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SINGULAR CONVERGENCE TO NONLINEAR DIFFUSION WAVES FOR SOLUTIONS TO THE CAUCHY PROBLEM FOR THE COMPRESSIBLE EULER EQUATIONS WITH DAMPING

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We investigate the singular limit for the solutions to the compressible gas dynamics equations with damping term, after a parabolic scaling, in the one-dimensional isentropic case. In particular, we study the convergence in Sobolev norms towards diffusive prophiles, in case of well-prepared initial data and small perturbations of them. The results are obtained by means of symmetrization and energy estimates.

Keywords: Symmetrizable hyperbolic systems; singular convergence; equations of compressible isentropic gas dynamics in a porous medium.

AMS Subject Classification: 35L55

1. Introduction

The aim of this paper is to study the singular convergence of a class of solutions to the one-dimensional compressible Euler flow through porous media. In particular, we consider solutions to the system

$$\begin{aligned} &\rho_\tau + (\rho u)_x = 0\,, \\ &u_\tau + u u_x + \frac{p(\rho)_x}{\rho} = -\frac{u}{\varepsilon}\,. \end{aligned} \tag{1.1}$$

After the time-scaling $\tau = t/\varepsilon$, we investigate the behaviour of the rescaled system

$$\begin{aligned} &\rho_t^\varepsilon + (\rho^\varepsilon u^\varepsilon)_y = 0\,, \\ &u_t^\varepsilon + u^\varepsilon u_y^\varepsilon + \frac{p(\rho^\varepsilon)_y}{\varepsilon^2 \rho^\varepsilon} = -\frac{u^\varepsilon}{\varepsilon^2}\,, \end{aligned} \tag{1.2}$$

as the parameter ε goes to zero, and show that, under some assumption on the initial data and on the limiting states at infinity, the density in (1.2) converges in a suitable norm to a self-similar solution to the Porous Medium equation

$$\tilde{\rho}_t = p(\tilde{\rho})_{yy} \,. \tag{1.3}$$

This topic was first studied by Marcati and Milani in Ref. 9, in order to contribute to the understanding of the hyperbolic nature of porous media flows. In the recent paper by Marcati and Rubino,[11] the case of inhomogeneous isentropic gas-dynamics is studied in the framework of the Hypebolic to Parabolic Relaxation Theory. We also mention the paper by Lattanzio and Yong,[7] where the hyperbolic–parabolic relaxation limits are studied in the framework of H^s-solutions, using the approach of singular limits. This technique is the most natural to analyse the so-called *initial layer*, which is one of the main issues of Ref. 7. Since we are not interested in this kind of problems, we avoid initial layer phenomena by prescribing well-prepared initial data.

The asymptotic behaviour of the damped compressible Euler flow in Lagrangian coordinates has been studied by Hsiao and Liu in Refs. 1 and 2 and by Nishihara in Ref. 12. Moreover, a result concerning the 2-D perturbation of this problem was proved by Lattanzio and Marcati in Ref. 5. Recently in Ref. 13, Nishihara, Wang and Yang proved a sharper result on the L_p-convergence $(2 \leq p \leq \infty)$ by means of a Green function technique. All the previously mentioned results deal with the asymptotic analysis in Lagrangian coordinates and cannot be used to investigate the asymptotic behaviour in the Eulerian framework. Our result can be seen as a description of the large-time behaviour in Eulerian coordinates by carrying out the parabolic scaling

$$x = \frac{y}{\varepsilon}, \qquad \tau = \frac{t}{\varepsilon^2}, \qquad u^\varepsilon(y,t) = \frac{1}{\varepsilon} u\left(\frac{y}{\varepsilon}, \frac{t}{\varepsilon^2}\right)$$

on the system

$$\begin{aligned} \rho_\tau + (\rho u)_x &= 0\,, \\ u_\tau + u u_x + \frac{p(\rho)_x}{\rho} &= -u\,. \end{aligned} \tag{1.4}$$

We remark that although in both the Eulerian and the Lagrangian cases the limiting prophiles satisfy the Porous Media equation, the two cases cover different physical situations.

In the next section we provide a detailed explanation of the problem and state the convergence results. In Sec. 3 we give the proof of the first theorem, concerning the convergence of the density in $L^\infty([0,T], L^\infty(\mathbb{R}))$ and of the velocity in $L^2([0,T], L^\infty(\mathbb{R}))$ for any $T > 0$. In the last section we carry out a small perturbation result for the Porous Medium equation, in order to extend the result in the first theorem to the case of small perturbated initial datum for the density.

2. Statement of the Problem and Results

Let us consider the one-dimensional, isentropic, compressible Euler equations through a porous medium in Eulerian coordinates. In case of smooth solutions, with $\rho > 0$, the system may be written as

$$\begin{aligned} &\partial_\tau \rho + u\partial_x \rho + \rho \partial_x u = 0\,, \\ &\partial_\tau u + u\partial_x u + \frac{p'(\rho)}{\rho}\partial_x \rho = -\frac{u}{\varepsilon}\,. \end{aligned} \tag{2.5}$$

Here, $\rho > 0$ is the density, u is the velocity, $x \in \mathbb{R}$, $\tau > 0$, $p : \mathbb{R} \to \mathbb{R}_+$ is a smooth function such that $p' > 0$, and $\varepsilon > 0$ is a small parameter. After the time scaling $\tau = \frac{t}{\varepsilon}$, $\rho^\varepsilon(x,t) = \rho(x, \frac{t}{\varepsilon})$, $u^\varepsilon(x,t) = \frac{1}{\varepsilon}u(x, \frac{t}{\varepsilon})$, the system (2.5) becomes

$$\begin{aligned} &\partial_t \rho^\varepsilon + u^\varepsilon \partial_x \rho^\varepsilon + \rho^\varepsilon \partial_x u^\varepsilon = 0\,, \\ &\partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + \frac{p'(\rho^\varepsilon)}{\varepsilon^2 \rho^\varepsilon}\partial_x \rho^\varepsilon = -\frac{u^\varepsilon}{\varepsilon^2}\,. \end{aligned} \tag{2.6}$$

Thus, as ε goes to 0, we expect the solutions to (2.6) to be described by the solutions to the following system

$$\begin{aligned} &\partial_t \tilde\rho + \partial_x(\tilde\rho \tilde u) = 0\,, \\ &\partial_x p(\tilde\rho) = -\tilde\rho \tilde u\,, \end{aligned} \tag{2.7}$$

which is equivalent to the Porous Medium equation

$$\tilde\rho_t = p(\tilde\rho)_{xx}\,, \tag{2.8}$$

where the relation between the pressure p and the velocity $\tilde u$ is given by the well-known Darcy's law

$$\tilde u = -\frac{p(\tilde\rho)_x}{\tilde\rho}\,. \tag{2.9}$$

For the system (2.6), we prescribe the following limiting conditions at infinity

$$\rho^\varepsilon(\pm\infty, t) = \rho^\pm \quad \text{for any } t \geq 0$$

$$u^\varepsilon(\pm\infty, 0) = u^\pm\,,$$

with ρ^+, $\rho^- > 0$. Since we expect the inertial terms of the second equation in (2.6) to decay faster than the others, in addition we require

$$u^\varepsilon(\pm\infty, t) = e^{-t/\varepsilon^2}u^\pm \quad \text{for any } t \geq 0\,.$$

Therefore, we assume the following behaviour at $x \to \pm\infty$ for the system (2.7)

$$\tilde\rho(\pm\infty, t) = \rho^\pm\,,$$

$$\tilde u(\pm\infty, t) = 0\,,$$

for any $t \geq 0$. The initial datum on the density of the hyperbolic problem (2.6) is assumed to be the same of (2.8), namely

$$\rho^\varepsilon(x, 0) = \tilde\rho(x, 0) = \tilde\rho_0(x)\,,$$

where $\tilde{\rho}_0$ is a bounded smooth function (e.g. $\tilde{\rho}_0 \in H^3(\mathbb{R})$) such that

$$0 < \mu_0 \leq \tilde{\rho}_0(x) \leq \mu_1 \,.$$

Moreover, we require the initial datum on the velocity u^ε to be given by the initial value of $\tilde{u}$ in the system (2.7) (which is determined by the Darcy's law) plus a small corrector, needed to match the limiting conditions, namely

$$u_0^\varepsilon(x) = -\frac{p'(\tilde{\rho}_0(x))}{\tilde{\rho}_0(x)}\tilde{\rho}_0'(x) + w^\varepsilon(x,0)\,. \tag{2.10}$$

The expression for the corrector w^ε is

$$w^\varepsilon(x,t) = e^{-t/\varepsilon^2}[u^- + (u^+ - u^-)\psi(x)]\,, \tag{2.11}$$

where

$$\psi(x) = \frac{\int_{-\infty}^{x} \phi(y)dy}{\int_{-\infty}^{+\infty} \phi(y)dy}\,,$$

for some $\phi \in C_c^\infty(\mathbb{R})$, $\phi \geq 0$. We observe that w^ε satisfies the equation

$$\partial_t w^\varepsilon = -\frac{1}{\varepsilon^2} w^\varepsilon\,.$$

This corrector does not affect the asymptotic analysis since it decays exponentially fast. The well-prepared initial data condition (2.10) is prescribed in order to avoid the problem of the initial layer.

As a consequence of the boundedness of $\tilde{\rho}_0$ and of the comparison principle for the parabolic Eq. (2.8), we have

$$\mu_0 \leq \tilde{\rho}(x,t) \leq \mu_1\,. \tag{2.12}$$

We will consider solutions $(\tilde{\rho}, \tilde{u})$ to (2.8) satisfying the time-asymptotic estimates

$$\begin{aligned}
\left|\frac{\partial^{\alpha+\beta}\tilde{\rho}(t)}{\partial x^\alpha \partial t^\beta}\right|_\infty &= O(\delta)\frac{1}{(t+1)^{\frac{\alpha}{2}+\beta}}\,, \quad \alpha,\beta > 0\\
\int_{-\infty}^{+\infty}\left|\frac{\partial^{\alpha+\beta}\tilde{\rho}(x,t)}{\partial x^\alpha \partial t^\beta}\right|^2 dx &= O(\delta^2)\frac{1}{(t+1)^{\alpha+2\beta-\frac{1}{2}}}\,, \quad \alpha,\beta > 0\\
\left|\frac{\partial^{\alpha+\beta}\tilde{u}(t)}{\partial x^\alpha \partial t^\beta}\right|_\infty &= O(\delta)\frac{1}{(t+1)^{\frac{\alpha}{2}+\beta+\frac{1}{2}}}\,, \quad \alpha,\beta \geq 0\,,\\
\int_{-\infty}^{+\infty}\left|\frac{\partial^{\alpha+\beta}\tilde{u}(x,t)}{\partial x^\alpha \partial t^\beta}\right|^2 dx &= O(\delta^2)\frac{1}{(t+1)^{\alpha+2\beta+\frac{1}{2}}}\,, \quad \alpha,\beta \geq 0\,,
\end{aligned} \tag{2.13}$$

where

$$\delta = |\rho^+ - \rho^-| + |u^+ - u^-|\,. \tag{2.14}$$

In particular, these estimates are satisfied both by the caloric self-similar solutions of (2.8) described in Refs. 1 and 12 and by a small perturbation of these solutions w.r.t. initial datum (as we will show in Theorem 2).

Our first result concerns the asymptotic behaviour as $\varepsilon \searrow 0$ of the scaled hyperbolic system (2.6) with Sobolev norms. The time interval where the asymptotic analysis is valid, is given by the condition $\varepsilon T^\alpha \ll 1$, for some constant $\alpha > 0$, which allows, for small ε, to include the solutions at large time.

Theorem 1. *Let $0 < \nu < 1/2$ be arbitrary. Suppose $\varepsilon T^{\frac{1+\nu}{2}} \ll 1$, $\varepsilon \ll 1$ and $\delta \ll 1$; then, there exists a fixed constant $\Delta > 0$ such that*

$$\sup_{0\le t\le T}\left\{\frac{1}{(t+1)^\nu}\left[\frac{1}{\varepsilon^2}\|\rho^\varepsilon(t)-\tilde\rho(t)\|^2_{H^{3\theta}} + \|u^\varepsilon(t)-\tilde u(t)-w^\varepsilon(t)\|^2_{H^{3\theta}} + \frac{1}{\varepsilon^2}\int_0^t \|u^\varepsilon(s)-\tilde u(s)-w^\varepsilon(s)\|^2_{H^{3\theta}} ds\right]\right\} \le \Delta\,, \tag{2.15}$$

for any $\theta \in (0,1)$.

Corollary 1. *Let $t > 0$ be arbitrary. Let $\beta > 0$ be arbitrarily small. Then, for small values of δ, we have*

$$\|\rho^\varepsilon(t)-\tilde\rho(t)\|^2_{L^\infty} + \|\rho^\varepsilon_x(t)-\tilde\rho_x(t)\|^2_{L^\infty} \le O(\varepsilon^{2-\beta})\,. \tag{2.16}$$

The proof of Theorem 1 will be given in Sec. 3.

The convergence result in Theorem 1 holds whenever $\tilde\rho$ is a caloric self-similar solution. Our next goal is to show that (2.15) is true also when $\tilde\rho$ is replaced by a small perturbation.

Let $\tilde\rho$ be the caloric self-similar solution of the Porous Medium equation (namely, $\tilde\rho(x,t)$ is a function of $(x/\sqrt{t+1})$) with limiting conditions $\tilde\rho(\pm\infty,t) = \rho^\pm$. Let us denote by $\check\rho$ the solution to the same equation with the same limiting conditions and with the initial datum given by a small perturbation of $\tilde\rho(x,0)$. Let us denote

$$r(x,t) = \check\rho(x,t) - \tilde\rho(x+x_0,t)\,,$$

where x_0 will be determined later on. By integrating w.r.t. x the equation satisfied by r, we get

$$\frac{d}{dt}\int_{-\infty}^{+\infty} r(x,t)dx = [p(\check\rho(x,t)) - p(\tilde\rho(x+x_0,t))]_x\Big|_{-\infty}^{+\infty} = 0\,.$$

Thus, if one has

$$\int_{-\infty}^{+\infty}[\check\rho_0(x) - \tilde\rho_0(x+x_0)]dx = 0\,,$$

it follows both

$$x_0 = \frac{1}{\rho^+ - \rho^-}\int_{-\infty}^{+\infty}[\check\rho_0(x) - \tilde\rho_0(x)]dx \tag{2.17}$$

and

$$\int_{-\infty}^{+\infty} r(x,t)dx = 0\,.$$

Let us define the primitive variable

$$R(x,t) = \int_{-\infty}^{x} r(\xi,t)d\xi\,, \tag{2.18}$$

which solves the following problem

$$\begin{aligned} R_t &= p(\tilde\rho + R_x)_x - p(\tilde\rho)_x\,, \\ R(x,0) &= \int_{-\infty}^{x} [\check\rho_0(\xi) - \tilde\rho_0(\xi + x_0)]d\xi\,, \\ R(\pm\infty, t) &= 0\,. \end{aligned} \tag{2.19}$$

Then, the small perturbation analysis with respect to the caloric self-similar solution is given by the following result:

Theorem 2. *Suppose that* $\|R(0)\|_5^2$ *is sufficiently small. Then, for any* $t \geq 0$, *we have*

$$\sum_{k=0}^{5} (t+1)^k \|R^{(k)}(t)\|^2 + \int_0^t (\tau+1)^k \|R^{(k+1)}(\tau)\|^2 d\tau \leq C\|R(0)\|_5^2\,. \tag{2.20}$$

The proof of Theorem 2 will be given in Sec. 4. As a consequence of (2.20), the result in Theorem 1 is also true when $\tilde\rho$ is replaced by $\check\rho$. Moreover, as a consequence of both Theorems 1 and 2, we have the following asymptotic result.

Theorem 3. *Let* $\tilde\rho(x,t)$ *be the caloric self-similar solution to*

$$\begin{aligned} \tilde\rho_t &= p(\tilde\rho)_{xx}\,, \\ \tilde\rho(x,0) &= \tilde\rho_0(x)\,, \\ \tilde\rho(\pm\infty,t) &= \rho^{\pm}\,. \end{aligned}$$

Let $(\rho^\varepsilon(x,t), u^\varepsilon(x,t))$ *be the solution to*

$$\begin{aligned} &\rho_t^\varepsilon + (\rho^\varepsilon u^\varepsilon)_x = 0\,, \\ &u_t^\varepsilon + u^\varepsilon u_x^\varepsilon + \frac{p(\rho^\varepsilon)_x}{\varepsilon^2\rho^\varepsilon} = -\frac{u^\varepsilon}{\varepsilon^2}\,, \\ &\rho^\varepsilon(x,0) = \rho_0(x) = \tilde\rho(x+x_0) + r_0(x)\,, \\ &u^\varepsilon(x,0) = -\frac{p(\rho_0(x))_x}{\rho_0(x)} + w^\varepsilon(x,0)\,, \\ &\rho^\varepsilon(\pm\infty,t) = \rho^{\pm}\,, \\ &u^\varepsilon(\pm\infty,t) = e^{-t/\varepsilon^2}u^{\pm}\,, \end{aligned}$$

with $w^\varepsilon(x,t)$ *given by* (2.11), x_0 *given by* (2.17). *Suppose that* $\|R(0)\|_5^2$, δ *and* ε *are sufficiently small* ($R(0)$ *defined by* (2.18)). *Then, there exists a fixed* $\Gamma > 0$ *such that*

$$\sup_{\gamma T(\varepsilon) \leq t \leq \Gamma T(\varepsilon)} \|\rho^\varepsilon(t) - \tilde\rho(t)\|_{L^\infty(\mathbb{R})} \leq O(\varepsilon^{\frac{1}{1+\nu}})\,, \tag{2.21}$$

where

$$T(\varepsilon) = \varepsilon^{-\frac{2}{1+\nu}} ,$$

$\nu > 0$ *is arbitrary small and* γ *is an arbitrary constant such that* $0 < \gamma < \Gamma$.

The proof of Theorem 3 is straightforward.

3. The Proof of the Main Theorem

We prove Theorem 1 by means of an iteration scheme. Let us define an approximating sequence $(\rho^\varepsilon_{(n)}, u^\varepsilon_{(n)})$ by setting,

$$\rho^\varepsilon_0 = \tilde\rho , \quad u^\varepsilon_0 = \tilde u + w^\varepsilon ,$$

and let, for any $n > 1$, $(\rho^\varepsilon_{(n)}, u^\varepsilon_{(n)})$ be the solution to the system

$$\begin{aligned}
&\partial_t \rho^\varepsilon_{(n)} + \rho^\varepsilon_{(n-1)} \partial_x u^\varepsilon_{(n)} + u^\varepsilon_{(n-1)} \partial_x \rho^\varepsilon_{(n)} = 0 ,\\
&\partial_t u^\varepsilon_{(n)} + u^\varepsilon_{(n-1)} \partial_x u^\varepsilon_{(n)} + \frac{p'(\rho^\varepsilon_{(n-1)})}{\varepsilon^2 \rho^\varepsilon_{(n-1)}} \partial_x \rho^\varepsilon_{(n)} = -\frac{u^\varepsilon_{(n)}}{\varepsilon^2} ,\\
&\rho^\varepsilon_{(n)}(x,0) = \tilde\rho_0(x) ,\\
&u^\varepsilon_{(n)}(x,0) = -\frac{p'(\tilde\rho_0(x))}{\tilde\rho_0(x)} \tilde\rho_0'(x) + w^\varepsilon(x,0) ,\\
&\rho^\varepsilon_{(n)}(\pm\infty, t) = \rho^\pm ,\\
&u^\varepsilon_{(n)}(\pm\infty, t) = e^{-t/\varepsilon^2} u^\pm .
\end{aligned} \tag{3.22}$$

We will prove the convergence of the approximating sequence $(\rho^\varepsilon_{(n)}, u^\varepsilon_{(n)})$ to the solution of the system (2.6) via the uniform boundedness of this sequence in some weighted high Sobolev norm (namely $H^3(\mathbb{R})$) and the contraction in some weighted L^2-norm. Thus, we obtain the desired estimate via interpolation. This strategy is used in Refs. 3, 4 and 8.

Denote, for any $T > 0$,

$$\begin{aligned}
E^n_\varepsilon(T) = \sup_{0\le t\le T} \Bigg\{ \frac{1}{(t+1)^\nu} \Bigg[\frac{1}{\varepsilon^2} \|(\rho^\varepsilon_{(n)} - \tilde\rho)(t)\|^2_{H^3} + \|(u^\varepsilon_{(n)} - \tilde u - w^\varepsilon)(t)\|^2_{H^3} \\
+ \frac{1}{\varepsilon^2} \int_0^t \|(u^\varepsilon_{(n)} - \tilde u - w^\varepsilon)(s)\|^2_{H^3} ds \Bigg] \Bigg\} .
\end{aligned} \tag{3.23}$$

Hence, we have the following result

Proposition 1. *Let us assume that* $\delta + \varepsilon + \varepsilon T^{\frac{1+\nu}{2}} \le \lambda$, *where* $\lambda \ll 1$. *Then, there exists a positive constant* $\Delta > 0$ *such that, for any* $n \in \mathbb{N}$,

$$E^n_\varepsilon(T) \le \Delta . \tag{3.24}$$

Proof. From now on, we denote

$$\rho = \rho^\varepsilon_{(n)}\,, \quad u = u^\varepsilon_{(n)}\,, \quad \hat\rho = \rho^\varepsilon_{(n-1)}\,, \quad \hat u = u^\varepsilon_{(n-1)}$$
$$\bar\rho = \rho - \tilde\rho\,, \quad \bar u = u - w^\varepsilon - \tilde u\,, \quad \hat{\hat\rho} = \rho^\varepsilon_{(n-2)}\,, \quad \hat{\hat u} = u^\varepsilon_{(n-2)}$$
$$\bar{\bar\rho} = \hat\rho - \tilde\rho\,, \quad \bar{\bar u} = \hat u - w^\varepsilon - \tilde u\,, \quad \bar{\bar{\bar u}} = \hat{\hat u} - \tilde u - w\,, \quad \pi(z) = \frac{p'(z)}{z}\,, \quad \text{for any } z \in \mathbb{R}_+\,.$$

The system (3.22) becomes

$$\bar\rho_t + \hat u\bar\rho_x + \hat\rho\bar u_x = -(\bar{\bar u} + w)\tilde\rho_x - \bar{\bar\rho}\tilde u_x - \hat\rho w_x\,, \tag{3.25}$$

$$\bar u_t + \hat u\bar u_x + \frac{1}{\varepsilon^2}\pi(\hat\rho)\bar\rho_x = -\bar u_t - \hat u(\hat u_x + w_x) - \frac{1}{\varepsilon^2}(\pi(\hat\rho) - \pi(\tilde\rho))\tilde\rho_x\frac{\bar u}{\varepsilon^2}\,. \tag{3.26}$$

We now assume that the estimate (3.24) holds for $(\hat\rho, \hat u)$ and show that it is true for (ρ, u). In particular, we assume

$$\sup_{0\le t\le T}\left\{\frac{1}{(t+1)^\nu}\left[\frac{1}{\varepsilon^2}\|\bar\rho(t)\|^2_{H^3} + \|\bar u(t)\|^2_{H^3} + \frac{1}{\varepsilon^2}\int_0^t \|\bar u(s)\|^2_{H^3}ds\right]\right\} \le \Delta\,, \tag{3.27}$$

for any $T > 0$, $\varepsilon > 0$ and $\delta > 0$ (δ defined by (2.14)) such that

$$\delta + \varepsilon + \varepsilon T^{\frac{1+\nu}{2}} \le \lambda\,, \quad \lambda \ll 1\,. \tag{3.28}$$

As usual in this framework, we determine the conditions on the constant Δ in the estimate at the nth step. As we will see, this constant depends only on the constant λ in (3.28). Let us multiply (3.25) by $(1/\varepsilon^2)\pi(\hat\rho)\bar\rho$ and (3.26) by $\hat\rho\bar u$. Then, via standard energy identity (as a consequence of symmetrization), we get

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty}\left[\frac{1}{\varepsilon^2}\pi(\hat\rho)\frac{\bar\rho^2}{2} + \hat\rho\frac{\bar u^2}{2}\right]dx \\
&\quad= \int_{-\infty}^{+\infty}(\pi'(\hat\rho)(\hat\rho_t + \hat\rho_x\hat u) + \pi(\hat\rho)\hat u_x)\frac{\bar\rho^2}{2\varepsilon^2}dx + \int_{-\infty}^{+\infty}(\hat\rho_t + \hat\rho_x\hat u + \hat\rho\hat u_x)\frac{\bar u^2}{2}dx \\
&\qquad + \int_{-\infty}^{+\infty}\frac{1}{\varepsilon^2}p''(\hat\rho)\hat\rho_x\bar\rho\bar u dx - \int_{-\infty}^{+\infty}\frac{1}{\varepsilon^2}\pi(\hat\rho)\bar\rho(\bar{\bar u} + \tilde\rho_x dx - \int_{-\infty}^{+\infty}\frac{1}{\varepsilon^2}\pi(\hat\rho)\bar\rho\bar{\bar\rho}\tilde u_x dx \\
&\qquad - \int_{-\infty}^{+\infty}\frac{1}{\varepsilon^2}\pi(\hat\rho)\bar\rho\hat\rho w_x dx - \int_{-\infty}^{+\infty}\hat\rho\tilde u_t\bar u dx - \int_{-\infty}^{+\infty}\hat\rho\bar u\hat u(\tilde u_x + w_x)dx \\
&\qquad - \int_{-\infty}^{+\infty}\frac{1}{\varepsilon^2}\hat\rho\bar u(\pi(\hat\rho) - \pi(\tilde\rho))\tilde\rho_x dx - \int_{-\infty}^{+\infty}\hat\rho\frac{\bar u^2}{\varepsilon^2}dx \\
&\quad=: \sum_{k=1}^{3}\tilde J_k(t) + \sum_{k=1}^{6}I_k(t) - \int_{-\infty}^{+\infty}\hat\rho\frac{\bar u^2}{\varepsilon^2}dx\,.
\end{aligned} \tag{3.29}$$

Remark 1. We remark that the function $\pi(z)$ satisfies

$$0 < c_0 \le \pi(z) \le c_1\,, \quad \text{as} \quad z \in (c_2, c_3)\,,$$

for some positive constants c_0, c_1, c_2, c_3. Now, from the assumption (3.27) and from (2.12), it follows that $\hat{\rho}$ satisfies

$$0 < \frac{\mu_0}{2} \le \hat{\rho}(x,t) \le \mu_1 + \frac{\mu_0}{2}\,, \quad \text{for any} \quad x \in \mathbb{R}\,, \quad 0 \le t \le T\,. \tag{3.30}$$

(where μ_0, μ_1 are defined in (2.12)) provided that

$$\varepsilon(T+1)^{\nu/2} \le \Delta^{-1/2}\frac{\mu_0}{2}\,,$$

(with Δ as in (3.27)). Thus, from the condition (3.28) and by requiring $\Delta < 1$ and $\lambda^{\frac{\nu}{1+\nu}}\varepsilon^{\frac{1}{1+\nu}} < \frac{\mu_0}{2}$ (i.e. $\varepsilon + \lambda \ll 1$), there exist C_1, C_2 fixed positive constants such that

$$C_1 \le \pi(\hat{\rho}) \le C_2\,. \tag{3.31}$$

Moreover, from (3.30) it follows that

$$\pi'(\hat{\rho}) + p''(\hat{\rho}) \le C_3\,, \tag{3.32}$$

for some positive fixed C_3.

Now, from (3.30)–(3.32), and after time integration of (3.29) in $[0,t]$, for $0 < t \le T$, T satisfying (3.28), it follows that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\|\bar{\rho}(t)\|^2 + \|\bar{u}(t)\|^2 + \frac{1}{\varepsilon^2}\int_0^t \|\bar{u}(s)\|^2 ds \\
&\quad \le O(1)\int_0^t [|\hat{\rho}_t|_\infty + |\hat{\rho}_x\hat{u}|_\infty + |\hat{u}_x|_\infty]\left[\frac{\|\bar{\rho}(s)\|^2}{\varepsilon^2} + \|\bar{u}(s)\|^2\right] ds \\
&\qquad + \frac{O(1)}{\varepsilon^2}\int_0^t |\hat{\rho}_x|_\infty \|\bar{\rho}(s)\|\|\bar{u}(s)\| ds + \int_0^t \sum_{k=1}^{6} I_k(s) ds \\
&\quad =: \sum_{h=1}^{2} J_h(t) + \sum_{k=1}^{6}\int_0^t I_k(s) ds\,.
\end{aligned} \tag{3.33}$$

We devote ourselves to the estimate of the terms $\int_0^t I_k(t)ds$, $k = 1, \ldots, 6$. In what follows, we exploit (3.27), (3.28), the time asymptotic estimates (2.13) and the estimates (2.12), (3.31), (3.32).

$$\begin{aligned}
\int_0^t I_1 ds &\le \frac{O(\delta)}{\varepsilon^2}\int_0^t \|\bar{\bar{u}}(s)\|\|\bar{\rho}(s)\|\frac{1}{(s+1)^{1/2}} ds \\
&\quad + \frac{O(1)}{\varepsilon^2}\int_0^t e^{-\frac{s}{\varepsilon^2}}\|\bar{\rho}(s)\|\|\tilde{\rho}_x(s)\| ds \\
&\le \frac{O(\delta)}{\varepsilon^2}\int_0^t \|\bar{\bar{u}}\|^2 ds + \frac{O(\delta)}{\varepsilon^2}\int_0^t \frac{\|\bar{\rho}(s)\|^2}{s+1} ds \\
&\quad + \frac{O(1)}{\varepsilon^2}\int_0^t e^{-\frac{s}{\varepsilon^2}}[\|\bar{\rho}(s)\|^2 + \|\tilde{\rho}_x(s)\|^2] ds
\end{aligned}$$

$$\le O(\delta)\left(\Delta(t+1)^\nu + \int_0^t \frac{E^n(s)}{(s+1)^{1-\nu}}ds + 1\right) + E^n(t)(t+1)^\nu\varepsilon^2$$

$$\le O(\delta)(\Delta(t+1)^\nu + 1) + E^n(t)(t+1)^\nu(O(\delta)+O(\varepsilon^2))\,.$$

$$\int_0^t I_2 ds \le \frac{O(\delta)}{\varepsilon^2}\int_0^t \|\bar\rho(s)\|\|\bar{\bar\rho}(s)\|\frac{1}{s+1}ds$$

$$\le \frac{O(\delta)}{\varepsilon^2}\int_0^t [\|\bar\rho(s)\|^2 + \|\bar{\bar\rho}(s)\|^2]\frac{1}{s+1}ds$$

$$\le O(\delta)\int_0^t \frac{E^n(s)}{(s+1)^{1-\nu}}ds + O(\delta)\int_0^t \Delta\frac{1}{(s+1)^{1-\nu}}ds$$

$$\le O(\delta)E^n(t)(t+1)^\nu + O(\delta)\Delta(t+1)^\nu\,.$$

$$\int_0^t I_3 ds = \int_0^t\int_{-\infty}^{+\infty}\frac{\pi(\hat\rho)}{\varepsilon^2}\bar\rho\bar{\bar\rho}w_x dxds + \int_0^t\int_{-\infty}^{+\infty}\frac{\pi(\hat\rho)}{\varepsilon^2}\tilde\rho\bar\rho w_x dxds$$

$$\le \frac{O(\delta)}{\varepsilon^2}\int_0^t \|\bar\rho(s)\|\|\bar{\bar\rho}(s)\|e^{-\frac{s}{\varepsilon^2}}ds + \frac{O(\delta)}{\varepsilon^2}\int_0^t\int_{\mathrm{supp}(\phi)}\bar\rho e^{-\frac{s}{\varepsilon^2}}dxds$$

$$\le O(\delta)E^n(t)(t+1)^\nu + O(\delta)\Delta(t+1)^\nu + \frac{O(\delta)}{\varepsilon^2}\int_0^t|\bar\rho|_\infty e^{-\frac{s}{\varepsilon^2}}ds$$

$$\le O(\delta) + E^n(t)O(\delta)(1+(t+1)^\nu) + O(\delta)\Delta(t+1)^\nu\,,$$

where in the second inequality we have used the estimate for $\int_0^t I_2$.

$$\int_0^t I_4 ds \le O(1)\int_0^t [\|\tilde u_t(s)\|^2 + \|\bar u(s)\|^2]ds$$

$$\le O(\delta^2)\int_0^t \frac{1}{(s+1)^{5/2}}ds + O(\varepsilon^2)E^n(t)(t+1)^\nu$$

$$\le O(\delta^2) + O(\varepsilon^2)E^n(t)(t+1)^\nu\,,$$

$$\int_0^t I_5 ds \le \int_0^t\int_{-\infty}^{+\infty}\bar{\bar u}\hat\rho\bar u(\tilde u_x + w_x)dxds + \int_0^t\int_{-\infty}^{+\infty}(\tilde u + w)(\tilde u_x + w_x)\hat\rho\bar u dxds$$

$$\le O(1)\int_0^t[\|\bar{\bar u}(s)\|^2 + \|\bar u(s)\|^2]ds + O(1)\int_0^t[\|\tilde u_x(s)\|^2 + \|\bar u(s)\|^2]ds$$

$$+ O(1)\int_0^t[\|w_x(s)\|^2 + \|\bar u(s)\|^2]ds$$

$$\le O(\varepsilon^2)(\Delta(t+1)^\nu + E^n(t)(t+1)^\nu + O(\delta^2)) + O(\delta^2)\int_0^t\frac{1}{(s+1)^{3/2}}ds$$

$$\le O(\delta^2) + O(\varepsilon^2)(\Delta(t+1)^\nu + E^n(t)(t+1)^\nu + O(\delta^2))\,,$$

$$
\begin{aligned}
\int_0^t I_6 ds &\le \frac{O(\delta)}{\varepsilon^2}\int_0^t \|\bar{\bar{\rho}}(s)\|\|\bar{u}(s)\|\frac{1}{(s+1)^{1/2}}ds\\
&\le \frac{O(\delta)}{\varepsilon^2}\int_0^t \|\bar{\bar{\rho}}(s)\|^2\frac{1}{s+1}ds + \frac{O(\delta)}{\varepsilon^2}\int_0^t \|\bar{u}(s)\|^2 ds\\
&\le O(\delta)(t+1)^\nu(\Delta + E^n(t))\,.
\end{aligned}
$$

Thus, by requiring $\Delta < 1$ and $\lambda \ll 1$, the estimates of these terms yield

$$
\sum_{k=1}^{6}\int_0^t I_k(t) \le O(\lambda)(\Delta + E^n(t))(t+1)^\nu + O(\lambda)(t+1)^\nu\,.
$$

Hence, we compute the integrals denoted by J_h, $h = 1, 2$.

$$
\begin{aligned}
J_1(t) &\le O(1)\int_0^t [|\hat{\hat{\rho}}(s)|_\infty|\hat{u}_x(s)|_\infty + |\hat{\hat{u}}(s)|_\infty|\hat{\rho}_x(s)|_\infty\\
&\quad + |\hat{\rho}_x(s)|_\infty|\hat{u}(s)|_\infty + |\hat{u}_x(s)|_\infty]\left[\frac{\|\bar{\rho}(s)\|^2}{\varepsilon^2} + \|\bar{u}(s)\|^2\right]ds\\
&\le O(1)\int_0^t [|\hat{u}_x(s)|_\infty + |\hat{\rho}_x(s)|_\infty(|\hat{u}(s)|_\infty + |\hat{\hat{u}}(s)|_\infty)]\left[\frac{\|\bar{\rho}(s)\|^2}{\varepsilon^2} + \|\bar{u}(s)\|^2\right]ds\\
&\le O(1)E^n(t)\int_0^t [|\hat{u}_x(s)|_\infty + |\hat{\rho}_x(s)|_\infty(|\hat{u}(s)|_\infty + |\hat{\hat{u}}(s)|_\infty)](s+1)^\nu ds\\
&\le O(1)E^n(t)\int_0^t [|\bar{\bar{u}}_x(s)|_\infty + |\tilde{u}_x(s)|_\infty + |w_x(s)|_\infty + (|\bar{\bar{\rho}}_x(s)|_\infty + |\tilde{\rho}_x(s)|_\infty)\\
&\quad \times (|\bar{\bar{u}}(s)|_\infty + |\bar{\bar{\bar{u}}}(s)|_\infty + |\tilde{u}(s)|_\infty + |w(s)|_\infty)](s+1)^\nu ds\,.
\end{aligned}
\tag{3.34}
$$

We now estimate separately the following terms.

$$
\begin{aligned}
\int_0^t |\bar{\bar{u}}_x(s)|_\infty(s+1)^\nu ds &\le \int_0^t \left(\lambda\frac{|\bar{\bar{u}}_x(s)|_\infty^2}{\varepsilon^2} + \frac{1}{\lambda}\varepsilon^2(s+1)^{2\nu}\right)ds\\
&\le \lambda\Delta(t+1)^\nu + \frac{1}{\lambda}\varepsilon^2(t+1)^{1+2\nu} \le O(\lambda)(t+1)^\nu\,,
\end{aligned}
\tag{3.35}
$$

where λ is the fixed constant in (3.28).

$$
\begin{aligned}
&\int_0^t (|\bar{\bar{\rho}}_x(s)|_\infty + |\tilde{\rho}_x(s)|_\infty)(|\bar{\bar{u}}(s)|_\infty + |\bar{\bar{\bar{u}}}(s)|_\infty)(s+1)^\nu ds\\
&\quad \le \int_0^t (\varepsilon\Delta^{1/2}(s+1)^{\frac{3\nu}{2}} + O(\delta)(s+1)^{-1/2+\nu})(|\bar{\bar{u}}(s)|_\infty + |\bar{\bar{\bar{u}}}(s)|_\infty)\\
&\quad \le (O(\lambda) + O(\delta))\int_0^t (|\bar{\bar{u}}(s)|_\infty + |\bar{\bar{\bar{u}}}(s)|_\infty) \le (O(\lambda) + O(\delta))(t+1)^\nu\,,
\end{aligned}
\tag{3.36}
$$

where the last inequality is justified by the preceding estimate (3.35), and where we used $\nu < 1/2$ and $\Delta < 1$.

$$\int_0^t (|\bar{\bar{\rho}}_x(s)|_\infty + |\tilde{\rho}_x(s)|_\infty)(|\tilde{u}(s)|_\infty + |w(s)|_\infty)(s+1)^\nu ds$$

$$\leq \int_0^t (\varepsilon\Delta^{1/2}(s+1)^{\frac{3\nu}{2}} + O(\delta)(s+1)^{-1/2+\nu})O(\delta)(s+1)^{-1/2} ds$$

$$+ O(\varepsilon)\Delta^{1/2}(t+1)^{\nu/2} + O(\delta)O(\varepsilon^2) \leq (O(\lambda) + O(\delta))(t+1)^\nu \,. \tag{3.37}$$

Now we can complete the estimate of the integral J_1 in (3.34), and obtain

$$J_1(t) \leq (O(\delta) + O(\lambda) + O(\varepsilon^2))(t+1)^\nu E^n(t)\,.$$

Let us estimate $J_2(t)$;

$$J_2(t) \leq O(1)\int_0^t [|\bar{\bar{\rho}}_x(s)|_\infty + |\tilde{\rho}_x(s)|_\infty]\frac{1}{\varepsilon^2}\|\bar{\rho}(s)\|\|\bar{u}(s)\| ds$$

$$\leq O(1)\int_0^t [\varepsilon\Delta^{1/2}(s+1)^{\nu/2} + O(\delta)(s+1)^{-1/2}]\frac{1}{\varepsilon^2}\|\bar{\rho}(s)\|\|\bar{u}(s)\| ds$$

$$\leq O(1)\varepsilon\Delta^{1/2}\int_0^t \left[\frac{\|\bar{\rho}(s)\|^2}{\lambda\varepsilon}(s+1)^\nu + \lambda\frac{\|\bar{u}(s)\|^2}{\varepsilon^3}\right] ds$$

$$+ O(\delta)\int_0^t \frac{\|\bar{\rho}(s)\|^2}{\varepsilon^2}(s+1)^{-1} ds + O(\delta)\int_0^t \frac{\|\bar{u}(s)\|^2}{\varepsilon^2} ds$$

$$\leq O(1)\frac{\varepsilon^2}{\lambda}(t+1)^{1+2\nu}E^n(t) + (O(\lambda) + O(\delta))E^n(t)(t+1)^\nu \,. \tag{3.38}$$

By combining all these estimates, dividing both sides of (3.33) by $(t+1)^\nu$, taking the $\sup_{0\leq t\leq T}$ and by suitably choosing Δ and λ such that

$$0 < \Delta < 1\,, \quad \Delta > \tilde{C}\lambda\,,$$

for a fixed constant $\tilde{C}$, we obtain the following:

Lemma 1. *Suppose $\delta + \varepsilon + \varepsilon T^{\frac{1+\nu}{2}} \leq \lambda \ll 1$. Then, there exists a positive fixed constant Δ such that*

$$\sup_{0\leq t\leq T}\left\{\frac{1}{(1+t)^\nu}\left[\frac{1}{\varepsilon^2}\|\bar{\rho}(t)\|^2 + \|\bar{u}(t)\|^2 + \frac{1}{\varepsilon^2}\int_0^t \|\bar{u}(s)\|^2 ds\right]\right\} \leq \frac{\Delta}{4}\,. \tag{3.39}$$

In a similar fashion, we can derive L_2 estimates for the derivatives of $\bar{\rho}$ and $\bar{u}$. By differentiatiting (3.25) and (3.26) w.r.t. x, we obtain

$$\bar{\rho}_{xt} + \hat{u}\bar{\rho}_{xx} + \hat{\rho}\bar{u}_{xx} = -\hat{u}_x\bar{\rho}_x - \hat{\rho}_x\bar{u}_x - (\bar{\bar{u}}_x + w_x)\tilde{\rho}_x - (\bar{\bar{u}} + w)\tilde{\rho}_{xx}$$

$$- \bar{\bar{\rho}}_x\tilde{u}_x - \bar{\bar{\rho}}\tilde{u}_{xx} - \hat{\rho}_x w_x - \hat{\rho} w_{xx}\,, \tag{3.40}$$

$$\begin{aligned}\bar{u}_{xt} + \hat{u}\bar{u}_{xx} + \frac{1}{\varepsilon^2}\pi(\hat{\rho})\bar{\rho}_{xx} &= -\hat{u}_x\bar{u}_x - \frac{1}{\varepsilon^2}\pi'(\hat{\rho})\hat{\rho}_x\bar{\rho}_x - \tilde{u}_{xt} - \hat{u}_x(\tilde{u}_x + w_x)\\ &\quad - \hat{u}(\tilde{u}_{xx} + w_{xx}) - \frac{1}{\varepsilon^2}(\pi(\hat{\rho}) - \pi(\tilde{\rho}))\tilde{\rho}_{xx}\\ &\quad - \frac{1}{\varepsilon^2}(\pi'(\hat{\rho})\hat{\rho}_x - \pi'(\tilde{\rho})\tilde{\rho}_x)\tilde{\rho}_x - \frac{1}{\varepsilon^2}\bar{u}_x\,. \end{aligned} \tag{3.41}$$

It is clear that the system (3.40)–(3.41) has the same stucture as system (3.25)–(3.26). Thus, by multiplying the first equation by $\frac{1}{\varepsilon^2}\pi(\hat{\rho})\bar{\rho}_x$ and the second equation by $\hat{\rho}\bar{u}_x$, we obtain an energy identity similar to (3.29). Then, by integrating w.r.t. time, and from the same considerations as those in Remark 1, we obtain

$$\begin{aligned}&\frac{1}{\varepsilon^2}\|\bar{\rho}_x(t)\|^2 + \|\bar{u}_x(t)\|^2 + \frac{1}{\varepsilon^2}\int_0^t \|\bar{u}_x(s)\|^2 ds\\ &\quad\le O(1)\int_0^t [|\hat{\rho}_t|_\infty + |\hat{\rho}_x\hat{u}|_\infty + |\hat{u}_x|_\infty]\left[\frac{\|\bar{\rho}_x(s)\|^2}{\varepsilon^2} + \|\bar{u}_x(s)\|^2\right] ds\\ &\quad\quad + \frac{O(1)}{\varepsilon^2}\int_0^t |\hat{\rho}_x|_\infty\|\bar{\rho}_x(s)\|\|\bar{u}_x(s)\|ds + \int_0^t G(s)ds + \int_0^t F(s)ds\,,\end{aligned}$$

where we denoted all the integrals involving $\tilde{\rho}$, $\tilde{u}$ and w by $\int_0^t F(s)ds$, and where

$$\begin{aligned}\int_0^t G(s)ds &= \frac{O(1)}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\hat{u}_x\bar{\rho}_x^2dxds + \frac{O(1)}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty}\hat{\rho}_x\bar{u}_x\bar{\rho}_xdxds\\ &\quad + O(1)\int_0^t\int_{-\infty}^{+\infty}\hat{u}_x\bar{u}_x^2dxds\,.\end{aligned} \tag{3.42}$$

The terms (3.42) are to be treated as the terms $\sum_{h=1}^2 J_h$ of the previous lemma. The integrals denoted by $\int_0^t F(s)ds$ are made up by bilinear terms (in the variables marked by $^-$ and $^=$), where the terms estimated in L^∞ depends only on the corrector w and on the derivatives of asymptotic prophile $\tilde{\rho}$, as in the estimates of the integrals $\int_0^t I_k(s)$ of the previous lemma (we also have a faster decay for these terms, which involve second order derivatives). Hence, we easily obtain the following:

Lemma 2. *Suppose $\delta + \varepsilon + \varepsilon T^{\frac{1+\nu}{2}} \le \lambda \ll 1$. Then, there exists a positive fixed constant Δ such that*

$$\sup_{0\le t\le T}\left\{\frac{1}{(1+t)^\nu}\left[\frac{1}{\varepsilon^2}\|\bar{\rho}_x(t)\|^2 + \|\bar{u}_x(t)\|^2 + \frac{1}{\varepsilon^2}\int_0^t\|\bar{u}_x(s)\|^2ds\right]\right\} \le \frac{\Delta}{4}\,, \tag{3.43}$$

Remark 2. To complete the proof of Theorem 1, we differentiate w.r.t. x in order to get estimates for second and third derivatives of $(\bar{\rho}, \bar{u})$. The analogous of terms (3.42) behave the same as above (there is always a coefficient with order of derivation less than or equal to 2, to be estimated in $L^\infty(\mathbb{R})$). Since these computations are very similar to those concerning the preceding L_2 estimates, we skip the details about them. Hence, the proof of the proposition is complete. □

We now prove the contraction of the sequence $(\rho^\varepsilon_{(n)}, u^\varepsilon_{(n)})$ in the following:

Proposition 2. *Let us denote, for any* $\varepsilon > 0$, $n \in \mathbb{N}$, $0 < \nu < 1/2$,

$$F^n_\varepsilon(T) = \sup_{0\le t\le T} \left\{ \frac{1}{(t+1)^\nu} \left[\frac{1}{\varepsilon^2} \|\rho^\varepsilon_{(n)}(t) - \rho^\varepsilon_{(n-1)}(t)\|^2_{L_2} + \|u^\varepsilon_{(n)}(t) - u^\varepsilon_{(n-1)}(t)\|^2_{L_2} \right.\right.$$
$$\left.\left. + \frac{1}{\varepsilon^2}\int_0^t \|u^\varepsilon_{(n)}(s) - u^\varepsilon_{(n-1)}(s)\|^2_{L_2} ds \right] \right\}.$$

Then, under the condition $\delta + \varepsilon + \varepsilon T^{\frac{1+\nu}{2}} \le \lambda$, *for* $\lambda \ll 1$, *there exists a positive constant* $\mu < 1$ *such that*

$$F^n_\varepsilon(T) \le \mu F^{n-1}_\varepsilon(T)\,. \tag{3.44}$$

Proof. We denote

$$\begin{aligned}
&\rho^\varepsilon_{(n-2)} = \hat{\hat\rho}\,, && \rho^\varepsilon_{(n-1)} = \hat\rho\,, && \rho^\varepsilon_{(n)} = \rho\,,\\
&u^\varepsilon_{(n-2)} = \hat{\hat u}\,, && u^\varepsilon_{(n-1)} = \hat u\,, && u^\varepsilon_{(n)} = u\\
&\bar\rho = \rho - \hat\rho\,, && \bar{\bar\rho} = \hat\rho - \hat{\hat\rho}\,, && \bar u = u - \hat u\,, \quad \bar{\bar u} = \hat u - \hat{\hat u}\,.
\end{aligned}$$

With this notation, we can write system (3.22) as

$$\begin{aligned}
&\bar\rho_t + \hat\rho\bar u_x + \hat u\bar\rho_x = -\bar{\bar\rho}\hat u_x - \bar{\bar u}\hat\rho_x\,,\\
&\bar u_t + \hat u\bar u_x + \frac{1}{\varepsilon^2}\pi(\hat\rho)\bar\rho_x = -\bar{\bar u}\hat u_x - \left(\frac{1}{\varepsilon^2}\pi(\hat\rho) - \frac{1}{\varepsilon^2}\pi(\hat{\hat\rho})\right)\hat\rho_x - \frac{\bar u}{\varepsilon^2}\,.
\end{aligned} \tag{3.45}$$

As in the preceding proposition, we symmetrize the system (3.45) by

$$\begin{pmatrix} \frac{1}{\varepsilon^2}\pi(\hat\rho) & 0 \\ 0 & \hat\rho \end{pmatrix}$$

and obtain the standard energy identity

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty} \left[\frac{p'(\hat\rho)}{\varepsilon^2\hat\rho}\frac{\bar\rho^2}{2} + \hat\rho\frac{\bar u^2}{2}\right] dx\\
&\quad = \int_{-\infty}^{+\infty} \left[(\pi'(\hat\rho)(\hat\rho_t + \hat\rho_x\hat u) + \pi(\hat\rho)\hat u_x)\frac{\bar\rho^2}{2\varepsilon^2}\right] dx\\
&\qquad + \int_{-\infty}^{+\infty} \left[(\hat\rho_t + \hat\rho_x\hat u + \hat\rho\hat u_x)\frac{\bar u^2}{2} + \frac{1}{\varepsilon^2}p''(\hat\rho)\hat\rho_x\bar\rho\bar u\right] dx\\
&\qquad - \int_{-\infty}^{+\infty} \pi(\hat\rho)\bar\rho\,\bar{\bar u}\hat\rho_x dx - \int_{-\infty}^{+\infty} \pi(\hat\rho)\bar\rho\bar{\bar\rho}\hat u_x dx\\
&\qquad - \int_{-\infty}^{+\infty} \hat\rho\bar u\bar{\bar u}\hat u_x dx - \int_{-\infty}^{+\infty} \hat\rho\bar u(\pi(\hat\rho) - \pi(\hat{\hat\rho}))\hat\rho_x dx - \int_{-\infty}^{+\infty} \hat\rho\frac{\bar u^2}{\varepsilon}dx\,.
\end{aligned} \tag{3.46}$$

As a consequence of Proposition 1, after some considerations about the symmetrizing coefficients (same as those in Remark 1), we obtain the same estimates as (3.30)–(3.32), under the condition (3.28). After time integration in the interval $[0,t]$ for $0<t<T$, we obtain

$$\frac{1}{\varepsilon^2}\|\bar{\rho}(t)\|^2 + \|\bar{u}(t)\|^2 + \frac{1}{\varepsilon^2}\int_0^t \|\bar{u}(s)\|^2 ds$$

$$\leq O(1)\int_0^t [|\hat{\rho}_t|_\infty + |\hat{\rho}_x\hat{u}|_\infty + |\hat{u}_x|_\infty]\left[\frac{\|\bar{\rho}(s)\|^2}{\varepsilon^2} + \|\bar{u}(s)\|^2\right] ds$$

$$+ \frac{O(1)}{\varepsilon^2}\int_0^t |\hat{\rho}_x|_\infty \|\bar{\rho}(s)\|\|\bar{u}(s)\| ds + \frac{O(1)}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty} \bar{\rho}\bar{\bar{\rho}}\hat{u}_x dxds$$

$$+ \frac{O(1)}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty} \bar{\rho}\,\bar{\bar{u}}\hat{\rho}_x + O(1)\int_0^t\int_{-\infty}^{+\infty} \bar{\bar{u}}\hat{u}_x\bar{u} dxds$$

$$+ \frac{O(1)}{\varepsilon^2}\int_0^t\int_{-\infty}^{+\infty} \bar{\bar{\rho}}\bar{u}\hat{\rho}_x dxds =: \sum_{k=1}^{6} L_k(t)\,.$$

We now consider each term separately, using the result of Proposition 1.

$$L_1(t) \leq O(1)F^n(t)\int_0^t [|\hat{\rho}_x(s)|_\infty(|\hat{\hat{u}}(s)|_\infty + |\hat{u}(s)|_\infty) + |\hat{u}_x(s)|_\infty](s+1)^\nu ds$$

$$\leq O(1)F^n(t)\int_0^t [|(\hat{u}-\tilde{u}-w)_x(s)|_\infty + |\tilde{u}_x(s)|_\infty + |w_x(s)|_\infty$$

$$+ (|(\hat{\rho}-\tilde{\rho})_x(s)|_\infty + |\tilde{\rho}_x(s)|_\infty)(|(\hat{u}-\tilde{u}-w)(s)|_\infty$$

$$+ |(\hat{\hat{u}}-\tilde{u}-w)(s)|_\infty + |\tilde{u}(s)|_\infty + |w(s)|_\infty)](s+1)^\nu ds\,.$$

We estimate this term as in (3.35)–(3.37) of Proposition 1 and obtain

$$L_1(t) \leq (O(\delta) + O(\lambda) + O(\varepsilon^2))(t+1)^\nu F^n(t)\,.$$

Then, in a similar fashion, we estimate the term $L_2(t)$ as in (3.38). Let us compute the remaining terms:

$$L_3(t) \leq \frac{O(1)}{\varepsilon^2}\int_0^t |\hat{u}_x(s)|_\infty \|\bar{\rho}(s)\|\|\bar{\bar{\rho}}(s)\| ds$$

$$\leq \frac{O(1)}{\varepsilon^2}\int_0^t \left[\frac{\|\bar{\rho}(s)\|^2}{(s+1)^\nu} + \frac{\|\bar{\bar{\rho}}(s)\|^2}{(s+1)^\nu}\right]|\hat{u}_x(s)|_\infty (s+1)^\nu ds$$

$$\leq O(1)[F^n(t) + F^{n-1}(t)]\int_0^t \left[\frac{\lambda|(\hat{u}-\tilde{u}-w)_x|_\infty^2}{\varepsilon^2} + \frac{\varepsilon^2(s+1)^{2\nu}}{\lambda}\right.$$

$$\left. + O(\delta)(s+1)^{\nu-1}\right] ds \leq [O(\lambda) + O(\delta)](t+1)^\nu(F^n(t) + F^{n-1}(t))\,,$$

where we have used the condition (3.28).

$$\begin{aligned}L_4(t) &\le \frac{O(1)}{\varepsilon^2}\int_0^t |\hat\rho_x(s)|_\infty \|\bar\rho(s)\|\|\bar{\bar u}(s)\|ds\\ &\le O(1)\int_0^t\left[\frac{\lambda\|\bar{\bar u}(s)\|^2}{\varepsilon^2}+\frac{|\hat\rho_x(s)|_\infty^2\|\bar\rho(s)\|^2}{\lambda\varepsilon^2(s+1)^\nu}(s+1)^\nu\right]ds\\ &\le O(\lambda)F^{n-1}(t)(t+1)^\nu + O(1)\frac{1}{\lambda}F^n(t)\int_0^t[|(\hat\rho-\tilde\rho)_x(s)|_\infty^2(s+1)^\nu\\ &\quad + O(\delta)(s+1)^{\nu-1}]ds\\ &\le [O(\lambda)+O(\delta)](t+1)^\nu(F^n(t)+F^{n-1}(t))\,.\end{aligned}$$

The integrals $L_5(t)$ and $L_6(t)$ can be treated as above. Thus, by suitably choosing δ and λ small, the proof is complete. □

We now devote ourseqlves to the convergence of the approximating sequence. Let ν, ε and T be fixed in the usual way. Since $(\rho_\varepsilon^n, u_\varepsilon^n)$ is a Cauchy sequence in the norm expressed by F^n, by interpolation we have

$$\begin{aligned}\sup_{0\le t\le T}\Bigg\{\frac{1}{(t+1)^\nu}\Bigg[\frac{1}{\varepsilon^2}\|(\rho^\varepsilon_{(n)}-\rho^\varepsilon_{(m)})(t)\|^2_{H^{3\theta}}+\|(u^\varepsilon_{(n)}-u^\varepsilon_{(m)})(t)\|^2_{H^{3\theta}}\\ +\frac{1}{\varepsilon^2}\int_0^t\|(u^\varepsilon_{(n)}-u^\varepsilon_{(m)})(t)\|^2_{H^{3\theta}}ds\Bigg]\Bigg\}\to 0 \quad \text{as } n,m\to\infty \end{aligned} \tag{3.47}$$

for any $\theta\in(0,1)$. Thus, the approximating sequence $(\rho_\varepsilon^n, u_\varepsilon^n)$ converges in the norm expressed by (3.47) to (ρ^*, u^*). By choosing $\theta\in(0,1)$ large enough, we obtain

$$(\rho_\varepsilon^n, u_\varepsilon^n)\to(\rho^*,u^*) \quad \text{in } L^\infty([0,T];H^2(\mathbb{R}))\,. \tag{3.48}$$

Hence, we can identify the limit as the solution $(\rho_\varepsilon, u_\varepsilon)$ to the system (2.6) and carry out the limit as $n\to\infty$ in the estimate (3.24), with $H^{3\theta}$ in place of H^3, and the proof of Theorem 1 is complete.

4. The Proof of Theorem 2

In this section we prove the asymptotic stability result (2.20) by means of a continuation principle. We start with the a *priori* condition

$$\sup_{0\le t\le T}\sum_{k=0}^{5}(1+t)^k\|R^{(k)}(t)\|^2\le\sigma\,. \tag{4.49}$$

Lemma 3. *Suppose* $\sigma\ll 1$. *Then*

$$\|R(t)\|^2+\int_0^t\|R_x(s)\|^2es\le O(1)\|R_0\|^2\,. \tag{4.50}$$

Proof. By multiplying the first equation in (2.19) by R and after integration over $\mathbb{R}$, we obtain

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|R(t)\|^2 &= \int_{-\infty}^{+\infty} [p(\tilde\rho + R_x)_x - p(\tilde\rho)_x]R dx \\ &= -\int_{-\infty}^{+\infty} [p(\tilde\rho + R_x) - p(\tilde\rho)]R_x dx \\ &= -\int_{-\infty}^{+\infty} [p'(\tilde\rho)R_x^2 + R_1(p,\tilde\rho,R_x)R_x]dx \\ &\le -O(1)\|R_x(t)\|^2 + O(\sigma)\|R_x(t)\|^2\,.\end{aligned}$$

We denoted by $R_1(p,\tilde\rho,R_x)$ the remainder in the first order Taylor expansion of p' around $\tilde\rho$. The last inequality is due to the uniform boundedness of the coefficient $p''(\zeta)$ when $\zeta \in (\tilde\rho, \tilde\rho + R_x)$ (as a consequence of the maximum principle). Then, we integrate over $[0,t]$ and get the desired estimate (4.50). □

Lemma 4. *Suppose* $\sigma \ll 1$. *Then we have*

$$(1+t)\|r(t)\|^2 + \int_0^t (1+s)\|r_x(s)\|^2 ds \le O(1)\|R_0\|_1^2\,. \tag{4.51}$$

Proof. By differentiating Eq. (2.19) w.r.t. x, we get

$$\begin{aligned} r_t &= p(\check\rho)_{xx} - p(\tilde\rho)_{xx} = (p'(\check\rho)\check\rho_x - p'(\tilde\rho)\tilde\rho_x)_x \\ &= [p'(\check\rho)r_x + (p'(\check\rho) - p'(\tilde\rho))\tilde\rho_x]_x\,. \end{aligned} \tag{4.52}$$

We multiply (4.52) by $(1+t)r$ and integrate over $\mathbb{R}$ to obtain

$$\begin{aligned} &\frac{d}{dt}\left[(1+t)\frac{\|r(t)\|^2}{2}\right] - \|r(t)\|^2 \\ &\quad = (1+t)\int_{-\infty}^{+\infty} (p'(\check\rho)r_x)_x r dx + (1+t)\int_{-\infty}^{+\infty} ((p'(\check\rho) - p'(\tilde\rho))\tilde\rho_x)_x r dx \\ &\quad = -(1+t)\int_{-\infty}^{+\infty} p'(\tilde\rho)r_x^2 dx - (1+t)\int_{-\infty}^{+\infty} (p'(\check\rho) - p'(\tilde\rho))\tilde\rho_x r_x dx\,. \end{aligned}$$

Finally, by integrating w.r.t. time, we get

$$\begin{aligned} (1+t)\frac{1}{2}\|r(t)\|^2 + \int_0^t (1+s)\|r_x(s)\|^2 ds &\le O(1)\|r(0)\|^2 \\ &\quad + O(1)\int_0^t \|r(s)\|^2 ds + O(\delta)\int_0^t (1+s)^{1/2}\|r(s)\|\|r_x(s)\| ds \\ &\le O(1)\|R(0)\|_1^2 + O(1)\int_0^t [\|r(s)\|^2 + O(\delta)(1+s)\|r_x(s)\|^2] ds \\ &\le O(1)\|R(0)\|_1^2 + O(\delta)\int_0^t (1+s)\|r_x(s)\|^2 ds\,, \end{aligned}$$

where we have used (2.13) and Lemma 3. Thus, for $\delta \ll 1$, we have the desired estimate (4.51). □

Let us write the equation satisfied by r_x:

$$r_{xt} = (p'(\check{\rho})\check{\rho}_x - p'(\tilde{\rho})\tilde{\rho}_x)_{xx} \,. \tag{4.53}$$

Hence, we obtain the following lemma.

Lemma 5. *Let* $\sigma \ll 1$. *Then*

$$(1+t)^2\|r_x(t)\|^2 + \int_0^t (1+s)^2\|r_{xx}(s)\|^2 ds \leq O(1)\|R(0)\|_2^2 \,. \tag{4.54}$$

Proof. From (4.53) we have

$$r_{xt} - (p'(\check{\rho})r_x)_{xx} = [(p'(\check{\rho}) - p'(\tilde{\rho}))\tilde{\rho}_x]_{xx} \,.$$

We multiply by $(1+t)^2 r_x$ and integrate over $\mathbb{R}$ to get

$$\frac{d}{dt}\left[(1+t)^2\frac{1}{2}\|r_x(t)\|^2\right] - (1+t)\|r_x(t)\|^2$$
$$+ (1+t)^2\int_{-\infty}^{+\infty} p'(\check{\rho})r_{xx}^2 dx = -(1+t)^2\int_{-\infty}^{+\infty}[p'(\check{\rho})\check{\rho}_x r_x r_{xx}$$
$$+ (p'(\check{\rho}) - p'(\tilde{\rho}))\tilde{\rho}_{xx} r_{xx} + (p'(\check{\rho}) - p'(\tilde{\rho}))_x \tilde{\rho}_x r_{xx}]dx \,.$$

Then, by integrating w.r.t. time, we obtain

$$(1+t)^2\|r_x(t)\|^2 + \int_0^t (1+s)^2\|r_{xx}(s)\|^2 ds$$
$$\leq O(1)\|r_x(0)\|^2 + O(1)\int_0^t (1+s)\|r_x(s)\|^2 ds$$
$$+ O(1)\int_0^t (1+s)^2\int_{-\infty}^{+\infty}[r_x^2 r_{xx} + \tilde{\rho}_x r_x r_{xx} + r r_{xx}\tilde{\rho}_{xx}$$
$$+ r_x\tilde{\rho}_x r_{xx} + r\tilde{\rho}_x^2 r_{xx}]dxds \leq O(1)\|R(0)\|_2^2$$
$$+ O(1)\int_0^t (1+s)^{1/2}|r_x(s)|_\infty[(1+s)\|r_x(s)\|^2 + (1+s)^2\|r_{xx}(s)\|^2]ds$$
$$+ O(\delta)\int_0^t [(1+s)\|r_x(s)\|^2 + (1+s)^2\|r_{xx}(s)\|^2$$
$$+ \|r(s)\|^2 + (1+s)^2\|r_{xx}(s)\|^2 + O(\delta)(1+s)\|r_{xx}(s)\|^2]ds \,,$$

where we have used Young inequality. Thus, using (4.49)–(4.51), together with $\delta \ll 1$, we get

$$(1+t)^2\|r_x(t)\|^2 + \int_0^t (1+s)^2\|r_{xx}(s)\|^2 ds \le O(1)\|R(0)\|_2^2$$

$$+ O(\sigma)\int_0^t [(1+s)^2\|r_{xx}(s)\|^2 + (1+s)\|r_x(s)\|^2] ds\,.$$

Finally, by means of (4.51) and since $\sigma \ll 1$, we get the desired result (4.54). □

In order to complete the energy estimate (2.20) we have to carry out the time dacay estimates for the higher order derivatives r_{xx}, r_{xxx}, r_{xxxx} which can be done by following the same technique as above. We omit the details about these calculations.

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