J Comb Optim (2011) 22:378–391
DOI 10.1007/s10878-010-9300-3

On the performances of Nash equilibria in isolation games

Vittorio Bilò · Michele Flammini · Gianpiero Monaco · Luca Moscardelli

Published online: 19 February 2010

Abstract We study the performances of Nash equilibria in isolation games, a class of competitive location games recently introduced in Zhao et al. (Proc. of the 19th International Symposium on Algorithms and Computation (ISAAC), pp. 148–159, 2008). For all the cases in which the existence of Nash equilibria has been shown, we give tight or asymptotically tight bounds on the prices of anarchy and stability under the two classical social functions mostly investigated in the scientific literature, namely, the minimum utility per player and the sum of the players' utilities. Moreover, we prove that the convergence to Nash equilibria is not guaranteed in some of the not yet analyzed cases.

Keywords Nash equilibria · Price of anarchy and stability · Isolation games

V. Bilò (✉)
Dipartimento di Matematica "Ennio De Giorgi", Università del Salento, Provinciale Lecce-Arnesano, P.O. Box 193, 73100 Lecce, Italy
e-mail: vittorio.bilo@unile.it

M. Flammini
Dipartimento di Informatica, Università di L'Aquila, Via Vetoio, 67100 Coppito, L'Aquila, Italy
e-mail: flammini@di.univaq.it

G. Monaco
Mascotte joint project, INRIA/CNRS/UNSA, Sophia Antipolis, France
e-mail: gianpiero.monaco@sophia.inria.fr

L. Moscardelli
Dipartimento di Scienze, Università di Chieti-Pescara, Viale Pindaro 42, 65127 Pescara, Italy
e-mail: moscardelli@sci.unich.it

1 Introduction

Competitive location (Eiselt et al. 1993) is a multidisciplinary field of research and applications ranging from economy and geography to operations research, game theory and social sciences. It studies games in which players aim at choosing suitable locations or points in given metric spaces so as to maximize their utility or revenue. Depending on different parameters such as the underlying metric space, the number of players, the adopted equilibrium concept, the customers' behavior and so on, several scenarios arise. The foundations of competitive location date back to the beginning of the last century when Hotelling studied the so called "ice-cream vendor problem" (Hotelling 1929). With the recent flourishing of contributions on Algorithmic Game Theory, competitive location has attracted also the interest of computer scientists about the existence of equilibrium solutions, the complexity of their determination and the efficiency of these equilibria when compared with cooperative solutions optimizing a given social function measuring the overall welfare of the system.

In this paper we consider *isolation games* (Zhao et al. 2008), a class of competitive location games in which the utility of a player is defined as a function of her distances from the other ones. For example, one can define the utility of a player as being equal to the distance from the nearest one *(nearest-neighbor isolation game)*, or to the sum of the distances from all the other players *(total-distance isolation game)*, or to the distance from the ℓ-th nearest player *(ℓ-selection isolation game)*. More generally, denoted as k the number of players, for any player i, each strategy profile S yields a vector $\mathbf{f}^i(S) \in \mathbb{R}_{\geq 0}^{k-1}$ such that the j-th component of $\mathbf{f}^i(S)$ is the distance between the location chosen by player i and the one chosen by her j-th nearest player. The utility of player i can thus be defined as any convex combination of the elements of $\mathbf{f}^i(S)$, that is, given a vector $\mathbf{w} \in \mathbb{R}_{\geq 0}^{k-1}$, the utility of player i is defined as the scalar product between $\mathbf{f}^i(S)$ and $\mathbf{w}$.

Isolation games find a natural application in data clustering (Jain et al. 1999) and geometric sampling (Teng 1999). Moreover, as pointed out in Zhao et al. (2008), they can be used to obtain a good approximation of the strategy a player should compute in another competitive location game, called Voronoi game (Ahn et al. 2004; Cheong et al. 2004; Dürr and Thang 2007; Eaton and Lipsey 1975; Fekete and Meijer 2005; Mavronicolas et al. 2008), which is among the most studied competitive location games. Here the utility of a player is given by the total number of all points that are closer to her than to any other player (Voronoi area), where points which are equidistant to several players are split up evenly among the closest ones. Given a certain strategy profile, a player needs to compute the Voronoi area of any point in the space in order to play the game which can be very expensive in several situations. As an approximation, instead, each player may choose to simply maximize her nearest-neighbor distance or the total distance from any other players and this clearly gives rise to either a nearest-neighbor isolation game or a total-distance isolation game.

As another interesting field of application for isolation games, consider the following problem in non-cooperative wireless network design. We are given a set of users who have to select a spot where to locate an antenna so as to transmit and receive a radio signal. When more than just one antenna is transmitting contemporaneously,

Table 1 Results for ℓ-selection isolation games

Social function	Price of stability	Price of anarchy
MIN	1	2
SUM	∞	∞

interference between two or more signals may occur. More specifically, it may be the case that antenna i receive, at the same time, h different signals $r_1, \ldots, r_h$ of which only r_1 is really destined to antenna i. According to the main standard models in wireless network design, signal r_1 can be correctly received by antenna i if and only if the ratio between the power at which r_1 is received by antenna i and the sum of the powers at which the undesired $h-1$ signals are received by antenna i is greater than a certain threshold. If users are selfish players interested in minimizing the amount of interference their antenna will be subject to, they will decide to locate it as far as possible from the other ones thus giving rise to a particular isolation game.

Related work Isolation games were recently introduced in Zhao et al. (2008), where the authors give several results regarding existence of pure Nash equilibria (Nash 1950) and the convergence of better and best response dynamics to any of such equilibria. In particular they prove that in any symmetric space the nearest-neighbor and the total-distance isolation games are potential games, which implies existence of Nash equilibria and convergence of any better response dynamics. In case of asymmetric spaces, however, for any non-null vector $\mathbf{w}$, there always exists a game admitting no Nash equilibria. Moreover, in asymmetric spaces deciding whether the nearest-neighbor or the total-distance isolation games possess a Nash equilibrium is NP-complete. On symmetric spaces, for ℓ-selection isolation games they show that Nash equilibria always exist and that there always exists a better response dynamics converging to an equilibrium even though the game is not a potential one (hence, even best response dynamics may lead to cycles). For isolation games defined by $\mathbf{w} = (1, 1, 0, \ldots, 0)$ on symmetric spaces, determining whether there exists a Nash equilibrium is NP-complete. Finally, some existential results for general isolation games on certain particular Euclidean spaces are given.

Our contribution We analyze the efficiency of pure Nash equilibria for all the classes of isolation games introduced in Zhao et al. (2008) for which existence of such equilibria is always guaranteed, namely, the ℓ-selection isolation games for any $1 \leq \ell < k$ and the total-distance isolation games. Following the leading approach in Algorithmic Game Theory's literature, we use either the minimum utility among all player (*MIN*) and the sum of the players' utilities (*SUM*) as social functions measuring the overall welfare of a strategy profile and adopt the notions of price of anarchy (Koutsoupias and Papadimitriou 1999) and price of stability (Anshelevich et al. 2003) as a measure of the quality of pure Nash equilibria when compared with the strategy profiles maximizing the social functions. All our results, summarized in Tables 1 and 2, are tight or asymptotically tight. We then show that isolation games yielded by the weight vector $\mathbf{w} = (0, \ldots, 0, 1, \ldots, 1)$ are not potential games, that is, better response dynamics may not converge to Nash equilibria.

Table 2 Results for total-distance isolation games

Social function	Price of stability	Price of anarchy
MIN	Between 1 and $\frac{k+1}{k-1}$	Between 2 and $2\frac{k+1}{k-1}$
SUM	1	2

Paper organization Next section contains the necessary definitions and notation. Sections 3 and 4 cover the study of both the prices of anarchy and stability for ℓ-selection and total-distance isolation games, respectively. In Sect. 5 we show that the cases yielded by the weight vectors $\mathbf{w} = (0, \dots, 0, 1, \dots, 1)$ are not potential games, while in Sect. 6, we address open problems and further research.

2 Definitions and notation

For any $n \in \mathbb{N}$, let $[n] := \{1, \dots, n\}$. Given an n-tuple $A = (a_1, \dots, a_n)$, we write (A_{-i}, x) to denote the n-tuple obtained from A by replacing a_i with x. For the ease of notation, we write $x \in A$ when there exists an index $i \in [n]$ such that $a_i = x$.

A *metric space* is a pair (X, d) where X is a set of *points* or *locations* and $d : X \times X \mapsto \mathbb{R}_{\geq 0}$ is a *distance* or *metric function* such that for any $x, y, z \in X$ it holds (i) $d(x, y) \geq 0$, (ii) $d(x, y) = 0 \Leftrightarrow x = y$, (iii) $d(x, y) = d(y, x)$ *(symmetry)*, and (iv) $d(x, y) \leq d(x, z) + d(z, y)$ *(triangular inequality)*.

An instance $I = ((X, d), k)$ of an *isolation game* is defined by a metric space (X, d) and a set of k players $\{1, \dots, k\}$ aiming at selfishly maximizing their own utility. The *strategy set* of each player is given by the set X of all the locations and the set of strategy profiles is $\mathcal{S} := X^k$. A *strategy profile* $S \in \mathcal{S}$ is thus a k-tuple $S = (s_1, \dots, s_k)$ where, for any $i \in [k]$, s_i is the location chosen by player i in S.

Given a strategy profile S, for any player $i \in [k]$, define the *distance vector* of i in S as $\mathbf{b}^i(S) = (d(s_i, s_1), \dots, d(s_i, s_{i-1}), d(s_i, s_{i+1}), \dots, d(s_i, s_k))$ and the *ordered distance vector* of i in S as the vector $\mathbf{f}^i(S) \in \mathbb{R}^{k-1}_{\geq 0}$ obtained from $\mathbf{b}^i(S)$ by sorting its components in non-decreasing order. Roughly speaking, if player p is the j-th nearest player to i in S, the j-th component of $\mathbf{f}^i(S)$ is equal to the distance between s_i and s_p.

For any vector $\mathbf{w} \in \mathbb{R}^{k-1}_{\geq 0}$, called *weight vector*, the utility $u_i(S)$ of player i in S is given by the scalar product between $\mathbf{f}^i(S)$ and $\mathbf{w}$, that is, $u_i(S) = \mathbf{f}^i(S) \cdot \mathbf{w}$. Informally speaking, w_ℓ denotes how much the distance from the ℓ-th furthest player influences the utility of any player. For instance, for the weight vector $\mathbf{w} = (1, 0, \dots, 0)$, the utility of a player is simply given by the distance from the nearest one.

In this paper, we consider the following weight vectors:

- **ℓ-selection vector** $(0, \dots, 0, 1, 0, \dots, 0)$, in which all components are equal to zero except for the ℓ-th one which is set to one. Therefore, the utility of a player is given by the distance from the ℓ-th nearest one. We call ℓ-selection isolation games, the games yielded by an ℓ-selection vector;
- **sum vector** $(1, \dots, 1)$, for which the utility of a player is given by the sum of the distances from all the other ones. We call total-distance isolation games, the games yielded by the sum vector;

- **ℓ-suffix vector** $(0, \dots, 0, 1, \dots, 1)$, in which the last ℓ components are equal to one and the other ones are set to zero. Hence, the utility of a player is given by the sum of the distances from the ℓ furthest ones. We call ℓ-suffix isolation games, the games yielded by an ℓ-suffix vector.

Given a strategy profile S, player i can perform an *improving move* in S if and only if there exists a location $x \in X$ such that $u_i((S_{-i}, x)) > u_i(S)$. A strategy profile is a *Nash equilibrium* if and only if no player can perform an improving move in it.

Given a *social function* $SF : \mathcal{S} \mapsto \mathbb{R}_{\geq 0}$, for each instance I, let $OPT_I = \max_{S \in \mathcal{S}}\{SF(S)\}$ be the *social optimum* of I. Denoted as $\mathcal{N}_I$ the set of Nash equilibria of I, the *price of anarchy* of I (denoted as PoA_I) is the worst case ratio between the social optimum and the social value of a Nash equilibrium, i.e., $PoA_I = \sup_{S \in \mathcal{N}_I} \frac{OPT_I}{SF(S)}$. Moreover, the *price of stability* of I (denoted as PoS_I) is the best case ratio between the social optimum and the social value of a Nash equilibrium, i.e., $PoS_I = \inf_{S \in \mathcal{N}_I} \frac{OPT_I}{SF(S)}$. Let $\mathcal{I}_\mathcal{G}$ be the set of all instances of a given class of games $\mathcal{G}$. The prices of anarchy and stability of $\mathcal{G}$ (denoted as $PoA_\mathcal{G}$ and $PoS_\mathcal{G}$, respectively) are defined as $PoA_\mathcal{G} = \sup_{I \in \mathcal{I}_\mathcal{G}} PoA_I$ and $PoS_\mathcal{G} = \sup_{I \in \mathcal{I}_\mathcal{G}} PoS_I$.

We study the prices of anarchy and stability of ℓ-selection and total-distance isolation games under the two standard social functions adopted in the literature, namely, the minimum utility per player $MIN(S) = \min_{i \in [k]}\{u_i(S)\}$ and the sum of the utilities of all the players $SUM(S) = \sum_{i \in [k]} u_i(S)$.

3 ℓ-selection isolation games

As proved in Zhao et al. (2008), for these the existence of Nash equilibria is guaranteed for any value of $\ell \in [k-1]$, therefore, we study the performances of such equilibria by tightly bounding the prices of anarchy and stability for both social functions *MIN* (Sect. 3.1) and *SUM* (Sect. 3.2).

3.1 The social function *MIN*

In the next theorem, we show a tight bound on the price of anarchy of ℓ-selection isolation games for any possible value of ℓ.

Theorem 1 *For any $k \geq 2$ and $\ell \in [k-1]$, the price of anarchy of ℓ-selection isolation games under the social function MIN is* 2.

Proof For any $k \geq 2$ we provide an instance $I = ((X, d), k)$ for which there exists a Nash equilibrium of social value 1 in any possible ℓ-selection isolation game, while $OPT_I \geq 2$, thus establishing a lower bound of 2 on the price of anarchy.

Let (X, d) be a metric space such that $X = X_1 \cup X_2$ with $X_1 = \{x_1, \dots, x_k\}$ and $X_2 = \{y_1, \dots, y_k\}$. The distance function d is defined as follows: for any $i, j \in [k]$, $i \neq j$, $d(x_i, x_j) = 1$ and $d(y_i, y_j) = 2$. Moreover, for any $i, j \in [k]$ $d(x_i, y_j) = d(y_j, x_i) = 1$.

On the one hand, it is easy to check that in the isolation game played on instance I, the strategy profile $S = (x_1, \dots, x_k)$ is a Nash equilibrium in which each player is at distance 1 from all the other players. In fact, if any player $i \in [k]$ can change her strategy by selecting a different node in $x_j \in X_1$ or a node in X_2; in both cases, her distance from all other nodes is at most 1. Therefore, for any $\ell \in [k-1]$, S is a Nash equilibrium of social cost $MIN(S) = 1$; on the other hand, in the strategy profile $S^* = (y_1, \ldots, y_k)$ each player is at distance 2 from all the other ones, and therefore $OPT_I \geq MIN(S^*) = 2$.

In order to complete the proof, it remains to show that for any instance of an ℓ-selection isolation game the price of anarchy is at most 2.

Consider a generic instance of an ℓ-selection isolation game, and let $S^* = (s_1^*, \ldots, s_k^*)$ and $S = (s_1, \ldots, s_k)$ be an optimal solution and a Nash equilibrium for it, respectively. We want to prove that $MIN(S^*) \leq 2MIN(S)$.

For every player $i \in [k]$, let $G(i) \subseteq [k]$, $|G(i)| = \ell$, be the set of the ℓ players closest to location s_i^* in S, breaking ties arbitrarily. Let $j \in [k]$ be a player having utility $u_j(S) = MIN(S)$; we have that for any $i \in [k]$, $d(s_i^*, s_x) \leq MIN(S)$ for every $x \in G(i)$ (we call such fact *main property*). In fact, assume by contradiction that there exist $i \in [k]$ and $x \in G(i)$ such that $d(s_i^*, s_x) > MIN(S)$; then, player j can improve her utility by migrating to the location s_i^*: a contradiction, because S is a Nash equilibrium.

For each $i \in [k]$, let $h(i)$ be the number of sets $G(j)$, $j \in [k]$, such that $i \in G(j)$; the proof is now divided into two cases.

- If there exists $i \in [k]$ such that $h(i) \geq \ell + 1$, by the *main property* there must be at least $\ell + 1$ players in S^* having distance at most $MIN(S)$ from location s_i. Thus, by the *symmetry* and *triangular inequality* properties of d, it follows that all such players have the ℓ-th furthest player at distance at most $2MIN(S)$. Therefore, $MIN(S^*) \leq 2MIN(S)$.
- If for all players $i \in [k]$, it holds $h(i) \leq \ell$, since $\sum_{i=1}^{k} h(i) = k\ell$, we have that $h(i) = \ell$ for all players $i \in [k]$. Let $i \in [k]$ be a player such that $j \in G(i)$, then there must exist a player $j' \notin G(i)$ such that $d(s_i^*, s_{j'}) \leq MIN(S)$. In fact, assume by contradiction that for all $j'' \notin G(i)$, it holds $d(s_i^*, s_{j''}) > MIN(S)$; then, player j can improve her utility by migrating to the location s_i^*: a contradiction, because S is a Nash equilibrium. Thus, since $h(j') = \ell$ and there exists a player i such that $d(s_i^*, s_{j'}) \leq MIN(S)$ and $j' \notin G(i)$, by the *main property* there must be at least $\ell + 1$ players in S^* having distance at most $MIN(S)$ from location $s_{j'}$. The claim follows by the same arguments exploited in the previous case.

□

In order to prove that the price of stability is 1 for ℓ-selection isolation games with $\ell > 1$, we first have to show that such a result holds for the particular case of the nearest-neighbor isolation game, that is, when $\ell = 1$.

Theorem 2 *The price of stability of the nearest-neighbor isolation game under the social function MIN is* 1.

Proof Given an instance of the nearest-neighbor isolation game, let S^* be a strategy profile attaining the social optimum.

Since, as shown in Zhao et al. (2008), the nearest-neighbor isolation game is a potential game, any sequence of better responses starting from S^* has to lead to a Nash equilibrium. Let $S^* = S_0, S_1, \ldots, S_h = \hat{S}$ be one of such sequences, that is, such that $\hat{S}$ is a Nash equilibrium and denote as i_α the player performing the improving move in S_α. If there exists $\alpha \in \{0, \ldots, h-1\}$ such that $MIN(S_\alpha) > MIN(S_{\alpha+1})$, by the symmetry of the distance function, we have $u_{i_\alpha}(S_\alpha) > u_{i_\alpha}(S_{\alpha+1})$: a contradiction. Hence, for each $\alpha = 0, \ldots, h-1$, it must hold $MIN(S_\alpha) \le MIN(S_{\alpha+1})$. Therefore, $OPT = MIN(S^*) \le MIN(\hat{S})$. □

Now we can prove the bound on the price of stability for any ℓ-selection isolation game.

Theorem 3 *For any* $1 < \ell < k$, *the price of stability of* ℓ*-selection isolation games under the social function MIN is* 1.

Proof Given an instance $I = ((X, d), k)$ of an ℓ-selection isolation game, we construct an instance $I' = ((X, d), \lceil \frac{k}{\ell} \rceil)$ of the nearest-neighbor isolation game.

The *natural conversion* of a strategy profile $S' = (s'_1, \ldots, s'_{k'})$ for I' into a strategy profile S for I consists in partitioning the k players of I in k' groups, each containing ℓ players, except the last group that may contain less players, and in assigning the location s'_j to all the players in the j-th group for any $j \in [k']$.

By Theorem 2, there exists a Nash equilibrium $\hat{S}'$ for I' with social cost equal to the social optimum. Moreover, by Zhao et al. (2008),[1] the strategy profile $\hat{S}$, obtained by applying any natural conversion of $\hat{S}'$, is a Nash equilibrium for I.

In order to show that $\hat{S}$ is a Nash equilibrium for I such that $MIN(\hat{S}) = OPT_I$, thus proving the claim, it remains to show that:

1. $MIN(S') = MIN(S)$, where S is obtained by applying any natural conversion of S';
2. $OPT_I = OPT_{I'}$.

In order to prove (1.) it is sufficient to notice that, since for each player $j \in [k']$ in I', there are at most ℓ corresponding players placed on the same location of j in S, the utility of each of them (with respect to the ℓ-selection vector) equals the utility of the corresponding player in I' (with respect to the 1-selection vector).

We divide the proof of (2.) into two parts. Clearly, $OPT_I \ge OPT_{I'}$ since a solution for I' can be converted (by the natural conversion) into a solution for I having the same social value.

We now show that $OPT_{I'} \ge OPT_I$, thus concluding the proof of the theorem.

Let S^* be a strategy profile attaining the social optimum for I. We now construct a strategy profile S'^* for I' such that $MIN(S'^*) \ge MIN(S^*) = OPT_I$.

```
Let K := [k] and h := 0
while K ≠ ∅
  h := h + 1
```

[1]More precisely, we refer to the proof of Theorem 5 in the extended version of Zhao et al. (2008) available online as a Technical Report MSR-TR-2008-126 at http://research.microsoft.com/en-us/people/weic/msr-tr-2008-126.pdf.

Table 3 The distance function of the metric space used in the proof of Theorem 4

d	x_1	x_2	x_3	x_4
x_1	0	ϵ	1	1
x_2	ϵ	0	1	1
x_3	1	1	0	ϵ
x_4	1	1	ϵ	0

```
    Choose a player i ∈ K
    Let j_1,...,j_{ℓ-1} ∈ K be the ℓ-1 players in K nearest to
  i in S*
    K := K \ {i, j_1,...,j_{ℓ-1}}
    s'*_h := s*_i
  endwhile
  Return S'* = (s'*_1,...,s'*_h).
```

First of all, note that the final value of h is k'. It is possible to check that, at the h-th iteration of the `while` cycle, since the $\ell - 1$ players who are nearest to player i in S^* are deleted from K, the location $s_h'^*$ assigned to player h in I' is such that all the locations of the remaining nodes in K are at distance at least $MIN(S^*)$ from $s_h'^*$. Therefore, since the locations of the players in I' are chosen equal to some location of the players in $K \subseteq [k]$, it follows by an easy inductive argument that the minimum distance between two players in I' is at least $MIN(S^*)$, that is $MIN(S'^*) \geq MIN(S^*)$. □

3.2 The social function *SUM*

Theorem 4 *For any $\ell \geq 1$, there exists an instance of ℓ-selection isolation game for which the prices of anarchy and stability under the social function SUM are unbounded.*

Proof For any $\ell \geq 1$, consider an instance $I = ((X, d), 4\ell)$ for the ℓ-selection isolation game such that $X = \{x_1, x_2, x_3, x_4\}$ and d is defined as shown in Table 3, where $\epsilon > 0$ is arbitrarily small.

Observe that the strategy profiles in which there are a location x occupied by more than ℓ players and a location y occupied by less than ℓ players cannot be a Nash equilibrium, because each player located at x can improve her utility by migrating to y. Hence, the only possible Nash equilibria are represented by the strategy profiles in which each location is occupied by exactly ℓ players. The social value corresponding to any of these profiles is $4\ell\epsilon$. On the other hand, the strategy profile in which $4\ell - 1$ players are located at x_4 and the remaining one at x_1 achieves a social value equal to 1. Then, $PoA = PoS \geq \frac{1}{4\ell\epsilon}$. □

4 Total-distance isolation games

It has been proved in Zhao et al. (2008) that total-distance isolation games are potential games thus implying the existence of Nash equilibria and convergence to one such an equilibrium starting from any initial strategy profile. Therefore, in this section we study again the prices of anarchy and stability of Nash equilibria by giving tight bounds for the social function *SUM* (Sect. 4.1) and asymptotically tight bounds for the social function *MIN* (Sect. 4.2).

4.1 The social function *SUM*

In the next two theorems we show exact bounds for both the prices of anarchy and stability of total-distance isolation games under the social function *SUM*.

Theorem 5 *The price of stability of total-distance isolation games under the social function SUM is* 1.

Proof Consider the function $\Phi(S) = \sum_{1\le i,j\le k} d(s_i, s_j)$. It is shown in Zhao et al. (2008) that $\Phi(S)$ is a potential function for total-distance isolation games. Hence, for any instance I of the total-distance isolation game, there exists a Nash equilibrium S' such that $\Phi(S') = \max_{S\in\mathcal{S}}\{\Phi(S)\}$. On the other hand, let S^* be a strategy profile attaining the social optimum, that is, such that $SUM(S^*) = OPT_I$. Since for any $S \in \mathcal{S}$, it holds

$$\begin{aligned} SUM(S) &= \sum_{i\in[k]} u_i(S) \\ &= \sum_{i\in[k]} \sum_{j\in[k], j\neq i} d(s_i, s_j) \\ &= \sum_{1\le i,j\le k} d(s_i, s_j) \\ &= \Phi(S), \end{aligned}$$

we have that $SUM(S') = \Phi(S') \le \Phi(S^*) = SUM(S^*) = OPT_I$. □

Theorem 6 *For any $k \ge 2$, the price of anarchy of total-distance isolation games under the social function SUM is* 2.

Proof The lower bound of 2 can be easily derived by using the same instance described in the first part of the proof of Theorem 1, since it works also for the case of total-distance isolation games. It remains to show an upper bound of 2 to the price of anarchy.

For any instance of the total-distance isolation game, let $S^* = (s_1^*, \dots, s_k^*)$ be a strategy profile attaining the social optimum and $S = (s_1, \dots, s_k)$ be a Nash equilibrium. We define the complete bipartite graph $K_{k,k} = (U \cup V, E)$ where $U = \{u_1, \dots, u_k\}$ and $V = \{v_1, \dots, v_k\}$. We associate the weight $d(s_i^*, s_j)$ with each $(u_i, v_j) \in E$. Moreover, we associate to each node u_i location s_i^* and to each node v_i location s_i; with a little abuse of notation, given $u, u' \in U$ and $v, v' \in V$, we denote by $d(u, v)$ (respectively $d(u, u')$ and $d(v, v')$) the distance in the metric space between the locations corresponding to nodes u and v (respectively nodes u and u' and nodes v and v').

Consider the partition of E into k matchings $M_1, \dots, M_k$ and let M^* be the minimum weight one, that is, such that $\sum_{(u,v)\in M^*} d(u,v) \le \min_{i\in[k]}\{\sum_{(u,v)\in M_i} d(u,v)\}$.

In order to prove the claim, we need to show that $SUM(S^*) = \sum_{u,u'\in U} d(u,u') \le 2\sum_{v,v'\in V} d(v,v') = 2SUM(S)$.

By applying a standard averaging argument, we have that

$$(k-1)\sum_{(u,v)\in M^*} d(u,v) \le \sum_{(u,v)\in E\setminus M^*} d(u,v). \tag{1}$$

Moreover, since S is a Nash equilibrium, it must hold $\sum_{v'\in V} d(v,v') \ge \sum_{v\neq v'\in V} d(u,v')$ for any $v\in V$ and $u\in U$. For any bijection $m: U\to V$, by summing up the k different inequalities that can be obtained for any disjoint pair of vertices $(u, m(u))$, we get

$$\sum_{v,v'\in V} d(v,v') \ge \sum_{u\in U}\sum_{m(u)\neq v'\in V} d(u,v'). \tag{2}$$

Consider the bijection m^* induced by M^*. Since d satisfies the triangular inequality, for any pair of nodes $u,u'\in U$, we have that $d(u,u')$ cannot be more than $d(u,m^*(u)) + d(u',m^*(u))$.

By summing up over all possible pairs of nodes in U, we obtain

$$\begin{aligned}
\sum_{u,u'\in U} d(u,u') &\le (k-1)\sum_{u\in U} d(u,m^*(u)) + \sum_{(u,v\neq m^*(u))} d(u,v)\\
&= (k-1)\sum_{(u,v)\in M^*} d(u,v) + \sum_{(u,v)\in E\setminus M^*} d(u,v)\\
\{\text{Because of Inequality 1}\} &\le 2\sum_{(u,v)\in E\setminus M^*} d(u,v)\\
&= 2\sum_{u\in U}\sum_{m^*(u)\neq v\in V} d(u,v)\\
\{\text{Because of Inequality 2}\} &\le 2\sum_{v,v'\in V} d(v,v').
\end{aligned}$$

□

4.2 The social function *MIN*

In order to prove that the price of anarchy is at most $2\frac{k+1}{k-1}$ and the price of stability is at most $\frac{k+1}{k-1}$, we first show a general property relating the two social values of certain Nash equilibria for total-distance isolation games.

Lemma 1 *For any instance I of total-distance isolation games, let S^* and $\overline{S}^*$ be two strategy profiles attaining the social optimum for I under the social functions*

MIN and SUM respectively. For any Nash equilibrium S such that $SUM(S) \geq \frac{1}{\alpha} \cdot SUM(\overline{S}^*)$, *it holds* $\frac{MIN(S^*)}{MIN(S)} \leq \alpha \cdot \frac{k+1}{k-1}$.

Proof Let $d = d(s_i, s_j)$ be the distance between s_i and s_j. Since S is a Nash equilibrium, we have that

$$u_j(S) \geq u_i(S) - d, \tag{3}$$

otherwise player j could obtain a better utility by moving to location s_i.

Moreover, by the triangular inequality property $d \leq d(s_i, s_h) + d(s_h, s_j)$ for all $h \in [k]$, $h \neq i, j$. By summing over all $h \in [k], h \neq i, j$, we obtain that $\sum_{h\in[k],h\neq i,j} d \leq \sum_{h\in[k],h\neq i,j} d(s_i, s_h) + \sum_{h\in[k],h\neq i,j} d(s_h, s_j)$. Therefore, $(k-2)d \leq (u_i(S) - d) + (u_j(S) - d)$. By inequality (3), we obtain

$$u_j(S) \geq \frac{k-1}{2} \cdot d. \tag{4}$$

Thus using (3) and (4), we obtain that $\forall i, j \in [k]$

$$\begin{aligned}
\frac{u_i(S)}{u_j(S)} &= \frac{u_i(S) - u_j(S)}{u_j(S)} + 1 \\
&\leq \frac{d}{u_j(S)} + 1 \\
&\leq \frac{2d}{(k-1)d} + 1 \\
&= \frac{k+1}{k-1}.
\end{aligned} \tag{5}$$

Since $SUM(\overline{S}^*) \geq SUM(S^*) \geq k \cdot MIN(S^*)$, we have that

$$\frac{SUM(\overline{S}^*)}{k} \geq MIN(S^*). \tag{6}$$

Moreover, since $SUM(S) = \sum_{i=1}^{k} u_i(S)$, there exists a player $j \in [k]$ such that

$$u_j(S) \geq \frac{SUM(S)}{k}. \tag{7}$$

Recalling that $SUM(S) \geq \frac{1}{\alpha} \cdot SUM(\overline{S}^*)$ and using inequalities (5), (6) and (7), we obtain that $\forall i \in [k]$

$$\begin{aligned}
u_i(S) &\geq u_j(S) \cdot \frac{k-1}{k+1} \\
&\geq \frac{SUM(S)}{k} \cdot \frac{k-1}{k+1} \\
&\geq \frac{SUM(\overline{S}^*)}{k \cdot \alpha} \cdot \frac{k-1}{k+1} \\
&\geq \frac{MIN(S^*)}{\alpha} \cdot \frac{k-1}{k+1}.
\end{aligned}$$

□

By exploiting Lemma 1, it is possible to obtain asymptotically tight bounds on the prices of anarchy and of stability of total-distance isolation games under the social function *MIN*.

Theorem 7 *The price of anarchy of total-distance isolation games under the social function MIN is between* 2 *and* $2\frac{k+1}{k-1}$.

Proof The lower bound of 2 can be easily derived by using the same instance described in the first part of the proof of Theorem 1, since it works also for the case of total-distance isolation games.

In order to prove an upper bound of $2\frac{k+1}{k-1}$ it is sufficient to apply Lemma 1 with $\alpha = 2$, since, by Theorem 6, the price of anarchy of total-distance isolation games under the social function *SUM* is 2. □

Theorem 8 *The price of stability of total-distance isolation games under the social function MIN is between* 1 *and* $\frac{k+1}{k-1}$.

Proof As in the previous theorem, the proof directly follows from Lemma 1 with $\alpha = 1$, since, by Theorem 5, the price of stability of total-distance isolation games under the social function *SUM* is 1. □

5 ℓ-suffix isolation games

In this section we consider the games induced by the ℓ-suffix vector $(0, \dots, 0, 1, \dots, 1)$ for any $2 \le \ell \le k - 2$. We recall that the utility of a player is given by the sum of the distances from the ℓ furthest ones. These games, representing a natural generalization of the total-distance ones, have not been considered in Zhao et al. (2008).

In the following theorem, we show that convergence to a Nash equilibrium is not guaranteed in this case.

Theorem 9 *The* ℓ*-suffix isolation games are not potential games.*

Proof We show that ℓ-suffix isolation games are not, in general, potential games by providing an instance with $k = 4$ in which there exists an infinite sequence of better-response moves.

Consider the instance $I = ((X, d), 4)$ such that $X = \{x_1, x_2, x_3, x_4\}$ and d is defined as shown in Table 4, where $\epsilon > 0$ is arbitrarily small. Since $k = 4$, the only possible ℓ-suffix isolation game which can be defined on I is the one yielded by the weight vector $(0, 1, 1)$. Let $S = (x_1, x_2, x_3, x_4)$ be the initial strategy profile. We show that after six improving moves, the game may end up again in S, thus creating an infinite cycle of migrations.

To this aim, consider the following triple of improving moves starting from S.

- First move: player 3 migrates from x_3 (with utility $4.5 + \epsilon$) to x_1 (with new utility 5). The new strategy profile is $S_1 = (x_1, x_2, x_1, x_4)$.

Table 4 The distance function of the metric space used in the proof of Theorem 9

d	x_1	x_2	x_3	x_4
x_1	0	2.5	3	2.5
x_2	2.5	0	$1.5+\epsilon$	$3+2\epsilon$
x_3	3	$1.5+\epsilon$	0	$1.5+\epsilon$
x_4	2.5	$3+2\epsilon$	$1.5+\epsilon$	0

- Second move: player 4 migrates from x_4 (with utility $5.5+2\epsilon$) to x_3 (with new utility 6). The new strategy profile is $S_2=(x_1,x_2,x_1,x_3)$.
- Third move: player 3 migrates from x_1 (with utility 5.5) to x_4 (with new utility $5.5+2\epsilon$). The new strategy profile is $S_3=(x_1,x_2,x_4,x_3)$.

As a result of the transition from S to S_3, we have that players 3 and 4 have swapped their locations. Hence, by exchanging the roles of players 3 and 4 in the above triple of moves and setting S_3 as the starting profile, it is possible to come back to S. □

6 Conclusions and open problems

We have studied the efficiency of pure Nash equilibria in both ℓ-selection and total-distance isolation games, that is, in all the cases among the ones analyzed in Zhao et al. (2008) for which such equilibria have been proven to exist. We achieved tight bounds in all the cases, even if for the case of total-distance isolation games under the social function *MIN* they are asymptotically optimal in the sense that they get tight when the number of players goes to infinity. Getting matching lower and upper bounds for low values of k is an interesting left open issue worth to be investigated. Moreover, for the social function *SUM* we have shown that for any $\ell \geq 1$ there exists an instance of ℓ-selection isolation game for which the prices of anarchy and stability are unbounded; an arising open question is that of studying what happens for any possible combination of k and $\ell \in [k-1]$.

As a natural generalization of total-distance isolation games, one can consider ℓ-prefix and ℓ-suffix isolation games. For ℓ-prefix games, Zhao et al. (2008) have proven that Nash equilibria may not exist even in the simple basic case yielded by the weight vector $(1,1,0,\dots,0)$. For the left over ℓ-suffix games, we have shown that even the simple basic case yielded by the weight vector $(0,1,1)$ may not be a potential game, thus encompassing all the possible ℓ-suffix isolation games which can be defined on an instance with four players. It is left open, however, extending such a result to best-response dynamics (the proof exploits better-response moves) and to general values of k and ℓ $(2\leq \ell \leq k-2)$, as well as establishing whether Nash equilibria are guaranteed to exist.

Concerning other possible future research directions, studying games yielded by more general and complicated weight vectors seems to be a very challenging task. Moreover, restricting to particular metric spaces, such as Euclidean spaces, looks intriguing and promising also from an application point of view (see for instance interferences in wireless networks as mentioned in the introduction). Clearly, all the positive results, that is the existence of equilibria and the upper bounds on both the prices of anarchy and stability, carry over also to these cases. It would be interesting to determine which properties a metric space should satisfy in order for the price of anarchy of the ℓ-selection and the total-distance isolation games to get lower than 2.

References

Ahn HK, Cheng SW, Cheong O, Golin MJ, Oostrum R (2004) Competitive facility location: the Voronoi game. Theor Comput Sci 310(1–3):457–467

Anshelevich E, Dasgupta A, Tardos E, Wexler T (2003) Near-optimal network design with selfish agents. In: Proc of the 35th annual ACM symposium on theory of computing (STOC). ACM, New York, pp 511–520

Cheong O, Har-Peled S, Linial N, Matousek J (2004) The one-round Voronoi game. Discrete Comput Geom 31:125–138

Dürr C, Thang NK (2007) Nash equilibria in Voronoi games on graphs. In: Proc of the 15th annual European symposium on algorithms (ESA). LNCS, vol 4698. Springer, Berlin, pp 17–28

Eaton BC, Lipsey RG (1975) The principle of minimum differentiation reconsidered: Some new developments in the theory of spatial competition. Rev Econ Stud 42(129):27–49

Eiselt HA, Laporte G, Thisse JF (1993) Competitive location models: A framework and bibliography. Transp Sci 27(1):44–54

Fekete SP, Meijer H (2005) The one-round Voronoi game replayed. Comput Geom Theory Appl 30:81–94

Hotelling H (1929) Stability in competition. Comput Geom Theory Appl 39(153):41–57

Jain AK, Murty MN, Flynn PJ (1999) Data clustering: a review. ACM Comput Surv 31(3)

Koutsoupias E, Papadimitriou CH (1999) Worst-case equilibria. In: Proc of the 16th international symposium on theoretical aspects of computer science (STACS). LNCS, vol 1653. Springer, Berlin, pp 404–413

Mavronicolas M, Monien B, Papadopoulou VG, Schoppmann F (2008) Voronoi games on cycle graphs. In: Proc. of the 33rd international symposium on mathematical foundations of computer science (MFCS). LNCS, vol 5162. Springer, Berlin, pp 503–514

Nash J (1950) Equilibrium points in n-person games. In: Proc of the national academy of sciences, vol 36, pp 48–49

Teng SH (1999) Low energy and mutually distant sampling. J Algorithms 30(1):42–67

Zhao Y, Chen W, Teng SH (2008) The isolation game: A game of distances. In: Proc of the 19th international symposium on algorithms and computation (ISAAC). LNCS, vol 5369. Springer, Berlin, pp 148–159