) is studied in [55] for $d = 1$ and in [65, 71] for $d \geqslant 2$, provided the initial data are sufficiently small. In general, the damping term in (0.14) is not strong enough to prevent the formation of shocks if the initial data are large although it is true for small initial data. Moreover, the convergence of the solutions to the constant equlibria is also obtained in [65] for $d = 3$ and in [71] for $d \geqslant 2$. The convergence rate in $L^p(\mathbb{R}^d)$ is $t^{-\frac{d}{2}(1-\frac{1}{p})}$ as $t \to +\infty$ for $p > 1$ and initial data in $L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

Diffusive large-time behaviors

Consider the linear relaxation system

$$\partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u + Bu = 0, \tag{0.16}$$

where $(x,t) \in \mathbb{R}^d \times [0,+\infty)$, $u = u(x,t) \in \mathbb{R}^n$, A_j and B are matrices in $\mathbb{R}^{n\times n}$ for $j \in \{1,\dots,d\}$. Let u be a solution to the Cauchy problem for (0.16) with an initial datum u_0. We then discuss about large-time behaviors of u.

For the case where A_j for $j \in \{1,\dots,d\}$ and B are symmetric matrices, let B be positive semi-definite and let (0.16) satisfy the Shizuta–Kawashima condition: there is no eigenvector of $A(\xi) = \sum_{j=1}^d \xi_j A_j$ in $\ker B$ for $\xi = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d \backslash \{0\}$. Following from [64], we have

$$\|u\|_{L^2} \leqslant C(1+t)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{2})}\|u_0\|_{L^q} + Ce^{-ct}\|u_0\|_{L^2} \tag{0.17}$$

for $t \geqslant 0$, $u_0 \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $q \in [1,2]$.

The result is later generalized with an unnecessarily symmetric matrix B in [5]. Assume that (0.16) can be transformed into the *conservative-dissipative* form i.e. A_j for $j \in \{1,\dots,d\}$ is symmetric and $B = \operatorname{diag}(O, D)$, where O is the null matrix in $\mathbb{R}^{m\times m}$ and D is not necessarily symmetric but a positive-definite matrix in $\mathbb{R}^{(n-m)\times(n-m)}$ for an integer $m \in [1, n)$. If the Shizuta–Kawashima condition holds, then u is decomposed into a *conservative part* $u^{(1)}$ and a *dissipative part* $u^{(2)}$ such that

$$\|u^{(1)}\|_{L^p} \leqslant Ct^{-\frac{d}{2}(1-\frac{1}{p})}\|u_0\|_{L^1}$$

and that

$$\|u^{(2)}\|_{L^2} \leqslant Ce^{-ct}\|u_0\|_{L^2} \tag{0.18}$$

for all $t \geqslant 1$, $p \geqslant \min\{d, 2\}$ and $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Moreover, if we consider $u = (v, w) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ based on the form of B, then

$$\|v - U\|_{L^p} \leqslant Ct^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}\|u_0\|_{L^1} \qquad \forall t \geqslant 1,$$

where $U \in \mathbb{R}^m$ solves the Cauchy problem for

$$\partial_t U + \sum_{j=1}^{d} C_j \partial_{x_j} U - \sum_{j=1}^{d}\sum_{h=1}^{d} D_{jh}\partial_{x_j x_h} U = 0 \tag{0.19}$$

with the initial datum $U_0 = L_+ u_0$. In (0.19), the coefficient $m \times m$ matrices

$$C_j := L_+ A_j R_+ \qquad \text{and} \qquad D_{jh} := L_+ A_j R_- D^{-1} L_- A_h R_+$$

for j, $h \in \{1, \dots, d\}$. Moreover, $L_+ \in \mathbb{R}^{m\times n}$ and $R_+ \in \mathbb{R}^{n\times m}$ are obtained from the eigenprojection $P_+ \in \mathbb{R}^{n\times n}$ onto $\ker B$ by rank factorization. Similarly, $L_- \in \mathbb{R}^{(n-m)\times n}$ and $R_- \in \mathbb{R}^{n\times(n-m)}$ are obtained from $I - P_+$ by rank factorization, where I is the identity matrix in $\mathbb{R}^{n\times n}$.

The system (0.19) is indeed obtained by applying the Chapman–Enskog expansion to (0.16) and taking advantage of the conservative-dissipative form. For some specific cases such as the Goldstein–Kac systems, a more precise U can be obtained by using the Kirchhoff's matrix-tree theorem from the graph theory (see [49]).

Noting that in a general situation where B is not symmetric, u does not necessarily have an exponentially decaying part as in (0.17) and (0.18). This part arises in the Fourier transform $\hat{u} = \hat{u}(\xi, t)$ of u for large ξ. In [67], there exist cases where the decay rate of the high-frequency part is at most polynomial, provided the initial datum u_0 is regular enough, for instance, the Euler–Maxwell systems.

Let $\lambda(i\xi)$ be a representation of the eigenvalues of $E(i\xi) := B + i\sum_{j=1}^{d} \xi_j A_j$ for $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, the difference between the two types can be explained due to estimates for the real part of λ. Indeed, for the exponential type, there is a constant $\theta > 0$ such that

$$\operatorname{Re}\lambda(i\xi) \geqslant \theta\frac{|\xi|^2}{1+|\xi|^2} \qquad \forall \xi \neq 0. \tag{0.20}$$

On the other hand, for the polynomial type, one has

$$\operatorname{Re}\lambda(i\xi) \geqslant \theta\frac{|\xi|^2}{(1+|\xi|^2)^2} \qquad \forall \xi \neq 0. \tag{0.21}$$

Since $e^{-\lambda(i\xi)t} \sim e^{-\theta t}$ in (0.20) and $e^{-\lambda(i\xi)t} \sim e^{-\frac{\theta}{|\xi|^2}t}$ in (0.21) as $|\xi| \to +\infty$, the case (0.21) is at most polynomial and requires u_0 to be regular enough.

Noting also that as soon as the Shizuta–Kawashima condition does not hold, counterexamples of the decay of u may exist (see Subsection 2.1.1 of Chapter 2).

Based on the Goldstein–Kac 2×2 system (0.5), where the unique solution has the optimal heat-decay rate $t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}$ in $L^p(\mathbb{R})$ as $t \to +\infty$ for initial data in $L^q(\mathbb{R})$ and $1 \leqslant q \leqslant p \leqslant \infty$, we expect to obtain a similar L^p-L^q estimate as in (0.8) for the solution u to (0.16) in the case of (0.20). The main aim of this thesis is to prove the following results as below. For more general and detailed discussions, one see Chapter 3 of this thesis.

We primarily consider the eigenprojection $P_0^{(0)} \in \mathbb{R}^{n\times n}$ and the reduced resolvent coefficient $Q_0^{(0)} \in \mathbb{R}^{n\times n}$ associated with the eigenvalue 0 of B, namely

$$P_0^{(0)} := -\frac{1}{2\pi i}\int_{\Gamma_0}(B-zI)^{-1}\,dz \text{ and } Q_0^{(0)} := \frac{1}{2\pi i}\int_{\Gamma_0} z^{-1}(B-zI)^{-1}\,dz,$$

where Γ_0, in the resolvent set of B, is an oriented closed curve enclosing 0 except for the other eigenvalues of B. On the other hand, we consider

$$P_{0j}^{(1)} := -P_0^{(0)}A_jQ_0^{(0)} - Q_0^{(0)}A_jP_0^{(0)}, \qquad j = 1,\dots,d. \tag{0.22}$$

In the one-dimensional space where $d = 1$, assume that

Condition $\mathscr{A}$ (Hyperbolicity)**.** *The matrix* $A := A_1$ *is diagonalizable with real eigenvalues.*

Condition $\mathscr{B}$ (Partial dissipation)**.** *The spectrum of* B *is decomposed into* $\sigma(B) = \{0\}\cup\sigma_+$ *where* 0 *is semi-simple and* $\sigma_+ \subseteq \{\lambda\in\mathbb{C} : \mathrm{Re}\,\lambda > 0\}$.

Condition $\mathscr{C}$ (Reduced hyperbolicity)**.** *The matrix* $C := P_0^{(0)}AP_0^{(0)}$ *considered in* $\ker B$ *is diagonalizable with real eigenvalues.*

Condition $\mathscr{D}$ (Uniform dissipation)**.** *There is a positive constant* θ *such that*

$$\mathrm{Re}\,\lambda(i\xi) \geqslant \frac{\theta|\xi|^2}{1+|\xi|^2} \quad \textit{for all} \quad \xi\neq 0,$$

where $\lambda(i\xi)$ *is any eigenvalue of the operator* $E(i\xi) = B + i\xi A$ *for* $\xi\in\mathbb{R}$.

We consider the Cauchy problem

$$\begin{cases} \partial_t U + C\partial_x U - D\partial_{xx}U = 0,\\ U|_{t=0} = P_0^{(0)}u_0 \end{cases} \tag{0.23}$$

and the Cauchy problem

$$\begin{cases} \partial_t V + A\partial_x V + \Pi_A(B)V = 0,\\ V|_{t=0} = u_0, \end{cases} \tag{0.24}$$

where $(x,t)\in\mathbb{R}\times[0,+\infty)$, $U = U(x,t)\in\mathbb{R}^n$ and $V = V(x,t)\in\mathbb{R}^n$. In (0.23), the $n\times n$ matrix

$$D := -\sum_{h=1}^{s} P_h^{(0)}(P_{01}^{(1)}BP_{01}^{(1)} + P_0^{(0)}AP_{01}^{(1)} + P_{01}^{(1)}AP_0^{(0)})P_h^{(0)},$$

where $P_h^{(0)}\in\mathbb{R}^{n\times n}$ is the eigenprojection associated with $c_h\in\sigma(C,\ker B)$ for $h\in\{1,\dots,s\}$ and $\sigma(C,\ker B)$ is the spectrum of C considered in $\ker B$ with the cardinality s. In (0.24), the $n\times n$ matrix

$$\Pi_A(B) := \sum_{h=1}^{r}\Pi_h^{(0)}B\Pi_h^{(0)},$$

where $\Pi_h^{(0)}\in\mathbb{R}^{n\times n}$ is the eigenprojection associated with $\alpha_h\in\sigma(A)$ for $h\in\{1,\dots,r\}$ and $\sigma(A)$ is the spectrum of A with the cardinality r.

Theorem 0.1 (L^p-L^q decay estimates [52]). *For* $u_0 \in L^q(\mathbb{R})$*, let* u, U *and* V *be respectively solutions to* (0.16), (0.23) *and* (0.24)*. If the conditions* $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ *and* $\mathscr{D}$ *hold, then for* $1 \leqslant q \leqslant p \leqslant \infty$ *and* $t \geqslant 1$*, there are positive constants* c *and* C *such that*

$$\|u - U - V\|_{L^p} \leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}}\|u_0\|_{L^q}. \qquad (0.25)$$

Moreover, one has

$$\|U\|_{L^p} \leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_{L^q} \qquad \textit{and} \qquad \|V\|_{L^q} \leqslant Ce^{-ct}\|u_0\|_{L^q}. \qquad (0.26)$$

In the multi-dimensional space where $d \geqslant 1$, assume that

Condition $\mathscr{A}^*$ (Hyperbolicity). *For* $w = (w_1, \ldots, w_d) \in \mathbb{S}^{d-1}$*, the unit sphere in* $\mathbb{R}^d$, $A = A(w) := \sum_{j=1}^{d} w_j A_j$ *is uniformly diagonalizable with real linear eigenvalues i.e. there are an invertible matrix* $R = R(w)$ *and a constant* $C > 0$ *such that*

$$\sup_{w\in\mathbb{S}^{d-1}} |R(w)||R^{-1}(w)| \leqslant C < +\infty$$

for a matrix norm $|\cdot|$*. Moreover,* $R^{-1}AR$ *is a diagonal matrix whose nonzero entries are real linear in* $w \in \mathbb{S}^{d-1}$*.*

Condition $\mathscr{R}^*$ (Diagonalizing matrix). *There is a matrix* R *uniformly diagonalizing* A *such that* $R^{-1}BR$ *is a constant matrix independent from* $w \in \mathbb{S}^{d-1}$*.*

Condition $\mathscr{B}^*$ (Partial dissipation). *The spectrum of* B *is decomposed into* $\sigma(B) = \{0\} \cup \sigma_+$ *where* 0 *is simple and* $\sigma_+ \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$*.*

Condition $\mathscr{D}^*$ (Uniform dissipation). *There is a positive constant* θ *such that*

$$\operatorname{Re}\lambda(i\xi) \geqslant \frac{\theta|\xi|^2}{1+|\xi|^2} \quad \textit{for all} \quad \xi \neq 0,$$

where $\lambda(i\xi)$ *is any eigenvalue of the operator* $E(i\xi) = B + i\sum_{j=1}^{d} \xi_j A_j$ *for* $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$*.*

We consider the Cauchy problem

$$\begin{cases} \partial_t U + \mathbf{c}\cdot\nabla_x U - \operatorname{div}(\mathbf{D}\nabla_x U) = 0, \\ U|_{t=0} = P_0^{(0)} u_0, \end{cases} \qquad (0.27)$$

where $(x, t) \in \mathbb{R}^d \times [0, +\infty)$, $\cdot$ denotes the scalar product on $\mathbb{R}^d$ and $U = U(x, t) \in \mathbb{R}^n$. On the other hand, the vector $\mathbf{c} = (c_h)_{h\in\{1,\ldots,d\}} \in \mathbb{R}^d$ and the matrix $\mathbf{D} = (D_{h\ell})_{h,\ell\in\{1,\ldots,d\}} \in \mathbb{R}^{d\times d}$ have

$$c_h := \operatorname{tr}\left(A_h P_0^{(0)}\right) \quad \text{and} \quad D_{h\ell} := \frac{1}{2}\operatorname{tr}\left(A_h P_0^{(0)} A_\ell Q_0^{(0)} + A_h Q_0^{(0)} A_\ell P_0^{(0)}\right).$$

Here, tr denotes the trace.

Theorem 0.2 (L^p-L^q decay estimates [54]). *Let* u *be a solution to the Cauchy problem* (0.16) *with an initial datum* $u_0 \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ *for* $1 \leqslant q \leqslant \infty$*. Under the assumptions* $\mathscr{A}^*$, $\mathscr{R}^*$, $\mathscr{B}^*$ *and* $\mathscr{D}^*$, u *is decomposed into*

$$u(x, t) = u^{(1)}(x, t) + u^{(2)}(x, t),$$

where

$$u^{(1)}(x, t) := \mathcal{F}^{-1}(e^{-Et}P_0\chi) * u_0(x),$$

$u^{(2)}$ *is the remainder,* P_0 *is the eigenprojection associated with the eigenvalue* λ_0 *of* E*,* $\lambda_0(i\xi)$ *converges to* 0 *as* $|\xi| \to 0$ *and* χ *is a cut-off function valued in* $[0,1]$ *with support contained in the ball* $B(0,\varepsilon) \subset \mathbb{R}^d$ *for small* $\varepsilon > 0$*.*

Moreover, for $1 \leqslant q \leqslant p \leqslant \infty$*, there are constants* $c > 0$ *and* $C > 0$ *such that*

$$\|u^{(1)} - U\|_{L^p} \leqslant Ct^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|u_0\|_{L^q} \qquad \forall t \geqslant 1 \tag{0.28}$$

and $u^{(2)}$ *satisfies*

$$\|u^{(2)}\|_{L^2} \leqslant Ce^{-ct}\|u_0\|_{L^2} \qquad \forall t \geqslant 0, \tag{0.29}$$

where U *which is a solution to* (0.27) *satisfies*

$$\|U\|_{L^p} \leqslant Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_{L^q} \qquad \forall t \geqslant 1. \tag{0.30}$$

In general, the rate $t^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}$ in (0.25) and (0.28) cannot be exceeded except for some cases where the following conditions hold in addition
Condition $\mathscr{E}$ (Equilibrium manifold)**.** *The eigenvalue* 0 *of* B *is simple.*
Condition $\mathscr{S}$ (Symmetry)**.** *There is an invertible matrix* $S = S(w)$ *such that*

$$SA = -AS \qquad \text{and} \qquad SB = BS,$$

where $A = A(w) = \sum_{j=1}^{d} w_j A_j$ *for* $w = (w_1, \dots, w_d) \in \mathbb{S}^{d-1}$*.*

Indeed, we primarily modify the systems (0.23) and (0.27) by

$$\begin{cases} \partial_t U - \operatorname{div}(\mathbf{D}\nabla_x U) = 0, \\ U|_{t=0} = P_0^{(0)} u_0 + \mathbf{P}_0^{(1)} \cdot \nabla_x u_0, \end{cases} \tag{0.31}$$

where $(x,t) \in \mathbb{R}^d \times [0, +\infty)$, $\cdot$ denotes the scalar product on $\mathbb{R}^d$, $\mathbf{D}$ is similar to the one in (0.27), $\mathbf{P}_0^{(1)} := (P_{0j}^{(1)})_{j \in \{1,\dots,d\}}$ and $P_{0j}^{(1)}$ is given by (0.22) for $j \in \{1, \dots, d\}$.

Theorem 0.3 (Increasing decay rates [52,54])**.** *Under the same hypotheses in Theorem 0.1 for* $d = 1$ *(resp. in Theorem 0.2 for* $d \geqslant 2$*), if* $u_0 \in W^{1,q}(\mathbb{R}^d)$ *and the conditions* $\mathscr{E}$ *and* $\mathscr{S}$ *hold additionally, then the decay rate in* (0.25) *(resp. in* (0.28)*) is increased to* $t^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-1}$ *for* $1 \leqslant q \leqslant p \leqslant \infty$ *and* $t \geqslant 1$*.*

The above results in fact generalize [5,49,64,78] at the linear level. Moreover, they can be applied to linear systems arising in the linearization (about constant equilibria) of, for instance, the Broadwell system (0.4), the Goldstein–Kac systems (0.6), the Jin–Xin systems (0.13) and the isentropic compressible Euler equations with damping (0.14). The significant novelties here are

i) the large-time asymptotic profile U of u is sharp;

ii) the decay rate $t^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-\alpha}$ for $\alpha \in \{0, \frac{1}{2}, 1\}$ with general exponents p, $q \in [1,\infty]$ and $d \geqslant 1$ is obtained;

iii) the assumption about L^2-initial data is relaxed in the one-dimensional space where $d = 1$ since the exponentially decaying singular part V of u is subtracted;

iv) the strictly optimal case where $\alpha = 1$ is established for a class of symmetry linear relaxation systems including the Goldstein–Kac 2×2 system (0.5) and the linearized isentropic compressible Euler equations with damping.

The proofs of these results are based on expansions of solutions to (0.16) in the Fourier space and an interpolation argument between the L^∞-L^1 and L^p-L^p estimates for $1 \leqslant p \leqslant \infty$. However, since the expansions require sufficiently uniform properties, it slightly explains why we need the simplicity of the eigenvalue 0 of the matrix B and why singular terms as V in (0.25) cannot be subtracted in the multi-dimensional space where $d \geqslant 2$.

Thesis organization

The thesis is organized as follows.

Chapter 1 (The Cauchy problem for first-order linear systems with constant coefficients). We study the global well-posedness of the Cauchy problem for (0.16) with initial data in $H^s(\mathbb{R}^d)$ where $s \in \mathbb{R}$ and the integer $d \geqslant 1$. *Hyperbolic operators* with examples and their relevant features are then considered. The global well-posedness with initial data in $L^p(\mathbb{R}^d)$ for $p \in [1, \infty]$ is also discussed.

Chapter 2 (Linear stability of constant equilibria). We invoke historic and recent works dealt with stability structures in order for solutions to (0.16) to decay to zero as time tends to infinity. We discuss the key role Shizuta–Kawashima condition. Moreover, useful tools for constructing large-time asymptotic profiles of the solutions are introduced. The tools include the Chapman–Enskog expansion and the asymptotic expansion in the Fourier space.

Chapter 3 (L^p-L^q decay estimates). Based on the stability conditions discussed in Chapter 2, we then study large-time behaviors of solutions to (0.16). This chapter is divided into three sections. The first section is to study the case of the one-dimensional space, the second section is to study the case of the multi-dimensional space and the last one is dealt with a class of symmetry linear relaxation systems in any space dimension. In each section, we study the L^∞-L^1 estimate, the L^p-L^p estimate for $p \in [1, \infty]$ and we finally give proofs of the main results of the section by an interpolation argument.

Appendix A (Lebesgue spaces and the Fourier transform). The first section is to recall the Lebesgue space $L^p(\mathbb{R}^d)$ where $p \in [1, \infty]$ in the sense of Bochner for matrix-valued or vector-valued functions or distributions on $\mathbb{R}^d$. The well-known Young inequality and the Riesz–Thorin complex interpolation inequality are also considered. The second section is to evoke the usual Fourier transform defined on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and its distributional generalization on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and the dual $\mathcal{S}'(\mathbb{R}^d)$. Particularly, we recall the useful Carleson–Beurling multiplier estimate. Such a multiplier estimate is useful for proofs of the L^p-L^p estimate with $p \in [1, \infty]$.

Appendix B (Perturbation theory for linear operators). Let

$$T(z) = T^{(0)} + zT^{(1)} + z^2T^{(2)} + \dots, \qquad z \in \mathbb{C}.$$

The aim of this chapter is to study the behavior of the eigenvalues of T near *exceptional points*, where the exceptional points are in $\mathbb{C}$ such that the eigenvalues intersect there. The eigenprojections and eigennilpotents associated with the eigenvalues near the exceptional points are also studied. Most of the materials in this chapter are in [36]. These results are applied to the previous chapters.

Chapter 1

The Cauchy problem for first-order linear systems with constant coefficients

For integers $n \geqslant 1$ and $d \geqslant 1$, we study the global (in time) existence and uniqueness of a solution to the Cauchy problem

$$\begin{cases} \partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u + Bu = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1.1}$$

Here, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $t \geqslant 0$. The unknown $u = u(x,t)$ and given $u_0 = u_0(x)$ are vectors in $\mathbb{R}^n$. Moreover, A_j for $j \in \{1, \dots, d\}$ and B are constant matrices in $\mathbb{R}^{n \times n}$.

Indeed, we primarily find a solution u to (1.1) in $C([0,+\infty); X)$ for $u_0 \in X$, where X is an appropriate function space and $C([0,+\infty); X)$ is the space of all continuous maps from $[0,+\infty)$ to X. By defining the differential operator

$$\mathcal{L} := \sum_{j=1}^{d} A_j \partial_{x_j} + B, \tag{1.2}$$

we thus expect to be able to construct a strongly continuous semigroup associated with the differential operator $\mathcal{L}$ on X. Moreover, such a semigroup is denoted by $e^{-\mathcal{L}t}$ for $t \geqslant 0$. If it exists, it belongs to $L(X)$, the space of all linear maps from X to X, and the following hold

i) $e^{-\mathcal{L}t}|_{t=0} = \mathrm{Id}$, where Id is the identity operator;

ii) $e^{-\mathcal{L}(t+s)} = e^{-\mathcal{L}t} \circ e^{-\mathcal{L}s}$ for $t \geqslant 0$ and $s \geqslant 0$, where $\circ$ can be considered as the composition;

iii) $\lim_{t \to 0} \|e^{-\mathcal{L}t} u_0 - u_0\|_X = 0$ for $u_0 \in X$, where $\|\cdot\|_X$ is a suitable norm equipped with X.

Hence, the idea is that we can consider the Fourier transform $\mathcal{F}$, which is an automorphism of $L^2(\mathbb{R}^d)$. More generally, we can consider the generalized Fourier transform also denoted by $\mathcal{F}$, which is an automorphism of the

Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and the dual space $\mathcal{S}'(\mathbb{R}^d)$. Then, we define $e^{-\mathcal{L}t}$ as a pseudo-differential operator associated with a matrix-valued symbol e^{-Et}, namely

$$e^{-\mathcal{L}t}u_0 := \mathcal{F}^{-1}(e^{-Et}\hat{u}_0) \qquad \forall t \geqslant 0. \tag{1.3}$$

In (1.3), $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, the operator E from $\mathbb{C}^d$ to $\mathbb{C}^{n\times n}$ is defined by

$$E(i\xi) := B + i\sum_{j=1}^{d} \xi_j A_j \qquad \forall \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$$

and $\hat{u}_0 := \mathcal{F}(u_0)$ is the Fourier transform of u_0. Noting that

$$e^M := \sum_{h=0}^{+\infty} \frac{1}{h!} M^h \qquad \forall M \in \mathbb{C}^{n\times n}.$$

The global well-posedness problem then holds if $e^{-Et}\hat{u}_0$ for $u_0 \in X$ is bounded in X uniformly for all $t \geqslant 0$, where X is an appropriate function space of which the Fourier transform $\mathcal{F}$ is an automorphism.

1.1 Hyperbolic operators

To study the global well-posedness of the problem (1.1) in $C([0,+\infty); X)$ where X is a complete space, it is then sufficient to assume that $B = O$ where O is the null matrix in $\mathbb{R}^{n\times n}$.

Indeed, we argue as in [3]. If (1.1) admits a unique solution in $C([0,+\infty); X)$ for initial data in X, then for $u_0 \in X$ and $t \geqslant 0$, $e^{-\mathcal{L}t}u_0 := u(\cdot, t)$ defines a strongly continuous semigroup on X, where $\mathcal{L}$ is given by (1.2) and u is the unique solution to the Cauchy problem (1.1) with the initial datum u_0. In particular, it follows from [20] that $\|e^{-\mathcal{L}t}\|_{L(X)} \leqslant ce^{\omega t}$ for all $t \geqslant 0$ and some constants $c > 0$ and $\omega > 0$.

We now seek a solution to the Cauchy problem

$$\begin{cases} \partial_t u + \mathcal{L}u = B'u, \\ u|_{t=0} = u_0 \end{cases} \tag{1.4}$$

for any $B' \in \mathbb{R}^{n\times n}\backslash\{O\}$ and $u_0 \in X$ by the Picard iterative scheme, where $t \in [0, T]$ and $0 < T < +\infty$.

Based on the Duhamel formula, we define $H_k : C([0,T]; X) \to C([0,T]; X)$ for $k \in \{0, 1, 2, \dots\}$ by $H_{k+1} := H_0 \circ H_k$ where

$$H_0(u(\cdot, t)) := e^{-\mathcal{L}t}u_0 + \int_0^t e^{-\mathcal{L}(t-s)}B'u(\cdot, s)\,ds.$$

Furthermore, we can check easily by induction that since $\|e^{-\mathcal{L}t}\|_{L(X)} \leqslant ce^{\omega t}$, the function H_k is a contraction mapping from $C([0,T]; X)$ to $C([0,T]; X)$ for sufficiently large k. Hence, since X is complete, $C([0,T]; X)$ is also complete, and thus, there exists a unique $u \in C([0,T]; X)$ such that $H_k(u) = u$ for a large k due to the fixed-point theorem. By applying H_0 to this relation, we deduce that $H_0(u) = H_0 \circ H_k(u) = H_k \circ H_0(u)$. Therefore, by the uniqueness of the fixed point, $u = H_0(u)$, which solves (1.4) in $C([0,T]; X)$. Thus, since T is arbitrary, we obtain a global unique solution to the Cauchy problem (1.4).

Definition 1.1 (Hyperbolicity [3, 76])**.** For $t \geqslant 0$, the operator

$$\mathcal{P}(t) := \partial_t + \sum_{j=1}^{d} A_j \partial_{x_j} \tag{1.5}$$

is *hyperbolic* if for $A(\eta) := \sum_{j=1}^{d} \eta_j A_j$ with $\eta = (\eta_1, \dots, \eta_d) \in \mathbb{R}^d$, one has

$$\sup_{\eta \in \mathbb{R}^d} |e^{iA(\eta)}| < +\infty, \tag{1.6}$$

where $|\cdot|$ denotes a certain matrix norm.

Example 1.2. The simplest example is the one-dimensional scalar case where $n = d = 1$.

Example 1.3. The class of $\mathcal{P}(t)$ for $t \geqslant 0$ in which A_j for all $j \in \{1, \dots, d\}$ is diagonalizable with real eigenvalues and commutes with each other. Indeed, there is an invertible matrix R such that the matrices $A_1, \dots, A_d$ are simultaneously diagonalized by R. Hence, we have

$$e^{iA(\eta)} = R \operatorname{diag}(e^{i\alpha_1(\eta)}, \dots, e^{i\alpha_n(\eta)}) R^{-1} \qquad \forall \eta \in \mathbb{R}^d,$$

where

$$\alpha_h(\eta) = \alpha_h^1 \eta_1 + \cdots + \alpha_h^d \eta_d \qquad \forall \eta = (\eta_1, \dots, \eta_d) \in \mathbb{R}^d$$

and α_h^j is a real eigenvalue of A_j for $h \in \{1, \dots, n\}$ and $j \in \{1, \dots, d\}$.

We have the following characterization of hyperbolic operators from [3], which is also known as the *Kreiss's matrix* criterion.

Theorem 1.1 (Kreiss [39, 40], [3])**.** *Let M be a linear map from $\mathbb{R}^d$ to $\mathbb{C}^{n \times n}$, each of the following properties is equivalent to the others.*

i) $M(\eta)$ *is uniformly diagonalizable with purely imaginary eigenvalues for* $\eta \in \mathbb{R}^d$*, namely*

$$M(\eta) = iD(\eta)^{-1} \operatorname{diag}(m_1(\eta), \dots, m_n(\eta)) D(\eta) \tag{1.7}$$

with $m_h(\eta) \in \mathbb{R}$ *for* $h \in \{1, \dots, n\}$ *and*

$$|D(\eta)^{-1}||D(\eta)| \leqslant C \tag{1.8}$$

for a constant $C > 0$ *and all* $\eta \in \mathbb{R}^d$.

ii) *There exists a constant* $C > 0$ *such that*

$$|e^{tM(\eta)}| \leqslant C \qquad \forall \eta \in \mathbb{R}^d,\ t \geqslant 0. \tag{1.9}$$

iii) *There is a constant* $C > 0$ *such that*

$$|(zI - M(\eta))^{-1}| \leqslant \frac{C}{\operatorname{Re} z} \qquad \forall \eta \in \mathbb{R}^d,\ \operatorname{Re} z > 0. \tag{1.10}$$

A proof of Theorem 1.1 can be found in [3] and it is based on the work in [66]. As a consequence, by rewriting $e^{iA(\eta)} = e^{i|\eta|A(\eta/|\eta|)}$ for $\eta \in \mathbb{R}^d$, $\mathcal{P}(t)$ given by (1.5) for $t \geqslant 0$ is hyperbolic if it has the following property.

Corollary 1.2. *The operator* $\mathcal{P}(t)$ *given by* (1.5) *for* $t \geqslant 0$ *is a hyperbolic operator if and only if* $A(w) = \sum_{j=1}^{d} w_j A_j$ *is uniformly diagonalizable for* $w = (w_1, \dots, w_d) \in \mathbb{S}^{d-1}$ *i.e. for* $w \in \mathbb{S}^{d-1}$, $A(w)$ *is diagonalizable with real eigenvalues and the invertible diagonalizing matrix* $R(w)$ *satisfies*

$$\sup_{w \in \mathbb{S}^{d-1}} |R(w)^{-1}||R(w)| < +\infty. \tag{1.11}$$

Remark 1.4. Noting that the bound (1.11) does not necessarily hold if $A(w)$ is only diagonalizable with real eigenvalues for $w \in \mathbb{S}^{d-1}$. In fact, there are cases where $R(w)$ for $w \in \mathbb{S}^{d-1}$ is unbounded. For instance, one can consider the *Petrowski's example* for $d = 2$ and $n = 3$ where

$$A_1 := \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

More examples can be found in [41].

There are two important classes of hyperbolic operators.

Definition 1.5 (Symmetric hyperbolicity [24, 25])**.** The operator $\mathcal{P}(t)$ given by (1.5) for $t \geqslant 0$ is called *Friedrichs symmetrizable* or *symmetrizable* or *symmetric* briefly if there exists a positive-definite symmetric matrix A_0 such that $A_0 A_j$ is symmetric for all $j \in \{1, \dots, d\}$. Then, A_0 is called a *symmetrizer.*

Definition 1.6 (Constant hyperbolicity)**.** The operator $\mathcal{P}(t)$ given by (1.5) for $t \geqslant 0$ is called *constantly hyperbolic* if $A(w) = \sum_{j=1}^{d} w_j A_j$ is diagonalizable with real eigenvalues for $w = (w_1, \dots, w_d) \in \mathbb{S}^{d-1}$. Moreover, the algebraic multiplicities associated with the eigenvalues are constant and independent from w. Particularly, if the eigenvalues are simple (with algebraic multiplicities one), then $\mathcal{P}(t)$ is called *strictly hyperbolic.*

Example 1.7. Every operator $\mathcal{P}(t)$ defined by (1.5) for $t \geqslant 0$ with a generic set $\{A_1, \dots, A_d\}$ where A_j for all $j \in \{1, \dots, d\}$ is a symmetric matrix is symmetrizable with the symmetrizer $A_0 = I$. Here, I is the identity matrix in $\mathbb{R}^{n \times n}$.

Example 1.8. For $d = 1$, $n = 4$ and real numbers $0 < \alpha < \beta$, one sets

$$A_0 := \frac{1}{\beta^2 - \alpha^2} \begin{pmatrix} \beta^2 & 0 & -\alpha & 0 \\ 0 & \beta^2 & 0 & -\alpha \\ -\alpha & 0 & 1 & 0 \\ 0 & -\alpha & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_1 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta^2 & 0 & 0 & 0 \\ 0 & \beta^2 & 0 & 0 \end{pmatrix}.$$

It is easy to check that A_0 is a positive-definite symmetric matrix and $A_0 A_1$ is a symmetric matrix.

Example 1.9. For $d = 2$ and $n = 3$, we consider the matrices

$$A_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The matrix $A(w) = w_1 A_1 + w_2 A_2$ for $w = (w_1, w_2) \in \mathbb{S}^1$ has the spectrum $\{0, \pm 1\}$ independent from w. Hence, $\mathcal{P}(t) = \partial_t + \sum_{j=1}^{2} A_j \partial_{x_j}$ is a constantly (strictly indeed) hyperbolic operator.

Proposition 1.3 ([3]). *If* $\mathcal{P}(t)$ *given by* (1.5) *for* $t \geqslant 0$ *is symmetric or constantly hyperbolic, then it is hyperbolic.*

Sketch of proof. Recall that $A(w) = \sum_{j=1}^{d} w_j A_j$ for $w = (w_1, \dots, w_d) \in \mathbb{S}^{d-1}$. For the symmetric case, let A_0 be a symmetrizer of $\mathcal{P}(t)$ for $t \geqslant 0$, one has

$$A(w) = A_0^{-\frac{1}{2}}(A_0^{-\frac{1}{2}} A_0 A(w) A_0^{-\frac{1}{2}}) A_0^{\frac{1}{2}} \qquad \forall w \in \mathbb{S}^{d-1}.$$

Hence, since $A_0^{-\frac{1}{2}} A_0 A(w) A_0^{-\frac{1}{2}}$ is symmetric, there exist an orthogonal matrix $R(w)$ and a diagonal matrix $D(w)$ with real entries such that

$$A_0^{-\frac{1}{2}} A_0 A(w) A_0^{-\frac{1}{2}} = R(w)^t D(w) R(w) \qquad \forall w \in \mathbb{S}^{d-1},$$

where t denotes the transpose. Thus, since every orthogonal matrix, which is an isometric linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, preserves every matrix norm $|\cdot|$, we have

$$|A_0^{-\frac{1}{2}} R(w)^t||R(w) A_0^{\frac{1}{2}}| = |A_0^{-\frac{1}{2}}||A_0^{\frac{1}{2}}| < +\infty \qquad \forall w \in \mathbb{S}^{d-1}.$$

We consider the constantly hyperbolic case. Since A is a linear operator depending only on $w \in \mathbb{S}^{d-1}$ and its eigenvalues do not change their algebraic multiplicities, by the perturbation theory for linear operators in [36], the eigenspaces associated with the eigenvalues of A consist of continuous functions depending only on $w \in \mathbb{S}^{d-1}$. Therefore, on each simple domain of $\mathbb{S}^{d-1}$, we can choose continuously a basis of $\mathbb{C}^n$ from these eigenspaces to form an invertible continuous matrix R depending only on $w \in \mathbb{S}^{d-1}$ such that A is diagonalized by R. Furthermore, we can choose R on compact domains covering $\mathbb{S}^{d-1}$ in order to have a uniform bound of R. The proof is done. □

Remark 1.10. The continuous matrix R can be chosen continuously on each simple domain of $\mathbb{S}^{d-1}$ rather than continuously on the whole $\mathbb{S}^{d-1}$. For instance, with $n = d = 2$ and from [3], one sets

$$A_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad A_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, the eigenvalues of $A(w) = w_1 A_1 + w_2 A_2$ for $w = (w_1, w_2) \in \mathbb{S}^1$ are ± 1 with the associated eigenvectors $(\cos\theta/2, \sin\theta/2)$ and $(-\sin\theta/2, \cos\theta/2)$ respectively if we consider $w_1 = \cos\theta$ and $w_2 = \sin\theta$ for $\theta \in [0, 2\pi)$. Hence, since R is induced from these eigenvectors and we have

$$\cos(\theta/2 + k\pi) = (-1)^k \cos(\theta/2) \quad \text{and} \quad \sin(\theta/2 + k\pi) = (-1)^k \sin(\theta/2)$$

for $k \in \{0, 1, 2, \dots\}$, it cannot be formed continuously for all $w \in \mathbb{S}^1$ after each period of 2π when w varies continuously in $\mathbb{S}^1$.

Remark 1.11. In general, strictly hyperbolic operators exist rarely in the multi-dimensional space as showed in [3, 21, 43]. More precisely, we can meet them only if $n = 0, 1, 7 \pmod 8$ when $d \geqslant 3$. However, analyses of these operators are very useful for constantly hyperbolic operators appearing frequently when $n \geqslant 4$. Similarly, for $d = 2$ and $n = 3$, there exist non-symmetric operators as proved by Lax in [42].

Using the above hyperbolic conditions together with Theorem 1.1 and Corollary 1.2, we obtain the following

Theorem 1.4 (H^s-global well-posedness)**.** *For initial data in* $H^s(\mathbb{R}^d)$ *with* $s \in \mathbb{R}$*, the Cauchy problem for the first-order system*

$$\mathcal{P}(t)u = \partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u = 0 \tag{1.12}$$

has a unique solution $u \in C([0,+\infty); H^s(\mathbb{R}^d)) \cap C^1([0,+\infty); H^{s-1}(\mathbb{R}^d))$ *if and only if the matrix* $A(w) = \sum_{j=1}^{d} w_j A_j$ *for* $w = (w_1, \ldots, w_d) \in \mathbb{S}^{d-1}$ *is uniformly diagonalizable in the sense of* (1.11) *with real eigenvalues.*

Furthermore, if $s > k + d/2$ *for an integer* $k \geqslant 1$*, then* u *is a* C^k*-classical solution to* (1.12)*, namely* $u \in C^k(\mathbb{R}^d \times [0,+\infty))$*.*

Sketch of proof. For $u_0 \in H^s(\mathbb{R}^d)$ and $s \in \mathbb{R}$, it is easy to see that $u(\cdot,t) := \mathcal{F}^{-1}(e^{-iAt}\hat{u}_0)$ is well defined and satisfies (1.12), where $A = A(\xi) = \sum_{j=1}^{d} \xi_j A_j$ for $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, $t \geqslant 0$, $\mathcal{F}^{-1}$ is the inverse map of the Fourier transform $\mathcal{F}$ and $\hat{u}_0 := \mathcal{F}(u_0)$.

On the other hand, by the definition of the H^s-norm for $s \in \mathbb{R}$ and the fact that $\mathcal{P}(t)$ for $t \geqslant 0$ is hyperbolic due to Corollary 1.2, there is a positive constant C independent from t such that

$$\begin{aligned} \|u(\cdot,t)\|_{H^s}^2 &= \int_{\mathbb{R}^d} (1+|\xi|^2)^s |e^{-iA(\xi)t}\hat{u}_0(\xi)|^2 \, d\xi \\ &\leqslant C \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{u}_0(\xi)|^2 \, d\xi \qquad \forall t \geqslant 0. \end{aligned}$$

Hence, by the dominated convergence theorem, u is a continuous map from $[0,+\infty)$ to $H^s(\mathbb{R}^d)$ since $u_0 \in H^s(\mathbb{R}^d)$ and $e^{-iA(\xi)t}$ is continuous at every $t \geqslant 0$ for all $\xi \in \mathbb{R}^d$.

Similarly, since $e^{-iA(\xi)t}$ is a C^1-function in t for all $\xi \in \mathbb{R}^d$ and the derivative $\partial_t(e^{-iA(\xi)t}) = -iA(\xi)e^{-iA(\xi)t} = -ie^{-iA(\xi)t}A(\xi)$, we also have

$$\begin{aligned} \|\partial_t u(\cdot,t)\|_{H^{s-1}}^2 &= \int_{\mathbb{R}^d} (1+|\xi|^2)^{s-1} |A(\xi)e^{-iA(\xi)t}\hat{u}_0(\xi)|^2 \, d\xi \\ &\leqslant C \int_{\mathbb{R}^d} |\xi|^2 (1+|\xi|^2)^{s-1} |\hat{u}_0(\xi)|^2 \, d\xi \qquad \forall t \geqslant 0. \end{aligned}$$

Therefore, since $|\xi|^2(1+|\xi|^2)^{s-1} \leqslant (1+|\xi|^2)^s$ for all $\xi \in \mathbb{R}^d$ and $s \in \mathbb{R}$, $\partial_t u$ is a continuous map from $[0,+\infty)$ to $H^{s-1}(\mathbb{R}^d)$ also by the dominated convergence theorem.

Proving the necessary condition of the H^s-global well-posedness is more complicated. It can be done by examining the eigenvalues of A. On the other hand, the latter conclusion in the theorem is in fact induced from the Sobolev embedding $H^s(\mathbb{R}^d) \subset C^k(\mathbb{R}^d)$ for $s > k + d/2$. A complete proof of Theorem 1.4 can be found in many text books of First-order Hyperbolic Systems, for instance, in [3]. We omit the details here. □

Remark 1.12 (The backward Cauchy problem)**.** The previous results are stated only for the forward Cauchy problem ($t \geqslant 0$). However, the same results also hold for the backward problem ($t \leqslant 0$) since the backward problem is equivalent to the forward one due to the fact that the hyperbolicity (1.6) is invariant under the map $\eta \mapsto -\eta$ for $\eta \in \mathbb{R}^d$. Hence, we also have the global well-posedness for all $t \in \mathbb{R}$.

Remark 1.13 (Finite speed of propagation)**.** If $\mathcal{P}(t)$ in (1.12) for $t \geqslant 0$ is symmetric, then an even more interesting result will follow from [57, 58]. It implies the existence of an *influence domain* of any solution u to (1.12) with an initial datum $u_0 \in L^2(\mathbb{R}^d) \equiv H^0(\mathbb{R}^d)$. Outside this domain, u is identically zero. Precisely, one has

$$\operatorname{supp} u(\cdot, t) \subset \operatorname{supp} u_0 + t\mathcal{C} \qquad \forall t \geqslant 0, \tag{1.13}$$

where

$$\mathcal{C} := \{x \in \mathbb{R}^d : \lambda + x \cdot \nu \geqslant 0,\ (\lambda, \nu) \in \mathcal{V}\}, \tag{1.14}$$

$\mathcal{V}$ is a closed convex cone of all $(\lambda, \nu) \in \mathbb{R} \times \mathbb{R}^d$ satisfying $\lambda I + A(\nu)$ is positive semi-definite, $A(\nu) = \sum_{j=1}^{d} \nu_j A_j$ for $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{R}^d$ and I denotes the identity matrix in $\mathbb{R}^{n \times n}$.

Especially, if G is the fundamental solution associated with u, namely $\mathcal{P}(t)G = 0$ for $t \geqslant 0$ with the initial datum $G|_{t=0} = \delta I$ where δ is the Dirac distribution, then (1.13) and (1.14) imply immediately that

$$\operatorname{supp} G(\cdot, t) \subset \{x \in \mathbb{R}^d : C_{\min} \leqslant |x/t| \leqslant C_{\max}\} \qquad \forall t \geqslant 0,$$

where $C_{\min} := \min_{h \in \{1,\ldots,n\}}\{|\alpha_h|\}$, $C_{\max} := \max_{h \in \{1,\ldots,n\}}\{|\alpha_h|\}$ and the set $\{\alpha_1, \ldots, \alpha_n\}$ is the spectrum of $A(w) = \sum_{j=1}^{d} w_j A_j$ for $w = (w_1, \ldots, w_d) \in \mathbb{S}^{d-1}$.

Remark 1.14 (Nonhomogeneous problems)**.** Theorem 1.4 can be also extended to the nonhomogeneous Cauchy problem $\mathcal{P}(t)u + Bu = f$ where f depends only on $(x, t) \in \mathbb{R}^d \times [0, +\infty)$ and B may be different from O. In fact, it can be done by using the Duhamel formula similarly to the case where only B is considered i.e. (1.4).

1.2 Global well-posedness in L^p-spaces

Since $L^2(\mathbb{R}^d) \equiv H^0(\mathbb{R}^d)$ and the H^s-global well-posedness holds with $s \in \mathbb{R}$, the L^2-global well-posedness is obtained. Nonetheless, it is not always true in the case of $L^p(\mathbb{R}^d)$ with $p \neq 2$. In fact, if $\mathcal{P}(t)$ given by (1.5) for $t \geqslant 0$ is hyperbolic, then the L^p-global well-posedness of the Cauchy problem for

$$\mathcal{P}(t)u = \partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u = 0 \tag{1.15}$$

with initial data in $L^p(\mathbb{R}^d)$ and $p \neq 2$ does not necessarily hold due to the fact that the Fourier transform is not an automorphism of $L^p(\mathbb{R}^d)$, which is indeed required for the proof of the H^s-global well-posedness. It is known that the Fourier transform can be only extended continuously from $L^p(\mathbb{R}^d)$ to $L^{p'}(\mathbb{R}^d)$ where $1/p + 1/p' = 1$ and $1 \leqslant p \leqslant 2$, and so does the inverse Fourier transform. Hence, since $p' \geqslant 2$, the Fourier transform is an automorphism of $L^2(\mathbb{R}^d)$ only.

The scenario may be even worse when Brenner in [7, 8] showed that the Cauchy problem for (1.15) is ill-posed for every $p \neq 2$ except for the case where the matrices $A_1, \ldots, A_d$ commute with each other. The ill-posedness is based on the argument that $P \subset [1, 2]$ where P indicates the set of all $p \in [1, \infty]$

satisfying that the problem is L^p-global well-posed. Moreover, we can also see that P is symmetric via the map $p \mapsto p'$, P must be identical to the singleton-point set $\{2\}$ (see [3]).

In the case where the matrices $A_1, \ldots, A_d$ commute with each other, similarly to Example 1.3, it is then easy to see that since the matrices $A_1, \ldots, A_d$ are simultaneously diagonalizable, there exists a normalized basis of $\mathbb{C}^n$ such that any solution u to (1.15) with an initial datum u_0 is decoupled into n traveling waves

$$u(x,t) = \sum_{h=1}^{n} \ell_h u_0(x - \alpha_h t) r_h,$$

where ℓ_h and r_h are respectively the left eigenvector and the right eigenvector associated with every component of $\alpha_h := (\alpha_h^1, \ldots, \alpha_h^d)$ generated by the eigenvalues α_h^j of A_j for $j \in \{1, \ldots, d\}$ and $h \in \{1, \ldots, n\}$. Hence, it follows directly that u is a continuous map from $[0, +\infty)$ to $L^p(\mathbb{R}^d)$ if $u_0 \in L^p(\mathbb{R}^d)$ with $p \in [1, \infty]$.

Remark 1.15. For $\mathcal{P}(t) = \partial_t + \sum_{j=1}^{d} A_j \partial_{x_j} + B$ with $B \neq O$ and $A_1, \ldots, A_d$ commuting with each other, it does not require that A_j for $j \in \{1, \ldots, d\}$ commutes with B to have the L^p-well-posedness with $p \neq 2$.

Remark 1.16 ($d = 1$)**.** It follows immediately that in the one-dimensional space, the hyperbolicity of $\mathcal{P}(t)$ for $t \geqslant 0$ is a necessary and sufficient condition for the L^p-well-posedness of the Cauchy problem for $\mathcal{P}(t)u = 0$ with initial data in $L^p(\mathbb{R})$ for all $p \in [1, \infty]$.

Chapter 2

Linear stability of constant equilibria

We consider the Cauchy problem

$$\begin{cases} \partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u + Bu = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{2.1}$$

In (2.1), $(x, t) \in \mathbb{R}^d \times [0, +\infty)$, $u = u(x, t)$ and $u_0 = u_0(x)$ are vectors in $\mathbb{R}^n$. On the other hand, A_j for $j \in \{1, \dots, d\}$ and B are matrices in $\mathbb{R}^{n\times n}$. Let u be a solution to (2.1) with $u_0 \in L^2(\mathbb{R}^d)$, we study a stability structure in order for u to decay to zero in $L^2(\mathbb{R}^d)$ as $t \to +\infty$.

2.1 Dissipative structures

As discussed in the previous chapter, we see that B plays no role in the global well-posedness of (2.1). Nevertheless, let $\hat{u} = \mathcal{F}(u)$, where $\mathcal{F}$ is the Fourier transform. Since $\hat{u} = e^{-Et}\hat{u}_0$ and $E(i\xi) = B + A(i\xi) \to B$ as $|\xi| \to 0$ where $A(i\xi) = i\sum_{j=1}^{d} \xi_j A_j$ and $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, $\hat{u}$ behaves as $e^{-Bt}\hat{u}_0$ as $|\xi| \to 0$. Hence, it is necessary that the spectrum $\sigma(B)$ of B satisfies

$$\sigma(B) = \{0\} \cup \sigma_+ \qquad \text{and} \qquad \sigma_+ \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \tag{2.2}$$

unless $\hat{u}(\xi, t)$ for small ξ grows exponentially in time. Noting that the case where $\ker B = \{0\}$ is not relevant based on the natural existence of equilibria in applications. We call this prior constraint on B by *partial dissipation*.

2.1.1 Lack of dissipation

In what follows, we thus always assume that (2.1) is partially dissipative in the sense of (2.2). Let us begin with a very simple counterexample where the decay of the solution u to (2.1) does not hold. Consider the one-dimensional 2×2 system

$$\begin{cases} \partial_t u_1 - \partial_x u_1 = 0, \\ \partial_t u_2 + \partial_x u_2 = -u_2, \end{cases} \tag{2.3}$$

where $(x,t) \in \mathbb{R} \times [0,+\infty)$ and $(u_1,u_2)|_{t=0} = (u_0^1, u_0^2) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$. One can check easily that

$$\begin{cases} u_1(x,t) = u_0^1(x+t), \\ u_2(x,t) = e^{-t}u_0^2(x-t) \end{cases}$$

is a solution to (2.3). Moreover, since

$$\|(u_1,u_2)\|_{L^2} = (\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2)^{\frac{1}{2}} = (\|u_0^1\|_{L^2}^2 + e^{-t}\|u_0^2\|_{L^2}^2)^{\frac{1}{2}},$$

one deduces that (u_1,u_2) does not decay as $t \to +\infty$.

There exist also critical cases where the solution u to (2.1) is totally conservative. Indeed, we consider the following counterexample in [62]. For $\xi = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d$, one sets

$$\mathcal{N} := \bigcap_{k=0}^{+\infty} \ker(B(A(i\xi))^k) \qquad \text{where} \qquad A(i\xi) = i\sum_{j=1}^{d} \xi_j A_j. \tag{2.4}$$

If $\mathcal{N} \neq \{0\}$, let $v_0 \in \mathcal{N}\backslash\{0\}$ and $v := e^{-Et}v_0$ where $E(i\xi) = B + A(i\xi)$ for $\xi \in \mathbb{R}^d$ and $t \geqslant 0$. Thus, we have

$$\begin{aligned} v(\xi,t) &= \sum_{k=0}^{+\infty} \frac{(-t)^k}{k!}(B + A(i\xi))^k v_0(\xi) \\ &= \sum_{k=0}^{+\infty} \frac{(-t)^k}{k!}(A(i\xi))^k v_0(\xi) = e^{-A(i\xi)t}v_0(\xi). \end{aligned}$$

Hence, if $v_0 \in \mathcal{S}'(\mathbb{R}^d)$, $u := \mathcal{F}^{-1}(v)$ solves (2.1) with $Bu = 0$, namely

$$\partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u = 0.$$

Therefore, u is conservative for all $t \geqslant 0$.

Remark 2.1 ([62,67])**.** For $\xi \in \mathbb{R}^d\backslash\{0\}$, $\mathcal{N} = \{0\}$ is equivalent to the following

i) rank $[B, BA(i\xi), \dots, B(A(i\xi))^{n-1}] = n$, where $[M_1,\dots,M_n]$ denotes the $n \times n^2$ matrix obtained by placing M_{k+1} at the right-hand side of M_k for $k = 1,\dots,n-1$;

ii) If z is an eigenvector of $A(i\xi)$, then $z \notin \ker B$.

In fact, for $\xi \in \mathbb{R}^d\backslash\{0\}$, since rank $[B, BA(i\xi), \dots, B(A(i\xi))^{n-1}] = n$ is equivalent to

$$\bigcap_{k=0}^{n-1} \ker(B(A(i\xi))^k) = \{0\},$$

the equivalence of $\mathcal{N} = \{0\}$ and the rank condition (i) obviously holds due to the Cayley–Hamilton theorem.

We prove the equivalence of $\mathcal{N} = \{0\}$ and (ii). If $\mathcal{N} = \{0\}$ and z is an eigenvector associated with an eigenvalue denoted by $\alpha \in \mathbb{C}$ of $A(i\xi)$ such that $z \in \ker B$, then

$$Bz = 0 \quad \text{and} \quad B(A(i\xi))^k z = \alpha^k Bz = 0 \quad \forall k = 1,2,3,\dots$$

Thus, $z \in \mathcal{N} = \{0\}$ is a contradiction. Conversely, assume that $\mathcal{N} \neq \{0\}$ and there is no eigenvector of $A(i\xi)$ in $\ker B$. Moreover, one can check easily that $A(i\xi)\mathcal{N} \subset \mathcal{N}$ by the definition of $\mathcal{N}$. Hence, there is an eigenvector of $A(i\xi)$ in $\mathcal{N}$ unless $A(i\xi)z \neq \alpha z$ for all $\alpha \in \mathbb{C}$ and all $z \in \mathcal{N}$. Thus, it is a contradiction since $\mathcal{N} \subset \ker B$.

The rank condition (i) is exactly the *Kalman condition* in control theory, which is sufficient and necessary for the controllability of a finite-dimensional system $\dot{x}(t) = Mx(t) + Nu$ i.e. $\operatorname{rank}[N, MN, \ldots, M^{n-1}N] = n$, where $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$, $(M, N) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ and $\dot{x}$ denotes the derivative of x with respect to t. Moreover, the condition (ii) is well known as the *Shizuta–Kawashima condition.*

2.1.2 The Shizuta–Kawashima criterion

Due to the previous counterexamples, we can see that the conservative property of the solution u to (2.1) is generally caused by the restriction of $A(\nabla_x) := \sum_{j=1}^{d} A_j \partial_{x_j}$ to $\ker B$, where $A(\nabla_x)$ is a pseudo-differential operator associated with the symbol $A(i\xi) = i\sum_{j=1}^{d} \xi_j A_j$. Thus, the restriction should be compensated and the idea of Shizuta and Kawashima is that there is a pseudo-differential operator $K(\nabla_x)$ associated with a symbol $K(i\xi)$ such that $\operatorname{Re}\langle K(\nabla_x)u, A(\nabla_x)u\rangle > 0$ for all nonzero $u \in \ker B$, where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{C}^n$. Moreover, the decay of u then follows the *Lasalle's invariance principle*, namely there exists a continuously differentiable map $\mathfrak{L} := \mathfrak{L}(u)$ from $\mathbb{R}^n$ to $\mathbb{R}$ such that $\mathfrak{L}(u) \sim \|u\|^2$ for a suitable norm $\|\cdot\|$ and $\frac{d}{dt}\mathfrak{L} < 0$ for all $t \geqslant 0$.

In many applications, the existence of $K(\nabla_x)$ is guaranteed by the following criterion, which is equivalent to the fact that $\mathcal{N} = \{0\}$ (see (ii) in Remark 2.1) where $\mathcal{N}$ is given by (2.4).

Proposition 2.1 (Shizuta–Kawashima criterion [64])**.** *If A and B are real symmetric matrices and B is positive semi-definite, then there exists a real skew-symmetric matrix K such that $u^t(KA-AK+B)u > 0$ for all $u \in \mathbb{R}^n\backslash\{0\}$ if and only if there is no eigenvector of A in $\ker B$, where t denotes the transpose.*

The prototypal idea of the construction of such a real skew-symmetric matrix K arises in the work of Ellis and Pinsky in [19]. For any complex matrices A and B, if A is unitarily diagonalizable, then there is a complex matrix denoted by K such that one has the unique decomposition

$$B = \Pi_A(B) + [A, K],$$

where $[A, K] := AK - KA$ and Π_A is the orthogonal projection onto the space of all complex matrices commuting with A, namely $[A, \Pi_A(B)] = O$ and O is the null matrix. Especially, for all $u \in \mathbb{C}^n$, we have

$$\bar{u}^t\Pi_A(B)u = \sum_{j=1}^{r} \bar{u}^t\Pi_j B\Pi_j u = \sum_{j=1}^{r} (\overline{\Pi_j u})^t B(\Pi_j u),$$

where Π_j is the eigenprojection associated with $\alpha_j \in \sigma(A)$, $\sigma(A)$ is the spectrum of A with the cardinality $r \in [1, n]$, $\bar{u}^t$ and $(\overline{\Pi_j u})^t$ are respectively the conjugate transposes of u and $\Pi_j u$ for $j \in \{1, \ldots, r\}$. Moreover, if A and B

are real matrices and B is symmetric and positive semi-definite, then so do $\Pi_A(B)$ and $[A,K]$. Hence, one can check easily that $[A,K]=[A,K_{\text{skew}}]$ and $[A,K]=[A,K_{\text{real}}]$ where $K_{\text{skew}}:=(K-K^t)/2$, $K_{\text{real}}:=(K+\bar{K})/2$, K^t is the transpose of K and $\bar{K}$ is the complex-conjugate matrix of K. Thus, K can be chosen as a real skew-symmetric matrix. Once the existence of K is proved, the other properties in Proposition 2.1 can be checked easily.

In general, we can obtain a similar result as in Proposition 2.1 with a non-symmetric matrix B. Let us assume that there is a linear change of variables such that $B=\begin{pmatrix} O & O \\ C & D\end{pmatrix}$ in the new variable where $C\in\mathbb{R}^{(n-m)\times m}$ and invertible $D\in\mathbb{R}^{(n-m)\times(n-m)}$. We also assume that (2.1) has a symmetrizer A_0, i.e. A_0 is a symmetric positive-definite matrix and A_0A_j is symmetric for all $j\in\{1,\dots,d\}$, such that A_0B is positive semi-definite.

Let $U:=A_0^{-1}u$ for $u\in\mathbb{C}^n$, one has

$$\mathrm{Re}\,\bar{u}^t(BA_0^{-1}+(BA_0^{-1})^t)u=2\mathrm{Re}\,\bar{u}^tBA_0^{-1}u=2\mathrm{Re}\,\bar{U}^tA_0BU\geqslant 0.$$

Hence, the symmetric matrix $BA_0^{-1}+(BA_0^{-1})^t$ is positive semi-definite. Moreover, we have

$$BA_0^{-1}=\begin{pmatrix} O & O \\ C_* & D_*\end{pmatrix} \quad\text{and}\quad BA_0^{-1}+(BA_0^{-1})^t=\begin{pmatrix} O & C_*^t \\ C_* & D_*+D_*^t\end{pmatrix},$$

where $C_*\in\mathbb{R}^{(n-m)\times m}$ and $D_*\in\mathbb{R}^{(n-m)\times(n-m)}$. It then follows from [75] (Lemma 2.1, p. 94) that $C_*=O$. Therefore, (2.1) is rewritten as

$$A_0^{-1}\partial_t U+\sum_{j=1}^d A_jA_0^{-1}\partial_{x_j}U+\begin{pmatrix} O & O \\ O & D_*\end{pmatrix}U=0.$$

Furthermore, it is easy to see that $A_jA_0^{-1}$ is symmetric for all $j\in\{1,\dots,d\}$. In particular, the assumption that there is no eigenvector of $\sum_{j=1}^d w_jA_j$ in $\ker B$ is equivalent to the assumption that the intersection of $\left\{\begin{pmatrix} X \\ 0\end{pmatrix}: X\in\mathbb{C}^m\backslash\{0\}\right\}$ and $\ker(\lambda A_0^{-1}+\sum_{j=1}^d w_jA_jA_0^{-1})$ is empty for all $\lambda\in\mathbb{C}$ and $(w_1,\dots,w_d)\in\mathbb{S}^{d-1}$.

We obtain the following

Corollary 2.2. *If D_* is a positive-definite matrix, then for $w=(w_1,\dots,w_d)\in\mathbb{S}^{d-1}$, there is a real skew-symmetric matrix $K(w)$ such that*

$$u^t(K(w)A(w)A_0^{-1}-A(w)A_0^{-1}K(w)+BA_0^{-1}+A_0^{-1}B^t)u>0 \tag{2.5}$$

for all $u\in\mathbb{R}^n\backslash\{0\}$ if and only if there is no eigenvector of $A(w)$ in $\ker B$, where $A(w)=\sum_{j=1}^d w_jA_j$.

Proof. For $w\in\mathbb{S}^{d-1}$, we apply Proposition 2.1 to the matrices $A(w)A_0^{-1}$ and $\mathrm{diag}\,(O,D_*+D_*^t)$. The proof is done. □

Remark 2.2. The real skew-symmetric matrix K can be chosen such that $K\in C^\infty(\mathbb{S}^{d-1})$ and $K(-w)=-K(w)$ for $w\in\mathbb{S}^{d-1}$ [67, 77].

Once the existence of $K(\nabla_x)$ is proved, the decay of the solution u to (2.1) in $L^2(\mathbb{R}^d)$ is then guaranteed by the Lyapunov function

$$\mathfrak{L}(u):=\|u\|_{L^2}^2+\alpha\int_{\mathbb{R}^d}\langle K(\nabla_x)u,u\rangle+\beta\|\nabla_x u\|_{L^2}^2,$$

where α and β can be chosen later. Here, we consider an other approach which allows us to obtain an explicit rate of the decay of u. More precisely, we show the following important bounds of the Fourier transform $\hat{u}$ of u and the real parts of the eigenvalues of $E(i\xi) = B + A(i\xi)$, where $A(i\xi) = i\sum_{j=1}^{d} \xi_j A_j$, for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \backslash \{0\}$.

Based on the previous discussions, we consider

$$A_0 \partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u + Bu = 0, \tag{2.6}$$

where A_0 is symmetric and positive definite, A_j for all $j \in \{1, \dots, d\}$ is symmetric and B (not necessarily symmetric) is positive semi-definite. Moreover, we assume that for $w \in \mathbb{S}^{d-1}$, there is a real skew-symmetric matrix $K(w)$ such that

$$u^t(K(w)A(w) - A(w)K(w) + B + B^t)u > 0 \tag{2.7}$$

for all $u \in \mathbb{R}^n \backslash \{0\}$, where $A(w) = \sum_{j=1}^{d} w_j A_j$ for $w = (w_1, \dots, w_d) \in \mathbb{S}^{d-1}$.

Proposition 2.3. *If u is a solution to* (2.6) *with an initial datum u_0, then there are constants $c > 0$ and $C > 0$ such that*

$$|\hat{u}(\xi, t)| \leqslant Ce^{-\frac{c|\xi|^2}{1+|\xi|^2}t} |\hat{u}_0(\xi)| \qquad \forall \xi \in \mathbb{R}^d \backslash \{0\}, t \geqslant 0. \tag{2.8}$$

Moreover, if $\lambda = \lambda(i\xi)$ satisfies $(A(i\xi) + B)z = \lambda A_0 z$ for a $z \in \mathbb{C}^n \backslash \{0\}$, then there is a constant $\theta > 0$ such that

$$\operatorname{Re} \lambda(i\xi) \geqslant \frac{\theta |\xi|^2}{1 + |\xi|^2} \qquad \forall \xi \in \mathbb{R}^d \backslash \{0\}. \tag{2.9}$$

Remark 2.3. It follows immediately from Proposition 2.3 that if (2.1) can be transformed into or is already in the form of (2.6) through $u \mapsto A_0 u$ and the above properties of (2.6) hold, then the Fourier transform $\hat{u}$ of the solution u to (2.1) satisfies (2.8) and for any eigenvalue $\lambda(i\xi)$ of $E(i\xi) = B + A(i\xi)$, the real part of λ satisfies (2.9).

Proof of Proposition 2.3. The proof is based on the energy estimate method in the Fourier space introduced in [70]. For $\xi \in \mathbb{R}^d \backslash \{0\}$, let $w = \xi/|\xi| \in \mathbb{S}^{d-1}$.

Consider (2.6) in the Fourier space, namely

$$A_0 \partial_t \hat{u} + (i|\xi| A(w) + B)\hat{u} = 0. \tag{2.10}$$

Taking the $\mathbb{C}^n$-inner product $\langle \cdot, \cdot \rangle$ between (2.10) and $\hat{u}$, we have

$$\frac{1}{2} \partial_t \langle A_0 \hat{u}, \hat{u} \rangle + \operatorname{Re} \langle B\hat{u}, \hat{u} \rangle = 0. \tag{2.11}$$

Similarly, multiplying (2.10) by $-i\alpha|\xi|K(w)$ where $\alpha > 0$ is small enough and will be chosen later. Taking the $\mathbb{C}^n$-inner product $\langle \cdot, \cdot \rangle$ between the new equation and $\hat{u}$, we obtain

$$\begin{aligned} -\frac{1}{2}\alpha|\xi| \partial_t \langle iK(w)A_0 \hat{u}, \hat{u} \rangle &+ \alpha|\xi|^2 \operatorname{Re} \langle K(w)A(w)\hat{u}, \hat{u} \rangle \\ &- \alpha|\xi| \operatorname{Re} \langle iK(w)B\hat{u}, \hat{u} \rangle = 0. \end{aligned} \tag{2.12}$$

Dividing (2.12) by $1+|\xi|^2$ and then adding the new equation to (2.12), we have

$$\partial_t \mathfrak{L}(\hat{u}) + \mathfrak{D}_1(\hat{u}) = \mathfrak{D}_2(\hat{u}), \tag{2.13}$$

where

$$\mathfrak{L}(\hat{u}) := \frac{1}{2}\Big(\langle A_0\hat{u}, \hat{u}\rangle - \frac{\alpha|\xi|}{1+|\xi|^2}\langle iK(w)A_0\hat{u}, \hat{u}\rangle\Big), \tag{2.14}$$

$$\begin{aligned}\mathfrak{D}_1(\hat{u}) := {} & \frac{\alpha|\xi|^2}{1+|\xi|^2}\mathrm{Re}\,\langle (K(w)A(w)+B)\hat{u}, \hat{u}\rangle \\ & + \Big(1-\frac{\alpha|\xi|^2}{1+|\xi|^2}\Big)\mathrm{Re}\,\langle B\hat{u}, \hat{u}\rangle\end{aligned} \tag{2.15}$$

and

$$\mathfrak{D}_2(\hat{u}) := \frac{\alpha|\xi|}{1+|\xi|^2}\mathrm{Re}\,\langle iK(w)B\hat{u}, \hat{u}\rangle. \tag{2.16}$$

On the other hand, let P_0 be the orthogonal projection onto $\ker B$ and I be the identity matrix. By the Cauchy–Schwarz inequality, for any $\varepsilon > 0$, one has

$$|\mathrm{Re}\,\langle i\varepsilon^{-\frac{1}{2}}\alpha^{\frac{1}{2}}K(w)B\hat{u}, \varepsilon^{\frac{1}{2}}\alpha^{\frac{1}{2}}|\xi|\hat{u}\rangle| \leqslant \varepsilon\alpha|\xi|^2|\hat{u}|^2 + \alpha\varepsilon^{-1}|(I-P_0)\hat{u}|^2. \tag{2.17}$$

Moreover, since B is positive semi-definite, there is a constant $\beta > 0$ such that

$$\mathrm{Re}\,\langle B\hat{u}, \hat{u}\rangle \geqslant \beta|(I-P_0)\hat{u}|^2. \tag{2.18}$$

Since (2.7) holds, there is a constant $\gamma > 0$ such that

$$\mathrm{Re}\,\langle (K(w)A(w)+B)\hat{u}, \hat{u}\rangle \geqslant \gamma|\hat{u}|^2. \tag{2.19}$$

Hence, by choosing $\varepsilon < \gamma/2$ and $\alpha < \beta\varepsilon/(2+\varepsilon\beta)$, it follows from (2.13) - (2.19) that $|\mathfrak{D}_2| \leqslant \mathfrak{D}_1/2$. Thus, we obtain

$$\partial_t \mathfrak{L}(\hat{u}) \leqslant -\frac{1}{2}\mathfrak{D}_1(\hat{u}) \leqslant -\frac{\alpha\gamma}{2}\frac{|\xi|^2}{1+|\xi|^2}|\hat{u}|^2. \tag{2.20}$$

Furthermore, since α is small and A_0 is positive definite, one deduces from (2.14) that $C^{-1}|\hat{u}|^2 \leqslant \mathfrak{L}(\hat{u}) \leqslant C|\hat{u}|^2$ for a constant $C > 0$. We then obtain (2.8) with $c = \alpha\gamma/4$.

The proof of (2.9) is similar to before with the same computations. More precisely, $\partial_t\hat{u}$ is substituted by $(-\lambda)\hat{u}$, where λ satisfies $(A(i\xi)+B)\hat{u} = \lambda A_0\hat{u}$ for a certain $\hat{u} \in \mathbb{C}^n\backslash\{0\}$, $A(i\xi) = i|\xi|A(w)$ and $\xi \in \mathbb{R}^d$. Hence, we have $\partial_t\mathfrak{L}(\hat{u}) = -(\mathrm{Re}\,\lambda)\mathfrak{L}(\hat{u})$ and

$$\begin{cases}(\mathrm{Re}\,\lambda)\mathfrak{L}(\hat{u}) \leqslant C^{-1}(\mathrm{Re}\,\lambda)|\hat{u}|^2 & \text{if } \mathrm{Re}\,\lambda < 0,\\ (\mathrm{Re}\,\lambda)\mathfrak{L}(\hat{u}) \leqslant C(\mathrm{Re}\,\lambda)|\hat{u}|^2 & \text{if } \mathrm{Re}\,\lambda \geqslant 0.\end{cases}$$

It then follows from (2.20) that there is $\theta > 0$ such that

$$(\mathrm{Re}\,\lambda)|\hat{u}|^2 \geqslant \frac{\theta|\xi|^2}{1+|\xi|^2}|\hat{u}|^2. \tag{2.21}$$

Thus, since $\hat{u} \in \mathbb{C}^n\backslash\{0\}$, dividing (2.21) by $|\hat{u}|^2$, we have $\mathrm{Re}\,\lambda$ is positive and satisfies (2.9) for all $\xi \neq 0$. The proof is done. □

By the Plancherel formula, it follows from (2.8) immediately that

Corollary 2.4 (Decay in $L^2(\mathbb{R}^d)$)**.** *Let* $u_0 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$*, if* u *is a solution to* (2.1) *and* (2.8) *satisfies, there are constants* $c > 0$ *and* $C > 0$ *such that*

$$\|u\|_{L^2} \leqslant C(1+t)^{-\frac{d}{4}}\|u_0\|_{L^1} + Ce^{-ct}\|u_0\|_{L^2} \qquad \forall t \geqslant 0. \tag{2.22}$$

Remark 2.4. Noting that the property (2.9) of the real parts of the eigenvalues of $E(i\xi) = B + A(i\xi)$ sometimes can be calculated directly.

Remark 2.5. The properties (2.8) and (2.9) depend strongly on (2.7), which is satisfied by many systems including the Goldstein–Kac systems and the linearized isentropic compressible Euler equations with damping.

Nevertheless, there are cases where (2.7) holds in a weaker sense. i.e. there is a real matrix L satisfying LA_0 is symmetric, the symmetric part of $LB + L$ is positive semi-definite and shares the same kernel with B and there is sufficiently small $\alpha > 0$ such that for $w \in \mathbb{S}^{d-1}$, we have

$$u^t(\alpha(K(w)A(w) - A(w)K(w)) \\ + LB + (LB)^t + B + B^t)u > 0$$

for all $u \in \mathbb{R}^n \backslash \{0\}$.

Then, (2.8) and (2.9) hold if $\operatorname{Re}\langle iLA(w)u, u\rangle \geqslant 0$ for all $u \in \mathbb{C}^n$ and $w \in \mathbb{S}^{d-1}$. On the other hand, if $\operatorname{Re}\langle iLA(w)u, u\rangle \geqslant 0$ for all $u \in \ker(B + B^t)$ and $w \in \mathbb{S}^{d-1}$, the following holds

$$|\hat{u}(\xi, t)| \leqslant Ce^{-\frac{c|\xi|^2}{(1+|\xi|^2)^2}t}|\hat{u}_0(\xi)| \qquad \forall \xi \in \mathbb{R}^d\backslash\{0\}, \forall t \geqslant 0. \tag{2.23}$$

Moreover, if $\lambda = \lambda(i\xi)$ satisfies $(A(i\xi) + B)z = \lambda A_0 z$ for a $z \in \mathbb{C}^n\backslash\{0\}$, then

$$\operatorname{Re}\lambda(i\xi) \geqslant \frac{\theta|\xi|^2}{(1+|\xi|^2)^2} \qquad \forall \xi \in \mathbb{R}^d\backslash\{0\}. \tag{2.24}$$

The properties (2.23) and (2.24) are also induced from the energy estimate method in the Fourier space similarly to the proof of Proposition 2.3. Such kind of weaker properties implies the decay (in $L^2(\mathbb{R}^d)$) of the high-frequency part of the solution u to (2.1) is at most polynomial, provided the initial datum u_0 is regular enough. For references, one sees [67–69] and the examples therein.

2.2 Asymptotic-profile construction methods

In the spirit of (2.22), the solution u to (2.1) is divided into two parts with two different behaviors as $t \to +\infty$. One part decays diffusively and the other one decays exponentially.

The slower part arises in the low-frequency part of u ($|\xi| < 1$) and the faster part arises in the high-frequency part ($|\xi| \geqslant 1$). In fact, for small ξ, from (2.8), the Fourier transform $\hat{u}$ of u behaves similarly to a heat kernel associated with the symbol $e^{-c|\xi|^2 t}$. For large ξ, $\hat{u}$ is bounded by $Ce^{-\frac{c}{2}t}|\hat{u}_0(\xi)|$ for some constants $c > 0$ and $C > 0$. As $t \to +\infty$, the exponentially decaying part can be negligible and it allows us to investigate parabolic approximations of u only.

Here, we introduce two useful tools that one can construct large-time asymptotic profiles of u: the *Chapman–Enskog expansion* and the *asymptotic expansion in the Fourier space*.

2.2.1 Chapman–Enskog expansion

The Chapman–Enskog expansion method was proposed firstly and independently by Chapman and Enskog between 1910 and 1920 in order to derive the Navier–Stokes equation from the Boltzmann equation as the Knudsen number vanishes. The main idea is to justify the expansion

$$f_\varepsilon(x,t) := f_0(x,t) + \varepsilon f_1(x,t) + \varepsilon^2 f_2(x,t) + \dots$$

approximating a solution f to the Boltzmann equation around an unperturbed Maxwellian state f_0, where $0 < \varepsilon \ll 1$ is the Knudsen number. Hence, each finite sum $\sum_{h=0}^{k} \varepsilon^k f_h$ is then a k-order approximation of f for $k \in \mathbb{N}$.

We now consider the *singular limit* of (2.1) (as $\varepsilon \to 0^+$) under the hyperbolic scaling $(x,t) \mapsto (x/\varepsilon, t/\varepsilon)$ with $0 < \varepsilon \ll 1$. Under the scaling, (2.1) is reformulated by the system

$$\partial_t u + \sum_{j=1}^{d} A_j \partial_{x_j} u + \frac{1}{\varepsilon} B u = 0. \tag{2.25}$$

Assume that the eigenvalue 0 of B is semi-simple with the algebraic multiplicity $m \in [1, n)$. Let $L_+ \in \mathbb{R}^{m \times n}$ and $R_+ \in \mathbb{R}^{n \times m}$ be obtained from the eigenprojection $P_+ \in \mathbb{R}^{n \times n}$ onto $\ker B$ by rank factorization. Similarly, $L_- \in \mathbb{R}^{(n-m) \times n}$ and $R_- \in \mathbb{R}^{n \times (n-m)}$ are obtained from $I - P_+$ by rank factorization.

Let $v := L_+ u$ and $w := L_- u$, (2.25) is decomposed into

$$\begin{cases} \partial_t v + \sum_{j=1}^{d} L_+ A_j R_+ \partial_{x_j} v + \sum_{j=1}^{d} L_+ A_j R_- \partial_{x_j} w = 0, & (2.26) \\ \partial_t w + \sum_{j=1}^{d} L_- A_j R_+ \partial_{x_j} v + \sum_{j=1}^{d} L_- A_j R_- \partial_{x_j} w + \frac{1}{\varepsilon} L_- B R_- w = 0. & (2.27) \end{cases}$$

Substituting the approximation

$$w_\varepsilon(x,t) := \varepsilon w_1 + \varepsilon^2 w_2 + \dots \tag{2.28}$$

for w in (2.27), one has

$$w_1 = -\sum_{j=1}^{d} (L_- B R_-)^{-1} L_- A_j R_+ \partial_{x_j} v. \tag{2.29}$$

Finally, substituting (2.28) and (2.29) for w in (2.26), we obtain

$$\partial_t v + \sum_{j=1}^{d} C_j \partial_{x_j} v - \varepsilon \sum_{j=1}^{d} \sum_{h=1}^{d} D_{jh} \partial_{x_j x_h} v = 0. \tag{2.30}$$

In (2.30), C_j and D_{jh} for j, $h \in \{1, \dots, d\}$ are matrices in $\mathbb{R}^{n \times n}$ such that

$$C_j := L_+ A_j R_+ \qquad \text{and} \qquad D_{jh} := L_+ A_j R_- (L_- B R_-)^{-1} L_- A_h R_+.$$

Since the singular-limit problem can be seen as a complementary problem of the large-time asymptotic behavior problem at a small fixed parameter, it gives rise to the idea that a large-time approximation of (2.1) can be constructed by this manner. From (2.27) with $\varepsilon = 1$, one has

$$w = -(L_-BR_-)^{-1}\Big(\partial_t w + \sum_{j=1}^{d} L_-A_jR_+\partial_{x_j}v + \sum_{j=1}^{d} L_-A_jR_-\partial_{x_j}w\Big). \tag{2.31}$$

Substituting (2.31) for w in (2.26), we obtain

$$\partial_t v + \sum_{j=1}^{d} C_j\partial_{x_j}v - \sum_{j=1}^{d}\sum_{h=1}^{d} D_{jh}\partial_{x_jx_h}v = 0,$$

which coincides (2.30) with $\varepsilon = 1$.

2.2.2 Asymptotic expansion in the Fourier space

Consider the Cauchy problem

$$\begin{cases}\partial_t\hat{G} + E\hat{G} = 0,\\ \hat{G}|_{t=0} = I,\end{cases} \tag{2.32}$$

where $(\xi, t) \in \mathbb{R}^d \times [0, +\infty)$, $E(i\xi) = B + i\sum_{j=1}^{d}\xi_jA_j$ for $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and I is the identity matrix in $\mathbb{R}^{n\times n}$.

A solution $\hat{G}$ to (2.32) is given by

$$\hat{G}(\xi, t) = e^{-E(i\xi)t} \qquad \forall(\xi, t) \in \mathbb{R}^d \times [0, +\infty). \tag{2.33}$$

We consider asymptotic expansions of $\hat{G}$ based on the perturbation theory for linear operators in [36]. Since $\hat{G}$ is the Fourier transform of the fundamental solution G to (2.1), the expansions of $\hat{G}$ allow us to obtain more exact large-time asymptotic profiles of the solution u to (2.1) than the Chapman–Enskog expansion. Nevertheless, the expansions require more uniform properties.

To this end, for $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, we consider

$$E = E(z) = B + zA(w),$$

where

$$z = i|\xi|, \quad w_j = \frac{\xi_j}{|\xi|} \quad \forall j = 1, \ldots, d \quad \text{and} \qquad A(w) = \sum_{j=1}^{d} w_jA_j.$$

The operator E can be expanded as z tends to a specific point $z_0 \in \mathbb{C}$ including the point $z_0 = \infty$. Specifically, since we know that large-time asymptotic profiles of u arise in its low-frequency part as $|\xi| \to 0$, we thus consider the expansion of E as $|z| \to 0$.

As mentioned before, since the method requires uniform properties, we primarily begin with the one-dimensional space where $d = 1$. We also assume that the eigenvalue 0 of B is semi-simple similarly to before and the restriction of $A := A_1$ to $\ker B$ is diagonalizable i.e. $C := P_0^{(0)}AP_0^{(0)}$ considered in $\operatorname{ran} P_0^{(0)} = \{z \in \mathbb{C}^n : z = P_0^{(0)}z\}$ is diagonalizable, where $P_0^{(0)}$ is the eigenprojection associated with the eigenvalue 0 of B.

The above conditions are very technical. Nonetheless, we can see that the singular limit (2.30) of the solution u to (2.1) via the Chapman–Enskog expansion is indeed governed by the reduced system

$$\partial_t v + C\partial_x v \approx 0, \tag{2.34}$$

where $v := P_0^{(0)} u \in \ker B$ and $C = P_0^{(0)} A P_0^{(0)}$. It amounts to saying that the hyperbolicity condition is necessary for (2.34) to be global well-posedness i.e. the diagonalizability with real eigenvalues of the matrix C.

Recall from (B.3) in Appendix B that

$$P_0^{(0)} = -\frac{1}{2\pi i}\int_{\Gamma_0}(B - zI)^{-1}\,dz \text{ and } Q_0^{(0)} = \frac{1}{2\pi i}\int_{\Gamma_0} z^{-1}(B - zI)^{-1}\,dz \tag{2.35}$$

are respectively the eigenprojection and the reduced resolvent coefficient associated with the eigenvalue 0 of B, where Γ_0, in the resolvent set of B, is an oriented closed curve enclosing 0 except for the other eigenvalues of B.

One also recall the formula (B.4) in Appendix B that the coefficient

$$P_0^{(1)} := -P_0^{(0)} A Q_0^{(0)} - Q_0^{(0)} A P_0^{(0)}. \tag{2.36}$$

One sets

$$D := -\sum_{h=1}^{s} P_h^{(0)}(P_0^{(1)} B P_0^{(1)} + P_0^{(0)} A P_0^{(1)} + P_0^{(1)} A P_0^{(0)}) P_h^{(0)}, \tag{2.37}$$

where $P_h^{(0)}$ is the eigenprojection associated with $c_h \in \sigma(C, \ker B)$ for $h \in \{1, \ldots, s\}$, $\sigma(C, \ker B)$ is the spectrum of C considered in $\ker B$ with the cardinality s.

Noting that

$$P_h^{(0)} = -\frac{1}{2\pi i}\int_{\Gamma_h}(C_\alpha - zI)^{-1}\,dz,$$

where $C_\alpha := C + \alpha P_0^{(0)}$ for $\alpha > \max\{|\lambda| : \lambda \in \sigma(C)\}$, $\sigma(C)$ is the spectrum of C and Γ_h, in the resolvent set of C_α, is an oriented closed curve enclosing $c_h + \alpha$ except for the other eigenvalues of C_α.

Let $\sigma(T, \mathcal{D})$ be the spectrum of a certain matrix T considered in a domain $\mathcal{D}$ with $\sigma(T) \equiv \sigma(T, \mathbb{C}^n)$, we have the following

Proposition 2.5 (Low frequency - one-dimensional space). *Assume that* $0 \in \sigma(B)$ *is semi-simple and* $C = P_0^{(0)} A P_0^{(0)}$ *considered in* $\ker B$ *is diagonalizable. For small* ξ, $E(i\xi) = B + i\xi A$ *is approximated by* $E = E_1 + E_2$ *where*

$$E_1(i\xi) = \sum_{h=1}^{s}\sum_{\ell=1}^{s_h}\big((ic_h\xi + d_{h\ell}\xi^2)I + \xi^2 N_{h\ell}^{(0)} + \mathcal{O}(|\xi|^3)\big)\big(P_{h\ell}^{(0)} + \mathcal{O}(|\xi|)\big) \tag{2.38}$$

and

$$E_2(i\xi) = \sum_{k=1}^{s'}\big(b_k I + M_k^{(0)} + \mathcal{O}(|\xi|)\big)\big(F_k^{(0)} + \mathcal{O}(|\xi|)\big). \tag{2.39}$$

In (2.38), $c_h \in \sigma(C, \operatorname{ran} P_0^{(0)})$ *with the associated eigenprojection* $P_h^{(0)}$, $d_{h\ell} \in \sigma(P_h^{(0)} D P_h^{(0)}, \operatorname{ran} P_h^{(0)})$ *with the associated eigenprojection* $P_{h\ell}^{(0)}$ *and eigennilpotent* $N_{h\ell}^{(0)}$, s *and* s_h *are respectively the cardinalities of* $\sigma(C, \operatorname{ran} P_0^{(0)})$ *and*

$\sigma(P_h^{(0)} D P_h^{(0)}, \operatorname{ran} P_h^{(0)})$. *In* (2.39), $b_k \in \sigma(B)\backslash\{0\}$ *with the associated eigenprojection* $F_k^{(0)}$ *and eigennilpotent* $M_k^{(0)}$ *and* s' *is the cardinality of* $\sigma(B)\backslash\{0\}$.

Proof. Assume that $\sigma(B) = \{0\} \cup \sigma_0$ where $\sigma_0 = \{b_1, \dots, b_{s'}\} \subseteq \mathbb{C}\backslash\{0\}$. It follows from Appendix B that the eigenvalues of B gives rise to the *total projections* $P_0 = P_0(i\xi)$ and $F_k = F_k(i\xi)$ for $k \in \{1, \dots, s'\}$ satisfying

$$P_0(i\xi) = P_0^{(0)} + \mathcal{O}(|\xi|) \qquad \text{and} \qquad F_k(i\xi) = F_k^{(0)} + \mathcal{O}(|\xi|)$$

for small ξ, where $P_0^{(0)}$ and $F_k^{(0)}$ are respectively the eigenprojections associated with the eigenvalues 0 and b_k of B for $k \in \{1, \dots, s'\}$. Each of the total projections associates with a group of eigenvalues of $E(i\xi)$, where the elements of each group converge to the same eigenvalue of B as $|\xi| \to 0$.

One sets $E_1 := EP_0$ and $E_2 := \sum_{k=1}^{s'} EF_k$. Due to the semi-simplicity of the eigenvalue 0 of B and the eigenvalues of $C = P_0^{(0)} A P_0^{(0)}$ considered in $\ker B$, one obtains (2.38) by applying two times of the reduction process introduced in Lemma B.3. On the other hand, (2.39) is obtained directly from the multiplication EF_k for $k \in \{1, \dots, s'\}$. The proof is done. □

Noting that since E_1 and E_2 commute with the total projections and their subprojections as showed in Appendix B (Lemma B.3), it follows from (2.33) that for small ξ, the solution $\hat{G}$ to (2.32) is decomposed into $\hat{G} = \hat{G}_1 + \hat{G}_2$ where

$$\hat{G}_1(\xi, t) := \sum_{h=1}^{s} \sum_{\ell=1}^{s_h} e^{-(ic_h \xi + d_{h\ell} \xi^2)t} e^{-\left(\xi^2 N_{h\ell}^{(0)} + \mathcal{O}(|\xi|^3)\right)t} \left(P_{h\ell}^{(0)} + \mathcal{O}(|\xi|)\right) \quad (2.40)$$

and

$$\hat{G}_2(\xi, t) := \sum_{k=1}^{s'} e^{-b_k t} e^{-\left(M_k^{(0)} + \mathcal{O}(|\xi|)\right)t} \left(F_k^{(0)} + \mathcal{O}(|\xi|)\right). \quad (2.41)$$

If $\operatorname{Re} b_k > 0$ for all $k \in \{1, \dots, s'\}$, from (2.40) and (2.41), $\hat{G}_2$ decays exponentially as $t \to +\infty$. Hence, large-time asymptotic profiles of $\hat{G}$ can be chosen as

$$\hat{K}(\xi, t) := \sum_{h=1}^{s} \sum_{\ell=1}^{s_h} e^{-(ic_h \xi + d_{h\ell} \xi^2)t} e^{-\xi^2 N_{h\ell}^{(0)} t} P_{h\ell}^{(0)} \quad (2.42)$$

if $\operatorname{Re} d_{h\ell} > 0$ for all $h \in \{1, \dots, s\}$ and $\ell \in \{1, \dots, s_h\}$.

The case of the multi-dimensional space with $d \geqslant 2$ is more complicated due to lack of uniform properties. Hence, we consider here only that $0 \in \sigma(B)$ is simple.

Recall $P_0^{(0)}$ and $Q_0^{(0)}$ in (2.35). For $h, \ell \in \{1, \dots, d\}$, one sets

$$c_h := \operatorname{tr}\left(A_h P_0^{(0)}\right) \text{ and } D_{h\ell} := \frac{1}{2} \operatorname{tr}\left(A^h P_0^{(0)} A^\ell Q_0^{(0)} + A_h Q_0^{(0)} A^\ell P_0^{(0)}\right), \quad (2.43)$$

where tr denotes the trace. We also set for $h \in \{1, \dots, d\}$, the matrix

$$P_{0h}^{(1)} := -P_0^{(0)} A_h Q_0^{(0)} - Q_0^{(0)} A_h P_0^{(0)}. \quad (2.44)$$

Proposition 2.6 (Low frequency - multi-dimensional space). *If* $0 \in \sigma(B)$ *is simple, then* $E(i\xi) = B + i\sum_{j=1}^{d}\xi_j A_j$ *for small* $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ *is approximated by*

$$E(i\xi) = \lambda_0(i\xi)P_0(i\xi) + \sum_{k=1}^{s'} E_k(i\xi)F_k(i\xi). \tag{2.45}$$

In (2.45)*, one has*

$$\lambda_0(i\xi) = i\sum_{h=1}^{d} c_h\xi_h + \sum_{h=1}^{d}\sum_{\ell=1}^{d} D_{h\ell}\xi_h\xi_\ell + \mathcal{O}(|\xi|^3), \tag{2.46}$$

and

$$P_0(i\xi) = P_0^{(0)} + i\sum_{h=1}^{d}\xi_h P_{0h}^{(1)} + \mathcal{O}(|\xi|^2). \tag{2.47}$$

Moreover, one has

$$E_k(i\xi) = b_k I + M_k^{(0)} + \mathcal{O}(|\xi|) \tag{2.48}$$

and

$$F_k(i\xi) = F_k^{(0)} + \mathcal{O}(|\xi|), \tag{2.49}$$

where $b_k \in \sigma(B)\backslash\{0\}$ *with the associated eigenprojection* $F_k^{(0)}$ *and eigennilpotent* $M_k^{(0)}$ *and* s' *is the cardinality of* $\sigma(B)\backslash\{0\}$.

Proof. For $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, one lets $\zeta = |\xi| \in [0, +\infty)$ and $w = (w_1, \dots, w_d) = (\xi_1, \dots, \xi_d)/|\xi| = \xi/|\xi| \in \mathbb{S}^{d-1}$. One has $E(i\xi) = E(i\zeta, w) = B + i\zeta A(w)$ where $A(w) := \sum_{j=1}^{d} w_j A_j$.

We primarily consider the 0-*group* of $E(i\zeta, w)$ for small $\zeta > 0$, where the 0-group contains the eigenvalues of $E(i\zeta, w)$ which converge to 0 as $\zeta \to 0$. Since $0 \in \sigma(B)$ is simple, the eigennilpotent $N_0^{(0)}$ associated with 0 is the null matrix and one obtains from (B.2) and (B.4) - (B.6) that the total projection $P_0(i\zeta, w)$ associated with the 0-group of $E(i\zeta, w)$ is approximated by

$$P_0(i\zeta, w) = P_0^{(0)} + i\zeta P_0^{(1)}(w) + \mathcal{O}(\zeta^2), \tag{2.50}$$

where $P_0^{(0)}$ is the eigenprojection associated with 0 and

$$\begin{aligned} P_0^{(1)}(w) &= -P_0^{(0)}A(w)Q_0^{(0)} - Q_0^{(0)}A(w)P_0^{(0)} \\ &= -\sum_{h=1}^{d} w_h(P_0^{(0)}A_h Q_0^{(0)} + Q_0^{(0)}A_h P_0^{(0)}). \end{aligned} \tag{2.51}$$

On the other hand, by (B.8) and (B.9) in Lemma B.1, the 0-group of $E(i\zeta, w)$ consists of one single eigenvalue $\lambda_0(i\zeta, w)$ approximated by

$$\lambda_0(i\zeta, w) = i\zeta\lambda_0^{(1)}(w) - \zeta^2\lambda_0^{(2)}(w) + \mathcal{O}(\zeta^3), \tag{2.52}$$

where

$$\lambda_0^{(1)}(w) = \operatorname{tr}(A(w)P_0^{(0)}) = \sum_{h=1}^{d}\operatorname{tr}(A_h P_0^{(0)})w_h \tag{2.53}$$

and

$$\lambda_0^{(2)}(w) = \frac{1}{2}\mathrm{tr}\,(A(w)P_0^{(1)}(w))$$
$$= -\frac{1}{2}\sum_{h=1}^{d}\sum_{\ell=1}^{d}\mathrm{tr}\,(A_h P_0^{(0)} A^\ell Q_0^{(0)} + A_h Q_0^{(0)} A^\ell P_0^{(0)}) w_h w_\ell. \tag{2.54}$$

We consider the other groups of $E(i\zeta, w)$ for small $\zeta > 0$. Recall $b_k \in \sigma(B)\backslash\{0\}$ is the k-th nonzero eigenvalue of B for $k \in \{1, \dots, s'\}$ where s' is the cardinality of $\sigma(B)\backslash\{0\}$.

One deduces directly from (B.2) that the approximation of the total projection $F_k(i\zeta, w)$ associated with the b_k-group of $E(i\zeta, w)$, which contains the eigenvalues of $E(i\zeta, w)$ converging to b_k as $\zeta \to 0$, is given by the following

$$F_k(i\zeta, w) = F_k^{(0)} + \mathcal{O}(\zeta), \tag{2.55}$$

where $F_k^{(0)}$ is the eigenprojection associated with b_k. Moreover, due to the discussions above (B.7), the b_k-group of $E(i\zeta, w)$ is equivalent to the eigenvalues of $E_k(i\zeta, w) = E(i\zeta, w)F_k(i\zeta, w)$ in $\mathrm{ran}\, F_k(i\zeta, w)$. Furthermore, one has

$$E_k(i\zeta, w) = (B + i\zeta A(w))(F_k^{(0)} + \mathcal{O}(\zeta))$$
$$= BF_k^{(0)} + \mathcal{O}(\zeta) = b_k I + M_k^{(0)} + \mathcal{O}(\zeta), \tag{2.56}$$

where $M_k^{(0)} = (B - b_k I)F_k^{(0)}$ is the eigennilpotent associated with b_k.

Finally, since $P_0 + \sum_{k=1}^{s'} F_k(i\zeta, w) = I$ the identity matrix, one has

$$E(i\zeta, w) = E(i\zeta, w)P_0(i\zeta, w) + \sum_{k=1}^{s'} E(i\zeta, w)F_k(i\zeta, w)$$
$$= \lambda_0(i\zeta, w)P_0(i\zeta, w) + \sum_{k=1}^{s'} E_k(i\zeta, w)F_k(i\zeta, w). \tag{2.57}$$

We thus obtain (2.45) - (2.49) by considering (2.50) - (2.57) in $\xi \in \mathbb{R}^d$. The proof is done. $\square$

One sets

$$\mathbf{c} := (c_h)_{h\in\{1,\dots,d\}} \in \mathbb{R}^d \quad \text{and} \quad \mathbf{D} := (D_{h\ell})_{h,\ell\in\{1,\dots,d\}} \in \mathbb{R}^{d\times d}, \tag{2.58}$$

where c_h and $D_{h\ell}$ are given by (2.43) for h, $\ell \in \{1, \dots, d\}$. We also set

$$\mathbf{P}_0^{(1)} := \big(P_{0h}^{(1)}\big)_{h\in\{1,\dots,d\}} \in (\mathbb{R}^{n\times n})^d, \tag{2.59}$$

where $P_{0h}^{(1)}$ is given by (2.44) for $h \in \{1, \dots, d\}$. Similarly to the one-dimensional space, for small ξ, the solution $\hat{G}$ to (2.32) is decomposed into $\hat{G} = \hat{G}_1 + \hat{G}_2$ where

$$\hat{G}_1(\xi, t) := e^{-(i\mathbf{c}\cdot\xi + \xi\cdot\mathbf{D}\xi + \mathcal{O}(|\xi|^3))t}\big(P_0^{(0)} + i\xi\cdot\mathbf{P}_0^{(1)} + \mathcal{O}(|\xi|^2)\big) \tag{2.60}$$

and

$$\hat{G}_2(\xi, t) := \sum_{k=1}^{s'} e^{-b_k t} e^{-\left(M_k^{(0)} + \mathcal{O}(|\xi|)\right)t}\big(F_k^{(0)} + \mathcal{O}(|\xi|)\big). \tag{2.61}$$

Therefore, if $\operatorname{Re} b_k > 0$ for all $k \in \{1, \dots, s'\}$, $\hat{G}_2$ decays exponentially as $t \to +\infty$. Hence, large-time asymptotic profiles of $\hat{G}$ can be chosen as

$$\hat{K}(\xi, t) := e^{-(i\mathbf{c}\cdot\xi + \xi\cdot\mathbf{D}\xi)t} P_0^{(0)} \tag{2.62}$$

if the matrix $\mathbf{D}$ is positive definite.

Remark 2.6 (High frequency - one-dimensional space)**.** We can also obtain the high-frequency expansion of $E(i\xi) = B + i\xi A$ by reformulating $E(i\xi) = i\eta^{-1}T(i\eta)$ where $T(i\eta) := A - i\eta B$ and $\eta := \xi^{-1} \in \mathbb{R}$. In fact, $|\eta| \to 0$ as $|\xi| \to +\infty$, and thus, one can apply the previous computations for $T(i\eta)$.

Proposition 2.7 (High-frequency approximation)**.** *If A is diagonalizable, then for large ξ, $E(i\xi)$ is approximated by*

$$E(i\xi) = \sum_{h=1}^{r}\sum_{\ell=1}^{r_h}\big((i\alpha_h\xi + \beta_{h\ell})I + \Theta_{h\ell}^{(0)} + \mathcal{O}(|\xi|^{-1})\big)\big(\Pi_{h\ell}^{(0)} + \mathcal{O}(|\xi|^{-1})\big), \tag{2.63}$$

where $\alpha_h \in \sigma(A)$ with the associated eigenprojection $\Pi_h^{(0)}$, the coefficient $\beta_{h\ell} \in \sigma(\Pi_h^{(0)}B\Pi_h^{(0)}, \operatorname{ran}\Pi_h^{(0)})$ with the associated eigenprojection $\Pi_{h\ell}^{(0)}$ and eigennilpotent $\Theta_{h\ell}^{(0)}$, r and r_h are respectively the cardinalities of $\sigma(A)$ and $\sigma(\Pi_h^{(0)}B\Pi_h^{(0)}, \operatorname{ran}\Pi_h^{(0)})$.

Proof. The expansion of $T(i\eta)$ as $|\eta| \to 0$ follows directly from Lemma B.3 for each semi-simple eigenvalue α_h of A since A is diagonalizable, where $h \in \{1, \dots, r\}$ and r is the cardinality of $\sigma(A)$. Then, we deduce the expansion of $E(i\xi)$ as $|\xi| \to +\infty$ by using the formula $E(i\xi) = i\xi T(i\xi^{-1})$. The proof is done. ☐

Remark 2.7 (High frequency - multi-dimensional space)**.** Similarly to the case of small ξ, $E(i\xi)$ for large ξ cannot be expanded uniformly in the multi-dimensional space. Moreover, since the case of strict hyperbolicity rarely appear in the multi-dimensional space, we will not assume that. Hence, we consider here only the case where the matrix $A(w) = \sum_{j=1}^{d} w_j A_j$ for $w = (w_1, \dots, w_d) \in \mathbb{S}^{d-1}$ is uniformly diagonalizable with linear eigenvalues, namely the matrix $R(w)$ diagonalizing $A(w)$ satisfies

$$\sup_{w\in\mathbb{S}^{d-1}} |R(w)||R^{-1}(w)| < C$$

for a constant $C > 0$ and the eigenvalues of $A(w)$ are linear functions in w.

Thus, we consider the ℓ-th diagonal element of $R^{-1}AR$ for $\ell \in \{1, \dots, n\}$ as the linear function

$$\nu_\ell(w) := \nu_\ell^{(0)} + \sum_{h=1}^{d} \nu_\ell^{(h)} w_h, \tag{2.64}$$

where the coefficient $\nu_\ell^{(h)} \in \mathbb{C}$ for all $h \in \{0, 1, \dots, d\}$. One sets

$$\boldsymbol{\nu}_\ell := (\nu_\ell^{(0)}, \dots, \nu_\ell^{(d)}) \in \mathbb{C}^{d+1}$$

be the coefficient vector associated with ν_ℓ for $\ell \in \{1, \dots, n\}$. One also sets

$$\mathcal{S}_1 := \{\ell \in \{1, \dots, n\} : \boldsymbol{\nu}_\ell = \boldsymbol{\nu}_1\}.$$

For $i_j := \min\{\{1,\dots,n\}\backslash \cup_{h=1}^{j-1} \mathcal{S}_h\}$, one defines

$$\mathcal{S}_j := \{\ell \in \{1,\dots,n\} : \boldsymbol{\nu}_\ell = \boldsymbol{\nu}_{i_j}\}, \qquad j = 2,3,\dots$$

This procedure will stop at a finite step $r \leqslant n$ and $\mathcal{S} := \{\mathcal{S}_1,\dots,\mathcal{S}_r\}$ is a partition of $\{1,\dots,n\}$. One denotes by [j] the representation of the elements of $\mathcal{S}_j$ for $j \in \{1,\dots,r\}$.

Lemma 2.8 (Measure zero set)**.** *There is a Lebesgue measure zero set in* $\mathbb{S}^{d-1}$ *such that except for this set, the number of the distinct eigenvalues of* $A(w)$ *for* $w \in \mathbb{S}^{d-1}$ *is* r.

Proof. Assume that there are $i, j \in \{1,\dots,r\}$ such that $i \neq j$ and $\nu_{[i]}(w_0) = \nu_{[j]}(w_0)$ for a $w_0 \in \mathbb{S}^{d-1}$. By (2.64), w_0 belongs to the intersection of the affine hyperplane

$$(\nu_{[i]}^{(0)} - \nu_{[j]}^{(0)}) + \sum_{h=1}^{d} (\nu_{[i]}^{(h)} - \nu_{[j]}^{(h)}) x_h = 0, \qquad (x_1,\dots,x_d) \in \mathbb{R}^d,$$

and the unit sphere $\mathbb{S}^{d-1}$. Moreover, the dimension of the hyperplane is at most $d-1$ since the coefficient vectors $\boldsymbol{\nu}_{[i]}$ and $\boldsymbol{\nu}_{[j]}$ satisfy $\boldsymbol{\nu}_{[i]} \neq \boldsymbol{\nu}_{[j]}$ for any $i \neq j$. Thus, the dimension of the intersection is at most $d-2$. Therefore, w_0 belongs to a Lebesgue measure zero set in $\mathbb{R}^{d-1}$.

Thus, $\nu_{[i]}(w) \neq \nu_{[j]}(w)$ for any $i \neq j$ and for $w \in \mathbb{S}^{d-1}$ except for a Lebesgue measure zero set. Finally, since the repeated eigenvalues of $A(w)$ are $\nu_\ell(w)$ in (2.64) for $\ell \in \{1,\dots,n\}$, it follows immediately that the number of the distinct eigenvalues of $A(w)$ for $w \in \mathbb{S}^{d-1}$ is r excluding a Lebesgue measure zero set. We finish the proof. □

One sets, for $j \in \{1,\dots,r\}$, the projection

$$(\Pi_j^{(0)})_{h\ell} := \begin{cases} 1 & \text{if } h = \ell \in \mathcal{S}_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.65}$$

Proposition 2.9 (High-frequency approximation)**.** *Assume that* $A(w)$ *is uniformly diagonalizable with linear eigenvalues by an invertible matrix* $R(w)$ *for* $w \in \mathbb{S}^{d-1}$*. If* $R^{-1}(w)BR(w)$ *does not depend on* w*, then for large* $\xi \in \mathbb{R}^d$, $E(i\xi)$ *is almost everywhere approximated by*

$$E(i\xi) = R(\xi/|\xi|) \sum_{h=1}^{r} \sum_{\ell=1}^{r_h} \Upsilon_{h\ell}(i\xi) \Pi_{h\ell}(i\xi) R^{-1}(\xi/|\xi|), \tag{2.66}$$

where

$$\Upsilon_{h\ell}(i\xi) = (\alpha_h(i\xi) + \beta_{h\ell}) I + \Theta_{h\ell}^{(0)} + \mathcal{O}(|\xi|^{-1}) \tag{2.67}$$

and

$$\Pi_{h\ell}(i\xi) = \Pi_{h\ell}^{(0)} + \mathcal{O}(|\xi|^{-1}). \tag{2.68}$$

Here, $\alpha_h(i\xi) = i|\xi|\nu_{[h]}(\xi/|\xi|)$ *for* $\nu_{[h]}$ *given by the formula* (2.64)*, the coefficient* $\beta_{h\ell} \in \sigma(\Pi_h^{(0)} R^{-1}(\xi/|\xi|) B R(\xi/|\xi|) \Pi_h^{(0)}, \operatorname{ran} \Pi_h^{(0)})$ *with the associated eigenprojection* $\Pi_{h\ell}^{(0)}$ *and eigennilpotent* $\Theta_{h\ell}^{(0)}$*,* r *and* r_h *are the cardinalities of* $\sigma(A)$ *and* $\sigma(\Pi_h^{(0)} R^{-1}(\xi/|\xi|) B R(\xi/|\xi|) \Pi_h^{(0)}, \operatorname{ran} \Pi_h^{(0)})$ *respectively.*

Proof. By Lemma 2.8, for almost everywhere, the spectrum of the matrix $R^{-1}(w)A(w)R(w)$ for $w \in \mathbb{S}^{d-1}$ is the set $\{\alpha_1(w), \dots, \alpha_r(w)\}$ where $\alpha_h(w) = \nu_{[h]}(w)$ given by (2.64) for $h \in \{1, \dots, r\}$.

For $h \in \{1, \dots, r\}$, we study the $\alpha_h(w)$-group of $T(i\eta, w)$ for small $\eta > 0$, where

$$T(i\eta, w) := R^{-1}(w)A(w)R(w) - i\eta R^{-1}(w)BR(w)$$

for $(\eta, w) \in [0, +\infty) \times \mathbb{S}^{d-1}$.

One obtains from (B.2), (B.4), (B.5) and (B.6) that the total projection $\Pi_h(i\eta, w)$ associated with the $\alpha_h(w)$-group is approximated by

$$\Pi_h(i\eta, w) = \Pi_h^{(0)}(w) + \mathcal{O}(\eta),$$

where $\Pi_h^{(0)}(w)$ is the eigenprojection associated with $\alpha_h(w)$. Moreover, by the definition of the eigenprojection, if Γ_h is an oriented closed curve in the resolvent set of $R^{-1}(w)A(w)R(w)$ enclosing $\alpha_h(w)$ except for the other eigenvalues of $R^{-1}(w)A(w)R(w)$, then

$$\begin{aligned}\Pi_h^{(0)}(w) &= -\frac{1}{2\pi i}\int_{\Gamma_h} \operatorname{diag}\left(\nu_1(w) - \mu)^{-1}, \dots, (\nu_n(w) - \mu)^{-1}\right) d\mu \\ &= \operatorname{diag}\left(-\frac{1}{2\pi i}\int_{\Gamma_h} (\nu_1(w) - \mu)^{-1}\, d\mu, \dots, -\frac{1}{2\pi i}\int_{\Gamma_h} (\nu_n(w) - \mu)^{-1}\, d\mu\right)\end{aligned}$$

and it coincides (2.65) for almost everywhere since by Lemma 2.8, for almost everywhere, one has

$$-\frac{1}{2\pi i}\int_{\Gamma_h} (\nu_\ell(w) - \mu)^{-1}\, d\mu = \begin{cases} 1 & \text{if } \ell \in \mathcal{S}_h, \\ 0 & \text{if } \ell \notin \mathcal{S}_h. \end{cases}$$

Therefore, one can expand $T(i\eta, w)$ as $\eta \to 0$ by Lemma B.3 for almost everywhere. Thus, we deduce the expansion of $E(i\xi)$ as $|\xi| \to +\infty$ for almost everywhere by using the formula

$$E(i\xi) = i\eta R(w)T(i\eta, w)R^{-1}(w),$$

where $\eta := |\xi|^{-1}$ and $w = \xi/|\xi|$ since $\eta \to 0$ as $|\xi| \to +\infty$. The proof is done. □

Remark 2.8 (Intermediate frequency)**.** In this framework, no expansion of $E(i\xi)$ for $\varepsilon \leqslant |\xi| \leqslant \rho$ with $\varepsilon > 0$ and $\rho < +\infty$ is used. However, there are a finite number of *exceptional curves* of $E(i\xi)$ for $\varepsilon \leqslant |\xi| \leqslant \rho$. At the exceptional curves, the eigenvalues of $E(i\xi)$ intersect. In the domain excluding these curves, the number of the distinct eigenvalues of $E(i\xi)$ and their algebraic multiplicities are constant (see [5, 36]).

Chapter 3

L^p-L^q decay estimates

Consider the Goldstein–Kac 2×2 system in the one-dimensional space

$$\begin{cases} \partial_t u_1 - \partial_x u_1 = -\dfrac{1}{2}u_1 + \dfrac{1}{2}u_2, \\ \partial_t u_2 + \partial_x u_2 = \dfrac{1}{2}u_1 - \dfrac{1}{2}u_2, \end{cases}$$

which can be written in the relaxation form of (2.1) with

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

One can check easily that $v_\pm := u_1 \pm u_2$ satisfy the Cauchy problem for the linear damped wave equation

$$\begin{cases} \partial_{tt} v - \partial_{xx} v + \partial_t v = 0, \\ (v, \partial_t v)|_{t=0} = (v_0, v_1), \end{cases} \tag{3.1}$$

where (v_0, v_1) can be computed by initial data of (u_1, u_2).

On the other hand, it follows from [48] that the solution v to (3.1) satisfies the following L^p-L^q *decay estimate*

$$\left\| v - \phi - e^{-\frac{t}{2}} \frac{v_0(\cdot + t) + v_0(\cdot - t)}{2} \right\|_{L^p} \leqslant Ct^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p}) - 1} \|(v_0, v_1)\|_{L^q} \tag{3.2}$$

for all $1 \leqslant q \leqslant p \leqslant \infty$ and $t \geqslant 1$, where ϕ is a solution to the Cauchy problem

$$\begin{cases} \partial_t \phi - \partial_{xx} \phi = 0, \\ \phi|_{t=0} = v_0 + v_1 \end{cases} \tag{3.3}$$

and the term $(v_0(\cdot + t) + v_0(\cdot - t))/2$ is a solution to the Cauchy problem

$$\begin{cases} \partial_{tt} \psi - \partial_{xx} \psi = 0, \\ (\psi, \partial_t \psi)|_{t=0} = (0, v_0). \end{cases}$$

Moreover, it is also well known that ϕ satisfies the L^p-L^q decay estimate

$$\|\phi\|_{L^p} \leqslant Ct^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p})} \|\phi_0\|_{L^q}$$

for all $1 \leqslant q \leqslant p \leqslant \infty$ and $t > 0$ (see [27]). It amounts to saying that if the initial datum (v_0, v_1) is regular enough, then the error term $e^{-\frac{t}{2}}(v_0(\cdot + t) + v_0(\cdot - t))/2$, which may contain singularities, decays exponentially in time to zero and can be neglected. Thus, the solution v behaves like the diffusion wave ϕ for large time. Therefore, since $(u_1, u_2) = (v_+ + v_-, v_+ - v_-)/2$, it allows us to expect such L^p-L^q decay estimates hold for (u_1, u_2) and for general systems.

3.1 One-dimensional space

We consider the Cauchy problem

$$\begin{cases} \partial_t u + A\partial_x u + Bu = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{3.4}$$

where $(x, t) \in \mathbb{R} \times [0, +\infty)$, $u = u(x, t)$ and $u_0 = u_0(x)$ are vectors in $\mathbb{R}^n$, A and B are matrices in $\mathbb{R}^{n\times n}$.

Moreover, based on the discussions in Chapter 2, we consider the following dissipative structure.

Condition $\mathscr{A}$ (Hyperbolicity). *The matrix A is diagonalizable with real eigenvalues.*

Condition $\mathscr{B}$ (Partial dissipation). *The spectrum of B is decomposed into $\sigma(B) = \{0\} \cup \sigma_+$ where 0 is semi-simple and $\sigma_+ \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$.*

Condition $\mathscr{C}$ (Reduced hyperbolicity). *The matrix $C = P_0^{(0)} A P_0^{(0)}$ considered in $\ker B$ is diagonalizable with real eigenvalues, where $P_0^{(0)}$ is the eigenprojection associated with the eigenvalue 0 of B.*

On the other hand, a requisite condition for the decay of solutions to (3.4) is that for any eigenvalue $\lambda(i\xi)$ of $E(i\xi) = B + i\xi A$ for $\xi \in \mathbb{R}$, the real part of λ satisfies

Condition $\mathscr{D}$ (Uniform dissipation). *There is a positive constant θ such that*

$$\operatorname{Re}\lambda(i\xi) \geqslant \frac{\theta|\xi|^2}{1+|\xi|^2} \quad \textit{for all} \quad \xi \neq 0.$$

Moreover, we introduce large-time asymptotic profiles of solutions to (3.4). Recall that the eigenprojection $P_0^{(0)}$ and the reduced resolvent coefficient $Q_0^{(0)}$ associated with the eigenvalue 0 of B are given by

$$P_0^{(0)} = -\frac{1}{2\pi i}\int_{\Gamma_0} (B - zI)^{-1}\, dz \text{ and } Q_0^{(0)} = \frac{1}{2\pi i}\int_{\Gamma_0} z^{-1}(B - zI)^{-1}\, dz,$$

where Γ_0, in the resolvent set of B, is an oriented closed curve enclosing 0 except for the other eigenvalues of B. One sets

$$P_0^{(1)} = -P_0^{(0)} A Q_0^{(0)} - Q_0^{(0)} A P_0^{(0)}. \tag{3.5}$$

We also set

$$D = -\sum_{h=1}^{s} P_h^{(0)}(P_0^{(1)} B P_0^{(1)} + P_0^{(0)} A P_0^{(1)} + P_0^{(1)} A P_0^{(0)}) P_h^{(0)}, \tag{3.6}$$

where $P_h^{(0)}$ is the eigenprojection associated with $c_h \in \sigma(C, \ker B)$ for $h \in \{1, \ldots, s\}$ and $\sigma(C, \ker B)$ is the spectrum of C considered in $\ker B$ with the cardinality s.

Noting that

$$P_h^{(0)} = -\frac{1}{2\pi i}\int_{\Gamma_h} (C_\alpha - zI)^{-1}\, dz,$$

where $C_\alpha = C + \alpha P_0^{(0)}$ for $\alpha > \max\{|\lambda| : \lambda \in \sigma(C)\}$, $\sigma(C)$ is the spectrum of C and Γ_h, in the resolvent set of C_α, is an oriented closed curve enclosing $c_h + \alpha$ except for the other eigenvalues of C_α.

Furthermore, one sets

$$\Pi_A(B) = \sum_{h=1}^{r} \Pi_h^{(0)} B \Pi_h^{(0)},$$

where $\Pi_h^{(0)}$ is the eigenprojection associated with $\alpha_h \in \sigma(A)$ for $h \in \{1, \dots, r\}$ and $\sigma(A)$ is the spectrum of A with the cardinality r. Noting that following from [19], if A is unitarily equivalent to a diagonal matrix, then $\Pi_A(B)$ is exactly the projection of B onto the space of all matrices commuting with A, namely there is a unique matrix K such that

$$B = \Pi_A(B) + [A, K],$$

where $[A, K] = AK - KA \neq O$ and $[A, \Pi_A(B)] = A\Pi_A(B) - \Pi_A(B)A = O$.

We consider the Cauchy problem

$$\begin{cases} \partial_t U + C\partial_x U - D\partial_{xx} U = 0, \\ U|_{t=0} = P_0^{(0)} u_0 \end{cases} \tag{3.7}$$

and the Cauchy problem

$$\begin{cases} \partial_t V + A\partial_x V + \Pi_A(B)V = 0, \\ V|_{t=0} = u_0, \end{cases} \tag{3.8}$$

where $(x, t) \in \mathbb{R} \times [0, +\infty)$, $U = U(x, t) \in \mathbb{R}^n$ and $V = V(x, t) \in \mathbb{R}^n$.

Theorem 3.1 (L^p-L^q decay estimates [52])**.** *For* $u_0 \in L^q(\mathbb{R})$*, let* u*,* U *and* V *be respectively solutions to* (3.4)*,* (3.7) *and* (3.8)*. If the conditions* $\mathscr{A}$*,* $\mathscr{B}$*,* $\mathscr{C}$ *and* $\mathscr{D}$ *hold, then for* $1 \leqslant q \leqslant p \leqslant \infty$ *and* $t \geqslant 1$*, there are positive constants* c *and* C *such that*

$$\|u - U - V\|_{L^p} \leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} \|u_0\|_{L^q}. \tag{3.9}$$

Moreover, one has

$$\|U\|_{L^p} \leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_0\|_{L^q} \qquad \textit{and} \qquad \|V\|_{L^q} \leqslant Ce^{-ct} \|u_0\|_{L^q}. \tag{3.10}$$

The proof of Theorem 3.1 is based on three steps: the L^∞-L^1 estimate, the L^p-L^p estimate for $1 \leqslant p \leqslant \infty$ and a complex interpolation argument given by Lemma A.2. Furthermore, we also divide each step into the low frequency, the intermediate frequency and the high frequency in order to use asymptotic expansions of the Fourier transform $\hat{G}$ of the fundamental solution denoted by G to (3.4). Noting that $\hat{G}$ satisfies (2.32) and is given by (2.33).

Example 3.1 (The one-dimensional linearized Broadwell 3×3 system)**.** We consider the linearized one-dimensional Broadwell 3×3 system fitting in the class (3.4) with

$$A = \begin{pmatrix} \nu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\nu \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} a & -2b & c \\ -\dfrac{a}{2} & b & -\dfrac{c}{2} \\ a & -2b & c \end{pmatrix} \tag{3.11}$$

for positive real numbers a, b, c and ν such that $b^2 = ac$.

One observes that the condition $\mathscr{A}$ holds obviously. On the other hand, we have

$$\det(B - \lambda I) = -\lambda^2(\lambda - (a + b + c)).$$

Particularly, $\ker B = \operatorname{span}\{(2b, a, 0), (-c, 0, a)\}$ with dimension 2. Hence, the condition $\mathscr{B}$ holds.

We check the condition $\mathscr{C}$. The eigenprojection $P_0^{(0)}$ and the reduced resolvent coefficient $Q_0^{(0)}$ associated with the eigenvalue 0 of B are given by

$$P_0^{(0)} = \frac{1}{a+b+c}\begin{pmatrix} b+c & 2b & -c \\ \frac{a}{2} & a+c & \frac{c}{2} \\ -a & 2b & a+b \end{pmatrix} \tag{3.12}$$

and

$$Q_0^{(0)} = \frac{1}{(a+b+c)^2}\begin{pmatrix} a & -2b & c \\ -\frac{a}{2} & b & -\frac{c}{2} \\ a & -2b & c \end{pmatrix}. \tag{3.13}$$

Hence, $C{=}P_0^{(0)}AP_0^{(0)}$ is given by

$$C = \frac{\nu}{(a+b+c)^2}\begin{pmatrix} (b+c)^2 - ac & 2b(b+2c) & c(a-c) \\ \frac{a(b+2c)}{2} & b(a-c) & -\frac{c(2a+b)}{2} \\ a(a-c) & -2b(b+2a) & ac-(a+b)^2 \end{pmatrix}.$$

By easy calculations and the fact that $b^2 = ac$, the eigenvalues of C are roots of the polynomial

$$\lambda\Big((a+b+c)\lambda^2 + \nu(c-a)\lambda - \nu^2 b(a+b+c)\Big) = 0.$$

Therefore, the eigenvalues are respectively given by

$$c_0 = 0 \quad \text{and} \quad c_\pm = \frac{-\nu(c-a) \pm \nu\sqrt{(c-a)^2 + 4b(a+b+c)}}{2(a+b+c)}.$$

Since $\dim \operatorname{ran}(I - P_0^{(0)}) = 1$ and $C(I - P_0^{(0)}) = 0$ due to the fact that $P_0^{(0)}(I - P_0^{(0)}) = O$, there is only one single eigenvalue $c_0 = 0$ of C considered in $\operatorname{ran}(I - P_0^{(0)})$. Thus, $c_\pm$, which are simple eigenvalues, are the eigenvalues of C considered in $\operatorname{ran} P_0^{(0)} = \ker B$ (since 0 is a semi-simple eigenvalue of B). The condition $\mathscr{C}$ is proved.

We check the condition $\mathscr{D}$ by the results in [33]. By scaling $(x, t) \mapsto (x/c, t/c)$, we can assume that $(a, b, c) = (\alpha^2, \alpha, 1)$ with $\alpha = (a/c)^{\frac{1}{2}} > 0$. Let $\xi \mapsto \xi/\nu$, we also assume $\nu = 1$. It then follows from Lemma 1 in [33] that there are three distinct analytic eigenvalues of $E(i\xi) = B + i\xi A$ denoted by λ_j for $j = 1, 2, 3$ except for a finite number of points in the complex plane.

Moreover, there are $\beta_1 > 0$ and $\beta_2 > 0$ such that for $|\xi| < \beta_1$, one has the convergent expansions

$$\begin{cases} \lambda_1(i\xi) = c_1 i\xi + d_1(i\xi)^2 + \mathcal{O}(|\xi|^3), \\ \lambda_2(i\xi) = c_2 i\xi + d_2(i\xi)^2 + \mathcal{O}(|\xi|^3), \\ \lambda_3(i\xi) = \mu_0 + \mathcal{O}(|\xi|), \end{cases} \tag{3.14}$$

where $c_1 > 0$, $c_2 < 0$, $d_1 < 0$, $d_2 < 0$ and $\mu_0 > 0$ with explicit formulas. Noting that $\mu_0 = a+b+c$, $c_1 = c_+$ and $c_2 = c_-$ indeed. In particular, d_1 and d_2 are respectively the only nonzero eigenvalues of $P_1^{(0)}DP_1^{(0)}$ and $P_2^{(0)}DP_2^{(0)}$, where D is computed by (3.6) by using A and B in (3.11), $P_0^{(0)}$ and $Q_0^{(0)}$ in (3.12) and (3.13), $P_0^{(1)}$ in (3.5) and $P_h^{(0)}$ for $h = 1, 2$ given by

$$P_h^{(0)} = \frac{\operatorname{adj}(C - c_h I)}{\operatorname{tr}(\operatorname{adj}(C - c_h I))}, \qquad h = 1, 2. \tag{3.15}$$

Here, adj denotes the adjunct matrix and tr denotes the trace.

On the other hand, for $|\xi|^{-1} < \beta_2$, from [33], we also have

$$\begin{cases} \lambda_1(i\xi) = i\xi + \alpha^2 + \mathcal{O}(|\xi|^{-1}), \\ \lambda_2(i\xi) = \alpha + \mathcal{O}(|\xi|^{-1}), \\ \lambda_3(i\xi) = -i\xi + 1 + \mathcal{O}(|\xi|^{-1}). \end{cases} \tag{3.16}$$

In particular, from (3.14) and (3.16), we have

$$\begin{cases} \operatorname{Re}\lambda_j(i\xi) \geqslant -\min\{d_1, d_2\}|\xi|^2 & \forall|\xi| < \beta_1,\ j = 1, 2, \\ \operatorname{Re}\lambda_3(i\xi) \geqslant \mu_0 & \forall|\xi| < \beta_1, \\ \operatorname{Re}\lambda_j(i\xi) \geqslant \max\{1, \alpha, \alpha^2\} & \forall|\xi|^{-1} < \beta_2,\ j = 1, 2, 3. \end{cases}$$

Hence, by the continuity of $\lambda_j(i\xi)$ and the fact that $\operatorname{Re}\lambda_j(i\xi) = 0$ if and only if $\xi = 0$ and $\lambda_1(0) = \lambda_2(0) = 0$, we deduce that $\operatorname{Re}\lambda_j(i\xi) > 0$ for any $\xi \neq 0$. Thus, there is a constant $\theta > 0$ such that

$$\operatorname{Re}\lambda_j(i\xi) \geqslant \begin{cases} -\min\{d_1, d_2\}\dfrac{|\xi|^2}{1+|\xi|^2}(1+|\xi|^2) & \forall|\xi| < \beta_1,\ j = 1, 2, \\ \mu_0\dfrac{|\xi|^2}{1+|\xi|^2}(1+|\xi|^{-2}) & \forall|\xi| < \beta_1,\ j = 3, \\ \theta\dfrac{|\xi|^2}{1+|\xi|^2}(1+|\xi|^{-2}) & \forall|\xi| \geqslant \beta_1,\ j = 1, 2, 3, \end{cases}$$

which implies the condition $\mathscr{D}$ since $1+|\xi|^2 \geqslant 1$ and $1+|\xi|^{-2} \geqslant 1$ for all $\xi \in \mathbb{R}$.

Example 3.2 (The one-dimensional linearized Jin–Xin systems)**.** We consider the $2n \times 2n$ system (3.4) with

$$A = \begin{pmatrix} O & I \\ \alpha^2 I & O \end{pmatrix} \qquad \text{and} \qquad B = \frac{1}{\varepsilon}\begin{pmatrix} O & O \\ -\Lambda & I \end{pmatrix}, \tag{3.17}$$

where α, $\varepsilon > 0$, $\Lambda \in \mathbb{R}^{n\times n}$, O and I are respectively the null matrix and the identity matrix in $\mathbb{R}^{n\times n}$.

By considering $u = (v, w) \in \mathbb{R}^n \times \mathbb{R}^n$ and by applying the Chapman–Enskog expansion

$$w = \Lambda v + \varepsilon w_1 + \dots$$

to (3.4), we obtain the approximation parabolic system

$$\partial_t v + \Lambda\partial_x v - \varepsilon(\alpha^2 I_n - \Lambda^2)\partial_{xx} v = 0.$$

Hence, we can assume that Λ is diagonalizable with real eigenvalues and the stability condition

$$\sup\{|\lambda| : \lambda \in \sigma(\Lambda)\} < \alpha \tag{3.18}$$

holds, where $\sigma(\Lambda)$ is the spectrum of Λ.

Particularly, let $Q \in \mathbb{R}^{n\times n}$ be the invertible matrix diagonalizing Λ i.e. $Q\Lambda Q^{-1} = D$ with a diagonal matrix $D \in \mathbb{R}^{n\times n}$. Let $P := \operatorname{diag}(Q, Q)$, we have $PAP^{-1} = A$ and $PBP^{-1} = \left(\begin{smallmatrix} O & O \\ -D/\varepsilon & I \end{smallmatrix}\right)$. Therefore, from (3.17), we can assume that Λ is a symmetric matrix as well.

We check the conditions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$. Without loss of generality, we assume that $\varepsilon = 1$. Let I be the identity matrix in $\mathbb{R}^{2n\times 2n}$, one has

$$\det(A - \lambda I) = (\lambda^2 - \alpha^2)^n.$$

Thus, A has two distinct eigenvalues $\pm\alpha$ with the same algebraic multiplicities n. Moreover, the associated eigenspaces are $\{\left(\begin{smallmatrix} X \\ \pm\alpha X \end{smallmatrix}\right) : X \in \mathbb{C}^n\}$ with dimensions n. It implies that the condition $\mathscr{A}$ holds.

On the other hand, we have

$$\det(B - \lambda I) = (-\lambda)^n(\lambda - 1)^n.$$

Hence, B has two distinct eigenvalues 0 and 1 with the same algebraic multiplicities n. Furthermore, the eigenspaces are $\{\left(\begin{smallmatrix} X \\ \Lambda X \end{smallmatrix}\right) : X \in \mathbb{C}^n\}$ and $\{\left(\begin{smallmatrix} 0 \\ X \end{smallmatrix}\right) : X \in \mathbb{C}^n\}$ with dimensions n. Hence, the condition $\mathscr{B}$ follows directly.

We check the condition $\mathscr{C}$. The eigenprojection $P_0^{(0)}$ and the reduced resolvent coefficient $Q_0^{(0)}$ associated with the eigenvalue 0 of B are

$$P_0^{(0)} = \begin{pmatrix} I & O \\ \Lambda & O \end{pmatrix} \qquad \text{and} \qquad Q_0^{(0)} = \begin{pmatrix} O & O \\ -\Lambda & I \end{pmatrix},$$

where O and I are respectively the null matrix and the identity matrix in $\mathbb{R}^{n\times n}$. It implies that

$$C = P_0^{(0)} A P_0^{(0)} = \begin{pmatrix} \Lambda & O \\ \Lambda^2 & O \end{pmatrix}.$$

Hence, we have

$$\det(C - \lambda I_{2n}) = (-\lambda)^n \det(\Lambda - \lambda I_n).$$

Since $\dim \operatorname{ran}(I_{2n} - P_0^{(0)}) = n$ and $C(I_{2n} - P_0^{(0)}) = O_{2n}$ due to the fact that $P_0^{(0)}(I_{2n} - P_0^{(0)}) = O_{2n}$, C has n eigenvalues 0 considered in $\operatorname{ran}(I_{2n} - P_0^{(0)})$. Therefore, the eigenvalues of C considered in $\operatorname{ran} P_0^{(0)} = \ker B$ are the eigenvalues of Λ. Thus, by the assumption that Λ is symmetric, we obtain the condition $\mathscr{C}$.

Finally, we examine the condition $\mathscr{D}$. By Remark 2.3, it is then sufficient to prove that (3.4) can be transformed into the form (2.6) such that (2.7) holds via $u \mapsto A_0 u$ with a symmetric positive-definite matrix A_0.

One sets

$$A_0 := \begin{pmatrix} I & \Lambda \\ \Lambda & \alpha^2 I \end{pmatrix}.$$

Then, A_0 is symmetric since Λ is symmetric. Moreover, for every $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) \in \mathbb{R}^{2n}\backslash\{0\}$ where $X, Y \in \mathbb{R}^n$, we have

$$\begin{aligned}(\, X^t \; Y^t \,) A_0 \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) &= X^t X + X^t \Lambda Y + Y^t \Lambda X + \alpha^2 Y^t Y \\ &= (X + \Lambda Y)^t (X + \Lambda Y) + Y^t(\alpha^2 I - \Lambda^2) Y > 0\end{aligned}$$

due to the subcharacteristic condition (3.18): $\alpha^2 I - \Lambda^2$ is positive definite. Hence, A_0 is also positive definite.

Furthermore, under $u \mapsto A_0 u$, (3.4) becomes

$$A_0 \partial_t u + A A_0 \partial_x u + B A_0 u = 0,$$

where

$$AA_0 = \begin{pmatrix} \Lambda & \alpha^2 I \\ \alpha^2 I & \alpha^2 \Lambda \end{pmatrix} \qquad \text{and} \qquad BA_0 = \begin{pmatrix} O & O \\ O & \alpha^2 I - \Lambda^2 \end{pmatrix}.$$

Therefore, since there is no eigenvector of AA_0 in $\ker B$, (2.7) holds and Proposition 2.3 implies the condition $\mathscr{D}$.

3.1.1 Preliminaries

Before going to further details, we have the following

Lemma 3.2. *If X is a constant nilpotent matrix in $\mathbb{C}^{n\times n}$, then for all $\varepsilon > 0$, there exists $C > 0$ such that*

$$\left| e^{cX+Y} - e^{cX} \right| \leqslant C e^{\varepsilon|c| + C|Y|} |Y| \tag{3.19}$$

and

$$\left| e^{cX+Y} - e^{cX} - de^{cX} Y \right| \leqslant C e^{\varepsilon|c| + C|Y|} |Y|^2, \tag{3.20}$$

where $c \in \mathbb{C}$, $Y \in \mathbb{C}^{n\times n}$ and de is the first derivative of $X \mapsto e^X$.

Proof. For any matrix $X \in \mathbb{C}^{n\times n}$, there is an induced norm $|\cdot|_*$ such that $|X|_* \leqslant \rho(X) + \varepsilon$ for any $\varepsilon > 0$, where $\rho(X)$ is the spectral radius of X [63]. Moreover, since every norms in $\mathbb{C}^{n\times n}$ are equivalent, $C^{-1}|X|_* \leqslant |X| \leqslant C|X|_*$ for any matrix norm $|\cdot|$ and a constant $C > 0$. Thus, we can consider $|\cdot|_*$, which implies that $|X|_* \leqslant \varepsilon$ for any $\varepsilon > 0$ if X is a nilpotent matrix. Then, (3.19) and (3.20) follow from the Taylor expansion of the application $X \mapsto e^X$. In fact, we have

$$|e^{cX+Y} - e^{cX}|_* \leqslant C \Big| \sup_{t\in[0,1]} de^{cX+tY} Y \Big|_* \leqslant C e^{\varepsilon|c| + |Y|_*} |Y|_*$$

and

$$|e^{cX+Y} - e^{cX} - de^{cX} Y|_* \leqslant C \Big| \sup_{t\in[0,1]} d^2 e^{cX+tY} Y^2 \Big|_* \leqslant C e^{\varepsilon|c| + |Y|_*} |Y|_*^2.$$

The proof is done. □

Remark 3.3. If $X \in \mathbb{C}^{n\times n}$ is a nilpotent matrix, then we can always assume that $|X| \leqslant \varepsilon$ for any $\varepsilon > 0$ to bound e^X.

Lemma 3.3. *Under the conditions $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$, the real part of $d_{h\ell}$ in (2.38) is bounded from below by a $\theta > 0$ for $h \in \{1, \dots, s\}$ and $\ell \in \{1, \dots, s_h\}$. Similarly, if the conditions $\mathscr{A}$ and $\mathscr{D}$ hold, the real part of $\beta_{h\ell}$ in (2.63) is bounded from below by a $\theta > 0$ for $h \in \{1, \dots, r\}$ and $\ell \in \{1, \dots, r_h\}$.*

Proof. Recall that $\sigma(\mathsf{T}, \mathcal{D})$ denotes the spectrum of a matrix T considered in a domain $\mathcal{D}$ with $\sigma(\mathsf{T}) \equiv \sigma(\mathsf{T}, \mathbb{C}^n)$. Recall the matrix $\mathsf{C} = \mathsf{P}_0^{(0)}\mathsf{A}\mathsf{P}_0^{(0)}$ where $\mathsf{P}_0^{(0)}$ is the eigenprojection associated with the eigenvalue 0 of B. We also recall the matrix D in (3.6).

For small ξ, since the conditions $\mathscr{B}$ and $\mathscr{C}$ hold, the expansion (2.38) of $\mathsf{E}(i\xi) = \mathsf{B} + i\xi\mathsf{A}$ is validated. Thus, the eigenvalues of $\mathsf{E}(i\xi)$ that converge to 0 as $|\xi| \to 0$ are approximated by

$$\lambda_{h\ell}(i\xi) = ic_h\xi + d_{h\ell}\xi^2 + o(|\xi|^2)$$

for $h \in \{1, \dots, s\}$ and $\ell \in \{1, \dots, s_h\}$, where $c_h \in \sigma(\mathsf{C}, \operatorname{ran} \mathsf{P}_0^{(0)})$, $d_{h\ell} \in \sigma(\mathsf{P}_h^{(0)}\mathsf{D}\mathsf{P}_h^{(0)}, \operatorname{ran} \mathsf{P}_h^{(0)})$, the integers s and s_h are respectively the cardinalities of $\sigma(\mathsf{C}, \ker \mathsf{B})$ and $\sigma(\mathsf{P}_h^{(0)}\mathsf{D}\mathsf{P}_h^{(0)}, \operatorname{ran} \mathsf{P}_h^{(0)})$.

Hence, it follows from the conditions $\mathscr{C}$ and $\mathscr{D}$ that c_h is real for all $h \in \{1, \dots, s\}$ and there is a positive constant θ such that

$$\frac{\theta}{1 + |\xi|^2} \leqslant \operatorname{Re}(d_{h\ell}) + \varepsilon$$

for small $0 < |\xi| < \varepsilon$, $h \in \{1, \dots, s\}$ and $\ell \in \{1, \dots, s_h\}$. Let $\varepsilon \to 0$, one obtains $\operatorname{Re}(d_{h\ell}) \geqslant \theta > 0$ for all $h \in \{1, \dots, s\}$ and $\ell \in \{1, \dots, s_h\}$.

Similarly, for large ξ, since the condition $\mathscr{A}$ holds, the expansion (2.63) of $\mathsf{E}(i\xi)$ is validated. Then, the eigenvalues of $\mathsf{E}(i\xi)$ that converge to $\alpha_h \in \sigma(\mathsf{A})$ as $|\xi| \to +\infty$ for $h \in \{1, \dots, r\}$, where r is the cardinality of $\sigma(\mathsf{A})$, are approximated by

$$\lambda_{h\ell}(i\xi) = i\alpha_h\xi + \beta_{h\ell} + o(1)$$

for $h \in \{1, \dots, r\}$ and $\ell \in \{1, \dots, r_h\}$, where $\beta_{h\ell} \in \sigma(\Pi_h^{(0)}\mathsf{B}\Pi_h^{(0)}, \operatorname{ran} \Pi_h^{(0)})$ and the integer r_h is the cardinality of $\sigma(\Pi_h^{(0)}\mathsf{B}\Pi_h^{(0)}, \operatorname{ran} \Pi_h^{(0)})$.

Therefore, it follows from the conditions $\mathscr{A}$ and $\mathscr{D}$ that α_h is real for all $h \in \{1, \dots, r\}$ and there is a positive constant θ such that

$$\frac{\theta}{1 + |\xi|^{-2}} \leqslant \operatorname{Re}(\beta_{h\ell}) + \varepsilon$$

for small $0 < |\xi|^{-1} < \varepsilon$, $h \in \{1, \dots, r\}$ and $\ell \in \{1, \dots, r_h\}$. Let $\varepsilon \to 0$, one obtains $\operatorname{Re}(\beta_{h\ell}) \geqslant \theta > 0$ for all $h \in \{1, \dots, r\}$ and $\ell \in \{1, \dots, r_h\}$. The proof is thus done. □

Let χ_i for $i = 1, 2, 3$ be smooth cut-off functions valued in $[0, 1]$ with supports contained in $\{\xi \in \mathbb{R} : |\xi| < \varepsilon\}$, $\{\xi \in \mathbb{R} : \varepsilon \leqslant |\xi| \leqslant \rho\}$ and $\{\xi \in \mathbb{R} : |\xi| > \rho\}$ respectively for small ε and large ρ such that $\chi_1 + \chi_2 + \chi_3 = 1$.

The solution $\hat{\mathsf{G}}$ to (2.32) then satisfies the following

$$\hat{\mathsf{G}}(\xi, t)\chi_1(\xi) = \hat{\mathsf{G}}_1(\xi, t)\chi_1(\xi) + \hat{\mathsf{G}}_2(\xi, t)\chi_1(\xi), \tag{3.21}$$

where

$$\hat{\mathsf{G}}_1(\xi, t) := \sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-(ic_h\xi + d_{h\ell}\xi^2)t} e^{-\left(\xi^2 \mathsf{N}_{h\ell}^{(0)} + \mathcal{O}(|\xi|^3)\right)t}\left(\mathsf{P}_{h\ell}^{(0)} + \mathcal{O}(|\xi|)\right) \tag{3.22}$$

and

$$\hat{G}_2(\xi,t) := \sum_{k=1}^{s'} e^{-b_k t} e^{-\left(M_k^{(0)}+\mathcal{O}(|\xi|)\right)t}\left(F_k^{(0)} + \mathcal{O}(|\xi|)\right). \tag{3.23}$$

In (3.22), $c_h \in \sigma(C, \operatorname{ran} P_0^{(0)})$ with the associated eigenprojection $P_h^{(0)}$, $d_{h\ell} \in \sigma(P_h^{(0)} D P_h^{(0)}, \operatorname{ran} P_h^{(0)})$ with the associated eigenprojection $P_{h\ell}^{(0)}$ and eigennilpotent $N_{h\ell}^{(0)}$, s and s_h are respectively the cardinalities of $\sigma(C, \operatorname{ran} P_0^{(0)})$ and $\sigma(P_h^{(0)} D P_h^{(0)}, \operatorname{ran} P_h^{(0)})$. In (3.23), $b_k \in \sigma(B)\backslash\{0\}$ with the associated eigenprojection $F_k^{(0)}$ and eigennilpotent $M_k^{(0)}$ and s' is the cardinality of $\sigma(B)\backslash\{0\}$.

On the other hand, we also have

$$\begin{aligned}&\hat{G}(\xi,t)\chi_3(\xi)\\&\quad= \sum_{h=1}^{r}\sum_{\ell=1}^{r_h} e^{-(i\alpha_h\xi+\beta_{h\ell})t} e^{-\left(\Theta_{h\ell}^{(0)}+\mathcal{O}(|\xi|^{-1})\right)t}\left(\Pi_{h\ell}^{(0)} + \mathcal{O}(|\xi|^{-1})\right),\end{aligned} \tag{3.24}$$

where $\alpha_h \in \sigma(A)$ with the associated eigenprojection $\Pi_h^{(0)}$, the coefficient $\beta_{h\ell} \in \sigma(\Pi_h^{(0)} B \Pi_h^{(0)}, \operatorname{ran} \Pi_h^{(0)})$ with the associated eigenprojection $\Pi_{h\ell}^{(0)}$ and eigennilpotent $\Theta_{h\ell}^{(0)}$, the integers r and r_h are respectively the cardinalities of $\sigma(A)$ and $\sigma(\Pi_h^{(0)} B \Pi_h^{(0)}, \operatorname{ran} \Pi_h^{(0)})$.

One also sets

$$\hat{K}(\xi,t) := \sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-(ic_h\xi+d_{h\ell}\xi^2)t} e^{-\xi^2 N_{h\ell}^{(0)} t} P_{h\ell}^{(0)} \tag{3.25}$$

and one sets

$$\hat{W}(\xi,t) := \sum_{h=1}^{r}\sum_{\ell=1}^{r_h} e^{-(i\alpha_h\xi+\beta_{h\ell})t} e^{-\Theta_{h\ell}^{(0)} t} \Pi_{h\ell}^{(0)}, \tag{3.26}$$

where the coefficients are introduced as before.

3.1.2 L^∞-L^1 estimate

Proposition 3.4 (Low frequency)**.** *For* $g \in L^1(\mathbb{R})$*, if the conditions* $\mathscr{A}$*,* $\mathscr{B}$*,* $\mathscr{C}$ *and* $\mathscr{D}$ *hold, there is a constant* $C > 0$ *such that for* $t \geqslant 1$*, one has*

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_1) * g\|_{L^\infty} \leqslant C t^{-1} \|g\|_{L^1}. \tag{3.27}$$

Proof. By the Young inequality in Lemma A.1 and the fact that $\mathcal{F}^{-1} : L^1 \to L^\infty$, it is sufficient to estimate the L^1-norm of $(\hat{G} - \hat{K} - \hat{W})\chi_1$ under the conditions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$.

We primarily estimate $(\hat{G} - \hat{K})\chi_1 = I + J$ where $I := (\hat{G}_1 - \hat{K})\chi_1$, $J := \hat{G}_2\chi_1$, $\hat{G}_1$ is given by (3.22) and $\hat{G}_2$ is given by (3.23). Moreover, we have $I = I_1 + I_2$ where

$$\begin{aligned}&I_1(\xi,t)\\&\quad:= \sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-(ic_h\xi+d_{h\ell}\xi^2)t}\left(e^{-(\xi^2 N_{h\ell}^{(0)}+\mathcal{O}(|\xi|^3))t} - e^{-\xi^2 N_{h\ell}^{(0)} t}\right) P_{h\ell}^{(0)}\end{aligned} \tag{3.28}$$

and

$$I_2(\xi,t) := \sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-(ic_h\xi + d_{h\ell}\xi^2)t} e^{-(\xi^2 N_{h\ell}^{(0)} + \mathcal{O}(|\xi|^3))t}\mathcal{O}(|\xi|). \qquad (3.29)$$

By the condition $\mathscr{C}$, Lemma 3.3, (3.19), (3.22) and (3.25), there are constants $c > 0$ and $C > 0$ such that for $t \geqslant 0$, we have

$$\begin{aligned} |I_1(\xi,t)| &\leqslant C\sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-\mathrm{Re}\,(d_{h\ell})|\xi|^2 t}\left| e^{-(\xi^2 N_{h\ell}^{(0)} + \mathcal{O}(|\xi|^3))t} - e^{-\xi^2 N_{h\ell}^{(0)} t}\right| |\chi_1(\xi)| \\ &\leqslant Ce^{-c|\xi|^2 t}|\xi|^3 t|\chi_1(\xi)|. \end{aligned}$$

Similarly, we estimate I_2. Indeed, by Remark 3.3, we also have

$$\begin{aligned} |I_2(\xi,t)| &\leqslant C\sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-\mathrm{Re}\,(d_{h\ell})|\xi|^2 t} e^{|\xi|^2|N_{h\ell}^{(0)}|t + C|\xi|^3 t}|\xi||\chi_1(\xi)| \\ &\leqslant Ce^{-c|\xi|^2 t}|\xi||\chi_1(\xi)|. \end{aligned}$$

On the other hand, by the condition $\mathscr{B}$, Remark 3.3 and from (3.23), there are constants $c > 0$ and $C > 0$ such that for $t \geqslant 0$, we have

$$|J(\xi,t)| \leqslant C\sum_{k=1}^{s'} e^{-\mathrm{Re}\,(b_k)t} e^{(|M_k^{(0)}| + C|\xi|)t}|\chi_1(\xi)| \leqslant Ce^{-ct}|\chi_1(\xi)|.$$

Hence, it implies that

$$\begin{aligned} \|(\hat{G} - \hat{K})\chi_1\|_{L^1} &\leqslant C\int_{\mathbb{R}} e^{-c|\xi|^2 t}(|\xi|^3 t + |\xi|)|\chi_1(\xi)|\, d\xi \\ &\qquad\qquad + Ce^{-ct}\int_{\mathbb{R}} |\chi_1(\xi)|\, d\xi \\ &\leqslant C(1+t)^{-1} + Ce^{-ct} \qquad \forall t \geqslant 0. \end{aligned} \qquad (3.30)$$

We estimate $\hat{W}\chi_1$. By the condition $\mathscr{A}$, Remark 3.3, Lemma 3.3 and from (3.26), there are constants $c > 0$ and $C > 0$ such that for $t \geqslant 0$, one has

$$|\hat{W}(\xi,t)\chi_1(\xi)| \leqslant C\sum_{h=1}^{r}\sum_{\ell=1}^{r_h} e^{-\mathrm{Re}\,(\beta_{h\ell})t} e^{|\Theta_{h\ell}^{(0)}|t}|\chi_1(\xi)| \leqslant Ce^{-ct}|\chi_1(\xi)|.$$

Hence, we deduce

$$\|\hat{W}\chi_1\|_{L^1} \leqslant Ce^{-ct}\int_{\mathbb{R}} |\chi_1(\xi)|\, d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 0. \qquad (3.31)$$

It therefore follows from (3.30) and (3.31) that

$$\begin{aligned} \|(\hat{G} - \hat{K} - \hat{W})\chi_1\|_{L^1} &\leqslant \|(\hat{G} - \hat{K})\chi_1\|_{L^1} + \|\hat{W}\chi_1\|_{L^1} \\ &\leqslant C(1+t)^{-1} + Ce^{-ct} \leqslant Ct^{-1} \qquad \forall t \geqslant 1. \end{aligned}$$

We finish the proof. □

Proposition 3.5 (Intermediate frequency)**.** *For* $g \in L^1(\mathbb{R})$*, if the conditions* $\mathscr{A}$*,* $\mathscr{B}$*,* $\mathscr{C}$ *and* $\mathscr{D}$ *hold, there are constants* $c > 0$ *and* $C > 0$ *such that for* $t \geqslant 0$*, one has*

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_2) * g\|_{L^\infty} \leqslant Ce^{-ct}\|g\|_{L^1}. \tag{3.32}$$

Proof. Similarly, by the Young inequality in Lemma A.1 and the fact that $\mathcal{F}^{-1} : L^1 \to L^\infty$, it is sufficient to estimate the L^1-norm of $(\hat{G}-\hat{K}-\hat{W})\chi_2$ under the conditions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$.

We estimate $\hat{G}\chi_2$ firstly where $\hat{G}(\xi, t) = e^{-E(i\xi)t}$. Since the condition $\mathscr{D}$ holds, $\mathrm{Re}\,\lambda(i\xi) > 0$ for any eigenvalue $\lambda(i\xi)$ of $E(i\xi)$ and $\xi \neq 0$. Thus, the operator $e^{-E(i\xi)}$ has the spectral radius $\rho(e^{-E(i\xi)}) < 1$ for any $\xi \neq 0$. It follows from [63] that there is an induced norm $|\cdot|$ such that

$$0 < \varphi := \operatorname*{ess\,sup}_{\xi\in\mathbb{R}} |e^{-E(i\xi)}| < 1.$$

Hence, for any $t \geqslant 0$ with the integer part denoted by m, since $\log\varphi < 0$, there are constants $c > 0$ and $C > 0$ such that one has

$$\begin{aligned}
|\hat{G}(\xi, t)\chi_2(\xi)| &\leqslant |e^{-E(i\xi)}|^m |e^{-E(i\xi)(t-m)}||\chi_2(\xi)| \\
&\leqslant \varphi^m e^{|E(i\xi)|}|\chi_2(\xi)| \\
&\leqslant \varphi^{-1} e^{(m+1)\log\varphi} e^{|E(i\xi)|}|\chi_2(\xi)| \\
&\leqslant Ce^{-ct} e^{|E(i\xi)|}|\chi_2(\xi)|.
\end{aligned}$$

Thus, since every norms in finite-dimensional spaces are equivalent, we obtain

$$\|\hat{G}\chi_2\|_{L^1} \leqslant Ce^{-ct}\int_{\mathbb{R}} e^{|E(i\xi)|}|\chi_2(\xi)|\,d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 0. \tag{3.33}$$

We estimate $\hat{K}\chi_2$. By the condition $\mathscr{C}$, Remark 3.3, Lemma 3.3 and from (3.25), there are constants $c > 0$ and $C > 0$ such that

$$\begin{aligned}
&|\hat{K}(\xi, t)\chi_2(\xi)| \\
&\qquad \leqslant C\sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-\mathrm{Re}\,(d_{h\ell})|\xi|^2 t} e^{|\xi|^2|N_{h\ell}^{(0)}|t}|\chi_2(\xi)| \leqslant Ce^{-c|\xi|^2 t}|\chi_2(\xi)|.
\end{aligned}$$

Therefore, we have

$$\|\hat{K}\chi_2\|_{L^1} \leqslant C\int_{\mathbb{R}} e^{-c|\xi|^2 t}|\chi_2(\xi)|\,d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 0. \tag{3.34}$$

We estimate $\hat{W}\chi_2$. By the condition $\mathscr{A}$, Remark 3.3, Lemma 3.3 and from (3.26), there are constants $c > 0$ and $C > 0$ such that

$$|\hat{W}(\xi, t)\chi_2(\xi)| \leqslant C\sum_{h=1}^{r}\sum_{\ell=1}^{r_h} e^{-\mathrm{Re}\,(\beta_{h\ell})t} e^{|\Theta_{h\ell}^{(0)}|t}|\chi_2(\xi)| \leqslant Ce^{-ct}|\chi_2(\xi)|.$$

It follows that

$$\|\hat{W}\chi_2\|_{L^1} \leqslant Ce^{-ct}\int_{\mathbb{R}} |\chi_2(\xi)|\,d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 0. \tag{3.35}$$

From (3.33) - (3.35), for $t \geqslant 0$, we obtain

$$\|(\hat{G} - \hat{K} - \hat{W})\chi_2\|_{L^1} \leqslant \|\hat{G}\chi_2\|_{L^1} + \|\hat{K}\chi_2\|_{L^1} + \|\hat{W}\chi_2\|_{L^1} \leqslant Ce^{-ct}.$$

The proof is done. □

Proposition 3.6 (High frequency)**.** *For* $g \in L^1(\mathbb{R})$*, if the conditions* $\mathscr{A}$*,* $\mathscr{B}$*,* $\mathscr{C}$ *and* $\mathscr{D}$ *hold, there are constants* $c > 0$ *and* $C > 0$ *such that for* $t \geqslant 1$*, one has*

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_3) * g\|_{L^\infty} \leqslant Ce^{-ct}\|g\|_{L^1}. \tag{3.36}$$

Proof. We start with the estimate for $\mathcal{F}^{-1}(\hat{K}\chi_3)$. By the condition $\mathscr{C}$, Remark 3.3, Lemma 3.3 and from 3.25, there are constants $c > 0$ and $C > 0$ such that

$$|\hat{K}(\xi,t)\chi_3(\xi)| \leqslant C\sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-\mathrm{Re}\,(d_{h\ell})|\xi|^2 t} e^{|\xi|^2|N_{h\ell}^{(0)}|t}|\chi_3(\xi)| \leqslant Ce^{-c|\xi|^2 t}|\chi_3(\xi)|.$$

Thus, we deduce that

$$\|\hat{K}\chi_3\|_{L^1} \leqslant C\int_{\mathbb{R}} e^{-c|\xi|^2 t}|\chi_3(\xi)|\,d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 1.$$

By the Young inequality in Lemma A.1 and the fact that $\mathcal{F}^{-1} : L^1 \to L^\infty$, we then obtain for all $t \geqslant 1$ that

$$\begin{aligned}\|\mathcal{F}^{-1}(\hat{K}\chi_3) * g\|_{L^\infty} &\leqslant C\|\mathcal{F}^{-1}(\hat{K}\chi_3)\|_{L^\infty}\|g\|_{L^1}\\ &\leqslant C\|\hat{K}\chi_3\|_{L^1}\|g\|_{L^1} \leqslant Ce^{-ct}\|g\|_{L^1}.\end{aligned} \tag{3.37}$$

To estimate $\mathcal{F}^{-1}((\hat{G} - \hat{W})\chi_3)$, we divide this one into two cases where $|x| \leqslant Ct$ and $|x| > Ct$ for $C > 0$ and for all $t \geqslant 0$. Moreover, it follows from (3.24) and (3.26) that $(\hat{G} - \hat{W})\chi_3 = I + J$ where

$$\begin{aligned}&I(\xi,t)\\ &:= \sum_{h=1}^{r}\sum_{\ell=1}^{r_h} e^{-(i\alpha_h\xi+\beta_{h\ell})t}\Big(e^{-(\Theta_{h\ell}^{(0)}+\mathcal{O}(|\xi|^{-1}))t} - e^{-\Theta_{h\ell}^{(0)}t}\Big)\Pi_{h\ell}^{(0)}\chi_3(\xi)\end{aligned} \tag{3.38}$$

and

$$J(\xi,t) := \sum_{h=1}^{r}\sum_{\ell=1}^{r_h} e^{-(i\alpha_h\xi+\beta_{h\ell})t}e^{-(\Theta_{h\ell}^{(0)}+\mathcal{O}(|\xi|^{-1}))t}\mathcal{O}(|\xi|^{-1})\chi_3(\xi). \tag{3.39}$$

Furthermore, from (3.38) and by applying the Taylor expansion to the application $X \to e^X$, one can decompose $I = I_1 + I_2 + I_3$ with

$$I_1(\xi,t) := t\sum_{h=1}^{r}\sum_{\ell=1}^{r_h} \frac{e^{-i\alpha_h\xi t}}{i\xi}e^{-\beta_{h\ell}t}de^{-\Theta_{h\ell}^{(0)}t}M\Pi_{h\ell}^{(0)}\chi_3(\xi), \tag{3.40}$$

where M is the coefficient in $\mathcal{O}(|\xi|^{-1})$ associated with $(i\xi)^{-1}$, and

$$I_2(\xi,t) := t\sum_{h=1}^{r}\sum_{\ell=1}^{r_h} e^{-(i\alpha_h\xi+\beta_{h\ell})t}de^{-\Theta_{h\ell}^{(0)}t}\mathcal{O}(|\xi|^{-2})\Pi_{h\ell}^{(0)}\chi_3(\xi) \tag{3.41}$$

and

$$\begin{aligned}I_3(\xi,t) := \sum_{h=1}^{r}\sum_{\ell=1}^{r_h} &e^{-(i\alpha_h\xi+\beta_{h\ell})t}\Big(e^{-(\Theta_{h\ell}^{(0)}+\mathcal{O}(|\xi|^{-1}))t}\\ &- e^{-\Theta_{h\ell}^{(0)}t} - de^{-\Theta_{h\ell}^{(0)}t}\mathcal{O}(|\xi|^{-1})t\Big)\Pi_{h\ell}^{(0)}\chi_3(\xi).\end{aligned} \tag{3.42}$$

On the other hand, from (3.39), one also has $J = J_1 + J_2$ where

$$J_1(\xi, t) := \sum_{h=1}^{r} \sum_{\ell=1}^{r_h} \frac{e^{-i\alpha_h \xi t}}{i\xi} e^{-\beta_{h\ell} t} e^{-(\Theta_{h\ell}^{(0)} + \mathcal{O}(|\xi|^{-1}))t} M\chi_3(\xi) \tag{3.43}$$

and

$$J_2(\xi, t) := \sum_{h=1}^{r} \sum_{\ell=1}^{r_h} e^{-(i\alpha_h \xi + \beta_{h\ell})t} e^{-(\Theta_{h\ell}^{(0)} + \mathcal{O}(|\xi|^{-1}))t} \mathcal{O}(|\xi|^{-2}) \chi_3(\xi). \tag{3.44}$$

We primarily estimate $\mathcal{F}^{-1}(I_1) = \sum_{h=1}^{r} \mathcal{F}_h^{-1}(I_1)$ where

$$\mathcal{F}_h^{-1}(I_1) := t \sum_{\ell=1}^{r_h} \mathcal{F}^{-1}\Big(\frac{e^{-i\alpha_h \xi t}}{i\xi} \chi_3(\xi) \Big) e^{-\beta_{h\ell} t} d e^{-\Theta_{h\ell}^{(0)} t} M \Pi_{h\ell}^{(0)}. \tag{3.45}$$

For $h \in \{1, \dots, r\}$, by the condition $\mathscr{A}$, Remark 3.3 and Lemma 3.3, we have

$$\begin{aligned} |\mathcal{F}_h^{-1}(I_1)(x,t)| &\leqslant Ct \sum_{\ell=1}^{r_h} e^{-\mathrm{Re}\,(\beta_{h\ell})t} e^{|\Theta_{h\ell}^{(0)}|t} \Big| \int_{-\infty}^{-\rho} + \int_{\rho}^{+\infty} \frac{e^{i(x-\alpha_h t)\xi}}{i\xi}\, d\xi \Big| \\ &\leqslant Ct \sum_{\ell=1}^{r_h} e^{-\mathrm{Re}\,(\beta_{h\ell})t} e^{|\Theta_{h\ell}^{(0)}|t} \Big| 2 \int_{\rho}^{+\infty} \frac{\sin((x-\alpha_h t)\xi)}{\xi}\, d\xi \Big| \\ &\leqslant Cte^{-ct} |x - \alpha_h t|. \end{aligned} \tag{3.46}$$

Hence, if $|x| \leqslant Ct$ for a constant $C > 0$, then we have

$$\|\mathcal{F}_h^{-1}(I_1)\|_{L^\infty} \leqslant Ct^2 e^{-ct} \leqslant Ce^{-ct} \quad \text{for } h \in \{1, \dots, r\}, \forall t \geqslant 0. \tag{3.47}$$

We now estimate $\mathcal{F}^{-1}(I_1)$ in the case where $|x| > Ct$. Noting that we can assume that C is large enough and in this case, we have

$$e^{x\alpha_h} \leqslant e^{|x||\alpha_h|} \leqslant e^{\frac{|x|^2}{t}|\alpha_h||x|^{-1}t} \leqslant e^{\varepsilon \frac{|x|^2}{t}}$$

for $h \in \{1, \dots, r\}$ and small $\varepsilon > 0$. Moreover, for $h \in \{1, \dots, r\}$, we also have

$$\begin{aligned} &|\mathcal{F}_h^{-1}(I_1)(x,t)| \\ &\qquad \leqslant Ct \sum_{\ell=1}^{r_h} e^{-\mathrm{Re}\,(\beta_{h\ell})t} e^{|\Theta_{h\ell}^{(0)}|t} \Big| \int_{-\infty}^{-\rho} + \int_{\rho}^{+\infty} \frac{e^{i(x-\alpha_h t)\xi}}{i\xi}\, d\xi \Big|. \end{aligned} \tag{3.48}$$

We estimate the integral

$$\begin{aligned} H &:= \int_{-\infty}^{-\rho} + \int_{\rho}^{+\infty} \frac{e^{i(x-\alpha_h t)\xi}}{i\xi}\, d\xi \\ &= \lim_{\delta \to +\infty} \int_{-\delta}^{-\rho} + \int_{\rho}^{\delta} \frac{e^{i(x-\alpha_h t)\xi}}{i\xi}\, d\xi = H_1 + H_2. \end{aligned} \tag{3.49}$$

Due to the fact that the integrand is holomorphic, we can estimate H_2 by considering $\xi = \zeta + i\eta \in \mathbb{C}$ and by changing the path $\{(\zeta, 0) : \zeta \text{ from } \rho \text{ to } \delta\}$ to the path $\gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3$ in the complex plane where

$$\gamma_1 := \{(\zeta, \eta) : \zeta = \rho, \eta \text{ from } 0 \text{ to } x/t\},$$

$$\gamma_2 := \{(\zeta,\eta) : \zeta \text{ from } \rho \text{ to } \delta, \eta = x/t\}$$

and

$$\gamma_3 := \{(\zeta,\eta) : \zeta = \delta, \eta \text{ from } x/t \text{ to } 0\}.$$

Then, by parameterizing $\gamma_1(s) = \rho + i\frac{x}{t}s$ for $s \in [0,1]$, since $|x| > Ct$ and $t \geqslant 1$, we have

$$\begin{aligned}
\Big|\lim_{\delta\to+\infty}\int_{\gamma_1}\frac{e^{i(x-\alpha_h t)\xi}}{i\xi}\,d\xi\Big| &= \Big|\int_0^1 \frac{e^{i(x-\alpha_h t)\rho + x\alpha_h s - \frac{|x|^2}{t}s}}{\rho + i\frac{x}{t}s}\frac{x}{t}\,ds\Big| \\
&\leqslant \frac{C}{\rho}\int_0^1\Big(\frac{|x|}{t} + \frac{|x|^2}{t^2}\Big)e^{\varepsilon\frac{|x|^2}{t}s}e^{-\frac{|x|^2}{t}s}\,ds \\
&\leqslant \frac{C}{\rho}\Big(\frac{1}{|x|} + \frac{1}{t}\Big)\Big(1 - e^{-\frac{|x|^2}{2t}}\Big) \leqslant \frac{C}{\rho}. \qquad (3.50)
\end{aligned}$$

On the other hand, noting that for all $(\eta,\zeta) \in \mathbb{R}^2$, we have

$$\frac{1}{-\eta + i\zeta} = \frac{1}{i\zeta} - \eta\left(\frac{1}{\zeta^2+\eta^2} + \frac{1}{i\zeta}\frac{\eta}{\zeta^2+\eta^2}\right).$$

Thus, for $|x| > Ct$ and $t \geqslant 1$, there is $c > 0$ such that we also have

$$\begin{aligned}
\Big|\lim_{\delta\to+\infty}\int_{\gamma_2}\frac{e^{i(x-\alpha_h t)\xi}}{i\xi}\,d\xi\Big| &= \Big|\int_\rho^{+\infty}\frac{e^{ix\zeta - i\alpha_h\zeta t - \frac{|x|^2}{t} + x\alpha_h}}{-\frac{x}{t} + i\zeta}\,d\zeta\Big| \\
&\leqslant e^{-\frac{|x|^2}{2t}}\Big|\int_\rho^{+\infty} e^{ix\zeta}\Big(\frac{1}{i\zeta} - \frac{x}{t}\Big(\frac{1}{\zeta^2 + \frac{|x|^2}{t^2}} + \frac{1}{i\zeta}\frac{\frac{x}{t}}{\zeta^2+\frac{|x|^2}{t^2}}\Big)\Big)\,d\zeta\Big| \\
&\leqslant Ce^{-\frac{|x|^2}{2t}}\Big(\Big|\int_\rho^{+\infty}\frac{e^{ix\zeta}}{i\zeta}\,d\zeta\Big| + \Big(\frac{|x|}{t} + \frac{|x|^2}{t^2}\Big)\int_\rho^{+\infty}\frac{1}{\zeta^2}\,d\zeta\Big) \\
&\leqslant Ce^{-\frac{|x|^2}{2t}}\frac{|x|^2}{2t}\Big(\frac{t}{|x|} + \frac{1}{|x|} + \frac{1}{t}\Big) \leqslant Ce^{-\frac{|x|^2}{ct}}. \qquad (3.51)
\end{aligned}$$

Similarly, we consider $\gamma_3(s) = \delta + i\frac{x}{t}(1-s)$ for $s \in [0,1]$, we have

$$\Big|\lim_{\delta\to+\infty}\int_{\gamma_3}\frac{e^{i(x-\alpha_h t)\xi}}{i\xi}\,d\xi\Big| = \Big|\lim_{\delta\to+\infty}\int_0^1\frac{e^{i(x-\alpha_h t)\delta + x\alpha_h(1-s) - \frac{|x|^2}{t}(1-s)}}{\delta + i\frac{x}{t}(1-s)}\frac{x}{t}\,ds\Big|.$$

On the other hand, for $\delta > 0$ and $t \geqslant 1$, we have

$$\begin{aligned}
\Big|\int_0^1\frac{e^{i(x-\alpha_h t)\delta + x\alpha_h(1-s) - \frac{|x|^2}{t}(1-s)}}{\delta + i\frac{x}{t}(1-s)}\frac{x}{t}\,ds\Big| &\leqslant \frac{C}{\delta}\int_0^1\Big(\frac{|x|}{t} + \frac{|x|^2}{t^2}\Big)e^{-\frac{|x|^2}{2t}(1-s)}\,ds \\
&= \frac{C}{\delta}\Big(\frac{1}{|x|} + \frac{1}{t}\Big)e^{-\frac{|x|^2}{2t}}\Big(e^{\frac{|x|^2}{2t}} - 1\Big) \leqslant \frac{C}{\delta}.
\end{aligned}$$

One deduces that

$$\lim_{\delta\to+\infty}\int_0^1\frac{e^{i(x-\alpha_h t)\delta + x\alpha_h(1-s) - \frac{|x|^2}{t}(1-s)}}{\delta + i\frac{x}{t}(1-s)}\frac{x}{t}\,ds = 0.$$

Hence, it implies

$$\Big|\lim_{\delta\to+\infty}\int_{\gamma_3}\frac{e^{i(x-\alpha_h t)\xi}}{i\xi}\,d\xi\Big| = 0. \qquad (3.52)$$

Moreover, one can estimate H_1 similarly by substituting ρ and δ by $-\rho$ and $-\delta$ respectively. Therefore, from (3.48), (3.49), (3.50), (3.51) and (3.52), one obtains

$$\left\|\mathcal{F}_h^{-1}(I_1)\right\|_{L^\infty} \leqslant Ce^{-ct} \qquad \forall t \geqslant 1$$

for $|x| > Ct$ and for $h \in \{1, \dots, r\}$, where C is large enough.

It then implies that

$$\|\mathcal{F}^{-1}(I_1)\|_{L^\infty} \leqslant C\sum_{h=1}^{r} \|\mathcal{F}_h^{-1}(I_1)\|_{L^\infty} \leqslant Ce^{-ct} \qquad \forall t \geqslant 1.$$

We estimate $\mathcal{F}^{-1}(I_2)$ and $\mathcal{F}^{-1}(I_3)$ where I_2 and I_3 are given by (3.41) and (3.42) respectively. Since $\mathcal{F}^{-1} : L^1 \to L^\infty$, one has

$$\left\|\mathcal{F}^{-1}(I_j)\right\|_{L^\infty} \leqslant C\left\|I_j\right\|_{L^1}, \qquad j = 2, 3. \tag{3.53}$$

Hence, we only need to estimate I_2 and I_3 in L^1.

From (3.41), we have

$$|I_2(\xi, t)| \leqslant C\sum_{h=1}^{r}\sum_{\ell=1}^{r_h} e^{-\mathrm{Re}\,(\beta_{h\ell})t} e^{|\Theta_{h\ell}^{(0)}|t}|\xi|^{-2}t|\chi_3(\xi)|.$$

Thus, by Remark 3.3 and Lemma 3.3, we obtain

$$\|I_2\|_{L^1} \leqslant Cte^{-ct}\int_{\mathbb{R}} |\xi|^{-2}|\chi_3(\xi)|\,d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 0.$$

From (3.42) and by Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned} |I_3(\xi, t)| \leqslant & \sum_{h=1}^{r}\sum_{\ell=1}^{r_h} e^{-\mathrm{Re}\,(\beta_{h\ell})t}\Big|e^{-(\Theta_{h\ell}^{(0)}+\mathcal{O}(|\xi|^{-1}))t} \\ & \qquad - e^{-\Theta_{h\ell}^{(0)}t} - de^{-\Theta_{h\ell}^{(0)}t}\mathcal{O}(|\xi|^{-1})t\Big||\chi_3(\xi)| \\ \leqslant & \; Ct^2e^{-ct}|\xi|^{-2}|\chi_3(\xi)|. \end{aligned}$$

Thus, one obtains

$$\|I_3\|_{L^1} \leqslant Ct^2e^{-ct}\int_{\mathbb{R}} |\xi|^{-2}|\chi_3(\xi)|\,d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 0.$$

Therefore, for $t \geqslant 1$, we have

$$\|\mathcal{F}^{-1}(I)\|_{L^\infty} \leqslant \|\mathcal{F}^{-1}(I_1)\|_{L^\infty} + \|\mathcal{F}^{-1}(I_2)\|_{L^\infty} + \|\mathcal{F}^{-1}(I_3)\|_{L^\infty} \leqslant Ce^{-ct}.$$

We now estimate $\mathcal{F}^{-1}(J)$ where J is given by (3.39). Then, we can estimate $\mathcal{F}^{-1}(J_1)$ similarly to $\mathcal{F}^{-1}(I_1)$ and $\mathcal{F}^{-1}(J_2)$ similarly to $\mathcal{F}^{-1}(I_2)$ and $\mathcal{F}^{-1}(I_3)$, where J_1 and J_2 are respectively given by (3.43) and (3.44). Thus, we obtain

$$\|\mathcal{F}^{-1}(J)\|_{L^\infty} \leqslant \|\mathcal{F}^{-1}(J_1)\|_{L^\infty} + \|\mathcal{F}^{-1}(J_2)\|_{L^\infty} \leqslant Ce^{-ct} \qquad \forall t \geqslant 1.$$

By the Young inequality in Lemma A.1, it follows that

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{W})\chi_3) * g\|_{L^\infty} \leqslant \left(\|\mathcal{F}^{-1}(I)\right\|_{L^\infty} + \left\|\mathcal{F}^{-1}(J)\|_{L^\infty}\right)\|g\|_{L^1}$$

$$\leqslant Ce^{-ct}\|g\|_{L^1} \qquad \forall t \geqslant 1. \tag{3.54}$$

Finally, from (3.37) and (3.54), we have

$$\begin{aligned}\|\mathcal{F}^{-1}((\hat{G}-\hat{K}-\hat{W})\chi_3) * g\|_{L^\infty} &\leqslant \|\mathcal{F}^{-1}(\hat{K}\chi_3) * g\|_{L^\infty} \\ &\quad + \|\mathcal{F}^{-1}((\hat{G}-\hat{W})\chi_3) * g\|_{L^\infty} \\ &\leqslant Ce^{-ct}\|g\|_{L^1} \qquad \forall t \geqslant 1.\end{aligned}$$

The proof is done. □

3.1.3 L^p-L^p estimates

We establish the L^p-L^p estimate for $1 \leqslant p \leqslant \infty$. By the Young inequality, for any $g \in L^p(\mathbb{R})$, we have

$$\|\mathcal{F}^{-1}(\hat{G}-\hat{K}-\hat{W}) * g\|_{L^p} \leqslant \|\mathcal{F}^{-1}(\hat{G}-\hat{K}-\hat{W})\|_{L^1}\|g\|_{L^p}. \tag{3.55}$$

In this case, we cannot use the estimates for $\hat{G}-\hat{K}-\hat{W}$ in the proofs of the L^∞-L^1 estimate since the very sharp estimate

$$|(\hat{G}_1(\xi,t)-\hat{K}(\xi,t))\chi_1(\xi)| \leqslant Ce^{-c|\xi|^2 t}|\xi|^3 t|\chi_1(\xi)|$$

is not L^1-integrable in $x \in \mathbb{R}$, where $\hat{G}_1$ is given by (3.22). Based on [5, 78], we can change paths of integrals such that the inverse Fourier transform of $\hat{G}-\hat{K}-\hat{W}$ is bounded from above by an L^1-integrable function in $x \in \mathbb{R}$. It can be done since $\hat{G}-\hat{K}-\hat{W}$ are holomorphic functions in $z = i\xi \in \mathbb{C}$.

Case $|x| \leqslant Ct$

Proposition 3.7 (Low frequency)**.** *For $g \in L^p(\mathbb{R})$, if the conditions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ hold, then there is a constant $C > 0$ such that*

$$\|\mathcal{F}^{-1}((\hat{G}-\hat{K}-\hat{W})\chi_1) * g\|_{L^p} \leqslant Ct^{-\frac{1}{2}}\|g\|_{L^p} \qquad \forall t \geqslant 1. \tag{3.56}$$

Proof. Recall that $(\hat{G}-\hat{K})\chi_1 = I + J$ where $I = (\hat{G}_1-\hat{K})\chi_1$, $J = \hat{G}_2\chi_1$, $\hat{G}_1$ is given by (3.22) and $\hat{G}_2$ is given by (3.23). Moreover, we consider $I = I_1 + I_2$ where I_1 and I_2 are respectively given by (3.28) and (3.29).

We estimate $\mathcal{F}^{-1}(I_1)$. One primarily has

$$\begin{aligned}\mathcal{F}^{-1}(I_1)(x,t) = \sum_{h=1}^{s}\sum_{\ell=1}^{s_h}\int_{-\varepsilon}^{\varepsilon} & e^{i(x-c_h t)\xi - d_{h\ell}\xi^2 t} \\ &\cdot \left(e^{-(\xi^2 N_{h\ell}^{(0)} + \mathcal{O}(|\xi|^3))t} - e^{-\xi^2 N_{h\ell}^{(0)} t}\right) P_{h\ell}^{(0)}\chi_1(\xi)\, d\xi.\end{aligned}$$

On the other hand, for $h \in \{1,\dots,s\}$ and $\ell \in \{1,\dots,s_h\}$, let $z := e^{i\phi/2}\xi$, where $\phi := \arg(d_{h\ell}) \in (-\pi/2,\pi/2)$ since $\mathrm{Re}(d_{h\ell}) > 0$ due to Lemma 3.3, one obtains

$$\begin{aligned}\mathcal{F}^{-1}(I_1)(x,t) = \sum_{h=1}^{s}\sum_{\ell=1}^{s_h}\int_{\gamma} & \chi_1(e^{-i\phi/2}z)e^{i(x-c_h t)e^{-i\phi/2}z - |d_{h\ell}|z^2 t} \\ &\cdot \left(e^{-(e^{-i\phi}z^2 N_{h\ell}^{(0)} + \mathcal{O}(|e^{-i\phi/2}z|^3))t} - e^{-e^{-i\phi}z^2 N_{h\ell}^{(0)} t}\right) P_{h\ell}^{(0)} e^{-i\phi/2}\, dz,\end{aligned}$$

where $\gamma := \{z \in \mathbb{C} : z = e^{i\phi/2}\xi \text{ for } \xi \in [-\varepsilon, \varepsilon]\}$. Then, we will estimate each summand by letting $\eta := \min\{\frac{|x-c_h t|}{2|d_{h\ell}|t}, \frac{\varepsilon}{2}\}$. Furthermore, since the integrand is holomorphic, we can change γ by $\gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3$ in the complex plane with respect to z where

$$\gamma_1 := \{-\varepsilon e^{i\phi/2} + i\operatorname{sgn}(x - c_h t)\eta e^{-i\phi/2} s : s \in [0, 1]\},$$

$$\gamma_2 := \{\zeta e^{i\phi/2} + i\operatorname{sgn}(x - c_h t)\eta e^{-i\phi/2} : \zeta \in [-\varepsilon, \varepsilon]\}$$

and

$$\gamma_3 := \{\varepsilon e^{i\phi/2} + i\operatorname{sgn}(x - c_h t)\eta e^{-i\phi/2}(1 - s) : s \in [0, 1]\}.$$

On the other hand, we have

$$\begin{aligned}&\left|e^{i(x-c_h t)e^{-i\phi/2}z-|d_{h\ell}|z^2 t}\right| \\ &\qquad = e^{-(x-c_h t)(\cos(\phi/2)\operatorname{Im} z-\sin(\phi/2)\operatorname{Re} z)}e^{-|d_{h\ell}|((\operatorname{Re} z)^2-(\operatorname{Im} z)^2)t}.\end{aligned}$$

Moreover, by Lemma 3.2, we also have

$$\left|e^{-(e^{-i\phi}z^2 N_{h\ell}^{(0)}+\mathcal{O}(|e^{-i\phi/2}z|^3))t} - e^{-e^{-i\phi}z^2 N_{h\ell}^{(0)} t}\right| \leqslant C|z|^3 t e^{\varepsilon'|z|^2 t + C|z|^3 t}$$

for any $\varepsilon' > 0$.

We estimate on γ_1. For $z \in \gamma_1$ and $s \in [0, 1]$, we have

$$\begin{aligned}\operatorname{Re} z(s) &= -\varepsilon\cos(\phi/2) + \operatorname{sgn}(x - c_h t)\eta\sin(\phi/2)s,\\ \operatorname{Im} z(s) &= -\varepsilon\sin(\phi/2) + \operatorname{sgn}(x - c_h t)\eta\cos(\phi/2)s.\end{aligned}$$

Hence, since $\cos(\phi) > 0$, $\eta \leqslant \varepsilon/2$, $\eta^2 s^2 \leqslant \varepsilon^2/4$ for $s \in [0, 1]$ and small $|z| \leqslant 3\varepsilon$, we obtain that $|z|^3 \leqslant \varepsilon'|z|^2/C$ and by choosing $\varepsilon' = |d_{h\ell}|\cos(\phi)/8$, there is $c > 0$ such that for $t \geqslant 0$, we have

$$\begin{aligned}\left|\int_{\gamma_1}\right| &\leqslant C\int_0^1 e^{-|x-c_h t|\eta\cos(\phi)s} \\ &\qquad \cdot e^{-|d_{h\ell}|\cos(\phi)(\varepsilon^2-\eta^2 s^2)t}e^{2\varepsilon'\varepsilon^2 t}\varepsilon^4 t\, ds \leqslant Ce^{-ct}. \quad (3.57)\end{aligned}$$

We estimate on γ_2. For $z \in \gamma_2$ and $\zeta \in [-\varepsilon, \varepsilon]$, we have

$$\begin{aligned}\operatorname{Re} z(\zeta) &= \zeta\cos(\phi/2) + \operatorname{sgn}(x - c_h t)\eta\sin(\phi/2),\\ \operatorname{Im} z(\zeta) &= \zeta\sin(\phi/2) + \operatorname{sgn}(x - c_h t)\eta\cos(\phi/2).\end{aligned}$$

Hence, since $|z| \leqslant 2(|\zeta| + |\eta|) \leqslant 3\varepsilon$ small enough, $|z|^3 \leqslant \varepsilon'|z|^2/C$ and one has

$$\left|\int_{\gamma_2}\right| \leqslant C\int_{-\varepsilon}^{\varepsilon} e^{-|x-c_h t|\eta\cos(\phi)}e^{-|d_{h\ell}|\cos(\phi)(\zeta^2-\eta^2)t}e^{2\varepsilon'(|\zeta|+|\eta|)^2 t}(|\zeta|+|\eta|)^3 t\, d\zeta.$$

If $\eta = \frac{|x-c_h t|}{2|d_{h\ell}|t}$, then since $(|\zeta| + |\eta|)^2 \leqslant 2(|\zeta|^2 + |\eta|^2)$ and by choosing $\varepsilon' = |d_{h\ell}|\cos(\phi)/8$, there is $c > 0$ such that

$$\begin{aligned}\left|\int_{\gamma_2}\right| &\leqslant C\int_{-\varepsilon}^{\varepsilon} e^{-\frac{|x-c_h t|^2}{4|d_{h\ell}|t}\cos(\phi)}e^{-|d_{h\ell}|\cos(\phi)\zeta^2 t}e^{4\varepsilon'\zeta^2 t+\varepsilon'\frac{|x-c_h t|^2}{|d_{h\ell}|^2 t}} \\ &\qquad \cdot \left(|\zeta|^3 t + 3|\zeta|^2\frac{|x-c_h t|}{2|d_{h\ell}|} + 3|\zeta|\frac{|x-c_h t|^2}{4|d_{h\ell}|^2 t} + \frac{|x-c_h t|^3}{8|d_{h\ell}|^3 t^2}\right) d\zeta\end{aligned}$$

$$\leqslant C\sum_{k=0}^{3} e^{-\frac{|x-c_h t|^2}{8|d_{h\ell}|t}\cos(\phi)}(t^{-\frac{1}{2}}|x-c_h t|)^k$$
$$\cdot\int_{-\varepsilon}^{\varepsilon} e^{-\frac{1}{2}|d_{h\ell}|\cos(\phi)\zeta^2 t}|\zeta|^{3-k}t^{1-\frac{k}{2}}\, d\zeta$$
$$\leqslant Ct^{-1}e^{-\frac{|x-c_h t|^2}{c|d_{h\ell}|t}} \qquad \forall t\geqslant 1. \tag{3.58}$$

If $\eta=\varepsilon/2$, then since $|x-c_h t|\geqslant \varepsilon|d_{h\ell}|t$ by the definition of η and since $|z|<\varepsilon$, by choosing $\varepsilon'=|d_{h\ell}|\cos(\phi)/16$, we have

$$\Big|\int_{\gamma_2}\Big| \leqslant C\int_{-\varepsilon}^{\varepsilon} e^{-|x-c_h t|\eta\cos(\phi)}e^{-|d_{h\ell}|\cos(\phi)(\zeta^2-\eta^2)t}e^{2\varepsilon'\varepsilon^2 t}\varepsilon^3 t\, d\zeta$$
$$\leqslant Ce^{-\frac{1}{8}\varepsilon^2|d_{h\ell}|\cos(\phi)t}\int_{-\varepsilon}^{\varepsilon} e^{-|d_{h\ell}|\cos(\phi)\zeta^2 t}t\, d\zeta$$
$$\leqslant Ct^{\frac{1}{2}}e^{-\frac{1}{8}\varepsilon^2|d_{h\ell}|\cos(\phi)t}\leqslant Ce^{-ct} \qquad \forall t\geqslant 0. \tag{3.59}$$

We estimate on γ_3. For $z\in\gamma_3$ and $s\in[0,1]$, we have

$$\operatorname{Re} z(s)=\varepsilon\cos(\phi/2)+\operatorname{sgn}(x-c_h t)\eta\sin(\phi/2)(1-s),$$
$$\operatorname{Im} z(s)=\varepsilon\sin(\phi/2)+\operatorname{sgn}(x-c_h t)\eta\cos(\phi/2)(1-s).$$

Thus, similarly to γ_1, by choosing $\varepsilon'=|d_{h\ell}|\cos(\phi)/8$, there is $c>0$ such that, for $t\geqslant 1$, we have

$$\Big|\int_{\gamma_3}\Big| \leqslant C\int_0^1 e^{-|x-c_h t|\eta\cos(\phi)(1-s)}$$
$$\cdot e^{-|d_{h\ell}|\cos(\phi)(\varepsilon^2-\eta^2(1-s)^2)t}e^{2\varepsilon'\varepsilon^2 t}\varepsilon^4 t\, ds\leqslant Ce^{-ct}. \tag{3.60}$$

On the other hand, for large $\delta>0$ and $t\geqslant 0$, since $|x|\leqslant Ct$, one has

$$e^{-ct}=e^{-ct}e^{\frac{|x-c_h t|^2}{\delta|d_{h\ell}|t}}e^{-\frac{|x-c_h t|^2}{\delta|d_{h\ell}|t}}\leqslant e^{-ct}e^{\frac{(C^2+2c_h C+c_h^2)}{\delta|d_{h\ell}|}t}e^{-\frac{|x-c_h t|^2}{\delta|d_{h\ell}|t}}$$
$$\leqslant e^{-\frac{c}{2}t}e^{-\frac{|x-c_h t|^2}{\delta|d_{h\ell}|t}}. \tag{3.61}$$

Thus, from (3.57) - (3.61), there is $c>0$ such that

$$|\mathcal{F}^{-1}(I_1)(x,t)|\leqslant Ct^{-1}\sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-\frac{|x-c_h t|^2}{c|d_{h\ell}|t}} \qquad \forall t\geqslant 1.$$

By the same way, for $\mathcal{F}^{-1}(I_2)$ and $\mathcal{F}^{-1}(J)$, there is $c>0$ such that

$$|\mathcal{F}^{-1}(I_2)(x,t)|,\ |\mathcal{F}^{-1}(J)(x,t)|\leqslant Ct^{-1}\sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-\frac{|x-c_h t|^2}{c|d_{h\ell}|t}} \qquad \forall t\geqslant 1.$$

Hence, by the Young inequality in Lemma A.1, it is sufficient to estimate the L^1-norm of $\mathcal{F}^{-1}((\hat{G}-\hat{K}-\hat{W})\chi_1)$. We thus obtain for $t\geqslant 1$ that

$$\|\mathcal{F}^{-1}((\hat{G}-\hat{K}-\hat{W})\chi_1)\|_{L^1}\leqslant \|\mathcal{F}^{-1}(I_1)\|_{L^1}+\|\mathcal{F}^{-1}(I_2)\|_{L^1}+\|\mathcal{F}^{-1}(J)\|_{L^1}$$
$$\leqslant Ct^{-\frac{1}{2}}\sum_{h=1}^{s}\sum_{\ell=1}^{s_h}\int_{|x|\leqslant Ct} t^{-\frac{1}{2}}e^{-\frac{|x-c_h t|^2}{c|d_{h\ell}|t}}\, dx\leqslant Ct^{-\frac{1}{2}}.$$

The proof is done. □

Proposition 3.8 (Intermediate frequency). *For* $g \in L^p(\mathbb{R})$, *if the conditions* $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ *and* $\mathscr{D}$ *hold, then there are constants* $c > 0$ *and* $C > 0$ *such that*

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_2) * g\|_{L^p} \leqslant Ce^{-ct}\|g\|_{L^p} \qquad \forall t \geqslant 0. \tag{3.62}$$

Proof. Due to the proof of Proposition 3.5, there are constants $c > 0$ and $C > 0$ such that we have

$$\begin{aligned}&|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_2)(x,t)| \\ &\qquad \leqslant \int_{\mathbb{R}} (|\hat{G}(\xi,t)| + |\hat{K}(\xi,t)| + |\hat{W}(\xi,t)|)|\chi_2(\xi)|\, d\xi \leqslant Ce^{-ct}.\end{aligned}$$

It follows that

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_2)\|_{L^1} \leqslant Ce^{-ct}\int_{|x| \leqslant Ct} dx \leqslant Cte^{-ct} \leqslant Ce^{-ct} \qquad \forall t \geqslant 0.$$

We finish the proof. □

Proposition 3.9 (High frequency). *For* $g \in L^p(\mathbb{R})$, *if the conditions* $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ *and* $\mathscr{D}$ *hold, then there are constants* $c > 0$ *and* $C > 0$ *such that*

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_3) * g\|_{L^p} \leqslant Ce^{-ct}\|g\|_{L^p} \qquad \forall t \geqslant 1. \tag{3.63}$$

Proof. Recall the decomposition $(\hat{G} - \hat{W})\chi_3 = I + J$ where I is defined by I_1 in (3.40) and J is the remainder. Hence, by (3.46), there is $c > 0$ such that for $|x| \leqslant Ct$ and $t \geqslant 0$, we have

$$|\mathcal{F}^{-1}((\hat{G} - \hat{W})\chi_3)(x,t)| \leqslant C\sum_{h=1}^{r} te^{-ct}|x - \alpha_h t| + Ce^{-ct} \leqslant Ce^{-ct}.$$

Moreover, we also have

$$|\mathcal{F}^{-1}(\hat{K}\chi_3)(x,t)| \leqslant C\int_{\mathbb{R}} e^{-c|\xi|^2 t}|\chi_3(\xi)|\, d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 1.$$

It follows that

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_3)\|_{L^1} \leqslant Ce^{-ct}\int_{|x| \leqslant Ct} dx \leqslant Cte^{-ct} \leqslant Ce^{-ct} \qquad \forall t \geqslant 1.$$

The proof is done. □

Case $|x| > Ct$

Proposition 3.10. *For* $g \in L^p(\mathbb{R})$, *if the conditions* $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ *and* $\mathscr{D}$ *hold, then there are constants* $c > 0$ *and* $C > 0$ *such that*

$$\|\mathcal{F}^{-1}(\hat{G} - \hat{K} - \hat{W}) * g\|_{L^p} \leqslant Ce^{-ct}\|g\|_{L^p} \qquad \forall t \geqslant 1. \tag{3.64}$$

Proof. Since $|x| > Ct$, we can assume that C is large enough. We estimate

$$\mathcal{F}^{-1}(\hat{G} - \hat{W})(x,t) = \lim_{\rho \to +\infty}\int_{-\rho}^{\rho} (\hat{G}(\xi,t) - \hat{W}(\xi,t))e^{ix\xi}\, d\xi. \tag{3.65}$$

Since $\hat{G}(\xi,t) = e^{-E(i\xi)t}$ where $E(i\xi) = B + i\xi A$ for $\xi \in \mathbb{R}$ and from (3.26), $\hat{G}-\hat{W}$ is holomorphic on the complex plane. Thus, by considering $\xi = \zeta + i\eta \in \mathbb{C}$, one can change the path of the integral from $\gamma := \{(\zeta,0) : \zeta \text{ from } -\rho \text{ to } \rho\}$ to the path $\gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3$ in the complex plane where

$$\gamma_1 := \{(\zeta,\eta) : \zeta = -\rho, \eta \text{ from } 0 \text{ to } x/t\},$$

$$\gamma_2 := \{(\zeta,\eta) : \zeta \text{ from } -\rho \text{ to } \rho, \eta = x/t\}$$

and

$$\gamma_3 := \{(\zeta,\eta) : \zeta = \rho, \eta \text{ from } x/t \text{ to } 0\}.$$

Furthermore, since ρ and $|x|/t$ are large, $\hat{G}-\hat{W}$ has the representation of the high-frequency part $(\hat{G}-\hat{W})\chi_3$ along γ. Therefore, by the same computation as in (3.50) - (3.52) and letting $\rho \to +\infty$, for some $c > 0$ and $C > 0$, we have

$$|\mathcal{F}^{-1}(\hat{G}-\hat{W})(x,t)| \leqslant Ce^{-ct}e^{-\frac{|x|^2}{ct}} \qquad \forall t \geqslant 1. \tag{3.66}$$

We estimate $\mathcal{F}^{-1}(\hat{K})$. From (3.25), it can be checked easily that

$$\mathcal{F}^{-1}(\hat{K})(x,t) = \sum_{h=1}^{s}\sum_{\ell=1}^{s_h} \frac{1}{2\sqrt{\pi d_{h\ell}t}} e^{-\frac{|x-c_h t|^2}{4d_{h\ell}t}} \sum_{k=0}^{m_{h\ell}-1} M_{kh\ell}\frac{|x-c_h t|^{2k}}{(d_{h\ell}t)^k},$$

where $m_{h\ell} \geqslant 1$ is the algebraic multiplicity associated with $d_{h\ell}$ and $M_{kh\ell}$ is some suitable matrix. Hence, since $|x| > Ct$ for sufficiently large C, we obtain

$$\begin{aligned}
|\mathcal{F}^{-1}(\hat{K})(x,t)| &\leqslant Ct^{-\frac{1}{2}} \sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-\frac{|x-c_h t|^2}{c|d_{h\ell}|t}} \\
&\leqslant Ct^{-\frac{1}{2}} \sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-\frac{|x-c_h t|^2}{2c|d_{h\ell}|t}} e^{-\left(\frac{|x-c_h t|^2}{2c|d_{h\ell}|t} - \frac{|Ct-c_h t|^2}{2c|d_{h\ell}|t}\right)} e^{-\frac{|C-c_h|^2 t}{2c|d_{h\ell}|}} \\
&\leqslant Ce^{-ct} \sum_{h=1}^{s}\sum_{\ell=1}^{s_h} e^{-\frac{|x-c_h t|^2}{c|d_{h\ell}|t}}
\end{aligned} \tag{3.67}$$

for some constants $c > 0$, $C > 0$ and for all $t \geqslant 1$.

Therefore, it follows from (3.66) and (3.67) that

$$\|\mathcal{F}^{-1}(\hat{G}-\hat{K}-\hat{W})\|_{L^1} \leqslant Ce^{-ct} \qquad \forall t \geqslant 1.$$

The proof is done. □

3.1.4 Proof of Theorem 3.1

We are now going to give a detailed proof for Theorem 3.1.

Proof of Theorem 3.1. For $u_0 \in L^q(\mathbb{R})$, let u, U and V be solutions to (3.4), (3.7) and (3.8) respectively. Recall the cut-off function χ_j for $j \in \{1,2,3\}$, it can be checked easily that

$$u - U - V = \sum_{j=1}^{3} \mathcal{F}^{-1}((\hat{G}-\hat{K}-\hat{W})\chi_j) * u_0, \tag{3.68}$$

where $\hat{G}$ is a solution to (2.32), $\hat{K}$ is given by (3.25) and $\hat{W}$ is given by (3.26).

Hence, by Proposition 3.4, Proposition 3.5 and Proposition 3.6, there is a constant $C > 0$ such that for all $t \geqslant 1$, from (3.68), one has

$$\|u - U - V\|_{L^\infty} \leqslant \sum_{j=1}^{3} \|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_j) * u_0\|_{L^\infty} \leqslant Ct^{-1}\|u_0\|_{L^1}. \quad (3.69)$$

On the other hand, by Proposition 3.7, Proposition 3.8, Proposition 3.9 and Proposition 3.10, there is a constant $C > 0$ such that for $1 \leqslant p \leqslant \infty$ and $t \geqslant 1$, from (3.68), we also have

$$\|u - U - V\|_{L^p} \leqslant \sum_{j=1}^{3} \|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_j) * u_0\|_{L^p} \leqslant Ct^{-\frac{1}{2}}\|u_0\|_{L^p}. \quad (3.70)$$

Thus, by Lemma A.2 in Appendix A, it follows from (3.69) and (3.70) that

$$\|u - U - V\|_{L^p} \leqslant Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|u_0\|_{L^q}$$

for $1 \leqslant q \leqslant p \leqslant \infty$ and for all $t \geqslant 1$.

The L^p-L^q decay estimate for U is accomplished similarly by applying the complex interpolation lemma A.2 in Appendix A once the L^∞-L^1 estimate and the L^p-L^p estimate for $1 \leqslant p \leqslant \infty$ were constructed. Moreover, the L^q-L^q estimate for V follows directly from Remark 3.3, Lemma 3.3 and the formula (3.26). The proof is thus done. □

3.2 Multi-dimensional space

Consider the Cauchy problem

$$\begin{cases} \partial_t u + \mathbf{A} \cdot \nabla_x u + Bu = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (3.71)$$

where $(x, t) \in \mathbb{R}^d \times [0, +\infty)$, $\cdot$ denotes the scalar product on $\mathbb{R}^d$, $u = u(x, t)$ and $u_0 = u_0(x)$ are vectors in $\mathbb{R}^n$, $\mathbf{A} := (A_1, \ldots, A_d) \in (\mathbb{R}^{n\times n})^d$ and $B \in \mathbb{R}^{n\times n}$.

Let u be a solution to (3.71). We study the L^p-L^q decay estimates for u. Nevertheless, as mentioned in [5], one cannot expect that the estimate

$$\|u\|_{L^1} \leqslant C\|u_0\|_{L^1} \quad (3.72)$$

holds in general. Indeed, for large time, L_+u behaves like a solution to the reduced system

$$\partial_t v + L_+\mathbf{A}R_+ \cdot \nabla_x v = 0. \quad (3.73)$$

In (3.73), $v = v(x, t) \in \mathbb{R}^m$, $L_+ \in \mathbb{R}^{m\times n}$ and $R_+ \in \mathbb{R}^{n\times m}$ are respectively a left and a right eigenprojection associated with the eigenvalue 0 of B and the integer $m \in [1, n)$ is the algebraic multiplicity of 0. Thus, it follows from [9] that (3.72) is not true in general for $d \geqslant 2$. Hence, we satisfy ourselves with the case where 0 is a simple eigenvalue of B in this section. In this case, (3.73) is a scalar equation since $m = 1$.

Noting that in the one-dimensional space, the singular term V in (3.9) is obtained based on the subtraction of $\hat{W}$ given by (3.26) from the high-frequency expansion of $\hat{G}$ given by (3.24) such that the remainder $\hat{G} - \hat{W}$ is L^1-integrable for large $\xi \in \mathbb{R}$. Such construction in the multi-dimensional space with $d \geqslant 2$ cannot be always achieved due to the fact that we cannot perform a uniform expansion of $\hat{G}$ for large $\xi \in \mathbb{R}^d$ (see Remark 2.7 in Chapter 2) which indeed can be established in the cases of the linear damped wave equations in any spatial dimension (see [31, 53, 56]).

Therefore, the aim of this section is only to study the decomposition $u = u^{(1)} + u^{(2)}$ of the solution u to (3.71) with $d \geqslant 2$, where $u^{(1)}$ satisfies the L^p-L^q decay of a parabolic kernel and $u^{(2)}$ decays in $L^2(\mathbb{R}^d)$ exponentially as $t \to +\infty$. Moreover, we also construct a large-time asymptotic profile of $u^{(1)}$.

For $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, we recall the operators

$$E(z) = B + A(z) \qquad \text{and} \qquad A(z) = \mathbf{A} \cdot z := \sum_{j=1}^{d} z_j A_j, \tag{3.74}$$

where $\mathbf{A} = (A_1, \dots, A_d) \in (\mathbb{R}^{n \times n})^d$ and $B \in \mathbb{R}^{n \times n}$. We start with the following reasonable assumptions.

Condition $\mathscr{A}^*$ (Hyperbolicity)**.** *For* $w \in \mathbb{S}^{d-1}$, $A = A(w)$ *is uniformly diagonalizable with real linear eigenvalues i.e. there are an invertible matrix* $R = R(w)$ *and a constant* $C > 0$ *such that*

$$\sup_{w \in \mathbb{S}^{d-1}} |R(w)||R^{-1}(w)| < C$$

for a matrix norm $|\cdot|$ *and* $R^{-1}AR$ *is a diagonal matrix whose nonzero entries are real linear in* $w \in \mathbb{S}^{d-1}$.

Condition $\mathscr{R}^*$ (Diagonalizing matrix)**.** *There is a matrix* R *uniformly diagonalizing* A *such that* $R^{-1}BR$ *is a constant matrix independent from* $w \in \mathbb{S}^{d-1}$.

Condition $\mathscr{B}^*$ (Partial dissipation)**.** *The spectrum of* B *is decomposed into* $\sigma(B) = \{0\} \cup \sigma_+$ *where* 0 *is simple and* $\sigma_+ \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\}$.

Condition $\mathscr{D}^*$ (Uniform dissipation)**.** *There is a constant* $\theta > 0$ *such that for any eigenvalue* $\lambda(i\xi)$ *of* $E(i\xi)$ *with* $\xi \in \mathbb{R}^d$, *one has*

$$\mathrm{Re}\,\lambda(i\xi) \geqslant \frac{\theta|\xi|^2}{1 + |\xi|^2} \qquad \forall \xi \neq 0.$$

Let Γ_0 be an oriented closed curve in the resolvent set of B such that it encloses zero except for the other eigenvalues of B. One recalls

$$P_0^{(0)} = -\frac{1}{2\pi i} \int_{\Gamma_0} (B - zI)^{-1}\, dz \text{ and } Q_0^{(0)} = \frac{1}{2\pi i} \int_{\Gamma_0} z^{-1}(B - zI)^{-1}\, dz, \tag{3.75}$$

which are the eigenprojection and the reduced resolvent coefficient associated with the eigenvalue zero of B. We consider the Cauchy problem

$$\begin{cases} \partial_t U + \mathbf{c} \cdot \nabla_x U - \mathrm{div}\,(\mathbf{D} \nabla_x U) = 0, \\ U|_{t=0} = P_0^{(0)} u_0, \end{cases} \tag{3.76}$$

where $U = U(x, t) \in \mathbb{R}^n$, $\mathbf{c} = (c_h)_{h \in \{1, \dots, d\}} \in \mathbb{R}^d$ and $\mathbf{D} = (D_{h\ell})_{h,\ell \in \{1, \dots, d\}} \in \mathbb{R}^{d \times d}$ with entries

$$c_h = \mathrm{tr}\left(A_h P_0^{(0)}\right) \text{ and } D_{h\ell} = \frac{1}{2} \mathrm{tr}\left(A_h P_0^{(0)} A_\ell Q_0^{(0)} + A_h Q_0^{(0)} A_\ell P_0^{(0)}\right). \tag{3.77}$$

Theorem 3.11 (L^p-L^q decay estimates [54])**.** *Let* u *be a solution to the Cauchy problem* (3.71) *with an initial datum* $u_0 \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ *for* $1 \leqslant q \leqslant \infty$*. Under the assumptions* $\mathscr{A}^*$*,* $\mathscr{R}^*$*,* $\mathscr{B}^*$ *and* $\mathscr{D}^*$*, the solution* u *is decomposed into*

$$u(x,t) = u^{(1)}(x,t) + u^{(2)}(x,t),$$

where

$$u^{(1)}(x,t) := \mathcal{F}^{-1}(e^{-Et}P_0\chi) * u_0(x)$$

and $u^{(2)}$ *is the remainder,* $P_0(i\xi)$ *is the eigenprojection associated with the eigenvalue* $\lambda_0(i\xi)$ *of* $E(i\xi)$*,* $\lambda_0(i\xi)$ *converges to* 0 *as* $|\xi| \to 0$ *and* χ *is a cut-off function valued in* $[0,1]$ *with support contained in the ball* $B(0,\varepsilon) \subset \mathbb{R}^d$ *for small* $\varepsilon > 0$*.*

Moreover, for any $1 \leqslant q \leqslant p \leqslant \infty$*, there are constants* $c > 0$ *and* $C > 0$ *such that one has*

$$\|u^{(1)} - U\|_{L^p} \leqslant Ct^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|u_0\|_{L^q} \qquad \forall t \geqslant 1 \tag{3.78}$$

and $u^{(2)}$ *satisfies*

$$\|u^{(2)}\|_{L^2} \leqslant Ce^{-ct}\|u_0\|_{L^2} \qquad \forall t \geqslant 0, \tag{3.79}$$

where U *which is a solution to* (3.76) *satisfies*

$$\|U\|_{L^p} \leqslant Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_{L^q} \qquad \forall t \geqslant 1. \tag{3.80}$$

Similarly to the one-dimensional space with $d = 1$, the proof of Theorem 3.11 is based on the L^∞-L^1 estimate, the L^p-L^p estimate for $1 \leqslant p \leqslant \infty$ and a complex interpolation argument given by Lemma A.2. Hence, we also divide each step of the proof into the low frequency, the intermediate frequency and the high frequency in order to use asymptotic expansions of the Fourier transform $\hat{G}$ of the fundamental solution denoted by G to (3.71). Noting that $\hat{G}$ satisfies (2.32) and is given by (2.33).

Remark 3.4 (Finite speed of propagation)**.** In the case where the solution u to (3.71) has finite speed of propagation, since the fundamental solution G has compact support contained in the wave cone $\{x \in \mathbb{R}^d : |x/t| \leqslant C\}$ for a constant $C > 0$, we can decompose u into $u = u^{(1)} + u^{(2)}$ where

$$u^{(1)}(x,t) := \mathcal{F}^{-1}(e^{-Et}\chi) * u_0(x)$$

and $u^{(2)}$ is the remainder. Here, χ is a cut-off function valued in $[0,1]$ with support contained in the ball $B(0,\rho) \subset \mathbb{R}^d$ for any $\rho > 0$. Then, the estimates (3.78) and (3.79) still hold. This fact will be proved in the subsequent subsections. For instance, it is the case where (3.71) is Friedrichs symmetrizable. However, in the one-dimensional space with $d = 1$, the case where $|x/t| > C$ can be treated since the Cauchy integral theorem holds for the whole complex plane. Thus, one can use the estimates for the high-frequency expansion of $\hat{G}$ after changing paths of integrals of holomorphic functions.

Remark 3.5 (The simplicity of the eigenvalue 0 of B)**.** Based on the discussions from the beginning of this section, if the simplicity of the eigenvalue 0 of B is relaxed by the semi-simplicity with the algebraic multiplicity $m \geqslant 2$, the

L^p-L^q decay estimate given by (3.78) does not hold for general $p, q \in [1, \infty]$. This case is very restrictive indeed. Following from [5], we have the following

$$\|u - U\|_{L^p \cap L^2} \leqslant Ct^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}} \|u_0\|_{L^1 \cap L^2} \qquad \forall p \geqslant \min\{d, 2\}, \tag{3.81}$$

where the initial datum $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and U is a solution to a parabolic system induced from the Chapman–Enskog expansion.

Remark 3.6 (Relaxing the conditions $\mathscr{A}^*$ and $\mathscr{R}^*$)**.** The requirements of the linearity of the eigenvalues of the matrix A satisfying the condition $\mathscr{A}^*$ and the existence of the matrix R satisfying the condition $\mathscr{R}^*$ can be omitted by considering the dissipative structures proposed in Proposition 2.3 and in [5,67]. Indeed, from (2.8), the Fourier transform $\hat{u}$ of the solution u to (3.71) satisfies

$$|\hat{u}(\xi, t)| \leqslant Ce^{-\frac{c|\xi|^2}{1+|\xi|^2}t} |\hat{u}_0(\xi)| \tag{3.82}$$

for all $\xi \neq 0$ and $t \geqslant 0$. It implies immediately that one can choose $u^{(2)} := \mathcal{F}^{-1}(\hat{u}(1 - \chi))$ decaying exponentially in $L^2(\mathbb{R}^d)$ as $t \to +\infty$, where χ is a cut-off function with support compact contained in $\{\xi \in \mathbb{R}^d : |\xi| < \varepsilon\}$ for sufficiently small ε. In general, we cannot obtain (3.82) without any additional assumptions. Here, the advantage of the linearity of the eigenvalues of A and the existence of R satisfying the condition $\mathscr{R}^*$ is that we can construct the high-frequency expansion of $\hat{G}$, provided a suitable Lebesgue measure zero set is subtracted. Then, the exponential decay of $u^{(2)}$ in $L^2(\mathbb{R}^d)$ as $t \to +\infty$ can be proved under the condition $\mathscr{D}^*$.

Moreover, in order to have the L^p-L^q decay estimate for the remainder $u^{(1)} := \mathcal{F}^{-1}(\hat{u}\chi)$, we also need that u has finite speed of propagation (see Remark 3.4). This condition still holds for the structures proposed in Proposition 2.3 and in [5, 67] since the matrix A is Friedrichs symmetrizable.

Example 3.7 (The generalized Goldstein–Kac systems)**.** Consider (3.71) with

$$A_j := \operatorname{diag}(\nu_1^j, \ldots, \nu_n^j) \qquad \text{and} \qquad B := (-\mu_{h\ell})_{h,\ell \in \{1,\ldots,n\}},$$

where ν_h^j and $\mu_{h\ell} \geqslant 0$ are respectively real numbers indicating the speed of the density u_h in the direction x_j and the transition rate from the speed set $\{\nu_h^1, \ldots, \nu_h^d\}$ to the speed set $\{\nu_\ell^1, \ldots, \nu_\ell^d\}$ such that $\mu_{hh} = -\sum_{\ell \neq h} \mu_{h\ell}$ for $h, \ell \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, d\}$. Here, B is not necessarily symmetric as in [49].

We will prove that the above system satisfies the conditions $\mathscr{A}^*$, $\mathscr{R}^*$, $\mathscr{B}^*$ and $\mathscr{D}^*$ if there are $h, \ell \in \{1, \ldots, n\}$ satisfying $\nu_h^j \neq \nu_\ell^j$ and B is an irreducible matrix. Noting that by the Gershgorin circle theorem, every eigenvalue of B has a nonnegative real part.

The conditions $\mathscr{A}^*$ and $\mathscr{R}^*$ are obvious. We prove the condition $\mathscr{B}^*$. Let $\nu > \max_{h \in \{1,\ldots,n\}}\{-\mu_{hh}\} > 0$, one deduces that $B + \nu I$ is a nonnegative irreducible matrix with eigenvalues having positive real parts. Hence, by the Perron–Frobenius theorem, $\rho(B + \nu I)$ which is the spectral radius of $B + \nu I$ is a simple eigenvalue of $B + \nu I$.

Moreover, due to the Gershgorin circle theorem and the definition of the spectral radius ρ, one has $\rho(B + \nu I) = \nu$ unless there is a fixed $h \in \{1, \ldots, n\}$

such that $\sum_{\ell\neq h}\mu_{h\ell} < |\nu+\mu_{hh}| < |\nu'+\mu_{hh}| \leqslant \sum_{\ell\neq h}\mu_{h\ell}$ which is a contradiction if $\nu' := \rho(B+\nu I) > \nu$.

On the other hand, since one has

$$\det(B-\lambda I) = \det(B+\nu I-\tau I),$$

where $\lambda=\tau-\nu$ and I is the identity matrix in $\mathbb{R}^{n\times n}$, it follows from the fact that $\tau=\nu$ is a simple eigenvalue of $B+\nu I$ that $\lambda=0$ is a simple eigenvalue of B. Thus, the condition $\mathscr{B}^*$ is satisfied.

We check the condition $\mathscr{D}^*$. We observe that $B+B^t$ inherits all of the properties of B, namely

$$(B+B^t)_{h\ell} \geqslant 0 \quad \forall h\neq\ell, \quad (B+B^t)_{hh} = -\sum_{\ell\neq h}(B+B^t)_{h\ell}$$

and $B+B^t$ is an irreducible matrix. Hence, by the same arguments as before where ν is substituted by $\nu > \max_{h\in\{1,\dots,n\}}\{-(B+B^t)_{hh}\}$ and B is substituted by $B+B^t$, we deduce that $B+B^t$ is a positive semi-definite matrix. Moreover, $\ker(B+B^t) = \operatorname{span}\{(1,\dots,1)\}$.

Thus, by Proposition 2.1, there is a real skew-symmetric matrix $K=K(w)$ such that

$$u^t(K(w)A(w)-A(w)K(w)+B+B^t)u > 0 \qquad \forall u\in\mathbb{R}^n\backslash\{0\},$$

where $A(w)=\sum_{j=1}^d w_jA_j$ and $w=(w_1,\dots,w_d)\in\mathbb{S}^{d-1}$. Indeed, any eigenvector of $A(w)$ does not belong to $\ker(B+B^t)$ for $w\in\mathbb{S}^{d-1}$ since there are $h,\ \ell\in\{1,\dots,n\}$ such that $v_h^j\neq v_\ell^j$.

Furthermore, $2\mathrm{Re}\,\bar{u}^tBu = \mathrm{Re}\,\bar{u}^t(B+B^t)u \geqslant 0$ for any $u\in\mathbb{C}^n$ i.e. B is also a positive semi-definite matrix (not necessarily symmetric). Therefore, the condition $\mathscr{D}^*$ follows Proposition 2.3 with $A_0=I$.

Example 3.8 (The two-dimensional linearized isentropic compressible Euler equations with damping)**.** For $\alpha>0$ and $x\in\mathbb{R}^2$, we consider (3.71) with

$$A_1 := \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \quad A_2 := \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix} \quad\text{and}\quad B := \begin{pmatrix}0&0&0\\0&\alpha&0\\0&0&\alpha\end{pmatrix}.$$

The conditions $\mathscr{A}^*$, $\mathscr{B}^*$ and $\mathscr{R}^*$ hold obviously due to the fact that a diagonalizing matrix $R(w)$ of $A(w)$ is given by

$$R(w) = \frac{1}{2}\begin{pmatrix}1&0&1\\-w_1&2w_2&w_1\\-w_2&-2w_1&w_2\end{pmatrix}.$$

The condition $\mathscr{D}^*$ also holds. Indeed, it can be checked easily that there is no eigenvector of $A(w)=w_1A_1+w_2A_2$ in $\ker B$ for $w=(w_1,w_2)\in\mathbb{S}^1$. Thus, by Proposition 2.1, the conditions in Proposition 2.3 hold with $A_0=I$, where I is the identity matrix in $\mathbb{R}^{3\times3}$. Hence, the condition $\mathscr{D}^*$ follows.

Noting that in this case, we can compute directly the eigenvalues of $E(i\xi) = B+A(i\xi)$ where $A(i\xi)=i\xi_1A_1+i\xi_2A_2$ for $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$. In fact, $E(i\xi)$ has three eigenvalues

$$\lambda_0(i\xi)=\alpha \quad\text{and}\quad \lambda_\pm(i\xi) = \begin{cases}\dfrac{\alpha\pm\sqrt{\alpha^2-4|\xi|^2}}{2} & \text{if } \alpha-4|\xi|^2\geqslant 0,\\[2ex] \dfrac{\alpha\pm i\sqrt{4|\xi|^2-\alpha^2}}{2} & \text{if } \alpha-4|\xi|^2<0,\end{cases}$$

where $\lambda_\pm = \alpha/2$ for $|\xi| = \alpha/2$.

Hence, by applying the Taylor expansion of $\sqrt{1-z}$ for $|z| < 1$, we have

$$\begin{cases} \lambda_+(i\xi) = \alpha - \dfrac{|\xi|^2}{\alpha} + \mathcal{O}(|\xi|^4) \\ \lambda_-(i\xi) = \dfrac{|\xi|^2}{\alpha} + \mathcal{O}(|\xi|^4) \end{cases} \qquad \forall |\xi| < \alpha/2.$$

On the other hand, we also have

$$\begin{cases} \lambda_+(i\xi) = i|\xi| + \dfrac{\alpha}{2} + \mathcal{O}(|\xi|^{-1}) \\ \lambda_-(i\xi) = -i|\xi| + \dfrac{\alpha}{2} + \mathcal{O}(|\xi|^{-1}) \end{cases} \qquad \forall |\xi| > \alpha/2.$$

For the cases where the space-dimension $d \geqslant 3$, computations are similar (see [65] as an example with $d = 3$).

3.2.1 Preliminaries

Let $\mathcal{I}$ be an index set given by $\mathcal{I} := \{i_1, \dots, i_s\}$ with possibly repeated $i_\ell \in \{1, \dots, d\}$ i.e. we allow $i_h = i_\ell$ for $h \neq \ell$. For a partition $\{\mathcal{I}_j : j = 1, \dots, r\}$ of $\mathcal{I}$ with $r \in \{1, \dots, s\}$ and $\mathcal{I}_j := \{i^j_1, \dots, i^j_{s_j}\}$, we define the partial derivative $\partial_{\mathcal{I}_j}$ of a smooth scalar function $q = q(x,t)$ on $\mathbb{R}^d \times [0, +\infty)$ with respect to $x \in \mathbb{R}^d$ by the partial derivative of order s_j

$$\partial_{\mathcal{I}_j} q(x,t) := \partial^{s_j}_{x_{i^j_1} \dots x_{i^j_{s_j}}} q(x,t).$$

Noting that we do not consider any $\mathcal{I}_j = \emptyset$ i.e. $s_j \geqslant 1$ for all $j \in \{1, \dots, r\}$.

On the other hand, for a fixed $\alpha \in \mathbb{N}^d$, if $|\alpha| = 0$ i.e. $\alpha = 0$, we set $\mathcal{I}_\alpha := \emptyset$ with the cardinality $|\mathcal{I}_\alpha| := 0$. If $|\alpha| = s \in \mathbb{Z}_+$, α determines an index set $\mathcal{I}_\alpha = \{i_1, \dots, i_s\} \neq \emptyset$ with possibly repeated indices. In fact, if $\alpha = (\alpha_1, \dots, \alpha_d)$, we define uniquely the index set $\mathcal{I}_\alpha$ having α_ℓ indices $\ell \in \{1, \dots, d\}$. We also set the cardinalities $|\mathcal{I}_\alpha| := s \geqslant 1$ and $|\mathcal{I}_j| := s_j \geqslant 1$ for $j \in \{1, \dots, r\}$ if $\mathcal{I}_\alpha \neq \emptyset$.

Lemma 3.12 (Partial derivatives)**.** *Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \geqslant 0$, for any smooth scalar function $q = q(x,t)$ on $\mathbb{R}^d \times [0, +\infty)$, we have*

$$\partial^\alpha e^{q(x,t)} = \sum_{\{\mathcal{I}_j : j=1,\dots,r\}, r \leqslant |\alpha|} \partial_{\mathcal{I}_1} q(x,t) \dots \partial_{\mathcal{I}_r} q(x,t) e^{q(x,t)}, \tag{3.83}$$

where $\{\mathcal{I}_j : j = 1, \dots, r\}$ is any possible partition of the set $\mathcal{I}_\alpha$ determined by α.

Proof. We prove by induction. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| = 0$. Since $\mathcal{I}_\alpha = \emptyset$, there is no partition of $\mathcal{I}_\alpha$ to be considered. Thus, $\partial^0 e^{q(x,t)} = e^{q(x,t)}$.

Let $\alpha \in \mathbb{N}^d$ with $|\alpha| = 1$. By the definition of ∂^α, we have

$$\partial^\alpha e^{q(x,t)} = \partial^1_{x_i} e^{q(x,t)} = \partial^1_{x_i} q(x,t) e^{q(x,t)} \tag{3.84}$$

if $\alpha_i = 1$ and $\alpha_\ell = 0$ for all $\ell \neq i$. On the other hand, the index set $\mathcal{I}_\alpha$ determined by α in this case is $\mathcal{I}_\alpha = \{i\}$ since $\alpha_i = 1$. Thus, $\mathcal{I}_\alpha$ has only one possible partition which is itself. Hence, (3.84) coincides (3.83).

Given an integer $s \geqslant 1$, assume that (3.83) holds for any $\alpha \in \mathbb{N}^d$ with $|\alpha| = s$. For any $\beta \in \mathbb{N}^d$ with $|\beta| = s + 1$, $\beta = (\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_d)$ for some $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and $i \in \{1, \dots, d\}$. Hence, since $\partial^\beta e^{q(x,t)} = \partial^1_{x_i}\partial^\alpha e^{q(x,t)}$, we have

$$\begin{aligned}\partial^\beta e^{q(x,t)} = &\sum_{\{\mathcal{I}_j : j=1,\dots,r\}, r\leqslant s} \sum_{\ell=1}^{r} \partial_{\mathcal{I}_1} q(x,t) \dots \partial^1_{x_i}\partial_{\mathcal{I}_\ell} q(x,t) \dots \partial_{\mathcal{I}_r} q(x,t) e^{q(x,t)} \\ &+ \sum_{\{\mathcal{I}_j : j=1,\dots,r\}, r\leqslant s} \partial_{\mathcal{I}_1} q(x,t) \dots \partial_{\mathcal{I}_r} q(x,t) \partial^1_{x_i} q(x,t) e^{q(x,t)}, \quad (3.85)\end{aligned}$$

where $\{\mathcal{I}_j : j = 1, \dots, r\}$ is any possible partition of $\mathcal{I}_\alpha$ determined by α.

We then consider all of possible partitions of $\mathcal{I}_\beta$. The first possibilities are the partitions $\{\{\mathcal{I}_j : j = 1, \dots, r\}, \{i\}\}$ since $\mathcal{I}_\beta$ has $\alpha_i + 1$ indices i. The last choices are that for each partition $\{\mathcal{I}_j : j = 1, \dots, r\}$ of $\mathcal{I}_\alpha$, we generate the partition $\{\mathcal{I}'_j : j = 1, \dots, r\}$ of $\mathcal{I}_\beta$ by putting i into $\mathcal{I}_\ell$ and let $\mathcal{I}'_j = \mathcal{I}_j$ for all $j \neq \ell$ and $\ell \in \{1, \dots, r\}$. Thus, since r varies such that $r \leqslant s$, there is no other possible partition of $\mathcal{I}_\beta$ to take part in. Therefore, we obtain from (3.85) that

$$\partial^\beta e^{q(x,t)} = \sum_{\{\mathcal{I}'_j : j=1,\dots,r'\}, r'\leqslant s+1} \partial_{\mathcal{I}'_1} q(x,t) \dots \partial_{\mathcal{I}'_r} q(x,t) e^{q(x,t)},$$

where the sum is made on all possible partitions $\{\mathcal{I}'_j : j = 1, \dots, r'\}$ of $\mathcal{I}_\beta$ determined by β. We thus proved (3.83). □

Remark 3.9. Lemma 3.12 is applied only to the case where $q = q(x,t)$ is scalar on $(x,t) \in \mathbb{R}^d \times [0, +\infty)$, the matrix case is a challenge due to the lack of commutativity of q and its partial derivatives.

Lemma 3.13 (Derivative estimates)**.** *Let $p = p(x)$ be a scalar polynomial on $\mathbb{R}^d$ such that the lowest order of p is $h \geqslant 1$ and let $\alpha \in \mathbb{N}^d$ with $|\alpha| \geqslant 0$. There is a constant $C > 0$ such that for small $x \in \mathbb{R}^d$ and $t \geqslant 0$, we have*

$$|\partial^\alpha e^{p(x)t}| \leqslant C \sum_{\{\mathcal{I}_j : j=1,\dots,r\},\, r\leqslant|\alpha|} |x|^{\sum_{k=1}^{h-1} k m_k} t^{\ell + \sum_{k=0}^{h-1} m_k} |e^{p(x)t}|, \qquad (3.86)$$

where the integer $m_k \geqslant 0$ is the cardinality of $\{j \in \{1, \dots, r\} : |I_j| = h - k\}$ for each $k \in \{0, \dots, h-1\}$, the integer $\ell \geqslant 0$ satisfies

$$h\ell < |\alpha| - \sum_{k=0}^{h-1} (h-k) m_k \qquad (3.87)$$

and $\{\mathcal{I}_j : j = 1, \dots, r\}$ is any possible partition of the set $\mathcal{I}_\alpha$ determined by α.

Proof. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \geqslant 0$ and $p = p(x)$ be a polynomial on $\mathbb{R}^d$ such that the lowest order of p is $h \geqslant 1$. For any partition $\{\mathcal{I}_j : j = 1, \dots, r\}$ of $\mathcal{I}_\alpha$ determined by α, by the definition of $\partial_{\mathcal{I}_j}$, there is a constant $C > 0$ such that

$$|\partial_{\mathcal{I}_j} p(x)| \leqslant C \cdot \begin{cases} 1 & \text{if } |\mathcal{I}_j| \geqslant h, \\ |x|^k & \text{if } |\mathcal{I}_j| = h - k \end{cases}$$

for $k \in \{0, \dots, h-1\}$ and small $x \in \mathbb{R}^d$, where $|\mathcal{I}_j|$ is the cardinality of the index set $\mathcal{I}_j$ with possibly repeated indices. Noting that $\sum_{j=1}^{r} |\mathcal{I}_j| = |\mathcal{I}_\alpha| = |\alpha|$

by definition. It implies that there is a constant $C > 0$ such that for small $x \in \mathbb{R}^d$ and $t \geqslant 0$, we have

$$|\partial_{\mathcal{I}_1}(p(x)t)| \dots |\partial_{\mathcal{I}_r}(p(x)t)| \leqslant C|x|^{\sum_{k=1}^{h-1} k m_k} t^{\ell + \sum_{k=0}^{h-1} m_k}, \tag{3.88}$$

where $m_k \geqslant 0$ is the cardinality of $\{j \in \{1, \dots, r\} : |\mathcal{I}_j| = h - k\}$ for $k \in \{0, \dots, h-1\}$ and $\ell \geqslant 0$ is the cardinality of $\mathcal{J} := \{j \in \{1, \dots, r\} : |\mathcal{I}_j| > h\}$. Moreover, we have

$$h\ell < \sum_{j \in \mathcal{J}} |\mathcal{I}_j| = |\alpha| - \sum_{k=0}^{h-1} (h-k) m_k. \tag{3.89}$$

We thus obtain (3.86) and (3.87) from (3.83), (3.88) and (3.89). The proof is done. □

Lemma 3.14. *If the conditions $\mathscr{B}^*$ and $\mathscr{D}^*$ hold, $\mathbf{D} \in \mathbb{R}^{d \times d}$ with the entries $D_{h\ell}$ given by (3.77) for $h, \ell \in \{1, \dots, d\}$ is positive definite. On the other hand, if the conditions $\mathscr{A}^*$, $\mathscr{R}^*$ and $\mathscr{D}^*$ hold, the coefficient $\beta_{h\ell}$ for $h \in \{1, \dots, r\}$ and $\ell \in \{1, \dots, r_h\}$ in the expansion (2.67) is bounded from below by $\theta > 0$.*

Proof. Recall the operator E in (3.74). For small ξ, since the condition $\mathscr{B}^*$ holds, the expansion (2.46) holds, namely $E(i\xi)$ has one single eigenvalue $\lambda_0(i\xi)$ that converges to 0 as $|\xi| \to 0$ and is approximated by

$$\lambda_0(i\xi) = i \sum_{h=1}^{d} c_h \xi + \sum_{h=1}^{d} \sum_{\ell=1}^{d} D_{h\ell} \xi_h \xi_\ell + \mathcal{O}(|\xi|^2),$$

where c_h and $D_{h\ell}$ are given by (3.77).

Hence, if the assumption $\mathscr{D}^*$ holds, then since $c_h \in \mathbb{R}$ for all $h \in \{1, \dots, d\}$, there is a constant $\theta > 0$ such that for small $\xi \neq 0$, one has

$$\frac{\theta|\xi|^2}{1 + |\xi|^2} \leqslant \mathrm{Re}\, \lambda_0(i\xi) \leqslant \mathrm{Re}\,(\xi \cdot \mathbf{D}\xi) + C|\xi|^3.$$

As $|\xi| \to 0$, one has $\mathrm{Re}\,(w \cdot \mathbf{D}w) \geqslant \theta > 0$ for all $w \in \mathbb{S}^{d-1}$. Therefore, for any $z \neq 0$, one has $\mathrm{Re}\,(z \cdot \mathbf{D}z) = |z|^2 \mathrm{Re}\,((z/|z|) \cdot \mathbf{D}(z/|z|)) > 0$.

Similarly, for large ξ, since the conditions $\mathscr{A}^*$ and $\mathscr{R}^*$ hold, the expansions (2.66) - (2.68) of E is validated. Thus, the eigenvalues of E that converge to $\alpha_h \in \sigma(A)$ as $|\xi| \to +\infty$ for $h \in \{1, \dots, r\}$, where r is the cardinality of $\sigma(A)$, are approximated by

$$\lambda_{h\ell}(i\xi) = \alpha_h(i\xi) + \beta_{h\ell} + \mathcal{O}(1)$$

for $h \in \{1, \dots, r\}$ and $\ell \in \{1, \dots, r_h\}$. Here, $\beta_{h\ell} \in \sigma(\Pi_h^{(0)} R^{-1} B R \Pi_h^{(0)}, \mathrm{ran}\, \Pi_h^{(0)})$ and the integer r_h is the cardinality of $\sigma(\Pi_h^{(0)} R^{-1} B R \Pi_h^{(0)}, \mathrm{ran}\, \Pi_h^{(0)})$, where $R = R(\xi/|\xi|)$ is an invertible matrix satisfying the condition $\mathscr{R}^*$ and $\Pi_h^{(0)}$ is given by (2.65).

Therefore, it follows from the conditions $\mathscr{A}^*$ and $\mathscr{D}^*$ that α_h is real for all $h \in \{1, \dots, r\}$ and there is a positive constant θ such that

$$\frac{\theta}{1 + |\xi|^{-2}} \leqslant \mathrm{Re}\,(\beta_{h\ell}) + \varepsilon$$

for small $0 < |\xi|^{-1} < \varepsilon$, $h \in \{1, \dots, r\}$ and $\ell \in \{1, \dots, r_h\}$. Let $\varepsilon \to 0$, one obtains $\mathrm{Re}\,(\beta_{h\ell}) \geqslant \theta > 0$ for all $h \in \{1, \dots, r\}$ and $\ell \in \{1, \dots, r_h\}$. The proof is thus done. □

Recall χ_i for $i = 1, 2, 3$ which are smooth cut-off functions valued in $[0, 1]$ with supports contained in $\{\xi \in \mathbb{R} : |\xi| < \varepsilon\}$, $\{\xi \in \mathbb{R} : \varepsilon \leqslant |\xi| \leqslant \rho\}$ and $\{\xi \in \mathbb{R} : |\xi| > \rho\}$ respectively for small ε and large ρ such that $\chi_1+\chi_2+\chi_3 = 1$.

The solution $\hat{G}$ to (2.32) then satisfies $\hat{G}\chi_1 = \hat{G}_1\chi_1 + \hat{G}_2\chi_2$ where

$$\hat{G}_1(\xi, t) = e^{-i\mathbf{c}\cdot\xi t - \xi\cdot\mathbf{D}\xi t + \mathcal{O}(|\xi|^3)t}(P_0^{(0)} + \mathcal{O}(|\xi|)) \tag{3.90}$$

and

$$\hat{G}_2(\xi, t) = \sum_{k=1}^{s'} e^{-b_k t} e^{-M_k^{(0)} t + \mathcal{O}(|\xi|)t}(F_k^{(0)} + \mathcal{O}(|\xi|)), \tag{3.91}$$

where $\mathbf{c} = (c_h)_{h\in\{1,\dots,d\}} \in \mathbb{R}^d$ and $\mathbf{D} = (D_{h\ell})_{h,\ell\in\{1,\dots,d\}} \in \mathbb{R}^{d\times d}$ with the entries given by (3.77), $P_0^{(0)}$ is the eigenprojection associated with $0 \in \sigma(B)$, $b_k \in \sigma(B)\backslash\{0\}$ with the associated eigenprojection $F_k^{(0)}$ and eigennilpotent $M_k^{(0)}$ and s' is the cardinality of $\sigma(B)\backslash\{0\}$.

Moreover, for almost everywhere, we also have

$$\begin{aligned} &\hat{G}(\xi, t)\chi_3(\xi) \\ &= R \sum_{h=1}^{r} \sum_{\ell=1}^{r_h} e^{-(\alpha_h(i\xi)+\beta_{h\ell})t} e^{-\left(\Theta_{h\ell}^{(0)}+\mathcal{O}(|\xi|^{-1})\right)t}\big(\Pi_{h\ell}^{(0)} + \mathcal{O}(|\xi|^{-1})\big)R^{-1}. \end{aligned} \tag{3.92}$$

In (3.92), $\alpha_h(i\xi) = i|\xi|\nu_{[h]}(\xi/|\xi|)$ for $\nu_{[h]}$ given by (2.64), the coefficient $\beta_{h\ell} \in \sigma(\Pi_h^{(0)} R^{-1}BR\Pi_h^{(0)}, \mathrm{ran}\,\Pi_h^{(0)})$ with the associated eigenprojection $\Pi_{h\ell}^{(0)}$ and eigennilpotent $\Theta_{h\ell}^{(0)}$, the integers r and r_h are respectively the cardinalities of $\sigma(A)$ and $\sigma(\Pi_h^{(0)} R^{-1}BR\Pi_h^{(0)}, \mathrm{ran}\,\Pi_h^{(0)})$ where $R = R(\xi/|\xi|)$ satisfies the condition $\mathscr{R}^*$ and $\Pi_h^{(0)}$ is given by (2.65).

One sets

$$\hat{K}(\xi, t) := e^{-\mathbf{c}\cdot i\xi t - \xi\cdot\mathbf{D}\xi t} P_0^{(0)}, \tag{3.93}$$

where the coefficients are introduced as before.

3.2.2 L^∞-L^1 estimate

Due to the discussions from the beginning of this section, since we cannot subtract any suitable error term $\hat{W}$ from the high-frequency part of $\hat{G}$ in order to obtain the L^∞-L^1 estimate as the space-dimension d increases, we consider here only the low-frequency part and the intermediate-frequency part of $\hat{G}$.

Proposition 3.15 (Low frequency)**.** *For $g \in L^1(\mathbb{R}^d)$, if the assumptions $\mathscr{B}^*$ and $\mathscr{D}^*$ hold, then there is a constant $C > 0$ such that for $t \geqslant 1$, we have*

$$\|\mathcal{F}^{-1}((\hat{G}P_0 - \hat{K})\chi_1) * g\|_{L^\infty} \leqslant Ct^{-\frac{d}{2}-\frac{1}{2}}\|g\|_{L^1}, \tag{3.94}$$

where $P_0(i\xi)$ is the eigenprojection associated with the eigenvalue $\lambda_0(i\xi)$ of $E(i\xi)$ and $\lambda_0(i\xi)$ converges to 0 as $|\xi| \to 0$.

On the other hand, we also have

$$\|\mathcal{F}^{-1}(\hat{G}(I - P_0)\chi_1) * g\|_{L^\infty} \leqslant Ce^{-ct}\|g\|_{L^1}. \tag{3.95}$$

Proof. By the Young inequality in Lemma A.1 and the fact that $\mathcal{F}^{-1} : L^1 \to L^\infty$, it is sufficient to estimate the L^1-norm of $(\hat{G}P_0 - \hat{K})\chi_1$ and $\hat{G}(I - P_0)\chi_1$ under the conditions $\mathscr{B}^*$ and $\mathscr{D}^*$. Moreover, by changing the coordinates $(x, t) \mapsto (x - \mathbf{c}t, t)$, one can always assume that $\mathbf{c} = 0$ without loss of generality.

Since $(P_0)^2 = P_0$ and $P_0(I - P_0) = (I - P_0)P_0 = O$ the null matrix, one has $(\hat{G}P_0 - \hat{K})\chi_1 = (\hat{G}_1 - \hat{K})\chi_1$ and $\hat{G}(I - P_0)\chi_1 = \hat{G}_2\chi_1$, where $\hat{G}_1$ is given by (3.90) and $\hat{G}_2$ is given by (3.91).

Moreover, $(\hat{G}_1 - \hat{K})\chi_1 = I + J$ where

$$I(\xi, t) := e^{-\xi\cdot\mathbf{D}\xi t}(e^{\mathcal{O}(|\xi|^3)t} - 1)P_0^{(0)}\chi_1(\xi) \tag{3.96}$$

and

$$J(\xi, t) := e^{-\xi\cdot\mathbf{D}\xi t + \mathcal{O}(|\xi|^3)t}\mathcal{O}(|\xi|)\chi_1(\xi). \tag{3.97}$$

Then, there are constants $c > 0$ and $C > 0$ such that

$$|(\hat{G}(\xi, t)P_0(\xi) - \hat{K}(\xi, t))\chi_1(\xi)| \leqslant |I(\xi, t)| + |J(\xi, t)| \leqslant Ce^{-c|\xi|^2 t}(|\xi|^3 t + |\xi|)|\chi_1(\xi)|.$$

Hence, it implies for all $t \geqslant 0$ that

$$\|(\hat{G}P_0 - \hat{K})\chi_1\|_{L^1} \leqslant C\int_{\mathbb{R}^d} e^{-c|\xi|^2 t}(|\xi|^3 t + |\xi|)|\chi_1(\xi)|\, d\xi \leqslant C(1 + t)^{-\frac{d}{2} - \frac{1}{2}}.$$

On the other hand, by the condition $\mathscr{B}^*$ and Remark 3.3, from (3.91), there are constants $c > 0$ and $C > 0$ such that we have

$$|\hat{G}(\xi, t)(I - P_0(\xi))\chi_1(\xi)| \leqslant C\sum_{k=1}^{s'} e^{-\mathrm{Re}\,(b_k)t}e^{|M_k^{(0)}|t + C|\xi|t}|\chi_1(\xi)| \leqslant Ce^{-ct}|\chi_1(\xi)|.$$

Hence, we have

$$\|\hat{G}(I - P_0)\chi_1\|_{L^1} \leqslant Ce^{-ct}\int_{\mathbb{R}^d} |\chi_1(\xi)|\, d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 0.$$

The proof is done. □

Proposition 3.16 (Intermediate frequency)**.** *For* $g \in L^1(\mathbb{R}^d)$*, if the assumptions* $\mathscr{B}^*$ *and* $\mathscr{D}^*$ *hold, then there are constants* $c > 0$ *and* $C > 0$ *such that for* $t \geqslant 0$*, we have*

$$\|\mathcal{F}^{-1}(\hat{G}\chi_2) * g\|_{L^\infty} + \|\mathcal{F}^{-1}(\hat{K}\chi_2) * g\|_{L^\infty} \leqslant Ce^{-ct}\|g\|_{L^1}. \tag{3.98}$$

Proof. Similarly, by the Young inequality in Lemma A.1 and the fact that $\mathcal{F}^{-1} : L^1 \to L^\infty$, it is sufficient to estimate the L^1-norm of $\hat{G}\chi_2$ and $\hat{K}\chi_2$ under the conditions $\mathscr{B}^*$ and $\mathscr{D}^*$.

We estimate $\hat{G}\chi_2$ firstly where $\hat{G}(\xi, t) = e^{-E(i\xi)t}$. Since the condition $\mathscr{D}^*$ holds, $\mathrm{Re}\,\lambda(i\xi) > 0$ for any eigenvalue $\lambda(i\xi)$ of $E(i\xi)$ and $\xi \neq 0$. Thus, the operator $e^{-E(i\xi)}$ has the spectral radius $\rho(e^{-E(i\xi)}) < 1$ for any $\xi \neq 0$. It follows from [63] that there is an induced norm $|\cdot|$ such that

$$0 < \varphi := \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^d} |e^{-E(i\xi)}| < 1.$$

Hence, for any $t \geqslant 0$ with the integer part denoted by m, since $\log\varphi < 0$, there are constants $c > 0$ and $C > 0$ such that one has

$$\begin{aligned}|\hat{G}(\xi,t)\chi_2(\xi)| &\leqslant |e^{-E(i\xi)}|^m |e^{-E(i\xi)(t-m)}||\chi_2(\xi)| \\ &\leqslant \varphi^m e^{|E(i\xi)|}|\chi_2(\xi)| \\ &\leqslant \varphi^{-1} e^{(m+1)\log\varphi} e^{|E(i\xi)|}|\chi_2(\xi)| \\ &\leqslant Ce^{-ct}e^{|E(i\xi)|}|\chi_2(\xi)|.\end{aligned}$$

Thus, since every norms in finite-dimensional spaces are equivalent, we obtain

$$\|\hat{G}\chi_2\|_{L^1} \leqslant Ce^{-ct}\int_{\mathbb{R}^d} e^{|E(i\xi)|}|\chi_2(\xi)|\,d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 0. \tag{3.99}$$

We estimate $\hat{K}\chi_2$. By Lemma 3.14 and from (3.93), there are constants $c > 0$ and $C > 0$ such that

$$|\hat{K}(\xi,t)\chi_2(\xi)| \leqslant Ce^{-c|\xi|^2 t}|\chi_2(\xi)|.$$

Therefore, we have

$$\|\hat{K}\chi_2\|_{L^1} \leqslant C\int_{\mathbb{R}^d} e^{-c|\xi|^2 t}|\chi_2(\xi)|\,d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 0. \tag{3.100}$$

From (3.99) and (3.100), for $t \geqslant 0$, we obtain

$$\|\hat{G}\chi_2\|_{L^1} + \|\hat{K}\chi_2\|_{L^1} \leqslant Ce^{-ct}.$$

The proof is done. □

Proposition 3.17 (High frequency). *For $g \in L^1(\mathbb{R}^d)$, if the assumptions $\mathscr{B}^*$ and $\mathscr{D}^*$ hold, then there are constants $c > 0$ and $C > 0$ such that for $t \geqslant 1$, we have*

$$\|\mathcal{F}^{-1}(\hat{K}\chi_3) * g\|_{L^\infty} \leqslant Ce^{-ct}\|g\|_{L^1}. \tag{3.101}$$

Proof. By Lemma 3.14 and from (3.93), there are constants $c > 0$ and $C > 0$ such that

$$|\hat{K}(\xi,t)\chi_3(\xi)| \leqslant Ce^{-c|\xi|^2 t}|\chi_3(\xi)|.$$

Therefore, we have

$$\|\hat{K}\chi_3\|_{L^1} \leqslant C\int_{\mathbb{R}^d} e^{-c|\xi|^2 t}|\chi_3(\xi)|\,d\xi \leqslant Ce^{-ct} \qquad \forall t \geqslant 1.$$

The proof is thus done by applying the Young inequality in Lemma A.1. □

3.2.3 L^p-L^p estimates

The aim of this subsection is to introduce the L^p-L^p estimate for $1 \leqslant p \leqslant \infty$ and the L^2-L^2 estimate. Similarly to the one-dimensional space with $d = 1$, we need to estimate the L^1-norm of $\mathcal{F}^{-1}((\hat{G}P_0 - \hat{K})\chi_1)$, where $P_0(i\xi)$ is the eigenprojection associated with the eigenvalue $\lambda_0(i\xi)$ of $E(i\xi)$ and $\lambda_0(i\xi)$ converges to 0 as $|\xi| \to 0$. However, since for $d \geqslant 2$, the Cauchy integral theorem does not hold in the whole complex plane in general, we cannot treat the case where $|x| > Ct$ similarly to the one in the one-dimensional space. Hence, the estimates should be retained in the variable $\xi \in \mathbb{R}^d$ and be obtained via Fourier multiplier estimates.

Proposition 3.18 (Low frequency)**.** *For $g \in L^p(\mathbb{R}^d)$ where $1 \leqslant p \leqslant \infty$, if the assumptions $\mathscr{B}^*$ and $\mathscr{D}^*$ hold, then there is a constant $C > 0$ such that for $t \geqslant 1$, we have*

$$\|\mathcal{F}^{-1}((\hat{G}P_0 - \hat{K})\chi_1) * g\|_{L^p} \leqslant Ct^{-\frac{1}{2}}\|g\|_{L^p}, \tag{3.102}$$

where $P_0(i\xi)$ is the eigenprojection associated with the eigenvalue $\lambda_0(i\xi)$ of $E(i\xi)$ and $\lambda_0(i\xi)$ converges to 0 as $|\xi| \to 0$.

On the other hand, we also have

$$\|\mathcal{F}^{-1}(\hat{G}(I - P_0)\chi_1) * g\|_{L^2} \leqslant Ce^{-ct}\|g\|_{L^2}. \tag{3.103}$$

Furthermore, if the kernel $G = \mathcal{F}^{-1}(\hat{G})$ has compact support contained in the cone $\{x \in \mathbb{R}^d : |x/t| \leqslant C\}$ for a constant $C > 0$, then

$$\|\mathcal{F}^{-1}(\hat{G}(I - P_0)\chi_1) * g\|_{L^p} \leqslant Ce^{-ct}\|g\|_{L^p}. \tag{3.104}$$

Proof. Recall that $(\hat{G}P_0 - \hat{K})\chi_1 = (\hat{G}_1 - \hat{K})\chi_1$ where $\hat{G}_1$ and $\hat{K}$ are respectively given by (3.90) and (3.93). Moreover, since we can assume that $\mathbf{c} = 0$, we can recall the decomposition $(\hat{G}_1 - \hat{K})\chi_1 = I + J$ where I and J are respectively given by (3.96) and (3.97).

In the spirit of Lemma A.7, we then need to estimate the L^2-norm of $\partial^\alpha I$ and $\partial^\alpha J$ for $\alpha \in \mathbb{N}^d$. We consider I firstly. By the Leibniz rule, one has

$$\begin{aligned}\partial^\alpha I(\xi, t) &= \sum_{\nu \leqslant \alpha}\sum_{\tau \leqslant \nu} \binom{\alpha}{\nu}\binom{\nu}{\tau} \partial^\tau e^{-\xi\cdot\mathbf{D}\xi t}\partial^{\nu-\tau}(e^{\mathcal{O}(|\xi|^3)t} - 1)\partial^{\alpha-\nu}\chi_1(\xi)\\ &= I^{(1)}(\xi, t) + I^{(2)}(\xi, t),\end{aligned} \tag{3.105}$$

where

$$I^{(1)}(\xi, t) := \sum_{\nu \leqslant \alpha} \binom{\alpha}{\nu} \partial^\nu e^{-\xi\cdot\mathbf{D}\xi t}(e^{\mathcal{O}(|\xi|^3)t} - 1)\partial^{\alpha-\nu}\chi_1(\xi) \tag{3.106}$$

and

$$I^{(2)}(\xi, t) := \sum_{\nu \leqslant \alpha}\sum_{\tau < \nu} \binom{\alpha}{\nu}\binom{\nu}{\tau} \partial^\tau e^{-\xi\cdot\mathbf{D}\xi t}\partial^{\nu-\tau}e^{\mathcal{O}(|\xi|^3)t}\partial^{\alpha-\nu}\chi_1(\xi). \tag{3.107}$$

By the estimate (3.86) in Lemma 3.13 and Lemma 3.14, since $\chi_1 \in C^\infty(\mathbb{R}^d)$ and $\mathbf{D} \in \mathbb{R}^{d\times d}$ is positive definite, there are constants $c > 0$ and $C > 0$ such that one has

$$\begin{aligned}|I^{(1)}(\xi, t)| &\leqslant C\sum_{\nu \leqslant \alpha} |\partial^\nu e^{-\xi\cdot\mathbf{D}\xi t}||e^{\mathcal{O}(|\xi|^3)t} - 1||\partial^{\alpha-\nu}\chi_1(\xi)|\\ &\leqslant C\sum_{\nu \leqslant \alpha}\sum_{\{\mathcal{I}_j : j=1,\dots,r\}, r \leqslant |\nu|} |\xi|^{m_1+3}t^{\ell+m_0+m_1+1}e^{-c|\xi|^2 t}|\partial^{\alpha-\nu}\chi_1(\xi)|,\end{aligned}$$

where $m_k \geqslant 0$ is the cardinality of the set $\{j \in \{1, \dots, r\} : |\mathcal{I}_j| = |\nu| - k\}$ for $k = 0, 1$ and $\ell \geqslant 0$ satisfies

$$2\ell < |\nu| - 2m_0 - m_1$$

and $\{\mathcal{I}_j : j = 1, \dots, r\}$ is any possible partition of the index set $\mathcal{I}_\nu$ determined by ν. Thus, we have

$$
\begin{aligned}
&\|I^{(1)}\|_{L^2}^2 \\
&\leqslant C \sum_{\nu \leqslant \alpha} \sum_{\{\mathcal{I}_j : j=1,\dots,r\}, r \leqslant |\nu|} \int_{\mathbb{R}^d} |\xi|^{2(m_1+3)} t^{2(\ell+m_0+m_1+1)} e^{-2c|\xi|^2 t} |\partial^{\alpha-\nu} \chi_1(\xi)| \, d\xi, \\
&\leqslant C \sum_{\nu \leqslant \alpha} \sum_{\{\mathcal{I}_j : j=1,\dots,r\}, r \leqslant |\nu|} (1+t)^{-\frac{d}{2}-1+2m_0+m_1+2\ell} \\
&\leqslant C \sum_{\nu \leqslant \alpha} (1+t)^{|\nu|-\frac{d}{2}-1} \leqslant C(1+t)^{|\alpha|-\frac{d}{2}-1} \qquad (3.108)
\end{aligned}
$$

since $|\nu| \leqslant |\alpha|$ for all $\nu \leqslant \alpha$.

Similarly, we then consider $I^{(2)}$ in (3.107). Since $\chi_1 \in C^\infty(\mathbb{R}^d)$ and $\mathbf{D} \in \mathbb{R}^{d\times d}$ is positive definite, from (3.107) and the estimate (3.86) in Lemma 3.13, there are constants $c, c' > 0$ and $C > 0$ such that

$$
\begin{aligned}
&|I^{(2)}(\xi, t)| \\
&\leqslant C \sum_{\nu \leqslant \alpha} \sum_{\tau < \nu} |\partial^\tau e^{-\xi\cdot \mathbf{D}\xi t}| |\partial^{\nu-\tau} e^{\mathcal{O}(|\xi|^3)t}| |\partial^{\alpha-\nu}\chi_1(\xi)| \\
&\leqslant C \sum_{\substack{\nu \leqslant \alpha \\ \tau < \nu}} \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\} \\ \{\mathcal{I}'_j : j=1,\dots,r'\} \\ r \leqslant |\tau|, r' \leqslant |\nu-\tau|}} |\partial^{\alpha-\nu}\chi_1(\xi)| |\xi|^{m_1+m_1'+2m_2'} \cdot t^{\ell+\ell'+m_0+m_1+m_0'+m_1'+m_2'} e^{-c|\xi|^2 t + c'|\xi|^3 t},
\end{aligned}
$$

where $m_k \geqslant 0$ is the cardinality of the set $\{j \in \{1, \dots, r\} : |\mathcal{I}_j| = |\tau| - k\}$ for $k = 0, 1$ and $m_k' \geqslant 0$ is the cardinality of the set $\{j \in \{1, \dots, r'\} : |\mathcal{I}'_j| = |\nu - \tau| - k\}$ for $k = 0, 1, 2$ and $\ell, \ell' \geqslant 0$ satisfies

$$
2\ell < |\tau| - 2m_0 - m_1, \qquad 3\ell' < |\nu - \tau| - 3m_0' - 2m_1' - m_2'
$$

and $\{\mathcal{I}_j : j = 1, \dots, r\}$ is any possible partition of the index set $\mathcal{I}_\tau$ determined by τ and $\{\mathcal{I}'_j : j = 1, \dots, r'\}$ is any possible partition of the index set $\mathcal{I}_{\nu-\tau}$ determined by $\nu - \tau$. Hence, we have

$$
\begin{aligned}
&\|I^{(2)}\|_{L^2}^2 \\
&\leqslant C \sum_{\substack{\nu \leqslant \alpha \\ \tau < \nu}} \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\} \\ \{\mathcal{I}'_j : j=1,\dots,r'\} \\ r \leqslant |\tau|, r' \leqslant |\nu-\tau|}} \int_{\mathbb{R}^d} |\partial^{\alpha-\nu}\chi_1(\xi)| |\xi|^{2(m_1+m_1'+2m_2')} \cdot t^{2(\ell+\ell'+m_0+m_1+m_0'+m_1'+m_2')} e^{-2c|\xi|^2 t} \, d\xi \\
&\leqslant C \sum_{\substack{\nu \leqslant \alpha \\ \tau < \nu}} \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\}, r \leqslant |\tau| \\ \{\mathcal{I}'_j : j=1,\dots,r'\}, r' \leqslant |\nu-\tau|}} (1+t)^{-\frac{d}{2}+2m_0+2m_0'+m_1+m_1'+2\ell+2\ell'} \\
&\leqslant C \sum_{\substack{\nu \leqslant \alpha \\ \tau < \nu}} \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\}, r \leqslant |\tau| \\ \{\mathcal{I}'_j : j=1,\dots,r'\}, r' \leqslant |\nu-\tau|}} (1+t)^{-\frac{d}{2}+|\tau|+|\nu-\tau|-(\ell'+m_0'+m_1'+m_2')} \\
&\leqslant C \sum_{\substack{\nu \leqslant \alpha \\ \tau < \nu}} \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\}, r \leqslant |\tau| \\ \{\mathcal{I}'_j : j=1,\dots,r'\}, r' \leqslant |\nu-\tau|}} (1+t)^{|\nu|-\frac{d}{2}-r'} \leqslant C(1+t)^{|\alpha|-\frac{d}{2}-1} \qquad (3.109)
\end{aligned}
$$

since $|\tau|+|\nu-\tau| = |\nu| \leqslant |\alpha|$ for any $\tau \leqslant \nu \leqslant \alpha$ and $\ell'+m_0'+m_1'+m_2' = r' \geqslant 1$ by definition and $\tau < \nu$.

We continue to estimate J in (3.97). By the Leibniz rule, one has

$$\begin{aligned} \partial^\alpha J(\xi,t) &= \sum_{\nu\leqslant\alpha}\sum_{\tau\leqslant\nu}\binom{\alpha}{\nu}\binom{\nu}{\tau}\partial^\tau e^{-\xi\cdot\mathbf{D}\xi t+\mathcal{O}(|\xi|^3)t}\partial^{\nu-\tau}\mathcal{O}(|\xi|)\partial^{\alpha-\nu}\chi_1(\xi) \\ &= J^{(1)}(\xi,t)+J^{(2)}(\xi,t), \end{aligned} \tag{3.110}$$

where

$$J^{(1)}(\xi,t) := \sum_{\nu\leqslant\alpha}\binom{\alpha}{\nu}\partial^\nu e^{-\xi\cdot\mathbf{D}\xi t+\mathcal{O}(|\xi|^3)t}\mathcal{O}(|\xi|)\partial^{\alpha-\nu}\chi_1(\xi) \tag{3.111}$$

and

$$\begin{aligned} &J^{(2)}(\xi,t) \\ &\quad := \sum_{\nu\leqslant\alpha}\sum_{\tau<\nu}\binom{\alpha}{\nu}\binom{\nu}{\tau}\partial^\tau e^{-\xi\cdot\mathbf{D}\xi t+\mathcal{O}(|\xi|^3)t}\partial^{\nu-\tau}\mathcal{O}(|\xi|)\partial^{\alpha-\nu}\chi_1(\xi). \end{aligned} \tag{3.112}$$

We then begin with $J^{(1)}$. Since $\chi_1 \in C^\infty(\mathbb{R}^d)$ and $\mathbf{D} \in \mathbb{R}^{d\times d}$ is positive definite, from (3.111) and the estimate (3.86) in Lemma 3.13, there are constants $c, c' > 0$ and $C > 0$ such that

$$\begin{aligned} |J^{(1)}(\xi,t)| &\leqslant C\sum_{\nu\leqslant\alpha}|\partial^\nu e^{-\xi\cdot\mathbf{D}\xi t+\mathcal{O}(|\xi|^3)t}||\mathcal{O}(|\xi|)||\partial^{\alpha-\nu}\chi_1(\xi)| \\ &\leqslant C\sum_{\nu\leqslant\alpha}\sum_{\{\mathcal{I}_j:j=1,\ldots,r\}}|\xi|^{m_1+1}t^{\ell+m_0+m_1}e^{-c|\xi|^2t+c'|\xi|^3t}|\partial^{\alpha-\nu}\chi_1(\xi)|, \end{aligned}$$

where $m_k \geqslant 0$ is the cardinality of the set $\{j \in \{1,\ldots,r\} : |\mathcal{I}_j| = |\nu|-k\}$ for $k = 0,1$ and $\ell \geqslant 0$ satisfies

$$2\ell < |\nu| - 2m_0 - m_1$$

and $\{\mathcal{I}_j : j = 1,\ldots,r\}$ is any possible partition of the index set $\mathcal{I}_\nu$ determined by ν. It then implies that

$$\begin{aligned} \|J^{(1)}\|_{L^2}^2 &\leqslant C\sum_{\nu\leqslant\alpha}\sum_{\{\mathcal{I}_j:j=1,\ldots,r\}}\int_{\mathbb{R}^d}|\xi|^{2(m_1+1)}t^{2(\ell+m_0+m_1)}e^{-2c|\xi|^2t}|\partial^{\alpha-\nu}\chi_1(\xi)|\,d\xi \\ &\leqslant C\sum_{\nu\leqslant\alpha}\sum_{\{\mathcal{I}_j:j=1,\ldots,r\}}(1+t)^{-\frac{d}{2}-1+2\ell+2m_0+m_1} \\ &\leqslant C\sum_{\nu\leqslant\alpha}(1+t)^{|\nu|-\frac{d}{2}-1} \leqslant C(1+t)^{|\alpha|-\frac{d}{2}-1} \end{aligned} \tag{3.113}$$

since $|\nu| \leqslant |\alpha|$ for all $\nu \leqslant \alpha$.

We estimate $J^{(2)}$ in (3.112). Since $\chi_1 \in C^\infty(\mathbb{R}^d)$ and $\mathbf{D} \in \mathbb{R}^{d\times d}$ is positive definite, from (3.112) and the estimate (3.86) in Lemma 3.13, there are constants $c, c' > 0$ and $C > 0$ such that

$$\begin{aligned} &|J^{(2)}(\xi,t)| \\ &\quad\leqslant C\sum_{\nu\leqslant\alpha}\sum_{\tau<\nu}|\partial^\tau e^{-\xi\cdot\mathbf{D}\xi t+\mathcal{O}(|\xi|^3)t}||\partial^{\nu-\tau}\mathcal{O}(|\xi|)||\partial^{\alpha-\nu}\chi_1(\xi)| \\ &\quad\leqslant C\sum_{\nu\leqslant\alpha}\sum_{\tau<\nu}\sum_{\{\mathcal{I}_j:j=1,\ldots,r\},r\leqslant|\tau|}|\xi|^{m_1}t^{\ell+m_0+m_1}e^{-c|\xi|^2t+c'|\xi|^3t}|\partial^{\alpha-\nu}\chi_1(\xi)|, \end{aligned}$$

where $m_k \geqslant 0$ is the cardinality of the set $\{j \in \{1, \ldots, r\} : |\mathcal{I}_j| = |\tau| - k\}$ for $k = 0, 1$ and $\ell \geqslant 0$ satisfies

$$2\ell < |\tau| - 2m_0 - m_1$$

and $\{\mathcal{I}_j : j = 1, \ldots, r\}$ is any possible partition of the index set $\mathcal{I}_\tau$ determined by τ. Thus, we have

$$\begin{aligned}
&\|J^{(2)}\|_{L^2}^2 \\
&\leqslant C \sum_{\nu \leqslant \alpha} \sum_{\tau < \nu} \sum_{\{\mathcal{I}_j : j=1,\ldots,r\}, r \leqslant |\tau|} \int_{\mathbb{R}^d} \begin{aligned} &|\xi|^{2m_1} t^{2(\ell+m_0+m_1)} \\ &\cdot e^{-2c|\xi|^2 t + 2c'|\xi|^3 t} |\partial^{\alpha-\nu}\chi_1(\xi)| \, d\xi \end{aligned} \\
&\leqslant C \sum_{\nu \leqslant \alpha} \sum_{\tau < \nu} \sum_{\{\mathcal{I}_j : j=1,\ldots,r\}, r \leqslant |\tau|} (1+t)^{-\frac{d}{2} + 2\ell + 2m_0 + m_1} \\
&\leqslant C \sum_{\nu \leqslant \alpha} \sum_{\tau < \nu} (1+t)^{|\tau| - \frac{d}{2}} \\
&\leqslant C \sum_{\nu \leqslant \alpha} (1+t)^{|\nu| - \frac{d}{2} - 1} \leqslant C(1+t)^{|\alpha| - \frac{d}{2} - 1}
\end{aligned} \tag{3.114}$$

since $|\tau| \leqslant |\nu| - 1$ for all $\tau < \nu$ and $|\nu| \leqslant |\alpha|$ for all $\nu \leqslant \alpha$.

Therefore, from (3.108), (3.109), (3.113) and (3.114), one has

$$\|\partial^\alpha((\hat{G}P_0 - \hat{K})\chi_1)\|_{L^2} \leqslant C(1+t)^{\frac{|\alpha|}{2} - \frac{d}{4} - \frac{1}{2}} \qquad \forall t \geqslant 0, \alpha \in \mathbb{N}^d. \tag{3.115}$$

Let $s \in \mathbb{Z}_+$ satisfying $s > d/2$, then by the Carlson–Beurling inequality (A.1) in Lemma A.7 and (3.115), we obtain for $t \geqslant 0$ and $1 \leqslant p \leqslant \infty$ that

$$\begin{aligned}
&\|(\hat{G}P_0 - \hat{K})\chi_1\|_{M_p} \\
&\qquad \leqslant \|(\hat{G}P_0 - \hat{K})\chi_1\|_{L^2}^{1-\frac{d}{2s}} \Big(\sum_{|\alpha|=s} \|\partial^\alpha((\hat{G}P_0 - \hat{K})\chi_1)\|_{L^2} \Big)^{\frac{d}{2s}} \\
&\qquad \leqslant C(1+t)^{-(\frac{d}{4}+\frac{1}{2})(1-\frac{d}{2s}) + (\frac{s}{2} - \frac{d}{4} - \frac{1}{2})\frac{d}{2s}} \leqslant C(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{3.116}$$

It follows from (3.116) and the definition of the M_p-norm that for any $g \in L^p(\mathbb{R}^d)$, one has

$$\|\mathcal{F}^{-1}((\hat{G}P_0 - \hat{K})\chi_1) * g\|_{L^p} \leqslant \|(\hat{G}P_0 - \hat{K})\chi_1\|_{M_p} \|g\|_{L^p} \leqslant C(1+t)^{-\frac{1}{2}} \|g\|_{L^p}$$

for all $1 \leqslant p \leqslant \infty$ and $t \geqslant 0$.

We estimate the remain parts. By the condition $\mathscr{B}^*$ and Remark 3.3, from (3.91), there are constants $c > 0$ and $C > 0$ such that we have

$$|\hat{G}(\xi, t)(I - P_0(\xi))\chi_1(\xi)| \leqslant C \sum_{k=1}^{s'} e^{-\mathrm{Re}\,(b_k)t} e^{|M_k^{(0)}|t + C|\xi|t} |\chi_1(\xi)| \leqslant Ce^{-ct}|\chi_1(\xi)|.$$

Thus, by the Plancherel theorem, one has

$$\|\mathcal{F}^{-1}(\hat{G}(I - P_0)\chi_1) * g\|_{L^2} = \|\hat{G}_2\chi_1\hat{g}\|_{L^2} \leqslant Ce^{-ct}\|g\|_{L^2} \qquad \forall t \geqslant 0.$$

Moreover, if $G = \mathcal{F}^{-1}(\hat{G})$ has compact support contained in $\{x \in \mathbb{R}^d : |x/t| \leqslant C\}$ for a constant $C > 0$. By the Young inequality in Lemma A.1 that there are constants $c' > 0$, $c > 0$ and $C > 0$ such that for $1 \leqslant p \leqslant \infty$ and $t \geqslant 0$, one has

$$\begin{aligned}\|\mathcal{F}^{-1}(\hat{G}(I-P_0)\chi_1) * g\|_{L^p} &\leqslant C\|\mathcal{F}^{-1}(\hat{G}(I-P_0)\chi_1)\|_{L^1}\|g\|_{L^p} \\ &\leqslant C\Big(\int_{|x|\leqslant Ct}\Big|\int_{\mathbb{R}^d} e^{ix\cdot\xi}\hat{G}_2(\xi,t)\chi_1(\xi)\,d\xi\Big|\,dx\Big)\|g\|_{L^p} \\ &\leqslant Ce^{-c't}t^d\|g\|_{L^p} \leqslant Ce^{-ct}\|g\|_{L^p}.\end{aligned}$$

The proof is done. □

Proposition 3.19 (Intermediate frequency). *For $g \in L^p(\mathbb{R}^d)$ where $1 \leqslant p \leqslant \infty$, if the assumptions $\mathscr{B}^*$ and $\mathscr{D}^*$ hold, then there are constants $c > 0$ and $C > 0$ such that for $t \geqslant 0$, we have*

$$\|\mathcal{F}^{-1}(\hat{K}\chi_2) * g\|_{L^p} \leqslant Ce^{-ct}\|g\|_{L^p}. \tag{3.117}$$

On the other hand, for $g \in L^2(\mathbb{R}^d)$, we also have

$$\|\mathcal{F}^{-1}(\hat{G}\chi_2) * g\|_{L^2} \leqslant Ce^{-ct}\|g\|_{L^2}. \tag{3.118}$$

Furthermore, if the kernel $G = \mathcal{F}^{-1}(\hat{G})$ has compact support contained in the cone $\{x \in \mathbb{R}^d : |x/t| \leqslant C\}$ for a constant $C > 0$, then

$$\|\mathcal{F}^{-1}(\hat{G}\chi_2) * g\|_{L^p} \leqslant Ce^{-ct}\|g\|_{L^p}. \tag{3.119}$$

Proof. We primarily consider $\hat{K}\chi_2$. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \geqslant 0$ and let $\mathbf{D} \in \mathbb{R}^{d\times d}$ be positive definite, by the formula (3.83) in Lemma 3.12, we have

$$\partial^\alpha(e^{-\xi\cdot\mathbf{D}\xi t}) = \sum_{\{\mathcal{I}_j : j=1,\dots,r\}, r\leqslant|\alpha|} \partial_{\mathcal{I}_1}(-\xi\cdot\mathbf{D}\xi t)\dots\partial_{\mathcal{I}_r}(-\xi\cdot\mathbf{D}\xi t)e^{-\xi\cdot\mathbf{D}\xi t},$$

where $\{\mathcal{I}_j : j = 1,\dots,r\}$ is any possible partition of the index set $\mathcal{I}_\alpha$ determined by α.

On the other hand, by the definition of $\partial_{\mathcal{I}_j}$, there is a constant $C > 0$ such that

$$|\partial_{\mathcal{I}_j}(-\xi\cdot\mathbf{D}\xi t)| \leqslant C\cdot\begin{cases}0 & \text{if } |\mathcal{I}_j| > 2,\\ t & \text{if } |\mathcal{I}_j| = 2,\\ |\xi|t & \text{if } |\mathcal{I}_j| = 1,\end{cases}$$

where $|\mathcal{I}_j|$ is the number of elements of $\mathcal{I}_j$ with possibly repeated indices for $j \in \{1,\dots,r\}$. We are then not interested in the cases where $|\mathcal{I}_j| > 2$ for $j \in \{1,\dots,r\}$. Thus, we can consider only the partitions $\{\mathcal{I}_j : j = 1,\dots,r\}$ of $\mathcal{I}_\alpha$ where $1 \leqslant |\mathcal{I}_j| \leqslant 2$. Hence, we have

$$|\partial_{\mathcal{I}_1}(-\xi\cdot\mathbf{D}\xi t)|\dots|\partial_{\mathcal{I}_r}(-\xi\cdot\mathbf{D}\xi t)| \leqslant C|\xi|^m t^{m+\ell},$$

where $m \geqslant 0$ is the cardinality of the set $\{j \in \{1,\dots,r\} : |\mathcal{I}_j| = 1\}$ and $\ell \geqslant 0$ is the cardinality of the set $\{j \in \{1,\dots,r\} : |\mathcal{I}_j| = 2\}$. Moreover, by definition, one has $m + 2\ell = |\mathcal{I}_\alpha| = |\alpha|$, where $|\mathcal{I}_\alpha| = \sum_{j=1}^r |\mathcal{I}_j|$ is the number of elements of the index set $\mathcal{I}_\alpha$ determined by α with possibly repeated indices.

Thus, since $\mathbf{D}$ is positive definite, there are constants $c > 0$ and $C > 0$ such that

$$|\partial^\alpha e^{-\xi\cdot\mathbf{D}\xi t}| \leqslant C \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\}, r\leqslant|\alpha| \\ 1\leqslant|\mathcal{I}_j|\leqslant 2}} |\xi|^m t^{m+\ell} e^{-c|\xi|^2 t}, \tag{3.120}$$

where $m + 2\ell = |\alpha|$ for all m and ℓ.

Therefore, from (3.93), (3.120), Lemma 3.14 and the fact that we can assume $\mathbf{c} = 0$, by the Leibniz formula, one has

$$\begin{aligned}|\partial^\alpha(\hat{K}\chi_2)| &\leqslant C \sum_{\nu\leqslant\alpha} |\partial^\nu e^{-\xi\cdot\mathbf{D}\xi t}||\partial^{\alpha-\nu}\chi_2(\xi)| \\ &\leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\}, r\leqslant|\nu| \\ 1\leqslant|\mathcal{I}_j|\leqslant 2}} |\xi|^m t^{m+\ell} e^{-c'|\xi|^2 t}|\partial^{\alpha-\nu}\chi_2(\xi)|,\end{aligned}$$

where $\{\mathcal{I}_j : j = 1, \dots, r\}$ is any possible partition of the index set $\mathcal{I}_\nu$ determined by ν and $m + 2\ell = |\nu|$.

Hence, we have

$$\begin{aligned}\|\partial^\alpha(\hat{K}\chi_2)\|_{L^2}^2 &\leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\}, r\leqslant|\nu| \\ 1\leqslant|\mathcal{I}_j|\leqslant 2}} \int_{\mathbb{R}^d} |\xi|^{2m} t^{2m+2\ell} e^{-2c|\xi|^2 t}|\partial^{\alpha-\nu}\chi_2(\xi)|\,d\xi \\ &\leqslant Ce^{-ct} \qquad \forall t \geqslant 0.\end{aligned} \tag{3.121}$$

By the Carlson–Beurling inequality (A.1) in Lemma A.7, one has

$$\|\hat{K}\chi_2\|_{M_p} \leqslant C\|\hat{K}\chi_2\|_{L^2}^{1-\frac{d}{2s}} \Big(\sum_{|\alpha|=s} \|\partial^\alpha(\hat{K}\chi_2)\|_{L^2}\Big)^{\frac{d}{2s}} \leqslant Ce^{-ct}$$

for any integer $s > d/2$, $1 \leqslant p \leqslant \infty$ and $t \geqslant 0$. Therefore, by the definition of the M_p-norm, we have the L^p-L^p estimate

$$\|\mathcal{F}^{-1}(\hat{K}\chi_2) * g\|_{L^p} \leqslant Ce^{-ct}\|g\|_{L^p}$$

for any $1 \leqslant p \leqslant \infty$ and $t \geqslant 0$.

We now consider $\hat{G}\chi_2$. By a similar argument as in the proof of Proposition 3.16, we have

$$|\hat{G}(\xi, t)\chi_2(\xi)| \leqslant Ce^{-ct} e^{|E(i\xi)|}|\chi_2(\xi)|. \tag{3.122}$$

It then follows from the Plancherel theorem that for $t \geqslant 0$, one has

$$\|\mathcal{F}^{-1}(\hat{G}\chi_2) * g\|_{L^2} \leqslant C\|\hat{G}\chi_2\|_{L^\infty}\|\hat{g}\|_{L^2} \leqslant Ce^{-ct}\|g\|_{L^2}$$

for some constants $c > 0$ and $C > 0$.

Moreover, if $G = \mathcal{F}^{-1}(\hat{G})$ has compact support contained in $\{x \in \mathbb{R}^d : |x/t| \leqslant C\}$ for a constant $C > 0$. From (3.122) and the Young inequality, there are $c' > 0$, $c > 0$ and $C > 0$ such that for $1 \leqslant p \leqslant \infty$, one has

$$\begin{aligned}\|\mathcal{F}^{-1}(\hat{G}\chi_2) * g\|_{L^p} &\leqslant C\|\mathcal{F}^{-1}(\hat{G}\chi_2)\|_{L^1}\|g\|_{L^p} \\ &\leqslant C\Big(\int_{|x|\leqslant Ct}\Big|\int_{\mathbb{R}^d} e^{ix\cdot\xi}\hat{G}(\xi, t)\chi_2(\xi)\,d\xi\Big|\,dx\Big)\|u_0\|_{L^p} \\ &\leqslant Ce^{-c't}t^d\|g\|_{L^p} \leqslant Ce^{-ct}\|g\|_{L^p} \qquad \forall t \geqslant 0.\end{aligned}$$

We finish the proof. □

Proposition 3.20 (High frequency)**.** *For* $g \in L^p(\mathbb{R}^d)$ *where* $1 \leqslant p \leqslant \infty$*, if the assumptions* $\mathscr{B}^*$ *and* $\mathscr{D}^*$ *hold, then there are constants* $c > 0$ *and* $C > 0$ *such that for* $t \geqslant 1$*, we have*

$$\|\mathcal{F}^{-1}(\hat{K}\chi_3) * g\|_{L^p} \leqslant Ce^{-ct}\|g\|_{L^p}. \tag{3.123}$$

Moreover, for $g \in L^2(\mathbb{R}^d)$*, we have*

$$\|\mathcal{F}^{-1}(\hat{G}\chi_3) * g\|_{L^2} \leqslant Ce^{-ct}\|g\|_{L^2} \tag{3.124}$$

if the conditions $\mathscr{A}^*$*,* $\mathscr{R}^*$ *and* $\mathscr{D}^*$ *hold.*

Proof. The estimate for $\hat{K}\chi_3$ is similarly to the estimate for $\hat{K}\chi_2$ in the proof of the previous proposition where $t \geqslant 1$. We only need to estimate $\hat{G}\chi_3$.

Under the assumptions $\mathscr{A}^*$, $\mathscr{R}^*$ and $\mathscr{D}^*$, for almost everywhere, from (3.92), Remark 3.3 and Lemma 3.14, we have

$$\begin{aligned}|\hat{G}(\xi, t)\chi_3(\xi)| &\leqslant C\sum_{h=1}^{r}\sum_{\ell=1}^{r_h} e^{-\mathrm{Re}\,(\beta_{h\ell})t}e^{|\Theta^{(0)}_{h\ell}|t+C|\xi|^{-1}t}(1+|\xi|^{-1})|\chi_3(\xi)| \\ &\leqslant Ce^{-ct}(1+|\xi|^{-1})|\chi_3(\xi)|.\end{aligned}$$

Therefore, by the Plancherel theorem, we have

$$\|\mathcal{F}^{-1}(\hat{G}\chi_3) * g\|_{L^2} = \|\hat{G}\chi_3\hat{g}\|_{L^2} \leqslant C\|\hat{G}\chi_3\|_{L^\infty}\|\hat{g}\|_{L^2} \leqslant Ce^{-ct}\|g\|_{L^2}$$

for some constants $c > 0$ and $C > 0$ and for all $t \geqslant 0$. The proof is done. □

3.2.4 Proof of Theorem 3.11

Proof of Theorem 3.11. For $u_0 \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, let u and U be solutions to (3.71) and (3.76) respectively. Recall the cut-off function χ_j for $j \in \{1, 2, 3\}$ and the eigenprojection $P_0(i\xi)$ associated with the eigenvalue $\lambda_0(i\xi)$ of $E(i\xi)$, where $\lambda_0(i\xi) \to 0$ as $|\xi| \to 0$. One sets

$$u^{(1)} := \mathcal{F}^{-1}(\hat{G}P_0\chi_1) * u_0 \tag{3.125}$$

and

$$u^{(2)} := \mathcal{F}^{-1}(\hat{G}(I - P_0)\chi_1) * u_0 + \sum_{j=2}^{3} \mathcal{F}^{-1}(\hat{G}\chi_j) * u_0, \tag{3.126}$$

where $\hat{G}$ is a solution to (2.32). It is then easily to check that $u = u^{(1)} + u^{(2)}$.

On the other hand, from (3.125), one has

$$u^{(1)} - U = \mathcal{F}^{-1}((\hat{G}P_0 - \hat{K})\chi_1) * u_0 + \sum_{j=2}^{3} \mathcal{F}^{-1}(\hat{K}\chi_j) * u_0, \tag{3.127}$$

where $\hat{K}$ is given by (3.93). Hence, by Proposition 3.15, Proposition 3.16 and Proposition 3.17, we obtain

$$\|u^{(1)} - U\|_{L^\infty} \leqslant Ct^{-\frac{d}{2}-\frac{1}{2}}\|u_0\|_{L^1} \qquad \forall t \geqslant 1. \tag{3.128}$$

By Proposition 3.18, Proposition 3.19 and Proposition 3.20, for $1 \leqslant p \leqslant \infty$, we also have

$$\|u^{(1)} - U\|_{L^p} \leqslant Ct^{-\frac{1}{2}}\|u_0\|_{L^p} \qquad \forall t \geqslant 1. \tag{3.129}$$

Therefore, by Lemma A.2 in Appendix A, it follows from (3.128) and (3.129) that

$$\|u^{(1)} - U\|_{L^p} \leqslant Ct^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|u_0\|_{L^q}$$

for $1 \leqslant q \leqslant p \leqslant \infty$ and for all $t \geqslant 1$.

Also by Proposition 3.18, Proposition 3.19 and Proposition 3.20, from (3.126), we have

$$\|u^{(2)}\|_{L^2} \leqslant Ce^{-ct}\|u_0\|_{L^2} \qquad \forall t \geqslant 0.$$

The proof is done since the estimate for $U = \mathcal{F}^{-1}(\hat{K}) * u_0$ is similar. □

Remark 3.10 (Proof of the case of finite speed of propagation)**.** In the case where $G = \mathcal{F}^{-1}(\hat{G})$ has compact support contained in the wave cone $\{x \in \mathbb{R}^d : |x/t| \leqslant C\}$ for a constant $C > 0$, also by Proposition 3.15 - Proposition 3.20, $u^{(1)}$ can be refined by the following

$$u^{(1)} = \mathcal{F}^{-1}(e^{-Et}\chi) * u_0,$$

where χ is a cut-off function valued in $[0, 1]$ with support contained in the ball $B(0, \rho) \subset \mathbb{R}^d$ for any $\rho > 0$. The proof is then similar to the above proof.

3.3 Symmetry systems

3.3.1 Motivative examples

Recall the Goldstein–Kac 2×2 system in the one-dimensional space

$$\begin{cases} \partial_t u_1 - \partial_x u_1 = -\frac{1}{2}u_1 + \frac{1}{2}u_2, \\ \partial_t u_2 + \partial_x u_2 = \frac{1}{2}u_1 - \frac{1}{2}u_2, \end{cases} \tag{3.130}$$

which can be written in the relaxation form with

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Moreover, the initial datum $(u_1, u_2)|_{t=0} = (u_0^1, u_0^2)$ is considered.

We can check easily that (3.130) satisfies the conditions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ in Section 3.1. Thus, it follows from Theorem 3.1 that $u = (u_1, u_2)$ satisfies

$$\|u - U - V\|_{L^p} \leqslant Ct^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}\|(u_0^1, u_0^2)\|_{L^q} \tag{3.131}$$

for $1 \leqslant q \leqslant p \leqslant \infty$, $t \geqslant 1$ and $(u_0^1, u_0^2) \in L^q(\mathbb{R}) \times L^q(\mathbb{R})$.

In (3.131), $U = (U_1, U_2)$ is a solution to the heat equation

$$\partial_t U - \partial_{xx} U = 0 \tag{3.132}$$

with the initial datum

$$U|_{t=0} = \frac{1}{2}(u_0^1 + u_0^2, u_0^1 + u_0^2). \tag{3.133}$$

Furthermore, $V = (V_1, V_2)$ is a solution to the system

$$\begin{cases} \partial_t V_1 - \partial_x V_1 = -\dfrac{1}{2} V_1, \\ \partial_t V_2 + \partial_x V_2 = -\dfrac{1}{2} V_2 \end{cases} \tag{3.134}$$

with the initial datum $(V_1, V_2)|_{t=0} = (u_0^1, u_0^2)$.

Recall that $v_\pm = u_1 \pm u_2$ satisfy to the linear damped wave equation

$$\partial_{tt} v - \partial_{xx} v + \partial_t v = 0 \tag{3.135}$$

with the initial data $(v, \partial_t v)|_{t=0} = (v_0, v_1)$ given by

$$\begin{cases} v_+|_{t=0} = u_0^1 + u_0^2, \\ \partial_t v_+|_{t=0} = \partial_x (u_0^1 - u_0^2) \end{cases} \quad \text{and} \quad \begin{cases} v_-|_{t=0} = u_0^1 - u_0^2, \\ \partial_t v_-|_{t=0} = \partial_x (u_0^1 + u_0^2) - (u_0^1 - u_0^2). \end{cases}$$

Recall also from [48] that the solution v to (3.135) satisfies the L^p-L^q decay estimate

$$\left\| v - \phi - e^{-\frac{t}{2}} \frac{v_0(\cdot + t) + v_0(\cdot - t)}{2} \right\|_{L^p} \leqslant C t^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p}) - 1} \|(v_0, v_1)\|_{L^q} \tag{3.136}$$

for all $1 \leqslant q \leqslant p \leqslant \infty$, $t \geqslant 1$ and $(v_0, v_1) \in L^q(\mathbb{R}) \times L^q(\mathbb{R})$, where ϕ is a solution to the Cauchy problem

$$\begin{cases} \partial_t \phi - \partial_{xx} \phi = 0, \\ \phi|_{t=0} = v_0 + v_1 \end{cases} \tag{3.137}$$

and the term $(v_0(\cdot + t) + v_0(\cdot - t))/2$ is a solution to the Cauchy problem

$$\begin{cases} \partial_{tt} \psi - \partial_{xx} \psi = 0, \\ (\psi, \partial_t \psi)|_{t=0} = (0, v_0). \end{cases}$$

Noting that (3.134) can be solved explicitly by a vector V of the two signals

$$V_1(x, t) = e^{-\frac{t}{2}} u_0^1(x + t) \qquad \text{and} \qquad V_2(x, t) = e^{-\frac{t}{2}} u_0^2(x - t),$$

which may contain singularities coming from the initial datum (u_0^1, u_0^2), has exactly the same decay rate $e^{-\frac{t}{2}}$ as the one obtained from (3.136).

However, there are differences between the L^p-L^q decay estimates for $v_\pm$ and for the solution (u_1, u_2) to (3.130).

i) Due to the difference between one-way hyperbolic equations and two-way hyperbolic equations, the singular term of v_+ (respectively v_-) is the average of the two signals $e^{-\frac{t}{2}}(u_0^1 + u_0^2)$ (respectively $e^{-\frac{t}{2}}(u_0^1 - u_0^2)$) propagating in the two contrary directions along the characteristic curves $x = \pm t$;

ii) From (3.136) and (3.131), we also recognize that there is a difference (about 1/2) between the decay rates of $v_\pm$ and (u_1, u_2);

iii) Furthermore, from (3.132) and (3.133), the large-time asymptotic profiles of $u_1 \pm u_2$ are $\phi_\pm := U_1 \pm U_2$ satisfying

$$\partial_t \phi - \partial_{xx} \phi = 0$$

with the initial data

$$\phi_+|_{t=0} = (u_0^1 + u_0^2) \qquad \text{and} \qquad \phi_-|_{t=0} = 0,$$

while from (3.137), the large-time asymptotic profiles of $v_\pm = u_1 \pm u_2$ as $t \to +\infty$ are solutions to the heat equations with the initial data

$$\phi_+|_{t=0} = (u_0^1 + u_0^2) + \partial_x(u_0^1 - u_0^2) \quad \text{and} \quad \phi_-|_{t=0} = \partial_x(u_0^1 + u_0^2). \tag{3.138}$$

We can explain (ii) and (iii) by the fact that when we subtract U in (3.131), we do not subtract the convolutions of the heat kernel and the first-order derivatives of the initial data with respect to the variable $x \in \mathbb{R}$. These convolutions are equivalent to the first-order derivatives of the heat solution with respect to the variable $x \in \mathbb{R}$. Thus, they have decay rates never exceeding $t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}$ for all $t \geqslant 0$ (see [27]).

On the other hand, from Section 3.1, the decay rate $t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}$ is also caused by the term $\mathcal{O}(|\xi|^3 t)$ in the expansion $\hat{G}(\xi,t)P_0(\xi) \sim e^{-c\xi^2 t + \mathcal{O}(|\xi|^3)t}$ for small $\xi \in \mathbb{R}$ and a $c \in \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\}$. Here, $\hat{G}$ is the Fourier transform of the fundamental solution G to (3.130), $P_0(i\xi)$ is the total projection associated with the 0-group of $E(i\xi) = B + i\xi A$ and the elements of the 0-group are the eigenvalues of $E(i\xi)$ that converge to 0 as $|\xi| \to 0$.

Nevertheless, in the case of (3.130), we can check easily that $\hat{G}(\xi,t)P_0(\xi) \sim e^{-c\xi^2 t + \mathcal{O}(|\xi|^4)t}$ instead. Thus, we only need to subtract from $u = (u_1, u_2)$ the first-order derivatives of the heat solution with respect to the variable $x \in \mathbb{R}$ in this case.

Similarly, consider the two-dimensional linearized isentropic Euler equations with damping

$$\begin{cases} \partial_t \rho + \operatorname{div} v = 0, \\ \partial_t v + \nabla_x \rho = -v, \end{cases} \tag{3.139}$$

which can be written in the relaxation form where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $u = (\rho, v_1, v_2)$ with the initial datum $u_0 = (\rho_0, v_0^1, v_0^2)$.

Moreover, the system (3.139) satisfies the conditions $\mathscr{A}^*$, $\mathscr{R}^*$, $\mathscr{B}^*$ and $\mathscr{D}^*$ in Section 3.2, where the matrix $R = R(w)$ satisfying the condition $\mathscr{R}^*$ is

$$R(w_1, w_2) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ -w_1 & 2w_2 & w_1 \\ -w_2 & -2w_1 & w_2 \end{pmatrix} \qquad \forall w = (w_1, w_2) \in \mathbb{S}^1.$$

Theorem 3.11 then implies that the solution $U = (U_1, U_2, U_3)$ to the heat equation

$$\begin{cases} \partial_t U - \Delta_x U = 0, \\ U|_{t=0} = (\rho_0, 0, 0), \end{cases}$$

is a large-time asymptotic profile of u. Moreover, $u = u^{(1)} + u^{(2)}$ where $u^{(1)} - U$ decays in $L^p(\mathbb{R}^2)$ at the rate $t^{-(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}$ and $u^{(2)}$ decays in $L^2(\mathbb{R}^2)$ exponentially as $t \to +\infty$ for any $1 \leqslant q \leqslant p \leqslant \infty$ and $u_0 \in L^q(\mathbb{R}^2)$.

On the other hand, $\rho \in \mathbb{R}$ satisfying (3.139) also satisfies the linear damped wave equation

$$\begin{cases} \partial_{tt}\rho - \Delta_x \rho + \partial_t \rho = 0, \\ \rho|_{t=0} = \rho_0, \\ \partial_t \rho|_{t=0} = -\partial_{x_1} v_0^1 - \partial_{x_2} v_0^2. \end{cases} \tag{3.140}$$

The proofs of Theorem 2.1 in [31] implies that without regarding any error term V decaying exponentially in $L^2(\mathbb{R}^2)$, $\rho - \phi$ decays in $L^p(\mathbb{R}^2)$ at the rate $t^{-(\frac{1}{q}-\frac{1}{p})-1}$ as $t \to +\infty$ for any $1 \leqslant q \leqslant p \leqslant \infty$ and $(\rho_0, \partial_t \rho_0) \in L^q(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$, where ϕ is a solution to the heat equation

$$\begin{cases} \partial_t \phi - \Delta_x \phi = 0, \\ \phi|_{t=0} = \rho_0 - \partial_{x_1} v_0^1 - \partial_{x_2} v_0^2. \end{cases}$$

Thus, there is a difference (about 1/2) between the decay rates of ρ in (3.139) and ρ in (3.140). Therefore, to have the decay rate $t^{-(\frac{1}{q}-\frac{1}{p})-1}$ for $u^{(1)} - U$, we need to modify U by subtracting from the solution u to (3.139) the first-order derivatives of the heat solution with respect to the variable $x \in \mathbb{R}^2$. Particularly, in this case, we also have $\hat{G}(\xi, t)P_0(\xi) \sim e^{-c\xi^2 t + \mathcal{O}(|\xi|^4)t}$ for small ξ.

Noting that a general relaxation system has $\hat{G}(\xi, t)P_0(\xi) \sim e^{-c\xi^2 t + \mathcal{O}(|\xi|^4)t}$ for small ξ, if it satisfies the following conditions

Condition $\mathscr{E}$ (Equilibrium manifold)**.** *The eigenvalue* 0 *of* B *is simple.*

Condition $\mathscr{S}$ (Symmetry)**.** *There is an invertible matrix* $S = S(w)$ *such that*

$$SA = -AS \qquad \textit{and} \qquad SB = BS,$$

where $A = A(w) = \sum_{j=1}^{d} w_j A_j$ *for* $w = (w_1, \ldots, w_d) \in \mathbb{S}^{d-1}$.

If the symmetry condition $\mathscr{S}$ holds, the solution u to the general relaxation systems under the map $x \mapsto -x$ is also a solution to the same systems up to a linear change of variables.

3.3.2 Increasing decay rate

In this section, we study the optimal decay rate $t^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-1}$ for $1 \leqslant q \leqslant p \leqslant \infty$ as $t \to +\infty$ for the general system (3.71) satisfying the symmetry properties similarly to the Goldstein–Kac 2×2 system (3.130) and the linearized isentropic Euler equations with damping (3.139), namely the conditions $\mathscr{E}$ and $\mathscr{S}$. Hence, we recall the Cauchy problem

$$\begin{cases} \partial_t u + \boldsymbol{A} \cdot \nabla_x u + Bu = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{3.141}$$

where $(x, t) \in \mathbb{R}^d \times [0, +\infty)$, $u = u(x, t)$ and $u_0 = u_0(x)$ are vectors in $\mathbb{R}^n$, $\boldsymbol{A} = (A_1, \ldots, A_d) \in (\mathbb{R}^{n \times n})^d$ and $B \in \mathbb{R}^{n \times n}$.

Let Γ_0 be an oriented closed curve in the resolvent set of B such that it encloses zero except for the other eigenvalues of B. We recall

$$P_0^{(0)} = -\frac{1}{2\pi i}\int_{\Gamma_0}(B - zI)^{-1}\,dz \text{ and } Q_0^{(0)} = \frac{1}{2\pi i}\int_{\Gamma_0} z^{-1}(B - zI)^{-1}\,dz, \quad (3.142)$$

which are the eigenprojection and the reduced resolvent coefficient associated with the eigenvalue zero of B. We consider the Cauchy problem

$$\begin{cases} \partial_t U - \operatorname{div}(\mathbf{D}\nabla_x U) = 0, \\ U|_{t=0} = P_0^{(0)}u_0 + \mathbf{P}_0^{(1)}\cdot\nabla_x u_0, \end{cases} \quad (3.143)$$

where $\mathbf{D} = (D_{h\ell})_{h,\ell\in\{1,\dots,d\}} \in \mathbb{R}^{d\times d}$ with scalar entries

$$D_{h\ell} = \frac{1}{2}\operatorname{tr}\left(A_h P_0^{(0)} A_\ell Q_0^{(0)} + A_h Q_0^{(0)} A_\ell P_0^{(0)}\right) \quad (3.144)$$

and $\mathbf{P}_0^{(1)} = (P_{0h}^{(1)}) \in (\mathbb{R}^{n\times n})^d$ with matrix entries

$$P_{0h}^{(1)} := -P_0^{(0)} A_h Q_0^{(0)} - Q_0^{(0)} A_h P_0^{(0)}. \quad (3.145)$$

We thus obtain the following results.

Recall the Cauchy problem

$$\begin{cases} \partial_t V + A\partial_x V + \Pi_A(B)V = 0, \\ V|_{t=0} = u_0. \end{cases} \quad (3.146)$$

In (3.146), $(x,t) \in \mathbb{R}\times[0,+\infty)$, $V = V(x,t) \in \mathbb{R}^n$ and the $n\times n$ matrix

$$\Pi_A(B) = \sum_{h=1}^{r} \Pi_h^{(0)} B \Pi_h^{(0)},$$

where $\Pi_h^{(0)} \in \mathbb{R}^{n\times n}$ is the eigenprojection associated with $\alpha_h \in \sigma(A)$ for $h \in \{1,\dots,r\}$ and $\sigma(A)$ is the spectrum of A with the cardinality r.

Theorem 3.21 (One-dimensional space [52])**.** *For* $u_0 \in W^{1,q}(\mathbb{R})$, *let* u, U *and* V *be respectively solutions to* (3.141), (3.143) *and* (3.146) *where* $d = 1$. *If the conditions* $\mathscr{A}$, $\mathscr{B}$, $\mathscr{D}$, $\mathscr{E}$ *and* $\mathscr{S}$ *hold, then for* $1 \leqslant q \leqslant p \leqslant \infty$ *and* $t \geqslant 1$, *there are positive constants* c *and* C *such that*

$$\|u - U - V\|_{L^p} \leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|u_0\|_{L^q}. \quad (3.147)$$

Moreover, one has

$$\|U\|_{L^p} \leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_{L^q} \qquad \text{and} \qquad \|V\|_{L^q} \leqslant Ce^{-ct}\|u_0\|_{L^q}. \quad (3.148)$$

Theorem 3.22 (Multi-dimensional space [54])**.** *Let* u *be a solution to the Cauchy problem* (3.141) *with the initial datum* $u_0 \in W^{1,q}(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ *for* $1 \leqslant q \leqslant \infty$. *Under the assumptions* $\mathscr{A}^*$, $\mathscr{R}^*$, $\mathscr{B}^*$, $\mathscr{D}^*$ *and* $\mathscr{S}$, *the solution* u *is decomposed into*

$$u(x,t) = u^{(1)}(x,t) + u^{(2)}(x,t),$$

where

$$u^{(1)}(x,t) := \mathcal{F}^{-1}(e^{-Et}P_0\chi) * u_0(x)$$

and $u^{(2)}$ *is the remainder,* $P_0(i\xi)$ *is the eigenprojection associated with the eigenvalue* $\lambda_0(i\xi)$ *of* $E(i\xi)$*,* $\lambda_0(i\xi)$ *converges to* 0 *as* $|\xi| \to 0$ *and* χ *is a cut-off function valued in* $[0,1]$ *with support contained in the ball* $B(0,\varepsilon) \subset \mathbb{R}^d$ *for small* $\varepsilon > 0$*.*

Moreover, for any $1 \leqslant q \leqslant p \leqslant \infty$*, there are constants* $c > 0$ *and* $C > 0$ *such that one has*

$$\|u^{(1)} - U\|_{L^p} \leqslant Ct^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-1}\|u_0\|_{L^q} \qquad \forall t \geqslant 1 \tag{3.149}$$

and $u^{(2)}$ *satisfies*

$$\|u^{(2)}\|_{L^2} \leqslant Ce^{-ct}\|u_0\|_{L^2} \qquad \forall t \geqslant 0, \tag{3.150}$$

where U *which is a solution to* (3.143) *satisfies*

$$\|U\|_{L^p} \leqslant Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_{L^q} \qquad \forall t \geqslant 1. \tag{3.151}$$

Remark 3.11 (Finite speed of propagation)**.** If the solution u to (3.141) has finite speed of propagation, one can decompose u into $u = u^{(1)} + u^{(2)}$ where

$$u^{(1)}(x,t) := \mathcal{F}^{-1}(e^{-Et}\chi) * u_0(x)$$

and $u^{(2)}$ is the remainder. Here, χ is a cut-off function valued in $[0,1]$ with support contained in the ball $B(0,\rho) \subset \mathbb{R}^d$ for any $\rho > 0$. Moreover, the estimates (3.149) and (3.150) also hold.

Example 3.12 (The one-dimensional Goldstein–Kac 2×2 system)**.** Recall (3.141) satisfied by $u = (u_1, u_2)$ with the initial datum $u_0 = (u_0^1, u_0^2)$ and the coefficient matrices given by

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It is easy to see that $S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies the condition $\mathscr{S}$. Hence, since 0 is a simple eigenvalue of B, the large-time asymptotic limit system satisfied by $U = (U_1, U_2)$ is refined to be

$$\begin{cases} \partial_t U - \partial_{xx} U = 0, \\ U|_{t=0} = U_0, \end{cases}$$

where

$$U_0 := \frac{1}{2}(u_0^1 + u_0^2, u_0^1 + u_0^2) + (\partial_x u_0^1, -\partial_x u_0^2).$$

In particular, $U_0^1 + U_0^2 = (u_0^1 + u_0^2) + \partial_x(u_0^1 - u_0^2)$ and $U_0^1 - U_0^2 = \partial_x(u_0^1 + u_0^2)$ coincide the ones in (3.138). Moreover, the decay rate of $u - U$ is also increased up to $t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-1}$ as $t \to +\infty$ for $1 \leqslant q \leqslant p \leqslant \infty$ and $u_0 \in W^{1,q}(\mathbb{R})$.

Example 3.13 (The two-dimensional linearized isentropic Euler equations with damping)**.** Recall (3.141) satisfied by $u = (\rho, v^1, v^2)$ with the initial datum $u_0 = (\rho_0, v_0^1, v_0^2)$ and the coefficient matrices given by

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, the matrix R satisfying the condition $\mathscr{R}^*$ and the matrix S satisfying the condition $\mathscr{S}$ are given by

$$R(w_1, w_2) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ -w_1 & 2w_2 & w_1 \\ -w_2 & -2w_1 & w_2 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, Theorem 3.22 implies that the asymptotic limit of u as $t \to +\infty$ is a solution denoted by U to the heat equations

$$\begin{cases} \partial_t U - \Delta_x U = 0, \\ U|_{t=0} = U_0, \end{cases}$$

where

$$U_0 = (\rho_0 - \partial_{x_1} v_0^1 - \partial_{x_2} v_0^2, -\partial_{x_1}\rho_0, -\partial_{x_2}\rho_0).$$

Furthermore, $u - U$ decays in $L^p(\mathbb{R}^2)$ at the optimal rate $t^{-(\frac{1}{q}-\frac{1}{p})-1}$ as $t \to +\infty$ for any $1 \leqslant q \leqslant p \leqslant \infty$ and $u_0 \in W^{1,q}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. This result can be comparable with [31] since as mentioned before, $\rho \in \mathbb{R}$ satisfies the two-dimensional linear damped wave equation

$$\begin{cases} \partial_{tt}\rho - \Delta_x \rho + \partial_t \rho = 0, \\ \rho|_{t=0} = \rho_0, \\ \partial_t \rho|_{t=0} = -\partial_{x_1} v_0^1 - \partial_{x_2} v_0^2. \end{cases}$$

Moreover, from [31], $\rho - \phi$ decays in $L^p(\mathbb{R}^2)$ at the rate $t^{-(\frac{1}{q}-\frac{1}{p})-1}$ as $t \to +\infty$ for any $1 \leqslant q \leqslant p \leqslant \infty$ and $(\rho_0, \partial_t \rho_0) \in L^q(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$, where ϕ is a solution to the heat equation

$$\begin{cases} \partial_t \phi - \Delta_x \phi = 0, \\ \phi|_{t=0} = \rho_0 - \partial_{x_1} v_0^1 - \partial_{x_2} v_0^2. \end{cases}$$

3.3.3 Proofs of Theorem 3.21 and Theorem 3.22

The proofs are based on the following proposition for the eigenvalues of $E(i\xi) = B + i\sum_{j=1}^d \xi_j A_j$ that converge to 0 as $|\xi| \to 0$ for $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$.

Proposition 3.23. *Under the condition $\mathscr{B}^*$, $\mathscr{D}^*$ and $\mathscr{S}$, for small $\xi \in \mathbb{R}^d$, there is only one eigenvalue*

$$\lambda_0(i\xi) = \xi \cdot \mathbf{D}\xi + \mathcal{O}(|\xi|^4) \tag{3.152}$$

of $E(i\xi)$ that converges to 0 as $|\xi| \to 0$, where $\mathbf{D}$ is given by (3.144)*. Furthermore, the eigenprojection P_0 associated with λ_0 is approximated by*

$$P_0(i\xi) = P_0^{(0)} + i\xi \cdot \mathbf{P}_0^{(1)} + \mathcal{O}(|\xi|^2), \tag{3.153}$$

where $P_0^{(0)}$ and $\mathbf{P}_0^{(1)}$ are respectively given by (3.142) *and* (3.145).

Proof. The proof follows directly from Proposition 2.6 in Chapter 2 and Corollary B.2 in Appendix B. We finish the proof. □

As a consequence, for small ξ, the Fourier transform $\hat{G}$ of the Green kernel G associated with (3.141) satisfies

$$\hat{G}_1(\xi,t) = \hat{G}(\xi,t)P_0(i\xi) = e^{-(\xi\cdot\mathbf{D}\xi+\mathcal{O}(|\xi|^4))t}(P_0^{(0)}+i\xi\cdot\mathbf{P}_0^{(1)}+\mathcal{O}(|\xi|^2)) \quad (3.154)$$

for $\xi \in \mathbb{R}^d$, where the related coefficients are introduced in (3.142) - (3.145).

One sets

$$\hat{K}(\xi,t) := e^{-\xi\cdot\mathbf{D}\xi t}(P_0^{(0)} + i\xi\cdot\mathbf{P}_0^{(1)}), \quad (3.155)$$

where the related coefficients are as before. We thus obtain

Proposition 3.24 (L^∞-L^1 estimate)**.** *For $g \in L^1(\mathbb{R}^d)$, if the assumptions $\mathscr{B}^*$, $\mathscr{D}^*$ and $\mathscr{S}$ hold, then there is a constant $C > 0$ such that for $t \geqslant 1$, we have*

$$\|\mathcal{F}^{-1}((\hat{G}P_0 - \hat{K})\chi_1) * g\|_{L^\infty} \leqslant Ct^{-\frac{d}{2}-1}\|g\|_{L^1}, \quad (3.156)$$

where χ_1 is a cut-off function valued in $[0,1]$ with support contained in the ball $B(0,\varepsilon) \subset \mathbb{R}^d$ for small $\varepsilon > 0$, $P_0(i\xi)$ is the eigenprojection associated with the eigenvalue $\lambda_0(i\xi)$ of $E(i\xi)$ and $\lambda_0(i\xi)$ converges to 0 as $|\xi| \to 0$.

Proof. One has $(\hat{G}_1 - \hat{K})\chi_1 = I + J$ where

$$I(\xi,t) := e^{-\xi\cdot\mathbf{D}\xi t}(e^{\mathcal{O}(|\xi|^4)t} - 1)(P_0^{(0)} + i\xi\cdot\mathbf{P}_0^{(1)})\chi_1(\xi) \quad (3.157)$$

and

$$J(\xi,t) := e^{-\xi\cdot\mathbf{D}\xi t+\mathcal{O}(|\xi|^4)t}\mathcal{O}(|\xi|^2)\chi_1(\xi). \quad (3.158)$$

Then, by Lemma 3.14, there are constants $c > 0$ and $C > 0$ such that

$$|(\hat{G}(\xi,t)P_0(\xi) - \hat{K}(\xi,t))\chi_1(\xi)| \leqslant |I(\xi,t)| + |J(\xi,t)| \leqslant Ce^{-c|\xi|^2t}(|\xi|^4t + |\xi|^2)|\chi_1(\xi)|.$$

Hence, it implies for all $t \geqslant 0$ that

$$\|(\hat{G}P_0 - \hat{K})\chi_1\|_{L^1} \leqslant C\int_{\mathbb{R}^d} e^{-c|\xi|^2t}(|\xi|^4t + |\xi|^2)|\chi_1(\xi)|\,d\xi \leqslant C(1+t)^{-\frac{d}{2}-1}.$$

The proof is done. □

Proposition 3.25 (L^p-L^p estimates)**.** *For $g \in L^p(\mathbb{R}^d)$, if the assumptions $\mathscr{B}^*$, $\mathscr{D}^*$ and $\mathscr{S}$ hold, then there is a constant $C > 0$ such that for $t \geqslant 1$ and $1 \leqslant p \leqslant \infty$, we have*

$$\|\mathcal{F}^{-1}((\hat{G}P_0 - \hat{K})\chi_1) * g\|_{L^p} \leqslant Ct^{-1}\|g\|_{L^p}, \quad (3.159)$$

where χ_1 is a cut-off function valued in $[0,1]$ with support contained in the ball $B(0,\varepsilon) \subset \mathbb{R}^d$ for small $\varepsilon > 0$, $P_0(i\xi)$ is the eigenprojection associated with the eigenvalue $\lambda_0(i\xi)$ of $E(i\xi)$ and $\lambda_0(i\xi)$ converges to 0 as $|\xi| \to 0$.

Proof. In the spirit of Lemma A.7 and the decomposition $(\hat{G}P_0 - \hat{K})\chi_1 = I + J$ where I and J are respectively given by (3.157) and (3.158), we then need to estimate the L^2-norm of $\partial^\alpha I$ and $\partial^\alpha J$ for $\alpha \in \mathbb{N}^d$.

We consider I firstly. By the Leibniz rule, one has

$$\begin{aligned}\partial^\alpha I(\xi,t) &= \sum_{\nu\leqslant\alpha}\sum_{\tau\leqslant\nu}\binom{\alpha}{\nu}\binom{\nu}{\tau}\partial^\tau e^{-\xi\cdot\mathbf{D}\xi t}\partial^{\nu-\tau}(e^{\mathcal{O}(|\xi|^4)t}-1)\partial^{\alpha-\nu}\chi_1(\xi)\\ &= I^{(1)}(\xi,t)+I^{(2)}(\xi,t),\end{aligned}\tag{3.160}$$

where

$$I^{(1)}(\xi,t) := \sum_{\nu\leqslant\alpha}\binom{\alpha}{\nu}\partial^\nu e^{-\xi\cdot\mathbf{D}\xi t}(e^{\mathcal{O}(|\xi|^4)t}-1)\partial^{\alpha-\nu}\chi_1(\xi)\tag{3.161}$$

and

$$I^{(2)}(\xi,t) := \sum_{\nu\leqslant\alpha}\sum_{\tau<\nu}\binom{\alpha}{\nu}\binom{\nu}{\tau}\partial^\tau e^{-\xi\cdot\mathbf{D}\xi t}\partial^{\nu-\tau}e^{\mathcal{O}(|\xi|^4)t}\partial^{\alpha-\nu}\chi_1(\xi).\tag{3.162}$$

By the estimate (3.86) in Lemma 3.13 and Lemma 3.14, since $\chi_1\in C^\infty(\mathbb{R}^d)$ and $\mathbf{D}\in\mathbb{R}^{d\times d}$ is positive definite, there are constants $c>0$ and $C>0$ such that one has

$$\begin{aligned}|I^{(1)}(\xi,t)| &\leqslant C\sum_{\nu\leqslant\alpha}|\partial^\nu e^{-\xi\cdot\mathbf{D}\xi t}||e^{\mathcal{O}(|\xi|^4)t}-1||\partial^{\alpha-\nu}\chi_1(\xi)|\\ &\leqslant C\sum_{\nu\leqslant\alpha}\sum_{\{\mathcal{I}_j:j=1,\dots,r\},r\leqslant|\nu|}|\xi|^{m_1+4}t^{\ell+m_0+m_1+1}e^{-c|\xi|^2t}|\partial^{\alpha-\nu}\chi_1(\xi)|,\end{aligned}$$

where $m_k\geqslant 0$ is the cardinality of the set $\{j\in\{1,\dots,r\}:|\mathcal{I}_j|=|\nu|-k\}$ for $k=0,1$ and $\ell\geqslant 0$ satisfies

$$2\ell<|\nu|-2m_0-m_1$$

and $\{\mathcal{I}_j:j=1,\dots,r\}$ is any possible partition of the index set $\mathcal{I}_\nu$ determined by ν. Thus, we have

$$\begin{aligned}&\|I^{(1)}\|_{L^2}^2\\ &\leqslant C\sum_{\nu\leqslant\alpha}\sum_{\{\mathcal{I}_j:j=1,\dots,r\},r\leqslant|\nu|}\int_{\mathbb{R}^d}|\xi|^{2(m_1+4)}t^{2(\ell+m_0+m_1+1)}e^{-2c|\xi|^2t}|\partial^{\alpha-\nu}\chi_1(\xi)|\,d\xi\\ &\leqslant C\sum_{\nu\leqslant\alpha}\sum_{\{\mathcal{I}_j:j=1,\dots,r\},r\leqslant|\nu|}(1+t)^{-\frac{d}{2}-2+2m_0+m_1+2\ell}\\ &\leqslant C\sum_{\nu\leqslant\alpha}(1+t)^{|\nu|-\frac{d}{2}-2}\leqslant C(1+t)^{|\alpha|-\frac{d}{2}-2}\end{aligned}\tag{3.163}$$

since $|\nu|\leqslant|\alpha|$ for all $\nu\leqslant\alpha$.

Similarly, we then consider $I^{(2)}$ in (3.162). Since $\chi_1\in C^\infty(\mathbb{R}^d)$ and $\mathbf{D}\in\mathbb{R}^{d\times d}$ is positive definite, from (3.162) and the estimate (3.86) in Lemma 3.13, there are constants $c,c'>0$ and $C>0$ such that

$$\begin{aligned}&|I^{(2)}(\xi,t)|\\ &\quad\leqslant C\sum_{\nu\leqslant\alpha}\sum_{\tau<\nu}|\partial^\tau e^{-\xi\cdot\mathbf{D}\xi t}||\partial^{\nu-\tau}e^{\mathcal{O}(|\xi|^4)t}||\partial^{\alpha-\nu}\chi_1(\xi)|\\ &\quad\leqslant C\sum_{\substack{\nu\leqslant\alpha\\ \tau<\nu}}\sum_{\substack{\{\mathcal{I}_j:j=1,\dots,r\}\\ \{\mathcal{I}'_j:j=1,\dots,r'\}\\ r\leqslant|\tau|,r'\leqslant|\nu-\tau|}}\begin{aligned}&|\partial^{\alpha-\nu}\chi_1(\xi)||\xi|^{m_1+m'_1+2m'_2+3m'_3}\\ &\cdot t^{\ell+\ell'+m_0+m_1+m'_0+m'_1+m'_2+m'_3}e^{-c|\xi|^2t+c'|\xi|^4t},\end{aligned}\end{aligned}$$

where $m_k \geqslant 0$ is the cardinality of the set $\{j \in \{1, \dots, r\} : |\mathcal{I}_j| = |\tau| - k\}$ for $k = 0, 1$ and $m'_k \geqslant 0$ is the cardinality of the set $\{j \in \{1, \dots, r'\} : |\mathcal{I}'_j| = |\nu - \tau| - k\}$ for $k = 0, 1, 2, 3$ and $\ell, \ell' \geqslant 0$ satisfies

$$2\ell < |\tau| - 2m_0 - m_1, \qquad 4\ell' < |\nu - \tau| - 4m'_0 - 3m'_1 - 2m'_2 - m'_3$$

and $\{\mathcal{I}_j : j = 1, \dots, r\}$ is any possible partition of the index set $\mathcal{I}_\tau$ determined by τ and $\{\mathcal{I}'_j : j = 1, \dots, r'\}$ is any possible partition of the index set $\mathcal{I}_{\nu-\tau}$ determined by $\nu - \tau$. Hence, we have

$$\begin{aligned}
&\|I^{(2)}\|_{L^2}^2 \\
&\leqslant C \sum_{\substack{\nu \leqslant \alpha \\ \tau < \nu}} \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\} \\ \{\mathcal{I}'_j : j=1,\dots,r'\} \\ r \leqslant |\tau|, r' \leqslant |\nu-\tau|}} \int_{\mathbb{R}^d} \begin{aligned} &|\partial^{\alpha-\nu}\chi_1(\xi)||\xi|^{2(m_1+m'_1+2m'_2+3m'_3)} \\ &\cdot t^{2(\ell+\ell'+m_0+m_1+m'_0+m'_1+m'_2+m'_3)} e^{-2c|\xi|^2 t} \, d\xi \end{aligned} \\
&\leqslant C \sum_{\substack{\nu \leqslant \alpha \\ \tau < \nu}} \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\}, r \leqslant |\tau| \\ \{\mathcal{I}'_j : j=1,\dots,r'\}, r' \leqslant |\nu-\tau|}} (1+t)^{-\frac{d}{2} + 2m_0 + 2m'_0 + m_1 + m'_1 - m'_3 + 2\ell + 2\ell'} \\
&\leqslant C \sum_{\substack{\nu \leqslant \alpha \\ \tau < \nu}} \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\}, r \leqslant |\tau| \\ \{\mathcal{I}'_j : j=1,\dots,r'\}, r' \leqslant |\nu-\tau|}} (1+t)^{-\frac{d}{2} + |\tau| + |\nu-\tau| - 2(\ell' + m'_0 + m'_1 + m'_2 + m'_3)} \\
&\leqslant C \sum_{\substack{\nu \leqslant \alpha \\ \tau < \nu}} \sum_{\substack{\{\mathcal{I}_j : j=1,\dots,r\}, r \leqslant |\tau| \\ \{\mathcal{I}'_j : j=1,\dots,r'\}, r' \leqslant |\nu-\tau|}} (1+t)^{|\nu| - \frac{d}{2} - 2r'} \leqslant C(1+t)^{|\alpha| - \frac{d}{2} - 2} \qquad (3.164)
\end{aligned}$$

since $|\tau| + |\nu - \tau| = |\nu| \leqslant |\alpha|$ for any $\tau \leqslant \nu \leqslant \alpha$ and $\ell' + m'_0 + m'_1 + m'_2 + m'_3 = r' \geqslant 1$ by definition and $\tau < \nu$.

We continue to estimate J in (3.158). By the Leibniz rule, one has

$$\begin{aligned}
\partial^\alpha J(\xi, t) &= \sum_{\nu \leqslant \alpha} \sum_{\tau \leqslant \nu} \binom{\alpha}{\nu} \binom{\nu}{\tau} \partial^\tau e^{-\xi \cdot \mathbf{D} \xi t + \mathcal{O}(|\xi|^4) t} \partial^{\nu-\tau} \mathcal{O}(|\xi|^2) \partial^{\alpha-\nu} \chi_1(\xi) \\
&= J^{(1)}(\xi, t) + J^{(2)}(\xi, t) + J^{(3)}(\xi, t), \qquad (3.165)
\end{aligned}$$

where

$$J^{(1)}(\xi, t) := \sum_{\nu \leqslant \alpha} \binom{\alpha}{\nu} \partial^\nu e^{-\xi \cdot \mathbf{D} \xi t + \mathcal{O}(|\xi|^4) t} \mathcal{O}(|\xi|^2) \partial^{\alpha-\nu} \chi_1(\xi), \qquad (3.166)$$

$$J^{(2)}(\xi, t) := \sum_{\nu \leqslant \alpha} \sum_{\substack{\tau < \nu \\ |\nu-\tau|=1}} \binom{\alpha}{\nu} \binom{\nu}{\tau} \begin{aligned} &\partial^\tau e^{-\xi \cdot \mathbf{D} \xi t + \mathcal{O}(|\xi|^4) t} \\ &\cdot \partial^{\nu-\tau} \mathcal{O}(|\xi|^2) \partial^{\alpha-\nu} \chi_1(\xi). \end{aligned} \qquad (3.167)$$

and

$$J^{(3)}(\xi, t) := \sum_{\nu \leqslant \alpha} \sum_{\substack{\tau < \nu \\ |\nu-\tau| \geqslant 2}} \binom{\alpha}{\nu} \binom{\nu}{\tau} \begin{aligned} &\partial^\tau e^{-\xi \cdot \mathbf{D} \xi t + \mathcal{O}(|\xi|^4) t} \\ &\cdot \partial^{\nu-\tau} \mathcal{O}(|\xi|^2) \partial^{\alpha-\nu} \chi_1(\xi). \end{aligned} \qquad (3.168)$$

We then begin with $J^{(1)}$. Since $\chi_1 \in C^\infty(\mathbb{R}^d)$ and $\mathbf{D} \in \mathbb{R}^{d \times d}$ is positive definite, from (3.166) and the estimate (3.86) in Lemma 3.13, there are constants $c, c' > 0$ and $C > 0$ such that

$$
\begin{aligned}
|J^{(1)}(\xi,t)| &\leqslant C \sum_{\nu\leqslant\alpha} |\partial^{\nu} e^{-\xi\cdot\mathbf{D}\xi t+\mathcal{O}(|\xi|^4)t}||\mathcal{O}(|\xi|^2)||\partial^{\alpha-\nu}\chi_1(\xi)| \\
&\leqslant C \sum_{\nu\leqslant\alpha} \sum_{\{\mathcal{I}_j : j=1,\dots,r\}} |\xi|^{m_1+2} t^{\ell+m_0+m_1} e^{-c|\xi|^2 t+c'|\xi|^4 t}|\partial^{\alpha-\nu}\chi_1(\xi)|,
\end{aligned}
$$

where $m_k \geqslant 0$ is the cardinality of the set $\{j \in \{1,\dots,r\} : |\mathcal{I}_j| = |\nu| - k\}$ for $k = 0, 1$ and $\ell \geqslant 0$ satisfies

$$
2\ell < |\nu| - 2m_0 - m_1
$$

and $\{\mathcal{I}_j : j = 1,\dots,r\}$ is any possible partition of the index set $\mathcal{I}_\nu$ determined by ν. It then implies that

$$
\begin{aligned}
\|J^{(1)}\|_{L^2}^2 &\leqslant C \sum_{\nu\leqslant\alpha} \sum_{\{\mathcal{I}_j : j=1,\dots,r\}} \int_{\mathbb{R}^d} |\xi|^{2(m_1+2)} t^{2(\ell+m_0+m_1)} \cdot e^{-2c|\xi|^2 t}|\partial^{\alpha-\nu}\chi_1(\xi)|\, d\xi \\
&\leqslant C \sum_{\nu\leqslant\alpha} \sum_{\{\mathcal{I}_j : j=1,\dots,r\}} (1+t)^{-\frac{d}{2}-2+2\ell+2m_0+m_1} \\
&\leqslant C \sum_{\nu\leqslant\alpha} (1+t)^{|\nu|-\frac{d}{2}-2} \leqslant C(1+t)^{|\alpha|-\frac{d}{2}-2}
\end{aligned}
\tag{3.169}
$$

since $|\nu| \leqslant |\alpha|$ for all $\nu \leqslant \alpha$.

We estimate $J^{(2)}$ in (3.167). Since $\chi_1 \in C^\infty(\mathbb{R}^d)$ and $\mathbf{D} \in \mathbb{R}^{d\times d}$ is positive definite, from (3.167) and the estimate (3.86) in Lemma 3.13, there are constants $c, c' > 0$ and $C > 0$ such that

$$
\begin{aligned}
&|J^{(2)}(\xi,t)| \\
&\leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\tau<\nu \\ |\nu-\tau|=1}} |\partial^{\tau} e^{-\xi\cdot\mathbf{D}\xi t+\mathcal{O}(|\xi|^4)t}||\partial^{\nu-\tau}\mathcal{O}(|\xi|^2)||\partial^{\alpha-\nu}\chi_1(\xi)| \\
&\leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\tau<\nu \\ |\nu-\tau|=1}} \sum_{\{\mathcal{I}_j : j=1,\dots,r\}, r\leqslant|\tau|} |\xi|^{m_1+1} t^{\ell+m_0+m_1} e^{-c|\xi|^2 t+c'|\xi|^4 t}|\partial^{\alpha-\nu}\chi_1(\xi)|,
\end{aligned}
$$

where $m_k \geqslant 0$ is the cardinality of the set $\{j \in \{1,\dots,r\} : |\mathcal{I}_j| = |\tau| - k\}$ for $k = 0, 1$ and $\ell \geqslant 0$ satisfies

$$
2\ell < |\tau| - 2m_0 - m_1
$$

and $\{\mathcal{I}_j : j = 1,\dots,r\}$ is any possible partition of the index set $\mathcal{I}_\tau$ determined by τ. Thus, we have

$$
\begin{aligned}
&\|J^{(2)}\|_{L^2}^2 \\
&\leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\tau<\nu \\ |\nu-\tau|=1}} \sum_{\{\mathcal{I}_j : j=1,\dots,r\}, r\leqslant|\tau|} \int_{\mathbb{R}^d} |\xi|^{2(m_1+1)} t^{2(\ell+m_0+m_1)} \cdot e^{-2c|\xi|^2 t+2c'|\xi|^4 t}|\partial^{\alpha-\nu}\chi_1(\xi)|\, d\xi \\
&\leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\tau<\nu \\ |\nu-\tau|=1}} \sum_{\{\mathcal{I}_j : j=1,\dots,r\}, r\leqslant|\tau|} (1+t)^{-\frac{d}{2}-1+2\ell+2m_0+m_1}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\tau<\nu \\ |\nu-\tau|=1}} (1+t)^{|\tau|-\frac{d}{2}-1} \\
&\leqslant C \sum_{\nu\leqslant\alpha} (1+t)^{|\nu|-\frac{d}{2}-2} \leqslant C(1+t)^{|\alpha|-\frac{d}{2}-2}
\end{aligned}
\tag{3.170}
$$

since $|\tau| = |\nu| - |\nu-\tau| = |\nu| - 1$ and $|\nu| \leqslant |\alpha|$ for all $\nu \leqslant \alpha$

We estimate $J^{(3)}$ in (3.168). Since $\chi_1 \in C^\infty(\mathbb{R}^d)$ and $\mathbf{D} \in \mathbb{R}^{d\times d}$ is positive definite, from (3.168) and the estimate (3.86) in Lemma 3.13, there are constants $c, c' > 0$ and $C > 0$ such that

$$
\begin{aligned}
&|J^{(3)}(\xi,t)| \\
&\quad \leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\tau<\nu \\ |\nu-\tau|\geqslant 2}} |\partial^\tau e^{-\xi\cdot\mathbf{D}\xi t+\mathcal{O}(|\xi|^4)t}||\partial^{\nu-\tau}\mathcal{O}(|\xi|^2)||\partial^{\alpha-\nu}\chi_1(\xi)| \\
&\quad \leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\tau<\nu \\ |\nu-\tau|\geqslant 2}} \sum_{\{\mathcal{I}_j:j=1,\ldots,r\},r\leqslant|\tau|} |\xi|^{m_1} t^{\ell+m_0+m_1} e^{-c|\xi|^2t+c'|\xi|^4t}|\partial^{\alpha-\nu}\chi_1(\xi)|,
\end{aligned}
$$

where $m_k \geqslant 0$ is the cardinality of the set $\{j \in \{1,\ldots,r\} : |\mathcal{I}_j| = |\tau| - k\}$ for $k = 0, 1$ and $\ell \geqslant 0$ satisfies

$$
2\ell < |\tau| - 2m_0 - m_1
$$

and $\{\mathcal{I}_j : j = 1,\ldots,r\}$ is any possible partition of the index set $\mathcal{I}_\tau$ determined by τ. Thus, we have

$$
\begin{aligned}
&\|J^{(3)}\|_{L^2}^2 \\
&\quad \leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\tau<\nu \\ |\nu-\tau|\geqslant 2}} \sum_{\{\mathcal{I}_j:j=1,\ldots,r\},r\leqslant|\tau|} \int_{\mathbb{R}^d} \begin{aligned}&|\xi|^{2m_1} t^{2(\ell+m_0+m_1)} \\ &\quad \cdot e^{-2c|\xi|^2t+2c'|\xi|^4t}|\partial^{\alpha-\nu}\chi_1(\xi)|\, d\xi\end{aligned} \\
&\quad \leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\tau<\nu \\ |\nu-\tau|\geqslant 2}} \sum_{\{\mathcal{I}_j:j=1,\ldots,r\},r\leqslant|\tau|} (1+t)^{-\frac{d}{2}+2\ell+2m_0+m_1} \\
&\quad \leqslant C \sum_{\nu\leqslant\alpha} \sum_{\substack{\tau<\nu \\ |\nu-\tau|\geqslant 2}} (1+t)^{|\tau|-\frac{d}{2}} \\
&\quad \leqslant C \sum_{\nu\leqslant\alpha} (1+t)^{|\nu|-\frac{d}{2}-2} \leqslant C(1+t)^{|\alpha|-\frac{d}{2}-2}
\end{aligned}
\tag{3.171}
$$

since $|\tau| = |\nu| - |\nu-\tau| \leqslant |\nu| - 2$ and $|\nu| \leqslant |\alpha|$ for all $\nu \leqslant \alpha$.

Therefore, from (3.163), (3.164), (3.169) and (3.170), one has

$$
\|\partial^\alpha((\hat{G}P_0 - \hat{K})\chi_1)\|_{L^2} \leqslant C(1+t)^{\frac{|\alpha|}{2}-\frac{d}{4}-1} \qquad \forall t \geqslant 0, \alpha \in \mathbb{N}^d. \tag{3.172}
$$

Let $s \in \mathbb{Z}_+$ satisfying $s > d/2$, then by the Carlson–Beurling inequality (A.1) in Lemma A.7 and (3.172), we obtain for $t \geqslant 0$ and $1 \leqslant p \leqslant \infty$ that

$$
\begin{aligned}
&\|(\hat{G}P_0 - \hat{K})\chi_1\|_{M_p} \\
&\quad \leqslant \|(\hat{G}P_0 - \hat{K})\chi_1\|_{L^2}^{1-\frac{d}{2s}} \Big(\sum_{|\alpha|=s} \|\partial^\alpha((\hat{G}P_0 - \hat{K})\chi_1)\|_{L^2} \Big)^{\frac{d}{2s}} \\
&\quad \leqslant C(1+t)^{-(\frac{d}{4}+1)(1-\frac{d}{2s})+(\frac{s}{2}-\frac{d}{4}-1)\frac{d}{2s}} \leqslant C(1+t)^{-1}.
\end{aligned}
\tag{3.173}
$$

It follows from (3.173) and the definition of the M_p-norm that for any $g \in L^p(\mathbb{R}^d)$, one has

$$\|\mathcal{F}^{-1}((\hat{G}P_0 - \hat{K})\chi_1) * g\|_{L^p} \leqslant \|(\hat{G}P_0 - \hat{K})\chi_1\|_{M_p}\|g\|_{L^p} \leqslant C(1+t)^{-1}\|g\|_{L^p}$$

for all $1 \leqslant p \leqslant \infty$ and $t \geqslant 0$. The proof is done. □

Remark 3.14 (Estimates for the remain parts)**.** Due to the proofs of the propositions in the previous subsections, the estimates for the remain parts of $\mathcal{F}^{-1}(\hat{G} - \hat{K} - \hat{W})$ in the one-dimensional space and $\mathcal{F}^{-1}(\hat{G})$ in the multi-dimensional space are similar to before.

In the one-dimensional space, we estimate $\mathcal{F}^{-1}(\hat{G}(I - P_0)\chi_1)$, $\mathcal{F}^{-1}(\hat{G}\chi_2)$, $\mathcal{F}^{-1}(\hat{W}(\chi_1+\chi_2))$, $\mathcal{F}^{-1}(\hat{K}(\chi_2+\chi_3))$ and $\mathcal{F}^{-1}((\hat{G}-\hat{W})\chi_3)$ separately for $|x| \leqslant Ct$. Moreover, we estimate $\mathcal{F}^{-1}(\hat{G} - \hat{W})$ and $\mathcal{F}^{-1}(\hat{K})$ separately for $|x| > Ct$ where $C > 0$ is large enough.

In the multi-dimensional space, we estimate $\mathcal{F}^{-1}(\hat{G}(I-P_0)\chi_1)$, $\mathcal{F}^{-1}(\hat{G}(\chi_2+\chi_3))$ and $\mathcal{F}^{-1}(\hat{K}(\chi_2 + \chi_3))$ independently.

Noting that in the one-dimensional space, the above terms are bounded by $Ce^{-ct}e^{-\frac{|x|^2}{ct}}$ for all $t \geqslant 1$ and some constants $c > 0$ and $C > 0$ due to the fact that they are bounded by Ce^{-ct} for $t \geqslant 1$ (see the proofs of Proposition 3.4, Proposition 3.5 and Proposition 3.6). Indeed, if $|x| \leqslant Ct$, one has $|x|^2/t \leqslant C^2t$ and

$$e^{-ct} = e^{-\frac{c}{2}t}e^{-\frac{c}{2C^2}C^2t} \leqslant e^{-\frac{c}{2}t}e^{-\frac{c}{2C^2}\frac{|x|^2}{t}} \qquad \forall t \geqslant 0.$$

Furthermore, the case $|x| > Ct$ is argued exactly by the same way as in the proof of Proposition 3.10.

In the multi-dimensional space, except $\mathcal{F}^{-1}(\hat{K}(\chi_2 + \chi_3))$ is bounded by $Ce^{-ct}e^{-\frac{|x|^2}{ct}}$ for $x \in \mathbb{R}^d$ with $d \geqslant 2$, the others cannot be bounded by an L^1-integrable function in $x \in \mathbb{R}^d$ since we cannot treat the case $|x| > Ct$ for large C as in the one-dimensional space.

Proof of Theorem 3.22. Recall the operators $\hat{K}$ and $\hat{W}$ given by (3.155) and (3.26) respectively. For $d = 1$ and $u_0 \in W^{1,q}(\mathbb{R})$, let u, U and V be solutions to (3.141), (3.143) and (3.146). Recall the cut-off function χ_j for $j \in \{1, 2, 3\}$, due to Proposition 3.23 under the conditions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{D}$, $\mathscr{E}$ and $\mathscr{S}$, we have

$$u - U - V = \sum_{j=1}^{3} \mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_j) * u_0, \tag{3.174}$$

where $\hat{G}$ which is the Fourier transform of the Green kernel G associated with (3.141) is a solution to (2.32).

We consider the L^∞-L^1 estimate. One has

$$\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_1) = \mathcal{F}^{-1}((\hat{G}P_0 - \hat{K})\chi_1) + \mathcal{F}^{-1}(\hat{G}(I - P_0)\chi_1) - \mathcal{F}^{-1}(\hat{W}\chi_1). \tag{3.175}$$

Due to the estimate (3.31), the Young inequality in Lemma A.1 and the fact that $\mathcal{F}^{-1} : L^1 \to L^\infty$, one primarily has

$$\|\mathcal{F}^{-1}(\hat{W}\chi_1) * u_0\|_{L^\infty} \leqslant C\|\hat{W}\chi_1\|_{L^1}\|u_0\|_{L^1} \leqslant Ce^{-ct}\|u_0\|_{L^1} \qquad \forall t \geqslant 0. \tag{3.176}$$

Hence, since the condition $\mathscr{E}$ implies the condition $\mathscr{C}$, from (3.95), (3.156) and (3.176), we obtain

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_1) * u_0\|_{L^\infty} \leqslant Ct^{-\frac{3}{2}}\|u_0\|_{L^1} \qquad \forall t \geqslant 1. \tag{3.177}$$

On the other hand, by Remark 3.14 and the Young inequality in Lemma A.1, we thus obtain

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_j) * u_0\|_{L^\infty} \leqslant Ce^{-ct}\|u_0\|_{L^1} \qquad \forall t \geqslant 1,\ j = 2, 3. \tag{3.178}$$

Therefore, it implies that

$$\|u - U - V\|_{L^\infty} \leqslant Ct^{-\frac{3}{2}}\|u_0\|_{L^1} \qquad \forall t \geqslant 1. \tag{3.179}$$

We construct the L^p-L^p estimate for $1 \leqslant p \leqslant \infty$. We primarily introduce the indicator function

$$\tilde{\chi}(x) := \begin{cases} 1 & |x| \leqslant Ct \\ 0 & |x| > Ct \end{cases} \qquad \forall x \in \mathbb{R},\ t \geqslant 0.$$

We have

$$\begin{aligned} \|u - U - V\|_{L^p} = \|&\mathcal{F}^{-1}(\hat{G} - \hat{K} - \hat{W})\tilde{\chi} * u_0\|_{L^p} \\ &+ \|\mathcal{F}^{-1}(\hat{G} - \hat{K} - \hat{W})(1 - \tilde{\chi}) * u_0\|_{L^p}. \end{aligned} \tag{3.180}$$

On the other hand, from (3.175), we have

$$\begin{aligned} &\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_1)\tilde{\chi} * u_0\|_{L^p} \\ &\quad = \|\mathcal{F}^{-1}((\hat{G}P_0 - \hat{K})\chi_1)\tilde{\chi} * u_0\|_{L^p} + \|\mathcal{F}^{-1}(\hat{G}(I - P_0)\chi_1)\tilde{\chi} * u_0\|_{L^p} \\ &\qquad + \|\mathcal{F}^{-1}(\hat{W}\chi_1)\tilde{\chi} * u_0\|_{L^p}. \end{aligned} \tag{3.181}$$

Moreover, by Remark 3.14, for all $t \geqslant 1$, we have

$$\|\mathcal{F}^{-1}(\hat{G}(I - P_0)\chi_1)\tilde{\chi}\|_{L^1} + \|\mathcal{F}^{-1}(\hat{W}\chi_1)\tilde{\chi}\|_{L^1} \leqslant Ce^{-ct}. \tag{3.182}$$

Thus, it follows from Proposition 3.25, the Young inequality in Lemma A.1 and (3.182) that

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_1)\tilde{\chi} * u_0\|_{L^p} \leqslant Ct^{-1}\|u_0\|_{L^p}. \tag{3.183}$$

Furthermore, by Remark 3.14, the Young inequality in Lemma A.1 and Proposition 3.10, for $1 \leqslant p \leqslant \infty$, we have the estimates

$$\|\mathcal{F}^{-1}((\hat{G} - \hat{K} - \hat{W})\chi_j)\tilde{\chi} * u_0\|_{L^p} \leqslant Ce^{-ct}\|u_0\|_{L^p} \quad \forall t \geqslant 1,\ j = 2, 3, \tag{3.184}$$

and the estimate

$$\|\mathcal{F}^{-1}(\hat{G} - \hat{K} - \hat{W})(1 - \tilde{\chi}) * u_0\|_{L^p} \leqslant Ce^{-ct}\|u_0\|_{L^p} \quad \forall t \geqslant 1. \tag{3.185}$$

It then follows from (3.180), (3.183), (3.184) and (3.185) that

$$\|u - U - V\|_{L^p} \leqslant Ct^{-1}\|u_0\|_{L^p} \qquad \forall t \geqslant 1,\ 1 \leqslant p \leqslant \infty. \tag{3.186}$$

The proof is thus done by applying the interpolation inequality in Lemma A.2 between (3.179) and (3.186). □

Proof of Theorem 3.22. For $u_0 \in W^{1,q}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, let u and U be solutions to (3.141) and (3.143) respectively. Recall the cut-off function χ_j for $j \in \{1, 2, 3\}$ and the eigenprojection $P_0(i\xi)$ associated with the eigenvalue $\lambda_0(i\xi)$ of $E(i\xi)$, where $\lambda_0(i\xi) \to 0$ as $|\xi| \to 0$. One sets

$$u^{(1)} := \mathcal{F}^{-1}(\hat{G}P_0\chi_1) * u_0 \tag{3.187}$$

and

$$u^{(2)} := \mathcal{F}^{-1}(\hat{G}(I - P_0)\chi_1) * u_0 + \sum_{j=2}^{3} \mathcal{F}^{-1}(\hat{G}\chi_j) * u_0, \tag{3.188}$$

where $\hat{G}$ is a solution to (2.32). It is then easily to check that $u = u^{(1)} + u^{(2)}$.

On the other hand, from (3.187), one has

$$u^{(1)} - U = \mathcal{F}^{-1}((\hat{G}P_0 - \hat{K})\chi_1) * u_0 + \sum_{j=2}^{3} \mathcal{F}^{-1}(\hat{K}\chi_j) * u_0, \tag{3.189}$$

where $\hat{K}$ is given by (3.155). Hence, by Proposition 3.24, Remark 3.14 and the Young inequality in Lemma A.1, we obtain

$$\|u^{(1)} - U\|_{L^\infty} \leqslant Ct^{-\frac{d}{2}-1}\|u_0\|_{L^1} \qquad \forall t \geqslant 1. \tag{3.190}$$

By Proposition 3.25 and the Young inequality in Lemma A.1, for $1 \leqslant p \leqslant \infty$, we also have

$$\|u^{(1)} - U\|_{L^p} \leqslant Ct^{-1}\|u_0\|_{L^p} \qquad \forall t \geqslant 1. \tag{3.191}$$

Therefore, by Lemma A.2 in Appendix A, it follows from (3.190) and (3.191) that

$$\|u^{(1)} - U\|_{L^p} \leqslant Ct^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-1}\|u_0\|_{L^q}$$

for $1 \leqslant q \leqslant p \leqslant \infty$ and for all $t \geqslant 1$.

Also by Proposition 3.18, Proposition 3.19 and Proposition 3.20, from (3.188), we have

$$\|u^{(2)}\|_{L^2} \leqslant Ce^{-ct}\|u_0\|_{L^2} \qquad \forall t \geqslant 0.$$

The proof is done since the estimate for $U = \mathcal{F}^{-1}(\hat{K}) * u_0$ is similar. □

Appendix A

Lebesgue spaces and the Fourier transform

This chapter is to briefly recall some elementary aspects of the Lebesgue spaces $L^p(\mathbb{R}^d)$ and the Fourier transform of functions in these spaces. Moreover, the Fourier transform considered here is an extension of the classical Fourier transform which is defined for $L^1 \cap L^2$-functions and can be applied to any L^p-functions for $p \geqslant 1$.

A.1 Lebesgue spaces

The aim of this section is not only to give the definition of L^p-spaces but also to introduce the two well-known inequalities which are the Young inequality and the complex interpolation inequality.

Definition A.1 (L^p-space)**.** Let u be a function from $\mathbb{R}^d$ to a Banach space equipped with a norm $|\cdot|$, we define the Lebesgue spaces $L^p(\mathbb{R}^d)$ for $1 \leqslant p \leqslant \infty$ consisting of functions u which satisfy

$$\|u\|_{L^p} := \Big(\int_{\mathbb{R}^d} |u(x)|^p \, dx\Big)^{\frac{1}{p}} < +\infty, \qquad 1 \leqslant p < \infty,$$

and satisfy

$$\|u\|_{L^\infty} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |u(x)| < +\infty.$$

Then, the following hold.

Lemma A.1 (Young's inequality)**.** *For* $1 \leqslant p, q, r \leqslant \infty$ *satisfying* $1/p + 1/q = 1 + 1/r$ *and any* $f \in L^p(\mathbb{R}^d)$ *and* $g \in L^q(\mathbb{R}^d)$*, one has* $f * g \in L^r(\mathbb{R}^d)$ *and the inequality*

$$\|f * g\|_{L^r} \leqslant \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. See the proof of Lemma 1.4 p. 5 in [1]. □

Lemma A.2 (Complex interpolation inequality)**.** *Consider a linear operator* T *which continuously maps* $L^{p_j}(\mathbb{R}^d)$ *into* $L^{q_j}(\mathbb{R}^d)$ *for* $1 \leqslant p_j, q_j \leqslant \infty$ *with* $j \in \{0, 1\}$*. Let* $\theta \in [0, 1]$ *be such that*

$$\Big(\frac{1}{p_\theta}, \frac{1}{q_\theta}\Big) := (1 - \theta)\Big(\frac{1}{p_0}, \frac{1}{q_0}\Big) + \theta\Big(\frac{1}{p_1}, \frac{1}{q_1}\Big),$$

then T *continuously maps* $L^{p_\theta}(\mathbb{R}^d)$ *into* $L^{q_\theta}(\mathbb{R}^d)$ *and one has*

$$\|T\|_{\mathcal{L}(L^{p_\theta};L^{q_\theta})} \leqslant \|T\|^{1-\theta}_{\mathcal{L}(L^{p_0};L^{q_0})}\|T\|^{\theta}_{\mathcal{L}(L^{p_1};L^{q_1})}.$$

Proof. See the proof of Corollary 1.12 p. 12 in [1]. □

A.2 The Fourier transform

Let $\alpha \in \mathbb{N}^d$ be the multi-index $\alpha := (\alpha_1, \dots, \alpha_d)$ with $\alpha_j \in \mathbb{N}$. Let

$$\partial^\alpha u := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}},$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_d$, be a higher-order partial derivative of a smooth function u on $\mathbb{R}^d$.

Definition A.2 (Schwartz space)**.** The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the set of smooth functions u on $\mathbb{R}^d$ such that for any $k \in \mathbb{N}$, we have

$$\|u\|_{k,\mathcal{S}} := \sup_{|\alpha|\leqslant k,\, x\in\mathbb{R}^d} (1+|x|)^k |\partial^\alpha u(x)| < +\infty.$$

Definition A.3 (Tempered distribution)**.** One denotes by $\mathcal{S}'(\mathbb{R}^d)$ the dual space of $\mathcal{S}(\mathbb{R}^d)$ and $u \in \mathcal{S}'(\mathbb{R}^d)$ is called a tempered distribution.

For $u \in \mathcal{S}(\mathbb{R}^d)$, the Fourier transform $\hat{u} = \mathcal{F}(u)$ is defined by the integral

$$\mathcal{F}(u)(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\, dx,$$

where $\cdot$ is the usual scalar product on $\mathbb{R}^d$. One has

Proposition A.3. $\mathcal{F}$ *is a linear continuous map from* $\mathcal{S}(\mathbb{R}^d)$ *into* $\mathcal{S}(\mathbb{R}^d)$. *Moreover, it is an automorphism of* $\mathcal{S}(\mathbb{R}^d)$ *and the inverse map which is denoted by* $\mathcal{F}^{-1}$ *satisfies that for* $u \in \mathcal{S}(\mathbb{R}^d)$

$$\mathcal{F}^{-1}(u)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} u(\xi)\, d\xi.$$

Proof. See the proof of Theorem 1.19 p. 17 in [1]. □

On the other hand, we can define the Fourier transform of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ by the product $\langle \cdot, \cdot \rangle$, namely

$$\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle = \int_{\mathbb{R}^d} u(\xi)\hat{\phi}(\xi)\, d\xi, \qquad \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

It is well defined due to the following

Proposition A.4. *For any linear continuous map* A *from* $\mathcal{S}(\mathbb{R}^d)$ *into* $\mathcal{S}(\mathbb{R}^d)$, *the product* $\langle u, A\phi \rangle$ *for* $u \in \mathcal{S}'(\mathbb{R}^d)$ *and* $\phi \in \mathcal{S}(\mathbb{R}^d)$ *defines a tempered distribution denoted by* $A^t u$ *such that* $\langle A^t u, \phi \rangle = \langle u, A\phi \rangle$ *for all* $\phi \in \mathcal{S}(\mathbb{R}^d)$ *and* A^t *is a linear continuous map from* $\mathcal{S}'(\mathbb{R}^d)$ *into* $\mathcal{S}'(\mathbb{R}^d)$ *in the sense that if* $\langle u_n, \phi \rangle \to \langle u, \phi \rangle$ *as* $n \to +\infty$ *for all* $\phi \in \mathcal{S}(\mathbb{R}^d)$, *then* $\langle A^t u_n, \phi \rangle \to \langle A^t u, \phi \rangle$ *as* $n \to +\infty$ *for all* $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Proof. See the proof of Proposition 1.23 p. 19 in [1]. □

Proposition A.5 (Plancherel formula)**.** *$\mathcal{F}$ is an automorphism of $\mathcal{S}'(\mathbb{R}^d)$ and the inverse map which is denoted by $\mathcal{F}^{-1}$ satisfies that for $u \in \mathcal{S}'(\mathbb{R}^d)$*

$$\langle \mathcal{F}^{-1}(u), \phi \rangle = \langle \nu, \mathcal{F}^{-1}(\phi) \rangle, \qquad \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

Moreover, $\mathcal{F}$ is an automorphism of $L^2(\mathbb{R}^d)$ and one has $\|\hat{u}\|_{L^2} = (2\pi)^{\frac{d}{2}} \|u\|_{L^2}$ for all $u \in L^2(\mathbb{R}^d)$.

Proof. See the proof of Theorem 1.25 p. 22 in [1]. □

On the other hand, we have the following important properties of the Fourier transform.

Proposition A.6. *For any $(u, \phi) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$, $\lambda \in \mathbb{R} \neq \{0\}$, $(a, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$, we have*

$$(i\partial)^\alpha \hat{u} = \mathcal{F}(x^\alpha u), \quad (i\xi)^\alpha \hat{u} = \mathcal{F}(\partial^\alpha u), \quad e^{-ia\cdot\omega}\hat{u} = \mathcal{F}(\tau_a u),$$
$$\tau_\omega \hat{u} = \mathcal{F}(e^{ix\cdot\omega} u), \quad \lambda^{-d}\hat{u}(\lambda^{-1}\xi) = \mathcal{F}(u(\lambda x)), \quad \mathcal{F}(u * \phi) = \hat{u}\hat{\phi},$$

where τ_a denotes the translation operator $\tau_a u = u(\cdot - a)$ and $u * \phi$ is the convolution of u and ϕ.

Proof. See the proof of Proposition 1.24 p. 21 in [1]. □

We finally introduce a powerful Fourier multiplier estimate which is the estimate (A.1) given by Lemma A.7. The multiplier estimates are very helpful to study the L^p-L^p estimate for $1 \leqslant p \leqslant \infty$.

Definition A.4. Let $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^d)$ consists of tempered distributions u such that $\hat{u} \in L^2_{loc}(\mathbb{R}^d)$ and

$$\|u\|_{H^s} := \Big(\int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \Big)^{1/2} < +\infty.$$

Definition A.5. Let $\rho \in \mathcal{S}'(\mathbb{R}^d)$, ρ is called a Fourier multiplier on $L^p(\mathbb{R}^d)$ for $1 \leqslant p \leqslant \infty$ if the convolution $\mathcal{F}^{-1}(\rho(\xi)) * \phi \in L^p(\mathbb{R}^d)$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ and if

$$\|\rho\|_{M_p} := \sup_{\|\phi\|_{L^p}=1} \|\mathcal{F}^{-1}(\rho(\xi)) * \phi\|_{L^p} < +\infty.$$

The linear space of all Fourier multipliers ρ is denoted by $M_p(\mathbb{R}^d)$ and is equipped with the norm $\|\cdot\|_{M_p}$.

Lemma A.7 (Carlson–Beurling)**.** *If $\rho \in H^s(\mathbb{R}^d)$ for $s > d/2$, $\rho \in M_p(\mathbb{R}^d)$ and for some constant $C > 0$, one has the estimate*

$$\|\rho\|_{M_p} \leqslant C \|\rho\|_{L^2}^{1-\frac{d}{2s}} \Big(\sum_{|\alpha|=s} \|\partial^\alpha \rho\|_{L^2} \Big)^{\frac{d}{2s}}, \qquad 1 \leqslant p \leqslant \infty. \tag{A.1}$$

Proof. See the proof of Lemma 6.1.5 p. 135 in [4]. □

Appendix B

Perturbation theory for linear operators

Consider the operator $T(z)$ for $z \in \mathbb{C}$ having the form

$$T(z) = T^{(0)} + zT^{(1)} + z^2T^{(2)} + \dots, \qquad T^{(j)} \in \mathbb{R}^{n \times n}. \tag{B.1}$$

Exceptional points of the analytic operator $T(z)$ for $z \in \mathbb{C}$ in (B.1) are defined to be points where the eigenvalues of $T(z)$ intersect. Nonetheless, they are of finite number in the plane. In the domain excluding these points, the operator $T(z)$ has p holomorphic distinct eigenvalues with constant algebraic multiplicities. Moreover, the p eigenprojections and the p eigennilpotents associated with them are also holomorphic. In fact, the eigenvalues of $T(z)$ are solutions to the dispersion polynomial

$$\det(T(z) - \mu I) = 0$$

with holomorphic coefficients. The eigenvalues of $T(z)$ are then branches of one or more than one analytic functions with algebraic singularities of at most order n. As a consequence, the number of eigenvalues of $T(z)$ is a constant except for a number of points which is finite in each compact set of the plane. The exceptional points can be either regular points of the analytic functions or branch-points of some eigenvalues of $T(z)$. In the former case, the eigenprojections and the eigennilpotents associated with the eigenvalues are bounded while in the latter case, they have poles at the exceptional points even if the eigenvalues are continuous there (see [36]).

We study the behavior of the eigenvalues of $T(z)$ and the associated eigenprojections and eigennilpotents near an exceptional point. Without loss of generality, we assume that the exceptional point is the point $0 \in \mathbb{C}$. Let $\lambda^{(0)}$ be an eigenvalue of $T^{(0)}$ with algebraic multiplicity $m \geqslant 1$ and let $P^{(0)}$ and $N^{(0)}$ be the associated eigenprojection and eigennilpotent. One has

$$T^{(0)}P^{(0)} = P^{(0)}T^{(0)} = P^{(0)}T^{(0)}P^{(0)} = \lambda^{(0)}P^{(0)} + N^{(0)},$$
$$\dim P^{(0)} = m, \quad (N^{(0)})^m = O, \quad P^{(0)}N^{(0)} = N^{(0)}P^{(0)}.$$

The eigenvalue $\lambda^{(0)}$ is in general split into several eigenvalues of $T(z)$ for small $z \neq 0$. The set of these eigenvalues is called the $\lambda^{(0)}$-group. The total projection $P(z)$ of this group is holomorphic at $z = 0$ and is approximated by

$$P(z) = P^{(0)} + zP^{(1)} + z^2P^{(2)} + \mathcal{O}(|z|^3), \tag{B.2}$$

where $P^{(j)}$ can be computed in terms of the coefficients $T^{(j)}$ in (B.1) and the coefficients $N^{(0)}$, $P^{(0)}$ and $Q^{(0)}$ given respectively by $N^{(0)} = (T^{(0)} - \lambda^{(0)} I)P^{(0)}$ and

$$P^{(0)} = -\frac{1}{2\pi i}\int_\Gamma (T^{(0)} - \mu I)^{-1}\, d\mu, \quad Q^{(0)} = \frac{1}{2\pi i}\int_\Gamma \mu^{-1}(T^{(0)} - \mu I)^{-1}\, d\mu, \tag{B.3}$$

where Γ, in the resolvent set of $T^{(0)}$, is an oriented closed curve enclosing $\lambda^{(0)}$ except for the other eigenvalues of $T^{(0)}$. In fact, from [36] (eq. (2.13) p. 76), one has

$$P^{(1)} = \sum_{i+j=1} X^{(i)} T^{(1)} X^{(j)}, \tag{B.4}$$

$$P^{(2)} = \sum_{i+j=1} X^{(i)} T^{(2)} X^{(j)} - \sum_{i+j+h=2} X^{(i)} T^{(1)} X^{(j)} T^{(1)} X^{(h)}, \tag{B.5}$$

where

$$X^{(0)} = -P^{(0)}, \qquad X^{(i)} = (Q^{(0)})^i, \qquad X^{(-i)} = -(N^{(0)})^i, \qquad \forall i \geqslant 1. \tag{B.6}$$

Moreover, the subspace $\operatorname{ran} P(z) := P(z)\mathbb{C}^n$ is m-dimensional and invariant under $T(z)$. The $\lambda^{(0)}$-group eigenvalues of $T(z)$ are identical with all the eigenvalues of $T(z)$ in $\operatorname{ran} P(z)$. In order to determine the $\lambda^{(0)}$-group eigenvalues, therefore, we only have to solve an eigenvalue problem in the subspace $\operatorname{ran} P(z)$, which is in general smaller than the whole space $\mathbb{C}^n$.

The eigenvalue problem for $T(z)$ in $\operatorname{ran} P(z)$ is equivalent to the eigenvalue problem for

$$T_r(z) = T(z)P(z) = P(z)T(z) = P(z)T(z)P(z). \tag{B.7}$$

Thus, the $\lambda^{(0)}$-group eigenvalues of $T(z)$ are exactly those eigenvalues of $T_r(z)$ which are different from 0, provided $|\lambda^{(0)}|$ is large enough to ensure that these eigenvalues do not vanish for the small z under consideration. The last condition does not restrict the generality, for $T^{(0)}$ could be replaced by $T^{(0)} + \alpha$ with a suitable scalar α without changing the nature of the problem (see [36]).

We also have the following result in [36].

Lemma B.1 (A simple case)**.** *If $T(z) = T^{(0)} + zT^{(1)}$ and $\lambda^{(0)}$ is a simple eigenvalue of $T^{(0)}$, the eigenvalue $\lambda(z)$ of $T(z)$ converging to $\lambda^{(0)}$ as $|z| \to 0$ and its associated eigenprojection $P(z)$ are holomorphic at $z = 0$. Moreover, for small $z \neq 0$, $P(z)$ is approximated by* (B.2) *with the coefficients $P^{(j)}$ for $j = 0, 1, 2, \dots$ and $\lambda(z)$ is approximated by*

$$\lambda(z) = \lambda^{(0)} + z\lambda^{(1)} + z^2\lambda^{(2)} + \mathcal{O}(|z|^3), \tag{B.8}$$

where

$$\lambda^{(j)} = \frac{1}{j}\operatorname{tr}(T^{(1)} P^{(j-1)}), \qquad j = 1, 2, 3, \dots \tag{B.9}$$

On the other hand, the eigennilpotent associated with $\lambda(z)$ which is $N(z) = \big(T(z) - \lambda(z)I\big)P(z)$ vanishes identically.

Proof. For any eigenvalue $\lambda^{(0)}$ of $T^{(0)}$ with algebraic multiplicity $m \geqslant 1$, one primarily considers the weighted mean of the $\lambda^{(0)}$-group defined by

$$\hat{\lambda}(z) := \frac{1}{m}\operatorname{tr}\big(T(z)P(z)\big) = \lambda^{(0)} + \frac{1}{m}\operatorname{tr}\big((T(z) - \lambda^{(0)} I)P(z)\big),$$

where $\mathsf{P}(z)$ is the total projection associated with the $\lambda^{(0)}$-group.

We study the asymptotic expansions of $\hat{\lambda}(z)$ and $\mathsf{P}(z)$ for small $z \neq 0$. The expansion of $\mathsf{P}(z)$ is in fact given by (B.2), and also following [36] (eq. (2.8) p. 76), the coefficient $\mathsf{P}^{(j)}$ in (B.2) is computed by

$$\mathsf{P}^{(j)} = -\frac{1}{2\pi i} \sum_{\substack{\nu_1+\cdots+\nu_p=j \\ \nu_i \geqslant 1,\, i=1,\dots,p}} (-1)^p \int_\Gamma \mathsf{R}^{(0)}(\zeta)\mathsf{T}^{(\nu_1)}\mathsf{R}^{(0)}(\zeta)\mathsf{T}^{(\nu_2)} \dots \mathsf{T}^{(\nu_p)}\mathsf{R}^{(0)}(\zeta)\, d\zeta, \tag{B.10}$$

where $\mathsf{T}^{(\nu_i)}$ for $i \in \{1,\dots,p\}$ are the coefficients in (B.1), $\mathsf{R}^{(0)}(\zeta) := (\mathsf{T}^{(0)} - \zeta I)^{-1}$ is the resolvent of $\mathsf{T}^{(0)}$ and Γ is a small positively-oriented circle around $\lambda^{(0)}$. On the other hand, following [36] (eq. (2.21) p.78 and eq. (2.30) p.79), the weighted mean $\hat{\lambda}(z)$ of the $\lambda^{(0)}$-group is approximated by

$$\hat{\lambda}(z) = \lambda^{(0)} + z\hat{\lambda}^{(1)} + z^2\hat{\lambda}^{(2)} + \mathcal{O}(|z|^3), \tag{B.11}$$

where the coefficient $\hat{\lambda}^{(j)}$ is given by

$$\hat{\lambda}^{(j)} = \frac{1}{2\pi i m} \mathrm{tr}\Big(\sum_{\substack{\nu_1+\cdots+\nu_p=j \\ \nu_i \geqslant 1,\, i=1,\dots,p}} \frac{(-1)^p}{p} \int_\Gamma \mathsf{T}^{(\nu_1)}\mathsf{R}^{(0)}(\zeta) \dots \mathsf{R}^{(0)}(\zeta)\mathsf{T}^{(\nu_p)}\mathsf{R}^{(0)}(\zeta)\, d\zeta \Big), \tag{B.12}$$

where the relative coefficients are introduced before.

In the case where $\mathsf{T}(z) = \mathsf{T}^{(0)} + z\mathsf{T}^{(1)}$, one has $\mathsf{T}^{(j)} = \mathsf{O}$, the null matrix, for $j \geqslant 2$. Furthermore, since ν_i in (B.10) and (B.12) satisfy $\nu_i \geqslant 1$, it implies that the relevances are $\nu_i = 1$ for all i. Hence, we obtain from (B.10) and (B.12) that $p = j$ and

$$\begin{aligned} \hat{\lambda}^{(j)} &= \frac{1}{mj} \mathrm{tr}\left(\mathsf{T}^{(1)}\Big(\frac{(-1)^j}{2\pi i} \int_\Gamma \mathsf{R}^{(0)}(\zeta)\mathsf{T}^{(1)} \dots \mathsf{T}^{(1)}\mathsf{R}^{(0)}(\zeta)\, d\zeta\Big)\right) \\ &= \frac{1}{mj} \mathrm{tr}\left(\mathsf{T}^{(1)}\mathsf{P}^{(j-1)}\right) \end{aligned} \tag{B.13}$$

since there are $j-1$ matrices $\mathsf{T}^{(1)}$ in the integrand in (B.13).

If $\lambda^{(0)}$ is a simple eigenvalue, one has $m = 1$ and $\lambda^{(0)}$ is not split into many eigenvalues of $\mathsf{T}(z)$. Thus, the $\lambda^{(0)}$-group contains only one single eigenvalue $\lambda(z)$ of $\mathsf{T}(z)$ converging to $\lambda^{(0)}$ as $|z| \to 0$. Hence, $\lambda(z) = \hat{\lambda}(z)$ and the eigenprojection associated with $\lambda(z)$ is exactly the total projection $\mathsf{P}(z)$ of the $\lambda^{(0)}$-group. Therefore, one obtains the expansion (B.8) from (B.11) and one obtains the formula (B.9) from (B.13) where $m = 1$. The eigenilpotent $\mathsf{N}(z)$ associated with $\lambda(z)$ is obviously null since $\lambda(z)$ is simple. The proof is done. □

Moreover, one obtains the following result from Lemma B.1.

Corollary B.2 (Symmetry)**.** *Under the same assumptions in Lemma B.1, if in addition, there is an invertible matrix* S *such that* $\mathsf{S}\mathsf{T}^{(1)} = -\mathsf{T}^{(1)}\mathsf{S}$ *and* $\mathsf{S}\mathsf{T}^{(0)} = \mathsf{T}^{(0)}\mathsf{S}$*, then* $\lambda^{(j)} = 0$ *for all* j *odd, where* $\lambda^{(j)}$ *is the* j*-th coefficient in the formulas* (B.8) *and* (B.9) *for* $j = 1, 2, \dots$

Proof. Recall $T(z) = T^{(0)} + zT^{(1)}$, one can study the eigenvalue problem for $T(z)$ by considering the operator

$$\begin{aligned} T_S(z) &:= ST(z)S^{-1} = ST^{(0)}S^{-1} + zST^{(1)}S^{-1} \\ &= T^{(0)} - zT^{(1)} = T_S^{(0)} + zT_S^{(1)}, \end{aligned} \tag{B.14}$$

where $T_S^{(0)} := T^{(0)}$ and $T_S^{(1)} := -T^{(1)}$. Thus, Lemma B.1 is applied to $T_S(z)$ since $\lambda^{(0)}$ is also a simple eigenvalue of $T_S^{(0)}$. It implies that the eigenvalue $\lambda_S(z)$ of $T_S(z)$ converging to $\lambda^{(0)}$ as $|z| \to 0$ and the associated eigenprojection $P_S(z)$ are holomorphic at $z = 0$. Moreover, for small $z \neq 0$, the expansion of $P_S(z)$ is given by the expansion (B.2) with coefficients denoted by $P_S^{(j)}$ for $j = 0, 1, 2, \ldots$ and $\lambda_S(z)$ is approximated by

$$\lambda_S(z) = \lambda^{(0)} + z\lambda_S^{(1)} + z^2\lambda_S^{(2)} + \mathcal{O}(|z|^3),$$

where

$$\lambda_S^{(j)} = \frac{1}{j}\mathrm{tr}\,(T_S^{(1)}P_S^{(j-1)}), \qquad j = 1, 2, 3, \ldots \tag{B.15}$$

On the other hand, the eigennilpotent $N_S(z)$ associated with $\lambda_S(z)$ vanishes identically.

Consider the total projection $P_S(z)$ associated with the $\lambda^{(0)}$-group of $T_S(z)$ in (B.2) with the coefficients $P_S^{(j)}$. We also consider the formula (B.10) of $P_S^{(j)}$, namely

$$P_S^{(j)} = -\frac{1}{2\pi i} \sum_{\substack{\nu_1+\cdots+\nu_p=j \\ \nu_i \geqslant 1,\, i=1,\ldots,p}} (-1)^p \int_\Gamma R_S^{(0)}(\zeta)T_S^{(\nu_1)}R_S^{(0)}(\zeta)T_S^{(\nu_2)} \ldots T_S^{(\nu_p)}R_S^{(0)}(\zeta)\,d\zeta,$$

where $T_S^{(\nu_i)}$ for $i \in \{1, \ldots, p\}$ are the coefficients in the expansion (B.1) of $T_S(z)$, $R_S^{(0)}(\zeta) := (T_S^{(0)} - \zeta I)^{-1}$ is the resolvent of $T_S^{(0)}$ and Γ is a small positively-oriented circle around $\lambda^{(0)}$. Then, since $T_S^{(\nu_i)} = O$ for all $\nu_i \geqslant 2$ and since $\nu_i \geqslant 1$ for all i, one has $p = j$ and

$$P_S^{(j)} = -\frac{1}{2\pi i}(-1)^j \int_\Gamma R_S^{(0)}(\zeta)T_S^{(1)}R_S^{(0)}(\zeta)T_S^{(1)} \ldots T_S^{(1)}R_S^{(0)}(\zeta)\,d\zeta,$$

where there are j matrices $T_S^{(1)}$ in the integrand.

Since $T_S^{(0)} = T^{(0)}$ and $T_S^{(1)} = -T^{(1)}$, it follows that for all j, one has

$$P_S^{(j)} = \begin{cases} P^{(j)} & \text{if } j \text{ is even}, \\ -P^{(j)} & \text{if } j \text{ is odd}, \end{cases} \tag{B.16}$$

where $P^{(j)}$ is the j-th coefficient in the expansion of the total projection $P(z)$ associated with the $\lambda^{(0)}$-group of $T(z) = T^{(0)} + zT^{(1)}$.

Hence, from (B.9), (B.15) and (B.16), we have

$$\lambda_S^{(j)} = \begin{cases} \lambda^{(j)} & \text{if } j \text{ is even}, \\ -\lambda^{(j)} & \text{if } j \text{ is odd}, \end{cases} \tag{B.17}$$

where λ_j is the j-th coefficient in the expansion of the eigenvalue $\lambda(z)$ of $T(z) = T^{(0)} + zT^{(1)}$ converging to $\lambda^{(0)}$ as $|z| \to 0$.

Finally, since $\lambda_S(z) \equiv \lambda(z)$ due to (B.14) and the fact that they are single eigenvalues, we deduce from (B.17) that $\lambda^{(j)} = -\lambda^{(j)} = 0$ for all j odd. We finish the proof. □

Let $\sigma(T, \mathcal{D})$ be the spectrum of T considered in a domain $\mathcal{D}$, we finish this chapter by introducing the reduction method in [36] which can be applied to the semi-simple-eigenvalue case.

Lemma B.3 (Reduction process). *Let $T(z)$ be in (B.1) with the coefficients $T^{(i)}$ for $i = 0, 1, 2, \dots$ and let $\lambda^{(0)}$ be a semi-simple eigenvalue of $T^{(0)}$. Let $P(z)$ in (B.2) with the coefficients $P^{(i)}$ for $i = 0, 1, 2, \dots$ be the total projection of the $\lambda^{(0)}$-group. The following holds for small $z \neq 0$*

$$T(z)P(z) = \sum_{j=1}^{p} (\lambda^{(0)} I + zT_j(z))P_j(z), \tag{B.18}$$

where $T_j(z)$ commutes with $P_j(z)$ and $P_j(z)$ satisfies

$$P_j(z)P_{j'}(z) = \delta_{jj'}P_j(z), \qquad \sum_{j=1}^{p} P_j(z) = P(z). \tag{B.19}$$

The expansions of $T_j(z)$ and $P_j(z)$ are

$$T_j(z) = \lambda_j^{(0)} I + N_j^{(0)} + \mathcal{O}(|z|) \tag{B.20}$$

and

$$P_j(z) = P_j^{(0)} + \mathcal{O}(|z|), \tag{B.21}$$

where $\lambda_j^{(0)} \in \sigma(P^{(0)}T^{(1)}P^{(0)}, \ker(T^{(0)} - \lambda^{(0)}I))$ with the associated eigenprojection $P_j^{(0)}$ and eigennilpotent $N_j^{(0)}$ for $j \in \{1, \dots, p\}$ and p is the cardinality of $\sigma(P^{(0)}T^{(1)}P^{(0)}, \ker(T^{(0)} - \lambda^{(0)}I))$.

Proof. Recall $T(z)$ and the coefficients $T^{(j)}$ in (B.1). Recall the expansion of the total projection $P(z)$ of the $\lambda^{(0)}$-group of $T(z)$ and the coefficients $P^{(j)}$ in (B.2), where the $\lambda^{(0)}$-group is generated by the eigenvalue $\lambda^{(0)}$ of $T^{(0)}$. If $\lambda^{(0)}$ is semi-simple, one obtains from (B.7) that $(T(z) - \lambda^{(0)}I)P(z) = z\tilde{T}(z)$, where

$$\tilde{T}(z) = \tilde{T}^{(0)} + z\tilde{T}^{(1)} + \mathcal{O}(|z|^2), \tag{B.22}$$

where

$$\tilde{T}^{(0)} := P^{(0)}T^{(1)}P^{(0)}, \tag{B.23}$$

$$\tilde{T}^{(1)} := P^{(1)}T^{(0)}P^{(1)} + P^{(1)}T^{(1)}P^{(0)} + P^{(0)}T^{(1)}P^{(1)}. \tag{B.24}$$

Thus, the eigenvalues of $\tilde{T}(z)$ in $\operatorname{ran} P(z)$ are considered and in general, they converge to the eigenvalues of $\tilde{T}^{(0)}$ in $\operatorname{ran} P^{(0)} = \ker(T^{(0)} - \lambda^{(0)}I)$ as $|z| \to 0$ (see Theorem 2.3 p. 82 in [36]). One denotes the distinct eigenvalues of $\tilde{T}^{(0)}$ considered in $\ker(T^{(0)} - \lambda^{(0)}I)$ by $\lambda_j^{(0)}$ for $j \in \{1, \dots, p\}$. Then, $\lambda_j^{(0)}$ generates the $\lambda_j^{(0)}$-group of $\tilde{T}(z)$ similarly to the $\lambda^{(0)}$-group of $T(z)$ generated by the eigenvalue $\lambda^{(0)}$ of $T^{(0)}$. Moreover, the total projection $P_j(z)$ of the $\lambda_j^{(0)}$-group commutes with $\tilde{T}(z)$, satisfies (B.19) and is approximated by (B.21).

Applying (B.7) where $\mathsf{T}(z)$ is substituted by $\tilde{\mathsf{T}}(z)$ and $\mathsf{P}(z)$ is substituted by $\mathsf{P}_j(z)$, it follows from (B.21) and (B.22) that

$$\tilde{\mathsf{T}}(z)\mathsf{P}_j(z) = \lambda_j^{(0)}\mathrm{I} + \mathsf{N}_j^{(0)} + \mathcal{O}(|z|), \tag{B.25}$$

where $\mathsf{N}_j^{(0)}$ is the eigennilpotent associated with $\lambda_j^{(0)}$. Let $\mathsf{T}_j(z) := \tilde{\mathsf{T}}(z)\mathsf{P}_j(z)$ and using (B.19), (B.25) and the fact that $\mathsf{T}(z)\mathsf{P}(z) = z\tilde{\mathsf{T}}(z)$, one obtains (B.18) and (B.20). We finish the proof. □

Remark B.1. The reduction process can be continued as soon as the coefficient $\lambda_j^{(0)}$ in (B.20) is semi-simple by applying the process to $\mathsf{T}_j(z)$ and $\mathsf{P}_j(z)$ given by (B.20) and (B.21) respectively.

Remark B.2. In general, if the simplicity of the eigenvalue $\lambda^{(0)}$ of $\mathsf{T}^{(0)}$ is relaxed by the semi-simplicity, then even in the special case of Remark B.1 where $\lambda_j^{(0)}$ is simple, the symmetry criterion in Corollary B.2 that there is an invertible matrix S satisfying $\mathsf{ST}^{(1)} = -\mathsf{T}^{(1)}\mathsf{S}$ and $\mathsf{ST}^{(0)} = \mathsf{T}^{(0)}\mathsf{S}$ is not sufficient for the vanishing of the coefficients associated with z^3 in the expansions of the eigenvalues of $\mathsf{T}(z) = \mathsf{T}^{(0)} + z\mathsf{T}^{(1)}$ converging to $\lambda^{(0)}$ as $|z| \to 0$ unless additional conditions are imposed.

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List of notations

Numbers

$\mathbb{F}$: the field of real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) numbers
$|c|$: the absolute value if $c \in \mathbb{R}$ and the modulus if $c \in \mathbb{C}$
$\mathbb{Z}_+$: the set of all positive integers
$\mathbb{N}$: the set of natural numbers including zero
$\operatorname{Re} z$: the real part of $z \in \mathbb{C}$
$\operatorname{Im} z$: the imaginary part of $z \in \mathbb{C}$

Indices

$\mathbb{N}^d$: the set of indices $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_\ell \in \mathbb{N}$
$|\alpha|$: the sum $\alpha_1 + \cdots + \alpha_d$
$\alpha \leqslant \beta$: if $\alpha_i \leqslant \beta_i$ for all $i \in \{1, \dots, d\}$
$\alpha < \beta$: if $\alpha_i < \beta_i$ for all $i \in \{1, \dots, d\}$
$\beta - \alpha$: the index $(\beta_1 - \alpha_1, \dots, \beta_d - \alpha_d)$ where $\alpha \leqslant \beta$
$\alpha!$: the factorial $\alpha_1! \dots \alpha_d!$
$\binom{\alpha}{\beta}$: the binomial coefficient $\frac{\beta!}{\alpha!(\beta-\alpha)!}$
x^α: $x_1^{\alpha_1} \dots x_d^{\alpha_d}$ where $x = (x_1, \dots, x_d)$ with $x_\ell \in \mathbb{F}$ for $\ell \in \{1, \dots, d\}$

Vectors

$\mathbb{F}^n$: the n-dimensional vector space over a field $\mathbb{F}$
$|u|$: a vector norm of u
$\mathbb{S}^{n-1}$: the unit sphere in $\mathbb{R}^n$
$\bar{u}^t$: the conjugate transpose of $u \in \mathbb{C}^n$
u^t: the transpose of $u \in \mathbb{R}^n$
$u \cdot v$: the scalar product on $\mathbb{R}^n$ of u and v in $\mathbb{R}^n$
$\langle u, v \rangle$: the inner product on $\mathbb{C}^n$ of u and v in $\mathbb{C}^n$

Matrices

$\mathbb{F}^{m \times n}$: the space of $m \times n$ matrices over a field $\mathbb{F}$
$|A|$: an induced matrix norm of A
$\bar{A}^t$: the conjugate transpose of $A \in \mathbb{C}^{m \times n}$
A^t: the transpose of $A \in \mathbb{R}^{m \times n}$
A_{sym}: the symmetric part of A
A_{skew}: the skew-symmetric part of A
$\sigma(A)$: the spectrum of A
$\sigma(A, \mathcal{D})$: the set of the eigenvalues of a matrix A with eigenvectors in a domain $\mathcal{D}$ where $\sigma(A, \mathbb{C}^n) \equiv \sigma(A)$
$\rho(A)$: the spectral radius of A
$(A - zI)^{-1}$: the resolvent of A
$\operatorname{ran} A$: the range space of A
$\ker A$: the kernel of A
$\operatorname{rank} A$: the rank of A

$\det A$: the determinant of A
$\mathrm{tr}\,(A)$: the trace of A
$\mathrm{adj}\,(A)$: the adjunct of A
$[A, B]$: the matrix defined by $AB - BA$
A^k: the k-th power of A with $k \in \mathbb{N}$ and $A^0 = I$ the identity matrix
$A^{(k)}$: a matrix A with an index $k \in \mathbb{N}$
e^A: the sum $\sum_{k=0}^{+\infty} \frac{A^k}{k!}$
$\mathrm{diag}\,(a_1, \dots, a_n)$: the diagonal matrix with diagonal entries given by numbers $a_1, \dots, a_n$
$\mathrm{diag}\,(A_1, \dots, A_n)$: the block diagonal matrix with diagonal entries given by matrices $A_1, \dots, A_n$
I: the identity matrix
O: the zero (or null) matrix
$z \cdot \mathbf{A}$: the sum $\sum_{j=1}^{d} z_j A_j$ of $z = (z_1, \dots, z_d) \in \mathbb{F}^d$ and $\mathbf{A} = (A_1, \dots, A_d)$ with $A_j \in \mathbb{F}^{n \times n}$ for $j \in \{1, \dots, d\}$

Derivatives

∂_x: the first derivative with respect to $x \in \mathbb{R}$
∂_{xx}: the second derivative with respect to $x \in \mathbb{R}$
∂_{x_i}: the first-order partial derivative along the direction $x_i \in \mathbb{R}$
$\partial_{x_i}^k$: the k-order partial derivative along the direction $x_i \in \mathbb{R}$ with $k \in \mathbb{N}$
∇_x: the gradient with respect to $x \in \mathbb{R}^d$
div: the divergence with respect to $x \in \mathbb{R}^d$
Δ_x: the Laplacian with respect to $x \in \mathbb{R}^d$
∂^α: the higher-order partial derivative $\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$ with $\alpha \in \mathbb{N}^d$
$\partial_{\mathcal{J}}$: the higher-order partial derivative $\partial_{x_{i_1}} \dots \partial_{x_{i_s}}$ where $\mathcal{J} = \{i_1, \dots, i_s\}$ with $i_\ell \in \{1, \dots, d\}$ for $\ell \in \{1, \dots, d\}$
f': the Jacobian of f
f'': the Hessian of f

Transforms

$\mathcal{F}$: the Fourier transform
$\mathcal{F}^{-1}$: the inverse Fourier transform
$\hat{f}$: the Fourier transform of f

Function spaces

$C^k(\mathbb{R}^d)$: the space of continuously differentiable functions on $\mathbb{R}^d$ up to order k where $C^0 \equiv C$
$C_c^\infty(\mathbb{R}^d)$: the space of smooth functions on $\mathbb{R}^d$ with compact support
$\mathcal{S}(\mathbb{R}^d)$: the space of smooth functions on $\mathbb{R}^d$ rapidly decreasing at infinity
$\mathcal{S}'(\mathbb{R}^d)$: the dual space of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$
$L^p(\mathbb{R}^d)$: the space of functions u on $\mathbb{R}^d$ such that $|u|^p$ is integrable if $1 \leqslant p < \infty$ and $|u|$ is bounded almost everywhere if $p = \infty$
$W^{1,p}(\mathbb{R}^d)$: the space of functions $u \in L^p(\mathbb{R}^d)$ with (weak) $\nabla_x u \in L^p(\mathbb{R}^d)$
$H^s(\mathbb{R}^d)$: the space of $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $(1 + |\cdot|^2)^s |\hat{u}|^2$ is integrable with $s \in \mathbb{R}$
$C^k([0, +\infty); X)$: the space of continuously differentiable functions from $[0, +\infty)$ to a function space X up to order $k \in \mathbb{N}$ where $C^0 \equiv C$
$L(X, Y)$: the space of all linear map from X to Y where X and Y are two function spaces and $L(X, X) \equiv L(X)$
$\|f\|_X$: the norm of $f \in X$ where X is a function space

Other notations

sgn: the sign function on $\mathbb{R}$
δ_{ij}: the Kronecker delta
$\emptyset$: the empty set
supp: support of a function
min: the minimum
max: the maximum
sup: the supremum
ess sup: the essential supremum
dim: dimension of a space
$o(|x|^\alpha)$: $f(x) = o(|x|^\alpha)$ if $\frac{|f(x)|}{|x|^\alpha} \to 0$ as $|x| \to 0$ for $\alpha > 0$
$\mathcal{O}(|x|^\alpha)$: $f(x) = \mathcal{O}(|x|^\alpha)$ if there is a constant $C > 0$ such that $|f(x)| \leqslant C|x|^\alpha$ for small x and $\alpha > 0$
$f * g$: the convolution of two functions f and g on $\mathbb{R}^d$
$\langle f, g \rangle$: the integral of the product of two functions f and g on $\mathbb{R}^d$