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The Zero Relaxation Limit for the Hydrodynamic Whitham Traffic Flow Model

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We study the relaxation problem for a hyperbolic system of balance laws which models the trafffic flow. In particular we show, as the relaxation parameter tends to zero, away from the vacuum, the strong convergence to the equilibrium solution, which satisfies Kružkov-type entropy conditions. A uniqueness result is provided.

1. INTRODUCTION

In this paper we analyze the relaxation of the weak solutions to the following system of balance laws,

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = \dfrac{1}{\tau}\,(Q(\rho) - m) \\ \rho(0, x) = \rho_0(x) \\ m(0, x) = m_0(x), \end{cases} \tag{1.1}$$

where $\tau > 0$; $p(\rho) = h\rho^\gamma$, $1 < \gamma < 3$, $h = \theta^2/\gamma$, $\theta = (\gamma - 1)/2$ and $Q(\rho) = a\rho(1-\rho)$, $a > 0$. The initial data $\rho_0(x)$, $m_0(x)$ are taken in $L^\infty(\mathbb{R})$. We want to prove this system relaxes, as $\tau \downarrow 0$, to the scalar conservation law

$$\rho_t + Q(\rho)_x = 0. \tag{1.2}$$

This kind of models has been introduced by Whitham in [16] in order to study traffic flow on highways. A simpler way to describe this phenomenon is given by the scalar conservation law

$$\rho_t + (\rho v)_x = 0, \tag{1.3}$$

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where the flux velocity v is a positive, non-decreasing function of the density ρ, namely:

$$v = V(\rho); \quad V(\rho) > 0; \quad V'(\rho) < 0. \tag{1.4}$$

The system (1.1) can be obtained by replacing the relation (1.4) with the following constitutive relation:

$$v_t + vv_x = -\frac{1}{\tau}\left\{v - V(\rho) + \tau p'(\rho)\,\frac{\rho_x}{\rho}\right\}, \tag{1.5}$$

which takes into account the reactions of the drivers to changes down the road (there is a sort of diffusion of informations along the road). Moreover, they are supposed to have a nonzero response-time τ. It is possible to prove global existence for classical solutions of the system (1.3)–(1.6) (see [14]), with a slightly different constitutive equation, namely

$$v_t + vv_x = -\frac{1}{\tau}\left\{v - V(\rho) + \nu\,\frac{\rho_x}{\rho}\right\}; \ \tau, \quad \nu > 0. \tag{1.6}$$

The relaxation phenomena are important since they arise in many relevant physical situations as the kinetic theories, elasticity with memory, chemically reacting flows and in the theory of linear and nonlinear waves. The first approach to this problem was done by Whitham in his book [16]. Later there have been several rigorous studies on this subject, as in the papers of Liu [10] and Chen, Levermore and Liu [1]. In particular, they consider, in ths case of 2×2 systems, the so-called *stability condition*, which turns out to be necessary in the general theory: the characteristic speed of the relaxed conservation law must be subcharacteristic with respect to characteristic speeds of the original system. Connections between the existence of invariant regions and the subcharacteristic condition have been investigate in [11], together with a partial result of convergence. Natalini in [13] studies the case of semilinear systems and prove the Kružkov entropy conditions for the limit function. Finally, some numerical schemes deduced from the relaxation models have been investigated in [7].

The plan of the paper is the following.

In the Section 2 we will get the fundamental L^∞ estimates by using the special geometry of the graph of $Q(\rho)$ and the theory of invariant regions due to Chueh, Conley and Smoller [4]. Subsequently we derive a special energy estimate which put into evidence the dissipative nature of the relaxation process and provides the rate of convergence toward the equilibrium as $\tau \downarrow 0$. This estimate is obtained under some mathematical restrictions, when the entropy conditions are given by the vanishing artificial viscosity (the kind of approximation used with the compensated compactness framework), while there are essentially no restrictions if we use the more reasonable (from the point of view of physics) vanishing Navier–Stokes viscosity. As a byproduct of these estimates we get the existence of the weak entropy solutions to the full hydrodynamic model, for any fixed $\tau > 0$.

In the Section 3 we will prove the convergence, as $\tau \downarrow 0$, of the weak entropy solutions of (1.1) to weak solutions of the scalar Eq. (1.2). The method used in this section is related to [1]. The convergence result is obtained away from vacuum.

The next section is devoted to study the validity of the entropy conditions to the relaxed conservation law (1.2). Although the flux $Q(\rho)$ is a concave function, the lack of the maximum principle for the approximating system make it difficult to show Oleinik type entropy inequality. Hence we prove Kružkov-type entropy inequalities. This result is obtained by a careful analysis of the regularized extensions of the Kružkov-type entropies for the Eq. (1.2) to entropies of the full system (1.1). The main mathematical difficulty which is necessary to overcome consists in finding a domain around the equilibrium curve, independent of k, where the entropies η^k are simultaneously strictly convex and verify $\eta^k_{mm} > 0$. In order to perform this task we will take advantage from some results on the linear hyperbolic entropy equation, which are exposed in the Appendix A.

The last section is dedicated to the uniqueness of the weak entropy solutions to the Eq. (1.2). It is well known (see [8, 6, 15]) that, for this purpose, the entropy conditions must be supplemented by the L^1 continuity of the semigroup in $t = 0$. The main difficulty is due to the formation of an initial layer which generates an extra entropy production. We prove the uniqueness when the initial data are in equilibrium.

We recall that the system (1.1) is hyperbolic with characteristic velocities

$$\lambda_{1,2}(\rho, m) = \frac{m}{\rho} \mp \sqrt{p'(\rho)} = \frac{m}{\rho} \mp \theta \rho^{\theta}$$

and Riemann's invariants

$$\omega_{1,2}(\rho, m) = \frac{m}{\rho} \pm \int^{\rho} \frac{\sqrt{p'(s)}}{s}\, ds = \frac{m}{\rho} \pm \rho^{\theta}.$$

Note that the system is not strictly hyperbolic since the characteristic velocities coalesce because of the vacuum formation. Hence, the stability condition

$$\lambda_1(\rho, Q(\rho)) < Q'(\rho) < \lambda_2(\rho, Q(\rho))$$

becomes in our case

$$\text{(SC)} \quad -\theta\rho^{(\gamma-3)/2} < a < \theta\rho^{(\gamma-3)/2}, \qquad 0 < \rho \leqslant 1. \tag{1.7}$$

2. A PRIORI ESTIMATES AND EXISTENCE

In this section we investigate the problem of the existence of the solution of system (2.2) for $\tau > 0$ fixed, under the stability condition (1.7). In particular we will prove the following theorem:

THEOREM 2.1. *Let $(\rho_0, m_0) \in \Sigma$ be such that*

$$\int_{-\infty}^{+\infty} \eta^*(\rho_0(x), m_0(x))\, dx < +\infty$$

and that

$$\int_{-\infty}^{+\infty} \bar{\eta}(\rho_0(x), m_0(x))\, dx < +\infty$$

and let $(\rho^\varepsilon, m^\varepsilon)$ be the solution of (2.3) with (ρ_0, m_0) as initial condition. Finally, assume the stability condition (1.7) holds. Then, passing if necessary to a subsequence $(\rho^\varepsilon, m^\varepsilon) \to (\rho, m)$ strongly in L^p_{loc} for every $p < +\infty$ as $\varepsilon \downarrow 0$, where (ρ, m) is a solution of (2.2), with $(\rho_0(x), m_0(x))$ as initial condition, which verifies the following entropy inequality:

$$\eta(\rho, m)_t + q(\rho, m)_x \leqslant \frac{1}{\tau}\, \eta_m(\rho, m)(Q(\rho) - m) \qquad \text{in } \mathscr{D}' \tag{2.1}$$

for every convex entropy η with flux q.

The proof of this theorem is postponed at the end of the section. We begin by collecting some preliminary estimates.

Let us consider the system

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho)\right)_x = \dfrac{1}{\tau}(Q(\rho) - m), \end{cases} \tag{2.2}$$

and its artificial viscosity approximation, namely the following parabolic system:

$$\begin{cases} \rho_t^\varepsilon + m_x^\varepsilon = \varepsilon \rho_{xx}^\varepsilon \\ m_t^\varepsilon + \left(\dfrac{(m^\varepsilon)^2}{\rho^\varepsilon} + p(\rho^\varepsilon) \right)_x = \dfrac{1}{\tau} (Q(\rho^\varepsilon) - m^\varepsilon) + \varepsilon m_{xx}^\varepsilon. \end{cases} \tag{2.3}$$

The following lemma provides the L^∞ bounds uniformly in ε and τ, by using the theory of invariant domains [4].

LEMMA 2.2. *Let* $(\rho^\varepsilon, m^\varepsilon)$ *be the solution of the system* (2.3), *with initial condition* $(\rho_0(x), m_0(x))$. *Assume there exists a constant* $C > 0$ *such that* $\max\{\|\rho_0\|_{L^\infty}, \|m_0\|_{L^\infty}\} \leqslant C$ *and the condition* (SC) *given in* (1.7) *holds, then* $(\rho^\varepsilon, m^\varepsilon)$ *is uniformly bounded in* L^∞ *with respect to* ε *and* τ.

Proof. It is well known (see for instance [4]) that, for some constant ω_{10} and ω_{20}, every region Σ of the form

$$\Sigma = \{(\rho, m) \colon \omega_1 \leqslant \omega_{10} \text{ and } \omega_2 \geqslant \omega_{20}\,,\ \rho > 0\} \tag{2.4}$$

is an invariant domain for the parabolic system

$$\begin{cases} \rho_t^\varepsilon + m_x^\varepsilon = \varepsilon \rho_{xx}^\varepsilon \\ m_t^\varepsilon + \left(\dfrac{(m^\varepsilon)^2}{\rho^\varepsilon} + p(\rho^\varepsilon) \right)_x = \varepsilon m_{xx}^\varepsilon, \end{cases}$$

where ω_1 and ω_2 denote the Riemann's invariant for the previous system (2.2). Now let us choose the constants ω_{10} and ω_{20} such that

$$\omega_1(1, 0) = \omega_{10} \qquad \text{and} \qquad \omega_2(1, 0) = \omega_{20}. \tag{2.5}$$

Then after the transformation

$$\begin{cases} R(y, s) = \rho(\tau y, \tau s) \\ M(y, s) = m(\tau y, \tau s), \end{cases}$$

the system (2.2) is transformed into the new system

$$\begin{cases} R_s + M_y = 0 \\ M_s + \left(\dfrac{M^2}{R} + p(R) \right)_y = Q(R) - M, \end{cases}$$

where the relaxation time has been eliminated. At this stage, let us consider the artificial viscosity approximation for the previous system, namely:

$$\begin{cases} R_s^\varepsilon + M_y^\varepsilon = \varepsilon R_{yy}^\varepsilon \\ M_s^\varepsilon + \left(\dfrac{(M^\varepsilon)^2}{R^\varepsilon} + p(R^\varepsilon)\right)_y = Q(R^\varepsilon) - M^\varepsilon + \varepsilon M_{yy}^\varepsilon. \end{cases} \tag{2.6}$$

As we said before, the homogeneous system associated with (2.6) has an invariant region Σ. Because of the particular choice for the constants done in (2.5), the curves $M = Q(R)$, $\omega_1(R, M) = \omega_{10}$ and $\omega_2(R, M) = \omega_{20}$ intersect with the R axis at the same points $R = 0$ and $R = 1$ (see Fig. 1). Let us consider now $M = M_1(R)$ and $M = M_2(R)$ be respectively the explicit form of the curves $\omega_1(R, M) = \omega_{10}$ and $\omega_2(R, M) = \omega_{20}$, as functions of R. Therefore, from the stability condition (1.7) we get

$$M_2'(0) < Q'(0) < M_1'(0), \tag{2.7}$$

and

$$M_1'(1) < Q'(1) < M_2'(1), \tag{2.8}$$

where the $'$ denotes the differentiation with respect to the R variable.

Indeed,

$$M_1'(1) = \omega_{10} - (\theta + 1) = -\theta = \lambda_1(1, 0),$$

since we took $\omega_{10} = 1$. Similarly, one can prove that $M_2'(1) = \theta = \lambda_2(1, 0)$ and so (2.8) is a direct consequence of (1.7). In a similar way, one can prove that $M_1'(0) = 1$ and $M_2'(0) = -1$ so (2.7) follows again from (1.7).

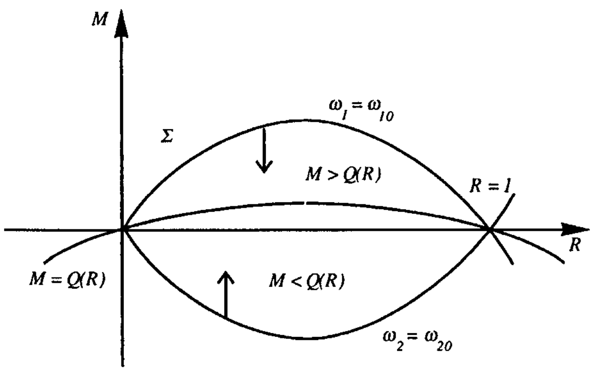

FIG. 1. The invariant region.

Since the graph of $M = Q(R)$ lies between the curves $M_1(R)$ and $M_2(R)$, then the vector field $(0, Q(R) - M)$ points inside Σ, and from [4] it follows that Σ is invariant for the non homogeneous system. ▮

The following result provides the existence of an entropy which is strictly convex close to the equilibrium curve and which allows to establish energy estimates needed to control the rate of convergence along the relaxation process. It plays the same role of the mechanical entropy for the Tartar-DiPerna arguments.

PROPOSITION 2.3. *Let us consider*

$$\bar{\eta} = \frac{1}{2}\left(\frac{m - Q(\rho)}{\rho}\right)^2 + g(\rho),$$

where $g''(\rho) = A(\rho) = \theta^2 \rho^{\gamma - 3} - a^2$; *moreover assume the stability condition* (SC) *given in* (1.7) *holds, then* $\bar{\eta}$ *is a* C^2 *positive entropy which satisfies*

$$D^2 \bar{\eta}(\rho, m) \geqslant 0$$

for all (ρ, m) *such that*

$$a\rho - \theta \rho^{(\gamma + 1)/2} \leqslant m \leqslant a\rho + \theta \rho^{(\gamma + 1)/2}.$$

Proof. The entropy equation for the system (2.2) in the $(\rho, u = m/\rho)$ variables is given by [9]:

$$\eta(\rho, u)_{\rho\rho} = \theta^2 \rho^{\gamma - 3} \eta(\rho, u)_{uu}. \tag{2.9}$$

Let us consider the following change of variables:

$$u = a(1 - \rho) + z.$$

In this new set of variables the previous equation becomes:

$$\eta(\rho, z)_{\rho\rho} + 2a\eta(\rho, z)_{\rho z} = A(\rho)\, \eta(\rho, z)_{zz}, \tag{2.10}$$

where $A(\rho) = \theta^2 \rho^{\gamma - 3} - a^2 > 0$ because of the stability condition (1.7). The Eq. (2.10) admits convex solutions, in the (ρ, z) variables, of the form:

$$\eta(\rho, z) = f(z) + g(\rho).$$

Thus, a convex solution in the (ρ, z) variables of (2.10) is given by

$$\bar{\eta}(\rho, z) = \tfrac{1}{2} z^2 + g(\rho),$$

where $g(\rho)$ is a function, which can be taken to be positive in $[0, 1]$, such that $g''(\rho) = A(\rho) > 0$. In order to study the convexity property in the (ρ, m) variables we compute:

$$\operatorname{Tr} D^2\bar{\eta}(\rho, m) = \bar{\eta}_{\rho\rho} + \bar{\eta}_{mm} = \theta^2 \rho^{\gamma - 3} + 3\,\frac{m^2}{\rho^4} - 2\,\frac{ma}{\rho^3} + \frac{1}{\rho^2}$$

$$= \theta^2 \rho^{\gamma - 3} + 2\,\frac{m^2}{\rho^4} + \frac{1}{\rho^2}\left(\frac{m}{\rho} - a\right)^2 + \frac{1 - a^2}{\rho^2},$$

and

$$\det D^2\bar{\eta}(\rho, m) = \bar{\eta}_{\rho\rho}\bar{\eta}_{mm} - \bar{\eta}^2_{\rho m} = \frac{1}{\rho^6}\,(\theta^2\rho^{\gamma + 1} - (m - a\rho)^2).$$

Therefore the function $\bar{\eta}$ is convex in the following domain

$$a\rho - \theta\rho^{(\gamma + 1)/2} \leqslant m \leqslant a\rho + \theta\rho^{(\gamma + 1)/2}. \quad \blacksquare \tag{2.11}$$

We shall denote by K_0, the set

$$K_0 = \{(\rho, m) \colon D^2\bar{\eta}(\rho, m) \geqslant 0\}. \tag{2.12}$$

The next result provides the basic energy estimate.

LEMMA 2.4. *Let* $(\rho^\varepsilon, m^\varepsilon)$ *be the solution of the parabolic system* (2.3) *with initial data* $(\rho_0, m_0) \in \Sigma$ *such that*

$$\int_{-\infty}^{+\infty} \bar{\eta}(\rho_0(x), m_0(x))\, dx < +\infty.$$

Finally, assume $(\rho^\varepsilon, m^\varepsilon)$ *belongs to the set* K_0 *defined in* (2.12) *and the stability condition* (1.7) *holds. Then*

$$\left\| \frac{Q(\rho^\varepsilon) - m^\varepsilon}{\rho^\varepsilon \sqrt{\tau}} \right\|_{L^2(\mathbb{R} \times [0, T])} \leqslant C, \tag{2.13}$$

where $C > 0$ *is independent from* ε, τ, *and* T.

Proof. From now on, we will omit the exponent ε. Let us consider the entropy $\bar{\eta}$ we have just found in the (ρ, m) variables:

$$\bar{\eta}(\rho, m) = \frac{1}{2}\left(\frac{m - Q(\rho)}{\rho}\right)^2 + g(\rho).$$

Since, denoting with $D^2\bar{\eta}[\rho_x, m_x]$ the quadratic form in (ρ_x, m_x) associated to the hessian matrix $D^2\bar{\eta}(\rho, m)$, one has

$$\bar{\eta}_t + \bar{q}_x = \varepsilon\bar{\eta}_\rho \rho_{xx} + \varepsilon\bar{\eta}_m m_{xx} + \frac{1}{\tau}\,\bar{\eta}_m(Q(\rho) - m)$$

$$= \varepsilon\bar{\eta}_{xx} - \varepsilon D^2\bar{\eta}[\rho_x, m_x] + \frac{1}{\tau}\,\bar{\eta}_m(Q(\rho) - m), \tag{2.14}$$

by integrating in x, t it follows

$$0 \leqslant \int_0^T \int_{-\infty}^{+\infty} \frac{(m - Q(\rho))^2}{\rho^2 \tau}\,dx\,dt \leqslant -\int_{-\infty}^{+\infty} \bar{\eta}(\rho(x, T), m(x, T))\,dx$$

$$+ \int_{-\infty}^{+\infty} \bar{\eta}(\rho_0(x), m_0(x))\,dx$$

$$\leqslant \int_{-\infty}^{+\infty} \bar{\eta}(\rho_0(x), m_0(x))\,dx \leqslant C. \quad \blacksquare$$

Remark 2.5. The estimate (2.13) and the L^∞ bound for ρ^ε imply that

$$\left\| \frac{Q(\rho^\varepsilon) - m^\varepsilon}{\sqrt{\tau}} \right\|_{L^2(\mathbb{R} \times [0, T])} = O(1),$$

uniformly in ε, τ and T. Therefore we have proved that $Q(\rho^\varepsilon) - m^\varepsilon$ tends to zero in $L^2(\mathbb{R} \times [0, T])$ as τ tends to zero with a rate which is equal to $\sqrt{\tau}$.

Now we want to examine a different approximation of the system (2.2): the physical viscosity approximation, namely the following system (one-dimensional Navier–Stokes system):

$$\begin{cases} \rho^\varepsilon_t + m^\varepsilon_x = 0 \\ m^\varepsilon_t + \left(\dfrac{(m^\varepsilon)^2}{\rho^\varepsilon} + p(\rho^\varepsilon) \right)_x = \dfrac{1}{\tau}\,(Q(\rho^\varepsilon) - m^\varepsilon) + \varepsilon \left(\dfrac{m^\varepsilon}{\rho^\varepsilon} \right)_{xx}. \end{cases} \tag{2.15}$$

In particular we show that the result of Lemma 2.4 can be obtained without the hypothesis $(\rho^\varepsilon, m^\varepsilon) \in K_0$. Indeed, let (ρ_0, m_0) be as in Lemma 2.4 and let $(\rho^\varepsilon, m^\varepsilon)$ be a solution of (2.15) with (ρ_0, m_0) as initial data. In this case, $(\rho^\varepsilon, m^\varepsilon)$ verifies a different entropy inequality, namely

$$\eta(\rho^\varepsilon, m^\varepsilon)_t + q(\rho^\varepsilon, m^\varepsilon)_x \leqslant \varepsilon(\nabla\eta(\rho^\varepsilon, m^\varepsilon)\,B(\rho^\varepsilon, m^\varepsilon)(\rho^\varepsilon_x, m^\varepsilon_x))_x$$

$$+ \frac{1}{\tau}\,\eta_m(\rho^\varepsilon_x, m^\varepsilon_x)(Q(\rho^\varepsilon) - m^\varepsilon), \tag{2.16}$$

for any η such that

$$D^2\eta(\rho, m)\; B(\rho, m) \geqslant 0, \tag{2.17}$$

where

$$B(\rho, m) = \begin{pmatrix} 0 & 0 \\ -\dfrac{m}{\rho^2} & \dfrac{1}{\rho} \end{pmatrix}.$$

In order to verify the condition (2.17) for the entropy $\bar{\eta}$, we observe that

$$\begin{aligned} \operatorname{Tr}(D^2\bar{\eta}(\rho, m)\; B(\rho, m)) &= \frac{1}{\rho^3}\left(\frac{2m^2}{\rho^2} - \frac{am}{\rho} + 1\right) \\ &= \frac{1}{\rho^3}\left[\left(\frac{m}{\rho} - \frac{a}{2}\right)^2 + 1 - \frac{a^2}{4}\right] + \frac{m^2}{\rho^5} \\ &\geqslant 0, \end{aligned} \tag{2.18}$$

when the stability condition (1.7) holds. Moreover, for any 2×2 matrix $M(\rho, m)$,

$$\det(M(\rho, m)\; B(\rho, m)) = 0 \tag{2.19}$$

and so $\bar{\eta}$ verifies (2.17). Hence, the inequality (2.16) holds for $\bar{\eta}$, for any (ρ, m), without the assumption $(\rho, m) \in K_0$. Therefore, integrating (2.16) in x and t as in Lemma 2.4, one can prove the following result:

LEMMA 2.6. *Let (ρ_0, m_0) be as in Lemma 2.4 and let $(\rho^\varepsilon, m^\varepsilon)$ be the solution of (2.15) with (ρ_0, m_0) as initial data. Assume $0 \leqslant \rho^\varepsilon$, $m^\varepsilon \in L^\infty$ uniformly in ε and τ and assume the stability condition (1.7) holds. Then*

$$\left\| \frac{Q(\rho^\varepsilon) - m^\varepsilon}{\rho^\varepsilon \sqrt{\tau}} \right\|_{L^2(\mathbb{R} \times [0, T])} \leqslant C, \tag{2.20}$$

where $C > 0$ is independent from ε, τ, and T.

Let us consider the mechanical entropy-flux pair $(\eta^*(\rho, m), q^*(\rho, m))$ for the system (2.2), namely

$$\eta^*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + \frac{h}{\gamma - 1}\,\rho^\gamma = \frac{1}{2}\frac{m^2}{\rho} + \sigma(\rho).$$

As usual, we have to prove an energy-type estimate for the hessian $D^2\eta^*$ in order to apply the compensated compactness techniques. This estimate must be independent on ε, but, at this stage, it can depend on τ.

LEMMA 2.7. *Let (ρ_0, m_0) be as in Lemma 2.4, such that*

$$\int_{-\infty}^{+\infty} \eta^*(\rho_0(x), m_0(x))\, dx < +\infty$$

and let $(\rho^\varepsilon, m^\varepsilon)$ be the solution of (2.3) with initial conditions (ρ_0, m_0) and assume the stability condition (1.7) holds. Then, for any fixed $\tau > 0$ and for any C_0^∞ function $\Phi(x) \geqslant 0$, we have

$$\varepsilon \int_0^T \int_{-\infty}^{+\infty} \Phi(x)\, D^2\eta^*[\rho_x^\varepsilon, m_x^\varepsilon]\, dx\, dt \leqslant C, \tag{2.21}$$

where the constant $C > 0$ depends only on T, τ and the initial conditions and it is independent from $0 < \varepsilon \leqslant 1$.

Proof. Multiplying the system (2.3) for a C_0^∞ function $\Phi(x) \geqslant 0$, we get:

$$\begin{aligned}(\Phi(x)\,\eta^*)_t + (\Phi(x)\, q^*)_x = {} & \varepsilon(\Phi(x)\,\eta_x^*)_x - \varepsilon\Phi(x)\, D^2\eta^*[\rho_x, m_x] + \Phi(x)'\, q^* \\ & - (\varepsilon\Phi(x)\,\eta^*)_x + \varepsilon\Phi(x)''\,\eta^*. \end{aligned} \tag{2.22}$$

Integrating (2.22) on x and t, in view of the convexity and the positiveness of η^*, the positiveness of Φ and using the L^∞ bounds for the solution, one has:

$$\begin{aligned} 0 & \leqslant \varepsilon \int_0^T \int_{-\infty}^{+\infty} \Phi(x)\, D^2\eta^*[\rho_x, m_x]\, dx\, dt \\ & \leqslant \int_{-\infty}^{+\infty} \Phi(x)\, \eta^*(\rho(0,x), m(0,x))\, dx + \int_0^T \int_{-\infty}^{+\infty} \Phi'(x)\, q^*\, dx\, dt \\ & \quad + \varepsilon \int_0^T \int_{-\infty}^{+\infty} \Phi(x)''\, \eta^*\, dx\, dt \\ & \quad + \frac{1}{\tau} \int_0^T \int_{-\infty}^{+\infty} \Phi(x)\, \frac{m}{\rho}\, (Q(\rho) - m)\, dx\, dt \\ & \leqslant C, \end{aligned} \tag{2.23}$$

where $C > 0$ is a positive constant independent from $0 < \varepsilon \leqslant 1$ (but depending on τ and T). ∎

The estimate (2.21) provides a "localized" version of the standard energy estimate, namely, let K be a compact subset of $\mathbb{R}$ and let Φ be any non-negative, C_0^∞ function such that $\Phi \equiv 1$ on K, it follows

$$\varepsilon \int_0^T \int_K D^2\eta^*[\rho_x, m_x]\, dx\, dt \leqslant \varepsilon \int_0^T \int_{-\infty}^{+\infty} \Phi(x)\, D^2\eta^*[\rho_x, m_x]\, dx\, dt$$

$$\leqslant C, \tag{2.24}$$

where the constant C is the above positive constant.

Remark 2.8. It is possible to prove a global version of the estimate (2.21) in the case of solutions $(\rho^\varepsilon, m^\varepsilon)$ of (2.3) which belong in the set K_0 defined in (2.12), by using the estimate (2.13), namely

$$\left\| \frac{Q(\rho)-m}{\rho\sqrt{\tau}} \right\|_{L^2(\mathbb{R}\times[0,T])} \leqslant C.$$

As usual, one has

$$\eta_t^* + q_x^* = \varepsilon \eta_{xx}^* - \varepsilon D^2\eta^*[\rho_x, m_x] + \frac{1}{\tau}\, \eta_m^*(Q(\rho)-m), \tag{2.25}$$

hence, by integrating in x and t, it follows

$$\int_{-\infty}^{+\infty} \eta^*(\rho(T,x), m(T,x))\, dx - \int_{-\infty}^{+\infty} \eta^*(\rho(0,x), m(0,x))\, dx$$

$$\leqslant \frac{1}{\tau} \int_0^T \int_{-\infty}^{+\infty} \frac{m}{\rho}\, (Q(\rho)-m)\, dx\, dt$$

$$\leqslant \frac{1}{\sqrt{\tau}} \left(\int_0^T \int_{-\infty}^{+\infty} \frac{m^2}{\rho}\, dx\, dt \right)^{1/2} \left(\frac{1}{\tau} \int_0^T \int_{-\infty}^{+\infty} \frac{(Q(\rho)-m)^2}{\rho}\, dx\, dt \right)^{1/2}$$

$$= O(1)\, \frac{1}{\sqrt{\tau}} \left(\int_0^T \int_{-\infty}^{+\infty} \frac{m^2}{\rho}\, dx\, dt \right)^{1/2}. \tag{2.26}$$

Therefore, let us define

$$F(T) = \int_0^T \int_{-\infty}^{+\infty} \eta^*(\rho, m)\, dx\, dt,$$

then

$$F'(T) = \frac{O(1)}{\sqrt{\tau}}\,(1 + F(T)^{1/2}).$$

As a consequence of this fact we have

$$1 + F(T) = \frac{O(1)}{\sqrt{\tau}}$$

and moreover

$$\varepsilon \int_0^T \int_{-\infty}^{+\infty} D^2\eta^*[\rho_x, m_x]\,dx\,dt + F(T) = \frac{O(1)}{\sqrt{\tau}}.$$

Now we can prove the main theorem of the section.

Proof of Theorem 2.1. At this point we are able to apply the standard compensated compactness arguments to show the strong convergence of the approximating sequence $(\rho^\varepsilon, m^\varepsilon)$. Indeed, $(1/\tau)\,\eta^*_m(\rho, m)(Q(\rho) - m) \in L^\infty$ for any fixed $\tau > 0$, then (2.24) allows us to conclude

$$\eta(\rho^\varepsilon, m^\varepsilon)_t + q(\rho^\varepsilon, m^\varepsilon)_x \in \text{comp } H^{-1}_{\text{loc}},$$

for all the entropy-entropy flux pairs (η, q), where η is a "weak entropy," namely, any η which satisfies (2.9) with $\eta(0, u) = 0$ and $\eta_\rho(0, u)$ prescribed (see [5, 9]). By using the compensated compactness methods, we get the strong convergence $\rho^\varepsilon \to \rho$ and $m_\varepsilon \to m$ in L^p_{loc} for every $p < +\infty$. Thus, by the same arguments used in [3, 5, 9], we get the existence result established in Theorem 2.1. ∎

3. ZERO RELAXATION LIMIT

In this section we want to study the behavior of the system (2.2) as the positive parameter τ tends to zero. In particular we are interested to the study of the limit function $\rho(x, t)$ and to show it is a weak solution of the relaxed conservation law

$$\rho_t + Q(\rho)_x = 0. \tag{3.1}$$

To do this, we restrict ourself to the solutions (ρ, m) of (2.2) which verify the following condition:

(K) $(\rho, m) \in \bar{K} \subset\subset K_0$, for a fixed $\bar{K}$.

Because of the estimate (2.13) obtained in the previous section, we get the following lemma, which provides the estimate to control the relaxation process:

LEMMA 3.1. *Let* (ρ_0, m_0) *be as in Theorem* 2.1 *and let us denote by* (ρ, m) *an entropy solution of the system* (2.2) *with* (ρ_0, m_0) *as initial condition. Assume the stability condition* (1.7) *and the condition* (K) *hold. Then*

$$\left\| \frac{Q(\rho) - m}{\rho} \right\|_{L^2(\mathbb{R} \times [0, T])} = O(1) \sqrt{\tau}. \tag{3.2}$$

The convergence result obtained in this section is close to the one proved in [1] (Theorem 4.1), however in our case we do not have any smallness condition. Clearly this type of convergence results do not provide any information concerning the entropy conditions and the uniqueness, which we will investigate in the following sections.

Remark 3.2. Let $\rho_0(x)$, $m_0(x)$ be L^∞ functions such that $\rho_0(x) \geqslant \hat{\rho}_0 > 0$ and $\rho_0(x)$, $m_0(x)$ approach the constant states $\bar{\rho}$, $\bar{m}$ at infinity and let us consider the functions $g(\rho)$ and $\sigma(\rho)$ appearing in the definition of $\bar{\eta}$ and η^*. Even in this case we can assume

$$\int_{-\infty}^{+\infty} \bar{\eta}(\rho_0(x), m_0(x))\, dx < +\infty$$

and

$$\int_{-\infty}^{+\infty} \eta^*(\rho_0(x), m_0(x))\, dx < +\infty,$$

just subtracting the linear part of g and σ at $\bar{\rho}$. We note that the "modified" functions will still be entropies, since linear perturbations do not affect the entropy equation and, due to the convexity of g and σ, they will still be positive.

THEOREM 3.3. *Let* (ρ^τ, m^τ) *be an entropy solution of* (2.2) *with initial data* $(\rho_0, m_0) \in \Sigma$ *such that* $\rho_0 \geqslant \hat{\rho}_0$ *and such that*

$$\int_{-\infty}^{+\infty} \eta^*(\rho_0(x), m_0(x)\, dx < +\infty$$

and

$$\int_{-\infty}^{+\infty} \bar{\eta}(\rho_0(x), m_0(x))\, dx < +\infty$$

and assume (ρ_0, m_0) *approaches the constant state* $(\bar{\rho}, \bar{m})$ *at infinity.*

Assume $\rho^\tau \geqslant \hat{\rho} > 0$ and the stability condition (1.7) *and the condition* (K) *hold. Then, extracting if necessary a subsequence, $\rho^\tau \to \rho$ as $\tau \downarrow 0$ strongly in L^p_{loc}, for all $p < +\infty$. Moreover, the limit function $\rho(x,t)$ turns out to be a weak solution of the relaxed conservation law*

$$\begin{cases} \rho_t + Q(\rho)_x = 0 \\ \rho(x,0) = \rho_0(x). \end{cases} \tag{3.3}$$

Proof. To achieve our goal it suffices to show (see, for instance, [2]) that

$$\rho_t + Q(\rho)_x \in \operatorname{comp} H^{-1}_{\mathrm{loc}}$$

and

$$Q(\rho)_t + \left(\int^{\rho} (Q'(s))^2 \, ds \right)_x \in \operatorname{comp} H^{-1}_{\mathrm{loc}}.$$

Let $\psi(\rho)$ be the entropy-flux associated to the entropy $g(\rho)$ of the relaxed conservation law (3.3). We have

$$\begin{aligned} g(\rho)_t + \psi(\rho)_x \leqslant{} & \bar{\eta}(\rho, Q(\rho))_t - \bar{\eta}(\rho, m)_t \\ & + \bar{q}(\rho, Q(\rho))_x - \bar{q}(\rho, m)_x \\ & + \frac{1}{\tau} (\bar{\eta}_m(\rho, m) - \bar{\eta}_m(\rho, Q(\rho)))(Q(\rho) - m) \\ I_1^\tau + I_2^\tau + I_3^\tau, \end{aligned}$$

where $\bar{q}(\rho, m)$ is the flux associated with the entropy $\bar{\eta}(\rho, m)$ of the system (2.2). By using the energy estimate (3.2) and the L^∞ estimate, we have

$$\begin{aligned} \|I_1^\tau\|_{H^{-1}_{\mathrm{loc}}} = & \sup_{\Phi \in H^1_0,\ \|\Phi\|_{H^1_0} \leqslant 1} \left| \int_{-\infty}^{+\infty} \int_0^T (\bar{\eta}(\rho, Q(\rho)) - \bar{\eta}(\rho, m))_t \, \Phi \, dx \, dt \right| \\ = & O(1) \left\| \frac{Q(\rho) - m}{\rho} \right\|_{L^\infty} \left\| \frac{Q(\rho) - m}{\rho} \right\|_{L^2} \|\Phi_t\|_{L^2} = O(1) \sqrt{\tau} \end{aligned}$$

and similarly

$$\|I_2^\tau\|_{H^{-1}_{\mathrm{loc}}} = O(1) \sqrt{\tau}.$$

Moreover, from (3.2)

$$\| I_3^\tau \|_{L^1} = O(1),$$

then $\{I_3^\tau\}$ is relatively compact in H_{loc}^{-1} by the Murat Lemma [12]. Therefore we can conclude that

$$g(\rho)_t + \psi(\rho)_x \in \text{comp } H_{\text{loc}}^{-1}.$$

Let $\varphi(\rho)$ be any function in C_0^∞ such that $\varphi(\rho) = Q(\rho)$ for $\rho \in [\hat{\rho}, 1]$. Then there exists a C^∞ entropy $\tilde{\eta}(\rho, m)$ of the system (2.2) such that

$$\begin{cases} \tilde{\eta}(\rho, 0) = \varphi(\rho) \\ \tilde{\eta}_z(\rho, 0) = 0 \end{cases}$$

(see appendix).

Let $\hat{\eta}(\rho, m) = \bar{\eta}(\rho, m) - \mu \tilde{\eta}(\rho, m)$. Hence, by condition (K), it follows that for a sufficiently small constant $\mu \neq 0$, $\hat{\eta}(\rho, m)$ remains strictly convex.

Moreover $\hat{\eta}_m(\rho, Q(\rho)) = \bar{\eta}_m(\rho, Q(\rho)) = 0$. By repeating the previous argument, we get

$$(g(\rho) - \mu Q(\rho))_t + \left(\psi(\rho) - \mu \int^\rho (Q'(s))^2 \, ds \right)_x \in \text{comp } H_{\text{loc}}^{-1}.$$

Therefore it follows

$$Q(\rho)_t + \left(\int^\rho (Q'(s))^2 \, ds \right)_x \in \text{comp } H_{\text{loc}}^{-1}.$$

We can show that

$$\rho_t + Q(\rho)_x = (Q(\rho) - m)_x \in \text{comp } H_{\text{loc}}^{-1}$$

in a similar way. Then the convergence result is a standard argument (see [2]). ∎

Now we turn to study another kind of weak entropy solutions to the system (2.2), namely the solutions given by the limit of the physical viscosity approximation. If we suppose that these solutions exist, it is possible to show a global convergence result as $\tau \downarrow 0$. Indeed, from Lemma 2.6, it follows that in this situations the energy estimate (3.2) is valid without the assumption (K). Therefore, the convergence result can be proved for all solutions $\rho, m \in L^\infty$ with $0 < \hat{\rho} \leqslant \rho$.

THEOREM 3.4. *Let* (ρ_0, m_0) *be as in Theorem* 3.3 *and let* (ρ^τ, m^τ) *be a weak solution of* (2.2), *given by the limit, as* $\varepsilon \downarrow 0$ *of the solutions of the physical viscosity system* (2.15), *with* ρ_0, m_0 *as initial data. Assume* ρ^τ, m^τ *are bounded in* L^∞ *uniformly with respect to* τ *and* $\rho^\tau \geqslant \hat{\rho} > 0$. *Finally, assume the stability condition* (1.7) *holds. Then, extracting if necessary a subsequence,* $\rho^\tau \to \rho$ *as* $\tau \downarrow 0$ *strongly in* L^p_{loc}, *for all* $p < +\infty$. *Moreover, the limit function* $\rho(x, t)$ *turns out to be a weak solution of the relaxed conservation law*

$$\begin{cases} \rho_t + Q(\rho)_x = 0 \\ \rho(x, 0) = \rho_0(x). \end{cases}$$

Proof. As in Theorem 3.3, we will show that

$$\rho_t + Q(\rho)_x \in \text{comp } H^{-1}_{\mathrm{loc}}$$

and

$$Q(\rho)_t + \left(\int^{\rho} (Q'(s))^2 \, ds \right)_x \in \text{comp } H^{-1}_{\mathrm{loc}}.$$

Since the energy estimate (3.2) is true for any solution (ρ, m), then, proceeding as in the proof of the previous theorem, we get

$$g(\rho)_t + \psi(\rho)_x \in \text{comp } H^{-1}_{\mathrm{loc}}.$$

Let $\tilde{\eta}$ be the same entropy considered above and let $\hat{\eta}(\rho, m) = \bar{\eta}(\rho, m) - \mu\tilde{\eta}(\rho, m)$. From the relations (2.18) and (2.19) established in Section 2, it follows that for a sufficiently small constant $\mu \neq 0$, depending only on $\hat{\rho}$ and on the L^∞ bounds of the solutions,

$$\mathrm{Tr}(D^2\hat{\eta}(\rho, m)\, B(\rho, m)) \geqslant 0.$$

Hence,

$$D^2\hat{\eta}(\rho, m)\, B(\rho, m)) \geqslant 0.$$

Therefore, by repeating the previous arguments for the entropy $\hat{\eta}$, one has

$$(g(\rho) - \mu Q(\rho))_t + \left(\psi(\rho) - \mu \int^{\rho} (Q'(s))^2 \, ds \right)_x \in \text{comp } H^{-1}_{\mathrm{loc}}.$$

The last part of the proof is just like that one of the Theorem 3.3 and it will be omitted. ∎

4. ENTROPY CONDITIONS

In this section we want to prove the validity of the Kružkov-type entropy conditions for the solution $\rho(x, t)$ with respect to the relaxed conservation law (3.1). The entropy inequality of the scalar conservation law will be obtained by extending in the (ρ, m) plane one parameter family Kružkov-type entropies. Hence the entropy inequality for the relaxed problem will follow from the entropy inequality of the relaxing system. Therefore we are required to extend the Kružkov-type entropies in such a way to preserve both the convexity and the dissipative properties. The nontrivial fact in this extension procedure is to find a suitable domain of the extended entropies where the previous properties hold, uniformly with respect to k. Moreover, since the relaxation limit is given in terms of L^p norms, $p < +\infty$, we must require that our solutions are confined in such a domain, uniformly in τ. We solve the former difficulty by using the results of the Appendix, but we are forced to assume that our solutions stay away from the vacuum. While the latter difficulty can be solved only in the case where $\mathscr{BV}$ estimates are available, which is not the case of our framework. In the paper [1] the authors do not investigate this problem. The following definition will be useful in what follows.

Definition 4.1. Let φ^k, $k \in [a, b]$ be a sequence of C_0^∞ functions and let η^k be the entropy of the system (2.2) which verifies

$$\begin{cases} \eta^k(\rho, Q(\rho)) = \varphi^k(\rho) \\ \eta_m^k(\rho, Q(\rho)) = 0. \end{cases}$$

We say that a sequence (ρ, m) of exact (or approximate) solutions of (2.2) is φ^k-stable if $(\rho, m) \in \Omega$, where Ω is a closed neighborhood (with nonempty interior) of the equilibrium curve $\{(\rho, m): m = Q(\rho)\}$ where $\eta^k(\rho, m)$ is strictly convex and verifies $\eta_{mm}^k(\rho, m) > 0$ for any $k \in [a, b]$. (This definition is well-posed in view of the results of the Appendix.)

Theorem 4.2. *Let $\varphi^k(\rho)$ be a C_0^∞ approximation of the Kružkov entropy $|\rho - k|$ (see the Appendix). Let (ρ^τ, m^τ) be a φ^k-stable sequence of entropy solutions of* (2.2) *with (ρ_0, m_0) as initial data. Assume $(\rho_0, m_0) \in \Sigma$ and $\rho^\tau \geqslant \hat{\rho} > 0$ and assume the condition* (K) *holds. Finally, assume (ρ_0, m_0) verifies:*

$$\int_{-\infty}^{+\infty} \eta^*(\rho_0(x), m_0(x))\, dx < +\infty$$

and

$$\int_{-\infty}^{+\infty} \bar{\eta}(\rho_0(x), m_0(x))\, dx < +\infty.$$

Moreover, assume (ρ_0, m_0) *approaches the constant state* $(\bar{\rho}, \bar{m})$ *at infinity and assume the stability condition* (1.7) *holds. Then the limit function* $\rho(x, t)$ *verifies*

$$\varphi^k(\rho)_t + \psi^k(\rho)_x \leqslant 0 \qquad \text{in } \mathcal{D}', \tag{4.1}$$

for every $k \in [\hat{\rho}, 1]$, *where* $\psi^k(\rho) = \int^\rho (\varphi^k(s))' \, Q'(s) \, ds$.

Proof. From the results of the Appendix, it follows that there exist C^∞ entropies $\eta^k(\rho, m)$ (with fluxes $q^k(\rho, m)$) for the system (2.2) provided by the equation (2.10) with the following initial conditions:

$$\begin{cases} \eta^k(\rho, 0) = \varphi^k(\rho) \\ \eta^k_z(\rho, 0) = 0. \end{cases}$$

Moreover, there exists a closed neighborhood (with nonempty interior) $A_{\hat{\rho}}$ of the curve $\{(\rho, m): m = Q(\rho)\}$ in which every $\eta^k(\rho, m)$ is strictly convex and verifies $\eta^k_{mm}(\rho, m) > 0$. For $k \in [\hat{\rho}, 1]$ and $(\rho, m) \in A_{\hat{\rho}}$, we have

$$\begin{aligned} &\varphi^k(\rho^\tau)_t + \psi^k(\rho^\tau)_x \\ &\quad \leqslant \eta^k(\rho^\tau, Q(\rho^\tau))_t - \eta^k(\rho^\tau, m^\tau)_t + q^k(\rho^\tau, Q(\rho^\tau))_x - q^k(\rho^\tau, m^\tau)_x \\ &\qquad + \frac{1}{\tau} (\eta^k_m(\rho^\tau, m^\tau) - \eta^k_m(\rho^\tau, Q(\rho^\tau)))(Q(\rho^\tau) - m^\tau) \\ &\quad = I_1 + I_2 + I_3. \end{aligned}$$

As before, from the condition (K) and the L^∞ bounds, I_1 and I_2 tend to zero in H^{-1}_{loc} and

$$I_3 = -\frac{1}{\tau} \eta^k_{mm}(\rho^\tau, \bar{m}^\tau)(Q(\rho^\tau) - m^\tau)^2 \leqslant 0$$

which concludes the proof. ∎

It is possible to show a similar result for the solution ρ of (3.3) obtained via the limit, as $\tau \downarrow 0$, of the weak solutions (ρ^τ, m^τ) of (2.2) which satisfy the zero Navier–Stokes viscosity entropy condition (see Section 3, Theorem 3.4). To apply the entropy inequality, in this case we do not need the convexity of the entropies η^k, but the so-called B-convexity (see (2.17)).

By (2.18) and (2.19), the point is to control, uniformly on k, the quantity

$$\operatorname{Tr}(D^2 \eta^k(\rho, m) \, B(\rho, m)).$$

It can be done by using similar arguments to those one used in the Appendix (cf. Proposition A.3). Therefore there exists a closed neighborhood (with nonempty interior) $O_{\hat{\rho}}$, depending on $\hat{\rho}$, but not on k, where η^k verifies $\eta^k_{mm}(\rho, m) > 0$ and

$$D^2\eta^k(\rho, m)\, B(\rho, m) \geqslant 0.$$

Hence, the following theorem holds:

THEOREM 4.3. *Let $\varphi^k(\rho)$ and (ρ_0, m_0) be as in Theorem 4.2 and let (ρ^τ, m^τ) be a weak solution of the system (2.2) given by the limit, as $\varepsilon \downarrow 0$ of the solution of the physical viscosity system (2.15). Assume ρ^τ, m^τ are bounded in L^∞ uniformly with respect to τ, $\rho^\tau \geqslant \hat{\rho} > 0$ and $(\rho^\tau, m^\tau) \in O_{\hat{\rho}}$. Finally, assume the stability condition (1.7) holds. Then the limit function $\rho(x, t)$ verifies*

$$\varphi^k(\rho)_t + \psi^k(\rho)_x \leqslant 0 \qquad \text{in} \quad \mathscr{D}',$$

for every $k \in [\hat{\rho}, 1]$, where $\psi^k(\rho) = \int^\rho (\varphi^k(s))'\, Q'(s)\, ds$.

This theorem shows in particular that the relaxation process provides always the same type of entropy conditions for the limit solution, both if one starts from the standard entropy solutions of (2.2) (given, for example, by the artificial viscosity approximation), and if one starts from solutions of (2.2) given by the physical viscosity approximation (the Navier–Stokes system), solutions which verify different entropy conditions.

5. UNIQUENESS

This section is devoted to the study of the uniqueness of the limit function $\rho(x, t)$. To this end, we assume the following hypothesis:

(E) the limit function $\rho(x, t)$ verifies

$$|\rho - k|_t + F^k(\rho)_x \leqslant 0 \qquad \text{in} \quad \mathscr{D}',$$

for every $k \in [\hat{\rho}, 1]$, where $F^k(\rho) = \int^\rho \operatorname{sign}(s-k)\, Q'(s)\, ds$.

It is well known (see, for instance [8]) that the Kružkov entropy conditions (E) are not enough to establish the uniqueness of the solution. Indeed, we need to supplement also a generalized L^1_{loc} continuity of the semigroup in $t = 0$. The connection between this kind of continuity property of the semigroup and the uniqueness has been analyzed very clearly in the papers of Kružkov [8], DiPerna [6] and Szepessy [15]. In any case the key point is to show some kind of contraction property in a suitable metric space (e.g. some L^p space). Of course this is natural once a similar property holds for the approximating problems (for instance this happens with the vanishing viscosity), but it is absolutely non-trivial when the approximating problem is very stiff, as in the case of the relaxation model under consideration.

The crucial difficulty in the analysis of the relaxation limit is the formation of an initial layer, unless we assume a compatibility condition on the initial data. This layer is created since in general $m_0(x)$ is different from $Q(\rho_0(x))$. This phenomenon generates an extra entropy production given by

$$J_0 = -\int_{-\infty}^{+\infty} \Phi(x)(\eta(\rho_0(x), Q(\rho_0(x))) - \eta(\rho_0(x), m_0(x)))\, dx,$$

which is nonnegative for any test function $\Phi \geqslant 0$ and for any convex entropy η, such that $\eta_m(\rho, Q(\rho)) = 0$.

In order to have the minimal contractivity necessary to implement the classical analysis leading to the uniqueness, we need to estimate J_0 in terms of the internal trace of the usual entropy production. Namely, if we denote by μ the weak limit in $L_1([0, T] \times \mathbb{R})$ (as $\varepsilon, \tau \downarrow 0$) of the sequence

$$\left\{ \varepsilon D^2 \eta v[\rho_x, m_x] + \frac{1}{\tau} \eta_m(\rho, m)(m - Q(\rho)) \right\},$$

It would be sufficient to show

$$J_0 - \frac{1}{T} \int_0^T \langle \mu, \chi_{[0, t]} \Phi(x) \rangle\, dt = o(1)$$

as $T \downarrow 0$. In our case we prove the uniqueness result under the compatibility condition $m_0 = Q(\rho_0)$, when no initial layer appears.

LEMMA 5.1. *Let $\varphi(\rho)$ be any C_0^∞ function equal to $|\rho - \bar{\rho}|$ in $[\hat{\rho}, 1]$ and let (ρ_0, m_0) be as in Theorem* 4.2. *Let (ρ^τ, m^τ) be a sequence of entropy solutions of* (2.2) (*with (ρ_0, m_0) as initial data*) *such that $\rho^\tau \geqslant \hat{\rho} > 0$ and assume the condition* (K) *holds. Moreover, assume $(\rho_0 - \bar{\rho}) \in L^1$ and $m_0 = Q(\rho_0)$ and assume the solution ρ^τ of the system* (2.2) *is the limit, as ε tends to zero, of the solutions $\rho^{\varepsilon, \tau}$ of the system* (2.3). *Finally, assume the stability condition* (1.7) *holds and assume the sequence $(\rho^{\varepsilon, \tau}, m^{\varepsilon, \tau})$ is ρ-stable* (*namely, we take $\varphi^k \equiv \varphi$ in the Definition* 4.1). *Then, for any nonnegative test function $\Phi(x)$, it follows:*

$$\int_0^T \int_{-\infty}^{+\infty} \Phi(x)\, |\rho(x, t) - \bar{\rho}|^2\, dx\, dt$$

$$\leqslant \int_0^T \int_{-\infty}^{+\infty} \Phi(x)\, |\rho_0(x) - \bar{\rho}|^2\, dx\, dt + O(1)\, T^2, \qquad (5.1)$$

uniformly in ε, τ.

Proof. It suffices to prove the relation 5.1 with $\varphi(\rho)$ instead of $|\rho - \bar{\rho}|^2$. Let $\Phi(x)$ be a nonnegative test function and let $(\eta(\rho, m), q(\rho, m))$ the related entropy-entropy flux pair with initial data

$$\begin{cases} \eta(\rho, Q(\rho)) = \varphi(\rho) \\ \eta_m(\rho, Q(\rho)) = 0. \end{cases}$$

Then, one has

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \Phi(x)\, \eta(\rho(x,t), m(x,t))\, dx - \int_{-\infty}^{+\infty} \Phi(x)\, \eta(\rho_0(x), m_0(x))\, dx \\
&= \frac{1}{\tau} \int_0^t \int_{-\infty}^{+\infty} \Phi(x)\, \eta_m(\rho(x,s), m(x,s))(Q(\rho(x,s)) - m(x,s))\, dx\, ds \\
&\quad - \varepsilon \int_0^t \int_{-\infty}^{+\infty} \Phi(x)\, D^2\eta[\rho_x, m_x]\, dx\, ds \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} \Phi'(x)\, q(\rho(x,s), m(x,s))\, dx\, ds \\
&\quad + \varepsilon \int_0^t \int_{-\infty}^{+\infty} \Phi''(x)\, \eta(\rho(x,s), m(x,s)\, dx\, ds \\
&\leqslant \int_0^t \int_{-\infty}^{+\infty} \Phi'(x)\, q(\rho(x,s), m(x,s))\, dx\, ds \\
&\quad + \varepsilon \int_0^t \int_{-\infty}^{+\infty} \Phi''(x)\, \eta(\rho(x,s), m(x,s))\, dx\, ds. \qquad (5.2)
\end{aligned}$$

Therefore, letting $\varepsilon \downarrow 0$ in (5.2) and integrating over $[0, T]$ one has

$$\begin{aligned}
&\int_0^T \int_{-\infty}^{+\infty} \Phi(x)\, \eta(\rho(x,t), m(x,t))\, dx\, dt \\
&\quad - \int_0^T \int_{-\infty}^{+\infty} \Phi(x)\, \eta(\rho_0(x), m_0(x))\, dx\, dt \leqslant C\, \|q\|_{L^\infty}\, T^2.
\end{aligned}$$

Hence, as $\tau \downarrow 0$, from the energy estimate (3.2) and from the relation $m_0 = Q(\rho_0)$, it follows

$$\begin{aligned}
&\int_0^T \int_{-\infty}^{+\infty} \Phi(x)\, \varphi(\rho(x,t))\, dx\, dt - \int_0^T \int_{-\infty}^{+\infty} \Phi(x)\, \varphi(\rho_0(x))\, dx\, dt \\
&\quad \leqslant C\, \|q\|_{L^\infty}\, T^2. \quad \blacksquare \qquad (5.3)
\end{aligned}$$

Remark 5.2. From the relation (5.1) we get also

$$\int_0^T \int_K |\rho(x,t)-\bar{\rho}|^2\,dx\,dt \leqslant \int_0^T \int_{-\infty}^{+\infty} |\rho_0(x)-\bar{\rho}|^2\,dx\,dt + O(1)\,T^2, \tag{5.4}$$

for every compact $K \subset \mathbb{R}$. Indeed, let $0 \leqslant \Phi(x) \leqslant 1$ be a C_0^∞ function such that $\Phi \equiv 1$ in K. Then (5.1) yields to the inequalities

$$\begin{aligned}\int_0^T \int_K |\rho(x,t)-\bar{\rho}|^2\,dx\,dt &\leqslant \int_0^T \int_{-\infty}^{+\infty} \Phi(x)\,|\rho(x,t)-\bar{\rho}|^2\,dx\,dt\\ &\leqslant \int_0^T \int_{-\infty}^{+\infty} \Phi(x)\,|\rho_0(x)-\bar{\rho}|^2\,dx\,dt + O(1)\,T^2\\ &\leqslant \int_0^T \int_{-\infty}^{+\infty} |\rho_0(x)-\bar{\rho}|^2\,dx\,dt + O(1)\,T^2.\end{aligned}$$

With the aid of (5.4) we can show the generalized L^1-continuity of $\rho(x,t)$ as $t \downarrow 0$, which is the last property needed to establish the uniqueness of the solution to (3.3).

THEOREM 5.3. *Let us assume the same hypothesis of Lemma* 5.1. *Then we have*

$$\lim_{T \to 0^+} \frac{1}{T} \int_0^T \int_K |\rho(x,t)\,\rho_0(x)|\,dx\,dt = 0, \tag{5.5}$$

for any compact $K \subset \mathbb{R}$. *In particular, if the condition* (E) *holds,* $\rho(x,t)$ *is the unique entropy solution* (*greater or equal than* $\hat{\rho}$) *of* (3.1) *with* $\rho_0(x)$ *as initial condition.*

Proof. To prove the relation (5.5) we proceed as in [15]. Let $d(\rho) = |\rho-\bar{\rho}|^2$ and let $D(\rho,\rho_0) = d(\rho) - d(\rho_0) - d'(\rho_0)(\rho-\rho_0) = (\rho-\rho_0)^2$. For any compact $K \subset \mathbb{R}$, the Schwarz inequality yields

$$\begin{aligned}\int_K |\rho-\rho_0|\,dx &\leqslant C_K \left(\int_K (\rho-\rho_0)^2\,dx \right)^{1/2}\\ &= C_K \left(\int_K D(\rho,\rho_0)\,dx \right)^{1/2}.\end{aligned}$$

Thus, from the Jensen's inequality it follows that

$$\frac{1}{T} \int_0^T \int_K |\rho-\rho_0|\,dx\,dt \leqslant C_K \left(\frac{1}{T} \int_0^T \int_K D(\rho,\rho^0)\,dx\,dt \right)^{1/2}. \tag{5.6}$$

Now, let f_n be a C_0^∞ approximation in L^1 of $d'(\rho_0)$. From (5.4) one has

$$\begin{aligned}\int_0^T\int_K D(\rho,\rho_0)\,dx\,dt \leqslant & \int_0^T\int_K f_n(\rho_0-\rho)\,dx\,dt \\ & + T(\|\rho\|_{L^\infty}+\|\rho_0\|_{L^\infty})\,\|d'(\rho_0)-f_n\|_{L^1} \\ & + \int_0^T\int_K |\rho-\bar{\rho}|^2\,dx\,dt \\ & - \int_0^T\int_K |\rho_0-\bar{\rho}|^2\,dx\,dt \\ \leqslant & \int_0^T\int_K f_n(\rho_0-\rho)\,dx\,dt \\ & + T(\|\rho\|_{L^\infty}+\|\rho_0\|_{L^\infty})\,\|d'(\rho_0)-f_n\|_{L^1} \\ & + \int_0^T\int_{\mathbb{R}-K} |\rho_0-\bar{\rho}|^2\,dx\,dt + CT^2.\end{aligned}$$

Let $\{K_i\}$ be a collection of compact set such that $K_i\subset K_{i+1}$ for every i, $K\subset K_1$ and $\bigcup_{i=1}^\infty K_i=\mathbb{R}$. Now, if we repeat the previous argument for all the K_i, since

$$\int_0^T\int_K D(\rho,\rho_0)\,dx\,dt \leqslant \int_0^T\int_{K_i} D(\rho,\rho_0)\,dx\,dt,$$

we get that

$$\begin{aligned}\frac{1}{T}\int_0^T\int_K D(\rho,\rho_0)\,dx\,dt \leqslant & \frac{1}{T}\int_0^T\int_{-\infty}^{+\infty} f_n(\rho_0-\rho)\,dx\,dt \\ & + (\|\rho\|_{L^\infty}+\|\rho_0\|_{L^\infty})\,\|d'(\rho_0)-f_n\|_{L^1} \\ & + CT. \end{aligned} \tag{5.7}$$

Hence, from (5.6) and (5.7) it follows that (5.5) is proved if

$$\lim_{T\to 0^+}\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty} f_n(\rho_0-\rho)\,dx\,dt=0. \tag{5.8}$$

Let $(\rho^{\varepsilon,\tau}, m^{\varepsilon,\tau})$ be the solution of the system (2.3). Now,

$$\begin{aligned}\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty} f_n(x)(\rho_0-\rho^{\varepsilon,\tau})\,dx\,dt &= -\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty}\int_0^t \rho_s^{\varepsilon,\tau} f_n(x)\,ds\,dx\,dt\\ &=\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty}\int_0^t (m_x^{\varepsilon,\tau}-\varepsilon\rho_{xx}^{\varepsilon,\tau}) f_n(x)\,ds\,dx\,dt\\ &=-\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty}\int_0^t (m^{\varepsilon,\tau}f_n'(x)+\varepsilon\rho^{\varepsilon,\tau}f_n''(x))\,ds\,dx\,dt\\ &\leqslant C_n T.\end{aligned}$$

Finally, from this last relation we get

$$\begin{aligned}\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty} f_n(\rho_0-\rho)\,dx\,dt &= \lim_{\varepsilon,\tau\to 0^+}\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty} f_n(\rho_0-\rho^{\varepsilon,\tau})\,dx\,dt\\ &\leqslant C_n T,\end{aligned}$$

which proves (5.8). ∎

APPENDIX: THE ENTROPY EQUATION

In this appendix we want to investigate some properties of the linear hyperbolic equation

$$\eta(\rho, z)_{\rho\rho}+2a\eta(\rho, z)_{\rho z}=A(\rho)\,\eta(\rho, z)_{zz}, \tag{A.1}$$

which provides the entropy functions for the system

$$\begin{cases}\rho_t+m_x=0\\ m_t+((m^2/\rho)+p(\rho))_x=1/\tau(Q(\rho)-m).\end{cases} \tag{A.2}$$

Let us replace the function $A(\xi)$ with a function $\tilde{A}(\xi)$ with the following properties:

1. $\tilde{A}(\xi)$ is $C^\infty(\mathbb{R})$ and $D^k\tilde{A}$ is bounded for all $k\geqslant 0$
2. $\tilde{A}(\xi)$ is constant for any $\xi\leqslant\hat{\rho}/2$
3. $\tilde{A}(\xi)\geqslant h>0$ for any $\xi\in\mathbb{R}$
4. $\tilde{A}(\xi)=A(\xi)$ for any $\xi\in[\hat{\rho}, 1]$
5. $\tilde{A}(\xi)=\tilde{A}(-\xi)$.

For any $k \in \mathbb{R}$, let $\varphi^k(\rho) \in \mathscr{D}(\mathbb{R})$ be a regularization of the Kružkov entropy $|\rho - k|$, strictly convex on a given fixed interval $[h_1, h_2]$, for some constants $h_1 < 0$, $h_2 > 1$. Let us denote with $\eta^k(\rho, z)$ the solution of the Cauchy problem

$$\begin{cases} \eta_{\rho\rho} + 2a\eta_{\rho z} = \tilde{A}(\rho)\, \eta_{zz} \\ \eta(\rho, 0) = \varphi^k(\rho) \\ \eta_z(\rho, 0) = 0. \end{cases} \tag{A.3}$$

Since $\tilde{A}(\rho) = A(\rho)$ for any $\rho \in [\hat{\rho}, 1]$, then $\eta^k(\rho, m)$ provides a solution to (A.1) on the strip $[\hat{\rho}, 1] \times \mathbb{R}$. Hence it is an entropy for the system (A.2).

LEMMA A.1. *Assume the stability condition*

$$-\theta\rho^{(\gamma-3)/2} < a < \theta\rho^{(\gamma-3)/2}, \qquad 0 < \rho \leqslant 1 \tag{A.4}$$

holds. Then for all $k \in [\hat{\rho}, 1]$, $s \geqslant 0$, $Z > 0$, *one has*

$$\|\eta^k(\rho, z)\|_{H^s(\mathbb{R}, [0, Z])} \leqslant C_s, \tag{A.5}$$

where C_s *depends only on* Z *and the initial conditions.*

Proof. Let us multiply the first equation of the system (A.3) for η_z and then, after integrating in $d\rho$, one has

$$\frac{d}{dz}\mathscr{E}_0(z) = 0,$$

where we set

$$\mathscr{E}_0(z) = \tfrac{1}{2}\int_{-\infty}^{+\infty} \{\tilde{A}(\rho)\, |D_z\eta(\rho, z)|^2 + |D_\rho\eta(\rho, z)|^2\}\, d\rho.$$

Since the entropy equation is independent on z, then it commutes with D_z^j and, as before, the following energies

$$\mathscr{E}_j(z) = \tfrac{1}{2}\int_{-\infty}^{+\infty} \{\tilde{A}(\rho)\, |D_z^{j+1}\eta(\rho, z)|^2 + |D_z^j D_\rho\eta(\rho\ , z)|^2\}\, d\rho$$

are conserved for any $j \geqslant 0$. Now, since the initial data are compactly supported C^∞ functions, we have $\mathscr{E}_j(z) = \mathscr{E}_j(0) < +\infty$. This relation, together with the positivity of $\tilde{A}$, provides the a priori bounds for some derivatives of η^k, namely $D_z^{j+1}\eta^k$ and $D_z^j D_\rho \eta^k$.`We prove the a priori estimates for the remaining derivatives by using the equation and its differentiated form with respect to ρ.

For instance, $\eta_{\rho\rho} = \tilde{A}(\rho)\,\eta_{zz} - 2a\eta_{\rho z}$ and so $\eta_{\rho\rho} \in L^2$ follows from $\eta_{\rho z} \in L^2$, $\eta_{zz} \in L^2$ and $\tilde{A}(\rho) \in L^\infty$. We omit the details of the remaining computations. ∎

The next result allows us to get a continuity property of the solutions $\eta^k(\rho, z)$ with respect to the parameter k in all the Sobolev norms $\|\cdot\|_{H^s(\mathbb{R},[0,Z])}$. This property is necessary to have a fixed closed neighborhood (with nonempty interior) of the curve $\{m = Q(\rho)\}$ in which all these functions are convex and satisfy $\eta^k_{mm}(\rho, m) > 0$, independently from k.

PROPOSITION A.2. *Assume the stability condition* (A.4) *holds. Then there exist constants* C_s, *depending only on* Z *and the initial data, such that for any* k, k_0 *one has*

$$\|\eta^k(\rho, z) - \eta^{k+k_0}(\rho, z)\|_{H^s(\mathbb{R},[0,Z])} \leqslant C_s\,|k_0|, \tag{A.6}$$

for any s.

Proof. From the linearity of the first equation of (A.3), it follows that $\psi^k(\rho, z) = \eta^k(\rho, z) - \eta^{k+k_0}(\rho, z)$ verifies

$$\begin{cases} \psi^k_{\rho\rho} + 2a\psi^k_{\rho z} = \tilde{A}(\rho)\,\psi^k_{zz} \\ \psi^k(\rho, 0) = \varphi^k(\rho) - \varphi^{k+k_0}(\rho) = \varphi^k(\rho) - \varphi^k(\rho - k_0) \\ \psi^k_z(\rho, 0) = 0. \end{cases}$$

Therefore, by using the conservation of energy, we have

$$\mathscr{E}_j(z) = \tfrac{1}{2}\int_{-\infty}^{+\infty} \{\tilde{A}(\rho)\,|D_z^{j+1}\psi^k(\rho, z)|^2 + |D_z^j D_\rho \psi^k(\rho, z)|^2\}\,d\rho = \mathscr{E}_j(0).$$

Now we need to show that $\mathscr{E}_j(0)$ can be estimated in terms of $|k_0|^2$. Indeed, by using (A.3) and its differentiated forms, one can prove for any $j \geqslant 0$ there exist two functions $A_j \in L^\infty$ and $F_j \in C_0^\infty$, depending only on A, its derivatives and the initial conditions, such that

$$\begin{aligned} \mathscr{E}_j(0) &= \tfrac{1}{2}\int_{-\infty}^{+\infty} \{\tilde{A}(\rho)\,|D_z^{j+1}\psi^k(\rho, 0)|^2 + |D_z^j\, D_\rho \psi^k(\rho, 0)|^2\}\,d\rho \\ &= \tfrac{1}{2}\int_{-\infty}^{+\infty} A_j(\rho)\,|F_j(\rho) - F_j(\rho - k_0)|^2\,d\rho \\ &\leqslant C_j\,|k_0|^2. \end{aligned}$$

As before, we need also the estimate of all the remaining derivatives of ψ, which can be obtained from the equation, because of the boundness of $\tilde{A}(\rho)$ and its derivatives. ∎

PROPOSITION A.3. *Under the previous hypotheses, there exists a closed neighborhood (with nonempty interior)* $\Omega \subset [\hat{\rho}, 1] \times \mathbb{R}$ *of the curve* $\{(\rho, m): m = Q(\rho)\}$ *such that* $\eta^k(\rho, m)$ *is strictly convex and verifies* $\eta^k_{mm}(\rho, m) > 0$ *in* Ω, *for any* $k \in [\hat{\rho}, 1]$.

Proof. The Proposition A.2 provides us the continuity of the partial derivatives of $\eta^k(\rho, z)$ with respect to the parameter k. Since the partial derivatives of $\eta^k(\rho, m)$ are linear combinations of the partial derivatives of $\eta^k(\rho, z)$, then also the partial derivatives of $\eta^k(\rho, m)$ depend continuously on k.

Thus, in particular, so it's true for the minimum eigenvalue $\lambda^k(\rho, m)$ of the hessian matrix $D^2\eta^k(\rho, m)$. Hence, due to the fact that $\lambda^k(\rho, Q(\rho))$ is positive for $k \in [\hat{\rho}, 1]$, there exists a positive constant λ^0 depending only on $\hat{\rho}$ such that

$$\lambda^k(\rho, Q(\rho)) \geqslant \lambda^0 \qquad \text{for any} \quad k \in [\hat{\rho}, 1].$$

Therefore, for all $k \in [\hat{\rho}, 1]$ we get

$$\begin{aligned}\langle v, D^2\eta^k(\rho, m)\, v\rangle &= \langle v, D^2\eta^k(\rho, Q(\rho))\, v\rangle \\ &\quad + \langle v, \partial_m D^2\eta^k(\rho, \bar{m})(m - Q(\rho))\, v\rangle \\ &\geqslant \lambda^0\, |v|^2 - \langle v, |\partial_m D^2\eta^k(\rho, \bar{m})|\, |m - Q(\rho)|\, v\rangle,\end{aligned}$$

where, $\langle v, D^2\eta^k(\rho, m)\, v\rangle$ denotes the quadratic form associated to $D^2\eta^k(\rho, m)$. Now, from Lemma A.1 it follows that there exists a positive constant C, such that

$$|D^3\eta^k(\rho, m)| \leqslant C \qquad \text{for all} \quad (\rho, m) \in [\hat{\rho}, 1] \times \mathbb{R}, k \in [\hat{\rho}, 1].$$

Then, for any′ $(\rho, m) \in [\hat{\rho}, 1] \times \mathbb{R}$, $k \in [\hat{\rho}, 1]$, it follows

$$\langle v, D^2\eta^k(\rho, m)\, v\rangle \geqslant \lambda^0\, |v|^2 - C\langle v, |m - Q(\rho)|\, v\rangle.$$

So we can conclude

$$\langle v, D^2\eta^k(\rho, m)\, v\rangle \geqslant \frac{\lambda^0}{2}\, |v|^2,$$

for any $(\rho, m) \in [\hat{\rho}, 1] \times \mathbb{R}$, $k \in [\hat{\rho}, 1]$ and $|m - Q(\rho)| \leqslant \lambda^0/2C$. Similarly, there exists a positive constant C such that, for any $(\rho, m) \in [\hat{\rho}, 1] \times \mathbb{R}$, $k \in [\hat{\rho}, 1]$ and $|m - Q(\rho)| \leqslant C$, one has $\eta^k_{mm}(\rho, m) > 0$. By the previous arguments, there exists a closed neighborhood (with nonempty interior) Ω in $[\hat{\rho}, 1] \times \mathbb{R}$ of $\{(\rho, m): m = Q(\rho)\}$, depending only on $\hat{\rho}$, where $\eta^k(\rho, m)$ is strictly convex and $\eta^k_{mm}(\rho, m) > 0$, for any $k \in [\hat{\rho}, 1]$. ∎

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