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J. Differential Equations 191 (2003) 445–469

Journal of Differential Equations

http://www.elsevier.com/locate/jde

The L^p-L^q estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media

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Received June 7, 2002

Abstract

We first obtain the L^p-L^q estimates of solutions to the Cauchy problem for one-dimensional damped wave equation

$$V_{tt} - V_{xx} + V_t = 0, \quad (V, V_t)|_{t=0} = (V_0, V_1)(x), \quad (x,t) \in \mathbf{R} \times \mathbf{R}_+,$$

corresponding to that for the parabolic equation

$$\phi_t - \phi_{xx} = 0 \quad \phi|_{t=0} = (V_0 + V_1)(x).$$

The estimates are shown by

$$\left\| (V-\phi)(\cdot,t) - e^{-t/2}\frac{V_0(\cdot+t)+V_0(\cdot-t)}{2} \right\|_{L^p} \leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1} \|V_0, V_1\|_{L^q}, \quad t \geqslant 1, \qquad (*)$$

etc. for $1 \leqslant q \leqslant p \leqslant \infty$. To show $(*)$, the explicit formula of the damped wave equation will be used. To apply the estimates to nonlinear problems is the second aim. We will treat the system of a compressible flow through porous media. The solution is expected to behave as the diffusion wave, which is the solution to the porous media equation due to the Darcy law. When the initial data has the same constant state at $\pm\infty$, a sharp L^p-convergence rate for

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[1] Supported in part by the Grant-in-Aid for Scientific Research (C)(2) 13640223 of Japan Society for the Promotion of Science.

doi:10.1016/S0022-0396(03)00026-3

$p \geqslant 2$ has been recently obtained by Nishihara (Proc. Roy. Soc. Edinburgh, Sect. A, 133A (2003), 1–20) by choosing a suitably located diffusion wave. We will show the L^1 convergence, applying $(*)$.

Keywords: Damped wave equation; Diffusive phenomenon; $L^p - L^q$ estimate; p-System with damping; Nonlinear diffusion wave.

1. Introduction

In this paper we first consider the Cauchy problem for the damped wave equations on one-dimensional space $\mathbf{R}^1$

$$\begin{cases} V_{tt} - V_{xx} + V_t = 0, \quad x \in \mathbf{R}^1, \quad t > 0, \\ (V, V_t)|_{t=0} = (V_0, V_1)(x), \end{cases} \tag{1.1}$$

corresponding to that for the parabolic equation

$$\begin{cases} \phi_t - \phi_{xx} = 0, \\ \phi|_{t=0} = (V_0 + V_1)(x). \end{cases} \tag{1.2}$$

Note that the initial data of ϕ is $V_0 + V_1$.

The damped wave equation has been indicated to have a diffusive structure as $t \to \infty$ by several authors [1,10], etc. Our first main purpose is to give the precise interpretation to this indication, which is shown by the $L^p - L^q$ estimates for $1 \leqslant q \leqslant p \leqslant \infty$. In fact, we obtain the following estimates.

Theorem 1.1. *Let V and ϕ be solutions in the distributional sense to* (1.1) *and* (1.2), *respectively. Then, the following estimates hold for $1 \leqslant q \leqslant p \leqslant \infty$ and $t \geqslant 2$:*

$$||\partial_x^\alpha \partial_t^\beta (V - \phi)(\cdot, t) - e^{-t/2} W_{\alpha,\beta}(\cdot, t)||_{L^p} \leqslant C t^{-\frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right) - 1 - \frac{\alpha}{2} - \beta} ||V_0, V_1||_{L^q} \tag{$1.3_{\alpha\beta}$}$$

for $\alpha, \beta \in \mathbf{Z}$ with $0 \leqslant \alpha + \beta \leqslant 2$. Here, by denoting

$$W(t)g = \frac{1}{2} \int_{x-t}^{x+t} g(y)\, dy, \tag{1.4}$$

$W_{\alpha,\beta}(x, t) = W_{\alpha,\beta}(x, t; V_0, V_1)$ are listed as

$$W_{0,0}(x, t) = \partial_t (W(t) V_0) = \frac{V_0(x + t) + V_0(x - t)}{2}, \tag{1.5}$$

$$W_{1,0}(x,t) = \partial_{tx}^2(W(t)V_0) + \left(\frac{t}{8}+\frac{1}{2}\right)\partial_x(W(t)V_0) + \partial_x(W(t)V_1), \tag{1.6}$$

$$W_{0,1}(x,t) = \frac{t}{8}\partial_t(W(t)V_0) + \partial_x(W(t)V_0) + \partial_t(W(t)V_1), \tag{1.7}$$

$$\begin{aligned} W_{2,0}(x,t) = {} & \partial_{txx}^3(W(t)V_0) + \left(\frac{t}{8}+\frac{1}{2}\right)\partial_{tt}^2(W(t)V_0) + \frac{t^2}{128}\partial_t(W(t)V_0) \\ & + \partial_{tx}^2(W(t)V_1) + \frac{t}{8}\partial_t(W(t)V_1), \end{aligned} \tag{1.8}$$

$$\begin{aligned} W_{1,1}(x,t) = {} & \partial_{ttx}^3(W(t)V_0) + \frac{t}{8}\partial_{tx}^2(W(t)V_0) + \left(\frac{t^2}{128}-\frac{1}{8}\right)\partial_t(W(t)V_0) \\ & + \partial_{tx}^2(W(t)V_1) + \left(\frac{t}{8}-1\right)\partial_t(W(t)V_1), \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} W_{0,2}(x,t) = {} & \partial_{ttt}^3(W(t)V_0) + \left(\frac{t}{8}-\frac{1}{2}\right)\partial_{tx}^2(W(t)V_0) + \left(\frac{t^2}{128}-\frac{t}{8}\right)\partial_t(W(t)V_0) \\ & + \partial_{tx}^2(W(t)V_1) + \left(\frac{t}{4}-1\right)\partial_t(W(t)V_1). \end{aligned} \tag{1.10}$$

For the parabolic equation (1.2) the $L^p - L^q$ estimates with $1 \leqslant q \leqslant p \leqslant \infty$

$$\|\partial_x^\alpha \partial_t^\beta \phi(\cdot,t)\|_{L^p} \leqslant C t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\alpha}{2}-\beta} \|V_0 + V_1\|_{L^q}, \quad t \geqslant 2 \tag{1.11}$$

are well-known. See e.g. [23]. Also, $w = W(t)g$ in (1.4) and $w = \partial_t(W(t)g)$ are solutions to

$$w_{tt} - w_{xx} = 0$$

with $(w, w_t)|_{t=0} = (0, g)$ and $(w, w_t)|_{t=0} = (g, 0)$, respectively, which are called the D'Alembert formulas. Hence, estimates (1.3) in Theorem 1.1 show that, if we subtract the term $e^{-t/2}W_{\alpha,\beta}(x,t)$, which may have the singularities coming from the initial data, then the solution ϕ to (1.2) is an asymptotic profile of the solution V to (1.1). However, if the initial data V_0 and V_1 have suitable regularities, then $e^{-t/2}W_{\alpha,\beta}(x,t)$ decay exponentially and can be neglected, so that we have the same $L^p - L^q$ estimates for V as same as ϕ. That is, ϕ is an asymptotic profile of V as $t \to \infty$.

These kinds of property have been recently obtained in [19] for three-dimensional damped wave equations, which have been applied to show the global existence theorem for semilinear damped wave equations.

The proof of Theorem 1.1 will be done by using the explicit formula of the solution, same as in [19]. By $S(t)g$ denote the solution $v(x,t)$ to the Cauchy problem

$$\begin{cases} v_{tt} - v_{xx} + v_t = 0, \\ (v, v_t)|_{t=0} = (0, g), \end{cases} \tag{1.12}$$

then

$$v(x,t) = S(t)g = \frac{1}{2}e^{-t/2} \int_{|z| \leqslant t} I_0\left(\frac{1}{2}\sqrt{t^2 - z^2}\right) g(x - z)\, dz \tag{1.13}$$

[2], where the modified Bessel function I_ν of order ν is given by

$$I_\nu(y) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(\nu + m + 1)}\left(\frac{y}{2}\right)^{2m+\nu} \tag{1.14}$$

with the Gamma function Γ. Then the solution V to (1.1) is

$$V(x,t) = S(t)(V_0 + V_1) + \partial_t(S(t)V_0). \tag{1.15}$$

Also, the solution $\phi =: P(t)(V_0 + V_1)$ to (1.2) has the form

$$\phi(x,t) = P(t)(V_0 + V_1) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-z)^2}{4t}} (V_0 + V_1)(z)\, dz, \tag{1.16}$$

which is estimated by (1.11). Hence

$$(V - \phi)(x,t) = (S(t) - P(t))(V_0 + V_1) + \partial_t(S(t)V_0). \tag{1.17}$$

To obtain (1.3) with $\alpha = \beta = 0$, for example, we need to estimate $||(S(t) - P(t))g||_{L^p} (g = V_0 + V_1)$ and $||\partial_t(S(t))V_0)||_{L^p}$ for $t \geqslant 2$ (see Proposition 2.1 in Section 2). Moreover, to apply the $L^p - L^q$ estimates to the nonlinear problem, we should also consider the inhomogeneous problem

$$\begin{cases} V_{tt} - V_{xx} + V_t = f(x,t), \\ (V, V_t)|_{t=0} = (0, 0), \end{cases}$$

whose solution V is given by

$$V(x,t) = \int_0^t S(t - \tau) f(x, \tau)\, d\tau. \tag{1.18}$$

Therefore, $S(t)g$ and its derivatives should be estimated for all $t \geqslant 0$. They are divided into two terms coming from the wave property and parabolic structure, and listed as follows:

$$\partial_x^\alpha \partial_t^\beta (S(t)g) = e^{-t/2}\omega_{\alpha\beta}(t)g + J_{\alpha\beta}(t)g \tag{1.19}$$

for $1 \leqslant \alpha + \beta \leqslant 3$, where, for $\alpha + \beta \leqslant 2$,

$$\omega_{10}(t)g = \frac{g(x+t) - g(x-t)}{2} (= \partial_x(W(t)g)), \tag{1.20$_{10}$}$$

$$\omega_{01}(t)g = \frac{g(x+t) + g(x-t)}{2} (= \partial_t(W(t)g)), \tag{1.20$_{01}$}$$

$$\begin{aligned} \omega_{20}(t)g =& \frac{t}{8}\frac{g(x+t) + g(x-t)}{2} + \frac{g'(x+t) - g'(x-t)}{2} \\ & \left(= \frac{t}{8}\partial_t(W(t)g) + \partial_x^2(W(t)g)\right), \end{aligned} \tag{1.20$_{20}$}$$

$$\begin{aligned} \omega_{11}(t)g =& \left(\frac{t}{8} - \frac{1}{2}\right)\frac{g(x+t) - g(x-t)}{2} + \frac{g'(x+t) + g'(x-t)}{2} \\ & \left(= \left(\frac{t}{8} - \frac{1}{2}\right)\partial_x(W(t)g) + \partial_x\partial_t(W(t)g)\right), \end{aligned} \tag{1.20$_{11}$}$$

$$\begin{aligned} \omega_{02}(t)g =& \left(\frac{t}{8} - 1\right)\frac{g(x+t) + g(x-t)}{2} + \frac{g'(x+t) - g'(x-t)}{2} \\ & \left(= \left(\frac{t}{8} - 1\right)\partial_t(W(t)g) + \partial_t^2(W(t)g)\right) \end{aligned} \tag{1.20$_{20}$}$$

and

$$J_{10}(t)g = \frac{1}{4}e^{-t/2} \int_{|z|\leqslant t} I_1\left(\frac{1}{2}\sqrt{t^2 - z^2}\right)\frac{-z}{\sqrt{t^2 - z^2}} g(x-z)\,dz, \tag{1.21$_{10}$}$$

$$\begin{aligned} J_{01}(t)g =& \frac{1}{4}e^{-t/2} \int_{|z|\leqslant t} \left[-I_0\left(\frac{1}{2}\sqrt{t^2 - z^2}\right) + I_1\left(\frac{1}{2}\sqrt{t^2 - z^2}\right)\frac{t}{\sqrt{t^2 - z^2}}\right] \\ & \times g(x-z)\,dz, \end{aligned} \tag{1.21$_{01}$}$$

$$\begin{aligned} J_{20}(t)g =& \frac{1}{4}e^{-t/2} \int_{|z|\leqslant t} \left[I_0\left(\frac{1}{2}\sqrt{t^2 - z^2}\right)\frac{z^2}{2(t^2 - z^2)}\cdot\right. \\ & \left. - I_1\left(\frac{1}{2}\sqrt{t^2 - z^2}\right)\left(\frac{1}{\sqrt{t^2 - z^2}} + \frac{2z^2}{(t^2 - z^2)\sqrt{t^2 - z^2}}\right)\right] g(x-z)\,dz, \end{aligned} \tag{1.21$_{20}$}$$

$$J_{11}(t)g = \frac{1}{4}e^{-t/2}\int_{|z|\leqslant t}\left[I_0\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\frac{-tz}{2(t^2-z^2)}\cdot\right.$$
$$\left.+I_1\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\left(\frac{2tz}{(t^2-z^2)\sqrt{t^2-z^2}}+\frac{z}{2\sqrt{t^2-z^2}}\right)\right]g(x-z)\,dz, \qquad (1.21_{11})$$

$$J_{02}(t)g = \frac{1}{4}e^{-t/2}\int_{|z|\leqslant t}\left[I_0\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\left(\frac{1}{2}+\frac{t^2}{2(t^2-z^2)}\right)\cdot\right.$$
$$\left.+I_1\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\left(\frac{-t+1}{\sqrt{t^2-z^2}}-\frac{2t^2}{(t^2-z^2)\sqrt{t^2-z^2}}\right)\right]g(x-z)\,dz, \quad (1.21_{02})$$

and, for $\alpha+\beta=3$,

$$\omega_{30}(t)g = \left(\frac{t^2}{128}-\frac{1}{8}\right)\frac{g(x+t)-g(x-t)}{2}+\frac{t}{8}\frac{g'(x+t)+g'(x-t)}{2}$$
$$+\frac{g''(x+t)-g''(x-t)}{2}$$
$$\left(=\left(\frac{t^2}{128}-\frac{1}{8}\right)\partial_x(W(t)g)+\frac{t}{8}\partial_x\partial_t(W(t)g)+\partial_x^3(W(t)g)\right) \qquad (1.20_{30})$$

$$\omega_{21}(t)g = \left(\frac{t^2}{128}-\frac{t}{8}\right)\frac{g(x+t)+g(x-t)}{2}+\left(\frac{t}{8}-\frac{1}{2}\right)\frac{g'(x+t)-g'(x-t)}{2}$$
$$+\frac{g''(x+t)+g''(x-t)}{2}$$
$$\left(=\left(\frac{t^2}{128}-\frac{1}{8}\right)\partial_x(W(t)g)+\left(\frac{t}{8}-\frac{1}{2}\right)\partial_x^2(W(t)g)+\partial_x^2\partial_t(W(t)g)\right) \quad (1.20_{21})$$

$$\omega_{12}(t)g = \omega_{30}(t)g-\omega_{11}(t)g, \qquad (1.20_{12})$$

$$\omega_{03}(t)g = \omega_{21}(t)g-\omega_{02}(t)g, \qquad (1.20_{03})$$

and

$$J_{30}(t)g = \frac{1}{4}e^{-t/2}\int_{|z|\leqslant t}\left[I_0\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\left(\frac{z}{2(t^2-z^2)}+\frac{z(t^2+z^2)}{(t^2-z^2)^2}\right)-I_1\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\right.$$
$$\left.\times\left(\frac{28z+z^3}{4(t^2-z^2)\sqrt{t^2-z^2}}-\frac{8z^3}{(t^2-z^2)^2\sqrt{t^2-z^2}}\right)\right]g(x-z)\,dz, \qquad (1.21_{30})$$

$$J_{21}(t)g = \frac{1}{4}e^{-t/2}\int_{|z|\leqslant t}\left[I_0\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\left(\frac{-2t-z^2}{4(t^2-z^2)}-\frac{2tz^2}{(t^2-z^2)^2}\right)+I_1\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\right.$$
$$\left.\times\left(\frac{1}{2\sqrt{t^2-z^2}}+\frac{4z^2+tz^2+8t}{4(t^2-z^2)\sqrt{t^2-z^2}}+\frac{8tz^2}{(t^2-z^2)^2\sqrt{t^2-z^2}}\right)\right]$$
$$\times g(x-z)\,dz. \tag{1.21$_{21}$}$$

$$J_{12}(t)g = J_{30}(t)g - J_{11}(t)g \tag{1.21$_{12}$}$$

$$J_{03}(t)g = J_{21}(t)g - J_{02}(t)g. \tag{1.21$_{03}$}$$

Here we have used $\partial_x\partial_t^2(S(t)g) = \partial_x^3(S(t)g) - \partial_x\partial_t(S(t)g)$ and $\partial_t^3(S(t)g) = \partial_t\partial_x^2(S(t)g) - \partial_t^2(S(t)g)$ for $\alpha+\beta=3$.

Then we have the following estimates.

Theorem 1.2. *For* $1\leqslant q\leqslant p\leqslant\infty$, *it holds that*

$$||S(t)g||_{L^p}\leqslant C(1+t)^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||g||_{L^q}, \quad t\geqslant 0, \tag{1.22}$$

$$||\partial_x^\alpha\partial_t^\beta(S(t)g) - e^{-t/2}\omega_{\alpha\beta}(t)g||_{L^p} = ||J_{\alpha\beta}(t)g||_{L^p}$$
$$\leqslant C(1+t)^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\alpha}{2}-\beta}||g||_{L^q}, \quad t\geqslant 0 \tag{1.23$_{\alpha\beta}$}$$

for $1\leqslant\alpha+\beta\leqslant 3$. *Moreover, for* $t\geqslant 2$,

$$||(S(t)-P(t))g||_{L^p}\leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}||g||_{L^q}, \tag{1.24}$$

$$||\partial_x^\alpha\partial_t^\beta(S(t)-P(t))g - e^{-t/2}\omega_{\alpha\beta}(t)g||_{L^p}$$
$$= ||(J_{\alpha\beta}(t)-\partial_x^\alpha\partial_t^\beta P(t))g||_{L^p}\leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1-\frac{\alpha}{2}-\beta}||g||_{L^q} \tag{1.25$_{\alpha\beta}$}$$

for $1\leqslant\alpha+\beta\leqslant 3$.

The proof of Theorem 1.1 is a direct consequence of Theorem 1.2. For example,

$$\begin{aligned}\partial_x^2(V-\phi)(x,t) &= \partial_x^2(S(t)-P(t))(V_0+V_1)+\partial_x^2\partial_t(S(t)V_0)\\ &= e^{-t/2}(\omega_{20}(t)(V_0+V_1)-\omega_{21}(t)V_0)\\ &\quad + (J_{20}(t)-\partial_x P(t))(V_0+V_1)+J_{21}(t)V_0,\end{aligned}$$

which yields (1.3) with $(\alpha,\beta)=(2,0)$ by $(1.23)_{20}$ and $(1.25)_{21}$.

As an application of Theorem 1.2, we investigate the Cauchy problem for a one-dimensional compressible flow through porous media

$$\begin{cases} v_t - u_x = 0, \quad x\in\mathbf{R}, \quad t>0,\\ u_t + p(v)_x = -\alpha u,\\ (v,u)|_{t=0} = (v_0,u_0)(x). \end{cases} \tag{1.26}$$

Here, $v(>0)$ denotes the specific volume, u is the velocity, $p(\cdot)$ denotes the pressure with $p_v(v)<0$ for $v>0$, and α is a positive constant.

Remark on the notation. The letter p has been used as a power order of the Lebesgue space. Here, $p(\cdot)$ denotes the pressure. Here and after, whenever p expresses the pressure, it is displayed as a function like $p(\cdot)$, $p'(\cdot)$, etc.

The initial data are assumed to have constant states at $x=\pm\infty$:

$$\lim_{x\to\pm\infty}(v_0,u_0)(x) = (\underline{v},0), \quad \underline{v}>0. \tag{1.27}$$

The solution (v,u) to (1.26) is expected to behave as that to

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0,\\ p(\bar{v})_x = -\alpha\bar{u}, \end{cases} \tag{1.28}$$

as t tends to infinity, which is due to the Darcy law. By (1.28), $(\bar{v},\bar{u})$, called the diffusion wave, is determined by the porous media equation

$$\begin{cases} \bar{v}_t + \frac{1}{\alpha}p(\bar{v})_{xx} = 0, \quad \bar{v}|_{x=\pm\infty} = \underline{v},\\ \bar{u} = -\frac{1}{\alpha}p(\bar{v})_x. \end{cases} \tag{1.29}$$

However, the solution to (1.29) is not unique (unique up to a shift), and so we determine $(\bar{v},\bar{u})$ uniquely by

$$\begin{cases} \begin{cases} \bar{v}_t + \frac{1}{\alpha}p(\bar{v})_{xx} = 0,\\ \bar{v}|_{t=0} = \underline{v} + \frac{\delta_{0v}}{\sqrt{4\pi a}}e^{-\frac{(x+x_0)^2}{4a}} =: \bar{v}_0(x;x_0), \quad a = \frac{-p'(\underline{v})}{\alpha}, \end{cases}\\ \bar{u} = -\frac{1}{\alpha}p(\bar{v})_x. \end{cases} \tag{1.30}$$

Here, x_0 is uniquely given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{x}(v_0(z)-\bar{v}_0(z;x_0))\,dz=-\frac{1}{\alpha}\delta_{0u} \tag{1.31}$$

and

$$(\delta_{0v},\delta_{0u})=\int_{-\infty}^{\infty}(v_0(x)-\underline{v},u_0(x))\,dx,\quad \delta_{0v}\neq 0. \tag{1.32}$$

Then, under the suitable initial conditions and smallness assumptions, the convergence rate

$$\|(v-\bar{v},u-\bar{u})(\cdot,t)\|_{L^p}=O\left(t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-1}\log t,t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{3}{2}}\log t\right),\quad p\geqslant 2 \tag{1.33}$$

has been recently obtained in [18], in which both the L^2-energy method and the Green function method are employed. See also [16,17,21]. Note that this estimate is rather sharp, though "$\log t$" could not be removed, because

$$\|(\bar{v},\bar{u})(\cdot,t)\|_{L^p}=O\left(t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}\log t,t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}\log t\right),\quad p\geqslant 1,$$

as $t\to\infty$. The key point was to choose the suitably located diffusion wave $(\bar{v},\bar{u})$ by (1.30)–(1.32).

However, the method in [18] will not yield the L^1-convergence. Our second main purpose is to have the convergence rate (1.33) for $p\geqslant 1$, applying the L^p-L^q estimates in Theorem 1.2.

Very recently, L^1-convergence in case of $v_+\neq v_-(v_\pm=\lim_{x\to\pm\infty}v_0(x))$ has been obtained by Wang and Yang [24], by employing the approximating Green function method. Their result is

$$\|v-\bar{v},u-\bar{u}\|_{L^p}=O\left(t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}},t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-1}\right).$$

Combining our present result with theirs, we can cover both cases $v_+\neq v_-$ and $v_+=v_-(=\underline{v})$.

Let us reformulate our problem. The diffusion wave $(\bar{v},\bar{u})$ is defined by (1.30) with (1.31)–(1.32), and so satisfies (1.28) with its initial data

$$(\bar{v},\bar{u})|_{t=0}=\left(\bar{v}_0(x),-\frac{1}{\alpha}p(\bar{v}_0(x))_x\right),\quad \bar{v}_0(x):=\bar{v}_0(x;x_0). \tag{1.34}$$

Introduce the auxiliary function

$$(\hat{v}, \hat{u})(x,t) = \left(-\frac{1}{\alpha} e^{-\alpha t} m_0'(x), e^{-\alpha t} m_0(x) \right) \tag{1.35}$$

for a function $m_0(x)$ satisfying

$$m_0 \in C_0^\infty(\mathbf{R}), \quad \int_{-\infty}^{\infty} m_0(x)\, dx = \delta_{0u} \tag{1.36}$$

so that $(\hat{v}, \hat{u})$ satisfies

$$\begin{cases} \hat{v}_t - \hat{u}_x = 0, \\ \hat{u}_t = -\alpha \hat{u}, \\ (\hat{v}, \hat{u})|_{t=0} = \left(-\frac{1}{\alpha} m_0'(x), m_0(x) \right). \end{cases} \tag{1.37}$$

Combining (1.26), (1.28) with (1.37), we have

$$\begin{cases} (v - \bar{v} - \hat{v})_t - (u - \bar{u} - \hat{u})_x = 0, \\ (u - \bar{u} - \hat{u})_t + (p(v) - p(\bar{v}))_x = -\alpha(u - \bar{u} - \hat{u}) + \frac{1}{\alpha} p(\bar{v})_{xt}, \\ (v - \bar{v} - \hat{v}, u - \bar{u} - \hat{u})|_{t=0} = \left(v_0 - \bar{v}_0 + \frac{1}{\alpha} m_0', u_0 + \frac{1}{\alpha} p(\bar{v}_0)' - m_0 \right)(x). \end{cases} \tag{1.38}$$

By the definitions of $(\bar{v}, \bar{u})$ and $(\hat{v}, \hat{u})$,

$$\frac{d}{dt} \int_{-\infty}^{\infty} (v - \bar{v} - \hat{v})\, dx = 0 \quad \text{and} \quad \frac{d}{dt} \int_{-\infty}^{\infty} (u - \bar{u} - \hat{u})\, dx + \alpha \int_{-\infty}^{\infty} (u - \bar{u} - \hat{u})\, dx = 0,$$

so that both $\int_{-\infty}^{\infty} (v - \bar{v} - \hat{v})\, dx$ and $\int_{-\infty}^{\infty} (u - \bar{u} - \hat{u})\, dx$ are conserved and their amounts are zero. Hence twice integration of $(1.38)_1$ yields

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{x} (v - \bar{v} - \hat{v})(z)\, dz\, dx \\ &\quad = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} (v_0 - \bar{v}_0)(z)\, dz + \frac{1}{\alpha} m_0(x) \right) dx = 0, \end{aligned} \tag{1.39}$$

which is a key point in our reformulation of the problem.

Thus, we reach to define the perturbation

$$w(x,t) = \int_{-\infty}^{x} \int_{-\infty}^{y} (v - \bar{v} - \hat{v})(z,t)\, dz\, dy \tag{1.40}$$

and

$$(v, u) = (\bar{v} + \hat{v} + w_{xx}, \bar{u} + \hat{u} + w_{xt}), \tag{1.40'}$$

so that w satisfies the Cauchy problem for the second order damped wave equation

$$\begin{cases} w_{tt} - (p(\bar{v}) - p(\bar{v} + \hat{v} + w_{xx})) + \alpha w_t = \dfrac{1}{\alpha}(p(\bar{v}) - p(\underline{v}))_t, \\ (w, w_t)|_{t=0} = (w_0, w_1)(x), \end{cases} \tag{1.41}$$

where

$$(w_0, w_1)(x) = \left(\int_{-\infty}^{x} \left(\int_{-\infty}^{y} (v_0 - \bar{v}_0)(z)\, dz + \frac{m_0(y)}{\alpha} \right) dy, \right. \\ \left. \int_{-\infty}^{x} (u_0 - m_0)(z)\, dz + \frac{p(\bar{v}_0(x)) - p(\underline{v})}{\alpha} \right). \tag{1.42}$$

In [18] we have obtained the following theorem.

Theorem 1.3 (Nishihara [18]). *Suppose that* $(w_0, w_1) \in H^3 \times H^2$ *and that* $||w_0, w_1||_{H^3 \times H^2} + \delta_0, \delta_0 = |\delta_{0v}| + |\delta_{0u}|$, *is small. Then there exists a unique solution* $w \in C^i([0, \infty); \ H^{3-i})$ *to* (1.41), *which satisfies*

$$\sum_{k=0}^{3} (1+t)^{\frac{k}{2}} ||\partial_x^k w(t)|| + \sum_{k=0}^{2} (1+t)^{1+\frac{k}{2}} ||\partial_x^k w_t(t)|| \\ + \sum_{k=0}^{1} (1+t)^{2+\frac{k}{2}} ||\partial_x^k w_{tt}(t)|| + (1+t)^{1+\frac{1}{2}} ||w_{ttt}(t)|| \\ \leqslant C(||w_0, w_1||_{H^3 \times H^2} + \delta_0).$$

Remark. The regularity theorem for $(w_0, w_1) \in H^s \times H^{s-1} (s \geqslant 3)$ is also obtained in [18].

We can now apply the $L^p - L^q$ estimate (1.22)–$(1.23)_{\alpha,\beta}$ in Theorem 1.2 to the solution w obtained in Theorem 1.3. Note that (1.41) is linearized with constant coefficients as

$$w_{tt} - a w_{xx} + \alpha w_t = F, \quad a = -p'(\underline{v}) > 0 \tag{1.43}$$

with (1.42), where

$$F = p(\bar{v}) - p(\bar{v} + \hat{v} + w_{xx}) - p'(\bar{v}) w_{xx} + (p'(\bar{v}) - p'(\underline{v})) w_{xx} + p'(\bar{v}) \bar{v}_t \\ = b(\bar{v} - \underline{v}) w_{xx} + O(|\hat{v}| + |\bar{v} - \underline{v}|^3 + |\bar{v}_t| + w_{xx}^2), \quad b = -p''(\underline{v})/2. \tag{1.44}$$

By $(1.40)'$ and (1.35), the estimates of (w_{xx}, w_{xt}) obtained later in Section 3 yield the following theorem.

Theorem 1.4. *Suppose that* $(w_0, w_1) \in (H^3 \times H^2) \cap (W^{2,1} \times W^{1,1}) =: Z_0$. *If both* $\|w_0, w_1\|_{Z_0} + \delta_0$ *is suitably small, then the solution* (v, u) to (1.26) *satisfies*

$$\|(v - \bar{v}, u - \bar{u})(\cdot, t)\|_{L^p} = O(t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-1} \log t, t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{3}{2}} \log t), \quad 1 \leqslant p \leqslant 2. \quad (1.45)$$

Remark. If $(w_0, w_1) \in (H^4 \times H^3) \cap (W^{2,1} \times W^{1,1})$, then (1.45) holds for $1 \leqslant p \leqslant \infty$, where $W^{m,p} = \{f \mid \partial_x^k f \in L^p,\ 0 \leqslant k \leqslant m\}$ and $W^{m,2} = H^m$.

In [18] $\bar{v}$ is proved to behave as

$$\|\partial_x^j(\bar{v}(\cdot, t) - \phi(\cdot, t; x_0))\|_{L^p} \leqslant C(1 + t)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{j}{2}-1} \log(2 + t) \quad (1.46)$$

for $j = 0, 1, 2, \dots$ and $p \geqslant 1$, where

$$\phi(x, t) := \phi(x, t; x_0) = \underline{v} + \phi_1(x, t; x_0) + \phi_2(x, t; x_0), \quad (1.47)$$

with

$$\phi_1(x, t; x_0) = \frac{\delta_{0v}}{\sqrt{4\pi a(t + 1)}} e^{-\frac{(x+x_0)^2}{4a(t+1)}} =: \delta_{0v} G(x + x_0, t + 1),$$
$$\phi_2(x, t; x_0) = \int_0^t \int_{-\infty}^{\infty} G(x - y, t - \tau) \cdot b\{\phi_1(y, \tau; x_0)\}^2_{yy}\, dy\, d\tau \quad (1.48)$$

and $a = \frac{-p'(\underline{v})}{\alpha}$, $b = \frac{-p''(\underline{v})}{\alpha}$. Thus, if we combine (1.46) with Theorem 1.4, then we have

Corollary 1.1. *Under the assumptions in Theorem* 1.4 *the solution* (v, u) *to* (1.26) *behaves as*

$$\left\|\left(v - \phi, u + \frac{1}{\alpha} p(\phi)_x\right)(\cdot, t)\right\|_{L^p} = O(t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-1} \log t, t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{3}{2}} \log t) \quad (1.49)$$

as $t \to \infty$ *for* $1 \leqslant p \leqslant 2$.

Finally, we state the related works. The behavior of solutions to (1.26) was first obtained by Hsiao and Liu [6]. After this work, several problems in some directions have been investigated by many authors [3–5,7–9,11–14,16–18,20–22,24]. See also the references therein. This paper is in the direction how fast the solutions converges to the diffusion waves.

Our plan of this paper is as follows. In Section 2 we devote to have the $L^p - L^q$ estimates. In Section 3 decay properties of the solution w to (1.43) will be investigated, which gives the desired result, Theorem 1.4.

2. The $L^p - L^q$ estimates

In this section we devote ourselves to the proof of Theorem 1.2.
Let us prepare the lemmas.

Lemma 2.1. *The modified Bessel function $I_v(y)$ defined by* (1.14) *satisfies*

$$I_0(y), \quad I_1(y)\frac{1}{y}, \quad \left(I_0(y) - \frac{2}{y}I_1(y)\right)\frac{1}{y^2}, \text{ and}$$

$$\left(\left(1-\frac{8}{y^2}\right)I_0(y) + \frac{16}{y^3}I_1(y)\right)\frac{1}{y^2} \text{ are bounded}, \tag{2.1}$$

$$I_0(0) = 1, \quad I_1(y)\frac{1}{y}\bigg|_{y=0} = \frac{1}{2}, \quad \left(I_0(y) - \frac{2}{y}I_1(y)\right)\frac{1}{y^2}\bigg|_{y=0} = \frac{1}{8}, \tag{2.2}$$

$$I_0'(y) = I_1(y), \quad I_1'(y) = I_0(y) - \frac{1}{y}I_1(y) \tag{2.3}$$

and, as $y \to \infty$,

$$\begin{aligned} I_v(y) = \sqrt{\frac{1}{2\pi y}}e^y\Bigg(1 &- \frac{(v-1/2)(v+1/2)}{2y} \\ &+ \frac{(v-1/2)(v-3/2)(v+3/2)(v+1/2)}{2!2^2y^2} - \cdots \\ &+ (-1)^k\frac{(v-1/2)\cdots(v-(k-1/2))(v+(k-1/2))\cdots(v+1/2)}{k!2^ky^k} \\ &+ O(y^{-k-1})\Bigg). \end{aligned} \tag{2.4}$$

For the proof, see e.g. [15].
Using (2.2), (2.3) in Lemma 2.1 we have

$$\begin{aligned} \partial_t(S(t)g) &= e^{-t/2}\frac{g(x+t)+g(x-t)}{2} \\ &\quad + \frac{1}{4}e^{-t/2}\int_{|z|\leqslant t}\left[I_1\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\frac{t}{\sqrt{t^2-z^2}} - I_0\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\right]g(x-z)\,dz \\ &= e^{-t/2}\omega_{01}(t)g + J_{01}(t)g, \end{aligned}$$

etc. To estimate $(S(t)-P(t))g$ and $J_{01}(t)g$, we apply not only Lemma 2.1 but also the Hausdorff–Young inequality.

Lemma 2.2. *For $p,q,r(1\leqslant p,q,r\leqslant\infty)$ satisfying $\frac{1}{q}-\frac{1}{p}=1-\frac{1}{r}$, the inequality*

$$||f*g||_{L^p}\leqslant C||f||_{L^r}||g||_{L^q}$$

holds, where $*$ denotes the convolution.

Our first basic estimates are the followings.

Proposition 2.1. *For $1\leqslant q\leqslant p\leqslant\infty$, it holds that*

$$||S(t)g||_{L^p}\leqslant C(1+t)^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||g||_{L^q},\quad t\geqslant 0, \tag{2.5}$$

$$||\partial_t(S(t)g)-e^{-t/2}\omega_{01}(t)g||_{L^p}=||J_{01}(t)g||_{L^p}\leqslant C(1+t)^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}||g||_{L^q},\quad t\geqslant 0 \tag{2.6}$$

and

$$||(S(t)-P(t))g||_{L^p}\leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}||g||_{L^q},\quad t\geqslant 2. \tag{2.7}$$

Proposition 2.1 is basic in the sense that (2.6) and (2.7) shows $(1.3)_{00}$ by (1.16) and (1.17).

Proof of Propositin 2.1. By (1.13) and (1.16), for $t\geqslant 2$

$$\begin{aligned}
&(S(t)-P(t))g\\
&\quad=\left(\int_{|z|\leqslant t^{2/3}}+\int_{t^{2/3}\leqslant|z|\leqslant t}\right)\left[\frac{1}{2}e^{-t/2}I_0\left(\frac{1}{2}\sqrt{t^2-z^2}\right)-\frac{e^{-z^2/t}}{\sqrt{4\pi t}}\right]g(x-z)\,dz\\
&\qquad+\int_{|z|>t}\frac{e^{-z^2/t}}{\sqrt{4\pi t}}g(x-z)\,dz\\
&\quad=:X_1+X_2+X_3.
\end{aligned}\tag{2.8}$$

We show the case $r<\infty$, that is, $(p,q)\neq(\infty,1)$. The case $r=\infty$ is shown similarly. First, note that both X_2 and X_3 decay exponentially. In fact, by Lemma 2.2

$$\begin{aligned}
||X_3||_{L^p}&\leqslant C\left(\int_t^\infty\left(\frac{e^{-z^2/4t}}{\sqrt{4\pi t}}\right)^r dz\right)^{1/r}||g||_{L^q}\\
&\leqslant Ce^{-\beta_0 t}||g||_{L^q}\quad(0<\beta_0<1/8).
\end{aligned}\tag{2.9}$$

Since $I_\nu(y)$ $(\nu = 0, 1, \ldots)$ is monotonically increasing by (1.14), the asymptotic expansion (2.4) yields, for $t^{2/3} \leqslant z \leqslant t$,

$$\begin{aligned}\frac{1}{2}e^{-t/2}I_\nu\left(\frac{1}{2}\sqrt{t^2-z^2}\right) &\leqslant \frac{1}{2}e^{-t/2}I_\nu\left(\frac{1}{2}\sqrt{t^2-t^{4/3}}\right)\\ &= \frac{1}{2}\sqrt{\frac{1}{\pi\sqrt{t^2-t^{4/3}}}}\,e^{-\frac{t}{2}+\frac{\sqrt{t^2-t^{4/3}}}{2}}(1+O(t^{-1}))\\ &= Ct^{-1/2}e^{-\frac{t^{4/3}}{2(t+\sqrt{t^2-t^{4/3}})}}(1+O(t^{-1}))\\ &\leqslant Ce^{-\beta_1 t^{1/3}}, \quad t \geqslant 2 \ (0<\beta_1<1/8)\end{aligned}$$

and

$$\frac{e^{-z^2/4t}}{\sqrt{4\pi t}} \leqslant \frac{e^{-\frac{1}{4}t^{1/3}}}{\sqrt{4\pi t}} \leqslant C\exp(-\beta_1 t^{1/3}), \quad t \geqslant 2.$$

Hence we have

$$||X_2||_{L^p} \leqslant Ce^{-\beta_1 t^{1/3}}||g||_{L^q}, \quad t \geqslant 2. \tag{2.10}$$

Main term is X_1. For $0 \leqslant z \leqslant t^{2/3}$, $t \geqslant 2$, noting that $(\frac{z}{t})^2 \geqslant (\frac{z}{t})^k (k \geqslant 2)$, we have

$$\begin{aligned}&\left(\frac{e^{-z^2/4t}}{\sqrt{4\pi t}}\right)^{-1}\frac{1}{2}e^{-t/2}I_0\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\\ &= \sqrt{4\pi t}\,\frac{1}{2}\sqrt{\frac{1}{\pi\sqrt{t^2-z^2}}}\,e^{\frac{z^2}{4t}-\frac{t}{2}+\frac{\sqrt{t^2-z^2}}{2}}\cdot\left(1+\frac{1}{4\sqrt{t^2-z^2}}+O(t^{-2})\right)\\ &= \left(1-\left(\frac{z}{t}\right)^2\right)^{-1/4}e^{-\frac{z^4}{4t^3(1+\sqrt{1-(z/t)^2})}}\cdot\left(1+\frac{1}{4t}+\frac{1}{t^2}O\left(1+\frac{z^2}{t}\right)\right)\\ &= \left(1+\frac{1}{t}O\left(1+\frac{z^2}{t}\right)\right)\left(1+\frac{1}{4t}+\frac{1}{t^2}O\left(1+\frac{z^2}{t}\right)\right)\\ &= 1+\frac{1}{t}O\left(1+\frac{z^2}{t}\right).\end{aligned} \tag{2.11}$$

Hence

$$X_1 = \int_{|z|\leqslant t^{2/3}}\frac{e^{-z^2/4t}}{\sqrt{4\pi t}}\left\{1+\frac{1}{t}O\left(1+\frac{z^2}{t}\right)-1\right\}g(x-z)\,dz \tag{2.12}$$

and, by Lemma 2.2,

$$\begin{aligned}||X_1||_{L^p} &\leqslant Ct^{-1}\left(\int\left(\frac{e^{-z^2/4t}}{\sqrt{4\pi t}}\right)^r O\left(1+\frac{z^2}{t}\right)^r dz\right)^{1/r}||g||_{L^q}\\ &\leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}||g||_{L^q}.\end{aligned} \tag{2.13}$$

By (2.9)–(2.11), we have (2.7). Moreover, by (1.11) with $\alpha=\beta=0$,

$$||S(t)g||_{L^p}\leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||g||_{L^q},\quad t\geqslant 2. \tag{2.14}$$

By (2.1) in Lemma 2.1, for $t\leqslant 2$,

$$||S(t)g||_{L^p}\leqslant C||g||_{L^q}$$

which shows (2.5).

Next, we estimate $J_{01}(t)$ in (2.10). Similar to (2.8)

$$\begin{aligned}J_{01}(t)g &= \left(\int_{|z|\leqslant t^{2/3}}+\int_{t^{2/3}\leqslant|z|\leqslant t}\right)\frac{e^{-t/2}}{4}\left[-J_0\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\right.\\ &\quad\left.+I_1\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\frac{t}{\sqrt{t^2-z^2}}\right]g(x-z)\,dz\\ &=: X_4+X_5,\end{aligned}$$

in which X_5 decays exponentially, same as (2.10). To estimate X_4, we prepare for $|z|\leqslant t^{2/3}$

$$\begin{aligned}I_0\left(\frac{1}{2}\sqrt{t^2-z^2}\right) &= \frac{e^{\frac{1}{2}\sqrt{t^2-z^2}}}{\sqrt{\pi\sqrt{t^2-z^2}}}\left(1+\frac{1}{4t}+\frac{1}{t^2}\left(\frac{9}{32}+\frac{1}{8}\frac{z^2}{t}\right).\right.\\ &\quad\left.+\frac{1}{t^3}O\left(1+\frac{z^2}{t}+\left(\frac{z^2}{t}\right)^2\right)\right),\end{aligned} \tag{2.15}$$

$$\begin{aligned}I_1\left(\frac{1}{2}\sqrt{t^2-z^2}\right) &= \frac{e^{\frac{1}{2}\sqrt{t^2-z^2}}}{\sqrt{\pi\sqrt{t^2-z^2}}}\left(1-\frac{3}{4t}-\frac{1}{t^2}\left(\frac{15}{32}+\frac{3}{8}\frac{z^2}{t}\right).\right.\\ &\quad\left.+\frac{1}{t^3}O\left(1+\frac{z^2}{t}+\left(\frac{z^2}{t}\right)^2\right)\right)\end{aligned} \tag{2.16}$$

and

$$\left(\frac{e^{-z^2/4t}}{\sqrt{4\pi t}}\right)^{-1}\frac{1}{2}e^{-t/2}\frac{e^{\frac{1}{2}\sqrt{t^2-z^2}}}{\sqrt{\pi\sqrt{t^2-z^2}}}=1+\frac{1}{t}O\left(1+\frac{z^2}{t}\right), \tag{2.17}$$

which are also available for estimates of higher order derivatives. Eqs. (2.15) and (2.16) follow from (2.4), and (2.17) is derived as same as in (2.11).

By (2.15)–(2.17)

$$\begin{aligned}X_4 &= \int_{|z|\leqslant t^{2/3}}\frac{1}{4}e^{-\frac{t}{2}}\frac{e^{\frac{1}{2}\sqrt{t^2-z^2}}}{\sqrt{\pi\sqrt{t^2-z^2}}}\left\{-1+\frac{1}{t}O\left(1+\frac{z^2}{t}\right)\right.\\ &\quad\left.+\left(-1+O\left(\frac{1}{t}\right)\right)\left(1+\frac{1}{t}O\left(1+\frac{z^2}{t}\right)\right)\right\}g(x-z)\,dz\\ &= \int_{|z|\leqslant t^{2/3}}\frac{e^{-z^2/4t}}{\sqrt{4\pi t}}\frac{1}{t}O\left(1+\frac{z^2}{t}\right)g(x-z)\,dz.\end{aligned}$$

Hence Lemma 2.2 yields

$$||X_4||_{L^p}\leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}||g||_{L^q}, \quad t\geqslant 2. \tag{2.18}$$

Since both $I_0(y)$ and $I_1(y)\frac{1}{y}$ are bounded for $0\leqslant y\leqslant y_0$ in Lemma 2.1, it is easy to see

$$||J_{01}(t)g||_{L^p}\leqslant C||g||_{L^q}, \quad t\leqslant 2. \tag{2.19}$$

Thus we have (2.6), which completes the proof of Proposition 2.1. □

Proof of Theorem 1.2. We only show $(1.23)_{20}$ and $(1.25)_{20}$. The other cases are treated in a same way, though the calculations are rather tedious. By (1.13) and Lemma 2.1

$$\begin{aligned}\partial_x(S(t)g) &= e^{-t/2}\frac{g(x+t)-g(x-t)}{2}\\ &\quad+\frac{1}{2}e^{-t/2}\int_{|x-y|\leqslant t}I_1\left(\frac{1}{2}\sqrt{t^2-(x-y)^2}\right)\frac{-(x-y)}{2\sqrt{t^2-(x-y)^2}}g(y)\,dy\\ &= e^{-t/2}\frac{g(x+t)-g(x-t)}{2}+\frac{1}{4}e^{-t/2}\int_{|z|\leqslant t}I_1\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\frac{-z}{\sqrt{t^2-z^2}}g(x-z)\,dz\\ &= e^{-t/2}\omega_{10}(t)g+J_{10}(t)g\end{aligned} \tag{2.20}$$

and

$$\partial_x^2(S(t)g) = e^{-t/2}(\frac{g'(x+t)-g'(x-t)}{2} + \frac{1}{4}\frac{t}{4}(g(x+t)+g(x-t))$$
$$+\frac{e^{-t/2}}{4}\int_{|x-y|\leqslant t}\left[I_1'\left(\frac{1}{2}\sqrt{t^2-(x-y)^2}\right)\frac{(x-y)^2}{2(t^2-(x-y)^2)}\right.$$
$$+I_1\left(\frac{1}{2}\sqrt{t^2-(x-y)^2}\left(\frac{-1}{\sqrt{t^2-(x-y)^2}}\right.\right.$$
$$\left.\left.\left.+\left(\frac{-(x-y)^2}{t^2-(x-y)^2\sqrt{t^2-(x-y)^2}}\right)\right)\right]g(y)\,dy$$

$$= e^{-t/2}\left(\frac{t}{8}\frac{g(x+t)+g(x-t)}{2} + \frac{g'(x+t)-g'(x-t)}{2}\right)$$
$$+\frac{e^{-t/2}}{4}\int_{|z|\leqslant t}\left[I_0\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\frac{z^2}{2(t^2-z^2)}\right.$$
$$\left.-I_1\left(\frac{1}{2}\sqrt{t^2-z^2}\right)\left(\frac{1}{\sqrt{t^2-z^2}}+\frac{2z^2}{(t^2-z^2)\sqrt{t^2-z^2}}\right)\right]g(x-z)\,dz$$
$$= e^{-t/2}\omega_{20}(t)g + J_{20}(t)g. \tag{2.21}$$

Same as $J_{01}(t)g$, we devide the integrand into two domains

$$J_{20}(t)g = \int_{|z|\leqslant t^{2/3}} + \int_{t^{2/3}\leqslant|z|\leqslant t} =: Y_1 + Y_2,$$

in which Y_2 decays exponentially. Using (2.15)–(2.17), we have

$$Y_1 = \int_{|z|\leqslant t^{2/3}} \frac{e^{-t/2}}{4}\frac{e^{\frac{1}{2}\sqrt{t^2-z^2}}}{\sqrt{\pi\sqrt{t^2-z^2}}}\left\{\left(1+\frac{1}{4t}+\frac{1}{t^2}O\left(1+\frac{z^2}{t}\right)\right)\cdot\frac{1}{2t}\frac{z^2}{t}\left(1+\frac{1}{t}O\left(\frac{z^2}{t}\right)\right)\right.$$
$$\left.-\left(1-\frac{3}{4t}+\frac{1}{t^2}O\left(1+\frac{z^2}{t}\right)\right)\cdot\frac{1}{t}(1+\frac{1}{t}O\left(\frac{z^2}{t}\right)\right\}g(x-z)\,dz$$
$$= \int_{|z|\leqslant t^{2/3}} \frac{e^{-z^2/t}}{\sqrt{4\pi t}}\left(1+\frac{1}{t}O\left(1+\frac{z^2}{t}\right)\right)\left\{-\frac{1}{2t}+\frac{1}{4t}\frac{z^2}{t}+\frac{1}{t^2}O\left(1+\frac{z^2}{t}\right)\right\}g(x-z)\,dz$$
$$= \int_{|z|\leqslant t^{2/3}} \frac{e^{-z^2/t}}{\sqrt{4\pi t}}\frac{1}{t}\left(-\frac{1}{2}+\frac{z^2}{4t}+\frac{1}{t}O\left(1+\frac{z^2}{t}\right)\right)g(x-z)\,dz. \tag{2.22}$$

Hence

$$||Y_1||_{L^p} \leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}||g||_{L^q}, \quad t \geqslant 2,$$

and so

$$||J_{20}(t)g||_{L^p} \leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}||g||_{L^q}, \quad t \geqslant 2. \tag{2.23}$$

For $0 \leqslant t \leqslant 2$, (2.1) in Lemma 2.1 gives

$$||J_{20}(t)g||_{L^p} \leqslant C||g||_{L^q} \tag{2.24}$$

in a same way as (2.19). Both (2.23) and (2.24) gives $(1.23)_{20}$.

On the other hand, since

$$\partial_x^2 P(t)g = \int_{\mathbf{R}} \frac{e^{-z^2/4t}}{\sqrt{4\pi t}}\left(-\frac{1}{2t} + \frac{1}{4t}\frac{z^2}{t}\right) g(x-z)\,dz, \tag{2.25}$$

we denote as

$$(J_{20}(t) - \partial_x^2 P(t))g = \int_{|z| \leqslant t^{2/3}} + \int_{t^{2/3} \leqslant |z| \leqslant t} + \int_{|z| \geqslant t} =: Y_3 + Y_4 + Y_5. \tag{2.26}$$

Then, Y_4 and Y_5 decay exponentially, and

$$Y_3 = \int_{|z| \leqslant t^{2/3}} \frac{e^{-z^2/4t}}{\sqrt{4\pi t}} \frac{1}{t^2} O\left(1 + \frac{z^2}{t}\right) g(x-z)\,dz$$

by (2.22) and (2.25), which yields

$$||Y_3|| \leqslant Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-2}||g||_{L^q}, \quad t \geqslant 2$$

and hence $(1.25)_{20}$.

Thus we have completed the proof of Theorem 1.2. □

3. L^1-Decay for quasilinear wave equation

In this section we apply (1.22) and (1.23) in Theorem 1.2 to the reformulated problem

$$\begin{cases} w_{tt} - w_{xx} + w_t = F(x,t), \\ (w, w_t)|_{t=0} = (w_0, w_1)(x) \end{cases} \tag{3.1}$$

with

$$F(x,t) = \{b(\bar{v} - \underline{v})w_{xx} + O(w_{xx}^2)\} + O(|\hat{v}| + |\bar{v} - \underline{v}|^3 + |\bar{v}_t|) =: F_1 + F_2, \quad (3.2)$$

coming from (1.41)–(1.44). Here, both a and α are normalized to be one.

By the Duhanmel principle the solution to (3.1) is given by

$$\begin{aligned} w(\cdot,t) &= S(t)(w_0 + w_1) + \partial_t(S(t)w_0) + \int_0^t S(t-\tau)F(\cdot,\tau)\,d\tau \\ &= \{e^{-t/2}\omega_{01}(t)w_0 + S(t)(w_0+w_1) + J_{01}(t)w_0\} + \int_0^t S(t-\tau)F(\cdot,\tau)\,d\tau \\ &=: w^{(0)} + K(\cdot,t). \end{aligned} \quad (3.3)$$

We estimate w in (3.3) to show the following theorem.

Theorem 3.1. *Let w be a solution to* (3.1) *obtained in Theorem* 1.3. *If* $(w_0, w_1) \in (H^3 \times H^2) \cap (W^{2,1} \times W^{1,1}) = Z_0$, *then w satisfies the estimate*

$$\|\partial_x^\alpha \partial_t^\beta w(\cdot,t)\|_{L^p} \leqslant C(1+t)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{\alpha}{2}-\beta} \log(2+t)(\|w_0,w_1\|_{Z_0} + \delta_0) \quad (3.4)$$

for $0 \leqslant \alpha + \beta \leqslant 2$ *and* $1 \leqslant p \leqslant 2$.

Remember that $(w_{xx}, w_{xt}) = (v - \bar{v} - \hat{v}, u - \bar{u} - \hat{v})$ and $(\hat{v}, \hat{u})$ decay exponentially. Hence, by taking $(\alpha, \beta) = (2,0)$ and $(1,1)$, Theorem 3.1 yields Theorem 1.4.

Proof of Theorem 3.1. By (1.22) and (1.23) it is clear that

$$\sum_{\alpha+\beta\leqslant 2} (1+t)^{\frac{1}{2}\left(1-\frac{1}{p}\right)+\frac{\alpha}{2}+\beta} \|\partial_x^\alpha \partial_t^\beta w^{(0)}(\cdot,t)\|_{L^p} \leqslant C\|w_0,w_1\|_{Z_0}. \quad (3.5)$$

To estimate the nonlinear term, assume a priori that

$$N(T) = \sup_{0\leqslant t\leqslant T} \left\{ \sum_{\alpha+\beta\leqslant 2} (1+t)^{\frac{1}{2}\left(1-\frac{1}{p}\right)+\frac{\alpha}{2}+\beta} (\log(2+t))^{-1} \|\partial_x^\alpha \partial_t^\beta w(\cdot,t)\|_{L^p} \right\} (\leqslant 1). \quad (3.6)$$

By (1.22)

$$\begin{aligned} \|K(\cdot,t)\|_{L^p} &= \left\| \int_0^t S(t-\tau)F(\cdot,\tau)\,d\tau \right\|_{L^p} \\ &\leqslant C \int_0^t (1+t-\tau)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \|F(\cdot,\tau)\|_{L^1}\,d\tau. \end{aligned}$$

Since

$$\begin{aligned}||F(\cdot,\tau)||_{L^1} \leqslant C\Bigg(&\sup_{\mathbf{R}} |\bar{v}(x,\tau)-\underline{v}|||w_{xx}(\cdot,\tau)||_{L^1}+||w_{xx}(\cdot,\tau)||_{L^2}^2 \\ &+||\hat{v}(\cdot,\tau)||_{L^1}+||\bar{v}(\cdot,\tau)-\underline{v}||_{L^3}^3+||\bar{v}_\tau(\cdot,\tau)||_{L^1}\Bigg) \\ &\leqslant C(\delta_0(1+\tau)^{-\frac{3}{2}}\log(2+\tau)N(t)+(1+\tau)^{-\frac{5}{2}}(\log(2+\tau))^2N(t)^2) \\ &+C\delta_0(1+\tau)^{-1},\end{aligned} \tag{3.7}$$

it holds that, for $t \leqslant T$,

$$\begin{aligned}||K(\cdot,t)||_{L^p} &\leqslant C\left|\left|\int_0^{t/2}+\int_{t/2}^t\right|\right|_{L^p} \\ &\leqslant C(1+t)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}(\delta_0N(T)+N(T)^2) \\ &\quad + C\delta_0(1+t)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}\log(2+t),\end{aligned} \tag{3.8}$$

and hence

$$\begin{aligned}&(1+t)^{\frac{1}{2}\left(1-\frac{1}{p}\right)}(\log(2+t))^{-1}||w(\cdot,t)||_{L^p} \\ &\leqslant C(||w_0,w_1||_{Z_0}+\delta_0+\delta_0N(T)+N(T)^2).\end{aligned} \tag{3.9}$$

Similarly, we estimate the derivatives of $K(\cdot,t)$ to have

$$\begin{aligned}||\partial_x^\alpha\partial_t^\beta K(\cdot,t)||_{L^p} &\leqslant C(1+t)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{\alpha}{2}-\beta}\log(2+t) \\ &\quad\times(\delta_0+(||w_0,w_1||_{Z_0}+\delta_0)N(T)+N(T)^2)\end{aligned} \tag{3.10}$$

for $1 \leqslant \alpha+\beta \leqslant 2$ and $1 \leqslant p \leqslant 2$. Then we have

$$\begin{aligned}&(1+t)^{\frac{1}{2}\left(1-\frac{1}{p}\right)+\frac{\alpha}{2}+\beta}(\log(2+t))^{-1}||\partial_x^\alpha\partial_t^\beta w(\cdot,t)||_{L^p} \\ &\leqslant C(||w_0,w_1||_{Z_0}+\delta_0+(||w_0,w_1||_{Z_0}+\delta_0)N(T)+N(T)^2).\end{aligned} \tag{3.11}$$

Combining (3.9) with (3.11) we have

$$N(T) \leqslant C(||w_0,w_1||_{Z_0}+\delta_0+(||w_0,w_1||_{Z_0}+\delta_0)N(T)+N(T)^2),$$

which shows that, if $||w_0, w_1||_{Z_0} + \delta_0 \ll 1$, then $N(T) \leqslant C(||w_0, w_1||_{Z_0} + \delta_0)$ and so (3.4).

We now need to show (3.11). Only the case $\alpha = 2$ will be shown. The other cases are omitted.

Differentiate (3.3) twice with respect to x:

$$\begin{aligned} w_{xx}(x,t) &= \partial_x^2 S(t)(w_0 + w_1) + \partial_{txx}^3 (S(t)w_0) + \int_0^t \partial_x^2 S(t-\tau)F(x,\tau)\,d\tau \\ &= \{e^{-t/2}(\omega_{20}(t)(w_0+w_1) + \omega_{21}(t)w_0) + J_{20}(t)(w_0+w_1) + J_{21}(t)w_0\} \\ &\quad + \int_0^t e^{-(t-\tau)/2}\omega_{20}(t-\tau)F(x,\tau)\,d\tau + \int_0^t J_{20}(t-\tau)F(x,\tau)\,d\tau \\ &=: w^{(20)}(x,t) + K_{20}^1(x,t) + K_{20}^2(x,t). \end{aligned} \tag{3.12}$$

By (1.23) it is clear that

$$||w^{(20)}(\cdot,t)||_{L^p} \leqslant C(1+t)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-1}||w_0, w_1||_{Z_0}, \tag{3.13}$$

which is included in (3.5). Note that first and second derivatives of w_0 are included in $\omega_{21}(t)w_0$ and first derivative of w_1 in $w_{20}(t)w_1$, and so $(w_0, w_1) \in W^{2,1} \times W^{1,1}$ is assumed.

The last term $K_{20}^2(x,t)$ is estimated as follows:

$$\begin{aligned} ||K_{20}^2(\cdot,t)||_{L^1} &\leqslant C\int_0^t (1+t-\tau)^{-1}||F(\cdot,\tau)||_{L^1}\,d\tau \\ &\leqslant C(1+t)^{-1}\log(2+t)(\delta_0 N(T) + N(T)^2), \end{aligned} \tag{3.14}$$

$$\begin{aligned} ||K_{20}^2(\cdot,t)||_{L^2} &\leqslant C\int_0^t (1+t-\tau)^{-5/4}||F_1(\cdot,\tau)||_{L^1}\,d\tau \\ &\quad + C\left\{\int_0^{t/2} (1+t-\tau)^{-5/4}||F_2(\cdot,\tau)||_{L^1}\,d\tau \right. \\ &\quad \left. + \int_{t/2}^t (1+t-\tau)^{-1}||F_2(\cdot,\tau)||_{L^2}\,d\tau\right\} \\ &\leqslant C(1+t)^{-5/4}\log(2+t)(\delta_0 N(T) + N(T)^2). \end{aligned} \tag{3.15}$$

Because

$$||F_2(\cdot,\tau)||_{L^2} \leqslant C(||\hat{v}||_{L^2} + ||\bar{v} - \underline{v}||_{L^6}^3 + ||\bar{v}_\tau||_{L^2}) \leqslant C\delta_0(1+\tau)^{-5/4}. \tag{3.16}$$

By $(1.20)_{20}$, $K^1_{20}(x,t)$ has the expression

$$K^1_{20}(x,t) = \int_0^t e^{-\frac{t-\tau}{2}} \left\{ \left(\frac{t-\tau}{8} - \frac{1}{2} \right) \frac{F(x+t-\tau,\tau) + F(x-t+\tau,\tau)}{2} + \frac{F_x(x+t-\tau,\tau) + F_x(x-t+\tau,\tau)}{2} \right\} d\tau \tag{3.17}$$

and

$$F_x(x \pm t \mp \tau, \tau) = \{ b\bar{v}_x w_{xx} + b(\bar{v} - \underline{v}) w_{xxx} + O(|w_{xx} w_{xxx}|) \} + O(|\hat{v}_x| + |\bar{v} - \underline{v}|^2 |\bar{v}_x| + |\bar{v}_{xt}|) =: F_{1x} + F_{2x}. \tag{3.18}$$

To estimate $||K^1_{20}(\cdot,t)||_{L^p}, 1 \leqslant p \leqslant 2$, we need $||F||_{L^1},\ ||F||_{L^2}$ and $||F_x||_{L^1},\ ||F_x||_{L^2}$. By virtue of Theorem 1.3,

$$\begin{aligned} ||F_1(\cdot,\tau)||_{L^2} &\leqslant C(||\bar{v} - \underline{v}||_{L^2} ||w_{xx}||_{L^2} + ||w_{xxx}||^{1/2} ||w_{xx}||^{3/2}_{L^2}) \\ &\leqslant C(1+\tau)^{-3/2} \log(2+\tau) \{ (||w_0, w_1||_{Z_0} + \delta_0) N(T) + N(T)^2 \}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} ||F_{1x}(\cdot,\tau)||_{L^1} &\leqslant C(||\bar{v}_x||_{L^\infty} ||w_{xx}||_{L^1} + ||\bar{v} - \underline{v}||_{L^2} ||w_{xxx}||_{L^2} + ||w_{xxx}||_{L^2} ||w_{xx}||_{L^2}) \\ &\leqslant C(||w_0, w_1||_{Z_0} + \delta_0) \{ (1+\tau)^{-2} \log(2+\tau) \cdot N(T) + (1+\tau)^{-7/4} \}, \end{aligned} \tag{3.20}$$

$$||F_{2x}(\cdot,\tau)||_{L^1} \leqslant C\delta_0 (1+\tau)^{-3/2}, \tag{3.21}$$

$$\begin{aligned} ||F_{1x}(\cdot,\tau)||_{L^2} &\leqslant C(||\bar{v}_x||_{L^\infty} ||w_{xx}||_{L^2} + ||\bar{v} - \underline{v}||_{L^\infty} ||w_{xxx}||_{L^2} + ||w_{xxx}||^{3/2} ||w_{xx}||^{1/2}_{L^2}) \\ &\leqslant C(||w_0, w_1||_{Z_0} + \delta_0) \{ (1+\tau)^{-9/4} \log(2+\tau) \cdot N(T) + (1+\tau)^{-2} \} \end{aligned} \tag{3.22}$$

and

$$||F_{2x}(\cdot,\tau)||_{L^2} \leqslant C\delta_0 (1+\tau)^{-7/4}. \tag{3.23}$$

Using (3.7), (3.16) and (3.19)–(3.23), we have

$$\begin{aligned} ||K^1_{20}(\cdot,t)||_{L^p} &\leqslant C(1+t)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-1} \log(2+\tau) \\ &\quad \cdot \{ ||w_0, w_1||_{Z_0} + \delta_0 + (||w_0, w_1||_{Z_0} + \delta_0) N(T) + N(T)^2 \}. \end{aligned} \tag{3.24}$$

Combining (3.13)–(3.15) with (3.24) we obtain (3.11), which completes the proof of Theorem 1.4. □

Remark. The estimate of $||w_{xxx}(\cdot,t)||_{L^2}$ obtained in Theorem 1.3 implies those of $||F||_{L^p}$ and $||F_x||_{L^p}$ only for $1 \leqslant p \leqslant 2$. The estimate $||w_{xxxx}(\cdot,t)||_{L^2}$ is enough to obtain $||w_{xx}(\cdot,t)||_{L^\infty}$ in our method, that is, $(w_0,w_1) \in (H^4 \times H^3) \cap (W^{2,1} \times W^{1,1})$ yields (1.33) for $1 \leqslant p \leqslant \infty$.

Acknowledgments

The original ideas in this paper arose when K. Nishihara visited Universitá degli Studi di L'Aquila in March and April, 2000. He expresses his sincere gratitude to Professors P. Marcati and B. Rubino for their kind invitation and hospitality.

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