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Analysis of Oscillations and Defect Measures for the Quasineutral Limit in Plasma Physics

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Abstract

We perform a rigorous analysis of the quasineutral limit for a hydrodynamical model of a viscous plasma represented by the Navier–Stokes–Poisson system in three dimensions. We show that as $\lambda \to 0$ the velocity field u^λ strongly converges towards an incompressible velocity vector field u and the density fluctuation $\rho^\lambda - 1$ weakly converges to zero. In general, the limit velocity field cannot be expected to satisfy the incompressible Navier–Stokes equation; indeed, the presence of high frequency oscillations strongly affects the quadratic nonlinearities and we have to take care of self-interacting wave packets. We provide a detailed mathematical description of the convergence process by using microlocal defect measures and by developing an explicit correctors analysis. Moreover, we were able to identify an explicit pseudo-parabolic PDE satisfied by the leading correctors terms. Our results include all the previous results in the literature; in particular, we show that the formal limit holds rigorously in the case of well prepared data.

1. Introduction and Plan of the Paper

1.1. Introduction

In this paper we perform a rigorous analysis of the quasineutral limit for a hydrodynamical model of a viscous plasma represented by the Navier–Stokes–Poisson system in three dimensions:

$$\partial_t \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0, \tag{1}$$

$$\partial_t(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + \nabla(\rho^\lambda)^\gamma = \mu \Delta u^\lambda + (\nu + \mu)\nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda, \tag{2}$$

$$\lambda^2 \Delta V^\lambda = \rho^\lambda - 1, \tag{3}$$

where $x \in \mathbb{R}^3$, $t \geqq 0$, denotes the space and time variable, $\rho(x,t)$ the *negative charge density*, $m(x,t) = \rho(x,t)u(x,t)$ the *current density*, $u(x,t)$ the *velocity field*, $V(x,t)$ the *electrostatic potential*, and μ, ν the *shear viscosity* and *bulk viscosity*, respectively. The parameter λ is the so called *Debye length* (up to a constant factor).

We show that as $\lambda \to 0$ the velocity field u^λ strongly converges towards an incompressible velocity vector field u and the density fluctuation $\rho^\lambda - 1$ weakly converges to zero. In general, the limit velocity field cannot be expected to satisfy the incompressible Navier–Stokes equation; indeed, the presence of high frequency oscillations strongly affects the quadratic nonlinearities and we have to take care of self-interacting wave packets. In the paper we shall provide a detailed analysis of the convergence process by using microlocal defect measures and by developing an explicit correctors analysis. Moreover, we will be able to identify an explicit pseudo-parabolic equation satisfied by by the leading correctors terms.

The system above can be seen as the coupling of the compressible Navier–Stokes equations (1), (2) with a Poisson equation (3), where in dimensionless units the coupling constant can be expressed in terms of a parameter λ representing the scaled Debye length, which is a characteristic physical parameter related to the phenomenon of the so called "Debye shielding", [14], studied by Peter Debye in 1912. Any charged particle inside a plasma attracts other particles with opposite charge and repels those with the same charge, thereby creating a net cloud of opposite charges around itself, this cloud shields the particle's own charge from external view and then causes the particle's Coulomb field to fall off exponentially at large radii, rather than falling off as $1/r^2$. So the physical meaning of the Debye length λ is the distance over which the usual Coulomb field is killed off exponentially by the polarization of the plasma. In terms of physical variables the Debye length can be expressed as

$$\lambda = \lambda_D / L \qquad \lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{\mathrm{e}^2 n_0}}, \tag{4}$$

where L is the macroscopic length scale, ε_0 is the vacuum permittivity, k_B the Boltzmann constant, T the average plasma temperature, e the absolute electron charge and n_0 the average plasma density. In many cases the Debye length is very small compared to the macroscopic length $\lambda_D \ll L$, so it makes sense to consider the quasineutral limit $\lambda \to 0$ of the system (1)–(3). In this situation the particle density is constrained to be close to the background density (equal to one in our case) of the oppositely charged particle. The limit $\lambda \to 0$ is called the quasineutral limit since the charge density almost vanishes identically. The velocity of the fluid then evolves according to the incompressible Navier–Stokes flow. This type of limit has been studied by many authors, in the case of the Euler Poisson system by CORDIER and GRENIER [4], GRENIER [16], CORDIER ET AL. [3], LOEPER [23], PENG ET AL. [24], in the case of the Navier–Stokes–Poisson system by WANG [31] and JIANG and WANG [32] and in the context of a combined quasineutral and relaxation time limit by GASSER and MARCATI [9–11]. This paper is still a mathematical theoretical approach to this complicated physical problem, but it removes many regularity and smallness assumptions of various papers in the literature, see for instance Wang [31] and Jiang and Wang [32]. In fact, Wang [31] studied the quasineutral limit for the smooth solution with well-prepared initial data. Wang and Jiang [32] studied the combined quasineutral and inviscid limit of the compressible Navier–Stokes–Poisson system for weak solutions and obtained the convergence of Navier–Stokes–Poisson systems to the incompressible Euler equations with general initial data. Moreover, in [32] the vanishing of the viscosity coefficient was required in order to take the quasineutral limit, and no convergence rate was derived therein. The authors in [6] investigated the quasineutral limit of the isentropic Navier–Stokes–Poisson system in the whole space and obtained the convergence of weak solutions of the Navier–Stokes–Poisson system to the weak solutions of the incompressible Navier–Stokes equations by means of dispersive estimates of Strichartz's type under the assumption that the Mach number is related to the Debye length. Ju et al. [18] studied the quasineutral limit of the isentropic Navier–Stokes–Poisson system both in the whole space and in the torus without the restriction on viscous coefficient with well prepared initial data. However, there is no analysis for the quasineutral limit for the Navier–Stokes–Poisson system in the context of weak solutions and in the framework of general ill prepared initial data. The common feature of this kind of limit in the ill prepared data framework is high plasma oscillations, that is, the presence of high frequency time oscillations along the acoustic waves. In these phenomena there are different behaviors of the various vector fields acting in our system. Particularly relevant is to understand the relationship between high frequency interacting waves, dispersive behavior and the different roles of time and space oscillations. In our analysis the velocity field both disperses and oscillates, however the dispersion behavior dominates on the high frequency time oscillations and Strichartz estimates are sufficient to pass into the limit of the convective term. The presence of quadratic terms on the electric field (for example $\rho^\lambda \nabla V^\lambda$) cannot be analyzed in the same way since the dispersive behavior no longer dominates on time high frequency wave packets. In the general case, these quadratic terms will not vanish in the limit as $\lambda \to 0$, unless we have well prepared initial data.

1.2. Plan of the Paper

The structure of this paper, as well as the main ingredients of our approach to this limiting process, can be summarized as follows.

- In Section 2 we collect many needed mathematical tools, including notations, Strichartz estimates and microlocal defect measures. Then in Section 3 we set up our problem.
- Section 4 is devoted to obtain a priori estimates independent of λ, namely standard energy bounds and dispersive estimates on the density fluctuation. The main idea here is based on the observation that the density fluctuation $\rho^\lambda - 1$ satisfies a Klein-Gordon equation, so acoustic waves analysis for the Navier–Stokes–Poisson system (1)–(3) follows by reading the system as a dispersive equation and we will get uniform estimates in λ by the use of he L^p-type estimates due to Strichartz [13,19,27]. The particular type of Strichartz estimates

for the Klein Gordon equation that we use here can be recovered from the seminal paper by STRICHARTZ [27] (where he studied the homogeneous equation) and by using Duhamel's principle.

- In the previous sections we get sufficient bounds in order to study the limiting behaviour of the velocity vector field. Therefore in Section 5 we analyze separately the limiting behaviour of the divergence free part and the gradient part of u^λ. Accordingly we obtain the strong convergence of the velocity field
- The next stumbling block is to get enough compactness for the electric field in order to pass into the limit in the quadratic term $\lambda\nabla V^\lambda \otimes \lambda\nabla V^\lambda$. Since $\lambda\nabla V^\lambda$ is bounded in $L_t^\infty L_x^2$ we can define microlocal defect measure ν^E introduced by GÈRARD [12] and by TARTAR (H-measure) in [28] with correctors E^+ and E^- to handle time oscillations at frequency $1/\lambda$. An analogous use of the P. Gèrard and L. Tartar ideas can be found in BRENIER and GRENIER [2] and GRENIER [15], regarding the Vlasov Poisson system. This will be done in Section 6.
- In Section 7 we will be able to prove our main Theorem 1.
- As a final step, in Section 8, we show that in the case of smooth solutions for the system (1)–(3) the class of correctors E^+ and E^- is not empty and they satisfy a "pseudo parabolic" type equation, see the Theorem 2.

2. Preliminaries

For convenience of the reader we establish some notations and recall some basic facts that will be useful in the sequel.

2.1. Notations

If F and G are functions, we denote by $F \lesssim G$ the fact that there exists $c \in \mathbb{R}$ such that $F \leqq G$. Then,

a) We denote by $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}_+)$ the space of test function $C_0^\infty(\mathbb{R}^d \times \mathbb{R}_+)$, by $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+)$ the space of Schwartz distributions and by $\langle \cdot, \cdot \rangle$ the duality bracket between $\mathcal{D}'$ and $\mathcal{D}$.

b) We denote by $W^{k,p}(\mathbb{R}^d) = (I - \Delta)^{-\frac{k}{2}} L^p(\mathbb{R}^d)$ and $H^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d)$ the nonhomogeneous Sobolev spaces, for any $1 \leqq p \leqq \infty$ and $k \in \mathbb{R}$, and by $\dot{W}^{k,p}(\mathbb{R}^d) = (-\Delta)^{-\frac{k}{2}} L^p(\mathbb{R}^d)$ and $\dot{H}^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d)$ the homogeneous Sobolev spaces. The notations $L_t^p L_x^q$ and $L_t^p W_x^{k,q}$ will abbreviate, respectively, the spaces $L^p([0,T]; L^q(\mathbb{R}^d))$, and $L^p([0,T]; W^{k,q}(\mathbb{R}^d))$.

c) We denote by $L_2^p(\mathbb{R}^d)$ the Orlicz space defined as

$$L_2^p(\mathbb{R}^d) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \mid |f|\chi_{|f| \leqq \frac{1}{2}} \in L^2(\mathbb{R}^d),\ |f|\chi_{|f| > \frac{1}{2}} \in L^p(\mathbb{R}^d)\}; \tag{5}$$

see [1,21] for more details.

d) We denote by $\mathcal{L}(\mathbb{R}^3)$ the space of bounded operators and by $\mathcal{K}(\mathbb{R}^3)$ the space of compact operators.

e) If X, Y are Banach spaces, $\mathcal{L}(X, Y)$ is the space of bounded operators.
f) We denote by Q and P, respectively, the Leray's projectors Q on the space of gradients vector fields and P on the space of divergence-free vector fields:

$$Q = \nabla \Delta^{-1} \operatorname{div}, \qquad P = I - Q. \tag{6}$$

It is well known that Q and P can be expressed in terms of Riesz multipliers, therefore they are bounded linear operators on every $W^{k,p}$ $(1 < p < \infty)$ space (see [26]).

Next, we recall the basic notations concerning pseudo-differential operators and symbols to be used later on. We refer to [30] for details. Assuming ρ, $\delta \in [0, 1]$, $m \in \mathbb{R}$, we denote by $S^m_{\rho,\delta}$ the set of C^∞ symbols satisfying

$$\left| D^\beta_x D^\alpha_\xi p(x, \xi) \right| \leqq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for all α, β, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. In this case the associated operator denoted by $OP(p(x, \xi))$ is given by

$$P(x, D) f(x) = \int p(x, \xi) \mathcal{F} f(\xi) \mathrm{e}^{ix\xi} \mathrm{d}\xi := OP(p(x, \xi)),$$

where $\mathcal{F} f(\xi) = (2\pi)^{-n} \int f(x) \mathrm{e}^{-ix\xi} \mathrm{d}x$ denotes the Fourier transform of the function f, and we say that it belongs to $OPS^m_{\rho,\delta}$. If there are smooth symbols $p_{m-j}(x, \xi)$, homogeneous in ξ of degree $m - j$ for $|\xi| \geqq 1$, that is $p_{m-j}(x, r\xi) = r^{m-j} p_{m-j}(x, \xi)$ for $r > 0$, $|\xi| \geqq 1$, and if

$$p(x, \xi) \sim \sum_{j \geqq 0} p_{m-j}(x, \xi)$$

in the sense that

$$p(x, \xi) - \sum_{j \geqq 0}^{N} p_{m-j}(x, \xi) \in S^{m-N}_{1,0}$$

for all N, then we say $p(x, \xi) \in S^m$ and $P(x, D)$ is polyhomogeneous of order m. If Ω is an open set in $\mathbb{R}^3$, we denote by $\psi^m_{\mathrm{comp}}(\Omega, \mathcal{L}(H))$, respectively $\psi_{\mathrm{comp}}(\Omega, \mathcal{K}(H))$, the space of polyhomogeneous pseudo-differential operators of order m on Ω, with values in $\mathcal{L}(H)$, respectively $\mathcal{K}(H)$, whose kernel is compactly supported in $\Omega \times \Omega$. Moreover, we recall that if $P \in \psi^m_{\mathrm{comp}}(\Omega, \mathcal{L}(H))$, then its symbol $p(x, \xi)$ is a linear application from $\psi^m_{\mathrm{comp}}(\Omega, \mathcal{L}(H))$ to $C^\infty_0(S^*\Omega, \mathcal{L}(H))$, where $S^*\Omega = S^{d-1} \times \Omega$.

Following P. Gèrard, we say that μ is the *microlocal defect measure* (or following L. Tartar, the *H-measure*) for a bounded sequence w_k in L^2 if, for any $A \in \psi^0_{\mathrm{comp}}(\omega, \mathcal{K}(H))$, one has (up to subsequences)

$$\lim_{k \to \infty} (A(w_k - w), (w_k - w)) = \int_{S^*\Omega} tr(a(x, \xi) \mu(\mathrm{d}x \mathrm{d}\xi)),$$

where $A = OP(a(x, \xi))$.

2.2. Technical Tools

2.2.1. Strichartz Estimates for Klein Gordon Equations Let us recall that if w is a solution of the following Klein Gordon equation in the space $[0, T] \times \mathbb{R}^d$

$$\left(-\frac{\partial^2}{\partial t} + \Delta - m^2\right) w(t, x) = F(t, x),$$

with Cauchy data

$$w(0, \cdot) = f, \quad \partial_t w(0, \cdot) = g,$$

where $m > 0$ is the mass and $0 < T < \infty$, then w satisfies the following Strichartz estimates (see [27]):

$$\|w\|_{L^q_{t,x}} + \|\partial_t w\|_{L^q_t W^{-1,q}_x} \lesssim \|f\|_{\dot{H}^{1/2}_x} + \|g\|_{\dot{H}^{-1/2}_x} + \|F\|_{L^p_{t,x}},$$

where (q, p), are *admissible pairs*, that is, they satisfy

$$\frac{2(n+1)}{n+3} \leqq p \leqq \frac{2(n+2)}{n+4} \qquad \frac{2(n+2)}{n} \leqq p \leqq \frac{2(n+1)}{n-1}.$$

In particular, in the case of $d = 3$, (q, p) are admissible if they satisfy

$$\frac{4}{3} \leqq p \leqq \frac{10}{7} \qquad \frac{10}{3} \leqq q \leqq \frac{4}{3}.$$

Moreover, by choosing $p = 4/3$ and $q = 4$ and by a standard application of Duhamel's principle, we have the following estimate:

$$\|w\|_{L^4_{t,x}} + \|\partial_t w\|_{L^4_t W^{-1,4}_x} \lesssim \|f\|_{\dot{H}^{1/2}_x} + \|g\|_{\dot{H}^{1/2}_x} + \|F\|_{L^1_t L^2_x}. \tag{7}$$

It is straightforward to observe that for any $s \geqq 0$, this estimate also holds:

$$\|w\|_{L^4_t W^{-s,4}_x} + \|\partial_t w\|_{L^4_t W^{-1-s,4}_x} \lesssim \|f\|_{H^{1/2-s}_x} + \|g\|_{H^{-1/2-s}_x} + \|F\|_{L^1_t H^{-s}_x}. \tag{8}$$

(It is sufficient to apply the operator $(I - \Delta)^{-s/2}$ to (7).)

2.2.2. Properties for Pseudo-Differential Operators We recall, here, two fundamental tools necessary for working with pseudodifferential operators (for more details see [12,29,30]).

Proposition 1. *If $A \in OPS^0$, then*

$$A : L^2_{\text{loc}}(\Omega, H) \to L^2(\Omega, H)$$

is bounded.

Proposition 2. (Generalized Rellich Theorem) *If $A \in \psi^m_{\text{comp}}(\Omega, \mathcal{K}(H))$ for some $m < 0$, then*

$$A : L^2_{\text{loc}}(\Omega, H) \to L^2(\Omega, H)$$

is compact, that is, if $w_k \rightharpoonup w$ weakly in L^2, then $\|Aw_k - Aw\| \to 0$ strongly.

2.2.3. Convolution Estimate and Compactness Theorems Here we state the following elementary lemma, which will be used later on.

Lemma 1. *Let us consider a smoothing kernel* $j \in C_0^\infty(\mathbb{R}^d)$, *such that* $j \geqq 0$, $\int_{\mathbb{R}^d} j\,\mathrm{d}x = 1$, *and let us define*

$$j_\alpha(x) = \alpha^{-d} j\left(\frac{x}{\alpha}\right).$$

Then for any $f \in \dot{H}^1(\mathbb{R}^d)$, *one has*

$$\|f - f * j_\alpha\|_{L^p(\mathbb{R}^d)} \leqq C_p \alpha^{1-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|\nabla f\|_{L^2(\mathbb{R}^d)}, \tag{9}$$

where

$$p \in [2, \infty) \quad \text{if } d = 2, \quad p \in [2, 6] \quad \text{if } d = 3.$$

Moreover, the following Young type inequality holds

$$\|f * j_\alpha\|_{L^p(\mathbb{R}^d)} \leqq C\alpha^{s-d\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{W^{-s,q}(\mathbb{R}^d)}, \tag{10}$$

for any $p, q \in [1, \infty]$, $q \leqq p$, $s \geqq 0$, $\alpha \in (0, 1)$.

We also recall the following compactness tool (see [25]).

Proposition 3. *Let* $\mathcal{F} \subset L^p([0, T]; B)$, $1 \leqq p < \infty$, *B be a Banach space.* $\mathcal{F}$ *is relatively compact in* $L^p([0, T]; B)$ *for* $1 \leqq p < \infty$, *or in* $C([0, T]; B)$ *for* $p = \infty$, *if and only if*

(i) $\left\{\int_{t_1}^{t_2} f(t)\mathrm{d}t,\ f \in \mathcal{F}\right\}$ *is relatively compact in B*, $0 < t_1 < t_2 < T$,
(ii) $\lim_{h\to 0} \|f(t+h) - f(t)\|_{L^p([0,T-h];B)} = 0$ *uniformly for any* $f \in \mathcal{F}$.

3. Statement of the Problem and Main Results

3.1. Basic Facts on the Navier–Stokes–Poisson System

In order set up our problem, we recall some results concerning the existence theory for the Navier–Stokes–Poisson system (1)–(3). For simplicity, we repeat the compressible Navier–Stokes equation coupled with the Poisson equation

$$\begin{cases} \partial_t \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0 \\ \partial_t(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + \nabla(\rho^\lambda)^\gamma = \mu\Delta u^\lambda \\ +(\nu+\mu)\nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda \\ \lambda^2 \Delta V^\lambda = \rho^\lambda - 1. \end{cases} \tag{11}$$

To simplify our notation, from now on we will set

$$\pi^\lambda = \frac{(\rho^\lambda)^\gamma - 1 - \gamma(\rho^\lambda - 1)}{(\gamma - 1)} \qquad \mu = \nu = 1.$$

The system (11) is endowed with the following initial conditions,

$$\begin{aligned}
&\rho^\lambda_{t=0} = \rho^\lambda_0 \geqq 0, \;\; V^\lambda|_{t=0} = V^\lambda_0, \\
&\rho^\lambda u^\lambda|_{t=0} = m^\lambda_0, \quad m^\lambda_0 = 0 \text{ on } \{x \in \mathbb{R}^3 \mid \rho^\lambda_0(x) = 0\}, \\
&\int_{\mathbb{R}^3} \left(\pi^\lambda|_{t=0} + \frac{|m^\lambda_0|^2}{2\rho^\lambda_0} + \lambda^2 |V^\lambda_0|^2 \right) \mathrm{d}x \leqq C_0.
\end{aligned} \tag{ID}$$

The existence of global weak solutions for fixed $\lambda > 0$ for the system (11) has been proved in the case of a bounded domain in [5] and in the case of the whole domain in [7] and [8]. We summarize this existence result in the following proposition.

Proposition 4. *Assume (ID), and let $\gamma > 3/2$; then there exists a global weak solution $(\rho^\lambda, u^\lambda, V^\lambda)$ to (11) such that $\rho^\lambda - 1 \in L^\infty((0,T); L^\gamma_2(\mathbb{R}^3))$, $\sqrt{\rho^\varepsilon} u^\varepsilon \in L^\infty((0,T); L^2(\mathbb{R}^3))$, $u^\lambda \in L^2((0,T); W^{1,2}(\mathbb{R}^3))$. Furthermore,*

– *The energy inequality holds for almost every $t \geqq 0$,*

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(\rho^\lambda \frac{|u^\lambda|^2}{2} + \pi^\lambda + \lambda^2 |\nabla V^\lambda|^2 \right) \mathrm{d}x \\
&\qquad + \int_0^t \int_{\mathbb{R}^3} \left(\mu |\nabla u^\lambda|^2 + (\nu + \mu) |\operatorname{div} u^\lambda|^2 \right) \mathrm{d}x ds \leqq C_0.
\end{aligned} \tag{12}$$

– *The continuity equation is satisfied in the sense of renormalized solutions, that is:*

$$\partial_t b(\rho^\lambda) + \operatorname{div}(b(\rho^\lambda) u) + (b'(\rho^\lambda)\rho^\lambda - b(\rho^\lambda)) \operatorname{div} u^\lambda = 0,$$

for any $b \in C^1(\mathbb{R}^3)$ such that

$$b'(z) = \text{constant}, \quad \text{for any } z \text{ large enough, say } z \geqq M.$$

– *The system (11) holds in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$.*

Besides the results on the existence of weak solutions for the Cauchy problem for the Navier–Stokes–Poisson system (11) there is a theory concerning the global existence of classical solutions of (11); see for example [20] for the H^s framework or [17] for global solutions in Besov spaces. We describe this global existence result in the following proposition.

Proposition 5. *Assume that $(\rho^\lambda_0 - 1, m_0) \in H^s(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $s \geqq 4$, with $\delta = \|(\rho^\lambda_0 - 1, m_0)\|_{H^s(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)}$ small. Then, there is a unique global classical solution $(\rho^\lambda, m^\lambda, V^\lambda)$ to the system (11) satisfying*

$$\begin{aligned}
&\rho^\lambda - 1 \in C^0(\mathbb{R}_+, H^s(\mathbb{R}^3)) \cap C^1(\mathbb{R}_+, H^{s-1}(\mathbb{R}^3)), \\
&m \in C^0(\mathbb{R}_+, H^s(\mathbb{R}^3)) \cap C^0(\mathbb{R}_+, H^{s-2}(\mathbb{R}^3)), \\
&\lambda V^\lambda \in C^0(\mathbb{R}_+, L^6(\mathbb{R}^3)) \;\; \lambda \nabla V^\lambda \in C^0(\mathbb{R}_+, H^{s+1}(\mathbb{R}^3)).
\end{aligned}$$

3.2. Main Results

Having collected all the preliminary material, we are now ready to state our main results. The first result concerns the convergence of solutions of the system (11) in the quasineutral regime.

Theorem 1. *Let* $(\rho^\lambda, u^\lambda, V^\lambda)$ *be a sequence of weak solutions in* $\mathbb{R}^3$ *of the system* (11)*, assume that the initial data satisfy (ID). Then*

(i) $\rho^\lambda \longrightarrow 1$ *weakly in* $L^\infty([0,T]; L^k_2(\mathbb{R}^3))$.
(ii) *There exists* $u \in L^\infty([0,T]; L^2(\mathbb{R}^3)) \cap L^2([0,T]; \dot{H}^1(\mathbb{R}^3))$ *such that*

$$u^\lambda \rightharpoonup u \quad \text{weakly in } L^2([0,T]; \dot{H}^1(\mathbb{R}^3)).$$

(iii) *The gradient component* Qu^λ *of the vector field* u^λ *satisfies*

$$Qu^\lambda \longrightarrow 0 \quad \text{strongly in } L^2([0,T]; L^p(\mathbb{R}^3)), \text{ for any } p \in [4,6).$$

(iv) *The divergence free component* Pu^λ *of the vector field* u^λ *satisfies*

$$Pu^\lambda \longrightarrow Pu = u \quad \text{strongly in } L^2([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^3)).$$

(v) *There exist correctors* E^+, E^- *in* $L^\infty((0,T), L^2(\mathbb{R}^3))$ *and a positive micro-local defect measure* ν^E *on* $\mathbb{R}^3 \times S^2$ *depending measurably on t, associated to the electric field* $E^\lambda = \nabla V^\lambda$*, such that for all pseudodifferential operators* $A \in \psi^0_{\mathrm{comp}}(\mathbb{R}^3, \mathcal{K}(\mathbb{R}^3))$*, and of symbol* $a(x,\xi)$ *and for all* $\phi \in \mathcal{D}(0,t)$*, one has*

$$\lim_{\lambda \to 0} \int \mathrm{d}t \phi(t) \lambda^2 (AE^\lambda, E^\lambda) = \int \mathrm{d}t \phi(t)(AE^+, E^+) + \int \mathrm{d}t \phi(t)(AE^-, E^-) + \int \mathrm{d}t \phi(t) \int_{\mathbb{R}^3 \times S^2} tr\left(a(x,\xi)\frac{\xi \otimes \xi}{|\xi|^2}\right) \mathrm{d}\nu^E. \tag{13}$$

(vi) $u = Pu$ *satisfies the following equation*

$$P\Big(\partial_t u - \Delta u + (u \cdot \nabla)u - \mathrm{div}(E^+ \otimes E^+ + E^- \otimes E^-) - \mathrm{div}\langle \nu^E, \frac{\xi \otimes \xi}{|\xi|^2}\rangle\Big) = 0, \tag{14}$$

in $\mathcal{D}'([0,T] \times \mathbb{R}^3)$.

Remark 1. In the previous theorem we constructed a defect measure ν^E and the correctors $E^\pm$. They correspond to the physical phenomenon of high frequency plasma oscillation. Notice that the correctors $E^\pm$ remain important as $\lambda \to 0$ and are not vanishing; in fact, we do not have the initial layer but, on the contrary, the effect of ill prepared initial data appears through $E^\pm$ and remains important for all times.

As we will see in the rest of paper, the construction of the defect measure ν^E will be done by using the theory developed by GÈRARD [12] and TARTAR [28]. The explicit construction of the correctors is not trivial and requires a smooth setting for the solutions. We will show this part in the next theorem.

Theorem 2. *Let* $(\rho^\lambda, u^\lambda, V^\lambda)$ *be a sequence of the Navier–Stokes–Poisson system, satisfying for* $s \geqq 4$

$$\|\rho^\lambda - 1\|_{L^\infty(0,T;H^s(\mathbb{R}^3))} \leqq C \qquad \|\lambda E^\lambda\|_{L^\infty(0,T;H^s(\mathbb{R}^3))} \leqq C. \tag{15}$$

Then, for all $s' < s-2$

$$u^\lambda - \frac{1}{i}\mathrm{e}^{-it/\lambda}E^+ - \frac{1}{i}\mathrm{e}^{it/\lambda}E^- \longrightarrow v \quad \textit{strongly in } C^0(0,T,H^{s'-1}_{\mathrm{loc}}(\mathbb{R}^3)). \tag{16}$$

$$\lambda(E^\lambda - \mathrm{e}^{-it/\lambda}E^+ - \mathrm{e}^{it/\lambda}E^-) \longrightarrow 0 \quad \textit{strongly in } C^0(0,T,H^{s'-1}_{\mathrm{loc}}(\mathbb{R}^3)), \tag{17}$$

and $E^\pm$ *satisfy*

$$\partial_t E^\pm - \Delta E^\pm + Q\,\mathrm{div}(v \otimes E^\pm) = 0, \qquad PE^\pm = 0. \tag{18}$$

In the Section 8 we will show in Proposition 14 the existence of solutions for the equation (18). The rest of the paper is devoted to proving the Theorems 1 and 2.

4. Uniform Estimates

In this section we wish to establish all the a priori estimates, independent on λ, for the solutions of the system (11) which are necessary to prove the Theorem 1.

4.1. Consequences of the Energy Estimate

We start by collecting all the a priori bounds that are a consequence of the energy inequality (12). Before going on, let us define the density fluctuation σ^λ as

$$\sigma^\lambda = \rho^\lambda - 1. \tag{19}$$

Proposition 6. *Let us consider the solution* $(\rho^\lambda, u^\lambda, V^\lambda)$ *of the Cauchy problem for the system* (11)*. Assume that the hypotheses (ID) hold; then it follows*

$$\sigma^\lambda \quad \textit{is bounded in } L^\infty([0,T];L^k_2(\mathbb{R}^3)), \textit{ where } k = \min(\gamma, 2), \tag{20}$$

$$\nabla u^\lambda \quad \textit{is bounded in } L^2([0,T]\times\mathbb{R}^3), \tag{21}$$

$$u^\lambda \quad \textit{is bounded in } L^2([0,T]\times\mathbb{R}^3) \cap L^2([0,T];L^6(\mathbb{R}^3)), \tag{22}$$

$$\sigma^\lambda u^\lambda \quad \textit{is bounded in } L^2([0,T];H^{-1}(\mathbb{R}^3)), \tag{23}$$

$$\lambda\nabla V^\lambda \quad \textit{is bounded in } L^\infty([0,T];L^2(\mathbb{R}^3)). \tag{24}$$

Proof. From (12) it follows that $\pi^\lambda \in L^\infty([0,T];L^1(\mathbb{R}^3))$. By taking into account that the function $z \to z^\gamma - 1 - \gamma(z-1)$ is convex and by following the same line of arguments as in [22], we get when $\gamma < 2$ that

$$\sup_{t\geqq 0}\int_{\mathbb{R}^3}\left\{|\rho^\lambda-1|^2\chi_{|\rho^\lambda-1|\leqq 1/2}+|\rho^\varepsilon-1|^\gamma\chi_{|\rho^\lambda-1|\geqq 1/2}\right\}(t,x)\mathrm{d}x \leqq C, \quad (25)$$

and when $\gamma \geqq 2$,

$$\sup_{t\geqq 0}\int_{\mathbb{R}^3}|\rho^\lambda-1|^2(t,x)\mathrm{d}x \leqq C, \quad (26)$$

so we can conclude that σ^λ is uniformly bounded in λ in $L^\infty([0,T];L^k_2(\mathbb{R}^3))$, where $k = \min(\gamma,2)$. (21) and (24) are a consequence of (12). The fact that $u^\lambda \in L^2([0,T];L^6(\mathbb{R}^3))$ follows from (21) and from Sobolev's embeddings. Now we prove $u^\lambda \in L^2([0,T]\times\mathbb{R}^3)$.

$$\begin{aligned}\int_{\mathbb{R}^3}|u^\lambda|^2\mathrm{d}x &= \int_{\mathbb{R}^3}\left\{|u^\lambda|^2\chi_{|\rho^\lambda-1|\leqq 1/2}+|u^\lambda|^2\chi_{|\rho^\varepsilon-1|\geqq 1/2}\right\}\mathrm{d}x\\ &\leqq 2\int_{\mathbb{R}^3}\rho^\lambda|u^\lambda|^2\mathrm{d}x + 2\|\rho^\varepsilon-1\|_{L^k_x}\|u^\lambda\|^2_{L_x^{2k/k-1}}\\ &\leqq C_0 + C_0\|u^\varepsilon\|^{2-\frac{3}{k}}_{L^2_x}\|\nabla u^\lambda\|^{\frac{3}{k}}_{L^2_x}. \quad (27)\end{aligned}$$

We then easily complete the proof by using (21). Recalling that $\gamma > 3/2$ and by interpolating, we get that $u^\lambda \in L^2([0,T];L^4(\mathbb{R}^3)\cap L^{2\gamma/(\gamma-1)}(\mathbb{R}^3))$. By using (20) we obtain that $\rho^\lambda u^\lambda$ is uniformly bounded in $L^2([0,T];L^{4/3}(\mathbb{R}^3)+L^{2k/(k+1)}(\mathbb{R}^3))$. Therefore, by Sobolev's embeddings we get (23). □

We want to complete this paragraph with a remark concerning the regularity of the initial data.

Remark 2. With the same procedure as for σ^λ, taking into account (ID), we get that σ_0^λ is bounded in $L^k_2(\mathbb{R}^3)$, hence in $H^{-1}(\mathbb{R}^3)$, since $\gamma > 3/2$. If we rewrite m_0^λ in the following way

$$m_0^\lambda = \frac{m_0^\lambda}{\sqrt{\rho_0^\lambda}}\sqrt{\rho_0^\lambda}\chi_{|\rho_0^\lambda-1|\leqq 1/2} + \frac{m_0^\lambda}{\sqrt{\rho_0^\lambda}}\frac{\sqrt{\rho_0^\lambda}}{\sqrt{|\rho_0^\lambda-1|}}\sqrt{|\rho_0^\lambda-1|}\chi_{|\rho_0^\lambda-1|>1/2},$$

we get that m_0^λ is bounded in $L^2(\mathbb{R}^3)+L^{2k/(k+1)}(\mathbb{R}^3)$ and hence in $H^{-1}(\mathbb{R}^3)$. Finally, we can conclude that

$$\sigma_0^\lambda,\ m_0^\lambda \qquad \text{are bounded in } H^{-1}(\mathbb{R}^3) \text{ uniformly in } \lambda. \quad (28)$$

4.2. Density Fluctuation Acoustic Equation

From the estimates of Proposition 6 we get only the weak convergence of the velocity field; unfortunately, this will be not sufficient to pass into the limit in the nonlinear terms (such as the convective term $\operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda)$) of the system (11). In particular, this weak convergence is induced by the rapid time oscillation of the acoustic waves or by the so called plasma oscillations. In order to overcome this problem, we will estimate the density fluctuation σ^λ uniformly with respect to λ, so we derive the so called acoustic equation which governs the time evolution of σ^λ. First, we rewrite the system (12) in the following way

$$\partial_t \sigma^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0 \tag{29}$$

$$\begin{aligned}\partial_t(\rho^\lambda u^\lambda) + \nabla\sigma^\lambda &= \mu\Delta u^\lambda + (\nu+\mu)\nabla \operatorname{div} u^\lambda - \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) \\ &\quad -(\gamma-1)\nabla\pi^\lambda + \sigma^\lambda \nabla V^\lambda + \nabla V^\lambda,\end{aligned} \tag{30}$$

$$\lambda^2 \Delta V^\lambda = \sigma^\lambda. \tag{31}$$

Then, by differentiating equation (29) with respect to time, taking the divergence of (30) and by using (31), we get that σ^λ satisfies the following equation

$$\begin{aligned}\partial_{tt}\sigma^\lambda - \Delta\sigma^\lambda + \frac{\sigma^\lambda}{\lambda^2} &= -\operatorname{div}(\mu\Delta u^\lambda + (\nu+\mu)\nabla \operatorname{div} u^\lambda) \\ &\quad + \operatorname{div}\left(\operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + (\gamma-1)\nabla\pi^\lambda + \sigma^\lambda\nabla V^\lambda\right).\end{aligned} \tag{32}$$

It turns out that (32) is a nonhomogeneous Klein Gordon equation with mass $1/\lambda$. In order to get some more uniform estimates on σ^λ, we apply the Strichartz estimates, (8), to (32). To renormalize the mass of equation (32) so it is easier to handle, we rescale the time and space variable, the density fluctuation, the velocity and the electric potential in the following way:

$$\tau = \frac{t}{\lambda}, \quad y = \frac{x}{\lambda} \tag{33}$$

$$\begin{aligned}\tilde{u}(y,\tau) &= u^\lambda(\lambda y, \lambda\tau), \quad \tilde{\rho}(y,t) = \rho^\lambda(\lambda y, \lambda\tau) \\ \tilde{\sigma}(y,\tau) &= \sigma^\lambda(\lambda y, \lambda\tau), \quad \tilde{V}(y,\tau) = V^\lambda(\lambda y, \lambda\tau).\end{aligned} \tag{34}$$

As a consequence of this scaling, the Klein Gordon equation (32) becomes of mass equal to one, namely

$$\begin{aligned}\partial_{\tau\tau}\tilde{\sigma} - \Delta\tilde{\sigma} + \tilde{\sigma} &= -\frac{1}{\lambda}\operatorname{div}(\mu\Delta\tilde{u} + (\nu+\mu)\nabla \operatorname{div}\tilde{u}) \\ &\quad + \operatorname{div}\left(\operatorname{div}(\tilde{\rho}\tilde{u}\otimes\tilde{u}) + (\gamma-1)\nabla\tilde{\pi} + \tilde{\sigma}\nabla\tilde{V}\right).\end{aligned} \tag{35}$$

Now we consider $\tilde{\sigma} = \tilde{\sigma}_1 + \tilde{\sigma}_2 + \tilde{\sigma}_3$, where $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$ solve the following Klein Gordon equations:

$$\begin{cases} \partial_{\tau\tau}\tilde{\sigma}_1 - \Delta\tilde{\sigma}_1 + \tilde{\sigma}_1 = -\frac{1}{\lambda}\operatorname{div}(\mu\Delta\tilde{u} + (\nu+\mu)\nabla\operatorname{div}\tilde{u}) = F_1 \\ \tilde{\sigma}_1(x,0) = \tilde{\sigma}(x,0) = \tilde{\sigma}_0 \quad \partial_\tau\tilde{\sigma}_1(x,0) = \partial_\tau\tilde{\sigma}(x,0) = \partial_t\tilde{\sigma}_0, \end{cases} \tag{36}$$

$$\begin{cases} \partial_{\tau\tau}\tilde{\sigma}_2 - \Delta\tilde{\sigma}_2 + \tilde{\sigma}_2 = \operatorname{div}(\operatorname{div}(\tilde{\rho}\tilde{u}\otimes\tilde{u}) + (\gamma-1)\nabla\tilde{\pi}) = F_2 \\ \tilde{\sigma}_2(x,0) = \partial_\tau\tilde{\sigma}_2(x,0) = 0, \end{cases} \tag{37}$$

$$\begin{cases} \partial_{\tau\tau}\tilde{\sigma}_3 - \Delta\tilde{\sigma}_3 + \tilde{\sigma}_3 = -\operatorname{div}(\tilde{\sigma}\nabla\tilde{V}) = F_3 \\ \tilde{\sigma}_3(x,0) = \partial_\tau\tilde{\sigma}_3(x,0) = 0. \end{cases} \tag{38}$$

We are able to prove the following estimate on σ^λ.

Theorem 3. *Let us consider the solutions $(\rho^\lambda, u^\lambda, V^\lambda)$ of the Cauchy problem for the system (11) with initial data satisfying (ID). Then, for any $s_0 \geqq 3/2$, the following estimate holds*

$$\begin{aligned} &\lambda^{-\frac{1}{2}}\|\sigma^\lambda\|_{L_t^4 W_x^{-s_0-2,4}} + \lambda^{-\frac{1}{2}}\|\partial_t\sigma^\lambda\|_{L_t^4 W_x^{-s_0-3,4}} \\ &\lesssim \lambda^{s_0-\frac{1}{2}}\|\sigma_0^\lambda\|_{H_x^{-3/2}} + \lambda^{s_0-\frac{1}{2}}\|m_0^\lambda\|_{H_x^{-5/2}} \\ &\quad + T\|\operatorname{div}(\operatorname{div}(\rho^\lambda u^\lambda\otimes u^\lambda) - (\gamma-1)\nabla\pi^\lambda)\|_{L_t^\infty H_x^{-s_0-2}} \\ &\quad + \lambda^{s_0}\|\operatorname{div}\Delta u^\lambda + \nabla\operatorname{div}u^\lambda\|_{L_t^2 H_x^{-2}} + T\|\operatorname{div}(\sigma^\lambda V^\lambda)\|_{L_t^\infty H_x^{-s_0-2}}. \end{aligned} \tag{39}$$

Proof. Since $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ and $\tilde{\sigma}_3$ are solutions of the equations (36), (37) and (38), we can apply the Strichartz estimate (8) with $(y,\tau) \in \mathbb{R}^3 \times (0, T/\lambda)$. We start with $\tilde{\sigma}_1$. From (21) we deduce that $F_1 \in L_t^2 H_x^{-2}$, so by using (8) with $s = 2$ we get

$$\begin{aligned} \|\tilde{\sigma}_1\|_{L_\tau^4 W_y^{-2,4}} + \|\partial_\tau\tilde{\sigma}_1\|_{L_\tau^4 W_y^{-3,4}} &\lesssim \|\tilde{\sigma}_0\|_{H_y^{-3/2}} + \|\partial_\tau\tilde{\sigma}_0\|_{H_y^{-5/2}} \\ &\quad + \lambda^{-1}T\|\lambda^{-1/2}\operatorname{div}(\Delta\tilde{u} + \nabla\operatorname{div}\tilde{u})\|_{L_\tau^2 H_x^{-2}}. \end{aligned} \tag{40}$$

From estimate (12) we have that $\tilde{\rho}|\tilde{u}|^2,\ \tilde{\pi} \in L_t^\infty L_x^1$, but L^1 is continuously embedded in H^{-s_0}, $s_0 \geqq 3/2$, so we have that $F_2 \in L_t^\infty H_x^{-s_0-2}$. If we apply (8) to $\tilde{\sigma}_2$, we obtain for any $s_0 \geqq 3/2$

$$\begin{aligned} &\|\tilde{\sigma}_2\|_{L_\tau^4 W_y^{-s_0-2,4}} + \|\partial_\tau\tilde{\sigma}_2\|_{L_\tau^4 W_y^{-s_0-3,4}} \\ &\quad \lesssim \lambda^{-1/2}T\|\operatorname{div}(\operatorname{div}(\tilde{\rho}\tilde{u}\otimes\tilde{u}) + \nabla\tilde{\pi})\|_{L_\tau^\infty H_y^{-s_0-2}}. \end{aligned} \tag{41}$$

By using the Poisson equation (31) we can rewrite F_3 as $F_3 = \operatorname{div}(\operatorname{div}(\nabla\tilde{V}\otimes\nabla\tilde{V}) + \frac{1}{2}\nabla|\nabla\tilde{V}|^2)$. Taking into account (12), as for F_2, we get $F_3 \in L_\tau^\infty H_x^{-s_0-2}$, for any $s_0 \geqq 3/2$. Hence $\tilde{\sigma}_3$ satisfies

$$\begin{aligned} &\|\tilde{\sigma}_3\|_{L_\tau^4 W_y^{-s_0-2,4}} + \|\partial_\tau\tilde{\sigma}_3\|_{L_\tau^4 W_y^{-s_0-3,4}} \\ &\quad \lesssim \lambda^{-1/2}T\|\operatorname{div}(\nabla\tilde{V}\otimes\nabla\tilde{V}) + \frac{1}{2}\nabla|\nabla\tilde{V}|^2\|_{L_\tau^\infty H_y^{-s_0-2}}. \end{aligned} \tag{42}$$

Summing up (40), (41) and (42), $\tilde{\sigma}$ verifies

$$
\begin{aligned}
&\|\tilde{\sigma}\|_{L^4_\tau W^{-s_0-2,4}_x} + \|\partial_\tau \sigma^\varepsilon\|_{L^4_t W^{-s_0-3,4}_y} \\
&\quad\lesssim \|\tilde{\sigma}_0\|_{H^{-3/2}_y} + \|\partial_\tau \tilde{\sigma}_0\|_{H^{-5/2}_y} \\
&\qquad + \lambda^{-1}\|\lambda^{-1/2}\operatorname{div}(\Delta\tilde{u} + \nabla \operatorname{div}\tilde{u})\|_{L^2_\tau H^{-2}_y} \\
&\qquad + \lambda^{-1} T\|\operatorname{div}(\operatorname{div}(\tilde{\rho}\tilde{u}\otimes\tilde{u}) + \nabla\tilde{\pi})\|_{L^\infty_\tau H^{-s_0-2}_y} \\
&\qquad + \lambda^{-1} T\|\operatorname{div}(\tilde{\sigma}\nabla\tilde{V})\|_{L^\infty_\tau H^{-s_0-1}_y}
\end{aligned} \tag{43}
$$

Finally, since

$$
\|\tilde{\sigma}\|_{L^q_\tau W^{k,p}_y} = \lambda^{-\frac{1}{q}+k-\frac{3}{p}} \|\sigma^\varepsilon\|_{L^p([0,T];L^q(\mathbb{R}^3))}
$$

and by using (28), we end up with (39). □

5. Strong Convergence of the Velocity Field

In this section we will study the strong convergence of the velocity field u^λ. This will be achieved by separately studying the convergence of the divergence free vector field Pu^λ and the gradient vector field Qu^λ.

5.1. Strong Convergence of Qu^λ

Here we prove the convergence of Qu^λ to 0. The main tool in this process lies on the fact that Qu^λ can be computed in terms of σ^λ, so we can use estimate (39) combined with the Young type inequalities (9) and (10).

Proposition 7. *Let us consider the solution $(\rho^\lambda, u^\lambda, V^\lambda)$ of the Cauchy problem for the system (11). Assume that the hypotheses (ID) hold. Then as $\lambda \downarrow 0$,*

$$
Qu^\lambda \longrightarrow 0 \quad \text{strongly in } L^2([0,T]; L^p(\mathbb{R}^3)) \text{ for any } p \in [4,6) \,. \tag{44}
$$

Proof. In order to prove Proposition 7 we split Qu^λ as follows

$$
\|Qu^\lambda\|_{L^2_t L^p_x} \leqq \|Qu^\lambda - Qu^\lambda * j_\alpha\|_{L^2_t L^p_x} + \|Qu^\lambda * j_\alpha\|_{L^2_t L^p_x} = J_1 + J_2,
$$

where j_α is the smoothing kernel defined in Lemma 1. Now we separately estimate J_1 and J_2. For J_1, by using (9) we get

$$
J_1 \leqq \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)} \|\nabla u^\lambda\|_{L^2_{t,x}}. \tag{45}
$$

To estimate J_2, we take into account the definition (19), and so we split J_2 as

$$
J_2 \leqq \|Q(\sigma^\lambda u^\lambda) * j_\alpha\|_{L^2_t L^p_x} + \|Q(\rho^\lambda u^\lambda) * j_\alpha\|_{L^2_t L^p_x} = J_{2,1} + J_{2,2}. \tag{46}
$$

For $J_{2,1}$ we use (23) and (10), so we have

$$J_{2,1} \leqq \alpha^{-1-3\left(\frac{1}{2}-\frac{1}{p}\right)} \|\sigma^\lambda u^\lambda\|_{L_t^2 H_x^{-1}}. \tag{47}$$

From the identity $Q(\sigma^\lambda u^\lambda) = \nabla\Delta^{-1}\partial_t\sigma^\lambda$ and by the inequality (10), we get that $J_{2,2}$ satisfies the following estimate

$$\begin{aligned} J_2 &= \lambda^{1/2}\|\lambda^{-1/2}\nabla\Delta^{-1}\partial_t\sigma^\lambda * j\|_{L_t^2 L_x^p} \\ &\leqq \lambda^{1/2}\alpha^{-s_0-4-3\left(\frac{1}{4}-\frac{1}{p}\right)}\|\lambda^{-1/2}\partial_t\sigma^\varepsilon\|_{L_t^2 W_x^{-s_0-4,4}} \\ &\leqq \lambda^{1/2}\alpha^{-s_0-4-3\left(\frac{1}{4}-\frac{1}{p}\right)} T^{1/2}\|\lambda^{-1/2}\partial_t\sigma^\varepsilon\|_{L_t^4 W_x^{-s_0-4,4}}. \end{aligned} \tag{48}$$

Now, summing up (46), (47) and (48) we get

$$\|Qu^\varepsilon\|_{L_t^2 L_x^p} \lesssim \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)} + C_T\lambda^{1/2}\alpha^{-s_0-4-3\left(\frac{1}{4}-\frac{1}{p}\right)}, \tag{49}$$

Finally, we choose α in terms of λ, for example, in a way that the two terms on the right-hand side of the inequality (49) are of the same order, namely

$$\alpha = \lambda^{\frac{2}{17+4s_0}}. \tag{50}$$

Therefore, we obtain

$$\|Qu^\lambda\|_{L_t^2 L_x^p} \leqq C_T\lambda^{\frac{6-p}{p(17+4s_0)}} \quad \text{for any } p \in [4, 6).$$

□

5.2. Strong Convergence of Pu^λ

It remains to prove the strong compactness of the incompressible component of the velocity field. To achieve this goal we need to recall, here, that compactness can be obtained by looking at some time regularity properties of Pu^λ and by using Proposition 3, but first we need to prove the following lemma.

Lemma 2. *Let us consider the solution $(\rho^\lambda, u^\lambda, V^\lambda)$ of the Cauchy problem for the system (11). Assume that the hypotheses (ID) hold. Then for all $h \in (0, 1)$, we have*

$$\|Pu^\lambda(t+h) - Pu^\lambda(t)\|_{L^2([0,T]\times\mathbb{R}^3)} \leqq C_T h^{2/5}. \tag{51}$$

Proof. Let us set $z^\lambda = u^\lambda(t+h) - u^\lambda(t)$. We have

$$\begin{aligned} \|Pu^\lambda(t+h) - Pu^\lambda(t)\|^2_{L^2_{t,x}} &= \int_0^T\int_{\mathbb{R}^3} \mathrm{d}t\mathrm{d}x(Pz^\lambda)\cdot(Pz^\lambda - Pz^\lambda * j_\alpha) \\ &\quad + \int_0^T\int_{\mathbb{R}^3} \mathrm{d}t\mathrm{d}x(Pz^\lambda)\cdot(Pz^\lambda * j_\alpha) = I_1 + I_2. \end{aligned} \tag{52}$$

By using (9) together with (22) we can estimate I_1 in the following way

$$I_1 \leqq \|Pz^\lambda\|_{L^2_{t,x}}\|Pz^\lambda(t) - (Pz^\lambda * j_\alpha)(t)\|_{L^2} \lesssim \alpha\|u^\lambda\|_{L^2_{t,x}}\|\nabla u^\lambda\|_{L^2_{t,x}}. \tag{53}$$

In order to estimate I_2, we split it as follows

$$
\begin{aligned}
I_2 &= \int_0^T \int_{\mathbb{R}^3} \mathrm{d}t\mathrm{d}x\, P(\rho^\lambda z^\lambda)\cdot(Pz^\lambda * j_\alpha) + \int_0^T \int_{\mathbb{R}^3} \mathrm{d}t\mathrm{d}x\, P(\sigma^\lambda z^\lambda)\cdot(Pz^\lambda * j_\alpha) \\
&= I_{2,1} + I_{2,2}.
\end{aligned} \tag{54}
$$

$I_{2,2}$ can be estimated by taking into account (22), (23) and $(11)_3$, so we have

$$
\begin{aligned}
I_{2,2} &= \lambda^2 \int_0^T \int_{\mathbb{R}^3} \mathrm{d}t\mathrm{d}x (\Delta V^\lambda z^\lambda (Pz^\lambda * j_\alpha) \\
&= \lambda^2 \int_0^T \int_{\mathbb{R}^3} \mathrm{d}t\mathrm{d}x \left[(\nabla V^\lambda z^\lambda)(\nabla Pz^\lambda * j_\alpha) + \nabla V^\lambda \nabla z^\lambda (Pz^\lambda * j_\alpha)\right] \\
&\leqq \lambda \|\lambda \nabla V^\lambda z^\lambda + \lambda \nabla V^\lambda \nabla u^\lambda\|_{L^2_t L^1_x} \|\nabla Pz^\lambda * j_\alpha\| \\
&\leqq \lambda \alpha^{-3/2} \|\nabla u^\varepsilon\|_{L^2_{t,x}} \|\lambda \nabla V^\lambda z^\lambda + \lambda \nabla V^\lambda \nabla u^\lambda\|_{L^2_t L^1_x}.
\end{aligned} \tag{55}
$$

Now we estimate $I_{2,1}$. Let us reformulate $P(\rho^\lambda z^\lambda)$ in integral form by using equation $(11)_2$ and the Poisson equation $(11)_3$, hence

$$
\begin{aligned}
I_{2,1} &\leqq \left| \int_0^T \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}x \int_t^{t+h} \mathrm{d}s (\operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + \Delta u^\lambda)(s,x)\cdot(Pz^\lambda * j_\alpha)(t,x) \right| \\
&\quad + \left| \int_0^T \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}x \int_t^{t+h} \mathrm{d}s\, P\left(\frac{\sigma^\lambda}{\lambda}\nabla V^\lambda\right)(s,x)\cdot(Pz^\lambda * j_\alpha)(t,x) \right| \\
&= \left| \int_0^T \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}x \int_t^{t+h} \mathrm{d}s (\operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + \Delta u^\lambda)\cdot(Pz^\lambda * j_\alpha)(t,x) \right| \\
&\quad + \left| \int_0^T \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}x \int_t^{t+h} \mathrm{d}s \lambda^2 \operatorname{div}(\nabla V^\lambda \otimes \nabla V^\lambda))(s,x)\cdot(Pz^\lambda * j_\alpha)(t,x) \right|.
\end{aligned} \tag{56}
$$

Then, by integrating by parts, by using (10) with $s = 0$, $p = \infty$, $q = 2$, we deduce

$$
\begin{aligned}
I_{2,1} &\leqq h\|\nabla u^\lambda\|^2_{L^2_{t,x}} \\
&\quad + C\alpha^{-3/2} T^{1/2} h \|\nabla u^\lambda\|_{L^2_{t,x}} \left(\|\rho^\lambda |u^\lambda|^2\|_{L^\infty_t L^1_x} + \||\lambda \nabla V^\lambda|^2\|_{L^\infty_t L^1_x} \right).
\end{aligned} \tag{57}
$$

Summing up I_1, $I_{2,1}$ and $I_{2,2}$ and by taking into account (12), we have

$$
\|Pu^\lambda(t+h) - Pu^\lambda(t)\|^2_{L^2([0,T]\times\mathbb{R}^3)} \leqq C(h^{2/5} + h + h\alpha^{-3/2}T^{1/2} + 2\alpha^{-3/2}\lambda),
$$

by choosing $\lambda < \alpha$ and $\alpha = h^{2/5}$, we end up with (51). □

Corollary 1. *Let us consider the solution $(\rho^\lambda, u^\lambda, V^\lambda)$ of the Cauchy problem for the system (11). Assume that the hypotheses (ID) hold. Then, as $\lambda \downarrow 0$,*

$$
Pu^\lambda \longrightarrow Pu, \qquad \textit{strongly in } L^2(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^3)). \tag{58}
$$

Proof. By using Lemma 2, Proposition 3 and Proposition 7, we get (5.2). □

6. Convergence of the Electric Field

This section addresses the convergence of the electric field $E^\lambda = \nabla V^\lambda$. By the a priori estimate (24) we know only that λE^λ is bounded in $L^\infty_t L^2_x$, which does not give enough information to pass into the limit in the quadratic term $\rho^\lambda \nabla V^\lambda = \mathrm{div}(\lambda E^\lambda \otimes \lambda E^\lambda) - 1/2\nabla|\lambda E^\lambda|^2$, appearing in the right-hand side of $(11)_2$. Hence, the problem is how to recover the weak continuity of quadratic forms in L^2. Since λE^λ is bounded in $L^\infty_t L^2_x$ we can define the so called microlocal defect measure introduced by GÈRARD [12] and by TARTAR [28] (H-measures), but in order to handle time oscillations we need to introduce correctors. In this section we will prove the following theorem.

Theorem 4. *Let be* $(\rho^\lambda, u^\lambda, E^\lambda)$ *a sequence of solutions of the Navier–Stokes–Poisson system* (11)*, then*

i) *there exists* E^+*,* E^- *in* $L^\infty((0,T), L^2(\mathbb{R}^3))$*,*
ii) *there exists a positive measure* ν^E *on* $\mathbb{R}^3 \times S^2$ *depending measurably on* t

such that for all pseudodifferential operators $A \in \psi^0_{\mathrm{comp}}(\mathbb{R}^3, \mathcal{K}(\mathbb{R}^3))$*, and of symbol* $a(x,\xi)$ *and for all* $\phi \in \mathcal{D}(0,t)$*, one has*

$$\lim_{\lambda\to 0} \int \mathrm{d}t\phi(t)\lambda^2(AE^\lambda, E^\lambda) = \int \mathrm{d}t\phi(t)(AE^+, E^+) + \int \mathrm{d}t\phi(t)(AE^-, E^-) + \int \mathrm{d}t\phi(t) \int_{\mathbb{R}^3\times S^2} tr\left(a(x,\xi)\frac{\xi\otimes\xi}{|\xi|^2}\right)\mathrm{d}\nu^E. \tag{59}$$

First, we rewrite (32) in terms of E^λ, namely

$$\lambda^2\partial_{tt}E^\lambda + E^\lambda = \mathrm{div}\,\Delta^{-1}\nabla\,\mathrm{div}\left(\rho^\lambda u^\lambda \otimes u^\lambda + (\rho^\lambda)^\gamma \mathbb{I} - \lambda^2 E^\lambda \otimes E^\lambda\right) + \frac{\lambda^2}{2}\mathrm{div}\left(|E^\lambda|^2\mathbb{I}\right) - 2\nabla\,\mathrm{div}\,u^\lambda = F^\lambda. \tag{60}$$

Then we observe that by using (39) and the uniqueness of the weak limit, we have

$$\lambda\nabla V^\lambda \rightharpoonup 0 \quad \text{weakly in } L^2(0,T; L^2(\mathbb{R}^3)). \tag{61}$$

By (61) we see that we are exactly in the framework described by P. Gèrard, but we have to pay attention to one fact. In our case in the quadratic form $\lambda^2\langle AE^\lambda, E^\lambda\rangle$, A is a pseudodifferential operator homogeneous only with respect to the x variable; in the general case we cannot extend it to a pseudodifferential operator homogeneous in (x,t). Hence, we have to work on λE^λ in order to isolate the components that oscillate rapidly in time; for that reason we introduce what we call the correctors of the electric field. By using (60) and Duhamel's formula, we can write the electric field E^λ as

$$E^\lambda(t,x) = \int_0^t \frac{F^\lambda(s,x)}{2i\lambda}\left(\mathrm{e}^{i\frac{t-s}{\lambda}} - \mathrm{e}^{-i\frac{t-s}{\lambda}}\right)\mathrm{d}s + \frac{\mathcal{E}_1^\lambda(x)}{\lambda}\mathrm{e}^{it/\lambda} + \frac{\mathcal{E}_2^\lambda(x)}{\lambda}\mathrm{e}^{-it/\lambda}, \tag{62}$$

where $\mathcal{E}_1^\lambda$ and $\mathcal{E}_2^\lambda$ are two functions in L_x^2 defined by the initial data of E^λ. In order to understand how to isolate the oscillating terms, let us consider the equation (60) in the case when F^λ does not depend on x and $\mathcal{E}_1^\lambda = \mathcal{E}_2^\lambda = 0$. Then, if we take the Fourier transform with respect to time we have ($\hat{E}$ denotes the Fourier transform with respect to time)

$$\lambda \hat{E^\lambda} = \frac{\lambda}{1 - \lambda^2|\tau|^2}\hat{F^\lambda},$$

where we can see that all the L^2-mass of λE^λ is concentrated in $\tau = \pm 1/\lambda$ as $\lambda \to 0$. This simple facts leads us to introduce correctors in time of order $1/\lambda$. So we define

$$E_+^\lambda = \lambda \mathrm{e}^{-it/\lambda} E^\lambda \qquad E_-^\lambda = \lambda \mathrm{e}^{it/\lambda} E^\lambda. \tag{63}$$

In particular, they take into account the L^2-mass of λE^λ around $1/\lambda$. By construction it easily follows that E_+^λ and E_-^λ are bounded in $L_{t,x}^2$ and converge weakly to E^+ and E^-, respectively. Moreover, we have

Lemma 3. *Let $(\rho^\lambda, u^\lambda, E^\lambda)$ be a sequence of solutions of the Navier–Stokes–Poisson system* (11) *which satisfy* (*ID*)*, then one has*

$$\mathcal{E}_1^\lambda(x) + \int_0^T \mathrm{d}s \frac{F^\lambda(s,x)}{2i} \mathrm{e}^{-is/\lambda} \rightharpoonup E^+ \quad in\ \mathcal{D}((0,T)\times\mathbb{R}^3).$$

The same holds for E^-.

Proof. The proof follows by using (62) and Proposition 6. □

So, if we look at the limit of $\lambda E^\lambda - \mathrm{e}^{it/\lambda}E^+ - \mathrm{e}^{-it/\lambda}E^-$ as $\lambda \to 0$, we expect to take away the L^2-mass of λE^λ which concentrates around $1/\lambda$. Now we can define

$$\widetilde{E^\lambda} = E^\lambda - \mathrm{e}^{it/\lambda}\frac{E^+}{\lambda} - \mathrm{e}^{-it/\lambda}\frac{E^-}{\lambda}, \tag{64}$$

then we can prove the following lemma.

Lemma 4. *Let $(\rho^\lambda, u^\lambda, E^\lambda)$ be a sequence of solutions of the Navier–Stokes–Poisson system* (11) *which satisfy* (*ID*)*, then it holds*

$$\lambda \widetilde{E^\lambda} \rightharpoonup 0 \qquad weakly\ in\ L^2(0,T,L^2(\mathbb{R}^3)).$$

Proof. The proof follows by taking into account (61) and that $\lambda\widetilde{E^\lambda}$ is bounded in $L_{t,x}^2$. □

At this point we can hope that the weak convergence of $\lambda\widetilde{E^\lambda}$ is caused only by spatial oscillations, which allow us to introduce the microlocal defect measure in space. In order to do this, since the solutions are defined only in $(0,T)$, we need to extend E^λ and F^λ to 0 out of this interval and to cut off the frequencies greater than a certain quantity. This will be done in the next proposition.

Proposition 8. *With the same assumptions as in Lemma 3 we have*

$$\int_{|\xi|\leqq R} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}t |\lambda \mathcal{F}\widetilde{E^\lambda}(t,x)| \to 0, \tag{65}$$

for any R independent on λ.

Proof. Let χ_R be the characteristic function on $B(0,R)$ and let $\mathcal{T}_R = \mathcal{F}^{-1}\chi_R\mathcal{F}$ be the operator that cuts the frequencies greater than R. Clearly $\mathcal{T}_R$ is a bounded operator from L^2 to H^s, for any $s \geqq 0$ and $\nabla\mathcal{T}_R = \mathcal{T}_R\nabla$. If we apply $\mathcal{T}_R$ to (62) we have

$$\begin{aligned}\mathcal{T}_R E^\lambda(t,x) = \int_0^t \frac{\mathcal{T}_R F^\lambda(s,x)}{2i\lambda} \sin\left(\frac{t-s}{\lambda}\right) \mathrm{d}s \\ + \frac{\mathcal{T}_R E_1^\lambda(x)}{\lambda} \mathrm{e}^{it/\lambda} + \frac{\mathcal{T}_R E_2^\lambda(x)}{\lambda} \mathrm{e}^{-it/\lambda},\end{aligned} \tag{66}$$

and, by the estimates of Proposition 6, we have that $\mathcal{T}_R F^\lambda, \mathcal{T}_R E_1^\lambda, \mathcal{T}_R E_2^\lambda$ are bounded in $L_t^\infty L_x^2$. Since the solutions that we consider are defined in the time interval $0 \leqq t \leqq T$, in order to use the Fourier transform in time we need to extend them to 0, so we get that $\mathcal{T}_R F^\lambda$ is bounded in $L_{t,x}^2$. Let us introduce

$$H_+^\lambda = \int_0^t \mathcal{T}_R F^\lambda(s,x) \mathrm{e}^{-is/\lambda} \mathrm{d}s + 2i\mathcal{T}_R E_1^\lambda(x). \tag{67}$$

If we compute the space and time Fourier transform of (67), for τ large enough we get

$$\mathcal{F}_{t,x} H_+^\lambda(\tau,\xi) = \frac{1}{\tau} \mathcal{F}_{t,x}\mathcal{T}_R F^\lambda\left(\tau + \frac{1}{\lambda}, \xi\right), \tag{68}$$

so $\mathcal{F}_{t,x}H_+^\lambda$ is in L^2 in a neighborhood of $|\tau| = \infty$. Moreover, we have that

$$\int_{\mathbb{R}^3}\int_{|\tau|\geqq A} |\mathcal{F}_{t,x}H_+^\lambda(\tau,\xi)|^2 \mathrm{d}\tau \leqq \frac{1}{A^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}} |\mathcal{T}_R F^\lambda(t,x)|^2 \mathrm{d}x\mathrm{d}t. \tag{69}$$

As a consequence we have that all the mass of $\lambda\mathcal{F}_{t,x}\mathcal{T}_R E^\lambda$ is concentrated in $1/\lambda$. In fact, with

$$\lambda\mathcal{F}_{t,x}\mathcal{T}_R E^\lambda = \frac{1}{2i}\left[\mathcal{F}_{t,x}H_+^\lambda\left(\tau + \frac{1}{\lambda}, \xi\right) - \mathcal{F}_{t,x}H_-^\lambda\left(\tau - \frac{1}{\lambda}, \xi\right)\right], \tag{70}$$

we have that for any $\eta > 0$ there exists A such that

$$\int_{\mathbb{R}^3_\xi}\int_{\left|\tau\pm\frac{1}{\lambda}\right|\geqq A} |\lambda\mathcal{F}_{t,x}\mathcal{T}_R E^\lambda|^2 \mathrm{d}\xi\mathrm{d}\tau \leqq \eta.$$

In order to prove (65) we take into account the decomposition of $\widetilde{E^\lambda}$ in (64) and the following properties. For any $\eta > 0$ there exists A such that for any $\lambda < 1$ one has

$$\int_{\mathbb{R}^3_\xi}\int_{\left|\tau\pm\frac{1}{\lambda}\right|\geqq A} |\lambda\mathcal{F}_{t,x}\mathrm{e}^{\pm it/\lambda}E^\pm|^2 \mathrm{d}\xi\mathrm{d}\tau \leqq \eta. \tag{71}$$

Moreover,

$$\int_{\mathbb{R}^3_\xi}\int_{\left|\tau-\frac{1}{\lambda}\right|\geqq A}|\mathcal{F}_{t,x}(\lambda\mathcal{T}_R\widetilde{E^\lambda}-\mathrm{e}^{it/\lambda}E^+)|\mathrm{d}\tau\mathrm{d}\xi$$
$$=\int_{\mathbb{R}^3_\xi}\int_{\tau|\geqq A}|\mathcal{F}_{t,x}((\lambda\mathcal{T}_R\widetilde{E^\lambda}\mathrm{e}^{-it/\lambda}-\mathcal{T}_RE^+)|^2\mathrm{d}\xi\mathrm{d}\tau, \tag{72}$$

and so for E^-. On the other hand, we know that

$$\lambda\mathrm{e}^{-it/\lambda}\mathcal{T}_R\widetilde{E^\lambda}=\lambda\mathrm{e}^{-it/\lambda}\mathcal{T}_RE^\lambda-\mathcal{T}_RE^+-\mathrm{e}^{-2it/\lambda}\mathcal{T}_RE^-. \tag{73}$$

The same holds for E^-. By taking into account (71), (72), (73) and Parseval's identity we conclude in the following way.

$$\int_{|\xi|\leqq R}\mathrm{d}\xi\int\mathrm{d}t|\lambda\mathcal{F}\widetilde{E^\lambda}|^2=\int_{|\xi|\leqq R}\mathrm{d}\xi\int\mathrm{d}\tau|\lambda\mathcal{F}_{t,x}\mathcal{F}^{-1}\chi_R\mathcal{F}\widetilde{E^\lambda}|^2$$
$$=\int_{|\xi|\leqq R}\mathrm{d}\xi\int\mathrm{d}\tau|\mathcal{F}_{t,x}\mathcal{T}_R\widetilde{E^\lambda}|^2\to 0. \tag{74}$$

□

Now we are ready to prove the existence of a microlocal defect measure for the electric field E^λ. We start by proving the L^2-orthogonality of E^+, E^- and $\widetilde{E^\lambda}$.

Proposition 9. *For any $A\in\psi^0_{\mathrm{comp}}(\mathbb{R}^3,\mathcal{K}(\mathbb{R}^3))$ and for any $\phi\in\mathcal{D}(0,T)$ it holds*

$$\lim_{\lambda\to 0}\lambda^2\int\mathrm{d}t\phi(t)(AE^\lambda,E^\lambda)=\lim_{\lambda\to 0}\lambda^2\int\mathrm{d}t\phi(t)(A\widetilde{E^\lambda},\widetilde{E^\lambda})$$
$$+\int\mathrm{d}t\phi(t)(AE^+,E^+)+\int\mathrm{d}t\phi(t)(AE^-,E^-). \tag{75}$$

Proof. First of all we observe that

$$\lim_{\lambda\to 0}\int\mathrm{d}t\phi(t)(AE^+\mathrm{e}^{it/\lambda},E^-\mathrm{e}^{-it/\lambda})=0. \tag{76}$$

Then we also have

$$\lim_{\lambda\to 0}\int\mathrm{d}t\phi(t)\lambda(A\widetilde{E^\lambda},E^+\mathrm{e}^{-it/\lambda})=0. \tag{77}$$

In fact, if we denote by A^* the adjoint operator of A, we have that A^*E^+ is bounded in L^2 and, as a consequence, for any $\eta>0$ there exists $B>0$ such that

$$\int_{|\xi|\geqq B}\int\mathrm{d}t|\mathcal{F}A^*E^+|^2\mathrm{d}t\leqq\eta. \tag{78}$$

Combining (78) with (65) we get (77). □

In order to prove the Theorem 4 and to get (59) it remains only to investigate

$$\lim_{\lambda\to 0}\int \mathrm{d}t\lambda^2\phi(t)(A\widetilde{E^\lambda}, \widetilde{E^\lambda}). \tag{79}$$

The sequence $\lambda\widetilde{E^\lambda}$ fits in the framework of microlocal defect measures of GÈRARD [12], but as already explained we need his proof for our sequence.

Proposition 10. *Let w^λ be a bounded sequence of functions of $L^2_{t,x}$ which converges weakly to 0, such that for every compact set $K\subset\mathbb{R}^3$ one has*

$$\lim_{\lambda\to 0}\int_K \mathrm{d}\xi\int \mathrm{d}t|\mathcal{F}w^\lambda(t,\xi)| = 0.$$

Then there exists a positive measure ν^{GT} on $\mathbb{R}\times\mathbb{R}^3\times S^2$, such that for any $A\in\psi^0_{\mathrm{comp}}(\mathbb{R}^3,\mathcal{K}(\mathbb{R}^3))$ and of principal symbol $a(x,\xi)$ and for any $\phi\in\mathcal{D}(\mathbb{R})$ it holds that

$$\lim_{\lambda\to 0}\int \mathrm{d}t\phi(t)(Aw^\lambda, w^\lambda) = \langle\nu^{GT}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}\xi), \phi(t)a(x,\xi)\rangle.$$

To prove the previous theorem we follow the same line of argument as in [12]; we start with the following lemma.

Lemma 5. *With the assumptions as in Proposition 10, it holds*

$$\lim_{\lambda\to 0}\Im\int \mathrm{d}t\phi(t)(Aw^\lambda, w^\lambda) = 0, \tag{80}$$

$$\lim_{\lambda\to 0}\Re\int \mathrm{d}t\phi(t)(Aw^\lambda, w^\lambda) \geqq 0. \tag{81}$$

Proof. Since A is Hermitian we have

$$\Im(Aw^\lambda, w^\lambda) = \frac{1}{2i}((A-A^*)w^\lambda, w^\lambda),$$

where $A-A^*\in\psi^{-1}_{\mathrm{comp}}(\mathbb{R}^3,\mathcal{K}(\mathbb{R}^3))$, so (80) follows by using Proposition (1). For the real part let $\delta>0$, then $a+\delta\in C^\infty(S^*\Omega,\mathcal{L}(\mathbb{R}^3)$ and we can extract the square root B, namely $b=\delta^{1/2}+b'$. Let B' be such that $B'=OPS(b')$ and B such that $B=\varphi(\delta^{1/2}+B')$, $\varphi\in C_0^\infty(\mathbb{R}^3)$, $\varphi=1$ on the support of a, then $OPS^0(B)=OPS^0(\varphi\delta^{1/2}+\varphi B')=\varphi b\in\psi^0_{\mathrm{comp}}(\mathbb{R}^3,\mathcal{L}(\mathbb{R}^3))$. So we have that

$$B^*B=|\varphi|^2\delta+A+R,\qquad R\in\psi^{-1}_{\mathrm{comp}}(\mathbb{R}^3,\mathcal{K}(\mathbb{R}^3)),$$

but then we have

$$\Re\int \mathrm{d}t\phi(t)(Aw^\lambda, w^\lambda)\geqq -\delta\|\varphi w^\lambda\|^2_{L^2}+\Re(Rw^\lambda, w^\lambda).$$

We end up with (81) by sending λ to 0 and by using Proposition (1) again. □

From now on, the proof of Proposition 10 follows the same line of arguments as in [12]. So we can apply Propositions 10 to the sequence $\lambda\widetilde{E^\lambda}$ and we can conclude that there exists a positive measure $\nu^{\widetilde{E}}$ such that

$$\lim_{\lambda\to 0}\int \mathrm{d}t\phi(t)(A\lambda\widetilde{E^\lambda}, \lambda\widetilde{E^\lambda}) = \langle \nu^{\widetilde{E}}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}\xi), \phi(t)a(x, \xi)\rangle.$$

If we apply the remark in Exercise 1.5 of [12], since $\lambda\widetilde{E^\lambda}$ is a gradient and we are in the finite dimensional case, we have that there exists a positive measure ν^E such that

$$\nu^{\widetilde{E^\lambda}}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}\xi) = \xi_i\xi_j\nu^E(\mathrm{d}t, \mathrm{d}x).$$

This ends the proof of the Theorem 4.

7. Proof of Theorem 1

(i) It follows from (20) and (24).
(ii) It follows from (22).
(iii) It is proved in Proposition 7.
(iv) By taking into account that we can decompose u^λ as $u^\lambda = Pu^\lambda + Qu^\lambda$ and by using Proposition 7 with Corollary 1, we get

$$Pu^\lambda \longrightarrow u \quad \text{strongly in } L^2([0, T]; L^2_{\mathrm{loc}}(\mathbb{R}^3)).$$

(v) It follows from Theorem 4.
(vi) First of all we apply the Leray projector P to the momentum equation of the system (11), then we have

$$\partial_t P(\rho^\lambda u^\lambda) + P\,\mathrm{div}(\rho^\lambda u^\lambda \otimes u^\lambda) = \mu\Delta Pu^\lambda + P\,\mathrm{div}(\lambda E^\lambda \otimes \lambda E^\lambda). \tag{82}$$

It is a straightforward computation to pass into the limit in the terms $\partial_t P(\rho^\lambda u^\lambda)$ and ΔPu^λ, so, for any $\varphi \in \mathcal{D}([0, T] \times \mathbb{R}^3)$ we obtain

$$\langle P(\partial_t(\rho^\lambda u^\lambda) - \mu\Delta u^\lambda), \varphi\rangle \longrightarrow \langle P(\partial_t u - \mu\Delta u), \varphi\rangle. \tag{83}$$

In order to study the convergence of the convective term we decompose it in this way

$$\begin{aligned}\langle P\,\mathrm{div}(\rho^\lambda u^\lambda \otimes u^\lambda), \varphi\rangle &= \langle P\,\mathrm{div}((\rho^\lambda - 1)u^\lambda \otimes u^\lambda), \varphi\rangle \\ &\quad + \langle P\,\mathrm{div}(u^\lambda \otimes u^\lambda), \varphi\rangle \\ &= I_1 + I_2.\end{aligned} \tag{84}$$

The term I_1 goes strongly to zero. In fact it is enough to take into account that $\rho^\lambda - 1$ goes weakly to zero in $L^\infty_t L^k_2$, while by interpolation u^λ is strongly convergent in $L^\infty_t L^{k'}_2$. Concerning I_2, we have as $\lambda \to 0$,

$$\begin{aligned} I_2 &= \langle \mathrm{div}(Pu^\lambda \otimes Qu^\lambda), P\varphi\rangle + \langle \mathrm{div}(Qu^\lambda \otimes Qu^\lambda), P\varphi\rangle \\ &\longrightarrow \langle \mathrm{div}(u \otimes u), P\varphi\rangle = \langle P\,\mathrm{div}(u \otimes u), \varphi\rangle.\end{aligned} \tag{85}$$

Finally, to establish the convergence of the term $P\,\mathrm{div}(\lambda\nabla V^\lambda \otimes \lambda\nabla V^\lambda)$ we have to take the limit of $\langle P\,\mathrm{div}(\lambda\nabla V^\lambda \otimes \lambda\nabla V^\lambda), \varphi\rangle$ for any $\varphi \in \mathcal{D}([0,T]\times\mathbb{R}^3)$, but we have no strong convergence for $\lambda\nabla V^\lambda$ as $\lambda \to 0$, so we apply the Theorem 4 and use the microlocal defect measure defined in (59) and we have as $\lambda \to 0$,

$$\begin{aligned}\langle P\,\mathrm{div}(\lambda\nabla V^\lambda \otimes \lambda\nabla V^\lambda), \varphi\rangle = {} & \langle \lambda\nabla V^\lambda \otimes \lambda\nabla V^\lambda, \nabla P\varphi\rangle \\ & + \int_0^T \int_{\mathbb{R}^3\times S^2} \nabla P\varphi \frac{\xi\otimes\xi}{|\xi|^2}\mathrm{d}\nu^E \mathrm{d}x\mathrm{d}t \\ & + \int_0^T \int_{\mathbb{R}^3} \nabla P\varphi(E^+\otimes E^+ \\ & + E^-\otimes E^-)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{86}$$

So, by using together (83), (84), (85) and (86), we have that u satisfies the following equation in $\mathcal{D}'([0,T]\times\mathbb{R}^3)$

$$\begin{aligned} & P\Big(\partial_t u - \Delta u + (u\cdot\nabla)u \\ & \quad - \mathrm{div}(E^+\otimes E^+ + E^-\otimes E^-) - \mathrm{div}\Big\langle \nu^E, \frac{\xi\otimes\xi}{|\xi|^2}\Big\rangle\Big) = 0. \end{aligned}$$

8. Equations for the Correctors: Proof of the Theorem 2

The purpose of this section is to show how to construct the correctors E^+ and E^- in the case of smooth solutions for the system (11). In particula,r we will perform the quasineutral limit in the framework of Proposition 5 and we will end up with the proof of the Theorem 2. We will divide the proof of the Theorem 2 in different steps. First of all we decompose the electric and the velocity fields in order to single out the oscillating parts, then we show the existence of the correctors and finally the equations that they satisfy. A similar analysis has been carried out in [16] in the case of a periodic domain for the Vlasov Poisson system.

For the computations we have to perform later on it is more convenient to rewrite the expression (62) for E^λ in terms of its Fourier transform,

$$\begin{aligned}\mathcal{F}E^\lambda(t,\xi) = {} & \int_0^t \frac{1}{\lambda}\mathcal{F}F(s,\xi)\sin\left(\frac{t-s}{\lambda}\right)\mathrm{d}s \\ & + \mathcal{F}E^\lambda(0,\xi)\cos\left(\frac{t}{\lambda}\right) + \lambda\mathcal{F}E^\lambda_t(0,\xi)\sin\left(\frac{t}{\lambda}\right). \end{aligned} \tag{87}$$

8.1. Step 1: Decomposition of the Electric Field

In this section we decompose the electric field in a way that isolates the time oscillations. First, we define the following operator that cuts off the oscillations in time. For any $\phi \in C^0(0,T,H^s)$, $s \geqq 0$ we set

$$\mathcal{H}^\lambda\phi(t,x) = \frac{1}{2\pi\lambda}\int_t^{t+2\pi\lambda}\phi(\sigma,x)\mathrm{d}\sigma, \qquad \mathcal{G}^\lambda = I - \mathcal{H}^\lambda. \tag{88}$$

Then we decompose E^λ in the following way:

$$E_1^\lambda = \mathcal{G}^\lambda E^\lambda \qquad E_2^\lambda = \mathcal{H}^\lambda E^\lambda. \tag{89}$$

Clearly, E_1^λ is the oscillatory part of E^λ, while E_2^λ is its averaged part. The following proposition holds.

Proposition 11. *Under the assumptions of Proposition 5 there exist three vector fields (which are gradients), E_1^λ, E_2^λ, and W^λ such that $E^\lambda = E_1^\lambda + E_2^\lambda$ with*

(i) $\|\lambda E_1^\lambda\|_{L_t^\infty H_x^{s-1}} \leqq C$,
(ii) $\partial_t W^\lambda = E_1^\lambda$, $\quad \|W^\lambda\|_{L_t^\infty H_x^{s-1}} \leqq C$ *and* $W^\lambda \rightharpoonup 0$ *weakly in* L^2,
(iii) $\|E_2^\lambda\|_{L_t^\infty H_x^{s-1}} \leqq C$.

Proof. Since F is uniformly bounded in $L_t^\infty H_x^{s-1}$ we have that $\mathcal{H}^\lambda E^\lambda$ is bounded in $L_t^\infty H_x^{s-1}$ and so we get (i) and (iii). Now, if we define

$$\begin{aligned}\mathcal{F}W^\lambda &= \lambda \mathcal{F}E^\lambda(0,\xi)\sin\left(\frac{t}{\lambda}\right) - \lambda\mathcal{F}E_t^\lambda(0,\xi)\cos\left(\frac{t}{\lambda}\right)\\ &\quad + \int_0^t \mathrm{d}\sigma \int_0^\sigma \mathrm{d}s\,\frac{\lambda F(s,\xi)}{\lambda}\sin\left(\frac{\sigma-s}{\lambda}\right) + \int_0^t \mathcal{F}\mathcal{H}^\lambda E^\lambda(\sigma,\xi)\mathrm{d}\sigma, \end{aligned}\tag{90}$$

we easily obtain (ii). □

8.2. Step 2: Decomposition and Limit System for the Velocity

Now we decompose the velocity field u^λ, in the following way

$$u^\lambda = v^\lambda + W^\lambda, \tag{91}$$

where W^λ is the corrector introduced in Proposition 11 and we can look at v^λ as the velocity field u^λ without its oscillatory part W^λ. For v^λ we can prove the following result.

Proposition 12. *Let $v^\lambda = u^\lambda - W^\lambda$, then for any $s' < s-2$, v^λ and ρ^λ converge in $C^0(0,T;H^{s'}(\mathbb{R}^3))$, respectively to v and 1 and there exists a function Π, such that v satisfies,*

$$\operatorname{div} v = 0 \tag{92}$$

$$\partial_t v + v\cdot\nabla v - \Delta v = \nabla\Pi. \tag{93}$$

Proof. First, we can observe that v^λ is bounded in $L_t^\infty H_x^{s-1}$ and $\partial_t v^\lambda$ is bounded in $L_t^\infty H_x^{s-2}$, so we have that

$$v^\lambda \longrightarrow v \quad \text{strongly in } C(0,T;H_{\mathrm{loc}}^{s'-1}), \text{ for any } s' < s. \tag{94}$$

Now we rewrite the second equation of (11) in terms of v^λ and W^λ.

$$\begin{aligned}&\rho\partial_t v^\lambda + \rho(v^\lambda + W^\lambda)\cdot\nabla(v^\lambda + W^\lambda) + \nabla(\rho^\lambda)^\gamma\\ &\quad = \Delta(v^\lambda + W^\lambda) + \nabla\operatorname{div}(v^\lambda + W^\lambda) + \rho E_2^\lambda.\end{aligned}\tag{95}$$

By the Poisson equation $\lambda^2 \operatorname{div} E^\lambda = \rho^\lambda - 1$, we have that

$$\rho^\lambda \longrightarrow 1 \quad \text{strongly in } C(0, T; H^{s-1}). \tag{96}$$

From this, (92) follows. Then, by using Proposition 11 we have that W^λ and $\partial_x W^\lambda$ are bounded in L^2 and converge weakly to 0, so we can pass into the limit in (95) and we conclude with (93). □

8.3. Step 3: Existence of the Correctors

In this section we will identify and establish the existence of the correctors. First, we introduce the operator $T^\lambda_\pm$; for any $\phi \in L^\infty_t L^2_x$, we set

$$\mathcal{F} T^\lambda_\pm \phi(t, \xi) = \mathrm{e}^{\mp t/\lambda} \mathcal{F}\phi(t, \xi). \tag{97}$$

By construction we have that $T^\lambda_\pm$ satisfies the following properties.

- (T1) $T^\lambda_\pm$ are selfadjoint, act isometrically on $L^\infty_t H^s_x$, for all s and $T^\lambda_+ T^\lambda_- = T^\lambda_- T^\lambda_+ = I$,
- (T2) if $\phi^\lambda \to \phi$, strongly in L^2, then, $T^\lambda_+ \rightharpoonup 0$, weakly in L^2,
- (T3) if $\phi, \psi \in L^\infty_t H^s_x$, $s > d/2$, $T^\lambda_+ \phi T^\lambda_- \psi \rightharpoonup 0$ weakly in $L^2_{t,x}$.

In the next proposition we prove the existence of correctors for the electric field E^λ and the velocity field u^λ.

Proposition 13. *There exist two functions E^+ and E^- in $C^0(H^{s-1}_{\mathrm{loc}})$ such that, for all $s' < s$,*

- (c1) $\|\lambda E^\lambda_1 - T^\lambda_- E^+ - T^\lambda_+ E^-\|_{C^0(H^{s'-1}_{\mathrm{loc}})} \to 0$,
- (c2) $\|W^\lambda - \frac{1}{i} T^\lambda_- E^+ - \frac{1}{i} T^\lambda_+ E^-\|_{C^0(H^{s'-1}_{\mathrm{loc}})} \to 0$.

Proof. We can split E^λ into two components, E^λ_+ and E^λ_-, in the following way:

$$\mathcal{F} E^\lambda_+(t, \xi) = \lambda \mathrm{e}^{it/\lambda} \left(\frac{\mathcal{F} E^\lambda(0, \xi)}{2} + \frac{\lambda \mathcal{F} E^\lambda_t(0, \xi)}{2i} + \int_0^t \frac{\mathcal{F} F(s, \xi)}{2i\lambda} \mathrm{e}^{-it/\lambda} \mathrm{d}s \right) \tag{98}$$

and we define E^λ_- in a similar way. We can easily verify that

$$\|T_\pm E^\lambda_\pm\|_{L^\infty_t H^{s-1}_x} = \|E^\lambda_\pm\|_{L^\infty_t H^{s-1}_x} \leqq C, \qquad \|\partial_t \mathcal{F} T_\pm E^\lambda_\pm\|_{L^\infty_t H^{s-1}_x} \leqq C. \tag{99}$$

So we can conclude that that there exist two curl free vectors E^+, E^- such that

$$T_\pm E^\lambda_\pm \to E^\pm \quad \text{strongly in } C^0_t H^{s'-1}_{x,\mathrm{loc}}, \text{ for all } s' < s. \tag{100}$$

Since we know that T^λ_+ is an isometry we have

$$E^\lambda_+ - T^\lambda_- E^+ \to 0 \quad \text{and} \quad E^\lambda_- - T^\lambda_+ E^- \to 0 \text{ strongly in } C^0_t H^{s'-1}_{x,\mathrm{loc}}. \tag{101}$$

By Proposition 11 we know that $E^\lambda = E^\lambda_1 + E^\lambda_2$, so by using (101) we get

$$\lambda E^\lambda_1 - T^\lambda_- E^+ - T^\lambda_+ E^- \longrightarrow 0 \quad \text{strongly in } C^0_t H^{s'-1}_{x,\mathrm{loc}}. \tag{102}$$

In order to prove (c2) we use for W^λ, defined by (90), a decomposition similar to (98), namely

$$\mathcal{F}W_+^\lambda(t,\xi) = \lambda e^{+it/\lambda}\left(\frac{\mathcal{F}E^\lambda(0,\xi)}{2i} - \frac{\lambda\mathcal{F}E_t^\lambda(0,\xi)}{2} - \int_0^t \frac{\mathcal{F}F(s,\xi)}{2\lambda} e^{-it/\lambda} ds\right), \tag{103}$$

and

$$\mathcal{F}W_0^\lambda = \int_0^t \mathcal{F}F(s,\xi)ds - \int_0^t \mathcal{H}^\lambda E^\lambda(s,\xi)ds. \tag{104}$$

We define W_-^λ in a similar way. From (90), (103), (104) we have $W^\lambda = W_0^\lambda + W_+^\lambda + W_-^\lambda$. Arguing as before we have that there exists W^+ and W^- such that

$$W_+^\lambda - T_-^\lambda W^+ \to 0 \quad \text{and} \quad W_-^\lambda - T_+^\lambda W^- \to 0 \quad \text{strongly in} \quad C_t^0 H_{x,\text{loc}}^{s'-1}, \tag{105}$$

and

$$W_0^\lambda \to 0 \text{ strongly in } L_t^\infty H_x^{s-1}.$$

So we get that

$$W^\lambda - T_-^\lambda W^+ - T_+^\lambda W^- \to 0 \quad \text{strongly in } C_t^0 H_{x,\text{loc}}^{s'-1}. \tag{106}$$

The last step is to identify $W^\pm$. Taking into account that $\mathcal{F}T_+^\lambda E_+^\lambda = i\mathcal{F}T_+^\lambda W_+^\lambda$, (100) and (105), we end up with

$$W^\pm = -iE^\pm. \tag{107}$$

□

8.4. Step 4: Equation of the Correctors

In this section we finish the proof of Theorem 2, and we will show that equation (18) is satisfied by the correctors. In order to do this, we take the equation (60)

$$\begin{aligned}\lambda^2\partial_{tt}E^\lambda + E^\lambda = \operatorname{div}\Delta^{-1}\nabla\operatorname{div}\left(\rho^\lambda u^\lambda \otimes u^\lambda + (\rho^\lambda)^\gamma\mathbb{I} - \lambda^2 E^\lambda \otimes E^\lambda\right)\\ + \frac{\lambda^2}{2}\operatorname{div}\left(|E^\lambda|^2\mathbb{I}\right) - 2\nabla\operatorname{div} u^\lambda.\end{aligned} \tag{108}$$

We substitute the decompositions obtained in the previous sections and we send λ to 0. Let $\phi \in \mathcal{D}((0,T)\times\mathbb{R}^3)$; then we have

$$\begin{aligned}(T_+^\lambda(\lambda^2\partial_{tt}E^\lambda + E^\lambda),\phi) &= (E^\lambda, \lambda^2\partial_{tt}T_-^\lambda\phi + T_-^\lambda\phi)\\ &= (T_-^\lambda T_+^\lambda E^\lambda, \lambda^2\partial_{tt}T_-^\lambda\phi + T_-^\lambda\phi)\\ &= (\lambda T_+^\lambda E^\lambda, \frac{1}{\lambda}(\lambda^2 T_+^\lambda\partial_{tt}T_-^\lambda\phi + \phi)).\end{aligned} \tag{109}$$

We know that $\lambda T_+^\lambda E^\lambda \rightharpoonup E^+$weakly in L^2 and we can compute

$$\mathcal{F}\left(\frac{\lambda^2 T_+^\lambda \partial_{tt} T_-^\lambda \phi + T_-^\lambda \phi + \phi}{\lambda}\right) = \lambda^2 \mathrm{e}^{-it/\lambda} \partial_{tt}(\mathrm{e}^{it/\lambda}\mathcal{F}\phi) + \frac{1}{\lambda}\mathcal{F}\phi$$
$$= 2i\partial_t \mathcal{F}\phi + \lambda \partial_{tt}\mathcal{F}\phi. \qquad (110)$$

So we have

$$\begin{aligned}(T_+^\lambda(\lambda^2 \partial_{tt} E^\lambda + E^\lambda), \phi) &= (\lambda \mathcal{F} T_+^\lambda E^\lambda, \mathcal{F}\frac{1}{\lambda}(\lambda^2 T_+^\lambda \partial_{tt} T_-^\lambda \phi + \phi)) \\ &= (\lambda \mathcal{F} T_+^\lambda E^\lambda, 2i\partial_t \mathcal{F}\phi + \lambda \partial_{tt}\mathcal{F}\phi) \\ &\to (\mathcal{F}E^+, 2i\partial_t \mathcal{F}\phi) = -2i(\partial_t E^+, \phi). \qquad (111)\end{aligned}$$

The next term to analyze is the convective term $\rho^\lambda u^\lambda \otimes u^\lambda = \rho^\lambda(v^\lambda + W^\lambda) \otimes (v^\lambda + W^\lambda)$, to which end it will be sufficient to analyse the terms of this sort: $\rho^\lambda v_i^\lambda v_k^\lambda$, $\rho^\lambda v_i^\lambda W_k^\lambda$, $\rho^\lambda W_i^\lambda W_k^\lambda$, $i, k = 1, \ldots, 3$. Since v_i^λ, v_k^λ are strongly convergent in L^2, by using the property (T2) we get

$$T_+^\lambda(\rho^\lambda v_i^\lambda v_k^\lambda) \rightharpoonup 0 \quad \text{weakly in } L^2. \qquad (112)$$

Taking into account (c2), we have

$$\begin{aligned}&\lim_{\lambda\to 0} T_+^\lambda(\operatorname{div} \Delta^{-1}\nabla \operatorname{div}(\rho^\lambda v_i^\lambda \otimes W_k^\lambda) \\ &= -i \lim_{\lambda\to 0} T_+^\lambda((\operatorname{div} \Delta^{-1}\nabla \operatorname{div}(\rho^\lambda v_i^\lambda T_+^\lambda E_k^- + \rho^\lambda v_i^\lambda T_-^\lambda E_k^+)) = I_1 + I_2.\end{aligned} \qquad (113)$$

For I_1 we have

$$\begin{aligned}I_1 = &-i \lim_{\lambda\to 0} T_+^\lambda(\operatorname{div} \Delta^{-1}\nabla \operatorname{div}(\rho^\lambda v_i^\lambda T_+^\lambda E_k^-)) \\ &-i \lim_{\lambda\to 0}(\mathrm{e}^{-it/\lambda}\mathcal{F}(\operatorname{div} \Delta^{-1}\nabla \operatorname{div}(\rho^\lambda v_i^\lambda \mathrm{e}^{-it/\lambda}\mathcal{F}E_k^-)) \to 0. \qquad (114)\end{aligned}$$

Concerning I_2 we have

$$\begin{aligned}I_2 &= -i(\mathcal{F}T_+^\lambda((\operatorname{div} \Delta^{-1}\nabla \operatorname{div}(\rho^\lambda v_i^\lambda T_-^\lambda E_k^+), \mathcal{F}\phi) \\ &= -i\mathrm{e}^{-it/\lambda}(\xi\mathcal{F}(\rho^\lambda v_i^\lambda) * \mathcal{F}T_-^\lambda E_k^+, \mathcal{F}\phi) = -i(\xi\mathcal{F}(\rho^\lambda v_i^\lambda) * \mathcal{F}E_k^+, \mathcal{F}\phi) \\ &= -i(\operatorname{div}(\rho^\lambda v_i^\lambda E_k^+), \phi) \to -i(\operatorname{div}(v_i E_k^+), \phi). \qquad (115)\end{aligned}$$

Finally we estimate the term $\rho^\lambda W_i^\lambda W_k^\lambda$; again, we use (c2) and we obtain

$$\begin{aligned}&\lim_{\lambda\to 0}(T_+^\lambda(\rho^\lambda W_i^\lambda W_k^\lambda), \phi) \\ &= -(\mathcal{F}T_+^\lambda[\rho^\lambda(T_+^\lambda E^- + T_-^\lambda E^+)_i(T_+^\lambda E^- + T_-^\lambda E^+)_k], \mathcal{F}\phi) \\ &= \lim_{\lambda\to 0}(\mathcal{F}(E^+ + \mathrm{e}^{-2it/\lambda}E^-)_i * \mathcal{F}(\mathrm{e}^{it/\lambda}E^+ + \mathrm{e}^{-it/\lambda}E^-)_k, \mathcal{F}\phi) \\ &= \lim_{\lambda\to 0}((E^+ + \mathrm{e}^{-2it/\lambda}E^-)_i(\mathrm{e}^{it/\lambda}E^+ + \mathrm{e}^{-it/\lambda}E^-)_k, \phi) = 0. \qquad (116)\end{aligned}$$

The next term that we have to estimate is $\nabla \operatorname{div} u^\lambda$; we have

$$\begin{aligned}\lim_{\lambda\to 0}(T_+^\lambda \nabla \operatorname{div} u^\lambda, \phi) &= \lim_{\lambda\to 0}\left[(T_+^\lambda \nabla \operatorname{div} v^\lambda, \phi) + (T_+^\lambda \nabla \operatorname{div} W^\lambda, \phi)\right]\\ &= -i \lim_{\lambda\to 0}(\mathcal{F}T_+^\lambda(T_-^\lambda E^+ + T_+^\lambda E^-), \mathcal{F}\phi)\\ &= -i \lim_{\lambda\to 0}(\xi_i\xi_j(\mathcal{F}E^+ + \mathrm{e}^{-2it/\lambda}\mathcal{F}E^-), \mathcal{F}\phi)\\ &= -i(\nabla \operatorname{div} E^+, \phi). \end{aligned} \tag{117}$$

By using (96) we get that $\operatorname{div} \Delta^{-1}\nabla \operatorname{div}((\rho^\lambda)^\gamma \mathbb{I})$ converges strongly to zero. It remains only to estimate the two terms $\operatorname{div} \Delta^{-1}\nabla \operatorname{div}(\lambda^2 E^\lambda \otimes E^\lambda)$ and $\operatorname{div}(|\lambda E^\lambda|^2\mathbb{I})$. Since the arguments work in a similar way, we estimate only the first one. Let us denote by $\mathcal{A}$ the operator $\operatorname{div} \Delta^{-1}\nabla \operatorname{div}$ and by $a(\xi)$ its principal symbol. Then we have

$$\begin{aligned}&(T_+^\lambda \mathcal{A}(\lambda^2 E^\lambda \otimes E^\lambda), \phi)\\ &\quad = (\mathrm{e}^{-it/\lambda} a(\xi)\mathcal{F}(T_-^\lambda E^+ + T_+^\lambda E^-) * \mathcal{F}(T_-^\lambda E^+ + T_+^\lambda E^-), \mathcal{F}\phi) = 0. \end{aligned} \tag{118}$$

If we sum up (111), (112), (111), (115), (114), (116), (117) and (118), we end up with the following equation for the corrector E^+

$$\partial_t E^+ + \operatorname{div}(v \otimes E^+) - \nabla \operatorname{div} E^+ = 0, \qquad PE^+ = 0. \tag{119}$$

Moreover, by projecting the equation (108) in divergence free space and following the same steps as before, we get as λ goes to 0, the following relation for E^+

$$P \operatorname{div}(v \otimes E^+) = 0. \tag{120}$$

As a consequence of (119) and (120) we get that E^+ satisfies the parabolic equation

$$\partial_t E^+ - \Delta E^+ + Q \operatorname{div}(v \otimes E^+) = 0. \tag{121}$$

On order to obtain the equation satisfied by E^- we can follow step by step what we have done for E^+. From the previous paragraph it is clear that the equation (121) holds in the sense of distribution. In the next proposition we can establish a more precise result on the existence of solutions for (121); in particular, we will see that the kernel of the Leray projector P is an invariant subspace for the flow of equation (121).

Proposition 14. *Let us consider the correctors equation*

$$\partial_t E^\pm - \Delta E^\pm + Q \operatorname{div}(v \otimes E^\pm) = 0, \tag{122}$$

where $v \in L^\infty(0, T; H^s(\mathbb{R}^3))$, $3/2 \leqq s \leqq 2$ and the initial data satisfy

$$E^\pm(0) \in L^2(\mathbb{R}^3), \qquad PE^\pm(0) = 0. \tag{123}$$

Then the Cauchy problem (122)–(123) has a unique solution $E^\pm \in L^\infty(0, T; L^2(\mathbb{R}^3))$, such that $PE^\pm(\cdot, t) = 0$, for any $t \in [0, T]$.

Proof. The proof follows by rewriting (122) in integral form and by using a standard fixed point argument (for more detail see [29], Chapter IV, Exercises 7.8 and 7.9). □

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