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Global well-posedness and relaxation limits of a model for radiating gas☆

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Abstract

We study the initial value problem for a hyperbolic–elliptic coupled system with arbitrary large discontinuous initial data. We prove existence and uniqueness for that model by means of L^1-contraction and comparison properties. Moreover, after suitable scalings, we study both the hyperbolic–parabolic and the hyperbolic–hyperbolic relaxation limits for that system.

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1. Introduction

We are interested in the study of the following hyperbolic–elliptic coupled system

$$\begin{cases} u_t + \frac{1}{2}(u^2)_x = -q_x, \\ -q_{xx} + q = -u_x \end{cases} \tag{1.1}$$

with u_0 as initial condition.

This system is known as the simplest mathematical model in the study of radiating gases. Indeed, in specific physical situation (see [Ham71] and the book of [VK65]), (1.1) gives a good approximation to the following Euler system which describes the

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motion of a radiating gas

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ \{\rho(e + \frac{u^2}{2})\}_t + \{\rho u(e + \frac{u^2}{2}) + pu + q\}_x = 0, \\ -q_{xx} + aq + b(\theta^4)_x = 0. \end{cases} \tag{1.2}$$

In (1.2), as usual, ρ, u, p, e and θ are, respectively, the mass density, velocity, pressure, internal energy and absolute temperature of the gas, while q represents the radiative heat flux and a and b are given positive constants depending on the gas itself. Systems (1.1) and (1.2) are both treated in [KNN98], where, with an appropriate stability condition and with energy estimates, it is proved the global existence of H^s solutions to those systems and its asymptotic behavior. In particular, for simpler system (1.1), it is proved that its solutions can be approximated, as $t \to \infty$, by the solutions of the viscous Burgers' equation

$$u_t + \tfrac{1}{2}(u^2)_x = u_{xx},$$

which can be obtained formally by neglecting the higher-order term q_{xx} in the elliptic equation in (1.1). Moreover, either shock waves and classical solutions for (1.1) are studied in [KN98,KN99a,KN99b], while the time asymptotic behavior of solutions to (1.1) with discontinuous initial data is investigated in [Nis00].

More recently, Serre [Ser] proved the stability of travelling waves for the hyperbolic–elliptic system (1.1), besides the case of the relaxation approximation. Moreover, system (1.1), which can be rewritten as a scalar balance law of the form

$$u_t + \tfrac{1}{2}(u^2)_x = -u + K * u, \tag{1.3}$$

where the kernel K is given by $\frac{1}{2}e^{-|x|}$ [KNN98,Ser], can be interpreted as a third way to approximate the scalar conservation law

$$u_t + \tfrac{1}{2}(u^2)_x = 0,$$

in addition to the classical vanishing viscosity and relaxation approximations.

In this paper we study the Cauchy problem for Eq. (1.3) and we prove the existence and the uniqueness of its weak entropy solutions. This result is obtained essentially by taking advantage of the dissipative nature of the source term which allows to prove L^1-contractivity and comparison principle for that equation. Moreover, driven by the previous results quoted above, we analyze both the hyperbolic–hyperbolic and the hyperbolic–parabolic relaxation limits for (1.1). More precisely, under the scaling $(x,t) \to (\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$, we prove the scaled solution converges strongly to the entropy solution of the inviscid Burgers' equation, while, under the scaling

$$u^{\varepsilon}(x,t) = \frac{1}{\varepsilon} u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right),$$

$$q^{\varepsilon}(x,t) = \frac{1}{\varepsilon^2} u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right),$$

the sequence u^{ε} converges to the solution of the viscous Burgers' equation, in agreement with the asymptotic results of [KNN98].

We emphasize that the singular limits analyzed here are not standard relaxation limits. Indeed, either in the hyperbolic and in the parabolic case, there is no initial layer in the limit solution, since there are no time derivatives in the second equation of (1.1). Therefore, we do not discuss genuine relaxation limits of 2×2 systems toward scalar equations, but we deal with singular limits of nonhomogeneous scalar equations. This feature becomes evident if we consider the reformulation of (1.1) given by Eq. (1.3). Concerning this equation, we notice that the particular form of the convolution kernel K comes from the original hyperbolic–elliptic system (1.1), but all the results we establish in this paper are valid for any convolution kernel K which is even and satisfies

$$K \geqslant 0, \quad \int_{-\infty}^{+\infty} K(x)\, dx = 1.$$

Indeed, the L^1-contraction property of the source term comes from the above assumptions and the control of the singular limits is based on the properties of the first two momenta of K (see Remarks 3.6 and 3.14). Finally, we remark that all the results presented here, except the hyperbolic–parabolic relaxation limit, can be carried out with a general C^2 flux $f(u)$ instead of the Burgers' flux $\frac{1}{2}u^2$. Indeed, as we pointed out before, in order to analyze the hyperbolic–parabolic relaxation limit, we must scale also the dependent variable u and therefore the form of the flux function takes a role in this limit. More precisely, considering a general, smooth flux $f(u)$ which verifies $f(0) = f'(0) = 0$, the limit equation is given by

$$u_t + \tfrac{1}{2} f''(0)(u^2)_x = u_{xx},$$

namely, it is again given by the viscous Burgers' equation. Moreover, due to the L^{∞} bound on u^{ε} we shall prove, even in this case, the analysis of the relaxation limit for a general flux can be carried out without significant modifications.

The rest of this paper is organized as follows. In the next section we prove the uniqueness of the entropy solution of our model and the existence of such solution, first when the initial datum is chosen in $L^{\infty}(\mathbb{R}) \cap L^1(\mathbb{R})$ and then for general $L^{\infty}(\mathbb{R})$ initial conditions. To perform this task, we use the vanishing viscosity method. The stability and compactness of the sequence issued by this method is proved with the aid of the L^1-contractivity and the comparison principle for (1.3). In the last section we analyze the two different relaxation limit and we prove the convergence of the relaxation sequence again using the L^∞ and L^1 estimates of the previous section. Finally, in the appendix we establish the local in time existence in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for the vanishing viscosity approximation of (1.3).

2. Global well-posedness of the model

In this section we treat the existence and the uniqueness of the weak entropy solution to the hyperbolic–elliptic model (1.1). We start with the study of the case of initial data $u_0(x)$ in $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. As we pointed out in the previous section, this system can be reformulated as a nonhomogeneous Burgers' equation as follows:

$$u_t + \tfrac{1}{2}(u^2)_x = -u + K * u, \tag{2.1}$$

where the convolution kernel K is given by $\frac{1}{2}e^{-|x|}$. As usual, a bounded measurable function u is a weak solution to (2.1) if it verifies this relation in distributional sense and the test functions are smooth functions with compact support, intersecting the line $t = 0$. Moreover, a weak solution is entropic if, in addition, it verifies the inequality

$$\begin{aligned}\int_0^T \int_{-\infty}^{+\infty} [\eta(u)\psi_t + q(u)\psi_x]\,dx\,dt + \int_{-\infty}^{+\infty} \eta(u_0(x))\psi(x,0)\,dx \\ \geqslant \int_0^T \int_{-\infty}^{+\infty} \eta'(u)[u - K*u]\psi\,dx\,dt,\end{aligned} \tag{2.2}$$

for any convex entropy η with flux q given by

$$q(u) = \int^u s\eta'(s)\,ds$$

and for any nonnegative Lipschitz continuous test function ψ on $\mathbb{R} \times [0, T]$ with compact support, intersecting the line $t = 0$. We stress that, as in the theory of homogeneous conservation laws, it suffices to require relation (2.2) only for a particular class of entropies η, namely, for $\eta(u) = \pm u$, together with the family $\eta_k(u) = (u-k)^+,\ k \in \mathbb{R}$ and $w^+ = \max\{w, 0\}$.

Remark 2.1. As in the study of scalar conservation laws with no source term, the existence and uniqueness of entropy solutions to (2.1) we will prove here do not rely on the particular form of the convection term, but it can be proved for a general, smooth flux function $f(u)$ instead of $\frac{1}{2}u^2$.

We start by proving the existence of weak entropy solutions to the Cauchy problem for (2.1). To perform this task, we employ the method of vanishing viscosity, namely, we approximate this equation with the parabolic equation

$$u_t + \tfrac{1}{2}(u^2)_x = -u + K * u + \mu u_{xx} \tag{2.3}$$

and we seek for a solution to (2.1) as limit, as $\mu \downarrow 0$, of solutions to (2.3). Concerning this equation, we can prove the following theorem.

Theorem 2.2. *Let u and $\bar{u}$ be solutions of* (2.3) *with initial data u_0, $\bar{u}_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, for any $t > 0$,*

$$\int_{-\infty}^{+\infty} (u(x,t) - \bar{u}(x,t))^+ \, dx \leqslant \int_{-\infty}^{+\infty} (u_0(x) - \bar{u}_0(x))^+ \, dx, \tag{2.4}$$

$$||u(\cdot,t) - \bar{u}(\cdot,t)||_{L^1(\mathbb{R})} \leqslant ||u_0(\cdot) - \bar{u}_0(\cdot)||_{L^1(\mathbb{R})}. \tag{2.5}$$

Moreover, if $u_0(x) \leqslant \bar{u}_0(x)$ a.e. on $\mathbb{R}$, then $u(x,t) \leqslant \bar{u}(x,t)$ a.e. on $\mathbb{R} \times [0,T]$. In addition, the range of both u and $\bar{u}$ is contained in $[-a,a]$, where

$$a = \max\{||u_0||_{L^\infty(\mathbb{R})}, ||\bar{u}_0||_{L^\infty(\mathbb{R})}\}.$$

Proof. Let $T > 0$ be such that Eq. (2.3), with initial data u_0, $\bar{u}_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, has smooth solutions such that $u(\cdot,t)$, $\bar{u}(\cdot,t) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for $t \in [0,T]$ (see Appendix A for details). We start by proving the relation (2.4) for any $t \in [0,T]$. For $\varepsilon > 0$, we define

$$\eta_\varepsilon(\xi) = \begin{cases} 0 & \text{if } -\infty < \xi \leqslant 0, \\ \frac{\xi^2}{4\varepsilon} & \text{if } 0 < \xi \leqslant 2\varepsilon, \\ \xi - \varepsilon & \text{if } 2\varepsilon < \xi < +\infty. \end{cases}$$

Since u and $\bar{u}$ verify (2.3), we have

$$\begin{aligned} &\eta_\varepsilon(u-\bar{u})_t + \tfrac{1}{2}[\eta'_\varepsilon(u-\bar{u})(u^2-\bar{u}^2)]_x - \tfrac{1}{2}\eta''_\varepsilon(u-\bar{u})(u^2-\bar{u}^2)(u-\bar{u})_x \\ &\quad = \mu\eta_\varepsilon(u-\bar{u})_{xx} - \mu\eta''_\varepsilon(u-\bar{u})[(u-\bar{u})_x]^2 - \eta'_\varepsilon[u-\bar{u}-K*(u-\bar{u})]. \end{aligned} \tag{2.6}$$

Integrating (2.6) in x and t and since

$$\mu\eta''_\varepsilon(u-\bar{u})[(u-\bar{u})_x]^2 \geqslant 0,$$

we end up with the following inequality:

$$\int_{-\infty}^{+\infty} \eta_\varepsilon(u(x,t)-\bar{u}(x,t))\,dx \leqslant \int_{-\infty}^{+\infty} \eta_\varepsilon(u_0(x)-\bar{u}_0(x))\,dx + \frac{1}{2}\int_0^t \int_{-\infty}^{+\infty} \eta''_\varepsilon(u-\bar{u})(u^2-\bar{u}^2)(u-\bar{u})_x\,dx\,ds - \int_0^t \int_{-\infty}^{+\infty} \eta'_\varepsilon[u-\bar{u}-K*(u-\bar{u})]\,dx\,ds. \tag{2.7}$$

Moreover, as $\varepsilon\downarrow 0$, we have, pointwise,

$$\eta_\varepsilon(u(x,t)-\bar{u}(x,t)) \to (u(x,t)-\bar{u}(x,t))^+,$$

$$\eta'_\varepsilon(u(x,t)-\bar{u}(x,t)) \to \mathrm{sgn}(u(x,t)-\bar{u}(x,t))^+,$$

$$\eta''_\varepsilon(u(x,t)-\bar{u}(x,t))(u^2-\bar{u}^2) \to 0.$$

Hence, letting $\varepsilon\downarrow 0$ in (2.7) we obtain

$$\int_{-\infty}^{+\infty} (u-\bar{u})^+\,dx \leqslant \int_{-\infty}^{+\infty} (u_0-\bar{u}_0)^+\,dx - \int_0^t \int_{-\infty}^{+\infty} (u-\bar{u})^+ - \mathrm{sgn}(u-\bar{u})^+ K*(u-\bar{u})]\,dx\,ds. \tag{2.8}$$

At this point, we estimate the convolution term in the above relation as follows:

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{+\infty} \mathrm{sgn}(u-\bar{u})^+ K*(u-\bar{u})]\,dx\,d\tau \\
&= \int_0^t \int_{-\infty}^{+\infty} \mathrm{sgn}(u-\bar{u})^+ \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x-y|}(u-\bar{u})(y)\,dy\,dx\,d\tau \\
&\leqslant \int_0^t \int_{-\infty}^{+\infty} \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x-y|}(u-\bar{u})^+(y)\,dy\,dx\,d\tau \\
&= \int_0^t \int_{-\infty}^{+\infty} (u-\bar{u})^+(y)\,dy\,d\tau
\end{aligned}$$

and therefore relation (2.8) reduces to

$$\int_{-\infty}^{+\infty} (u-\bar{u})^+\,dx \leqslant \int_{-\infty}^{+\infty} (u_0-\bar{u}_0)^+\,dx$$

which is exactly (2.4). Interchanging the roles of u and $\bar{u}$, from (2.4) we easily obtain (2.5). Moreover, the monotonicity property stated at the end of the theorem is again a consequence of (2.4). Now, it is not possible to prove the bound of the solutions in L^∞ using directly this monotonicity, because the constants are solutions of our equations, which do not belong to $L^1(\mathbb{R}_x)$. Indeed, it is still possible to prove this bound, thanks to an uniform control of the L^p norms of the solutions. Indeed, with standard arguments, we can multiply Eq. (2.3) by a regularization of $p|u|^{p-1}\mathrm{sgn}(u)$ and we obtain

$$\int_{-\infty}^{+\infty} |u|^p \, dx \leqslant \int_{-\infty}^{+\infty} |u_0|^p \, dx - \int_0^t \int_{-\infty}^{+\infty} p|u|^p \, dx \, ds + \int_0^t \int_{-\infty}^{+\infty} p|u|^{p-1}|K*u| \, dx \, ds. \tag{2.9}$$

We estimate once again the convolution integral

$$\int_{-\infty}^{+\infty} |u|^{p-1}|K*u| \, dx \leqslant ||u||_{L^p}^{p-1} ||K*u||_{L^p} \leqslant ||u||_{L^p}^{p-1} ||u||_{L^p} ||K||_{L^1} = ||u||_{L^p}^p,$$

because of the identity

$$\frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x|} \, dx = 1.$$

Hence, (2.9) becomes

$$||u||_{L^p} \leqslant ||u_0||_{L^p} \leqslant ||u_0||_{L^\infty}^{\frac{p-1}{p}} ||u_0||_{L^1}^{\frac{1}{p}}.$$

Thus, the L^∞ norm of u is estimated by

$$||u||_{L^\infty} \leqslant \limsup_{p\to+\infty} ||u||_{L^p} \leqslant \lim_{p\to+\infty} ||u_0||_{L^\infty}^{\frac{p-1}{p}} ||u_0||_{L^1}^{\frac{1}{p}} = ||u_0||_{L^\infty}$$

and the last assertion of the theorem follows easily. Finally, this a priori estimate, together with the L^1 estimate coming from (2.3), implies in particular the local-in-time solution of (2.3) is indeed global and all the estimates we have proved are global in time. □

As in the homogeneous case [Daf00], setting $\bar{u}(x,t) = u(x+h,t)$, the estimate (2.5) can be used to estimate, uniformly with respect to μ, the L^1-modulus of continuity of the solution u^μ to (2.3) with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Indeed, the following lemma holds.

Lemma 2.3. *Let u^μ be the solution to* (2.3) *with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ as initial datum. In particular*

$$\int_{-\infty}^{+\infty} |u_0(x+h) - u_0(x)| \, dx \leqslant \omega(|h|), \quad \textit{for any } h \in \mathbb{R}, \tag{2.10}$$

for some nondecreasing function ω on $[0,+\infty)$ with $\omega(r)\downarrow 0$ as $r\downarrow 0$. Then there exists a constant C, depending only on $||u_0||_{L^\infty(\mathbb{R})}$ such that, for any $t>0$,

$$\int_{-\infty}^{+\infty} |u^\mu(x+h,t) - u^\mu(x,t)| \, dx \leqslant \omega(|h|), \quad \textit{for any } h \in \mathbb{R} \tag{2.11}$$

and

$$\int_{-\infty}^{+\infty} |u^{\mu}(x,t+k) - u^{\mu}(x,t)|\,dx \leqslant C(k + k^{\frac{2}{3}} + \mu k^{\frac{1}{3}})||u_0||_{L^1(\mathbb{R})} + 4\omega(k^{\frac{1}{3}}), \tag{2.12}$$

for any $k > 0$.

Proof. For any fixed $t > 0$, applying (2.5) with $\bar{u}(x,t) = u(x+h,t)$, we obtain (2.11).

Fix now $k > 0$. Let ϕ be a smooth, bounded function on $\mathbb{R}$. Then, multiplying (2.3) by ϕ and integrating by parts one has

$$\begin{aligned}&\int_{-\infty}^{+\infty} \phi(x)[u(x,t+k) - u(x)]\,dx \\ &\quad = \int_t^{t+k}\int_{-\infty}^{+\infty} \left[\phi'(x)\frac{1}{2}u(x,\tau)^2 + \mu\phi''(x)u(x,\tau)\right. \\ &\qquad \left. + \phi(x)(-u(x,\tau) + (K*u)(x,\tau))\right]dx\,d\tau.\end{aligned} \tag{2.13}$$

Let us choose

$$\phi(x) = \int_{-\infty}^{+\infty} k^{-\frac{1}{3}}\rho\left(\frac{x-\xi}{k^{\frac{1}{3}}}\right)\mathrm{sgn}(u(\xi,t+k) - u(\xi,t))\,d\xi,$$

where ρ is a mollifier, namely a smooth, nonnegative function with support contained in $[-1,1]$ and total mass one. Since $|\phi| \leqslant 1$, $|\phi'| \leqslant c_1 k^{-\frac{1}{3}}$ and $|\phi''| \leqslant c_2 k^{-\frac{2}{3}}$, from (2.13) we obtain

$$\int_{-\infty}^{+\infty} \phi(x)(u(x,t+k) - u(x,t))\,dx \leqslant C(k + k^{\frac{2}{3}} + \mu k^{\frac{1}{3}})||u_0||_{L^1(\mathbb{R})}, \tag{2.14}$$

where the constant C depends only on $||u_0||_{L^\infty}$. Moreover,

$$\begin{aligned}&|u(x,t+k) - u(x,t)| - \phi(x)(u(x,t+k) - u(x,t)) \\ &\quad = \int_{-\infty}^{+\infty} k^{-\frac{1}{3}}\rho\left(\frac{x-\xi}{k^{\frac{1}{3}}}\right)[|u(x,t+k) - u(x,t)| \\ &\qquad - (u(x,t+k) - u(x,t))\mathrm{sgn}(u(\xi,t+k) - u(\xi,t))]\,d\xi \\ &\quad \leqslant 2\int_{-\infty}^{+\infty} k^{-\frac{1}{3}}\rho\left(\frac{x-\xi}{k^{\frac{1}{3}}}\right)|u(x,t+k) - u(x,t) - (u(\xi,t+k) - u(\xi,t))|\,d\xi.\end{aligned}$$

Finally, integrating in dx the above inequality, we get (2.12), in view of (2.11) and (2.14). □

In view of Lemma 2.3, the sequence u^μ is compact in L^1_{loc} (and bounded in L^∞) that is, passing if necessary to a subsequence, it converges strongly (and boundedly almost everywhere in $\mathbb{R} \times [0, +\infty)$) to a function $u \in L^1(\mathbb{R} \times [0, T]) \cap L^\infty(\mathbb{R} \times [0, T])$. Moreover, due to the strong convergence of the sequence and due to its boundedness in L^∞, it is easy to verify that u is an entropy solution to (2.1) with u_0 as initial data. Hence, the following theorem holds.

Theorem 2.4. *Let u^μ be the solution to* (2.3) *with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ as initial datum. Then, as $\mu \downarrow 0$, for any $T > 0$,*

$$u^\mu \to u, \quad \textit{strongly in } L^p_{\mathrm{loc}}(\mathbb{R} \times [0, T]), \; p < +\infty,$$

and $u \in L^1(\mathbb{R} \times [0, T]) \cap L^\infty(\mathbb{R} \times [0, T])$ is an entropy solution to (2.1) *with u_0 as initial datum.*

We pass now to the study of the uniqueness of the weak, entropy solutions to (2.1) with initial datum in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Theorem 2.5. *Let $u, \bar{u} \in L^\infty([0, T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ be weak entropy solutions of* (2.1) *with initial data u_0, $\bar{u}_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, for any $t \in [0, T]$,*

$$\int_{-\infty}^{+\infty} (u(x,t) - \bar{u}(x,t))^+ \, dx \leqslant \int_{-\infty}^{+\infty} (u_0(x) - \bar{u}_0(x))^+ \, dx; \tag{2.15}$$

$$||u(\cdot, t) - \bar{u}(\cdot, t)||_{L^1(\mathbb{R})} \leqslant ||u_0(\cdot) - \bar{u}_0(\cdot)||_{L^1(\mathbb{R})}. \tag{2.16}$$

Moreover, if $u_0(x) \leqslant \bar{u}_0(x)$ a.e. on $\mathbb{R}$, then $u(x,t) \leqslant \bar{u}(x,t)$ a.e. on $\mathbb{R} \times [0, T]$. In addition, the essential range of both u and $\bar{u}$ is contained in $[-a, a]$, where

$$a = \max\{||u_0||_{L^\infty(\mathbb{R})}, ||\bar{u}_0||_{L^\infty(\mathbb{R})}\}.$$

Proof. To prove the results and, in particular, (2.16), we proceed as in the homogeneous case [Kru70]; see also the book [Daf00]. To this end, we use the entropy $(u(x,t) - \bar{u}(y,s))^+$ either in the (x,t) and in the (y,s) variables and we choose an appropriate test function, converging to δ functions centered at $y = x$ and at $t = s$ to obtain the following relation:

$$\begin{aligned} &\int_0^T \int_{-\infty}^{+\infty} [\psi_t (u - \bar{u})^+ + \psi_x \, \mathrm{sgn}(u - \bar{u})^+ (f(u) - f(\bar{u}))] \, dx \, dt \\ &\quad + \int_{-\infty}^{+\infty} \psi(x, 0)(u_0 - \bar{u}_0)^+ \, dx \\ &\geqslant \int_0^T \int_{-\infty}^{+\infty} \psi[(u - \bar{u})^+ - \mathrm{sgn}(u - \bar{u})^+ K * (u - \bar{u})] \, dx \, dt, \end{aligned} \tag{2.17}$$

for any nonnegative Lipschitz continuous test function ψ on $\mathbb{R}\times[0,T]$ with compact support, intersecting the line $t=0$. Now we fix $R>0$, $t\in[0,T)$ and $\varepsilon>0$ and we write (2.17) for the test function $\psi(x,\tau)=\chi(x,\tau)\theta(\tau)$, where

$$\chi(x,\tau)=\begin{cases}1 & \text{if } 0\leqslant\tau<T,\ 0\leqslant|x|<R+s(t-\tau),\\ \frac{1}{\varepsilon}[R+s(t-\tau)-|x|]+1 & \text{if } 0\leqslant\tau<T,\\ & \quad R+s(t-\tau)\leqslant|x|<R+s(t-\tau)+\varepsilon,\\ 0 & \text{if } 0\leqslant\tau<T,\ |x|\geqslant R+s(t-\tau)+\varepsilon,\end{cases}$$

$$\theta(\tau)=\begin{cases}1 & \text{if } 0\leqslant\tau<t,\\ \frac{1}{\varepsilon}(t-\tau)+1 & \text{if } t\leqslant\tau<t+\varepsilon,\\ 0 & \text{if } t+\varepsilon\leqslant\tau<T,\end{cases}$$

and

$$s=\max\left\{\frac{f(u)-f(\bar{u})}{u-\bar{u}}\right\},\tag{2.18}$$

for u and $\bar{u}$ in the range of the solutions. Computing explicitly the derivatives of ψ, we get, as in the homogeneous case,

$$\begin{aligned}&\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{|x|<R}(u-\bar{u})^+\,dx\,d\tau\\ &\quad\leqslant\int_{|x|<R+st}(u_0-\bar{u}_0)^+\,dx-\frac{1}{\varepsilon}\int_0^T\int_{R+s(t-\tau)\leqslant|x|<R+s(t-\tau)+\varepsilon}\Big[s(u-\bar{u})^+\\ &\qquad+\frac{x}{|x|}\operatorname{sgn}(u-\bar{u})^+(f(u)-f(\bar{u}))\Big]\,dx\,d\tau\\ &\qquad-\int_0^T\int_{-\infty}^{+\infty}\chi(x,\tau)\theta(\tau)[(u-\bar{u})^+-\operatorname{sgn}(u-\bar{u})^+K*(u-\bar{u})]\,dx\,d\tau+O(\varepsilon)\\ &\quad\leqslant\int_{|x|<R+st}(u_0-\bar{u}_0)^+\,dx-I+O(\varepsilon).\end{aligned}\tag{2.19}$$

In the last line of (2.19), we used the special choice of the constant s and I stands for the extra term, due to the source,

$$I=\int_0^T\int_{-\infty}^{+\infty}\chi(x,\tau)\theta(\tau)[(u-\bar{u})^+-\operatorname{sgn}(u-\bar{u})^+K*(u-\bar{u})]\,dx\,d\tau$$

we have to estimate in our case. The special form of the test function we have chosen yields to

$$I=\int_0^t\int_{|x|<R+s(t-\tau)}[(u-\bar{u})^+-\operatorname{sgn}(u-\bar{u})^+K*(u-\bar{u})]\,dx\,d\tau+O(\varepsilon).$$

Hence, letting $\varepsilon\downarrow 0$, we end up to

$$\int_{|x|<R}(u-\bar{u})^+\,dx\leqslant\int_{|x|<R+st}(u_0-\bar{u}_0)^+\,dx-\int_0^t\int_{|x|<R+s(t-\tau)}(u-\bar{u})^+\,dx\,d\tau$$
$$+\int_0^t\int_{|x|<R+s(t-\tau)}\operatorname{sgn}(u-\bar{u})^+K*(u-\bar{u})]\,dx\,d\tau. \tag{2.20}$$

From the above inequality, with $R\to+\infty$, we obtain

$$\int_{-\infty}^{+\infty}(u-\bar{u})^+\,dx\leqslant\int_{-\infty}^{+\infty}(u_0-\bar{u}_0)^+\,dx-\int_0^t\int_{-\infty}^{\infty}(u-\bar{u})^+\,dx\,d\tau$$
$$+\int_0^t\int_{-\infty}^{+\infty}\operatorname{sgn}(u-\bar{u})^+K*(u-\bar{u})]\,dx\,d\tau. \tag{2.21}$$

Indeed, since u and $\bar{u}$ are in $L^\infty([0,T];L^1(\mathbb{R}))$,

$$\begin{aligned}
&\int_0^t\int_{-\infty}^{+\infty}\operatorname{sgn}(u-\bar{u})^+K*(u-\bar{u})]\,dx\,d\tau\\
&\quad=\int_0^t\int_{-\infty}^{+\infty}\operatorname{sgn}(u-\bar{u})^+\frac{1}{2}\int_{-\infty}^{+\infty}e^{-|x-y|}(u-\bar{u})(y)\,dy\,dx\,d\tau\\
&\quad\leqslant\int_0^t\int_{-\infty}^{+\infty}\frac{1}{2}\int_{-\infty}^{+\infty}e^{-|x-y|}(u-\bar{u})^+(y)\,dy\,dx\,d\tau\\
&\quad=\int_0^t\int_{-\infty}^{+\infty}(u-\bar{u})^+(y)\,dy\,d\tau.
\end{aligned}$$

Hence, (2.21) becomes

$$\int_{-\infty}^{+\infty}(u-\bar{u})^+\,dx\leqslant\int_{-\infty}^{+\infty}(u_0-\bar{u}_0)^+\,dx,$$

which is exactly (2.15). In addition, interchanging the roles of u and $\bar{u}$, from (2.15) we easily obtain (2.16). Moreover, the monotonicity property stated at the end of the theorem is again a consequence of (2.15). As in the case of the vanishing viscosity approximation, it is not possible to prove the bound of the solutions in L^∞ using directly this monotonicity, because the constants are solutions of our equations, which do not belong to $L^1(\mathbb{R}_x)$. However, we can prove also in this case an estimate for the L^p norm of the form

$$\int_{-\infty}^{+\infty}|u|^p\,dx\leqslant\int_{-\infty}^{+\infty}|u_0|^p\,dx-\int_0^t\int_{-\infty}^{+\infty}p|u|^p\,dx\,ds$$
$$+\int_0^t\int_{-\infty}^{+\infty}p|u|^{p-1}|K*u|\,dx\,ds$$

which yields the desired L^∞ estimate as we pointed out in the proof of Theorem 2.2.

Finally, the constant s defined in (2.18) depends only on $\max\{||u_0||_{L^\infty(\mathbb{R})}, ||\bar{u}_0||_{L^\infty(\mathbb{R})}\}$ and the proof is complete. □

The results of Theorem 2.5 clearly implies the following corollary.

Corollary 2.6. *There exists at most one entropy solution of* (2.1), *belonging in the space* $L^\infty([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, *with initial datum in* $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Remark 2.7. In view of the uniqueness result we have established, we can conclude that any weak, entropy solution to (2.1) which belongs to the space $L^\infty([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ verifies the estimate

$$\int_{-\infty}^{+\infty} |u(x,t+k) - u(x,t)|\, dx \leqslant C(k + k^{\frac{2}{3}})||u_0||_{L^1(\mathbb{R})} + 4\omega(k^{\frac{1}{3}}),$$

where ω represents the L^1-modulus of continuity of the initial datum, because it can be obtained as limit of the vanishing viscosity approximation (2.3). Thus, $u \in C([0,T]; L^1(\mathbb{R}))$ for any $T > 0$.

We conclude the section with the case of initial datum $u_0 \in L^\infty(\mathbb{R})$. From the previous results, it follows that the solutions of (2.1) form a Lipschitz semigroup S_t, defined in the space $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with the L^1-norm (see, for instance, Theorem 2.5). Our aim is to extend such semigroup to the space $L^\infty(\mathbb{R})$ endowed with the following norm:

$$|\,||u||\,| = \int_{-\infty}^{+\infty} \phi(x)|u(x)|\, dx, \tag{2.22}$$

where ϕ is a positive, smooth function satisfying

$$\begin{aligned} \phi(x) &= e^{-|x|} && \text{for any } |x| \geqslant 2, \\ \phi(x) &= 1 && \text{for any } |x| \leqslant 1. \end{aligned}$$

In order to extend the semigroup to the whole space $(L^\infty, |\,||\cdot||\,|)$, it suffices to prove it is continuous in its dense subspace $(L^\infty \cap L^1, |\,||\cdot||\,|)$. We show this property in the next theorem. Let us remark that $(L^\infty, |\,||\cdot||\,|)$ is not a Banach space and our procedure actually extends the semigroup to the larger space $L^1(\mathbb{R}, \phi\, dx)$.

Theorem 2.8. *Let* $u, \bar{u} \in L^\infty([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ *be weak entropy solutions of* (2.1) *with initial data* $u_0, \bar{u}_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *Then, for any* $t \in [0,T]$,

$$|\,||u(\cdot,t) - \bar{u}(\cdot,t)||\,| \leqslant e^{Ct}|\,||u_0 - \bar{u}_0||\,|, \tag{2.23}$$

where the constant C *depends only on* ϕ *and on* $||u_0||_{L^\infty}$, $||\bar{u}_0||_{L^\infty}$.

Proof. Due to our uniqueness result (Corollary 2.6), we can consider the solutions $u,\ \bar{u}$ as limit of solutions $u^\mu,\ \bar{u}^\mu$ of the vanishing viscosity approximation

$$u_t + \tfrac{1}{2}(u^2)_x = -u + K*u + \mu u_{xx}. \tag{2.24}$$

We multiply the equation for $u^\mu - \bar{u}^\mu$ by the function ϕ to obtain

$$\begin{aligned}[] [\phi(u^\mu - \bar{u}^\mu)]_t + \tfrac{1}{2}[\phi((u^\mu)^2 - (\bar{u}^\mu)^2)]_x = & -\phi[u^\mu - \bar{u}^\mu - K*(u^\mu - \bar{u}^\mu)] \\ & + \tfrac{1}{2}\phi'((u^\mu)^2 - (\bar{u}^\mu)^2) + \mu\phi(u^\mu - \bar{u}^\mu)_{xx}. \end{aligned} \tag{2.25}$$

Let $\eta_\varepsilon(\xi)$ be a regular approximation of $|\xi|$ (in the spirit of the function used in the proof of Theorem 2.2). Multiplying (2.25) by $\eta_\varepsilon(u^\mu - \bar{u}^\mu)$ we obtain

$$\begin{aligned} \phi\eta_{\varepsilon,t} + \tfrac{1}{2}[\phi\eta'_\varepsilon((u^\mu)^2 - (\bar{u}^\mu)^2)]_x = & -\phi\eta'_\varepsilon[u^\mu - \bar{u}^\mu - K*(u^\mu - \bar{u}^\mu)] \\ & + \tfrac{1}{2}\phi\eta''_\varepsilon((u^\mu)^2 - (\bar{u}^\mu)^2)(u^\mu - \bar{u}^\mu)_x \\ & + \tfrac{1}{2}\phi'\eta'_\varepsilon((u^\mu)^2 - (\bar{u}^\mu)^2) \\ & + \mu[\phi\eta'_\varepsilon(u^\mu - \bar{u}^\mu)_x]_x - \mu\phi'\eta'_\varepsilon(u^\mu - \bar{u}^\mu)_x \\ & - \mu\eta''_\varepsilon\phi[(u^\mu - \bar{u}^\mu)_x]^2. \end{aligned} \tag{2.26}$$

Thus, proceeding as in the proof of Theorem 2.2, we integrate (2.26) in dx and dt and we let $\varepsilon\downarrow 0$ to obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} \phi(x)|u^\mu - \bar{u}^\mu|\,dx \leqslant & \int_{-\infty}^{+\infty} \phi(x)|u_0 - \bar{u}_0|\,dx \\ & + C\int_{-\infty}^{+\infty}\int_0^t (|\phi'| + |\phi''|)|u^\mu - \bar{u}^\mu|\,ds\,dx, \end{aligned} \tag{2.27}$$

where the constant C depends only on $||u||_\infty$, $||u_0||_\infty$ for $\mu < 1$. Since $(|\phi'| + |\phi''|) \leqslant C_1\phi$, from (2.27) and by the Gronwall lemma, we get (2.23) as $\mu\downarrow 0$. $\square$

3. Analysis of relaxation limits

In the previous section, we discussed the existence and the uniqueness of weak entropy solutions to our hyperbolic–elliptic model

$$\begin{cases} U_s + \tfrac{1}{2}(U^2)_y = -Q_y, \\ -Q_{yy} + Q = -U_y. \end{cases} \tag{3.1}$$

Hence, we can discuss now the convergence of relaxation limits for this model.

3.1. Hyperbolic–hyperbolic relaxation limit

In order to obtain an hyperbolic-type limit, we perform the following scaling:

$$u^\varepsilon(x,t) = U\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big),$$

$$q^\varepsilon(x,t) = Q\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\Big).$$

Thus, system (3.1) becomes

$$\begin{cases} u^\varepsilon_t + \frac{1}{2}[(u^\varepsilon)^2]_x = -q^\varepsilon_x, \\ -\varepsilon^2 q^\varepsilon_{xx} + q^\varepsilon = -\varepsilon u^\varepsilon_x. \end{cases} \tag{3.2}$$

Moreover, for the sake of clearness, we give an initial datum $u_0(x)$ to this problem such that $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We postpone the discussion of L^∞ initial data at the end of the section.

Remark 3.1. In the construction of the Cauchy problem for (3.2), we scale only the terms for $t>0$, without scaling the initial datum, which is given a posteriori as a fixed function in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Indeed, the scaled initial datum is given by the sequence $u^\varepsilon_0(x) = U_0(\frac{x}{\varepsilon})$, which converges to zero strongly in $L^1(\mathbb{R})$ because $U_0 \in L^1(\mathbb{R})$, namely, in this way we can recover at the limit only the null solution. In other words, we do not investigate the relaxation limit of the scaled solution, but we use the scaling only to detect the terms which are physically negligible in the equations and we study the singular limit of the new Cauchy problem, with fixed datum.

Remark 3.2. As we pointed out in the previous section, the restriction to the Burgers' flux $\frac{1}{2}u^2$ is unnecessary even at this point and the analysis of the hyperbolic–hyperbolic relaxation limit can be carried out, without essential changes, for general, smooth fluxes $f(u)$.

Letting $\varepsilon \downarrow 0$ in (3.2), we see that formally we obtain $q = 0$ and the limit equation is given by

$$u_t + \tfrac{1}{2}(u^2)_x = 0. \tag{3.3}$$

To prove rigorously this limit, it is convenient once again to rewrite system (3.2) as the nonhomogeneous scalar conservation law

$$u^\varepsilon_t + \frac{1}{2}[(u^\varepsilon)^2]_x = -\frac{1}{\varepsilon}(u^\varepsilon - K^\varepsilon * u^\varepsilon), \tag{3.4}$$

where the convolution kernel is given by

$$K^{\varepsilon} = \frac{1}{2\varepsilon} e^{\frac{-|x|}{\varepsilon}}.$$

Thus, the good contraction properties of the source term we employed in the proofs of Theorems 2.2 and 2.5 are still valid for (3.4). More precisely, the new convolution kernel K^{ε} is scaled such that $||K^{\varepsilon}||_{L^1} = ||K||_{L^1} = 1$. Therefore, following step by step the lines of the proofs presented above, we obtain the following result.

Theorem 3.3. *Let* $u^{\varepsilon}, \ \bar{u}^{\varepsilon} \in L^{\infty}([0,T]; L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R}))$ *be weak entropy solutions of* (3.4) *with initial data* $u_0, \ \bar{u}_0 \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$. *Then, for any* $t \in [0,T]$,

$$\int_{-\infty}^{+\infty} (u^{\varepsilon}(x,t) - \bar{u}^{\varepsilon}(x,t))^{+} \, dx \leqslant \int_{-\infty}^{+\infty} (u_0(x) - \bar{u}_0(x))^{+} \, dx;$$

$$||u^{\varepsilon}(\cdot,t) - \bar{u}^{\varepsilon}(\cdot,t)||_{L^1(\mathbb{R})} \leqslant ||u_0(\cdot) - \bar{u}_0(\cdot)||_{L^1(\mathbb{R})}.$$

Moreover, if $u_0(x) \leqslant \bar{u}_0(x)$ *a.e. on* $\mathbb{R}$, *then* $u^{\varepsilon}(x,t) \leqslant \bar{u}^{\varepsilon}(x,t)$ *a.e. on* $\mathbb{R} \times [0,T]$. *In addition, the essential range of both* u^{ε} *and* $\bar{u}^{\varepsilon}$ *is contained in* $[-a,a]$, *where*

$$a = \max\{||u_0||_{L^{\infty}(\mathbb{R})}, ||\bar{u}_0||_{L^{\infty}(\mathbb{R})}\}.$$

As we pointed out in the study of the vanishing viscosity limit, the results of Theorem 3.3 give the necessary compactness to get the strong convergence of our relaxation limit. We collect these properties in the next lemma.

Lemma 3.4. *Let* $u^{\varepsilon} \in L^{\infty}([0,T]; L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R}))$ *be the weak, entropy solution to* (3.4) *with* $u_0 \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ *as initial datum. In particular*

$$\int_{-\infty}^{+\infty} |u_0(x+h) - u_0(x)| \, dx \leqslant \omega(|h|), \quad \textit{for any } h \in \mathbb{R},$$

for some nondecreasing function ω *on* $[0,+\infty)$ *with* $\omega(r) \downarrow 0$ *as* $r \downarrow 0$. *Then there exists a constant* C, *depending only on* $||u_0||_{L^{\infty}(\mathbb{R})}$ *such that, for any* $t > 0$,

$$\int_{-\infty}^{+\infty} |u^{\varepsilon}(x+h,t) - u^{\varepsilon}(x,t)| \, dx \leqslant \omega(|h|), \quad \textit{for any } h \in \mathbb{R} \tag{3.5}$$

and

$$\int_{-\infty}^{+\infty} |u^{\varepsilon}(x,t+k) - u^{\varepsilon}(x,t)| \, dx \leqslant Ck^{\frac{2}{3}} ||u_0||_{L^1(\mathbb{R})} + 4\omega(k^{\frac{1}{3}}), \tag{3.6}$$

for any $k > 0$.

Proof. As in the proof of Lemma 2.3, the results of Theorem 3.3 implies (3.5). Due to the time regularity of our weak, entropy solutions (Remark 2.7), which can be always viewed as limit of the vanishing viscosity approximation, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \phi(x)[u^\varepsilon(x,t+k) - u^\varepsilon(x)]\,dx \\ = \int_t^{t+k} \int_{-\infty}^{+\infty} \bigg[\phi'(x)\frac{1}{2}u^\varepsilon(x,\tau)^2 \\ + \frac{\phi(x)}{\varepsilon}(-u^\varepsilon(x,\tau) + (K^\varepsilon * u^\varepsilon)(x,\tau))\bigg]\,dx\,d\tau, \qquad (3.7)\end{aligned}$$

where $\phi(x)$ stands for the smooth function considered in the proof of Lemma 2.3. Hence, the only difference with the case $\varepsilon = 1$ is in the source term, which is singular, due to the coefficient $\frac{1}{\varepsilon}$. However, we can still control this term, taking advantage of its dissipative nature. Indeed,

$$\begin{aligned}&-\frac{1}{\varepsilon}\int_t^{t+k}\int_{-\infty}^{+\infty} \phi(x)(u^\varepsilon - K^\varepsilon * u^\varepsilon)\,dx\,d\tau \\ &= \frac{1}{2\varepsilon}\int_t^{t+k}\int_{-\infty}^{+\infty}\phi(x)\int_{-\infty}^{+\infty} e^{-|\xi|}(u^\varepsilon(\varepsilon\xi + x,\tau) - u^\varepsilon(x,\tau))\,d\xi\,dx\,d\tau \\ &= \frac{1}{2\varepsilon}\int_t^{t+k}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\phi(x-\varepsilon\xi) - \phi(x))e^{-|\xi|}u^\varepsilon(x,\tau)\,dx\,d\tau\,d\xi \\ &= -\varepsilon\frac{1}{2\varepsilon}\int_t^{t+k}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-|\xi|}u^\varepsilon(x,\tau)\phi'(\zeta)\xi\,dx\,dt\,d\xi \\ &\leqslant Ck^{\frac{2}{3}}||u_0||_{L^1(\mathbb{R})},\end{aligned}$$

where the constant C is independent from ε. Finally, relation (3.6) can be proved as before starting from (3.7) and the proof is complete. □

Now we are ready to prove our relaxation theorem.

Theorem 3.5. *Let* $u^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ *be the weak, entropy solution of* (3.4) *with initial datum* $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *Then, as* $\varepsilon \downarrow 0$,

$$u^\varepsilon \to u, \quad \textit{strongly in } L^p_{\mathrm{loc}}(\mathbb{R} \times [0,T]),\ p < +\infty,$$

and u *is the unique entropy solution to* (3.3), *belonging in the space* $L^\infty([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, *with* u_0 *as initial datum.*

Proof. Applying the results of Lemma 3.4, we get the sequence u^ε is uniformly bounded in L^∞ and it is compact in L^1_{loc}. Therefore, passing if necessary to subsequence, as $\varepsilon \downarrow 0$,

$$u^\varepsilon \to u, \quad \text{strongly in } L^p_{\text{loc}}(\mathbb{R} \times [0,T]), \; p < +\infty,$$

and boundedly almost everywhere on $\mathbb{R} \times [0, +\infty)$. Passing into the limit in the weak formulation of (3.4) and in its entropy inequality, we can conclude u is the unique entropy solution of (3.3) if, as $\varepsilon \downarrow 0$,

$$-\frac{1}{\varepsilon}(u^\varepsilon - K^\varepsilon * u^\varepsilon) \to 0 \tag{3.8}$$

in the sense of distributions. We remark that once we prove u is the unique entropy solution to (3.3), the whole sequence u^ε will converge. In order to prove (3.8), we proceed as in Lemma 3.4, controlling this time in a more accurate way the singularity of the source term. Let $\psi(x,t)$ be a smooth, compactly supported test function. Then,

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int_0^T \int_{-\infty}^{+\infty} \psi(x,t)(u^\varepsilon - K^\varepsilon * u^\varepsilon)\,dx\,dt \\
&= \frac{1}{2\varepsilon}\int_0^T \int_{-\infty}^{+\infty} \psi(x,t) \int_{-\infty}^{+\infty} e^{-|\xi|}(u^\varepsilon(\varepsilon\xi + x, t) - u^\varepsilon(x,t))\,d\xi\,dx\,dt \\
&= \frac{1}{2\varepsilon}\int_0^T \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi(x - \varepsilon\xi, t) e^{-|\xi|} u^\varepsilon(x,t)\,dx\,dt\,d\xi \\
&\quad - \frac{1}{2\varepsilon}\int_0^T \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi(x,t) e^{-|\xi|} u^\varepsilon(x,t)\,dx\,dt\,d\xi.
\end{aligned}$$

Moreover,

$$|\psi(x - \varepsilon\xi, t) - \psi(x,t) + \varepsilon\psi_x(x,t)\xi| \leqslant \frac{1}{2}\varepsilon^2\xi^2 ||\psi||_{C_0^2}$$

and hence we have

$$\begin{aligned}
&\left|\frac{1}{\varepsilon}\int_0^T \int_{-\infty}^{+\infty} \psi(x,t)(u^\varepsilon - K^\varepsilon * u^\varepsilon)\,dx\,dt\right| \\
&\quad \leqslant \left|-\varepsilon\frac{1}{2\varepsilon}\int_0^T \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-|\xi|} u^\varepsilon(x,t)\psi_x(x,t)\xi\,dx\,dt\,d\xi\right| \\
&\qquad + \varepsilon^2 \frac{1}{4\varepsilon} ||\psi||_{C_0^2} \int_0^T \int_{-\infty}^{+\infty} |u^\varepsilon(x,t)|\,dx\,dt \int_{-\infty}^{+\infty} \xi^2 e^{-|\xi|}\,d\xi \\
&\quad \leqslant \varepsilon C T ||\psi||_{C_0^2} ||u_0||_{L^1(\mathbb{R})} = O(\varepsilon),
\end{aligned}$$

because of the identity

$$\int_{-\infty}^{+\infty} e^{-|\xi|}\xi \, d\xi = 0,$$

which proves (3.8). $\square$

Remark 3.6. In the proof of the hyperbolic–hyperbolic relaxation limit, we take advantage of the L^1-contraction of the source of (3.4), which are due solely to the property of the convolution kernel K^ε, namely

- K^ε even,
- $K^\varepsilon \geqslant 0$,
- $||K^\varepsilon||_{L^1} = 1$.

Moreover, the above conditions also imply the first momentum of K^ε is zero and this feature allows to control the singular limit of the source in the sense of distribution (see the last line in the proof of Theorem 3.5). Thus, the results we obtained are still valid for any equation of form (3.4), with a convolution kernel satisfying the above conditions.

We conclude the section with the discussion of the case of L^∞ initial data. As we pointed out before, the only difference with the nonscaled equation lies in the source term, due to the singular coefficient $\frac{1}{\varepsilon}$ and the scaled kernel K^ε. However, this changements do not affect the monotonicity properties of the source and therefore we can repeat the argument of the previous section to prove the following theorem.

Theorem 3.7. *Let u^ε, $\bar{u}^\varepsilon \in L^\infty([0,T] \times \mathbb{R})$ be weak entropy solutions of* (3.4) *with initial data u_0, $\bar{u}_0 \in L^\infty(\mathbb{R})$. Then, for any $t \in [0,T]$,*

$$|\,||u^\varepsilon(\cdot,t) - \bar{u}^\varepsilon(\cdot,t)||\,| \leqslant e^{Ct} |\,||u_0 - \bar{u}_0||\,|, \tag{3.9}$$

where the constant C depends only on ϕ and on $||u_0||_{L^\infty}$, $||\bar{u}_0||_{L^\infty}$ and $|\,||\cdot||\,|$ is the norm defined in (2.22).

Finally, once we have the property (3.9), we can easily obtain the results of Lemma 3.4 even for the norm (2.22) and therefore we get the convergence of the relaxation limit for L^∞ solutions of (3.4).

Remark 3.8. As we pointed out in Remark 3.1, we consider only relaxation limits with fixed initial data, without scaling the function at $t = 0$. However, if we choose the initial data U_0 of (3.1) only in $L^\infty(\mathbb{R})$, then we can consider the genuine relaxation limit of the corresponding weak solution, by scaling also the initial data $u_0^\varepsilon(x) = U_0(\frac{x}{\varepsilon})$. Indeed, this time the sequence u_0^ε is only bounded in L^∞ and therefore the solutions u^ε of (3.2) will converge to the solution of its formal limit (3.3) with the weak-* limit in L^∞ of u_0^ε as initial condition, provided U_0 is also in $\mathscr{BV}(\mathbb{R})$. Indeed, in this case, the estimate (3.9) gives the necessary uniform control of the modulus of continuity of the sequence u^ε in the L^1-weighted norm even for the scaled initial data $u_0^\varepsilon(x) = U_0(\frac{x}{\varepsilon})$.

3.2. Hyperbolic–parabolic relaxation limit

In this section, we move to the study of the hyperbolic–parabolic relaxation limit for system (3.1). To get the parabolic behavior at the limit, we need the following transformation:

$$u^\varepsilon(x,t) = \frac{1}{\varepsilon} U\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right),$$

$$q^\varepsilon(x,t) = \frac{1}{\varepsilon^2} Q\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right).$$

Hence, this time system (3.1) becomes

$$\begin{cases} u_t^\varepsilon + \frac{1}{2}[(u^\varepsilon)^2]_x = -q_x^\varepsilon, \\ -\varepsilon^2 q_{xx}^\varepsilon + q^\varepsilon = -u_x^\varepsilon. \end{cases} \tag{3.10}$$

Once again, we consider an initial datum $u_0(x)$ to this problem such that $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. For the general L^∞ case, we refer to the end of the section.

Remark 3.9. As we pointed out in Remark 3.1, we perform the above scaling only to show which are the negligible terms in the equations and we do not consider also the scaled initial datum. With this scaling, namely, in the parabolic regime, it turns out that the negligible term is q_{xx}, as proposed in [KNN98]. Once again, we do not scale the initial datum, because, in this way, the sequence we end up is given by $\frac{1}{\varepsilon} U_0(\frac{x}{\varepsilon})$, which is uniformly bounded in $L^1(\mathbb{R})$, but not in $L^\infty(\mathbb{R})$, if $U_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, without further restrictions.

From the second line of (3.10), it is clear that the formal limit is given by the relation $q = -u_x$ and therefore the relaxed equation is

$$u_t + \tfrac{1}{2}(u^2)_x = u_{xx}. \tag{3.11}$$

In order to utilize the L^1-contractivity of the solutions of our problem, we transform (3.10) as follows:

$$u_t^\varepsilon + \frac{1}{2}[(u^\varepsilon)^2]_x = -\frac{1}{\varepsilon^2}(u^\varepsilon - K^\varepsilon * u^\varepsilon), \tag{3.12}$$

with the same convolution kernel of the previous case. Hence, we end up with the same structure of the hyperbolic–hyperbolic relaxation limit and therefore the same results apply to this case.

Remark 3.10. Conversely to the previous relaxation limit, the form of the Burgers' flux is somehow crucial in this case, because we scale also the dependent variable u. However, assuming the flux $f(u)$ is smooth and it verifies $f(0) = f'(0) = 0$, the limit equation is given by

$$u_t + \tfrac{1}{2} f''(0)(u^2)_x = u_{xx},$$

namely, it is again given by the viscous Burgers' equation. Moreover, due to the L^∞ bound on u^ε, the analysis of this limit does not differ to the one presented here and therefore the study of the general case is again inessential.

Theorem 3.11. *Let* $u^\varepsilon, \ \bar{u}^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ *be weak entropy solutions of* (3.12) *with initial data* $u_0, \ \bar{u}_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *Then, for any* $t \in [0,T]$,

$$\int_{-\infty}^{+\infty} (u^\varepsilon(x,t) - \bar{u}^\varepsilon(x,t))^+ \, dx \leqslant \int_{-\infty}^{+\infty} (u_0(x) - \bar{u}_0(x))^+ \, dx;$$

$$\|u^\varepsilon(\cdot,t) - \bar{u}^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R})} \leqslant \|u_0(\cdot) - \bar{u}_0(\cdot)\|_{L^1(\mathbb{R})}.$$

Moreover, if $u_0(x) \leqslant \bar{u}_0(x)$ *a.e. on* $\mathbb{R}$, *then* $u^\varepsilon(x,t) \leqslant \bar{u}^\varepsilon(x,t)$ *a.e. on* $\mathbb{R} \times [0,T]$. *In addition, the essential range of both* u^ε *and* $\bar{u}^\varepsilon$ *is contained in* $[-a,a]$, *where*

$$a = \max\{\|u_0\|_{L^\infty(\mathbb{R})}, \|\bar{u}_0\|_{L^\infty(\mathbb{R})}\}.$$

Once again, the above results yield the compactness of our relaxation approximation.

Lemma 3.12. *Let* $u^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ *be the weak, entropy solution to* (3.12) *with* $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *as initial datum. In particular*

$$\int_{-\infty}^{+\infty} |u_0(x+h) - u_0(x)| \, dx \leqslant \omega(|h|), \quad \textit{for any } h \in \mathbb{R},$$

for some nondecreasing function ω *on* $[0,+\infty)$ *with* $\omega(r) \downarrow 0$ *as* $r \downarrow 0$. *Then there exists a constant* C, *depending only on* $\|u_0\|_{L^\infty(\mathbb{R})}$ *such that, for any* $t > 0$,

$$\int_{-\infty}^{+\infty} |u^\varepsilon(x+h,t) - u^\varepsilon(x,t)| \, dx \leqslant \omega(|h|), \quad \textit{for any } h \in \mathbb{R} \tag{3.13}$$

and

$$\int_{-\infty}^{+\infty} |u^\varepsilon(x,t+k)-u^\varepsilon(x,t)|\,dx \leqslant C(k^{\frac{2}{3}}+k^{\frac{1}{3}})||u_0||_{L^1(\mathbb{R})}+4\omega(k^{\frac{1}{3}}), \tag{3.14}$$

for any $k>0$.

Proof. As in the proof of Lemma 3.4, (3.13) follows from Theorem 3.11. Moreover, our weak, entropy solution verifies

$$\begin{aligned}\int_{-\infty}^{+\infty} \phi(x)[u^\varepsilon(x,t+k)-u^\varepsilon(x)]\,dx \\ = \int_t^{t+k}\int_{-\infty}^{+\infty}\bigg[\phi'(x)\frac{1}{2}u^\varepsilon(x,\tau)^2 \\ + \frac{\phi(x)}{\varepsilon^2}(-u^\varepsilon(x,\tau)+(K^\varepsilon * u^\varepsilon)(x,\tau))\bigg]\,dx\,d\tau,\end{aligned} \tag{3.15}$$

where $\phi(x)$ stands for the smooth function considered in the proofs of Lemmas 2.3 and 3.4. Once again, we must control the singularity of the source term, which in this case is more stiff than in the hyperbolic limit. However, it can be controlled using the second derivative of the smooth function ϕ. Indeed,

$$\begin{aligned}&-\frac{1}{\varepsilon^2}\int_t^{t+k}\int_{-\infty}^{+\infty}\phi(x)(u^\varepsilon-K^\varepsilon * u^\varepsilon)\,dx\,d\tau \\ &=\frac{1}{2\varepsilon^2}\int_t^{t+k}\int_{-\infty}^{+\infty}\phi(x)\int_{-\infty}^{+\infty}e^{-|\xi|}(u^\varepsilon(\varepsilon\xi+x,\tau)-u^\varepsilon(x,\tau))\,d\xi\,dx\,d\tau \\ &=\frac{1}{2\varepsilon^2}\int_t^{t+k}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\phi(x-\varepsilon\xi)-\phi(x))e^{-|\xi|}u^\varepsilon(x,\tau)\,dx\,d\tau\,d\xi \\ &=-\varepsilon\frac{1}{2\varepsilon^2}\int_t^{t+k}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-|\xi|}u^\varepsilon(x,\tau)\phi'(x)\xi\,dx\,dt\,d\xi \\ &\quad+\varepsilon^2\frac{1}{4\varepsilon^2}\int_t^{t+k}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-|\xi|}u^\varepsilon(x,\tau)\phi''(\zeta)\xi^2\,d\tau\,dx\,d\xi \\ &\leqslant Ck^{\frac{1}{3}}||u_0||_{L^1(\mathbb{R})},\end{aligned}$$

because, as we pointed out before,

$$\int_{-\infty}^{+\infty} e^{-|\xi|}\xi\,d\xi = 0.$$

As before, the constant C is independent from ε and therefore relation (3.14) is an easy consequence of (3.15). The proof is complete. □

Finally, the convergence result is contained in the next theorem.

Theorem 3.13. *Let* $u^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ *be the weak, entropy solution of* (3.12) *with initial datum* $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *Then, as* $\varepsilon \downarrow 0$,

$$u^\varepsilon \to u, \quad \textit{strongly in } L^p_{\mathrm{loc}}(\mathbb{R} \times [0,T]), \; p < +\infty,$$

and u is the unique solution to (3.11) *with* u_0 *as initial datum.*

Proof. As for the proof of Theorem 3.5, we obtain, up to subsequences,

$$u^\varepsilon \to u, \quad \text{strongly in } L^p_{\mathrm{loc}}(\mathbb{R} \times [0,T]), \; p < +\infty,$$

and boundedly almost everywhere in $\mathbb{R} \times [0, +\infty)$, where $u \in L^1(\mathbb{R} \times [0,T]) \cap L^\infty(\mathbb{R} \times [0,T])$, thanks to the compactness coming from the results of Lemma 3.12. Hence, to conclude that u is the unique solution to (3.11) with u_0 as initial datum, we need to prove

$$-\frac{1}{\varepsilon^2}(u^\varepsilon - K^\varepsilon * u^\varepsilon) \to u_{xx} \tag{3.16}$$

in the sense of distributions. We perform this task proceeding as in the proof of Lemma 3.12. Also in this case, the uniqueness of solutions to this Cauchy problem implies that the whole sequence u^ε will converge. Let $\psi(x,t)$ be a smooth, compactly supported test function. Since

$$\psi(x - \varepsilon\xi, t) - \psi(x,t) = -\varepsilon\psi_x(x,t)\xi + \tfrac{1}{2}\varepsilon^2\xi^2\psi_{xx}(x,t) - \tfrac{1}{6}\varepsilon^3|\xi|^3\psi_{xxx}(\zeta^\varepsilon, t),$$

we have

$$\begin{aligned}
&-\frac{1}{\varepsilon^2}\int_0^T \int_{-\infty}^{+\infty} \psi(x,t)(u^\varepsilon - K^\varepsilon * u^\varepsilon)\,dx\,dt \\
&= \frac{1}{2\varepsilon^2}\int_0^T \int_{-\infty}^{+\infty} \psi(x,t) \int_{-\infty}^{+\infty} e^{-|\xi|}(u^\varepsilon(\varepsilon\xi + x, t) - u^\varepsilon(x,t))\,d\xi\,dx\,dt \\
&= \frac{1}{2\varepsilon^2}\int_0^T \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi(x - \varepsilon\xi, t)e^{-|\xi|}u^\varepsilon(x,t)\,dx\,dt\,d\xi \\
&\quad - \frac{1}{2\varepsilon^2}\int_0^T \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi(x,t)e^{-|\xi|}u^\varepsilon(x,t)\,dx\,dt\,d\xi \\
&= -\varepsilon\frac{1}{2\varepsilon^2}\int_0^T \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-|\xi|}u^\varepsilon(x,t)\psi_x(x,t)\xi\,dx\,dt\,d\xi
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon^2 \frac{1}{4\varepsilon^2}\int_0^T \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-|\xi|}u^\varepsilon(x,t)\psi_{xx}(x,t)\xi^2\,dx\,dt\,d\xi \\
&-\varepsilon^3 \frac{1}{12\varepsilon^2}\int_0^T \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \xi^3 e^{-|\xi|}u^\varepsilon(x,t)\psi_{xxx}(\zeta^\varepsilon,t)\,dx\,dt\,d\xi \\
=&\int_0^T \int_{-\infty}^{+\infty} u^\varepsilon(x,t)\psi_{xx}(x,t)\,dx\,dt + O(\varepsilon),
\end{aligned}
$$

because of the relations

$$
\int_{-\infty}^{+\infty} e^{-|\xi|}\xi\,d\xi = 0, \quad \frac{1}{4}\int_{-\infty}^{+\infty} e^{-|\xi|}\xi^2\,d\xi = 1,
$$

and the estimate

$$
\begin{aligned}
&\left|\varepsilon^3 \frac{1}{12\varepsilon^2}\int_0^T \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \xi^3 e^{-|\xi|}u^\varepsilon(x,t)\psi_{xxx}(\zeta^\varepsilon,t)\,dx\ \,dt\,d\xi\right| \\
&\leqslant \varepsilon \frac{1}{12}||\psi||_{C_0^3}\int_0^T \int_{-\infty}^{+\infty} |u^\varepsilon(x,t)|\,dx\,dt \int_{-\infty}^{+\infty} |\xi|^3 e^{-|\xi|}\,d\xi \\
&= \varepsilon CT||\psi||_{C_0^3}||u_0||_{L^1(\mathbb{R})}.
\end{aligned}
$$

Hence, passing into the limit in the above relation, we recover (3.16) and the proof is complete. □

Remark 3.14. As we pointed out in Remark 3.6, the results established above are valid for any convolution kernel which is even, nonnegative and such that $||K^\varepsilon||_{L^1} = 1$. In this case, the fact that the first momentum of K^ε is zero is used also in the control of the L^1-modulus of continuity in the t variable (see the proof of Lemma 3.12), while the second momentum of K^ε, necessarily positive, gives the double of the viscosity constant in the limit equation (see the relations above).

We pass now to the study of the case of L^∞ initial data. As we put into evidence in the hyperbolic–hyperbolic relaxation limit, our scaling introduces a singular, positive coefficient $\frac{1}{\varepsilon^2}$ in front of the source term and it slightly modifies the kernel, but it preserves the monotonicity of the source itself. Therefore, even in this case, repeating the previous arguments, we can prove the Lipschitz continuity in the weighted norm considered in the previous sections, which is the basis of our analysis.

Theorem 3.15. *Let* $u^\varepsilon,\ \bar{u}^\varepsilon \in L^\infty([0,T]\times\mathbb{R})$ *be weak entropy solutions of* (3.12) *with initial data* $u_0,\ \bar{u}_0 \in L^\infty(\mathbb{R})$. *Then, for any* $t\in[0,T]$,

$$
|\,||u^\varepsilon(\cdot,t) - \bar{u}^\varepsilon(\cdot,t)||\,| \leqslant e^{Ct}|\,||u_0 - \bar{u}_0||\,|, \tag{3.17}
$$

where the constant C depends only on ϕ *and on* $||u_0||_{L^\infty}$, $||\bar{u}_0||_{L^\infty}$ *and* $|\,||\cdot||\,|$ *is the norm defined in* (2.22).

Once we have proved (3.17), without significant modifications, we can recover the result of Lemma 3.12 in terms of norm (2.22) and hence we obtain the convergence of the relaxation limit even in the case of L^∞ solutions of (3.12).

Note added in proof. The authors have learned of two relevant references (S. Schochet, E. Tadmor, Arch. Rational Mech. Anal. 119 (1992) 95–107; H. Liu, E. Tadmor, SIAM J. Math. Anal 33 (2001) 930–945) that contain some ideas regarding the contraction properties used in this paper and some interesting results concerning the travelling wave analysis. We found no mention of these references in the literature quoted below.

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Appendix A. Local existence for the vanishing viscosity approximation

We prove here the local-in-time existence of the equation

$$u_t + \tfrac{1}{2}(u^2)_x = -u + K*u + \mu u_{xx} \tag{A.1}$$

when the initial data u_0 is chosen in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We show in particular that the time of existence depends on the L^1 and L^∞ norms of the initial data and therefore, by a continuation principle, the solution is defined globally in time, once we prove the following a priori estimates:

$$||u(t)||_{L^1} \leqslant ||u_0||_{L^1}$$

$$||u(t)||_{L^\infty} \leqslant ||u_0||_{L^\infty},$$

for any $t \in [0, T_0]$. As usual in this kind of problems (see, for instance, [Smo94]), we shall use a fixed point argument to prove the existence of such solutions. Therefore, let us consider the Banach space

$$\{C([0, T_0]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))\}$$

with the norm

$$|\,||u||\,| = \sup_{0 \leqslant t \leqslant T_0} \max\{||u(\cdot, t)||_{L^1}, ||u(\cdot, t)||_{L^\infty}\}$$

and let us consider its closed subset

$$\mathscr{B} = \{u \in C([0, T_0]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})) \text{ such that } |\,||u - G_\mu * u_0|\,|| \leqslant |\,||u_0||\,|\},$$

where

$$G_\mu = \frac{1}{\sqrt{4\pi\mu t}} e^{-\frac{x^2}{4\mu t}}$$

and hence $G_\mu * u_0$ is the solution of the linear heat equation $u_t = \mu u_{xx}$ with u_0 as initial datum. We will obtain the solution of (A.1) as fixed point of the following operator, defined on $\mathscr{B}$,

$$\begin{aligned}\mathscr{T} u = & \int_{-\infty}^{+\infty} G_\mu(x-y,t) u_0(y)\, dy + \int_0^t \int_{-\infty}^{+\infty} G_\mu(x-y,t-s) \frac{1}{2} (u(y,s)^2)_x \, dy\, ds \\ & - \int_0^t \int_{-\infty}^{+\infty} G_\mu(x-y,t-s)[u(y,s) - (K * u)(y,s)]\, dy\, ds \\ = & I_1 + I_2(u) + I_3(u).\end{aligned}$$

Remark A.1. Since $|\,||G_\mu * u_0||\,| \leqslant |\,||u_0||\,|$, $0 \in \mathscr{B}$ and $|\,||u||\,| \leqslant 2|\,||u_0||\,|$ for any $u \in \mathscr{B}$.

The local existence for the solutions to (A.1) comes from the next theorem.

Theorem A.2. *There exists a positive time $T_0 > 0$ such that the operator $\mathscr{T} : \mathscr{B} \to \mathscr{B}$ is a contraction. In particular, there exists an unique solution $u \in C([0, T_0]; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ to* (A.1) *with $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ as initial datum.*

Proof. In order to prove the theorem, we must find constants $T_0 > 0$ and $c_0 < 1$ such that

(i) $\mathscr{T} u \in \mathscr{B}$ for any $u \in \mathscr{B}$;
(ii) $|\,||\mathscr{T} u - \mathscr{T} v||\,| \leqslant c_0 |\,||u - v||\,|$ for any $u, v \in \mathscr{B}$.

Let us consider $u \in \mathscr{B}$. In order to control $|\,||\mathscr{T} u - G_\mu * u_0||\,|$, we have to estimate the L^1 and the L^∞ norms of I_2 and I_3 in the definition of $\mathscr{T}$. We have

$$||I_2(u)||_{L^\infty} \leqslant \frac{1}{2} ||u||_{L^\infty}^2 \frac{C}{\sqrt{\mu}} \sqrt{t} \leqslant C_\mu \sqrt{t} ||u||_{L^\infty}^2$$

and

$$||I_2(u)||_{L^1} \leqslant \frac{1}{2} ||u^2||_{L^1} \frac{C}{\sqrt{\mu}} \sqrt{t} \leqslant C_\mu \sqrt{t} ||u||_{L^\infty} ||u||_{L^1},$$

where C_μ depends only on μ and the L^1 norms of e^{-x^2} and $e^{-x^2}x$. Therefore

$$|\,||I_2(u)||\,| \leqslant C_\mu\sqrt{T_0}|\,||u||\,| \leqslant C_\mu\sqrt{T_0}4|\,||u_0||\,|^2.$$

Moreover,

$$||I_3(u)||_{L^\infty} \leqslant 2t||u||_{L^\infty}$$

and

$$||I_3(u)||_{L^1} \leqslant 2t||u||_{L^1},$$

that is

$$|\,||I_3(u)||\,| \leqslant 2T_0|\,||u||\,| \leqslant T_0 4|\,||u_0||\,|.$$

Thus, if

$$C_\mu\sqrt{T_0}4|\,||u_0||\,|^2 + T_0 4|\,||u_0||\,| \leqslant ||u_0|\,||, \tag{A.2}$$

(i) is satisfied. In order to fulfill (A.2), it is sufficient to choose

$$T_0 = \min\left\{\frac{1}{8}, \frac{1}{64C_\mu^2}\frac{1}{|\,||u_0||\,|^2}\right\}.$$

To show (ii), we have to estimate

$$|\,||\mathcal{T}u - \mathcal{T}v||\,| \leqslant |\,||I_2(u) - I_2(v)||\,| + |\,||I_3(u) - I_3(v)||\,|,$$

for any $u,\ v \in \mathcal{B}$. Since $I_3(u)$ is linear in u, we have

$$|\,||I_3(u) - I_3(v)||\,| = |\,||I_3(u-v)||\,| \leqslant 2T_0|\,||u-v||\,|.$$

Moreover,

$$||I_2(u) - I_2(v)||_{L^\infty} \leqslant \frac{1}{2}||u^2 - v^2||_{L^\infty}\frac{C}{\sqrt{\mu}}\sqrt{t} \leqslant C_\mu\sqrt{t}||u-v||_{L^\infty}||u+v||_{L^\infty}$$

and

$$||I_2(u) - I_2(v)||_{L^1} \leqslant \frac{1}{2}||u^2 - v^2||_{L^1}\frac{C}{\sqrt{\mu}}\sqrt{t} \leqslant C_\mu\sqrt{t}||u-v||_{L^1}||u+v||_{L^\infty},$$

with C_μ as before. Hence,

$$\begin{aligned}|\,||I_2(u) - I_2(v)||\,| &\leqslant 4C_\mu\sqrt{T_0}(|\,||u||\,| + |\,||v||\,|)|\,||u-v||\,| \\ &\leqslant 4C_\mu\sqrt{T_0}|\,||u_0||\,||\,||u-v||\,|.\end{aligned}$$

Therefore,

$$|||\mathcal{T}u - \mathcal{T}v|||\leqslant(4C_\mu\sqrt{T_0}|||u_0|||+2T_0)|||u-v|||\leqslant\tfrac{1}{2}|||u-v|||,$$

provided

$$T_0 = \frac{1}{2}\min\left\{\frac{1}{4},\frac{1}{64C_\mu^2}\frac{1}{|||u_0|||^2}\right\}.$$

Finally, the theorem is proved with the choice

$$T_0 = \min\left\{\frac{1}{8},\frac{1}{128C_\mu^2}\frac{1}{|||u_0|||^2}\right\}. \qquad \square$$

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