On the weak coupling limit of the periodic quantum Lorentz gas

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Abstract

The Lorentz gas is an important model in kinetic theory where one seeks to understand the behavior of a single particle interacting with a fixed environment. In certain scaling regimes, the complicated microscopic behavior of the particle can be well-approximated by an effective equation or stochastic process. A description of the effective behavior usually does not rely on all details of the microscopic dynamics. When the interactions are quantum mechanical, the model is referred to as the quantum Lorentz gas. In this thesis, we study the quantum Lorentz gas in a periodic, weakly-coupled environment. We show that for certain observables, the limiting effective behavior in this regime is given by a transport equation, while for certain other observables, we provide evidence that the existence of the limit hinges on regularity properties of a certain phase space distribution that we introduce.

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Notation.

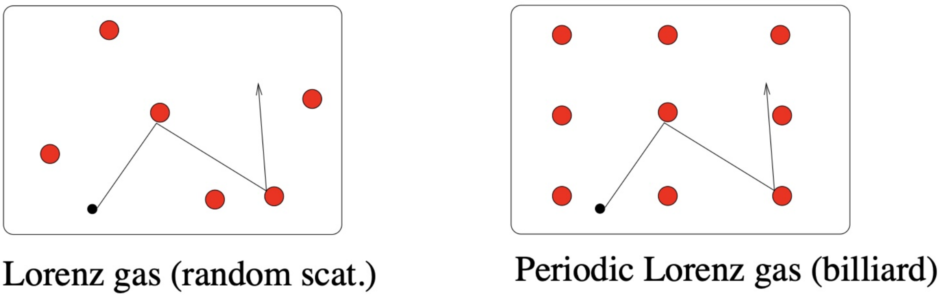
1. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
2. $\mathbb{R}_{\geqslant 0} := [0, +\infty), \mathbb{R}_+ := (0, +\infty)$.
3. $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$.
4. *For X a Banach space and $f: \mathbb{R} \longrightarrow X$, we will sometimes use the notation $f_t := f(t)$. Furthermore we define $\delta_{st} t: \mathbb{R} \times \mathbb{R} \longrightarrow X$ via*
$$\delta_{st} f := f_t - f_s.$$
5. *For a function $f \in L^1(\mathbb{R}^d)$ we use the following convention of the Fourier transform: for $\xi \in \mathbb{R}^d$*
$$\hat{f}(\xi) := \int_{\mathbb{R}^d} \mathrm{d}x e^{-2\pi i \xi \cdot x} f(\xi)$$
For a function $f \in L^1(\mathbb{T}^d)$ we use the following convention for the coefficients of the Fourier series: for $n \in \mathbb{Z}^d$
$$\hat{f}(n) := \int_{\mathbb{T}^d} \mathrm{d}x e^{-2\pi i n \cdot x} f(x)$$
6. *We use the following notation for the floor function $\lfloor \cdot \rfloor: \mathbb{R} \longrightarrow \mathbb{Z}, x \longrightarrow \lfloor x \rfloor$, taking any real number to the closest integer smaller than or equal to it. When applied to $x \in \mathbb{R}^d$, this is the floor function applied component-wise.*
7. *Let (S, Σ, μ) be a measure space. Let $p \in [1, \infty]$. We denote $L^p(S; \mu)$ to be the space of L^p functions on S with respect to the measure μ. For $d \in \mathbb{N}$, we denote $L^p(\mathbb{R}^d)$ to be the space of L^p functions on $\mathbb{R}^d$ with respect to the Lebesgue measure.*
8. *For $d \in \mathbb{N}$, $\mathscr{M}(\mathbb{R}^d)$ denotes the set of Radon measures on $\mathbb{R}^d$ and $\mathscr{M}_+(\mathbb{R}^d)$ denotes the set of positive Radon measures on $\mathbb{R}^d$. $C_b(\mathbb{R}^d)$ denotes the Banach space of bounded continuous functions on $\mathbb{R}^d$. $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ refer to the set of Schwartz functions and tempered distributions on $\mathbb{R}^d$ respectively.*
9. *By $a \lesssim b$, we mean that there exists $C > 0$ such that $a \leqslant Cb$. By $a \lesssim_\gamma b$, we mean that there exists $C(\gamma) > 0$ such that $a \leqslant C(\gamma) b$.*

Chapter 1

Introduction

1.1 Setup

The classical Lorentz gas is an important model in kinetic theory. Consider a single particle moving in an environment of fixed obstacles. Assume that the moving particle obeys Newton's laws of motion and collides elastically with the obstacles. This is like studying the trajectory of a moving cue ball that collides with an immovable configuration of billiard balls as in the pictures below. These images have been sourced from [Erd10].



Lorenz gas (random scat.)

Periodic Lorenz gas (billiard)

Figure 1.1. Two possible obstacle configurations for the Lorentz gas.

At this “microscopic” level, the time evolutions of the position and velocity of the cue ball are complicated, and depend on its initial position and momenta, and the position and geometry of the scatterers.

In kinetic theory, one is interested in situations where this complex behavior at the microscopic scale can be approximated by effective behavior on a larger “macroscopic” scale. This effective macroscopic behavior usually does not depend on all the details of the microscopic dynamics, and is therefore relatively easier to describe. For many models in kinetic theory, two particular scaling regimes are especially interesting, where an effective description is sometimes possible. They are the “low-density scaling” and the “weak-coupling scaling”. In the Lorentz gas model, the low-density scaling models a scenario where collisions happen rarely, but have a strong effect on the cue ball's trajectory when they do happen. On the other hand, the weak coupling scaling models a situation where the cue ball scatters frequently, but where each collision only has a weak effect on the trajectory of the particle.

There are situations in physics and probability theory where interactions in a system are either "strong and rare" or "weak and often" as above. In physics, the first situation is common when working with dilute gases, while the second is common when dealing with plasmas. In probability theory one has the Poisson and central limit theorems. Roughly speaking, the Poisson limit theorem says that if one has random variables that are usually zero, but sometimes non-zero, under an appropriate choice of the parameters, the sum of these random variables converges in distribution to the Poisson distribution. The central limit theorem says that a weighted sum of random variables with bounded variance converges to a normal distribution, when the weights ensure that each random variable contributes weakly to the sum. These are analogous to the low-density scaling and the weak-coupling scaling regimes of the Lorentz gas. In all these cases, one expects to use only a small amount of the total information in the "microscopic" system to be relevant to the description of the "macroscopic" system.

When the interactions of the particle with its environment are quantum mechanical, one refers to the model as the quantum Lorentz gas. Mathematically, the particle is modeled by the solution to a linear Schrödinger equation, with the environment represented by a real-valued potential term V. For $x \in \mathbb{R}^d, t \in \mathbb{R}_+$ consider

$$i\partial_t \varphi(t,x) = -\Delta_x \varphi(t,x) + \lambda V(x)\varphi(t,x). \tag{1.1}$$

The real constant $\lambda \geqslant 0$ is called the coupling constant. The solution to the free Schrödinger equation with $\lambda = 0$ can be written down explicitly. In general, when $\lambda > 0$, describing the precise behavior of the particle can be very challenging. However, when λ is very small, the potential only has a weak effect on the particle, and one expects that after a small period of microscopic time has elapsed, the evolution of the particle is very similar to that of the free evolution ($\lambda = 0$). If these weak interactions with the environment occur often, non-trivial behavior can be observed if one waits for a sufficient amount of time. This is the weak coupling regime for the quantum Lorentz gas.

Mathematically, one sets $\lambda = \varepsilon^{1/2}$ for some $0 < \varepsilon \ll 1$ and studies the following rescaled wavefunction, which is a function of macroscopic time and space:

$$\varphi_\varepsilon(t,x) := \varphi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \tag{1.2}$$

This satisfies the following rescaled Schrödinger equation: for $x \in \mathbb{R}^d, t \in \mathbb{R}_+$

$$i\partial_t \varphi_\varepsilon(t,x) = -\varepsilon \Delta_x \varphi_\varepsilon(t,x) + \varepsilon^{-1/2} V\left(\frac{x}{\varepsilon}\right)\varphi_\varepsilon(t,x). \tag{1.3}$$

Note that the effect of the potential is no longer weak here.

Recall that in quantum mechanics, the wavefunction cannot be measured. However $|\varphi(t,x)|^2$ corresponds to the probability of finding a particle at position x and time t, and is a quantity that can be estimated from experimental measurements. Before we characterize the measurable quantities that we are interested in, it is convenient for us to switch to the phase-space formulation of quantum mechanics. Unlike in classical mechanics, the Heisenberg uncertainty principle precludes the existence of a phase space density, i.e., one cannot represent a particle via a Dirac delta in phase space. An object that has many of the properties one requires from a phase space density is the Wigner transform W_φ of the wavefunction φ: for position $x \in \mathbb{R}^d$ and momentum $k \in \mathbb{R}^d$,

$$W_\varphi(x,k) := \int_{\mathbb{R}^d} e^{2\pi i k\cdot y} \varphi\Big(x - \frac{y}{2}\Big)\varphi^*\Big(x + \frac{y}{2}\Big)\mathrm{d}y.$$

For $\varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$, it is an easy computation to see that

$$\int_{\mathbb{R}^d} \mathrm{d}k\, W_\varphi(x,k) = |\varphi(x)|^2, \qquad \int_{\mathbb{R}^d} \mathrm{d}x\, W_\varphi(x,k) = |\hat{\varphi}(k)|^2,$$

where $\hat{\varphi}$ denotes the Fourier transform of φ. These two expressions correspond to physically measurable quantities, namely the probabilities of finding a particle at position x and momentum k respectively. An important class of measurable quantities at the microscopic scale are of the form

$$\int_{\mathbb{R}^{2d}} \mathrm{d}x\,\mathrm{d}k\, W_\varphi(t,x,k)F(x,k) = \int_{\mathbb{R}^{2d}} \mathrm{d}x\,\mathrm{d}k\, W_{\varphi(t)}(x,k)F(x,k),$$

for some Schwartz function $F \in \mathcal{S}(\mathbb{R}^{2d})$. To go from the microscopic to the macroscopic scale, we consider a rescaled version of the Wigner transform:

$$W^\varepsilon(t,x,k) := \varepsilon^{-d} W_\varphi\Big(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, k\Big). \tag{1.4}$$

The rescaled observables related to the rescaled Wigner transform take the form

$$\int_{\mathbb{R}^{2d}} \mathrm{d}x\,\mathrm{d}k\, W^\varepsilon(t,x,k)F(x,k). \tag{1.5}$$

For the quantum Lorentz gas, we are interested in answering the following

Question 1. Given a certain potential V, can one describe the time evolution of

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2d}} \mathrm{d}x\,\mathrm{d}k\, W^\varepsilon(t,x,k)F(x,k). \tag{1.6}$$

for any $F \in \mathcal{S}(\mathbb{R}^{2d})$?

This is equivalent to asking for a description of the weak limit (in the space of tempered distributions on $\mathbb{R}^{2d}$) of the rescaled Wigner transform, if it exists. One could then plausibly describe the dynamics with $0 < \varepsilon \ll 1$ as being a correction to this limit behavior.

In this thesis, our goal is to tackle this question when V is a smooth $\mathbb{Z}^d$-periodic function. We refer to this model as the periodic quantum Lorentz gas. Before describing our motivations for studying this problem with this choice of potential, we review some key results that have been obtained in the study of the Lorentz gas.

Remark 1.1. This is one possible way to study the time evolution of measurable quantities. Using the Weyl transform of the Wigner function, one could equivalently work with the time evolution of observables in the interaction picture of quantum mechanics, as is also commonly done in quantum kinetic theory, as, for instance, was done in [Spo77] and [GM19].

1.2 Literature review

The classical random Lorentz gas was first investigated in the low density limit in [Gal69] and then in greater generality in [Spo78] and [BBS83]. For $d \geqslant 2$, these results show that under suitable assumptions on the initial data and the random environment, the phase space density converges in the low density scaling limit to a solution of a linear Boltzmann equation, which is the forward equation of a Markov jump process. The weak coupling limit of the classical random Lorentz gas in $d \geqslant 3$ was studied in [KP80] and the case $d = 2$ was handled in [DGL87] and [KR06]. For $d \geqslant 2$ the weak coupling limit of the phase space density is given by the solution of a linear Landau equation. These results say that the complicated Schrödinger evolution is well approximated by the solution to a linear Boltzmann or linear Landau equation, in the low-density and weak-coupling regimes respectively. In both these situations, the limit equations are not time-reversible, unlike the microscopic dynamics.

It is important to note that the result of Gallavotti [Gal69] was a precursor to the first derivation of the (nonlinear) Boltzmann equation from a many-body system of hard spheres, by O. Lanford [Lan75].

For the quantum random Lorentz gas, in [Spo77] it was shown that for small enough times the weak coupling limiting behavior was given by a linear Boltzmann equation. This was improved in [EY99], where the authors show that, in the weak-coupling scaling, when V is a Gaussian random field with smooth covariance, that for a certain family of initial data, and for all $F \in \mathcal{S}(\mathbb{R}^{2d}), t \in \mathbb{R}_+$

$$\mathbb{E}\Big[\int_{\mathbb{R}^d} \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}x\, W^{\varepsilon}(t,x,k) F(x,k)\Big] \xrightarrow{\varepsilon \to 0} \int_{\mathbb{R}^d} \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}x\, f(t,x,k) F(x,k),$$

where f satisfies a linear Boltzmann equation and where W^{ε} is the rescaled Wigner transform introduced previously. The papers [Che06] and [But15] prove convergence of moments of all orders, for an analogous problem on the lattice.

The low density limiting behavior in the quantum random Lorentz gas is also given by a linear Boltzmann equation. The only difference from the weak coupling behavior is in the collision kernel of the limiting linear Boltzmann equation. This was proved in [EE05]. See [Mik23] for a recent generalization.

For the periodic Lorentz gas, fewer results are available, both in the classical and quantum cases. For the classical periodic Lorentz gas in the low density limit, the limiting behavior is not given by a linear Boltzmann equation. Instead, in [MS11], a non-Markovian flight process was derived in the low density limit. The low density limit of the periodic quantum Lorentz gas was investigated in [GM19] and [GM21]. Conditional on a certain version of the Berry–Tabor conjecture holding true, the authors derive the limit dynamics, and show that it is not given by a linear Boltzmann equation. However, if one introduces a damping term in the Schrödinger dynamics, one recovers a linear Boltzmann equation in the low density limit, as was shown in [Gri23].

To the best of our knowledge, the weak coupling limit in the full space for the observables as in expression (1.6) has not yet been established for the periodic quantum Lorentz gas. The following table summarizes the situation described above

	Classical		Quantum	
	Low density	Weak coupling	Low density	Weak coupling
Random	Lin. Boltzmann [Gal69], [BBS83], [Spo78]	Lin. Landau [KP80], [DGL87] [KR06]	Lin. Boltzmann [EE05]	Lin. Boltzmann [Spo77], [EY99], [Che06], [But15]
Periodic	Non-Markovian flight process, [MS11]	? -	Random flight process, [GM21]	? -

Table 1.1. Kinetic limits of the Lorentz gas

Below is a list of related investigations that is by no means exhaustive. We mention only the aspects of the work that we feel are most directly connected to the study of the periodic quantum Lorentz gas.

- A heuristic derivation of the linear Boltzmann equation in the weak coupling regime for the random quantum Lorentz gas is given in [BFPR99]. We will come back to this in Chapter 3, where we use this method to get an idea of what the limit should be for the periodic quantum Lorentz gas. In [PR04], the authors modify the equation for the rescaled Wigner transform in the case of the periodic quantum Lorentz gas, and provide a different connection to the linear Boltzmann equation. We also refer the reader to Chapter 8 of [Spo91], for more heuristics related to the derivation of kinetic equations for the Lorentz gas.

- In the works [Cas01] and [Cas02], a linear Boltzmann equation is obtained in the low-density limit from the von-Neumann equation with a periodic potential and a damping term. The period is taken to infinity before the damping is sent to 0. This is similar in spirit to the work [PR04], which we discuss in Chapter 3 of this thesis. In [CP02], it was shown that when working on the torus, in a certain scaling regime where non-trivial limiting behavior can be observed, the limiting behavior is not described by the linear Boltzmann equation. In [Cas99] a different weak coupling limit was studied for the von-Neumann equation on the torus, and the limiting behavior was shown to not coincide with that of the linear Boltzmann equation.

- The periodic Schrödinger equation has been investigated in the so-called semiclassical limit, which is when spacetime is rescaled as in the weak coupling scaling, but the potential remains strongly coupled. See [MMP94] and [GMMP98]. These and related works such as [LP93] were among the first to use the rescaled Wigner transform to pass from microscopic to macroscopic scales.

- There are also works investigating the random quantum Lorentz gas with time-dependent potentials. In these models the environment evolves in time, and such problems are of relevance to understanding the propagation of waves in random media. See for instance [BPR02] and [Gom13].

- One could ask about what happens to these systems on space and time scales larger than the kinetic space and time scales discussed above. We refer the reader to [ESY08] and [ESY07] where the authors derive a diffusion equation for the random quantum Lorentz gas. The lecture notes [Erd10] also explain the connection of this problem to the extended states conjecture of Anderson localization in $d \geqslant 3$. For the periodic quantum Lorentz gas, the long time behavior is known to be ballistic as was demonstrated in [AK98] and [dMS23]. We comment on this at the end of Chapter 4.

1.3 Motivation

We have two principal motivations for studying the periodic quantum Lorentz gas.

1. In the proof of the convergence to the linear Boltzmann equation in [EY99] the authors work with the Duhamel iterates of the solutions to the Schrödinger equation. They plug these expansions into the expressions for the rescaled Wigner transform, and the rescaled observables, and proceed to show that the resulting series remain summable in the scaling limit considered, by identifying which Duhamel integrals remain relevant in the limit. They carefully estimate the size and number of those integrals which do not contribute, in order to make the argument that the series containing all the terms is convergent, and converges to the Neumann series representation of the solution to a linear Boltzmann equation.

   Now, one can compute the evolution equation for the rescaled Wigner transform, to get that W^ε must solve

   $$\partial_t W^\varepsilon + k\cdot\nabla_x W^\varepsilon = i\varepsilon^{-1/2}\int_{\mathbb{R}^d} \mathrm{d}\xi\, e^{2\pi i\varepsilon^{-1}x\cdot\xi}\hat{V}_\omega(\xi)\left[W^\varepsilon(k+\xi) - W^\varepsilon(k-\xi)\right].$$

   However, since one starts with the linear Schrödinger equation and ends with the linear Boltzmann equation, one could ask if there is an argument that uses PDE tools to prove this convergence. One could expect that the right hand side in the expression above converges to the collision kernel of a linear Boltzmann equation, as a consequence of the oscillations in the integrand.

   But, proving this is challenging: in early computations, we found it difficult to work with the Fourier transform $\hat{V}_\omega$ of a Gaussian random field, which is not in any Lebesgue space.

Our approach was to switch to studying a random quantum Lorentz gas, but where the randomness was $(L\mathbb{Z})^d$-periodically repeated: certain computations were easier when working with a $(L\mathbb{Z})^d$-periodic smooth Gaussian random field. This is because such a field has, almost surely, a "nice" Fourier series. Here we show some realizations of such a field in two dimensions:

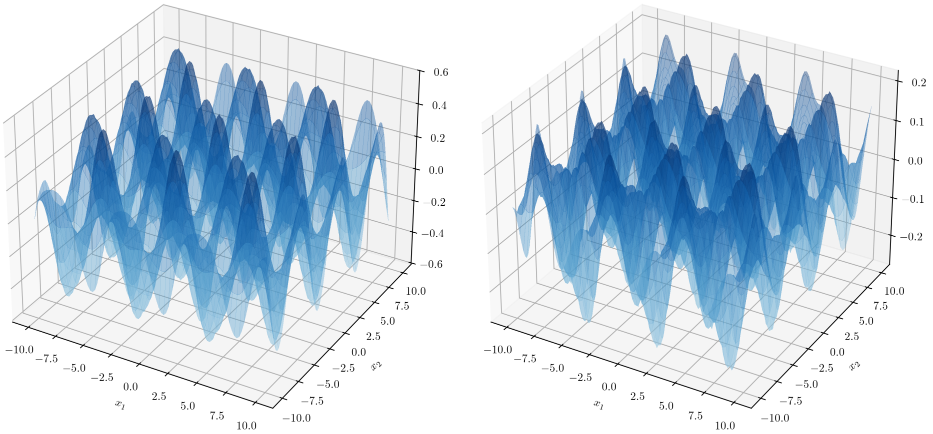


Figure 1.2. Two different realizations of a smooth zero-mean periodic Gaussian field in $d = 2$.

But this changes the physics of the problem: for a fixed period L, one intuitively expects the physics to be different since the same source of randomness is revisited by the particle. However, we expect that if one takes the period $L \longrightarrow \infty$ at an appropriate rate depending on ε, one should converge to the linear Boltzmann equation as in [EY99]. The study of the weak coupling limit for a fixed period is for us an intermediate step towards this larger goal. Within this context, we sought to fix a period and develop a proof that does not rely on the combinatorics of Duhamel iterations, and to then (hopefully) extend it to the case where the period of the potential grows depending on ε.

Since the results we were able to prove for this model apply path-wise (for almost every realization of the randomness), they apply also to the periodic quantum Lorentz gas. In this thesis, we work with a fixed $\mathbb{Z}^d$-periodic smooth potential, but many results can be applied also to almost every realization of a $\mathbb{Z}^d$-periodic smooth Gaussian random field. In particular, here we are looking at the challenges that arise when fixing $L = 1$ and sending $\varepsilon \longrightarrow 0$.

Summarizing: Looking for an alternative proof to [EY99] in the case of the random Lorentz gas, we switched from studying a smooth, non-periodic Gaussian field to a smooth, $(L\mathbb{Z})^d$-periodic Gaussian field for convenience, and sought to first investigate the limit $\varepsilon \longrightarrow 0$ fixing $L = 1$. We are able to prove certain statements for almost every realization of a smooth $\mathbb{Z}^d$-periodic Gaussian field, which is equivalent to proving statements for the periodic quantum Lorentz gas.

2. As can be seen from Table 1.1 above, the kinetic limits for the random Lorentz gas are much better understood than their periodic counterparts. However, the results [MS11] and [GM21] show that one can obtain non-trivial limiting stochastic processes that approximate the microscopic dynamics of the Lorentz gas, at least in the case of the low-density scaling. It is natural to ask if one can have effective descriptions of the periodic Lorentz gas also in the weak coupling scaling. In this thesis, we provide partial answers in the case of the weak coupling regime for the quantum mechanical version of this model. To the best of our knowledge, the weak coupling limit of the classical periodic Lorentz gas is also an open problem.

1.4 Structure of the thesis

In Chapter 2, which could be treated as an appendix, we collect some relevant information about:

1. The Bloch–Floquet–Zak transform, which plays a role analogous to the Fourier transform, when studying periodic operators. We will use it later to derive a representation formula for the Wigner transform.
2. The Wigner and rescaled Wigner transforms introduced above.
3. The sewing lemma, which plays a key role in the theory of rough paths. We also give a toy example of how we intend to use it in studying kinetic limits.
4. The linear Boltzmann equation's well-posedness theory, in particular the uniqueness theory, will be important to our arguments. We also make some comments on the result of [EY99].

In Chapter 3, we use an asymptotic expansion that was previously used to formally derive a linear Boltzmann equation from the random quantum Lorentz gas. In the case of the periodic quantum Lorentz gas, this method yields trivial transport equations in the limit for certain “non-resonant” observables (similar to the case $V = 0$), and hints at the difficulties one encounters when studying other observables.

In Chapter 4 we rigorously prove the first part of the previous statement. Here a version of the Bloch–Floquet–Zak transform is used to restate the problem in terms of “energy bands” (which are commonly used to describe the difference between conductors and insulators). We derive a limit equation for observables supported away from a certain zero measure set of momenta and comment on why this approach does not completely answer Question 1.

Chapter 5 contains the most novel aspects of this thesis. It is structured as follows:

1. Using the BFZ transform, we derive a representation formula for the Wigner transform. Motivated by this, we define a generalized Bloch–Wigner transform, and derive its evolution equation.
2. We show that this evolution equation has the structure of a rough equation for each $\varepsilon > 0$, and that in a certain topology one can obtain uniform in ε-estimates for the rough operators involved.

3. We then use the sewing lemma and the uniqueness theory of the linear Boltzmann equation in order to pass to the limit, in a certain topology, and explicitly characterize the limit equation.
4. We then comment on the weakness of this result, and why it does not permit to characterize the weak coupling limit of all the observables.
5. We attempt to use the Bloch–Wigner transform to study observables, and show that for observables supported on resonant momenta, the existence of the limit is tied to regularity properties of the Bloch–Wigner transform.

We stress that this approach differs conceptually from the approach used in [EY99] to derive the limit of the random quantum Lorentz gas. In particular it avoids diagrammatic expansions and combinatorics, and uses the sewing lemma instead.

We conclude by summarizing our results in Chapter 6 and highlight promising directions for future research.

Chapter 2

Our toolkit

2.1 The Bloch–Floquet–Zak decomposition

In this Section, we recall the Bloch–Floquet–Zak decomposition, introduced by J. Zak in [Zak67] and [Zak68], and which is now a well-known tool in solid state physics (see for instance [MP14]). For the reader's convenience, I collect here the properties of the Bloch–Floquet–Zak decomposition that will be of use for our study of the weak coupling limit of the periodic quantum Lorentz gas.

Definition 2.1. *For $\varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$, $\theta \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, define the* ***Bloch–Floquet–Zak decomposition of φ****, or BFZ decomposition of φ as:*

$$(\mathcal{U}_{\mathrm{BFZ}}\varphi)_\theta(x) := \tilde{\varphi}(\theta,x) := \sum_{m\in\mathbb{Z}^d} e^{2\pi i\theta\cdot(x-m)}\varphi(x-m). \tag{2.1}$$

Let us state a few basic properties of the BFZ decomposition:

1. $\tilde{\varphi}(\theta,x)$ is $\mathbb{Z}^d$-periodic in x: for $n\in\mathbb{Z}^d$,

$$\begin{aligned}\tilde{\varphi}(\theta,x+n) &= \sum_{m\in\mathbb{Z}^d} e^{2\pi i\theta\cdot(x+n-m)}\varphi(x+n-m)\\ &= \sum_{m'\in\mathbb{Z}^d} e^{2\pi i\theta\cdot(x-m')}\varphi(x-m') = \tilde{\varphi}(\theta,x).\end{aligned}$$

   Hence, we can identify $\tilde{\varphi}(\theta,x)$ with a complex valued function on $\mathbb{R}^d\times\mathbb{T}^d$. We also note that it is quasiperiodic in θ: for $n\in\mathbb{Z}^d$,

$$\begin{aligned}\tilde{\varphi}(\theta+n,x) &= \sum_{m\in\mathbb{Z}^d} e^{2\pi i(\theta+n)\cdot(x-m)}\varphi(x-m)\\ &= e^{2\pi i n\cdot x}\sum_{m\in\mathbb{Z}^d} e^{2\pi i\theta\cdot(x-m)}\varphi(x-m) = e^{2\pi i n\cdot x}\tilde{\varphi}(\theta,x).\end{aligned}$$

   By complex conjugation we have that

$$\tilde{\varphi}^*(\theta,x+n) = \tilde{\varphi}^*(\theta,x), \qquad \tilde{\varphi}^*(\theta+n,x) = e^{-2\pi i n\cdot x}\tilde{\varphi}^*(\theta,x).$$

   Note that, similar to the Fourier transform, the BFZ transform does not commute with complex conjugation.

2. We have $\mathbb{Z}^d$-periodicity in the θ-variable for the functions $e^{-2\pi i\theta\cdot x}\tilde{\varphi}(\theta,x)$ and $|\tilde{\varphi}(\theta,x)|^2$ since

   $$e^{-2\pi i(\theta+n)\cdot x}\tilde{\varphi}(\theta+n,x)=e^{-2\pi i(\theta+n)\cdot x}e^{2\pi in\cdot x}\tilde{\varphi}(\theta,x)=e^{-2\pi i\theta\cdot x}\tilde{\varphi}(\theta,x),$$

   and so

   $$|\tilde{\varphi}(\theta+n,x)|^2=\tilde{\varphi}^*(\theta+n,x)\,\tilde{\varphi}(\theta+n,x),$$

   $$\Longrightarrow|\tilde{\varphi}(\theta+n,x)|^2=e^{-2\pi in\cdot x}\tilde{\varphi}^*(\theta,x)\;e^{2\pi in\cdot x}\tilde{\varphi}(\theta,x)=|\tilde{\varphi}(\theta,x)|^2. \qquad (2.2)$$

3. For $\varphi\in\mathcal{S}(\mathbb{R}^d;\mathbb{C})$ we can invert the BFZ transform. One has that for all $\varphi\in\mathcal{S}(\mathbb{R}^d;\mathbb{C})$,

   $$\varphi(x)=\int_{\mathbb{T}^d}e^{-2\pi i\theta\cdot x}\tilde{\varphi}(\theta,x)\mathrm{d}\theta. \qquad (2.3)$$

   This can be seen from the following computation

   $$\int_{\mathbb{T}^d}e^{-2\pi i\theta\cdot x}\tilde{\varphi}(\theta,x)\mathrm{d}\theta=\int_{[0,1]^d}e^{-2\pi i\theta\cdot x}\tilde{\varphi}(\theta,x)\mathrm{d}\theta=\int_{[0,1]^d}\sum_{m\in\mathbb{Z}^d}e^{-2\pi i\theta\cdot m}\varphi(x-m)\mathrm{d}\theta$$

   which, by Fubini's theorem, is

   $$=\sum_{m\in\mathbb{Z}^d}\underbrace{\int_{[0,1]^d}e^{-2\pi i\theta\cdot m}\mathrm{d}\theta}_{\delta_{m,0}}\varphi(x-m)=\varphi(x).$$

4. If $\varphi\in\mathcal{S}(\mathbb{R}^d;\mathbb{C})$ then $\tilde{\varphi}\in L^2\left(\left[-\frac{1}{2},\frac{1}{2}\right]\times\mathbb{T}^d;\mathbb{C}\right)$ since

   $$\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x|\tilde{\varphi}(\theta,x)|^2=\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x\tilde{\varphi}(\theta,x)\,\tilde{\varphi}^*(\theta,x)$$

   $$=\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x\sum_{m,m'}e^{2\pi i\theta\cdot(x-m)}\varphi(x-m)\,e^{-2\pi i\theta\cdot(x-m')}\varphi^*(x-m')$$

   $$=\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}x\mathrm{d}\theta\sum_{m,m'}e^{-2\pi i\theta\cdot(m-m')}\varphi(x-m)\,\varphi^*(x-m').$$

   Since $\sum_m|\varphi(x-m)|\sum_{m'}|\varphi(x-m')|<\infty$, Fubini's theorem says this is

   $$=\int_{\mathbb{T}^d}\mathrm{d}x\sum_{m,m'}\underbrace{\int_{\mathbb{T}^d}\mathrm{d}\theta e^{-2\pi i\theta\cdot(m-m')}}_{\delta_{m-m'}}\varphi(x-m)\,\varphi^*(x-m')$$

   $$=\int_{\mathbb{T}^d}\mathrm{d}x\sum_m\varphi(x-m)\,\varphi^*(x-m)=\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}x\sum_m\varphi(x-m)\,\varphi^*(x-m)$$

   $$=\|\varphi\|^2_{L^2(\mathbb{R}^d)}.$$

5. For $\varphi\in\mathcal{S}(\mathbb{R}^d;\mathbb{C})$, we have that $\tilde{\varphi}\in C(\mathbb{R}^d\times\mathbb{T}^d;\mathbb{C})$: Fix $\theta\in\mathbb{R}^d,x\in\mathbb{T}^d,\varepsilon>0$. Then we then have that for $\delta_1,\delta_2\in\mathbb{R}^d$:

   $$\tilde{\varphi}(\theta+\delta_1,x+\delta_2)-\tilde{\varphi}(\theta,x)=\sum_{m\in\mathbb{Z}^d}(e^{2\pi i\delta_1\cdot(x-m)}e^{2\pi i\theta\cdot\delta_2}-1)\,e^{2\pi i\theta\cdot(x-m)}\varphi(x-m).$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$, we can define a ball B such that

$$\sum_{m\in\mathbb{Z}^d\cap B^c} |\varphi(x-m)| < \frac{\varepsilon}{4}.$$

This then gives us that

$$\begin{aligned}|\tilde{\varphi}(\theta+\delta,x)-\tilde{\varphi}(\theta,x)| &\leq \left|\sum_{m\in\mathbb{Z}^d\cap B}(e^{2\pi i\delta_1\cdot(x-m)}e^{2\pi i\theta\cdot\delta_2}-1)e^{2\pi i\theta\cdot(x-m)}\varphi(x-m)\right|\\ &+\left|\sum_{m\in\mathbb{Z}^d\cap B^c}(e^{2\pi i\delta_1\cdot(x-m)}e^{2\pi i\theta\cdot\delta_2}-1)e^{2\pi i\theta\cdot(x-m)}\varphi(x-m)\right|\\ &<\left|\sum_{m\in\mathbb{Z}^d\cap B}(e^{2\pi i\delta_1\cdot(x-m)}e^{2\pi i\theta\cdot\delta_2}-1)e^{2\pi i\theta\cdot(x-m)}\varphi(x-m)\right|+\frac{\varepsilon}{2}.\end{aligned}$$

One can then conclude by choosing δ_1, δ_2 small enough such that the first term is also smaller than $\varepsilon/2$. This can be done, since the sum is finite and the terms are bounded. Note that here we only needed the continuity and decay of the function itself, and not its derivatives.

6. Defining $\gamma_m u(x) := e^{2\pi i m\cdot x}u(x)$, one can show that the Bloch–Floquet–Zak transform extends to a unitary transformation (also called the Block–Floquet–Zak transform) of $L^2(\mathbb{R}^d;\mathbb{C})$ into $\mathscr{H}_\gamma$, the Hilbert space of $L^2(\mathbb{T}^d;\mathbb{C})$ valued γ-equivariant functions, i.e.,

   $$\mathscr{H}_\gamma := \{u\in L^2_{\mathrm{loc}}(\mathbb{R}^d;L^2(\mathbb{T}^d;\mathbb{C})) : \tilde{u}(\theta+n,\cdot)=\gamma_n\tilde{u}(\theta,\cdot), \forall\,\theta\in\mathbb{R}^d, n\in\mathbb{Z}^d\}.$$

   This is similar to how the Fourier transform extends from $\mathcal{S}(\mathbb{R}^d)$ to a unitary transformation on $L^2(\mathbb{R}^d)$. The above Hilbert space can be endowed with the scalar product

   $$\langle u,v\rangle_{\mathscr{H}_\gamma} := \int_{\mathbb{T}^d}\mathrm{d}\theta\langle\tilde{u}(\theta,\cdot),\tilde{v}(\theta,\cdot)\rangle_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d}\mathrm{d}\theta\int_{\mathbb{T}^d}\mathrm{d}x\,\tilde{u}^*(\theta,x)\,\tilde{v}(\theta,x),$$

   and with the norm

   $$\|u\|_{\mathscr{H}_\gamma} := \left(\int_{\mathbb{T}^d}\mathrm{d}\theta\|\tilde{u}(\theta,\cdot)\|^2_{L^2(\mathbb{T}^d)}\right)^{1/2} = \left(\int_{\mathbb{T}^d}\mathrm{d}\theta\int_{\mathbb{T}^d}\mathrm{d}x\,|\tilde{u}(\theta,x)|^2\right)^{1/2}.$$

   The inverse BFZ transform is explicitly given by

   $$(\mathscr{U}^{-1}_{\mathrm{BFZ}}\tilde{u})(x) = \int_{\mathbb{T}^d}\mathrm{d}\theta e^{-2\pi i\theta\cdot x}\tilde{u}(\theta,x).$$

   See [MP14] for more details.

Next we have a lemma that describes the regularity of the BFZ transform of a function with the regularity of a classical solution to the Schrödinger equation.

Lemma 2.2. *Let $\varphi\in C(\mathbb{R};H^2(\mathbb{R}^d;\mathbb{C}))\cap C^1(\mathbb{R};L^2(\mathbb{R}^d;\mathbb{C}))$, then $\tilde{\varphi}\in C(\mathbb{R};L^2_{\mathrm{loc}}(\mathbb{R}^d;H^2(\mathbb{T}^d;\mathbb{C})))\cap C^1(\mathbb{R};\mathscr{H}_\gamma)$. Furthermore, one has that*

$$\widetilde{\partial_t\varphi}(t) = \partial_t\tilde{\varphi}(t) \tag{2.4}$$

$$\widetilde{\Delta\varphi}(t,\theta,x) = (\Delta_x - 4\pi^2|\theta|^2 - 4\pi i\theta\cdot\nabla_x)\,\tilde{\varphi}(t,\theta,x) \tag{2.5}$$

and

$$(\widetilde{\Delta\varphi})^*(t,\theta,x) = (\Delta_x - 4\pi^2|\theta|^2 + 4\pi i\theta\cdot\nabla_x)\tilde{\varphi}^*(t,\theta,x) \tag{2.6}$$

as functions in $\mathscr{H}_\gamma$.

Proof. First, we have that since U_{BFZ} is a unitary transformation, it is a bounded linear operator from $L^2(\mathbb{R}^d;\mathbb{C})$ to H_γ, so in particular, it is a continuous linear operator. Hence using the linearity and continuity, one has that

$$\begin{aligned}\partial_t\tilde{\varphi}(t) = \partial_t U_{\mathrm{BFZ}}\varphi(t) &= \lim_{h\to 0}\frac{U_{\mathrm{BFZ}}\varphi(t+h) - U_{\mathrm{BFZ}}\varphi(t)}{h}\\ &= \lim_{h\to 0} U_{\mathrm{BFZ}}\left(\frac{\varphi(t+h)-\varphi(t)}{h}\right)\\ &= U_{\mathrm{BFZ}}\left(\lim_{h\to 0}\frac{\varphi(t+h)-\varphi(t)}{h}\right) = U_{\mathrm{BFZ}}(\partial_t\varphi(t)) = \widetilde{\partial_t\varphi}(t)\end{aligned}$$

The unitarity of U_{BFZ} also immediately implies that $\tilde{\varphi}\in C^1(\mathbb{R};\mathscr{H}_\gamma)$, since $\varphi\in C^1(\mathbb{R};L^2(\mathbb{R}^d;\mathbb{C}))$. Next, we will show that $\tilde{\varphi}\in C(\mathbb{R};L^2_{\mathrm{loc}}(\mathbb{R}^d;H^1(\mathbb{T}^d;\mathbb{C})))$. One can proceed by computing that for $\varphi(t)\in H^2(\mathbb{R}^d;\mathbb{C})$, $\psi\in C_c^\infty(\mathbb{R}^d;\mathbb{C})$, $j\in\{1,\dots,d\}$

$$\int_{\mathbb{R}^d}\partial_{x_j}\psi^*(x)\varphi(t,x)\mathrm{d}x = -\int_{\mathbb{R}^d}\psi^*(x)v_j(t,x)\mathrm{d}x = -\int_{\mathbb{T}^d\times\mathbb{T}^d}(\tilde{\psi})^*(\theta,x)\tilde{v}_j(t,\theta,x)\mathrm{d}\theta\mathrm{d}x$$

where $v_j(t) = \partial_{x_j}\varphi(t)$ weak is the unique function in $C(\mathbb{R};L^2(\mathbb{R}^d;\mathbb{C}))$ that satisfies the first identity, and $\tilde{v}_j(t,\theta,x)$ is its BFZ transform. The left hand side can also be written as

$$\int_{\mathbb{R}^d}\partial_{x_j}\psi^*(x)\varphi(t,x)\mathrm{d}x = \int_{\mathbb{T}^d\times\mathbb{T}^d}(\widetilde{\partial_{x_j}\psi})^*(\theta,x)\tilde{\varphi}(t,\theta,x)\mathrm{d}\theta\mathrm{d}x$$

since one has an explicit representation for the BFZ of $\partial_{x_j}\psi$, this is

$$= \int_{\mathbb{T}^d\times\mathbb{T}^d}\tilde{\varphi}(t,\theta,x)\sum_{m\in\mathbb{Z}^d} e^{-2\pi i\theta\cdot(x-m)}\partial_{x_j}\psi^*(x-m)$$

which by the chain rule is

$$\begin{aligned}&= \int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x\tilde{\varphi}(t,\theta,x)\partial_{x_j}\left(\sum_{m\in\mathbb{Z}^d} e^{-2\pi i\theta\cdot(x-m)}\psi^*(x-m)\right)\\ &- \int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x\,\tilde{\varphi}(t,\theta,x)\sum_{m\in\mathbb{Z}^d}\partial_{x_j}(e^{-2\pi i\theta\cdot(x-m)})\psi^*(x-m)\\ &= \int_{\mathbb{T}^d\times\mathbb{T}^d}\tilde{\varphi}(t,\theta,x)(\partial_{x_j}(\tilde{\psi})^*(\theta,x) + 2\pi i\theta_j(\tilde{\psi})^*(\theta,x))\mathrm{d}\theta\mathrm{d}x\end{aligned}$$

Hence for all $t\in\mathbb{R}$ one has

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{T}^d}\tilde{\varphi}(t,\theta,x)\partial_{x_j}(\tilde{\psi})^*(\theta,x)\mathrm{d}\theta\mathrm{d}x =\\ &- \int_{\mathbb{T}^d\times\mathbb{T}^d}[\tilde{v}_j(t,\theta,x) + 2\pi i\theta_j\tilde{\varphi}(t,\theta,x)](\tilde{\psi})^*(\theta,x)\mathrm{d}\theta\mathrm{d}x\end{aligned}$$

Since $\tilde{v}_j(t)$ and $2\pi i\theta_j\tilde{\varphi}(t)$ are functions in $\mathscr{H}_\gamma$ for all $t\in\mathbb{R}, j\in\{1,\dots,d\}$ one writes

$$\partial_{x_j}\tilde{\varphi}(t) = \tilde{v}_j(t) + 2\pi i\theta_j\tilde{\varphi}(t)\,\text{weak on}\,\mathscr{H}_\gamma$$

One deduces that $\tilde{\varphi} \in C(\mathbb{R}; L^2_{\mathrm{loc},\theta}(\mathbb{R}^d; H^1_x(\mathbb{T}^d;\mathbb{C})))$. Rearranging it, one has

$$\tilde{v}_j(t) = \widetilde{\partial_{x_j}\varphi}(t) = \partial_{x_j}\tilde{\varphi}(t) - 2\pi i\theta_j\tilde{\varphi}(t) \text{ weak on } \mathscr{H}_\gamma \tag{2.7}$$

Hence

$$\widetilde{\nabla\varphi}(t) = \tilde{v}(t) = \begin{pmatrix} \tilde{v}_1(t) \\ \vdots \\ \tilde{v}_d(t) \end{pmatrix} = \nabla\tilde{\varphi}(t) - 2\pi i\theta\tilde{\varphi}(t) \text{ weak on } \mathscr{H}_\gamma$$

and by conjugating, one has

$$(\widetilde{\nabla\varphi})^*(t) = \tilde{v}^*(t) = \nabla(\tilde{\varphi})^*(t) + 2\pi i\theta(\tilde{\varphi})^*(t) \text{ weak on } \mathscr{H}_\gamma \tag{2.8}$$

Similarly one can deduce that $\tilde{\varphi} \in C(\mathbb{R}; L^2_{\mathrm{loc},\theta}(\mathbb{R}^d; H^2(\mathbb{T}^d;\mathbb{C})))$. For $j,k \in \{1,\dots,d\}$ and $\psi \in C^\infty_c(\mathbb{R}^d;\mathbb{C})$

$$\int_{\mathbb{R}^d}\varphi(t,x)\partial^2_{x_jx_k}\psi^*(x)\mathrm{d}x = \int_{\mathbb{R}^d}v_{jk}(t,x)\psi^*(x)\mathrm{d}x = \int_{\mathbb{T}^d\times\mathbb{T}^d}\tilde{v}_{jk}(t,\theta,x)(\tilde{\psi})^*(\theta,x)\mathrm{d}\theta\mathrm{d}x$$

where $v_{jk}(t)$ is the unique function in $L^2(\mathbb{R}^d;\mathbb{C})$ that satisfies the first identity, and $\tilde{v}_{jk}(t)$ is its Bloch–Floquet–Zak transform. The left hand side can also be written as

$$\int_{\mathbb{R}^d}\varphi(t,x)\partial^2_{x_jx_k}\psi^*(x)\mathrm{d}x = \int_{\mathbb{T}^d\times\mathbb{T}^d}\tilde{\varphi}(t,\theta,x)\widetilde{(\partial_{x_j}\partial_{x_k}\psi)}^*(\theta,x)\mathrm{d}\theta\mathrm{d}x$$

$$= \int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x\tilde{\varphi}(t,\theta,x)(\partial_{x_j}\widetilde{(\partial_{x_k}\psi)}^*(\theta,x) + 2\pi i\theta_j\widetilde{(\partial_{x_k}\psi)}^*(\theta,x))$$

and using equation (2.8) above, one has

$$= \int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x\tilde{\varphi}(t,\theta,x) \times$$

$$\times(\partial^2_{x_jx_k}\tilde{\psi}^*(\theta,x) + 2\pi i\theta_k\partial_{x_j}\tilde{\psi}^*(\theta,x) + 2\pi i\theta_j\partial_{x_k}\tilde{\psi}^*(\theta,x) - 4\pi^2\theta_j\theta_k\tilde{\psi}^*(\theta,x))$$

Now, we use that

$$\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x\tilde{\varphi}(t,\theta,x)\partial_{x_j}\tilde{\psi}^*(\theta,x) = -\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x(\tilde{v}_j(t) + 2\pi i\theta_j\tilde{\varphi}(t))\tilde{\psi}^*(\theta,x)$$

To write the above as

$$\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x\tilde{\varphi}(t,\theta,x)\partial^2_{x_jx_k}\tilde{\psi}^*(\theta,x)$$

$$-2\pi i\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x(\theta_k(\tilde{v}_j(t) + 2\pi i\theta_j\tilde{\varphi}(t)) + \theta_j(\tilde{v}_k(t) + 2\pi i\theta_k\tilde{\varphi}(t)))\tilde{\psi}^*(\theta,x)$$

$$-4\pi^2\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x\theta_j\theta_k\tilde{\varphi}(t,\theta,x)\tilde{\psi}^*(\theta,x)$$

Hence

$$\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x\tilde{\varphi}(t,\theta,x)\partial^2_{x_jx_k}\tilde{\psi}^*(\theta,x) =$$

$$\int_{\mathbb{T}^d\times\mathbb{T}^d}\tilde{v}_{jk}(t,\theta,x)(\tilde{\psi})^*(\theta,x)\mathrm{d}\theta\mathrm{d}x$$

$$+2\pi i\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x(\theta_k(\tilde{v}_j(t) + 2\pi i\theta_j\tilde{\varphi}(t)) + \theta_j(\tilde{v}_k(t) + 2\pi i\theta_k\tilde{\varphi}(t)))\tilde{\psi}^*(\theta,x)$$

$$+4\pi^2\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathrm{d}\theta\mathrm{d}x\theta_j\theta_k\tilde{\varphi}(t,\theta,x)\tilde{\psi}^*(\theta,x)$$

from which one deduces that $\tilde{\varphi} \in C(\mathbb{R}; L^2_{\mathrm{loc}}(\mathbb{R}^d; H^2(\mathbb{T}^d; \mathbb{C})))$ and one has that

$$\partial^2_{x_j x_k} \tilde{\varphi}(t) = \tilde{v}_{jk}(t) + 2\pi i(\theta_k(\tilde{v}_j(t) + 2\pi i \theta_j \tilde{\varphi}(t)) + \theta_j(\tilde{v}_k(t) + 2\pi i \theta_k \tilde{\varphi}(t))) + 4\pi^2 \theta_j \theta_k \tilde{\varphi}(t)$$

weak on $\mathscr{H}_\gamma$. Hence

$$\begin{aligned}
\Delta\tilde{\varphi}(t) = \sum_j \, &(\tilde{v}_{jj}(t) + 2\pi i(\theta_j(\tilde{v}_j(t) + 2\pi i \theta_j \tilde{\varphi}(t)) + \theta_j(\tilde{v}_j(t) + 2\pi i \theta_j \tilde{\varphi}(t))) + 4\pi^2 \theta_j^2 \tilde{\varphi}(t)) \\
&= \sum_j \, (\tilde{v}_{jj}(t) + 2\pi i(2\theta_j \tilde{v}_j(t) + 4\pi i \theta_j^2 \tilde{\varphi}(t)) + 4\pi^2 \theta_j^2 \tilde{\varphi}(t)) \\
&= \sum_j \, (\tilde{v}_{jj}(t) + 4\pi i \theta_j \tilde{v}_j(t) - 8\pi^2 i \theta_j^2 \tilde{\varphi}(t) + 4\pi^2 \theta_j^2 \tilde{\varphi}(t)) \\
&= \sum_j \, (\tilde{v}_{jj}(t) + 4\pi i \theta_j \tilde{v}_j(t) - 4\pi^2 i \theta_j^2 \tilde{\varphi}(t))
\end{aligned}$$

Finally, plugging in the expression for $\tilde{v}_j(t)$, this is

$$\begin{aligned}
&= \sum_j \, (\tilde{v}_{jj}(t) + 4\pi i \theta_j(\partial_{x_j} \tilde{\varphi}(t) - 2\pi i \theta_j \tilde{\varphi}(t)) - 4\pi^2 i \theta_j^2 \tilde{\varphi}(t)) \\
&= \sum_j \, (\tilde{v}_{jj}(t) + 4\pi i \theta_j \partial_{x_j} \tilde{\varphi}(t) + 8\pi^2 \theta_j^2 \tilde{\varphi}(t) - 4\pi^2 i \theta_j^2 \tilde{\varphi}(t)) \\
&= \sum_j \, (\tilde{v}_{jj}(t) + 4\pi i \theta_j \partial_{x_j} \tilde{\varphi}(t) + 4\pi^2 \theta_j^2 \tilde{\varphi}(t))
\end{aligned}$$

Rewriting it, one has

$$\sum_j \, \tilde{v}_{jj}(t) = \widetilde{\Delta\varphi}(t) = \Delta_x \tilde{\varphi}(t) - 4\pi^2 |\theta|^2 \tilde{\varphi}(t) - 4\pi i \theta \cdot \nabla_x \tilde{\varphi}(t) \text{ weak on } \mathscr{H}_\gamma$$

which is equation (2.5), and complex conjugation yields equation (2.6). □

Remark 2.3. Alternatively, one could compute directly for $\varphi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$ that

$$\begin{aligned}
\widetilde{\Delta_x \varphi}(\theta, x) &= \sum_{m \in \mathbb{Z}^d} e^{2\pi i \theta \cdot (x-m)} \Delta_x \varphi(x) = \sum_{j=1}^d \sum_{m \in \mathbb{Z}^d} e^{2\pi i \theta \cdot (x-m)} \partial^2_{x_j x_j} \varphi(x) \\
&= \sum_{j=1}^d \partial_{x_j} \left(\sum_{m \in \mathbb{Z}^d} e^{2\pi i \theta \cdot (x-m)} \partial_{x_j} \varphi(x) \right) - 2\pi i \sum_{j=1}^d \theta_j \sum_{m \in \mathbb{Z}^d} e^{2\pi i \theta \cdot (x-m)} \partial_{x_j} \varphi(x) \\
&= \sum_{j=1}^d \partial^2_{x_j x_j} \left(\sum_{m \in \mathbb{Z}^d} e^{2\pi i \theta \cdot (x-m)} \varphi(x) \right) - 2\pi i \sum_{j=1}^d \theta_j \partial_{x_j} \left(\sum_{m \in \mathbb{Z}^d} e^{2\pi i \theta \cdot (x-m)} \varphi(x) \right) \\
&\quad - 2\pi i \sum_{j=1}^d \theta_j \partial_{x_j} \left(\sum_{m \in \mathbb{Z}^d} e^{2\pi i \theta \cdot (x-m)} \varphi(x) \right) - 4\pi^2 \sum_{j=1}^d \theta_j^2 \sum_{m \in \mathbb{Z}^d} e^{2\pi i \theta \cdot (x-m)} \varphi(x) \\
&= \sum_{j=1}^d \partial^2_{x_j x_j} \tilde{\varphi}(\theta, x) - 4\pi i \sum_{j=1}^d \theta_j \partial_{x_j} \tilde{\varphi}(\theta, x) - 4\pi^2 \sum_{j=1}^d \theta_j^2 \tilde{\varphi}(\theta, x) \\
&= \Delta_x \tilde{\varphi}(\theta, x) - 4\pi i \theta \cdot \nabla_x \tilde{\varphi}(\theta, x) - 4\pi^2 |\theta|^2 \tilde{\varphi}(\theta, x),
\end{aligned}$$

and make a density argument using the unitarity of the BFZ transform. One might shorten the computation for the second derivative by recursively using the computation for the first derivative.

Definition 2.4. *Let $\varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$. For $\theta, x \in \mathbb{R}^d$ define the classical Bloch–Floquet transform $\mathcal{U}_{\text{cl}}\varphi$ of φ by*

$$(\mathcal{U}_{\text{cl}}\varphi)(\theta,x) := \sum_{m\in\mathbb{Z}^d} e^{-2\pi i\theta\cdot m}\varphi(x-m) \tag{2.9}$$

One has that

$$(\mathcal{U}_{\text{BFZ}}\varphi)(\theta,x) = e^{2\pi i\theta\cdot x}(\mathcal{U}_{\text{cl}}\varphi)(\theta,x)$$

Remark 2.5. Note that $\mathcal{U}_{\text{cl}}\varphi$ is $\mathbb{Z}^d$-periodic in θ and $\mathbb{Z}^d$-quasiperiodic in x. For certain applications, it is more useful to work with $\mathcal{U}_{\text{cl}}$ than with $\mathcal{U}_{\text{BFZ}}$. The two operators can be related by a unitary operator, and have the same spectrum. See Remark 1 of [MP14] for the details.

We list some useful properties of $\mathcal{U}_{cl}\varphi$ here:

1. One has the following inversion formula for $\varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$:

$$\varphi(x) = \int_{\mathbb{T}^d} \mathrm{d}\theta (\mathcal{U}_{\text{cl}}\varphi)(\theta,x).$$

2. (Plancherel) One has for $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$,

$$\begin{aligned}
&\int_{\mathbb{T}^d}\mathrm{d}\theta\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}x(\mathcal{U}_{\text{cl}}\phi_1)(\theta,x)(\mathcal{U}_{\text{cl}}\phi_2)^*(\theta,x)\\
=&\int_{\mathbb{T}^d}\mathrm{d}\theta\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}x\sum_{m,m'\in\mathbb{Z}^d} e^{-2\pi i\theta\cdot m}\phi_1(x-m)e^{2\pi i\theta\cdot m'}\phi_2^*(x-m')\\
=&\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}x\sum_{m\in\mathbb{Z}^d}\phi_1(x-m)\phi_2^*(x-m) = \int_{\mathbb{R}^d}\mathrm{d}x\phi_1(x)\phi_2^*(x).
\end{aligned}$$

   One can extend $\mathcal{U}_{cl}$ to a unitary map from $L^2(\mathbb{R}^d;\mathbb{C}) \longrightarrow L^2_{\text{loc},x}(\mathbb{R}^d; L^2_\theta(\mathbb{T}^d;\mathbb{C}))$.

2.2 The Wigner and rescaled Wigner transforms

We recall the definition of the Wigner transform for $\varphi \in L^2(\mathbb{R}^d;\mathbb{C})$: for $x, k \in \mathbb{R}^d$

$$W_\varphi(x,k) := \int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi\Big(x-\frac{y}{2}\Big)\varphi^*\Big(x+\frac{y}{2}\Big)\mathrm{d}y$$

This defines a real-valued function on $\mathbb{R}^{2d}$.

Lemma 2.6. *Assume $\varphi \in L^2(\mathbb{R}^d;\mathbb{C})$. Then $W_\varphi \in L^\infty(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d})$. Furthermore,*

$$\int_{\mathbb{R}^d}\mathrm{d}x\, W_\varphi(x,k) = |\hat{\varphi}(k)|^2, \qquad \int_{\mathbb{R}^d}\mathrm{d}k\, W_\varphi(x,k) = |\varphi(x)|^2. \tag{2.10}$$

almost surely.

These are well-known, but we include a direct proof below for the reader's convenience.

Proof. We first show (2.6). To see this first assume $\varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$. Then

$$\int_{\mathbb{R}^d} \mathrm{d}x\, W_\varphi(x,k) = \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y e^{2\pi i k\cdot y} \varphi\Big(x - \frac{y}{2}\Big) \varphi^*\Big(x + \frac{y}{2}\Big).$$

Perform the change of variables: $u = x - \frac{y}{2}, v = x + \frac{y}{2}$. Then $x = \frac{u+v}{2}, y = v - u$. The Jacobian of this transformation is 1. Hence we rewrite

$$\int_{\mathbb{R}^d} \mathrm{d}x\, W_\varphi(x,k) = \int_{\mathbb{R}^d} \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}v e^{2\pi i k\cdot (v-u)} \varphi(u) \varphi^*(v)$$

$$= \int_{\mathbb{R}^d} \mathrm{d}u e^{-2\pi i k\cdot u} \mathrm{d}u \varphi(u) \int_{\mathbb{R}^d} \mathrm{d}v e^{2\pi i k\cdot v} \varphi^*(v) = |\hat{\varphi}(k)|^2.$$

Next, consider for $\varepsilon \in (0,1]$,

$$I_\varepsilon(x) = \int_{\mathbb{R}^d} \mathrm{d}k e^{-\varepsilon^2|k|^2} W_\varphi(x,k) = \int_{\mathbb{R}^d} \mathrm{d}k e^{-\varepsilon^2|k|^2} \int_{\mathbb{R}^d} \mathrm{d}y e^{2\pi i k\cdot y} \varphi\Big(x - \frac{y}{2}\Big) \varphi^*\Big(x + \frac{y}{2}\Big).$$

By the dominated convergence theorem, since $|e^{-\varepsilon^2|k|^2} W_\varphi(x,k)| \leq |W_\varphi(x,k)| \in L^1_k(\mathbb{R}^d)$ we have that

$$\lim_{\varepsilon \to 0} I_\varepsilon(x) = \int_{\mathbb{R}^d} \mathrm{d}k \lim_{\varepsilon \to 0} e^{-\varepsilon^2|k|^2} W_\varphi(x,k) = \int_{\mathbb{R}^d} \mathrm{d}k\, W_\varphi(x,k).$$

By Fubini's theorem,

$$I_\varepsilon(x) = \int_{\mathbb{R}^d} \mathrm{d}y \varphi\Big(x - \frac{y}{2}\Big) \varphi^*\Big(x + \frac{y}{2}\Big) \int_{\mathbb{R}^d} \mathrm{d}k e^{-\varepsilon^2 \pi |k|^2} e^{2\pi i k\cdot y}$$

$$= \varepsilon^{-d} \int_{\mathbb{R}^d} \mathrm{d}y \varphi\Big(x - \frac{y}{2}\Big) \varphi^*\Big(x + \frac{y}{2}\Big) e^{-\frac{\pi}{\varepsilon^2}|y|^2} = \int_{\mathbb{R}^d} \mathrm{d}z \varphi\Big(x - \frac{\varepsilon z}{2}\Big) \varphi^*\Big(x + \frac{\varepsilon z}{2}\Big) e^{-\pi|z|^2}.$$

Putting this together,

$$\int_{\mathbb{R}^d} \mathrm{d}k\, W_\varphi(x,k) = \lim_{\varepsilon \to 0} I_\varepsilon(x) = \int_{\mathbb{R}^d} \mathrm{d}z \lim_{\varepsilon \to 0} \varphi\Big(x - \frac{\varepsilon z}{2}\Big) \varphi^*\Big(x + \frac{\varepsilon z}{2}\Big) e^{-\pi|z|^2}$$

$$= |\varphi(x)|^2 \int_{\mathbb{R}^d} \mathrm{d}z e^{-\pi|z|^2} = |\varphi(x)|^2.$$

One can then use a density argument to extend this to $\varphi \in L^2(\mathbb{R}^d;\mathbb{C})$.

To see $W_\varphi \in L^\infty(\mathbb{R}^{2d})$, we compute

$$|W_\varphi(x,k)| = \Big|\int_{\mathbb{R}^d} \mathrm{d}y e^{2\pi i k\cdot y} \varphi\Big(x - \frac{y}{2}\Big) \varphi^*\Big(x + \frac{y}{2}\Big)\Big| \leq \int_{\mathbb{R}^d} \mathrm{d}y \Big|\varphi\Big(x - \frac{y}{2}\Big) \varphi^*\Big(x + \frac{y}{2}\Big)\Big|.$$

By using the Cauchy–Schwarz inequality, this is

$$\leq \Big(\int_{\mathbb{R}^d} \mathrm{d}y \Big|\varphi\Big(x - \frac{y}{2}\Big)\Big|^2\Big)^{1/2} \Big(\int_{\mathbb{R}^d} \mathrm{d}y \Big|\varphi^*\Big(x + \frac{y}{2}\Big)\Big|^2\Big)^{1/2} \lesssim \|\varphi\|^2_{L^2}$$

For the L^2 bound, consider $\varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$. Then

$$\int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}k |W_\varphi(x,k)|^2 = \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}k e^{-\varepsilon^2 \pi |k|^2} |W_\varphi(x,k)|^2$$

$$= \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}k e^{-\varepsilon^2 \pi |k|^2} \left| \int_{\mathbb{R}^d} \mathrm{d}y e^{2\pi i k \cdot y} \varphi\left(x - \frac{y}{2}\right) \varphi^*\left(x + \frac{y}{2}\right) \right|^2$$

$$= \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}z e^{2\pi i k \cdot (y-z)} e^{-\varepsilon^2 \pi |k|^2}$$

$$\varphi\left(x - \frac{y}{2}\right) \varphi^*\left(x + \frac{y}{2}\right) \varphi^*\left(x - \frac{z}{2}\right) \varphi\left(x + \frac{z}{2}\right).$$

By using Fubini's theorem and the dominated convergence theorem, this is

$$= \lim_{\varepsilon \longrightarrow 0} \varepsilon^{-d} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}z e^{-\varepsilon^{-2} \pi |z-y|^2} \varphi\left(x - \frac{y}{2}\right) \varphi^*\left(x + \frac{y}{2}\right) \varphi^*\left(x - \frac{z}{2}\right) \varphi\left(x + \frac{z}{2}\right)$$

$$= \lim_{\varepsilon \longrightarrow 0} \varepsilon^{-d} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}z' e^{-\varepsilon^{-2} \pi |z'|^2} \varphi\left(x - \frac{y}{2}\right) \varphi^*\left(x + \frac{y}{2}\right)$$

$$\varphi^*\left(x - \frac{y}{2} - \frac{z'}{2}\right) \varphi\left(x + \frac{y}{2} + \frac{z'}{2}\right)$$

$$= \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}r e^{-\pi |r|^2} \varphi\left(x - \frac{y}{2}\right) \varphi^*\left(x + \frac{y}{2}\right)$$

$$\varphi^*\left(x - \frac{y}{2} - \frac{\varepsilon r}{2}\right) \varphi\left(x + \frac{y}{2} + \frac{\varepsilon r}{2}\right)$$

$$= \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \left| \varphi\left(x - \frac{y}{2}\right) \right|^2 \left| \varphi^*\left(x + \frac{y}{2}\right) \right|^2$$

$$= \int_{\mathbb{R}^d} \mathrm{d}x |\varphi(x)|^2 \int_{\mathbb{R}^d} \mathrm{d}y |\varphi^*(x+y)|^2$$

$$= \int_{\mathbb{R}^d} \mathrm{d}x |\varphi(x)|^2 \|\varphi\|_{L^2}^2 = \|\varphi\|_{L^2}^4. \qquad \square$$

In kinetic theory, we pass from microscopic scales to macroscopic scales. As explained in Chapter 1, an important object in quantum kinetic theory is rescaled Wigner function W^ε from expression (1.4). As we will subsequently explain, one can equivalently work with the following definition:

$$W_\varphi^\varepsilon(t,x,k) := W_{\varphi\left(\frac{t}{\varepsilon}\right)}(x,k) = \int_{\mathbb{R}^d} e^{2\pi i k \cdot y} \varphi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} - \frac{y}{2}\right) \varphi^*\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} + \frac{y}{2}\right) \mathrm{d}y \tag{2.11}$$

Note that this differs from expression (1.4) by a factor of ε^{-d}.

Defining the scaling operator $S_\varepsilon f(x) = f(\frac{x}{\varepsilon})$, one has

$$\|S_\varepsilon f\|_{L^2(\mathbb{R}^d)} = \varepsilon^{d/2} \|f\|_{L^2(\mathbb{R}^d)}, \qquad \|S_\varepsilon f\|_{L^\infty(\mathbb{R}^d)} = \|f\|_{L^\infty(\mathbb{R}^d)}.$$

So. writing $S_{\varepsilon,x} f(t,x,k) := f\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, k\right)$ one has

$$\|W_\varphi^\varepsilon(t)\|_{L^2(\mathbb{R}^{2d})} = \|S_{\varepsilon,x} W_\varphi(t)\|_{L^2(\mathbb{R}^{2d})} = \varepsilon^{d/2} \left\| W_\varphi\left(\frac{t}{\varepsilon}\right) \right\|_{L^2(\mathbb{R}^{2d})} = \varepsilon^{d/2} \left\| \varphi\left(\frac{t}{\varepsilon}\right) \right\|_{L^2(\mathbb{R}^d;\mathbb{C})}^2.$$

On the other hand

$$\|W_\varphi^\varepsilon(t)\|_{L^\infty(\mathbb{R}^{2d})} = \|S_{\varepsilon,x} W_\varphi(t)\|_{L^\infty(\mathbb{R}^{2d})} = \left\| W_\varphi\left(\frac{t}{\varepsilon}\right)\right\|_{L^\infty(\mathbb{R}^{2d})} \leqslant \left\|\varphi\left(\frac{t}{\varepsilon}\right)\right\|^2_{L^2(\mathbb{R}^d;\mathbb{C})}.$$

An important aspect for problems in kinetic theory is the following: one wants to move from microscopic scales to macroscopic scales, and one needs to specify initial data that interpolates between these scales. For instance, if one fixes a microscopic data $\varphi \in C_c^\infty(\mathbb{R}^d)$ supported in $B_1(0)$, then on the macroscopic scale this looks like $\varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon})$ which is supported in $B_\varepsilon(0)$, and disappears in the limit.

Hence, in order to have a non-trivial macroscopic limit, one should not consider $W_\varphi^\varepsilon(x,k)$ for a fixed $\varphi \in L^2(\mathbb{R}^d;\mathbb{C})$. Instead, one needs to consider $W_{\varphi_\varepsilon}^\varepsilon(x,k)$ for a suitably chosen ε-dependent family $\varphi_\varepsilon(x)$.

Example 2.7. (WKB family) Let

$$\varphi_\varepsilon(x) = h(\varepsilon x) e^{2\pi i S(\varepsilon x)/\varepsilon}$$

for $h, S \in \mathcal{S}(\mathbb{R}^d)$. Here

$$\|\varphi_\varepsilon\|_{L^2}^2 = \int_{\mathbb{R}^d} \mathrm{d}x |h(\varepsilon x)|^2 = \varepsilon^{-d} \|h\|^2_{L^2(\mathbb{R}^d)}.$$

One then has that for any $f \in \mathcal{S}(\mathbb{R}^{2d})$

$$\lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^d} \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}x\, W_{\varphi_\varepsilon}^\varepsilon(x,k) f(x,k)$$

$$= \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^d} \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y e^{2\pi i k\cdot y} h\left(x - \frac{\varepsilon y}{2}\right) h\left(x + \frac{\varepsilon y}{2}\right) e^{2\pi i \varepsilon^{-1}(S(x - \frac{\varepsilon y}{2}) - S(x + \frac{\varepsilon y}{2}))} f(x,k)$$

Now, since $S \in \mathcal{S}(\mathbb{R}^d)$, one has that $S(x - \frac{\varepsilon y}{2}) - S(x + \frac{\varepsilon y}{2}) = -\varepsilon y \cdot \nabla S(x) + O(\varepsilon^2)$, so this is

$$= \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^d} \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y e^{2\pi i (k - \nabla S(x))\cdot y} e^{O(\varepsilon)} h\left(x - \frac{\varepsilon y}{2}\right) h\left(x + \frac{\varepsilon y}{2}\right) f(x,k).$$

By using Fubini's theorem and the dominated convergence theorem, this is

$$= \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y e^{-2\pi i \nabla S(x)\cdot y} e^{O(\varepsilon)} h\left(x - \frac{\varepsilon y}{2}\right) h\left(x + \frac{\varepsilon y}{2}\right) \mathcal{F}f(x,-y)$$

$$= \int_{\mathbb{R}^d} |h(x)|^2 \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y e^{-2\pi i \nabla S(x)\cdot y} \mathcal{F}f(x,-y)$$

$$= \int_{\mathbb{R}^d} |h(x)|^2 \mathrm{d}x f(x, \nabla S(x)).$$

This computation shows that $W_{\varphi_\varepsilon}^\varepsilon(x,k)$ converges as a tempered distribution to a positive Radon measure $|h(x)|^2 \delta_{\nabla S(x)}(\mathrm{d}k)$.

Example 2.8. (Mixture of states) Let us now introduce a source of randomness into the initial family, via a Gaussian random vector q in $\mathbb{R}^d$ with independent entries, and for some $h \in \mathcal{S}(\mathbb{R}^d)$ let

$$\varphi_\varepsilon(x,q) = h(\varepsilon x) e^{2\pi i q\cdot x}.$$

Here for any $q \in \mathbb{R}^d$,

$$\|\varphi_\varepsilon(\cdot,q)\|_{L^2}^2 = \int_{\mathbb{R}^d} \mathrm{d}x |h(\varepsilon x)|^2 = \varepsilon^{-d} \|h\|_{L^2(\mathbb{R}^d)}^2.$$

Now, defining the rescaled Wigner transform to average over the source of randomness via

$$W_{\varphi_\varepsilon}^\varepsilon(x,k) := \int_{\mathbb{R}^d} \mathrm{d}q e^{-\frac{\pi|q|^2}{2}} \varphi_\varepsilon^*\Big(\frac{x}{\varepsilon}+\frac{y}{2},q\Big)\varphi_\varepsilon\Big(\frac{x}{\varepsilon}-\frac{y}{2},q\Big),$$

one has

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0} W_{\varphi_\varepsilon}^\varepsilon(x,k)\\
=&\lim_{\varepsilon\longrightarrow 0}\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d}\mathrm{d}q e^{-\frac{\pi|q|^2}{2}} e^{2\pi i k\cdot y} h\Big(x-\frac{\varepsilon y}{2}\Big)h\Big(x+\frac{\varepsilon y}{2}\Big)e^{-2\pi i q\cdot(\frac{x}{\varepsilon}+\frac{y}{2})}e^{2\pi i q\cdot(\frac{x}{\varepsilon}-\frac{y}{2})}\\
=&\lim_{\varepsilon\longrightarrow 0}\int_{\mathbb{R}^d}\mathrm{d}y e^{2\pi i k\cdot y} h\Big(x-\frac{\varepsilon y}{2}\Big)h\Big(x+\frac{\varepsilon y}{2}\Big)\int_{\mathbb{R}^d}\mathrm{d}q e^{-\frac{\pi|q|^2}{2}} e^{-2\pi i q\cdot y}\\
=&\lim_{\varepsilon\longrightarrow 0} 2^{d/2}\int_{\mathbb{R}^d}\mathrm{d}y e^{2\pi i k\cdot y} h\Big(x-\frac{\varepsilon y}{2}\Big)h\Big(x+\frac{\varepsilon y}{2}\Big)e^{-2\pi|y|^2}.
\end{aligned}$$

By the dominated convergence theorem, this is

$$= |h(x)|^2 \lim_{\varepsilon\longrightarrow 0} 2^{d/2}\int_{\mathbb{R}^d}\mathrm{d}y e^{2\pi i k\cdot y} e^{-2\pi|y|^2} = |h(x)|^2 e^{-\frac{\pi|k|^2}{2}}$$

pointwise. This implies convergence in $L^2(\mathbb{R}^{2d})$ and $L^\infty(\mathbb{R}^{2d})$ of $W_{\varphi_\varepsilon}^\varepsilon$.

For the limits of other important families of initial data, see [LP93].

For families φ_ε such that $\varepsilon^{d/2}\|\varphi_\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{C})} \leq C$ for some C that is independent of ε, one has that the rescaled Wigner transforms $W_{\varphi_\varepsilon}^\varepsilon$ are uniformly bounded as tempered distributions, i.e., for any $f \in \mathscr{S}(\mathbb{R}^d)$, and

$$\langle W_{\varphi_\varepsilon}^\varepsilon, f\rangle \leq C,$$

uniformly in ε. One can thus take weak limits along subsequences, and one can show that any limit obtained must be a positive Radon measure. For instance, in [LP93], the authors identified the Banach space

$$\mathscr{A} := \{f \in C_0(\mathbb{R}_x^d \times \mathbb{R}_k^d) : \mathscr{F}_k f(x,\xi) \in L^1(\mathbb{R}_\xi^d; C_0(\mathbb{R}_x^d))\}, \tag{2.12}$$

which is a Banach algebra when equipped with the norm

$$\|\mathscr{F}_k f\|_{\mathscr{A}} := \int_{\mathbb{R}^d} \mathrm{d}\xi \sup_{x\in\mathbb{R}^d} |\mathscr{F}_k f|(x,\xi).$$

They show in Proposition III.1 that for $\varphi_\varepsilon(x) = u_\varepsilon(\varepsilon x)$, with $\|u_\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{C})} \leq C$ that

$$\begin{aligned}
W_{\varphi_\varepsilon}^\varepsilon(x,k) &= \int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi_\varepsilon\Big(\frac{x}{\varepsilon}-\frac{y}{2}\Big)\varphi_\varepsilon^*\Big(\frac{x}{\varepsilon}+\frac{y}{2}\Big)\mathrm{d}y\\
&= \int_{\mathbb{R}^d} e^{2\pi i k\cdot y} u_\varepsilon\Big(x-\frac{\varepsilon y}{2}\Big)u_\varepsilon^*\Big(x+\frac{\varepsilon y}{2}\Big)\mathrm{d}y
\end{aligned}$$

is uniformly bounded in $\mathscr{A}^*$. We include the explicit computation below for the reader's convenience. For $\phi \in \mathscr{A}$ consider

$$\int_{\mathbb{R}^{2d}} \mathrm{d}x\mathrm{d}k\, W^{\varepsilon}_{\varphi_\varepsilon}(x,k)\phi(x,k) = \int_{\mathbb{R}^{2d}} \mathrm{d}x\mathrm{d}k \int_{\mathbb{R}^d} e^{2\pi i k\cdot y} u_\varepsilon\Big(x - \frac{\varepsilon y}{2}\Big) u_\varepsilon^*\Big(x + \frac{\varepsilon y}{2}\Big) \mathrm{d}y \phi(x,k)$$

$$= \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y\, u_\varepsilon\Big(x - \frac{\varepsilon y}{2}\Big) u_\varepsilon^*\Big(x + \frac{\varepsilon y}{2}\Big) \mathscr{F}_k \phi(x,-y)$$

Hence

$$\Big|\int_{\mathbb{R}^{2d}} \mathrm{d}x\mathrm{d}k\, W^{\varepsilon}_{\varphi_\varepsilon}(x,k)\phi(x,k)\Big|$$

$$\leqslant \int_{\mathbb{R}^d} \mathrm{d}y \sup_{x\in\mathbb{R}^d} |\mathscr{F}_k\phi|(x,-y) \int_{\mathbb{R}^d} \mathrm{d}x \Big| u_\varepsilon\Big(x - \frac{\varepsilon y}{2}\Big) u_\varepsilon^*\Big(x + \frac{\varepsilon y}{2}\Big)\Big|$$

$$\leqslant \int_{\mathbb{R}^d} \mathrm{d}y \sup_{x\in\mathbb{R}^d} |\mathscr{F}_k\phi|(x,-y) \Big(\int_{\mathbb{R}^d} |\tau_{\frac{\varepsilon y}{2}} u_\varepsilon|^2(x)\mathrm{d}x\Big)^{1/2} \Big(\int_{\mathbb{R}^d} |\tau_{-\frac{\varepsilon y}{2}} u_\varepsilon|^2(x)\mathrm{d}x\Big)^{1/2}$$

$$= \int_{\mathbb{R}^d} \mathrm{d}y \sup_{x\in\mathbb{R}^d} |\mathscr{F}_k\phi|(x,-y) \|u_\varepsilon\|_{L^2}^2 \leqslant C_\varphi \|u_\varepsilon\|_{L^2}^2$$

In Theorem III.1 of [LP93], they show that any limit point of such a family $\{W^{\varepsilon}_{\varphi_\varepsilon}\}_{\varepsilon\in(0,1]}$ is a positive Radon measure.

They also show conditions for the convergence of the energy. For other related early results on rescaled Wigner transforms and their limits, see [GMMP98] and [Gér90].

Remark 2.9. There are some alternative definitions that one can find in the literature. Each of these definitions interpolates between the microscopic and macroscopic definitions in slightly different but ultimately equivalent ways.

For instance, in [EY99], the authors work with expression (1.4), which we introduced in Chapter 1.

$$W^{\varepsilon}(t,x,k) := \varepsilon^{-d} W_\varphi\Big(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, k\Big),$$

In this case one has

$$\|W^{\varepsilon}\|(t)_{L^\infty(\mathbb{R}^{2d})} = \varepsilon^{-d}\Big\|W_\varphi\Big(\frac{t}{\varepsilon}\Big)\Big\|_{L^\infty(\mathbb{R}^{2d})} \leqslant \varepsilon^{-d}\Big\|\varphi\Big(\frac{t}{\varepsilon}\Big)\Big\|^2_{L^2(\mathbb{R}^d;\mathbb{C})},$$

and

$$\|W^{\varepsilon}(t)\|^2_{L^2(\mathbb{R}^{2d})} = \varepsilon^{-2d} \int_{\mathbb{R}^d} \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}x \Big| \int_{\mathbb{R}^d} \mathrm{d}y\, e^{2\pi i k\cdot y} \varphi\Big(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} - \frac{y}{2}\Big) \varphi^*\Big(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} + \frac{y}{2}\Big)\Big|^2$$

$$= \varepsilon^{-d} \int_{\mathbb{R}^d} \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}x \Big| \int_{\mathbb{R}^d} \mathrm{d}y\, e^{2\pi i k\cdot y} \varphi\Big(\frac{t}{\varepsilon}, x - \frac{y}{2}\Big) \varphi^*\Big(\frac{t}{\varepsilon}, x + \frac{y}{2}\Big)\Big|^2$$

$$= \varepsilon^{-d} \Big\|W_\varphi\Big(\frac{t}{\varepsilon}\Big)\Big\|^2_{L^2(\mathbb{R}^{2d})} = \varepsilon^{-d}\Big\|\varphi\Big(\frac{t}{\varepsilon}\Big)\Big\|^4_{L^2(\mathbb{R}^d;\mathbb{C})}$$

$$\Longrightarrow \Big\|W^{1,\varepsilon}_\varphi\Big(\frac{t}{\varepsilon}\Big)\Big\|_{L^2(\mathbb{R}^{2d})} = \varepsilon^{-d/2}\Big\|\varphi\Big(\frac{t}{\varepsilon}\Big)\Big\|^2_{L^2(\mathbb{R}^d;\mathbb{C})}.$$

One has that

$$\|W^{1,\varepsilon}_\varphi(t)\|_{L^1(\mathbb{R}^{2d})} = \Big\|W_\varphi\Big(\frac{t}{\varepsilon}\Big)\Big\|_{L^1(\mathbb{R}^{2d})},$$

but the RHS cannot be controlled using the L^2-norm of φ. This inequality is thus not very useful for quantum mechanics, but has some applications in areas of time-frequency analysis. Another definition one can find in the literature (for instance in [Gom13]) is, for $\varphi_\varepsilon \in L^2(\mathbb{R}^d;\mathbb{C})$,

$$
\begin{aligned}
W^{2,\varepsilon}_{\varphi_\varepsilon}(t,x,k) &:= \int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi_\varepsilon\Big(t, x-\frac{\varepsilon y}{2}\Big)\varphi^*_\varepsilon\Big(t, x+\frac{\varepsilon y}{2}\Big)\mathrm{d}y \\
&= \varepsilon^{-d}\int_{\mathbb{R}^d} e^{2\pi i \varepsilon^{-1} k\cdot z}\varphi_\varepsilon\Big(t, x-\frac{z}{2}\Big)\varphi^*_\varepsilon\Big(t, x+\frac{z}{2}\Big)\mathrm{d}z = \varepsilon^{-d} W_{\varphi_\varepsilon}\Big(t, x, \frac{k}{\varepsilon}\Big).
\end{aligned}
$$

Once more one has that

$$
\| W^{2,\varepsilon}_{\varphi_\varepsilon}(t)\|_{L^\infty(\mathbb{R}^{2d})} = \varepsilon^{-d}\| W_{\varphi_\varepsilon}(t)\|_{L^\infty(\mathbb{R}^{2d})} \leq \varepsilon^{-d}\|\varphi_\varepsilon(t)\|^2_{L^2(\mathbb{R}^d;\mathbb{C})},
$$

and

$$
\begin{aligned}
\| W^{2,\varepsilon}_{\varphi_\varepsilon}(t)\|^2_{L^2(\mathbb{R}^{2d})} &= \varepsilon^{-2d}\int_{\mathbb{R}^d}\mathrm{d}k\int_{\mathbb{R}^d}\mathrm{d}x\left| W_{\varphi_\varepsilon}\Big(t,x,\frac{k}{\varepsilon}\Big)\right|^2 = \varepsilon^{-d}\int_{\mathbb{R}^d}\mathrm{d}k\int_{\mathbb{R}^d}\mathrm{d}x| W_{\varphi_\varepsilon}(t,x,k)|^2 \\
&= \varepsilon^{-d}\| W_{\varphi_\varepsilon}(t)\|^2_{L^2(\mathbb{R}^{2d})} = \varepsilon^{-d}\|\varphi_\varepsilon(t)\|^4_{L^2(\mathbb{R}^d;\mathbb{C})}.
\end{aligned}
$$

Hence

$$
\| W^{2,\varepsilon}_{\varphi_\varepsilon}(t)\|_{L^2(\mathbb{R}^{2d})} = \varepsilon^{-d/2}\|\varphi_\varepsilon(t)\|^2_{L^2(\mathbb{R}^d;\mathbb{C})}.
$$

Finally, $\| W^{2,\varepsilon}_{\varphi_\varepsilon}(t)\|_{L^1(\mathbb{R}^{2d})} = \| W_\varphi(t)\|_{L^1(\mathbb{R}^{2d})}$.

We conclude this section by considering the time evolution of the rescaled Wigner transform associated to the solution of the linear Schrödinger equation with a non-zero potential term. Assume $\varphi \in C(\mathbb{R}_{\geq 0}; H^2(\mathbb{R}^d;\mathbb{C})) \cap C^1(\mathbb{R}_+; L^2(\mathbb{R}^d;\mathbb{C}))$ satisfies the linear Schrödinger equation (1.1). One has from (2.11) that

$$
\begin{aligned}
\partial_t W^\varepsilon_\varphi(t,x,k) &= \varepsilon^{-1}\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\partial_t\varphi\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\Big)\varphi^*\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\Big)\mathrm{d}y \\
&\quad + \varepsilon^{-1}\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\Big)\partial_t\varphi^*\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\Big)\mathrm{d}y \\
&= \varepsilon^{-1}\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\Big(i\varepsilon^{-2}\Delta_x\varphi\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\Big) - i\lambda V\Big(\frac{x}{\varepsilon}-\frac{y}{2}\Big)\varphi\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\Big)\Big)\varphi^*\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\Big)\mathrm{d}y \\
&\quad + \varepsilon^{-1}\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\Big)\Big(-i\varepsilon^{-2}\Delta_x\varphi^*\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\Big) + \lambda V\Big(\frac{x}{\varepsilon}+\frac{y}{2}\Big)\varphi^*\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\Big)\Big)\mathrm{d}y.
\end{aligned}
$$

Noting that

$$
\partial_{x_j}\varphi\Big(\frac{x}{\varepsilon}-\frac{y}{2}\Big) = -\frac{\varepsilon^{-1}}{2}\partial_{y_j}\varphi\Big(\frac{x}{\varepsilon}-\frac{y}{2}\Big),
$$

the terms with the Laplacians are

$$
\begin{aligned}
&= \frac{i\varepsilon^{-1}}{4}\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\Delta_y\varphi\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\Big)\varphi^*\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\Big)\mathrm{d}y \\
&\quad - \frac{i\varepsilon^{-1}}{4}\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\Big)\Delta_y\varphi^*\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\Big)\mathrm{d}y.
\end{aligned}
$$

Integrating by parts in y and using the product rule, this is

$$
\begin{aligned}
&= -\frac{i\varepsilon^{-1}}{4}\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\nabla_y\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\right)\nabla_y\varphi^*\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\right)\mathrm{d}y\\
&+\frac{\varepsilon^{-1}}{4}2\pi k\cdot\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\nabla_y\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\right)\varphi^*\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\right)\mathrm{d}y\\
&+\frac{\varepsilon^{-1}}{4}\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\nabla_y\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\right)\nabla_y\varphi^*\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\right)\mathrm{d}y\\
&-\frac{\varepsilon^{-1}}{4}2\pi k\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\right)\nabla_y\varphi^*\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\right)\mathrm{d}y\\
&= -\pi k\cdot\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\nabla_x\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\right)\varphi^*\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\right)\mathrm{d}y\\
&-\pi k\cdot\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\right)\nabla_x\varphi^*\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\right)\mathrm{d}y\\
&= -\pi k\cdot\nabla_x W_\varphi^\varepsilon(t,x,k).
\end{aligned}
$$

Finally, the potential terms are

$$
= -i\lambda\varepsilon^{-1}\int_{\mathbb{R}^d}\mathrm{d}y\, e^{2\pi i k\cdot y}\left(V\left(\frac{x}{\varepsilon}-\frac{y}{2}\right)-V\left(\frac{x}{\varepsilon}+\frac{y}{2}\right)\right)\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\right)\varphi^*\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\right).
$$

Formally, using the Fourier transform, $V(x)=\int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x}\hat{V}(\xi)$ one has that this is

$$
\begin{aligned}
&= -i\lambda\varepsilon^{-1}\int_{\mathbb{R}^d}\mathrm{d}\xi\hat{V}(\xi)\,e^{2\pi i\varepsilon^{-1}\xi\cdot x}\\
&\int_{\mathbb{R}^d}\mathrm{d}y\left(e^{2\pi i\left(k-\frac{\xi}{2}\right)\cdot y}-e^{2\pi i\left(k+\frac{\xi}{2}\right)\cdot y}\right)\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\right)\varphi^*\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\right)\\
&= i\lambda\varepsilon^{-1}\int_{\mathbb{R}^d}\mathrm{d}\xi\hat{V}(\xi)\,e^{2\pi i\varepsilon^{-1}\xi\cdot x}\left(W_\varphi^\varepsilon\left(t,x,k+\frac{\xi}{2}\right)-W_\varphi^\varepsilon\left(t,x,k-\frac{\xi}{2}\right)\right).
\end{aligned}
$$

Hence, W_φ^ε satisfies the PDE

$$
\partial_t W_\varphi^\varepsilon+\pi k\cdot\nabla_x W_\varphi^\varepsilon = i\lambda\varepsilon^{-1}\int_{\mathbb{R}^d}\mathrm{d}\xi\hat{V}(\xi)\,e^{2\pi i\varepsilon^{-1}\xi\cdot x}\left(W_\varphi^\varepsilon\left(k+\frac{\xi}{2}\right)-W_\varphi^\varepsilon\left(k-\frac{\xi}{2}\right)\right). \tag{2.13}
$$

This equation is known to conserve the L^2-norm. Formally, this can be seen by multiplying this equation by W_φ^ε and integrating in x and k. One has that

$$
\int_{\mathbb{R}^{2d}}\mathrm{d}x\mathrm{d}k\,\partial_t W_\varphi^\varepsilon(t,x,k)\,W_\varphi^\varepsilon(t,x,k)=\frac{1}{2}\partial_t\|W_\varphi^\varepsilon\|_{L^2(\mathbb{R}^{2d})}^2,
$$

$$
\pi\int_{\mathbb{R}^{2d}}\mathrm{d}x\mathrm{d}k\,k\cdot\nabla_x W_\varphi^\varepsilon\,W_\varphi^\varepsilon=-\pi\int_{\mathbb{R}^{2d}}\mathrm{d}x\mathrm{d}k\,W_\varphi^\varepsilon\,k\cdot\nabla_x W_\varphi^\varepsilon=0
$$

and

$$
\begin{aligned}
&i\lambda\varepsilon^{-1}\int_{\mathbb{R}^{2d}}\mathrm{d}x\mathrm{d}k\int_{\mathbb{R}^d}\mathrm{d}\xi\hat{V}(\xi)\,e^{2\pi i\varepsilon^{-1}\xi\cdot x}W_\varphi^\varepsilon\left(t,x,k+\frac{\xi}{2}\right)W_\varphi^\varepsilon(t,x,k)\\
&= i\lambda\varepsilon^{-1}\int_{\mathbb{R}^{2d}}\mathrm{d}x\mathrm{d}k\int_{\mathbb{R}^d}\mathrm{d}\xi\hat{V}(\xi)\,e^{2\pi i\varepsilon^{-1}\xi\cdot x}W_\varphi^\varepsilon(t,x,k)\,W_\varphi^\varepsilon\left(t,x,k-\frac{\xi}{2}\right).
\end{aligned}
$$

Hence

$$
\|W_\varphi^\varepsilon(t)\|_{L^2(\mathbb{R}^{2d})}=\|W_\varphi^\varepsilon(0)\|_{L^2(\mathbb{R}^{2d})}.
$$

In certain situations, the weak limits in $L^2(\mathbb{R}^{2d})$ of the solutions of W^ε_φ as $\varepsilon \longrightarrow 0$ are known to satisfy the linear Boltzmann equation, for which the L^2-norm preservation property above does not hold. See for instance [Gom13], where the mixed states from Example 2.8 are used for the initial data.

2.3 The linear Boltzmann equation

In this Section we will collect some important aspects of the well-posedness theory of linear Boltzmann equations. Let μ be a positive Radon measure on $\mathbb{R}^d$ with $\mu(\{0\}) = 0$ and supported on the closed set V. Let $t \in \mathbb{R}_+, x \in \mathbb{R}^d, k \in V \subset \mathbb{R}^d$ and consider

$$\partial_t f(t,x,k) + k \cdot \nabla_x f(t,x,k) = \int_V \mathrm{d}\mu(l) K(l,k)(f(t,x,l) - (f(t,x,k))) \tag{2.14}$$

with initial data $f(0,x,k) = f_0(x,k)$. The function K will be referred to as the collision kernel. Equation (2.14) can be rewritten as

$$\partial_t f(t,x,k) = Tf(t,x,k) \tag{2.15}$$

with initial data $f_0(x,k)$ and $T = A + G - L$ where the terms A, G, L are the "transport", "gain" and "loss" terms which we now define.

$$A := -k \cdot \nabla_x.$$

$$Gf(t,x,k) := \int_V \mathrm{d}\mu(l) K(l,k) f(t,x,l).$$

The "loss" term L is defined as the multiplication operator with the following function:

$$L(k) = \Big(\int_V \mathrm{d}\mu(l) K(l,k) \Big).$$

Our primary references for this section are Chapter XXI of [DL12] for the theory of linear Boltzmann equations, and the book [EN00] for well known results from the theory of semigroups. We will focus here on the L^1 and L^∞ theories, as this will be required in Chapter 5.

We make the following assumptions:

1. L and K are measurable with respect to the measures $\mathrm{d}\mu$ and $\mathrm{d}\mu \otimes \mathrm{d}\mu$ respectively.
2. $K \geqslant 0$, and there exist M such that

$$\sup_{v' \in V} \int_V K(v',v) \mathrm{d}\mu(v) \leqslant M \tag{2.16}$$

Next, we introduce some definitions from the theory of semigroups.

Definition 2.10. *A semigroup on a Banach space X is a family of operators $S: \mathbb{R}_+ \longrightarrow \mathscr{L}(X)$ satisfying*

1. $S(0) = \mathrm{Id}_X$.
2. $S(t+s) = S(t)S(s), \forall s, t \in \mathbb{R}_+$.

Definition 2.11. *A $\mathscr{C}_0$-semigroup or strongly continuous semigroup on a Banach space X is a semigroup $S(t)$ such that $\forall x \in X$,*

$$\|S(t)x - x\|_X \xrightarrow{t \longrightarrow 0} 0.$$

Definition 2.12. *Let $S(t)$ be a $\mathscr{C}_0$-semigroup on a Banach space X. Let*

$$D(T) = \left\{ x \in X : \lim_{t \downarrow 0} \frac{S(t)x - x}{t} \text{exists} \right\}.$$

The infinitesimal generator $T: D(T) \longrightarrow X$ is defined by

$$Tx = \lim_{t \downarrow 0} \frac{S(t)x - x}{t}, \qquad x \in D(T)$$

Definition 2.13. *Let $S(t)$ be a $\mathscr{C}_0$-semigroup on a Banach space X. The adjoint semigroup associated to $S(t)$ is the semigroup on X^* formed by the adjoint operators $\{S^*(t)\}_{t \geqslant 0}$.*

Definition 2.14. *Let*

$$D(P) = \left\{ y \in X^* : \forall x \in X, \left\langle \lim_{t \longrightarrow 0} \frac{S^*(t)y - y}{t}, x \right\rangle \text{exists} \right\}$$

The weak infinitesimal generator of a weak*-continuous adjoint semigroup $S^*(t)$ is defined to be the map $P: D(P) \longrightarrow X^*$ defined by*

$$P(y) := \lim_{t \longrightarrow 0} \frac{S^*(t)y - y}{t}$$

Remark 2.15. The adjoint semigroup is always weak*-continuous, in the sense that for

$$\xi_{x,y}(t) = \langle S^*(t)y, x \rangle = \langle y, S(t)\, x \rangle,$$

the maps

$$t \longrightarrow \xi_{x,y}(t)$$

are continuous for all $x \in X, y \in X^*$. One has that

$$D(P) = \{ y \in X^* : \exists y' \in X^* : \langle y', x \rangle = \langle y, Tx \rangle \forall x \in D(T) \}$$

and $P = T^*$ where T is the generator of the $\mathscr{C}_0$-semigroup $S(t)$ whose adjoint semigroup is $S^*(t)$. See section 2.5 in [EN00] for a discussion of these facts.

Definition 2.16. *Let $\tau > 0$. Let T be the infinitesimal generator associated to a $\mathscr{C}_0$-semigroup $S(t)$.*

1. *A function $f \in C^1((0, \tau); X) \cap C([0, \tau]; X)$ is called a classical solution in X to equation (2.15) in $[0, \tau]$ with initial condition $f_0 \in X$, if $f(t) \in D(T), \forall t \in [0, \tau]$, $f(0) = f_0$ and (2.15) holds $\forall t \in (0, \tau)$.*

2. *A function* $f \in C([0,\tau];X)$ *is called a mild solution in* X *to the linear equation (2.15) in* $[0,\tau]$ *with initial condition* $f_0 \in X$, *if for all* $t \in [0,\tau]$ *it holds that* $\int_0^t f(s)\mathrm{d}s \in D(T)$ *and*
$$f(t) = T\int_0^t f(s)\mathrm{d}s + f_0.$$

Clearly, classical solutions are mild solutions.

Let us first consider in this subsection the Banach space $X = L^1(\mathbb{R}^d \times V; \mathrm{d}x \times \mathrm{d}\mu)$. For $\varphi \in C_c^\infty(\mathbb{R}^d \times V)$, define
$$Z(t)\varphi(x,v) := \varphi(x - vt, v).$$
This can be extended to a $\mathscr{C}_0$-semigroup on X with
$$\|Z(t)\varphi\|_X = \|\varphi\|_X,$$
and the infinitesimal generator of this group is A with the domain
$$D(A) = \{u \in X : \exists v \cdot \nabla u \in X\}.$$
Now for the perturbation result, which is Proposition 2 in [DL12].

Proposition 2.17. *Let* A *be the infinitesimal generator of a* $\mathscr{C}_0$*-semigroup in* $X, L \in L^\infty(\mathbb{R}^d \times V)$ *and* $G \in \mathscr{L}(X)$. *Then* $T = A + G - L, D(T) = D(A)$ *is the infinitesimal generator of a* $\mathscr{C}_0$*-semigroup in X.*

We now check that $G \in \mathscr{L}(X)$:
$$\|G\varphi\|_X = \int_{\mathbb{R}^d}\mathrm{d}x\int_V \mathrm{d}\mu(k)\left|\int_V \mathrm{d}\mu(l)\, K(l,k)\varphi(x,l)\right|.$$
By assumption (2.16) this
$$\le M\int_{\mathbb{R}^d}\mathrm{d}x\int_V \mathrm{d}\mu(l)|\varphi(x,l)| \le M\|\varphi\|_X.$$
Combining Propositions 6.2 and 6.4 in [EN00] with Proposition 2.17 above, we conclude that for every $f_0 \in X$, $f(t) = S(t)f_0$ is the unique mild solution in X to the linear Boltzmann equation. Furthermore, if $f_0 \in D(T)$, the mild solution is a classical solution.

Define $T^* := A^* + G^* - L$ with domain
$$D(T^*) := D(A^*) := \{u \in X^* : v \cdot \nabla u \in X^*\}.$$
i.e., the elements u in X^* whose weak derivative ∇u satisfies the property that $v \cdot \nabla u \in X^*$.

Definition 2.18. *We say that for* $\tau > 0$, $f \in C([0,\tau];X)$ *is a weak solution in* X *to the linear Boltzmann equation (2.15) in* $[0,\tau]$ *with initial condition* $f_0 \in X$ *if*
$$\langle f(\tau), \varphi(\tau)\rangle - \langle f_0, \varphi(0)\rangle = \int_0^\tau \mathrm{d}t \langle f(t), (\partial_t + T^*)\varphi(t)\rangle \tag{2.17}$$
for all $\varphi \in C^1((0,\tau);X^*) \cap C((0,\tau);D(T^*)) \cap C([0,\tau];X^*)$.

Lemma 2.19. *A function $f\in C([0,\tau];X)$ is a mild solution in X to the linear Boltzmann equation (2.15) in $[0,\tau]$ iff it is a weak solution in X to the linear Boltzmann equation (2.15) in $[0,\tau]$ with initial condition $f_0\in X$.*

Proof. Assume $f\in C([0,\tau];X)$ is a mild solution. Let $\varphi\in C^1((0,\tau);X^*)\cap C((0,\tau);D(T^*))\cap C([0,\tau];X^*)$. Then for any $t\in(0,\tau)$

$$\begin{aligned}\partial_t\langle f(t),\varphi(t)\rangle &= \partial_t\langle S(t)f_0,\varphi(t)\rangle = \partial_t\langle f_0,S^*(t)\varphi(t)\rangle\\ &= \langle f_0,S^*(t)(T^*+\partial_t)\varphi(t)\rangle = \langle S(t)f_0,(T^*+\partial_t)\varphi(t)\rangle\\ &= \langle f(t),(T^*+\partial_t)\varphi(t)\rangle.\end{aligned}$$

Integrating in time we see that f is indeed a weak solution. Now, assume f is a weak solution and consider $r(t)=f(t)-S(t)f_0$. Since we showed mild solutions are weak solutions, $r(t)$ is a weak solution with $r(0)=0$. Let $\psi\in D(T^*)$ and assume $\varphi(t)=S^*(\tau-t)\psi$. Clearly, $\varphi\in C^1((0,\tau);X^*)\cap C((0,\tau);D(T^*))\cap C([0,\tau];X^*)$. Now

$$\partial_t\varphi(t)=\partial_t S^*(\tau-t)\psi=-T^*S^*(\tau-t)\psi=-T^*\varphi(t)$$

so $(T^*(t)+\partial_t)\varphi(t)=0$. Hence one has that

$$\langle r(\tau),\varphi(\tau)\rangle-\langle r(0),\varphi(0)\rangle=0\Longrightarrow\langle r(\tau),\varphi(\tau)\rangle=0$$

But $\varphi(\tau)=\psi$ and so

$$\langle r(t),\psi\rangle=0$$

for a dense set of ψ in X^*. Hence $r(t)=0$ and so the weak solution $f(t)$ corresponds to $S(t)f_0$, the unique mild solution in X. □

Corollary 2.20. *Let $\tau>0$. There exists a unique weak solution $f(t)\in C([0,\tau];X)$ to the linear Boltzmann equation (2.15) in $[0,\tau]$ with initial condition $f_0\in X$.*

Now, we will work with the Banach space $Y=L^\infty(X\times V;\mathrm{d}x\times\mathrm{d}\mu)$. The point is that cannot have an analogue of Proposition 2.17 on Y, i.e., the transport operator does not generate a $\mathscr{C}_0$-semigroup on Y. Hence we cannot use Proposition 6.4 in [EN00] to conclude the existence of mild solutions in Y. Despite this, we will use the fact that we can construct $\mathscr{C}_0$-semigroups on X with weak$*$-continuous adjoint semigroups on Y, and use this to define a weak solution on Y.

Definition 2.21. *Define*

$$C_{w^*}([0,\tau];Y):=\{f:[0,\tau]\longrightarrow Y: t\longrightarrow\langle f(t),\psi\rangle\ \text{is continuous}\ \forall\psi\in X\}$$

Definition 2.22. *We say that for $\tau>0, f\in C_{w^*}([0,\tau];Y)$ is a weak solution in Y to the linear Boltzmann equation in $[0,\tau]$ with initial condition $f_0\in Y$ if*

$$\langle f(\tau),\varphi(\tau)\rangle-\langle f_0,\varphi(0)\rangle=\int_0^\tau\mathrm{d}t\langle f(t),(\partial_t+T^*)\varphi(t)\rangle \tag{2.18}$$

for all $\varphi\in C^1((0,\tau);X)\cap C((0,\tau);D(T^*))\cap C([0,\tau];X)$.

Note that in this definition, the test functions take values in X, and not Y^*.

Let us consider the dual equation

$$\partial_t f(t,x,k) = T^* f(t,x,k) \tag{2.19}$$

on X, with $T^* = A^* + G^* - L$ as before. This generates a $\mathscr{C}_0$-semigroup $S_2(t)$ on X via Proposition 2.17. There is an associated weak$*$-continuous adjoint semigroup $S_2^*(t)$ on Y with infinitesimal generator Q on the domain

$$D(Q) = \{y \in Y : \exists y' \in Y : \langle y', x\rangle = \langle y, Tx\rangle \forall x \in X\}.$$

In particular, for $\phi \in D(Q)$ one has that there exists $\phi' \in Y : \forall g \in D(T^*)$

$$\langle \phi', g\rangle = \langle \phi, A^* g\rangle = \int_{X\times V} \mathrm{d}x\,\mathrm{d}\mu(k)\,\phi(t,x,k)\,k\cdot\nabla_x g(t,x,k)$$

Integrating by parts, we have that

$$\phi' = -k\cdot\nabla_x \phi = A\phi$$

and so on $D(Q)$ we have that $Q = T$.

Proposition 2.23. *For any $\tau > 0$, there exists a unique weak solution in Y to the linear Boltzmann equation in $[0,\tau]$ with initial condition $f_0 \in Y$.*

Proof. (Uniqueness) Assume that there are two weak solutions f, f' with initial condition f_0. Then their difference $r = f - f'$ is a weak solution with initial condition 0. Hence

$$\langle r(\tau), \varphi(\tau)\rangle = \int_0^\tau \mathrm{d}t \langle r(t), (\partial_t + T^*)\varphi(t)\rangle$$

for all $\varphi \in C^1((0,\tau);X) \cap C((0,\tau);D(T^*)) \cap C([0,\tau];X)$. Since we have existence and uniqueness of mild solutions to equation (2.19) in X, let us consider $\varphi(t) = S_2(\tau - t)\psi$ for some $\psi \in D(T^*)$. Then $\varphi(t) \in D(T^*), \forall t \in [0,\tau]$ and for $t \in (0,\tau)$ one has that

$$\partial_t \varphi(t) = \partial_t S_2(\tau - t)\psi = -T^* S_2(\tau - t)\psi = -T^*\varphi(t).$$

Hence $(\partial_t + T^*)\varphi(t) = 0, \forall t$. This implies that

$$\langle r(\tau), \varphi(\tau)\rangle = \langle r(\tau), \psi\rangle = 0.$$

By a density argument, $r(\tau) = 0$, and since τ is arbitrary, $r = 0$ and we have uniqueness of weak solutions.

(Existence) Define, for $\psi \in X$,

$$\langle f(t), \psi\rangle := \langle f_0, S_2(t)\psi\rangle.$$

The map $t \longrightarrow \langle f_0, S_2(t)\psi\rangle$ is continuous. We now check condition (2.17). Let $\varphi \in C^1((0,\tau);X) \cap C((0,\tau);D(T^*)) \cap C([0,\tau];X)$. Consider

$$\langle f(t), (\partial_t + T^*)\varphi(t)\rangle = \langle f_0, S_2(t)(\partial_t + T^*)\varphi(t)\rangle.$$

Now since $\varphi \in C((0,\tau);D(T^*)) \cap C^1((0,\tau);X)$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(S_2(t)\varphi(t)) = S_2(t)T^*\varphi(t) + S_2(t)\partial_t\varphi(t).$$

So

$$\langle f(t),(\partial_t+T^*)\varphi(t)\rangle=\frac{\mathrm{d}}{\mathrm{d}t}\langle f_0,S_2(t)\varphi(t)\rangle.$$

Integrating in time, we have that

$$\begin{aligned}\int_0^\tau \mathrm{d}t\langle f(t),(\partial_t+T^*)\varphi(t)\rangle&=\langle f_0,S_2(\tau)\varphi(\tau)\rangle-\langle f_0,\varphi(0)\rangle\\&=\langle f(\tau),\varphi(\tau)\rangle-\langle f_0,\varphi(0)\rangle.\end{aligned}$$

Hence we have verified that f is a weak solution. □

Lemma 2.24. *Let* $\tau>0$*. Assume that* $f\in C_{w^*}([0,\tau];Y)$ *satisfies*

$$\langle f(t),\psi\rangle-\langle f_0,\psi\rangle=\int_0^t \mathrm{d}s\langle f(s),T^*\psi\rangle,\qquad \forall\psi\in D(T^*),$$

Then $f(t)$ *is a weak solution in* Y *to the linear Boltzmann equation (2.15) in* $[0,\tau]$ *with initial condition* $f_0\in Y$*.*

Proof. Assume $\phi\in C^1((0,\tau))\cap C([0,\tau])$. Since for all $\psi\in D(T^*)$ we have

$$\langle f(t),\psi\rangle-\langle f_0,\psi\rangle=\int_0^t \mathrm{d}s\langle f(s),T^*\psi\rangle,$$

we have that for a.e $t\in(0,\tau)$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f(t),\psi\rangle=\langle f(t),T^*\psi\rangle.$$

Furthermore, for a.e $t\in(0,\tau)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi(t)\langle f(t),\psi\rangle)=\phi'(t)\langle f(t),\psi\rangle+\phi(t)\langle f(t),T^*\psi\rangle$$

Integrating in time,

$$\langle f(\tau),\phi(\tau)\psi\rangle-\langle f_0,\phi(0)\psi\rangle=\int_0^\tau \mathrm{d}t\langle f(t),(\partial_t+T^*)\phi(t)\psi\rangle,$$

and hence we proved the statement for $\varphi(t)=\phi(t)\psi\in C^1((0,\tau);D(T^*))\cap C([0,\tau];D(T^*))$.

Now, let $\varphi\in C^1((0,\tau);X)\cap C((0,\tau);D(T^*))\cap C([0,\tau];X)$. For an arbitrary interval $[a,b]\subset(0,\tau)$ we have that $\varphi\in C([a,b];D(T^*)),\partial_t\varphi\in C([a,b];X)$. For any $\varepsilon>0$, φ can be approximated by functions of the form $\rho_\varepsilon(t)=\sum_{j=1}^{N_\varepsilon}\phi_j(t)\psi_j$ with $\phi_j\in C^1((a,b))\cap C([a,b])$ and $\psi_j\in D(T^*)$ such that

$$\sup_{s\in[a,b]}\|\varphi(s)-\rho_\varepsilon(s)\|_X\leqslant C\varepsilon,\qquad \sup_{s\in[a,b]}\|\partial_t\varphi(s)-\partial_t\rho_\varepsilon(s)\|_X\leqslant C\varepsilon$$

and

$$\sup_{s\in[a,b]}\|T^*\varphi(s)-T^*\rho_\varepsilon(s)\|_X\leqslant C\varepsilon$$

for some constant $C>0$. By construction of ρ_ε, it holds that

$$\langle f(b),\rho_\varepsilon(b)\rangle-\langle f(a),\rho_\varepsilon(a)\rangle=\int_a^b \mathrm{d}t\langle f(t),(\partial_t+T^*)\rho_\varepsilon(t)\rangle$$

and using the above information, one can pass to the limit in ε to get

$$\langle f(b),\varphi(b)\rangle - \langle f(a),\varphi(a)\rangle = \int_a^b \mathrm{d}t \langle f(t),(\partial_t + T^*)\varphi(t)\rangle.$$

Now we show the continuity of the map $t \longrightarrow \langle f(t),\varphi(t)\rangle$. By the continuity of $t \longrightarrow \langle f(t),\psi\rangle$,

$$\sup_{t\in[0,\tau]} |\langle f(t),\psi\rangle| < \infty.$$

By the uniform boundedness principle, $\sup_{t\in[0,\tau]}\|f(t)\|_Y < \infty$. Now, choose t_1, t_2: $|t_2 - t_1| \leqslant \delta$ and one has

$$\begin{aligned}&\langle f(t_2),\varphi(t_2)\rangle - \langle f(t_1),\varphi(t_1)\rangle \pm \langle f(t_2),\varphi(t_1)\rangle\\ =&\langle f(t_2),\varphi(t_2)-\varphi(t_1)\rangle + \langle f(t_2)-f(t_1),\varphi(t_1)\rangle\end{aligned}$$

and one uses $\sup_{t\in[0,\tau]}\|f(t)\|_Y < \infty$ and the continuity of φ for the first term, and the continuity of the maps $t \longrightarrow \langle f(t),\psi\rangle$ for all $\psi \in X$ for the second term.

Finally, one can take limits $a\downarrow 0$, $b\uparrow\tau$ to get that

$$\langle f(\tau),\varphi(\tau)\rangle - \langle f_0,\varphi(0)\rangle = \int_0^\tau \mathrm{d}t \langle f(t),(\partial_t + T^*)\varphi(t)\rangle,$$

thus showing that f is a weak solution. □

Remark 2.25. In the work [EY99], one encounters a linear Boltzmann equation written in the form

$$\partial_t f(t,x,k) + k\cdot\nabla_x f(t,x,k) = \int_{\mathbb{R}^d} \mathrm{d}l \hat{R}(l-k)\delta(|l|^2-|k|^2)(f(t,x,l)-f(t,x,k)).$$

where $R \in \mathcal{S}(\mathbb{R}^d)$ and initial condition $f_0 \in \mathscr{M}_+(\mathbb{R}^{2d})$. The initial condition requires the notion of a measure-valued solution to the linear Boltzmann equation. However note that the RHS does not make sense if $f(t) \in \mathscr{M}(\mathbb{R}^{2d})$, since we would be multiplying two measures.

The way to interpret the above equation is by duality. Above we introduced the idea of duality using $(Y,X) = (L^\infty(\mathbb{R}^{2d}), L^1(\mathbb{R}^{2d}))$. Here one requires $(Y,X) = (\mathscr{M}(\mathbb{R}^{2d}), C_b(\mathbb{R}^{2d}))$ and one needs to understand also the equation by duality. Consider

$$\partial_t f(t,x,k) = T^* f(t,x,k)$$

where $T^* = k\cdot\nabla_x + G^* - L^*$. The adjoint of the gain operator is given by

$$G^* f(t,x,k) = \int_{S^{d-1}_{|k|}} \mathrm{d}l \hat{R}(l-k) f(t,x,l),$$

and L^* is the multiplication operator defined by multiplication with the function $L^*(k)$ with

$$L^*(k) = \int_{S^{d-1}_{|k|}} \mathrm{d}l \hat{R}(l-k).$$

One can then check that this defines a $\mathscr{C}_0$-semigroup on $C_b(\mathbb{R}^{2d})$, and then give a notion to weak solutions on $\mathscr{M}(\mathbb{R}^{2d})$ by the duality arguments introduced above. For more details on this see the end of Section 3 in Chapter XXI in [DL12].

2.4 The sewing lemma

Rough path theory was a theory originally developed by Terry Lyons ([Lyo98]) as an alternative to Ito's construct solutions to SDEs. Recall that almost every realization of a standard Brownian motion is Hölder continuous with Hölder exponent less than $\frac{1}{2}$. Unlike Ito's theory, rough path theory uses this Hölder regularity and certain algebraic conditions in order to construct an integral solution to SDEs.

In this section, we first explain the statement of the sewing lemma in the real valued case, and provide a toy example of how we intend to use it in order to take kinetic limits. First, imagine that one seeks to give a meaning to the formal expression

$$I(T) = I(0) + \int_0^T Y(t)\mathrm{d}X(t)$$

when $Y \in C^\alpha([0,T])$ and $X \in C^1((0,T))$, for $\alpha > 0$. We could write $\delta I_{st} := I(t) - I(s)$ as

$$\begin{aligned}\delta I_{st} = \int_s^t Y(\tau)\dot{X}(\tau)\mathrm{d}\tau = \int_s^t Y(s)\dot{X}(\tau)\mathrm{d}\tau + \int_s^t (Y(\tau) - Y(s))\dot{X}(\tau)\mathrm{d}\tau \\ = Y(s)\delta X_{st} + r_{st}\end{aligned}$$

where

$$r_{st} = \int_s^t (Y(\tau) - Y(s))\dot{X}(\tau)\mathrm{d}\tau.$$

One has

$$|r_{st}| \leqslant \sup_{\tau \in [s,t]} |Y(\tau) - Y(s)| \delta X_{st} \leqslant C|t-s|^{1+\alpha}$$

which vanishes by the continuity of X, Y as $|t-s| \longrightarrow 0$, hence $|r_{st}| = o(|t-s|)$. Hence, for $A_{st} = Y_s \delta X_{st}$ we have an equation of the form

$$\delta I_{st} = A_{st} + r_{st}, \qquad |r_{st}| = o(|t-s|) \tag{2.20}$$

The sewing lemma says that if the “germ” A_{st} satisfies certain conditions, then there is a unique combination (I, r) satisfying the equation (2.20).

Let us define the simplices

$$\Delta_T^2 := \{(s,t) : [0,T]^2 : s \leqslant t\},$$

and

$$\Delta_T^3 := \{(s,u,t) : [0,T]^3 : s \leqslant u \leqslant t\},$$

Denote $\Delta^2 := \Delta_1^2, \Delta^3 := \Delta_1^3$. For $A \in C(\Delta_T^2)$ and $(s,u,t) \in \Delta_T^3$ define

$$\delta A_{sut} := A_{st} - A_{su} - A_{ut}.$$

Next, recall the definition of the Hölder seminorm

$$|||A|||_\gamma := \sup_{(s,t) \in \Delta_T^2, s<t} \frac{|A_{st}|}{|t-s|^\gamma}, \qquad \|\delta A\|_\gamma := \sup_{(s,u,t) \in \Delta_T^3, s<u<t} \frac{|\delta A_{sut}|}{|t-s|^\gamma},$$

and the Banach space of Hölder continuous functions starting at 0:

$$C^\gamma([0,T]) := \{f \in C([0,T]) : |||f|||_\gamma < \infty, f(0) = 0\}$$

with norm $\|f\|_\gamma = |||f|||_\gamma$. Define

$$C^{\alpha,\beta}(\Delta_T^2) := \{A\colon C(\Delta_T^2), A_{tt} = 0, \forall t \in [0,T], \|A\|_{\alpha,\beta} := |||A|||_\alpha + |||\delta A|||_\beta < \infty\}.$$

Theorem 2.26. *(Sewing lemma) Let $0 < \alpha \leqslant 1 < \beta$. Then there exists a unique continuous map $I\colon C^{\alpha,\beta}(\Delta_T^2) \longrightarrow C^\alpha([0,T])$ such that $(IA)_0 = 0$ and*

$$|\delta_{st}(IA) - A_{st}| \leqslant C_\beta |||\delta A|||_\beta |t-s|^\beta,$$

where C_β is a constant that only depends on β. Furthermore,

$$(IA)_t = \lim_{|\mathcal{P}| \longrightarrow 0} \sum_{i=0}^{\#\mathcal{P}-1} A_{t_i t_{i+1}}$$

along arbitrary sequences of partitions $\mathcal{P}$ of $[0,t]$ with vanishing mesh size.

This version of the sewing lemma is due to [Gub04]. For a proof, see Theorem 4.2 in [FH].

Remark 2.27. Defining $r_{st} := \delta_{st}(IA) - A_{st}$, one has

$$|r_{st}| \leqslant C_\beta |||\delta A|||_\beta |t-s|^\beta.$$

We see that

$$\delta r_{sut} := r_{st} - r_{su} - r_{ut} = -A_{st} + A_{su} + A_{ut} = -\delta A_{sut} \tag{2.21}$$

Putting this together, one has that

$$|||r|||_\beta \leqslant C_\beta |||\delta r|||_\beta \tag{2.22}$$

Remark 2.28. The sewing lemma applies to the germ in equation (2.20). Taking as a germ

$$A_{st} = Y_s \delta X_{st}$$

for $X \in C^1((0,T))$ and $Y \in C^\alpha([0,T])$, with $\alpha > 0$, we have for $(s,u,t) \in \Delta_T^3$ that

$$\begin{aligned}\delta A_{sut} = A_{st} - A_{su} - A_{ut} &= Y_s \delta X_{st} - Y_s \delta X_{su} - Y_u \delta X_{ut} \pm Y_s \delta X_{ut}\\ &= \delta Y_{su} \delta X_{ut}.\end{aligned}$$

Hence

$$\begin{aligned}\sup_{(s,u,t)\in\Delta_T^3} \frac{|\delta A_{sut}|}{|t-s|^{1+\alpha}} &\leqslant \sup_{(s,u,t)\in\Delta_T^3} \frac{|\delta Y_{su}|}{|t-s|^\alpha} \sup_{(s,u,t)\in\Delta_T^3} \frac{|\delta X_{ut}|}{|t-s|}\\ &\leqslant |||Y|||_\alpha \|\partial X\|\end{aligned}$$

The sewing lemma says there exists a unique map $\mathcal{I}_t = (IA)_t \in C^\alpha([0,T])$ such that for all $(s,t) \in \Delta_T^2$ one has that the remainder term $r_{st} := \delta \mathcal{I}_{st} - Y_s \delta X_{st}$ satisfies

$$|||r|||_{1+\alpha} \leqslant C_{1+\alpha} |||Y|||_\alpha \|\partial X\|.$$

Furthermore, since

$$\mathcal{I}_t = \lim_{N\to\infty} \sum_{i=0}^{N-1} A_{\frac{i}{N}\frac{i+1}{N}},$$

we have that

$$\mathcal{I}_t = \int_0^t Y(s)\dot{X}(s)\mathrm{d}s = \mathcal{I}_0 + \int_0^t Y(s)\dot{X}(s)\mathrm{d}s.$$

To see how one can give a meaning to the integral equation

$$Y_t = Y_0 + \int_0^t Y(s)\dot{X}(s)\mathrm{d}s$$

when $X \in C^\alpha([0,T])$ we refer the reader to Chapter 8 of [FH], but we will not need such results in this document.

The sewing lemma can be generalized from real-valued paths to paths taking values in an arbitrary Banach space. For details, we refer the reader to Corollary 2.4 in [GT10] or Theorem 4.2 in [FH]. We will omit rewriting the statement here, and will present instead how we will make use of the theory.

Lemma 2.29. *Let $T \in \mathbb{R}_+$. Consider a family of real valued functions $\{f^\varepsilon\}_{\varepsilon\in(0,1]}$, f^ε: $[0,T] \longrightarrow \mathbb{R}$. Let $C \in \mathbb{R}_+$ and $\gamma \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Assume one has the uniform bound*

$$\sup_{\varepsilon\in(0,1], t\in[0,1]} |f^\varepsilon(t)| \leqslant C \tag{2.23}$$

and that for any $0 \leqslant s < t \leqslant 1$ the increments $\delta f^\varepsilon_{st} := f^\varepsilon(t) - f^\varepsilon(s)$ satisfy

$$\delta f^\varepsilon_{st} = \mathbb{X}^{1,\varepsilon}_{st} f^\varepsilon_s + \mathbb{X}^{2,\varepsilon}_{st} f^\varepsilon_s + r^\varepsilon_{st} \tag{2.24}$$

for functions $\mathbb{X}^{1,\varepsilon}_{st}$, $\mathbb{X}^{2,\varepsilon}_{st}$ and r^ε_{st} in $C(\Delta^2_T)$ satisfying

$$|||\mathbb{X}^{1,\varepsilon}|||_\gamma \leqslant C, \qquad |||\mathbb{X}^{2,\varepsilon}|||_{2\gamma} \leqslant C, \qquad |||r^\varepsilon|||_{3\gamma} \leqslant C'_\varepsilon, \tag{2.25}$$

where C'_ε denotes an ε-dependent constant C' that need not be uniformly bounded in ε. Finally assume that Chen's relation holds

$$\delta\mathbb{X}^{1,\varepsilon}_{sut} = 0, \qquad \delta\mathbb{X}^{2,\varepsilon}_{sut} = \mathbb{X}^{1,\varepsilon}_{ut}\mathbb{X}^{1,\varepsilon}_{su}. \tag{2.26}$$

Then one has the following uniform in ε-bound for the remainder:

$$|||r^\varepsilon|||_{3\gamma} \leqslant 6C_{3\gamma}C^3.$$

where $C_{3\gamma}$ is the constant C_β appearing in Lemma 2.26 with $\beta = 3\gamma$.

Remark 2.30. This lemma is the core of our idea to use the sewing lemma in order to obtain kinetic limits. It shows that one can control a regular-in-time remainder term if one establishes

- An a-priori bound of the form $\|f^\varepsilon\| \leqslant C$ uniformly in ε and t;
- Uniform control of terms with low time regularity;
- Structure of a rough equation (Chen relation and Hölder bounds).

Once one has uniform estimates in all terms of (2.24), one can extract converging subsequences, and can then attempt to characterize all sub-sequential limits.

The lemma above could have been simplified if we had used more terminology from the theory of rough paths. In this thesis, we avoid discussing the general theory of rough paths, and focus only on the aspects that are most relevant to our study of kinetic limits. For the general theory, we refer the reader to [FH] and [CGZ].

Proof. Let

$$A^{\varepsilon}_{st} = \mathbb{X}^{1,\varepsilon}_{st} f^{\varepsilon}_{s} + \mathbb{X}^{2,\varepsilon}_{st} f^{\varepsilon}_{s}.$$

Then equation (2.24) is of the same form as equation (2.20), but with the bound on the remainders being non-uniform in ε. First restrict (s,t) to Δ^2.

One has that

$$\sup_{(s,t)\in\Delta^2_T} \frac{|A^{\varepsilon}_{st}|}{|t-s|^{\gamma}} \leqslant \frac{|\mathbb{X}^{1,\varepsilon}_{st}|}{|t-s|^{\gamma}} |f^{\varepsilon}_{s}| + \frac{|\mathbb{X}^{1,\varepsilon}_{st}|}{|t-s|^{2\gamma}} |f^{\varepsilon}_{s}|\,|t-s|^{\gamma}.$$

By our assumptions, we have

$$\sup_{(s,t)\in\Delta^2_T} \frac{|A^{\varepsilon}_{st}|}{|t-s|^{\gamma}} \leqslant 2C^2.$$

Next, we compute

$$\delta A^{\varepsilon}_{sut} = \mathbb{X}^{1,\varepsilon}_{st} f^{\varepsilon}_{s} + \mathbb{X}^{2,\varepsilon}_{st} f^{\varepsilon}_{s} - \mathbb{X}^{1,\varepsilon}_{su} f^{\varepsilon}_{s} - \mathbb{X}^{2,\varepsilon}_{su} f^{\varepsilon}_{s} - \mathbb{X}^{1,\varepsilon}_{ut} f^{\varepsilon}_{u} - \mathbb{X}^{2,\varepsilon}_{su} f^{\varepsilon}_{u}.$$

Adding and subtracting $\mathbb{X}^{1,\varepsilon}_{ut} f^{\varepsilon}_{s}$ and using the Chen relations, one has that

$$\delta A^{\varepsilon}_{sut} = -\mathbb{X}^{1,\varepsilon}_{ut} \delta f^{\varepsilon}_{su} - \mathbb{X}^{2,\varepsilon}_{ut} \delta f^{\varepsilon}_{su} + \mathbb{X}^{1,\varepsilon}_{ut} \mathbb{X}^{1,\varepsilon}_{su} f^{\varepsilon}_{s}$$

$$= -\mathbb{X}^{1,\varepsilon}_{ut} \mathbb{X}^{2,\varepsilon}_{su} f^{\varepsilon}_{s} - \mathbb{X}^{1,\varepsilon}_{ut} r^{\varepsilon}_{su} - \mathbb{X}^{2,\varepsilon}_{ut} \mathbb{X}^{1,\varepsilon}_{su} f^{\varepsilon}_{s} - \mathbb{X}^{2,\varepsilon}_{ut} \mathbb{X}^{2,\varepsilon}_{su} f^{\varepsilon}_{s} - \mathbb{X}^{2,\varepsilon}_{ut} r^{\varepsilon}_{su}.$$

Hence

$$\frac{|\delta A_{sut}|}{|t-s|^{3\gamma}} \leqslant \frac{|\mathbb{X}^{1,\varepsilon}_{ut}|}{|t-s|^{\gamma}} \frac{|\mathbb{X}^{2,\varepsilon}_{su}|}{|t-s|^{2\gamma}} |f^{\varepsilon}_{s}| + \frac{|\mathbb{X}^{1,\varepsilon}_{ut}|}{|t-s|^{\gamma}} \frac{|r^{\varepsilon}_{su}|}{|t-s|^{3\gamma}} |t-s|^{\gamma}$$

$$+ \frac{|\mathbb{X}^{2,\varepsilon}_{ut}|}{|t-s|^{2\gamma}} \frac{|\mathbb{X}^{1,\varepsilon}_{su}|}{|t-s|^{2\gamma}} |f^{\varepsilon}_{s}| + \frac{|\mathbb{X}^{2,\varepsilon}_{ut}|}{|t-s|^{2\gamma}} \frac{|\mathbb{X}^{2,\varepsilon}_{su}|}{|t-s|^{2\gamma}} |f^{\varepsilon}_{s}|\,|t-s|^{\gamma} + \frac{|\mathbb{X}^{2,\varepsilon}_{ut}|}{|t-s|^{\gamma}} \frac{|r^{\varepsilon}_{sul}|}{|t-s|^{3\gamma}} |t-s|^{\gamma}.$$

By using the bounds (2.25), this is

$$\frac{|\delta A_{sut}|}{|t-s|^{3\gamma}} \leqslant C^3 + C\,|||r^{\varepsilon}|||_{3\gamma} |t-s|^{\gamma} + C^3 + C^3 |t-s|^{\gamma} + C\,|||r^{\varepsilon}|||_{3\gamma} |t-s|^{\gamma}$$

$$\leqslant 3C^3 + 2C\,|||r^{\varepsilon}|||_{3\gamma} |t-s|^{\gamma}.$$

Hence $A^{\varepsilon} \in C^{\gamma,3\gamma}(\Delta^2)$. Applying the sewing lemma and using the facts from Remark 2.27, one has that

$$|||r^{\varepsilon}|||_{3\gamma} \leqslant 3C_{3\gamma}C^3 + 2C_{3\gamma} |||r^{\varepsilon}|||_{3\gamma} |t-s|^{\gamma}.$$

Choosing $I \subset [0,1]$ such that $2C_{3\gamma}|I| \leqslant \frac{1}{2}$ and restricting (s,t) to the 2-simplex defined by I, one has that

$$|||r^{\varepsilon}|||_{3\gamma} \leqslant 3C_{3\gamma}C^3 + \frac{1}{2} |||r^{\varepsilon}|||_{3\gamma}.$$

Thus

$$|||r^\varepsilon|||_{3\gamma} \leq 6 C_{3\gamma} C^3.$$

Covering the time domain $[0, T]$ with such I, we have the claim for the entire time interval. □

Chapter 3

A formal derivation of the limit

In [BFPR99] a heuristic argument was introduced in order to derive the weak coupling limit of a Schrödinger equation with a Gaussian random potential V_ω. The key to this heuristic derivation is a certain asymptotic expansion for the rescaled Wigner transform (that we introduced in Chapter 1). From the evolution equation of the rescaled Wigner transform, a set of equations follows for each term in the asymptotic expansion, that can be successively solved. The coefficients of the leading order term in the expansion satisfy a linear Boltzmann equation, whose collision kernel depends on the covariance of the field V_ω and has the energy conservation property.

In this chapter we adapt this heuristic method to the case of a weakly coupled $\mathbb{Z}^d$-periodic Gaussian field V_ω. Differently from the non-periodic case, for certain momenta, we obtain free transport in the weak coupling limit, while for certain other momenta, there is an obstruction to taking the limit. We characterize the problematic momenta in terms of energy bands associated to periodic Hamiltonians, using Bloch theory. More precisely, these momenta are demonstrated to be associated to energy band crossings of the free Hamiltonian. To conclude this section, we relate this heuristic approach to the work [PR04], in which the evolution equation for the rescaled Wigner transform is modified by the addition of a damping term, allowing the weak coupling limit to be taken in a strong sense.

In the work [BFPR99], the authors consider a Schrödinger equation with a deterministic $\mathbb{Z}^d$-periodic potential V' and a real, weakly coupled, zero-mean, translation invariant, time-independent Gaussian random field V_ω:

$$i\partial_t\varphi(t,x) = -\Delta_x\varphi(t,x) + V'(x)\varphi(t,x) - \varepsilon^{1/2}V_\omega(x)\varphi(t,x) \tag{3.1}$$

The covariance function of the Gaussian random field is characterized by

$$\mathbb{E}[\hat{V}_\omega(n)\hat{V}_\omega(m)] = \hat{\rho}(m)\delta(m+n), \qquad \mathbb{E}[\hat{V}_\omega(n)\hat{V}^*_\omega(m)] = \hat{\rho}(m)\delta(m-n) \tag{3.2}$$

for all $m, n \in \mathbb{R}^d$, and for some $\rho \in \mathcal{S}(\mathbb{R}^d)$ with $\hat{\rho} \geq 0, \hat{\rho}(\xi) = \hat{\rho}^*(-\xi), \forall \xi \in \mathbb{R}^d$. The key steps in their approach can be summarized as follows:

1. Write down the equation describing the time evolution of an asymmetric rescaled Wigner transform associated to the solution of the above Schrödinger equation (3.1).
2. Introduce a formal expansion in ε for the solution of this equation, and derive a system of equations order by order in ε.
3. Study the leading order terms in ε using the theory of Bloch bands for periodic potentials.

4. Obtain the dynamics of the leading order term in the expansion, which formally should be the limiting equation as $\varepsilon \longrightarrow 0$.

In the limit, they obtain a linear Boltzmann equation, for the coefficients σ_j of W_0 in a certain basis.

In this section, we follow this procedure, making two changes at the level of equation (3.1).

- We will set the background periodic potential $V' = 0$. This corresponds to the quantum Lorentz gas model we introduced in Chapter 1. This will allow us to use the Fourier transform in this section, and do computations explicitly, instead of relying on the Bloch theory used in [BFPR99].
- We will use a real, $\mathbb{Z}^d$-periodic zero-mean, translation invariant, time-independent Gaussian field V_ω instead of its non-periodic counterpart.

We note that if the covariance of V_ω is chosen to be sufficiently smooth, then almost every realization of V_ω is a smooth periodic function. Stated differently, making a statement about the periodic quantum Lorentz gas is equivalent to making a statement about almost every realization of a $\mathbb{Z}^d$-periodic Gaussian random field with sufficiently smooth covariance.

In Subsection 3.1, we work with a $\mathbb{Z}^d$-periodic random field rather than with a fixed periodic function, since in principle this gives us the option to make a weaker statement about averages, rather than almost sure statements. We will see however that there are some challenges to establishing even such weak statements.

In Subsection 3.2 we will see that if one modifies the equation for the rescaled Wigner transform in Step 1 of the procedure above by introducing a damping term for the high frequencies, that one can take limits in a strong sense, and the problems present in the undamped case can be overcome.

To improve readability, some computations that don't contribute to the main ideas are presented in the form of lemmas in Subsection 3.3.

3.1 A two-scale asymptotic expansion

We will start from the usual non-rescaled Schrödinger equation, with a real, weakly coupled, zero-mean, $\mathbb{Z}^d$-periodic Gaussian random field V_ω. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and consider the following Schrödinger equation: for $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$, $\varepsilon \in (0, 1]$,

$$i\partial_t \varphi(t,x) = -\Delta_x \varphi(t,x) + \varepsilon^{1/2} V_\omega(x) \varphi(t,x), \qquad \varphi(0,x) = \psi_\varepsilon(\varepsilon x) \tag{3.3}$$

By definition, one has that for all $x \in \mathbb{R}^d$

$$\mathbb{E}[V_\omega(x)] = 0, \qquad V_\omega(x+n) = V_\omega(x), \forall n \in \mathbb{Z}^d, \omega \in \Omega,$$

where $\mathbb{E}$ is the expectation with respect to the probability measure $\mathbb{P}$. Fix some $R \in C^\infty(\mathbb{T}^d)$ with $\hat{R} \geqslant 0, \hat{R}(n) = \hat{R}^*(-n), \forall n \in \mathbb{Z}^d$ and define the covariance of V_ω by

$$\mathbb{E}[\hat{V}_\omega(n) \hat{V}_\omega(m)] = \hat{R}(m) \delta(m+n), \qquad \mathbb{E}[\hat{V}_\omega(n) \hat{V}_\omega^*(m)] = \hat{R}(m) \delta(m-n), \tag{3.4}$$

for all $m,n\in\mathbb{Z}^d$. With this covariance, almost every realization of this Gaussian random field is a function in $\cap_{m\in\mathbb{N}_0}H^m(\mathbb{T}^d)$.

We assume that the initial data is given by the ε-dependent family $\psi_\varepsilon(\varepsilon x)$ where

1. The functions ψ_ε are uniformly bounded in $L^2(\mathbb{R}^d;\mathbb{C})$, i.e., $\exists C$ s.t. for any $\varepsilon\in(0,1]$,
$$\|\psi_\varepsilon\|_{L^2(\mathbb{R}^d;\mathbb{C})}\leqslant C \tag{3.5}$$
2. For any test function $\phi\in C_c(\mathbb{R}^d)$
$$\limsup_{\varepsilon\longrightarrow 0}\int_{|k|\geqslant R/\varepsilon}|\widehat{\phi\psi_\varepsilon}(k)|^2\mathrm{d}k\xrightarrow{R\longrightarrow+\infty}0 \tag{3.6}$$
3. Their tails satisfy the following condition
$$\limsup_{\varepsilon\longrightarrow 0}\int_{|x|\geqslant R}|\psi_\varepsilon(x)|^2\mathrm{d}x\xrightarrow{R\longrightarrow+\infty}0 \tag{3.7}$$

Such a family is called an ε-oscillatory family. These are known to be necessary and sufficient conditions to have weak convergence of the energy $\int_{\mathbb{R}^d}|\varphi(t,x)|^2\mathrm{d}x$. See Proposition 1 in [RPK97] for details.

Summarizing our setup, we have that for any $\varepsilon\in(0,1]$, $\varphi=\varphi_\varepsilon$ is the solution of a linear Schrödinger equation (3.3) with a weakly coupled potential $\varepsilon^{1/2}V_\omega$ and with initial condition $\varphi(0,x)=\psi_\varepsilon(\varepsilon x)$. We can view sending $\varepsilon\longrightarrow 0$ as moving from the microscopic to the macroscopic scale, and we need to rescale space and time appropriately in order to see a non-trivial limit.

We do this by studying the weak-coupling limit in phase space via the one-sided rescaled Wigner transform, defined as follows: for $t>0, x,k\in\mathbb{R}^d$
$$W_{\mathrm{OS},\varepsilon}(t,x,k)=\varepsilon^{-d}\int_{\mathbb{R}^d}e^{2\pi ik\cdot y}\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-y\right)\varphi^*\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\mathrm{d}y \tag{3.8}$$

The initial data for $W_{\mathrm{OS},\varepsilon}$ is obtained by replacing the φ in the above expression by ψ_ε. We will be interested in weak limits of $W_{\mathrm{OS},\varepsilon}$, as explained in the introduction.

It is known (see [RPK97] or [GL93]) that for any $t\in\mathbb{R}_+$, up to the extraction of a subsequence, a weak limit of $W_{\mathrm{OS},\varepsilon}(t)$ exists as a tempered distribution and coincides with the corresponding weak limit of the usual rescaled Wigner transform:
$$W^\varepsilon(t,x,k)=\varepsilon^{-d}\int_{\mathbb{R}^d}e^{2\pi ik\cdot y}\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}-\frac{y}{2}\right)\varphi^*\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}+\frac{y}{2}\right)\mathrm{d}y \tag{3.9}$$

Hence, as long as we care about weak limits, it is solely a matter of convenience which version we choose to work with. We use the one-sided version here, since it will produce a Laplacian term in the evolution equation for $W_{\mathrm{OS},\varepsilon}$, which will be convenient for the analysis.

A computation shows (see Lemma 3.6) that the rescaled Wigner transform satisfies the equation
$$(\partial_t+i\varepsilon\Delta_x+4\pi k\cdot\nabla_x)\,W_{\mathrm{OS},\varepsilon}(t,x,k) \tag{3.10}$$
$$=\frac{\varepsilon^{-1/2}}{i}\sum_{n\in\mathbb{Z}^d}e^{2\pi i\varepsilon^{-1}n\cdot x}\hat{V}_\omega(n)\,[\,W_{\mathrm{OS},\varepsilon}(t,x,k-n)-W_{\mathrm{OS},\varepsilon}(k)\,]. \tag{3.11}$$

Here is the key idea of this section: If one uses the two-scale asymptotic expansion that

$$\boxed{W_{\mathrm{OS},\varepsilon}(t,x,k)=W_0(t,x,z,k)+\varepsilon^{1/2}W_1(t,x,z,k)+\varepsilon W_2(t,x,z,k)+\cdots \qquad (3.12)}$$

where $z=\frac{x}{\varepsilon}$ and W_0 is assumed to be $\mathbb{Z}^d$-periodic in the fast variable z, then, replacing ∇_x in equations (3.10) and (3.11) by $\nabla_x+\varepsilon^{-1}\nabla_z$ and Δ_x by $\Delta_x+2\varepsilon^{-1}\nabla_x\cdot\nabla_z+\varepsilon^{-2}\Delta_z$ one has

$$\begin{aligned}
&\partial_t(W_0+\varepsilon^{1/2}W_1+\varepsilon W_2+\cdots)\\
&+i\varepsilon(\Delta_x+2\varepsilon^{-1}\nabla_x\cdot\nabla_z+\varepsilon^{-2}\Delta_z)(W_0+\varepsilon^{1/2}W_1+\varepsilon W_2+\cdots)\\
&+4\pi k\cdot(\nabla_x+\varepsilon^{-1}\nabla_z)(W_0+\varepsilon^{1/2}W_1+\varepsilon W_2+\cdots)\\
&=\frac{\varepsilon^{-1/2}}{i}\sum_{n\in\mathbb{Z}^d}e^{2\pi in\cdot z}\hat{V}(n)[W_0(k-n)+\varepsilon^{1/2}W_1(k+n)+\cdots-W_0(k)-\varepsilon^{1/2}W_1(k)-\cdots].
\end{aligned}$$

Matching terms of the same order of ε, we have at the leading orders of ε the following equations, for $\mathscr{L}(k):=i\Delta_z+4\pi k\cdot\nabla_z$,

$$\mathscr{L}(k)W_0=0, \tag{3.13}$$

$$\mathscr{L}(k)W_1(t,x,z,k)=\frac{1}{i}\sum_{n\in\mathbb{Z}^d}e^{2\pi in\cdot z}\hat{V}_\omega(n)[W_0(t,x,z,k-n)-W_0(t,x,z,k)], \tag{3.14}$$

$$(\partial_t+2i\nabla_x\cdot\nabla_z+4\pi k\cdot\nabla_x)W_0+\mathscr{L}(k)W_2=\frac{1}{i}\sum_{n\in\mathbb{Z}^d}e^{2\pi in\cdot z}\hat{V}_\omega(n)[W_1(k-n)-W_1(k)]. \tag{3.15}$$

For what follows, it will be useful to obtain the eigenfunctions of the operator $\mathscr{L}$. Let $C:=\left[-\frac{1}{2},\frac{1}{2}\right]^d$ and the Brillouin zone $B=\left(-\frac{1}{2},\frac{1}{2}\right)^d$. Now decompose the variable k in the Wigner transform as $k=\eta+\kappa$ for $\eta\in B,\kappa\in\mathbb{Z}^d$ and define for $m,n\in\mathbb{Z}^d$

$$Q_{mn}(z,\kappa)=\delta_{\kappa,m}e^{2\pi i(m-n)\cdot z},\qquad \delta_{\kappa,m}=\begin{cases}1 & \kappa=m\\ 0 & o.w.\end{cases}$$

Since W_0 is assumed to be periodic in z, it can be written in terms of the basis $\{Q_{mn}(z,\kappa)\}_{m,n\in\mathbb{Z}^d}$. This is equivalent to working with the usual basis for a Fourier series, and we use it in this form to make convenient comparisons with what we present in upcoming sections. Furthermore, Lemma 3.7 (see Subsection 3.3) says we have that $Q_{mn}(z,\kappa)$ are eigenfunctions of $\mathscr{L}(\kappa+\eta)$, with eigenvalue $i(E_m(\eta)-E_n(\eta))$, where $E_m(\eta):=4\pi^2|\eta+m|^2$.

Define $Q_m(z,\kappa):=Q_{mm}(z,\kappa)=\delta_{\kappa m}$. Assume that η is not involved in any energy-band crossings, i.e., we consider $\eta:\nexists\kappa,\kappa'\in\mathbb{Z}^d:E_\kappa(\eta)=E_{\kappa'}(\eta)$. Since W_0 satisfies (3.13), W_0 can be constructed from just the Q_m as follows

$$W_0(t,x,z,\eta+\kappa)=\sum_{m\in\mathbb{Z}^d}\sigma_m(t,x,\eta)Q_m(z,\kappa)=\sigma_\kappa(t,x,\eta). \tag{3.16}$$

Having constructed a solution to equation (3.13), we are now ready to use this information to construct a solution to equation (3.14). Since W_1 need not be periodic in z, we will expand it in a general Fourier basis as follows

$$W_1(t,x,z,k)=\sum_{m\in\mathbb{Z}^d}\int_B \mathrm{d}q P_m(z,q)c_m(t,x,k,q),$$

where for $m\in\mathbb{Z}^d, z\in\mathbb{R}^d, q\in B, P_m(z,q):=e^{2\pi i(m+q)\cdot z}$. This is the usual basis involved in the Fourier transform, except that we have decomposed the Fourier variable into an integer and non-integer part. Then since Lemma 3.7 says

$$\mathscr{L}(\kappa+\eta)P_m(z,q)=i(E_\kappa(\eta)-E_{\kappa-m}(\eta-q))P_m(z,q)$$

we can obtain the Fourier coefficients by multiplying equation (3.14) by $P_j^*(z,q_0)$ and integrating in z. The LHS gives

$$\begin{aligned}&\int_{\mathbb{R}^d}\mathrm{d}z\mathscr{L}(k)\,W_1(t,x,z,k)P_j^*(z,q_0)\\ &=\sum_{m\in\mathbb{Z}^d}\int_B\mathrm{d}q\int_{\mathbb{R}^d}\mathrm{d}z\mathscr{L}(k)P_m(z,q)c_m(t,x,k,q)P_j^*(z,q_0)\\ &=i\sum_{m\in\mathbb{Z}^d}\int_B\mathrm{d}q\int_{\mathbb{R}^d}\mathrm{d}z\,(E_\kappa(\eta)-E_{\kappa-m}(\eta-q))\,e^{2\pi i(m+q)\cdot z}c_m(t,x,\kappa+\eta,q)\,e^{-2\pi i(j+q_0)\cdot z}\end{aligned}$$

Using $\int\mathrm{d}z e^{2\pi ik\cdot z}=\delta_k$ in the sense of distributions, this is

$$\begin{aligned}&=i\sum_{m\in\mathbb{Z}^d}\int_B\mathrm{d}q\delta(m-j+q-q_0)\,(E_\kappa(\eta)-E_{\kappa-m}(\eta-q))\,c_m(t,x,\kappa+\eta,q)\\ &=i\sum_{m\in\mathbb{Z}^d}\int_B\mathrm{d}q\delta_{mj}\delta_{q_0}(q)\,(E_\kappa(\eta)-E_{\kappa-m}(\eta-q))\,c_m(t,x,\kappa+\eta,q)\\ &=i(E_\kappa(p)-E_{\kappa-j}(p-q_0))\,c_j(t,x,\kappa+\eta,q_0).\end{aligned}$$

The RHS on the other hand yields,

$$-i\int_{\mathbb{R}^d}\mathrm{d}z\sum_{n\in\mathbb{Z}^d}\hat{V}_\omega(n)\,(\sigma_{\kappa-n}(t,x,\eta)-\sigma_\kappa(t,x,\eta))\,e^{-2\pi i(j-n+q_0)\cdot z}.$$

To avoid dividing by 0 in an expression for the Fourier coefficients c, we introduce a regularization parameter $0<\theta\ll1$ which will be taken to zero at the end of the computations, and consider the regularized Fourier coefficients

$$c_j^\theta(t,x,\kappa+\eta,q_0)=-\int_{\mathbb{R}^d}\mathrm{d}z\sum_{n\in\mathbb{Z}^d}\hat{V}_\omega(n)\frac{(\sigma_{\kappa-n}(t,x,\eta)-\sigma_\kappa(t,x,\eta))}{E_\kappa(\eta)-E_{\kappa-j}(\eta-q_0)+i\theta}e^{-2\pi i(j-n+q_0)\cdot z}.$$

Similarly, defining W_1^θ to be the regularization of W_1 using the regularized coefficients c^θ in place of c, one has that

$$\begin{aligned}W_1^\theta(t,x,z,\eta+\kappa)=-\sum_{m\in\mathbb{Z}^d}\int_B\mathrm{d}q\int_{\mathbb{R}^d}\mathrm{d}z'\sum_{n\in\mathbb{Z}^d}e^{2\pi i(m+q)\cdot(z-z')}e^{2\pi in\cdot z'}\hat{V}_\omega(n)\\ \frac{(\sigma_{\kappa-n}(t,x,\eta)-\sigma_\kappa(t,x,\eta))}{E_\kappa(\eta)-E_{\kappa-m}(\eta-q)+i\theta}.\end{aligned}$$

We now consider equation (3.15), with W_1^θ in place of W_1. The left hand side when tested against $Q_j^*(z,\kappa)=\delta_{\kappa j}$, integrating in z, summing over κ and averaging over the noise yields (see Lemma 3.8 for the computation)

$$\partial_t\sigma_j(t,x,\eta)+\frac{1}{2\pi}\nabla_\eta E_j(\eta)\cdot\nabla_x\sigma_j(t,x,\eta).$$

The RHS is $I_1^\theta + I_2^\theta$ where

$$I_1^\theta = -i \sum_{\kappa\in\mathbb{Z}^d} \int_C \mathrm{d}z \sum_{n'\in\mathbb{Z}^d} e^{2\pi i n'\cdot z} \langle \hat{V}_\omega(n')[W_1^\theta(\eta+\kappa-n')]\rangle \delta_{\kappa j}, \tag{3.17}$$

$$I_2^\theta = i \sum_{\kappa\in\mathbb{Z}^d} \int_C \mathrm{d}z \sum_{n'\in\mathbb{Z}^d} e^{2\pi i n'\cdot z} \langle \hat{V}_\omega(n')[W_1^\theta(\eta+\kappa)]\rangle \delta_{\kappa j}. \tag{3.18}$$

Lemma 3.9 shows that

$$I_1^\theta + I_2^\theta = i \sum_{n'\in\mathbb{Z}^d} \hat{R}(n')(\sigma_{j+n'}(t,x,\eta) - \sigma_j(t,x,\eta))$$

$$\left(\frac{-1}{E_{j+n'}(\eta) - E_j(\eta) + i\theta} + \frac{1}{E_{j+n'}(\eta) - E_j(\eta) - i\theta}\right)$$

$$= -2 \sum_{n'\in\mathbb{Z}^d} \hat{R}(n')(\sigma_{j+n'}(t,x,\eta) - \sigma_j(t,x,\eta)) \frac{\theta}{(E_{j+n'}(\eta) - E_j(\eta))^2 + \theta^2}$$

Recall that θ was a regularization parameter that we introduced artificially, and we would like to take $\theta \longrightarrow 0$. Putting this together with the LHS one has

$$\partial_t \sigma_j(t,x,\eta) + \frac{1}{2\pi} \nabla_p E_j(p) \cdot \nabla_x \sigma_j(t,x,\eta)$$

$$= \lim_{\theta\longrightarrow 0} 2 \sum_{q\in\mathbb{Z}^d} \hat{R}(q)[\sigma_j(t,x,\eta) - \sigma_{j+q}(t,x,\eta)] \left[\frac{\theta}{(E_{j+q}(\eta) - E_j(\eta))^2 + \theta^2}\right]. \tag{3.19}$$

If η is not involved in energy-band crossings, i.e. $\nexists \kappa, \kappa' \in \mathbb{Z}^d : E_\kappa(\eta) = E_{\kappa'}(\eta)$, then

$$\lim_{\theta\longrightarrow 0} \frac{\theta}{(E_{j+q}(\eta) - E_j(\eta))^2 + \theta^2} = 0.$$

and there is no collisional-effect in the limit for these momenta. A rigorous version of this statement is part of Theorem 4.10 in the next Section.

The procedure above does not provide us with information about what happens for momenta involved in band crossings. Note that if $\eta : \exists j, q \in \mathbb{Z}^d : E_{j+q}(\eta) = E_j(\eta)$ then

$$\lim_{\theta\longrightarrow 0} \frac{\theta}{(E_{j+q}(\eta) - E_j(\eta))^2 + \theta^2} = +\infty.$$

so one cannot take the limit in (3.19) pointwise for these η. However since the observables in expression (1.5) involve integrating over all momenta, it is important to have some control also at these “resonant” momenta. We will deal with this issue at the end of Chapter 5.

Before we go to the next section, we make some remarks on the result, the connection to the non-periodic random field, the work of [BFPR99] and the regularization parameter θ.

Remark 3.1. Since V_ω is periodic, we have a Fourier series representation, and sum over discrete Fourier modes. If one replaces the sum in equation (3.19) by an integral over a continuum of Fourier modes and choose an appropriate $\hat{R} \in \mathcal{S}(\mathbb{R}^d)$ this would be

$$\begin{aligned}&\partial_t \sigma_j(t,x,\eta) + \frac{1}{2\pi}\nabla_\eta E_j(\eta) \cdot \nabla_x \sigma_j(t,x,\eta)\\ &= \lim_{\theta \to 0} 2\int_{\mathbb{R}^d} \mathrm{d}q \hat{R}(q)[\sigma_j(t,x,\eta) - \sigma_{j+q}(t,x,\eta)]\left[\frac{\theta}{(E_{j+q}(\eta) - E_j(\eta))^2 + \theta^2}\right]. \qquad (3.20)\end{aligned}$$

Hence since $\frac{\theta}{((E_{j+q}(\eta) - E_j(\eta))^2 + \theta^2)\pi}$ is an approximation of the delta distribution, one formally obtains the following linear Boltzmann equation

$$\begin{aligned}&\partial_t \sigma_j(t,x,\eta) + \frac{1}{2\pi}\nabla_\eta E_j(\eta) \cdot \nabla_x \sigma_j(t,x,\eta)\\ &= 2\pi \int_{\mathbb{R}^d} \mathrm{d}q \hat{R}(q)[\sigma_j(t,x,\eta) - \sigma_{j+q}(t,x,\eta)]\delta(E_{j+q}(\eta) - E_j(\eta)). \qquad (3.21)\end{aligned}$$

This is the equation derived in [Spo77] and [EY99]. For certain V' with no energy-band crossings, the authors of [BFPR99] use the asymptotic expansion (3.4) to derive appropriate linear Boltzmann equations, called radiative transport equations, when working with a weakly coupled, non-periodic, smooth, zero-mean Gaussian field. They also provide a formal derivation of the limit when multiplicities of the eigenvalue are greater than 1, but constant in p. This is relevant to the study of wave propagation in random media.

Remark 3.2. When working with a non-zero background potential $V' \neq 0$, the Fourier basis is not the most convenient for the analysis. In that case, one first finds solutions to the following eigenvalue problem:

$$(-\Delta_z + V')\psi(z,p) = E(p)\psi(z,p)$$

for $p \in \mathbb{R}^d$ fixed, and satisfying the following boundary conditions: for any $n \in \mathbb{Z}^d$, $i \in \{1, \ldots, d\}, z \in \mathbb{R}^d$

$$\psi(z+n,p) = e^{2\pi i n \cdot p}\psi(z,p), \qquad \partial_{z_i}\psi(z+n,p) = e^{2\pi i n \cdot p}\partial_{z_i}\psi(z,p).$$

It is known that there exists a countable, complete family of eigenfunctions $\{\psi_m(z, p)\}_{m \in \mathbb{Z}^d}$ satisfying this, that are orthonormal in $L^2_z(C)$, with corresponding eigenvalues $E_m(p)$. Then, one can construct Q_{mn} via

$$Q_{mn}(z,\kappa,\eta) := \int_C e^{2\pi i(\eta+\kappa)\cdot y}\psi_m(z-y,\eta)\psi_n^*(z,\eta)\mathrm{d}y$$

for $z \in \mathbb{T}^d$, $\kappa \in \mathbb{Z}^d$, $\eta \in B$, and one can check that Lemma 3.7 still holds. In general for $V' \neq 0$, the Q_m will be eigenfunctions of $\mathscr{L}$ that are periodic in z, and so unlike in the case $V' = 0$, the W_0 will not be constant in z. When $V' \neq 0$, the computations below can be suitably modified using some results from Bloch theory, in order to derive the limit equations. This is an important feature of the work [BFPR99]. When $V' = 0$ one has that ψ_m are plain waves: for $m \in \mathbb{Z}^d, z, p \in \mathbb{R}^d$, $\psi_m(z,p) = e^{2\pi i(m+p)\cdot z}$, with $E_m(p) = 4\pi^2|m+p|^2$.

Remark 3.3. To the best of our knowledge, this procedure has not been made rigorous for any time-independent potential. The only rigorous derivations of the linear Boltzmann equation from a Schrödinger equation with a time-independent Gaussian random field are based on the control of Duhamel expansions of high order (see [Spo77], [EY99], [Che06] and [But15]). See [Her24] for a recent approach where many diagrams are avoided. The difficulty is that the regularization parameter was introduced artificially, and is not part of the problem.

In the following Subsection we describe the main result of [PR04] where the authors introduce a damping term into the evolution equation for the Wigner, and this naturally introduces the parameter θ into the corrector equations. One can then derive a kinetic equation in the limit $\varepsilon \longrightarrow 0$, without taking the limit of $\theta \longrightarrow 0$, even for a deterministic smooth periodic potential. A further comment on time-dependent potentials is made at the end of the next subsection.

3.2 On the regularization parameter θ

In the above arguments we introduced a regularization parameter θ, and we see that there is an issue interpreting the limit as $\theta \longrightarrow 0$. In [PR04], the authors modify the equation satisfied by the usual rescaled Wigner transform as in (3.9) and study

$$\partial_t W^\varepsilon(t,x,k) + k\cdot\nabla_x W^\varepsilon(t,x,k) + \frac{\theta}{\varepsilon}(W_\varepsilon - \chi_\varepsilon * W_\varepsilon)(t,x,k) = \mathscr{L}_\varepsilon W_\varepsilon(t,x,k). \tag{3.22}$$

where the collision term is

$$\mathscr{L}_\varepsilon f(x,k) := i\varepsilon^{-1/2}\int_{\mathbb{R}^d} \hat{V}(p)e^{2\pi i p\cdot\frac{x}{\varepsilon}}\Big(f\Big(x,k+\frac{p}{2}\Big) - f\Big(x,k-\frac{p}{2}\Big)\Big)\mathrm{d}p,$$

and where $\chi_\varepsilon(x) = \varepsilon^{-d}\chi(\frac{x}{\varepsilon})$ for some nice mollifier $\chi \in \mathcal{S}(\mathbb{R}^d)$ satisfying

$$\chi(x) = \chi(|x|) \geqslant 0, \qquad \|\chi\|_{L^1(\mathbb{R}^d)} = 1, \qquad \hat{\chi} \in \mathbb{R}, |\hat{\chi}(p)| < 1, \forall p \neq 0, \hat{\chi}(0) = 1 \tag{3.23}$$

The term involving θ is introduced into the equation to cut-off high frequencies. Assume that the potential has a Fourier transform of the form

$$\hat{V}(p) = \sum_{j=1}^{\infty} \alpha_j[\delta(p-p_j) + \delta(p+p_j)] + \hat{\Phi}(p), \tag{3.24}$$

where $\alpha_j \in \mathbb{R}$ and $\hat{\Phi}(p)$ is smooth, decays rapidly and has $\hat{\Phi}(0) = 0$. Any smooth $\mathbb{Z}^d$-periodic potential can be written in this form since the Fourier transform of a smooth $\mathbb{Z}^d$-periodic function is a tempered distribution of the form $\mathscr{F}_x V(\xi) = \sum_{n\in\mathbb{Z}^d} \hat{V}(n)\delta(\xi - n)$, with $\hat{V}(n)$ the coefficients in the Fourier series of V. One needs to ensure that $\hat{V}(n) = \hat{V}(-n)$, which can be done by assuming radial symmetry of V so $V(x) = V(|x|)$. Assume also that

$$\sum_{j=1}^{\infty} \frac{|\alpha_j|}{|1-\hat{\chi}(p_j)|} < \infty, \tag{3.25}$$

$$\sum_{j,l=1}^{\infty} \frac{|\alpha_j||\alpha_l|}{|1-\hat{\chi}(p_l)||1-\hat{\chi}(p_j+p_l)|} + \sum_{j\neq l}^{\infty} \frac{|\alpha_j||\alpha_l|}{|1-\hat{\chi}(p_l)||1-\hat{\chi}(p_j-p_l)|} < \infty. \tag{3.26}$$

So $p_j = 0$ is forbidden. These conditions are satisfied if $\alpha_j \in l^1$ and there exists $\omega_0 > 0$ such that

$$|p_j| \geqslant \omega_0 > 0, \qquad |p_j \pm p_l| \geqslant \omega_0, j \neq l.$$

Conditions (3.24) and (3.25) are clearly satisfied by a smooth zero-mean $\mathbb{Z}^d$-periodic potential. Let

$$K_\theta(k,p) = \frac{2\theta(1-\hat{\chi}(p))}{\theta^2(1-\hat{\chi}(p))^2 + \left(\left(k+\frac{p}{2}\right)\cdot p\right)^2}.$$

This is analogous to the expression

$$\frac{2\theta}{\theta^2 + (E_{j+q}(\eta) - E_j(\eta))^2}$$

we derived by using the regularized W_1^θ. When modifying the Wigner equation, this regularization is introduced at the level of the equation, and one can then prove the following

Theorem 3.4. *Let the initial data $W_\varepsilon(0,x,k) = W_0(x,k)$ of equation (3.22) belong to $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and assume that $V \in \cap_{m \in \mathbb{N}_0} H^m(\mathbb{T}^d)$ is a zero mean $\mathbb{Z}^d$-periodic function. Assume that conditions (3.23) on the mollifier hold. Then $\mathscr{L}_\varepsilon$ is uniformly bounded on $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and the solution of equation (3.22) converges in $C([0,T]; L^2(\mathbb{R}^d \times \mathbb{R}^d))$ to the solution of the kinetic equation*

$$\frac{\partial W_0}{\partial t} + \frac{1}{2\pi} k \cdot \nabla_x W_0 = \sum_{q \in \mathbb{Z}^d \setminus \{0\}} |\alpha_q|^2 K_\theta(k, p_q)[W_0(k+p_q) - W_0(k)]. \tag{3.27}$$

This is a direct corollary of Theorem 2.1 in [PR04]. In this article, the authors can then also work with a series of potentials V^δ with their Fourier transforms supported on lattices with increasingly fine lattice spacing δ. Under some suitable assumptions on the family $\{V^\delta\}_\delta$ one can recover a linear Boltzmann equation (analogous to equation (3.21))by first taking $\delta \longrightarrow 0$ followed by taking the regularization parameter $\theta \longrightarrow 0$. The limit does not depend on the choice of χ. See Theorems 2.2 and 2.3 in [PR04] for the details. It is important to mention that there is no need to average over any randomness using this approach, and so one can also get path-wise results for random potentials whose Fourier transforms have almost surely the structure described above.

When studying the weak-coupling limit of a Schrödinger equation with a periodic potential, we cannot change the evolution equation for the rescaled Wigner transform, since this is dictated by the Schrödinger equation. It would however be interesting to see if a similar procedure can be made to work by introducing a suitable damping term at the level of the Schrödinger equation as, for instance, was done in [Gri23] for the low-density periodic quantum Lorentz gas. Furthermore, in Chapter 5 of this thesis we will present some results without introducing any additional regularizing parameter, using techniques differ from these asymptotic expansions.

The proof of Theorem 3.4 is similar in style to those in qualitative homogenization theory and follows closely some parts of the heuristic derivation we did. Roughly speaking, one makes an Ansatz of the form

$$W^{\varepsilon}(t,x,k)=W_0(t,x,k)+\sqrt{\varepsilon}\,W_1(t,x,z,k)+\varepsilon W_2(t,x,z,k)+R_{\varepsilon},$$

where $z=\frac{x}{\varepsilon}$ and W_0 satisfies the limit equation and one builds W_1 and W_2 one after the other from W_0. Then one shows that

$$R_{\varepsilon}=W^{\varepsilon}-W_0+\sqrt{\varepsilon}\,W_1+\varepsilon W_2 \tag{3.28}$$

is of order $\sqrt{\varepsilon}$. As a consequence of modifying the dynamics of the rescaled Wigner transform, W_1 is chosen to cancel terms that are formally of power $\varepsilon^{-1/2}$, and hence the equation for W_1 reads as:

$$k\cdot\nabla_z W_1+\theta(W_1-\chi * W_1)=i\int_{\mathbb{R}^d}\mathrm{d}n\,e^{2\pi i n\cdot z}\hat{V}(n)\Big[W_0\Big(t,x,k+\frac{n}{2}\Big)-W_0\Big(t,x,k-\frac{n}{2}\Big)\Big].$$

Recall that when working with the usual rescaled Wigner transform (3.9) instead of (3.8) one no longer has the Laplacian in the evolution equation, and the collision term involves symmetric shifts. Fourier transforming this in z, one has that

$$(2\pi i k\cdot\xi+\theta(1-\hat{\chi}(\xi)))\,\widehat{W_1}(\xi)=i\hat{V}(\xi)\left[W_0\left(t,x,k+\frac{\xi}{2}\right)-W_0\left(t,x,k-\frac{\xi}{2}\right)\right]$$

where ξ is the dual variable to z. So

$$W_1(t,x,z)=i\int_{\mathbb{R}^d}\mathrm{d}\xi\,e^{2\pi i z\cdot\xi}\frac{\hat{V}(\xi)\big[W_0\big(t,x,k+\frac{\xi}{2}\big)-W_0\big(t,x,k-\frac{\xi}{2}\big)\big]}{(2\pi i k\cdot\xi+\theta(1-\hat{\chi}(\xi)))}.$$

For the term involving W_2, a simple computation shows that

$$k\cdot\nabla_z W_2+\theta(W_2-\chi * W_2)=-(\partial_t W_0+k\cdot\nabla_x W_0)+\int_{\mathbb{R}^d}e^{2\pi i\xi\cdot z}G(t,x,k,\xi)\mathrm{d}\xi,$$

where

$$G(t,x,k,\xi)=\int_{\mathbb{R}^d}\mathrm{d}\xi'\,\hat{V}(\xi-\xi')\hat{V}(\xi')\frac{W_0\big(k-\frac{\xi}{2}+\xi'\big)-W_0\big(k-\frac{\xi}{2}\big)}{\theta(1-\hat{\chi}(\xi'))+i\Big(\Big(k-\frac{\xi-\xi'}{2}\Big)\cdot\xi'\Big)}$$

$$+\int_{\mathbb{R}^d}\mathrm{d}\xi'\,\hat{V}(\xi-\xi')\hat{V}(\xi')\frac{W_0\big(k+\frac{\xi}{2}+\xi'\big)-W_0\big(k+\frac{\xi}{2}\big)}{\theta(1-\hat{\chi}(\xi'))+i\Big(\Big(k+\frac{\xi-\xi'}{2}\Big)\cdot\xi'\Big)}$$

See expressions (3.2) and (3.7) in [PR04]. Fourier transforming in z yields

$$(2\pi i k\cdot\xi+\theta(1-\hat{\chi}(\xi)))\,\widehat{W_2}(\xi)=-(\partial_t W_0+k\cdot\nabla_x W_0)\delta(\xi)+G(\xi).$$

Averaging over the randomness one can show that

$$G(t,x,\xi,k)=G_0(t,x,k)\delta(\xi)+G'(t,x,k,\xi),$$

where $G'(t,x,k,\xi)$ is regular at $\xi=0$, and an explicit computation shows that $G_0(t,x,k)$ coincides with the right hand side of (3.27). Hence one has that the leading order term W_0 satisfies

$$\partial_t W_0+k\cdot\nabla_x W_0=G_0(t,x,k).$$

At this point, one has W_0 defined as the solution of (3.27), and has defined W_1 and W_2 in terms of it. One can bound W_1 and W_2 in L^2 in terms of $\|W_0\|_{L^2}$ due to assumptions (3.25) and (3.26). The goal then is to show that the remainder term (3.28) is small, and this involves writing an evolution equation for the remainder, using the cancellations in the resulting terms due to how W_1 and W_2 were constructed from W_0, and finally using the estimates on W_0, W_1 and W_2 in L^2. The details of this can be found in Section 4 of [PR04].

Remark 3.5. This program has also been carried out in the case of time dependent random potentials in [BPR02] using martingale tools, without modifying the equation for the rescaled Wigner transform, and using the randomness in time instead for the regularization. For further results in the case of time dependent random potentials, see [Gom13].

3.3 Relevant lemmas

Lemma 3.6. *When $\varphi\in C(\mathbb{R}_{\geqslant 0};H^2(\mathbb{R}^d;\mathbb{C}))\cap C^1(\mathbb{R}_+;L^2(\mathbb{R}^d;\mathbb{C}))$ solves the free Schrödinger equation from initial data in $H^2(\mathbb{R}^d;\mathbb{C})$, (which is equation (3.1), with $V'=V=0$), the time evolution of the rescaled one-sided Wigner transform $W_{\mathrm{OS},\varepsilon}(t,x,k)$ is given by*

$$\partial_t W_{\mathrm{OS},\varepsilon}(t,x,k)+i\varepsilon\Delta_x W_{\mathrm{OS},\varepsilon}(t,x,k)+4\pi k\cdot\nabla_x W_{\mathrm{OS},\varepsilon}(t,x,k)=0$$

Hence for φ satisfying (3.3), the time evolution of the rescaled one-sided Wigner transform $W_{\mathrm{OS},\varepsilon}(t,x,k)$ is given by equations (3.10) and (3.11).

Proof. We rewrite

$$W_{\mathrm{OS},\varepsilon}(t,x,k)=\int_{\mathbb{R}^d}e^{2\pi ik\cdot y}\varphi_\varepsilon(t,x-\varepsilon y)\varphi_\varepsilon^*(t,x)\mathrm{d}y,$$

where $\varphi_\varepsilon(t,x)=\varphi(\frac{t}{\varepsilon},\frac{x}{\varepsilon})$. Since φ satisfies the free Schrödinger equation we have that $\partial_t\varphi(t,x)=i\Delta_x\varphi(t,x)$, so

$$\begin{aligned}\partial_t\varphi_\varepsilon(t,x)=\partial_t(\varphi_\varepsilon(t,x))=\partial_t\left(\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\right)=\varepsilon^{-1}\partial_t\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\\ =i\varepsilon^{-1}\Delta_x\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)=i\varepsilon^{-1}\left(\varepsilon^2\Delta_x\left(\varphi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)\right)\right)=i\varepsilon\Delta_x\varphi_\varepsilon(t,x).\end{aligned}$$

Using this, we have that

$$\begin{aligned}&\partial_t W_{\mathrm{OS},\varepsilon}(t,x,k)\\ &=i\varepsilon\int_{\mathbb{R}^d}e^{2\pi ik\cdot y}\Delta_x\varphi_\varepsilon(t,x-\varepsilon y)\varphi_\varepsilon^*(t,x)\mathrm{d}y-i\varepsilon\int_{\mathbb{R}^d}e^{2\pi ik\cdot y}\varphi_\varepsilon(t,x-\varepsilon y)\Delta_x\varphi_\varepsilon^*(t,x)\mathrm{d}y.\end{aligned}$$

Now since

$$\begin{aligned}i\varepsilon\Delta_x W_\varepsilon(t,x,k)=i\varepsilon\int_{\mathbb{R}^d}e^{2\pi ik\cdot y}\Delta_x\varphi_\varepsilon(t,x-\varepsilon y)\varphi_\varepsilon^*(t,x)\mathrm{d}y\\ +i\varepsilon\int_{\mathbb{R}^d}e^{2\pi ik\cdot y}\varphi_\varepsilon(t,x-\varepsilon y)\Delta_x\varphi_\varepsilon^*(t,x)\mathrm{d}y+2i\varepsilon\int_{\mathbb{R}^d}e^{2\pi ik\cdot y}\nabla_x\varphi_\varepsilon(t,x-\varepsilon y)\nabla_x\varphi_\varepsilon^*(t,x)\mathrm{d}y,\end{aligned}$$

one has that

$$\begin{aligned}\partial_t W_\varepsilon(t,x,k) = -i\varepsilon\Delta_x W_\varepsilon(t,x,k) + 2i\varepsilon\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\Delta_x\varphi_\varepsilon(t,x-\varepsilon y)\varphi_\varepsilon^*(t,x)\mathrm{d}y\\ +2i\varepsilon\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\nabla_x\varphi_\varepsilon(t,x-\varepsilon y)\nabla_x\varphi_\varepsilon^*(t,x)\mathrm{d}y.\end{aligned}$$

Furthermore, since

$$\nabla_x\varphi_\varepsilon(t,x-\varepsilon y) = -\varepsilon^{-1}\nabla_y\varphi_\varepsilon(t,x-\varepsilon y), \qquad \Delta_x\varphi_\varepsilon(t,x-\varepsilon y) = \varepsilon^{-2}\Delta_y\varphi_\varepsilon(t,x-\varepsilon y),$$

one has that

$$\begin{aligned}\partial_t W_\varepsilon(t,x,k) &= -i\varepsilon\Delta_x W_\varepsilon(t,x,k) + 2i\varepsilon^{-1}\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\Delta_y\varphi_\varepsilon(t,x-\varepsilon y)\varphi_\varepsilon^*(t,x)\mathrm{d}y\\ &\quad -2i\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\nabla_y\varphi_\varepsilon(t,x-\varepsilon y)\nabla_x\varphi_\varepsilon^*(t,x)\mathrm{d}y\\ &= -i\varepsilon\Delta_x W_\varepsilon(t,x,k) - 2i\varepsilon^{-1}(2\pi i k)\cdot\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\nabla_y\varphi_\varepsilon(t,x-\varepsilon y)\varphi_\varepsilon^*(t,x)\mathrm{d}y\\ &\quad +2i(2\pi i k)\cdot\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi_\varepsilon(t,x-\varepsilon y)\nabla_x\varphi_\varepsilon^*(t,x)\mathrm{d}y\\ &= -i\varepsilon\Delta_x W_\varepsilon(t,x,k) - 4\pi k\cdot\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\nabla_x\varphi_\varepsilon(t,x-\varepsilon y)\varphi_\varepsilon^*(t,x)\mathrm{d}y\\ &\quad -4\pi k\cdot\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi_\varepsilon(t,x-\varepsilon y)\nabla_x\varphi_\varepsilon^*(t,x)\mathrm{d}y\\ &= -i\varepsilon\Delta_x W_\varepsilon(t,x,k) - 4\pi k\cdot\nabla_x W_\varepsilon(t,x,k).\end{aligned}$$

Hence, one has that

$$\partial_t W_\varepsilon(t,x,k) + i\varepsilon\Delta_x W_\varepsilon(t,x,k) + 4\pi k\cdot\nabla_x W_\varepsilon(t,x,k) = 0.$$

When one includes the potential from (3.3), the potential terms contribute two additional terms, and one has

$$\begin{aligned}&\partial_t W_{\mathrm{OS},\varepsilon}(t,x,k) + i\varepsilon\Delta_x W_{\mathrm{OS},\varepsilon} + 4\pi k\cdot\nabla_x W_{\mathrm{OS},\varepsilon}\\ &= -\frac{\varepsilon^{-1/2}}{i}V_\omega\Big(\frac{x}{\varepsilon}\Big)\varphi_\varepsilon^*(t,x)\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi_\varepsilon(t,x-\varepsilon y)\mathrm{d}y\\ &\quad +\frac{\varepsilon^{-1/2}}{i}\varphi_\varepsilon^*(t,x)\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}V_\omega\Big(\frac{x}{\varepsilon}-y\Big)\varphi_\varepsilon(t,x-\varepsilon y)\mathrm{d}y.\end{aligned}$$

By expanding the periodic potential in terms of its Fourier series, the RHS is

$$\begin{aligned}&= -\frac{\varepsilon^{-1/2}}{i}\sum_{n\in\mathbb{Z}^d} e^{2\pi i\varepsilon^{-1}n\cdot x}\hat{V}_\omega(n)\varphi_\varepsilon^*(t,x)\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\varphi_\varepsilon(t,x-\varepsilon y)\mathrm{d}y\\ &\quad +\frac{\varepsilon^{-1/2}}{i}\varphi_\varepsilon^*(t,x)\int_{\mathbb{R}^d} e^{2\pi i k\cdot y}\sum_{n\in\mathbb{Z}^d} e^{2\pi i\varepsilon^{-1}n\cdot(x-\varepsilon y)}\hat{V}_\omega(n)\varphi_\varepsilon(t,x-\varepsilon y)\mathrm{d}y,\end{aligned}$$

which lead to equations (3.10) and (3.11). □

Lemma 3.7. *For* $k = \eta + \kappa \in B + \mathbb{Z}^d$,

$$\mathscr{L}(\eta+\kappa)Q_{mn}(z,\kappa) = i(E_m(\eta) - E_n(\eta))Q_{mn}(z,\kappa),$$

and

$$\mathscr{L}(\eta+\kappa)P_m(z,q) = i(E_\kappa(\eta) - E_{\kappa-m}(\eta-q))P_m(z,q).$$

Proof. (of Lemma 3.7) The proof is a simple computation. We first compute the LHS

$$\begin{aligned}\mathscr{L}(\eta+\kappa)Q_{mn}(z,\kappa) &= i\Delta_z Q_{mn}(z,\kappa)+4\pi(\eta+\kappa)\cdot\nabla_z Q_{mn}(z,\kappa)\\ &=4\pi^2 i\delta_{\kappa,m}(-|\kappa-n|^2+2(\eta+\kappa)\cdot(\kappa-n))e^{2\pi i(m-n)\cdot z}\\ &=4\pi^2 i\delta_{\kappa,m}(|\kappa|^2-|n|^2+2\eta\cdot\kappa-2\eta\cdot n)e^{2\pi i(m-n)\cdot z}.\end{aligned}$$

Next, since $E_m(\eta)=4\pi^2|m+\eta|^2$, one has that

$$\begin{aligned}E_m(\eta)-E_n(\eta) &= 4\pi^2|m+\eta|^2-4\pi^2|n+\eta|^2\\ &=4\pi^2(|m|^2+|\eta|^2+2\eta\cdot m-|n|^2-|\eta|^2-2\eta\cdot n)\\ &=4\pi^2(|m|^2-|n|^2+2p\cdot m-2p\cdot n)\end{aligned}$$

from which we can conclude. Similarly,

$$\begin{aligned}\mathscr{L}(\eta+\kappa)P_m(z,q) &= (i\Delta_z+4\pi(\eta+\kappa)\cdot\nabla_z)e^{2\pi i(m+q)\cdot z}\\ &=(-4\pi^2 i|m+q|^2+8\pi^2 i(\eta+\kappa)\cdot(m+q))e^{2\pi i(m+q)\cdot z}\\ &=i(4\pi^2|\eta+\kappa|^2-4\pi^2|(\eta+\kappa)-(m+q)|^2)e^{2\pi i(m+q)\cdot z}\\ &=i(E_\kappa(\eta)-E_{\kappa-m}(\eta-q))P_m(z,q).\end{aligned}$$

□

Lemma 3.8. *The LHS of equation (3.15) when multiplied by $Q_j^*(z,\kappa)$, integrated in z, summed over κ yields the term*

$$\partial_t\sigma_j(t,x,\eta)+\frac{1}{2\pi}\nabla_\eta E_j(\eta)\cdot\nabla_x\sigma_j(t,x,\eta)$$

Proof. Consider the first term,

$$\sum_\kappa\int_C \mathrm{d}z\,\partial_t W_0(t,x,z,\eta+\kappa)Q_j^*(z,\kappa)=\sum_\kappa\int_C \mathrm{d}z\,\partial_t\sigma_\kappa(t,x,\eta)\delta_{\kappa j}=\partial_t\sigma_j(t,x,\eta).$$

The second term is 0 since W_0 does not depend on z. The third term, for $k=\eta+\kappa$ is

$$\begin{aligned}&\sum_\kappa\int_C \mathrm{d}z\,4\pi(\eta+\kappa)\cdot\nabla_x W_0(t,x,z,\eta+\kappa)Q_j^*(z,\kappa)\\ &=\sum_\kappa\int_C \mathrm{d}z\,4\pi(\eta+\kappa)\cdot\nabla_x\sigma_\kappa(t,x,\eta)\delta_{\kappa j}=4\pi(\eta+j)\cdot\nabla_x\sigma_j(t,x,\eta)\\ &=\frac{1}{2\pi}\nabla_\eta E_j(\eta)\cdot\nabla_x\sigma_j(t,x,\eta).\end{aligned}$$

The last term vanishes, since we can integrate by parts in z and $Q_j^*\in\ker\mathscr{L}$. This concludes the proof. □

Lemma 3.9. *One has that for I_1^θ, I_2^θ defined in (3.17), (3.18) that*

$$I_1^\theta=-i\sum_{n'\in\mathbb{Z}^d}\hat{R}(n')\frac{(\sigma_{j+n'}(t,x,\eta)-\sigma_j(t,x,\eta))}{E_{j+n'}(\eta)-E_j(\eta)+i\theta},$$

and

$$I_2^\theta = i \sum_{n' \in \mathbb{Z}^d} \hat{R}(n') \frac{(\sigma_{j+n'}(t,x,\eta) - \sigma_j(t,x,\eta))}{E_{j+n'}(\eta) - E_j(\eta) - i\theta}$$

Proof. Consider the second term

$$I_2^\theta = i \sum_{\kappa \in \mathbb{Z}^d} \int_C \mathrm{d}z \sum_{n' \in \mathbb{Z}^d} \hat{V}_\omega(n') e^{2\pi i n' \cdot z} W_1^\theta(t,x,z,\eta+\kappa)\delta_{\kappa j}$$

$$= -i \int_C \mathrm{d}z \sum_{n' \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} \int_B \mathrm{d}q \int_{\mathbb{R}^d} \mathrm{d}z' \sum_{n \in \mathbb{Z}^d} e^{2\pi i (m+q)\cdot(z-z')} e^{2\pi i n \cdot z'} \hat{V}_\omega(n) \hat{V}_\omega(n') e^{2\pi i n' \cdot z}$$

$$\frac{(\sigma_{j-n}(t,x,\eta) - \sigma_j(t,x,\eta))}{E_j(\eta) - E_{j-m}(\eta - q) + i\theta}.$$

Since $\langle \hat{V}_\omega(n) \hat{V}_\omega(n') \rangle = \hat{R}(n')\delta(n+n')$, by averaging over the noise and shifting in z' this becomes

$$= -i \sum_{n' \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} \int_B \mathrm{d}q \int_{\mathbb{R}^d} \mathrm{d}z' e^{-2\pi i (n'+m+q)\cdot z'} \hat{R}(n') \frac{(\sigma_{j+n'}(t,x,\eta) - \sigma_j(t,x,\eta))}{E_j(\eta) - E_{j-m}(\eta - q) + i\theta}.$$

Let $f_2(q) = \frac{1}{E_j(\eta) - E_{j-m}(\eta-q) + i\theta}$, then

$$\int_{\mathbb{R}^d} \mathrm{d}z' e^{-2\pi i (n'+m)\cdot z'} \int_B \mathrm{d}q f_2(q) e^{-2\pi i q \cdot z'} = \int_{\mathbb{R}^d} \mathrm{d}z' e^{-2\pi i (n'+m)\cdot z'} \int_{\mathbb{R}^d} f_2(q) \mathbb{I}_B(q) e^{-2\pi i q \cdot z'}$$

$$= \int_{\mathbb{R}^d} \mathrm{d}z' \widehat{f_2 \mathbb{I}_B}(z') e^{-2\pi i (n'+m)\cdot z'} = f_2(n'+m)\mathbb{I}_B(n'+m)$$

$$= f_2(0)\delta_{n'+m}.$$

Hence the above term is

$$= -i \sum_{n' \in \mathbb{Z}^d} \hat{R}(n') \frac{(\sigma_{j+n'}(t,x,\eta) - \sigma_j(t,x,\eta))}{E_j(\eta) - E_{j+n'}(\eta) + i\theta}$$

$$= i \sum_{n' \in \mathbb{Z}^d} \hat{R}(n') \frac{(\sigma_{j+n'}(t,x,\eta) - \sigma_j(t,x,\eta))}{E_{j+n'}(\eta) - E_j(\eta) - i\theta}.$$

Now, consider the first term

$$I_1^\theta = -i \sum_{\kappa \in \mathbb{Z}^d} \int_C \mathrm{d}z \sum_{n' \in \mathbb{Z}^d} \hat{V}_\omega(n') e^{2\pi i n' \cdot z} W_1^\theta(t,x,z,\eta+\kappa-n')\delta_{\kappa j}$$

$$= i \int_C \mathrm{d}z \sum_{n' \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} \int_B \mathrm{d}q \int_{\mathbb{R}^d} \mathrm{d}z' \sum_{n \in \mathbb{Z}^d} \hat{V}_\omega(n') e^{2\pi i n' \cdot z} e^{2\pi i (m+q)\cdot(z-z')} e^{2\pi i n \cdot z'} \hat{V}_\omega(n)$$

$$\frac{(\sigma_{j-n'-n}(t,x,\eta) - \sigma_{j-n'}(t,x,\eta))}{E_{j-n'}(\eta) - E_{j-n'-m}(\eta - q) + i\theta}.$$

Averaging once more over the noise makes this

$$= i \sum_{n' \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} \int_B \mathrm{d}q \int_{\mathbb{R}^d} \mathrm{d}z' \hat{R}(n') e^{-2\pi i (n'+m+q)\cdot z'} \frac{(\sigma_j(t,x,\eta) - \sigma_{j-n'}(t,x,\eta))}{E_{j-n'}(\eta) - E_{j-n'-m}(\eta - q) + i\theta}.$$

Let $f_1(q) = \frac{1}{E_{j-n'}(\eta) - E_{j-n'-m}(\eta - q) + i\theta}$, then as before

$$\int_{\mathbb{R}^d} \mathrm{d}z' e^{-2\pi i (n'+m)\cdot z'} \int_B \mathrm{d}q f_1(q) e^{-2\pi i q\cdot z'} = f_1(0)\,\delta_{n'+m}.$$

Hence the above expression becomes

$$= i \sum_{n'\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} \hat{R}(n') \frac{(\sigma_j(t,x,\eta) - \sigma_{j-n'}(t,x,\eta))}{E_{j-n'}(\eta) - E_{j-n'-m}(\eta) + i\theta} \delta_{n'+m}.$$

Assuming $\hat{R}(n') = \hat{R}(-n'), \forall n' \in \mathbb{Z}^d$, this is

$$= -i \sum_{n'\in\mathbb{Z}^d} \hat{R}(n') \frac{(\sigma_{j+n'}(t,x,\eta) - \sigma_j(t,x,\eta))}{E_{j+n'}(\eta) - E_j(\eta) + i\theta}.$$

This concludes the computation. □

Chapter 4
The non-resonant observables case

This Chapter will make precise the intuition gained from the previous heuristic argument, that for the non-problematic momenta, free transport holds in the weak coupling limit. This will be the content of Theorem 4.10. We will use the so-called Wigner series, which was originally introduced in [MMP94] in order to derive Newtonian equations, in the semiclassical scaling limit, from the Schrödinger equation with a fixed periodic potential. Finally, in Subsection 4.4, we discuss how our problem connects with the study of the long time behavior of the Schrödinger equation with a fixed periodic potential.

Schrödinger operators with periodic potentials are common in solid state physics. We refer the reader to Chapter 16 of [RS78] for an introduction to the functional analysis of such operators. Key roles are played by the Bloch–Floquet–Zak transform and the classical Bloch–Floquet transform introduced in Section 2. These tools have been previously used to study semiclassical scaling limits for Schrödinger equations with periodic potentials in [MMP94] and [GMMP98]. In this section we adapt these tools to the study of the weak coupling limit, and prove that for momenta not associated to energy band crossings of the Laplacian, one has free transport in the limit.

The contents of this chapter form the core of the preprint [San6+] currently in preparation.

4.1 Summary of the section

Let $C := \left[-\frac{1}{2}, \frac{1}{2}\right]^d, B = \left(-\frac{1}{2}, \frac{1}{2}\right)^d$, as in the previous section. Let V be a real valued $\mathbb{Z}^d$-periodic potential with the properties

$$V \in L^\infty(\mathbb{R}^d), \qquad V(x+n) = V(x), \forall n \in \mathbb{Z}^d. \tag{4.1}$$

Let $\varepsilon \in (0,1], \lambda \in [0,1]$, and define the Hamiltonian $H^{\varepsilon,\lambda}$ as follows

$$H^{\varepsilon,\lambda} := -\varepsilon^2 \Delta_x + \lambda V\left(\frac{x}{\varepsilon}\right).$$

Consider φ^λ satisfying the Schrödinger equation with $H^{1,\lambda}$:

$$i\partial_t \varphi^\lambda(t,x) = H^{1,\lambda} \varphi^\lambda(t,x), \qquad \varphi^\lambda_{0,\varepsilon}(x) = \psi_\varepsilon(\varepsilon x), \tag{4.2}$$

where $\psi_\varepsilon(x)$ is an ε-oscillatory family of initial data satisfying (3.5)- (3.7) as introduced in the previous section.

One can compute that the rescaled wavefunction $\varphi_\varepsilon^\lambda(t,x) := \varphi^\lambda(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$ satisfies

$$i\varepsilon\partial_t\varphi_\varepsilon^\lambda(t,x) = -\varepsilon^2\Delta_x\varphi_\varepsilon^\lambda(t,x) + \lambda V\Big(\frac{x}{\varepsilon}\Big)\varphi_\varepsilon^\lambda(t,x) = H^{\varepsilon,\lambda}\varphi_\varepsilon^\lambda(t,x) \tag{4.3}$$

with initial data

$$\varphi_{\varepsilon,0}(x) = \psi_\varepsilon(x). \tag{4.4}$$

Recall the definition of the Bloch–Floquet transform $\mathscr{U}_{\mathrm{cl}}$ (Definition 2.4). One can prove the following

Proposition 4.1. *Assume V satisfies (4.1). Let $\theta \in B$ and let $H_{cl}^0(\theta)$ be the operator $-\Delta$ on $L^2(C;\mathbb{C})$ with the boundary conditions*

$$\varphi(x+n) = e^{2\pi i n\cdot\theta}\varphi(x), \qquad \frac{\partial\varphi}{\partial x_l}(x+n) = e^{2\pi i n\cdot\theta}\frac{\partial\varphi}{\partial x_l}(x), \quad l\in\{1,\dots,d\} \tag{4.5}$$

Then

$$\mathscr{U}_{cl}H^{1,\lambda}\mathscr{U}_{cl}^{-1} = \int_B \mathrm{d}\theta H_{cl}^{1,\lambda}(\theta) \tag{4.6}$$

where

$$H_{cl}^{1,\lambda}(\theta) = H_{cl}^0(\theta) + \lambda V$$

This is Theorem X111.97 from [RS78].

Since $\mathscr{U}_{\mathrm{cl}}$ is unitary, the formula (4.6) says one can obtain spectral information of $H^{1,\lambda}$ by integrating spectral information of $H^{1,\lambda}(\theta)$ over the Brillouin zone.

From Theorem X111.98 in [RS78] (see also [Wil78] and [Gér90]), one has the following properties for the eigenfunctions and eigenvalues of $H^{1,\lambda}(\theta)$:

1. For each $\theta \in B, H^{1,\lambda}(\theta)$ has a complete set of measurable eigenfunctions $\{\Psi_m^\lambda(x,\theta)\}_{m\in\mathbb{N}}$ with associated eigenvalues $E_m^\lambda(\theta)$. For each $\theta\in B, \Psi_m^\lambda(x,\theta)$ are orthonormed in $L_x^2(C;\mathbb{C})$.
2. One can label the eigenvalues in increasing order, so one has
$$0 \leq E_1^\lambda(\theta) \leq E_2^\lambda(\theta) \leq \cdots \leq E_m^\lambda(\theta) \leq \cdots, \qquad \forall\theta\in B,$$
and moreover $E_m^\lambda(\theta) \xrightarrow{m\to\infty} \infty$. With this labeling, for each $m\in\mathbb{N}, E_m^\lambda$ is Lipschitz continuous. Clearly, the E_m^λ are $\mathbb{Z}^d$-periodic since they are eigenvalues of a $\mathbb{Z}^d$-periodic $H^{1,\lambda}(\theta)$, as a consequence of equation (4.5).
3. There exists a set of zero measure F_0^λ such that E_m^λ are analytic on $B\backslash F_0^\lambda$.

The function $\theta \longrightarrow E_n^\lambda(\theta)$ is called the n^{th}-Bloch band for the Hamiltonian $H^{1,\lambda}(\theta)$, and $\Psi_n^\lambda(x,\theta)$ is called the n^{th}-Bloch function. The eigenfunctions of $H^{\varepsilon,\lambda}$ can be obtained by rescaling the eigenfunctions of $H^{1,\lambda}$, as in the following

Lemma 4.2. *Define the rescaled Bloch function*

$$\Psi_m^{\varepsilon,\lambda}(x,\theta) := \varepsilon^{-d/2}\Psi_m^\lambda\Big(\frac{x}{\varepsilon},\theta\Big), \quad x\in\mathbb{R}^d, \theta\in B. \tag{4.7}$$

One has that

$$H^{\varepsilon,\lambda}\Psi_m^{\varepsilon,\lambda}(x,\theta) = E_m^\lambda(\theta)\Psi_m^{\varepsilon,\lambda}(x,\theta), \tag{4.8}$$

and for any $x\in\mathbb{R}^d, n\in\mathbb{Z}^d, l\in\{1,\dots,d\}$

$$\Psi_m^{\varepsilon,\lambda}(x+\varepsilon n,\theta)=e^{2\pi i n\cdot\theta}\Psi_m^{\varepsilon,\lambda}(x,\theta), \tag{4.9}$$

$$\frac{\partial\Psi_m^{\varepsilon,\lambda}}{\partial x_l}(x+\varepsilon n,\theta)=e^{2\pi i n\cdot\theta}\frac{\partial\Psi_m^{\varepsilon,\lambda}}{\partial x_l}(x,\theta). \tag{4.10}$$

The proof is postponed to Subsection 4.3. One has that the rescaled Bloch functions are orthonormal in $L_x^2(\varepsilon C;\mathbb{C})$. Using these eigenfunctions, one can define the so-called band-spaces:

Definition 4.3. *Let* $m\in\mathbb{N}$. *The* m^{th}*-band space* $S_m^{\varepsilon,\lambda}$ *is defined to be*

$$S_m^{\varepsilon,\lambda}:=\left\{f(x)\in L^2(\mathbb{R}^d;\mathbb{C}):f(x)=\int_B\sigma(\theta)\Psi_m^{\varepsilon,\lambda}(x,\theta)\mathrm{d}\theta:\sigma\in L^2(B)\right\} \tag{4.11}$$

This is a subset of $L^2(\mathbb{R}^d;\mathbb{C})$ which is invariant under the action of $H^{\varepsilon,\lambda}$. Furthermore, $S_{m_1}^{\varepsilon,\lambda}\perp S_{m_2}^{\varepsilon,\lambda}$ for $m_1\neq m_2$ (see Proposition 4.20 and Lemma 4.21 for the details). Let us denote the orthogonal projection onto this subspace by

$$\Pi_m^{\varepsilon,\lambda}:L^2(\mathbb{R}^d;\mathbb{C})\longrightarrow S_m^{\varepsilon,\lambda}$$

The action of the Hamiltonian in the subspaces $S_m^{\varepsilon,\lambda}$ is described by the following

Lemma 4.4. *Let* $\psi\in S_m^{\varepsilon,\lambda}$. *Then*

$$(H^{\varepsilon,\lambda}\psi)(x)=\sum_{n\in\mathbb{Z}^d}\mathscr{E}_m^\lambda(n)\,\psi(x+\varepsilon n),$$

where $\mathscr{E}_m^\lambda(n)$ *are the Fourier coefficients of* E_m^λ.

The proof is postponed to Subsection 4.3. This allows one to decompose the solution $\varphi_\varepsilon^\lambda$ to the rescaled Schrödinger equation (4.3) into a countable family of functions $\{\varphi_m^{\varepsilon,\lambda}\}_{m\in\mathbb{N}}$, one in each band space. Each $\varphi_m^{\varepsilon,\lambda}$ satisfies

$$i\varepsilon\partial_t\varphi_m^{\varepsilon,\lambda}(t,x)=-\varepsilon^2\Delta_x\varphi_m^{\varepsilon,\lambda}(t,x)+\lambda V\Big(\frac{x}{\varepsilon}\Big)\varphi_m^{\varepsilon,\lambda}(t,x)=H^{\varepsilon,\lambda}\varphi_m^{\varepsilon,\lambda}(t,x)$$

with initial conditions

$$\varphi_m^{\varepsilon,\lambda}(0,x)=\Pi_m^{\varepsilon,\lambda}\psi_\varepsilon(x). \tag{4.12}$$

with ψ_ε an ε-oscillatory family of initial data satisfying (3.5)-(3.4). The IVP for $\varphi_m^{\varepsilon,\lambda}$ can, using Lemma 4.4, be rewritten as

$$i\varepsilon\partial_t\varphi_m^{\varepsilon,\lambda}(t,x)=\sum_{n\in\mathbb{Z}^d}\mathscr{E}_m^\lambda(n)\,\varphi_m^{\varepsilon,\lambda}(t,x+\varepsilon n),\qquad \varphi_m^{\varepsilon,\lambda}(x,0)=\Pi_m^{\varepsilon,\lambda}\psi_\varepsilon(x). \tag{4.13}$$

As in the previous section, it is convenient to work in phase space. Here, in order to make use of the decomposition of the wavefunction, we work with the Wigner series. Define for $f,g\in\mathcal{S}(\mathbb{R}^d), x\in\mathbb{R}^d, \theta\in B$

$$w_m^\varepsilon(f,g)(x,\theta):=\sum_{\mu\in\mathbb{Z}^d}f\Big(x+\frac{\varepsilon}{2}\mu\Big)g\Big(x-\frac{\varepsilon}{2}\mu\Big)e^{2\pi i\theta\cdot\mu}.$$

And for $t > 0$, we will study the m^{th}-band Wigner function: for $x \in \mathbb{R}^d, \theta \in B, t > 0$,

$$w_m^{\varepsilon,\lambda}(t,x,\theta) := w_m^{\varepsilon}(\varphi_m^{\varepsilon,\lambda,*}(t), \varphi_m^{\varepsilon,\lambda}(t))(x,\theta).$$

The initial condition for m^{th} band Wigner functions is obtained by replacing $\varphi_m^{\varepsilon,\lambda}(t)$ by the initial wave functions $\varphi_m^{\varepsilon,\gamma}(0,x)$ from expression (4.12).

Remark 4.5. Recall that the usual rescaled Wigner transform (3.9) is defined in terms of the wavefunction φ^{λ}. Equivalently, one could define it in terms of the rescaled wavefunction as follows

$$W^{\varepsilon}(t,x,k) := \int_{\mathbb{R}^d} \mathrm{d}y e^{2\pi i k\cdot y} \varphi_{\varepsilon}^{\lambda}\Big(t, x - \frac{\varepsilon y}{2}\Big) \varphi_{\varepsilon}^{\lambda,*}\Big(t, x + \frac{\varepsilon y}{2}\Big)$$

and rescaling the initial conditions appropriately. Here the rescaled Wigner series is defined in terms of projections of the rescaled wavefunction, but this could also have been done using the non-rescaled wavefunction.

Remark 4.6. One has the following connection of the Wigner series to the usual Wigner transform: Define for $f, g \in L^2(\mathbb{R}^d), x, k \in \mathbb{R}^d$,

$$W^{\varepsilon}(f,g)(x,k) = \int_{\mathbb{R}^d} \mathrm{d}\mu f\Big(x + \frac{\varepsilon}{2}\mu\Big) g\Big(x - \frac{\varepsilon}{2}\mu\Big) e^{2\pi i k\cdot\mu}.$$

Then

$$w_m^{\varepsilon}(f,g)(x,\theta) = \sum_{\mu' \in \mathbb{Z}^d} W^{\varepsilon}(f,g)(x, \theta + \mu').$$

We give a proof in Lemma 4.22 when $f, g \in \mathcal{S}(\mathbb{R}^d)$.

Remark 4.7. One could also ask about the limit of the Wigner series with arguments from different band spaces $S^{\varepsilon,\lambda}$. For instance, why not consider

$$w_{s,ml}^{\varepsilon,\lambda}(t,x,\theta) := \sum_{\mu \in \mathbb{Z}^d} \varphi_m^{\varepsilon,\lambda,*}\Big(t, x + \frac{\varepsilon}{2}\mu\Big) \varphi_l^{\varepsilon,\lambda}\Big(t, x - \frac{\varepsilon}{2}\mu\Big) e^{2\pi i \theta\cdot\mu} \quad ?$$

The reason is that the limit for these objects for $\varepsilon \longrightarrow 0$ is 0 in $\mathcal{D}'$. This assertion is proved in Lemma 4.13 in [GMMP98]. We discuss this after the proof of Theorem 4.10, where the underlying mechanism becomes more transparent in light of the preceding argument.

Next, we define the Banach space

$$\mathcal{B} := \Big\{ \varphi(x,\theta) = \sum_{\mu \in \mathbb{Z}^d} \mathcal{F}_k \varphi(x,\mu) e^{2\pi i \mu\cdot\theta} : \mathcal{F}_\theta \varphi \in l^1(\mathbb{Z}^d; C_0(\mathbb{R}_x^d)) \Big\},$$

equipped with the norm

$$\|\varphi\|_{\mathcal{B}} := \sum_{\mu \in \mathbb{Z}^d} \|\mathcal{F}_\theta \varphi(\cdot,\mu)\|_{L^\infty(\mathbb{R}_x^d)}.$$

This is actually a Banach algebra. It is useful to consider this space, as a consequence of the following Lemma, which says that the rescaled band Wigner functions are uniformly bounded in the dual of $\mathcal{B}$.

Lemma 4.8. *Assume (4.12) holds. One then has the following uniform bound for* $w_m^{\varepsilon,\lambda}$*: for all* $m\in\mathbb{N}$, $t\geqslant 0$

$$\|w_m^{\varepsilon,\lambda}(t)\|_{\mathscr{B}^*}\leqslant D.$$

The proof is postponed to Subsection 4.3. The above bound is analogous to Proposition III.1 in [LP93].

Finally, consider the following transport equation: Let $\Omega\subset B$ open. For $x,v\in\mathbb{R}^d$, $\theta\in\Omega$, $t\in\mathbb{R}_+$ consider

$$\partial_t f(t,x,\theta)+v\cdot\nabla_x f(t,x,\theta)=0,\qquad f(0,x,\theta)=f_0(x,\theta). \tag{4.14}$$

Definition 4.9. *A function* $f\in D'_{\mathrm{per}}([0,\infty)\times\mathbb{R}_x^d\times\Omega)$ *is a distributional solution of (4.14) with initial condition* f_0 *if for any* $\phi\in C_c^\infty([0,\infty)\times\mathbb{R}_x^d\times\Omega)$ *one has that*

$$\int_{[0,\infty)\times\mathbb{R}_x^d\times\Omega}\mathrm{d}x\mathrm{d}\theta\mathrm{d}t f(t,x,\theta)\partial_t\varphi(t,x,\theta)+\int_{\mathbb{R}_x^d\times\Omega}\mathrm{d}x\mathrm{d}\theta f_0(x,\theta)\phi(0,x,\theta)$$
$$+\int_0^\infty\mathrm{d}t\int_{\mathbb{R}^d\times\Omega}\mathrm{d}x\mathrm{d}\theta f(t,x,\theta)v\cdot\nabla_x\phi(t,x,\theta)=0.$$

Finally recall that F_0^0 is the zero measure set where the $E_m^0(\theta)$ are not analytic.

We can now state our main result, which is the following

Theorem 4.10. *Let assumptions (4.1) and (4.12) hold and let* $\varepsilon\in(0,1]$ *be a sequence with limit zero. Let* $\lambda=\varepsilon^{1/2}$*. Then one has that*

$$w_{I,m}^{\varepsilon,\lambda}\xrightarrow{*,\varepsilon\to 0}w_{I,m}\geqslant 0,\qquad \text{in}\,\mathscr{B}^*,\forall m\in\mathbb{N},$$

$$w_m^{\varepsilon,\lambda}\xrightarrow{*,\varepsilon\to 0}w_m\geqslant 0,\qquad \text{in}\,L^\infty(\mathbb{R}_+;\mathscr{B}^*),\forall m\in\mathbb{N}.$$

For any $m\in\mathbb{N}$*, the limit* $w_m=w_m(t,x,\theta)$ *is the unique distributional solution in* $D'_{\mathrm{per}}([0,\infty)\times\mathbb{R}_x^d\times(\bar{B}-F_0^0))$ *in the sense of Definition 4.9, of the initial value problem*

$$\partial_t w_m(t,x,\theta)+\frac{1}{2\pi}\nabla_\theta E_m^0(\theta)\cdot\nabla_x w_m(t,x,\theta)=0,\qquad w_m(0,x,\theta)=w_{I,m}(x,\theta), \tag{4.15}$$

and such that w_m *is* $\mathbb{Z}^d$*-periodic in* θ*.*

The proof of this is the content of Section 4.2.

Remark 4.11. $E_m^0(\theta)$ are the ordered Bloch bands from the operator $H_{cl}^0(\theta)$, for which one can explicitly characterize the countable family of eigenfunctions (as we did in the previous Section when constructing Q_{mn}), which are just the plane waves: for $m\in\mathbb{Z}^d$, $\theta\in B$

$$\Psi_m^0(x,\theta)=e^{2\pi i(\theta+m)\cdot x}$$

which clearly form a complete orthonormal family in $L^2(C)$, with associated eigenvalues $E_m^0(\theta)=4\pi^2|\theta+m|^2$. Undoing the relabelling, one has that

$$\partial_t w_m(t,x,\theta)+4\pi(\theta+m)\cdot\nabla_x w_m(t,x,\theta)=0,$$

for any momenta θ, for which there do not exist $n \neq n' \in \mathbb{Z}^d : E_n(\theta) = E_{n'}(\theta)$. This is thus a rigorous statement summarizing the discussion following equation (3.19) from the previous Section. In the next chapter, we will introduce yet another phase space object related to the Wigner transform, for which one can view the problem of limits at energy band crossings as a problem of regularity of a different phase space object we will introduce.

Remark 4.12. In the article [MMP94] (on which this Subsection is primarily based), one has a fixed λ, and in the limit equation (4.15), one has $\nabla_k E_m^\lambda(\theta)$ in place of $\nabla_k E_m^0(\theta)$. The corresponding equations are known as semiclassical equations, since quantum effects disappear in this limit, leaving a family of decoupled transport equations in each band, with no possibility of tunneling from one band to another.

Furthermore, for certain families of mixed states, one can show uniform bounds for the initial rescaled Wigner series in $L^2_{x,k}(\mathbb{R}^d \times B)$, and for their evolutions in $L^\infty([0,T]; L^2_{x,k}(\mathbb{R}^d \times B))$. In this case, one can prove weak convergence of the rescaled Wigner series to the transport equation (4.15) in L^2, for test functions with support including the momenta associated to band crossings. Note that the result does not allow us to handle a general ε-oscillatory family of initial conditions such as the WKB family.

For fixed λ, under the assumption that one can obtain a mixture of states within a single band space $S_m^{\varepsilon,\lambda}$, and that there are no energy band crossings associated to E_m (uniformly in ε) one can also prove convergence of the rescaled Wigner transform associated to the wavefunction in the band. We remark that our mixture of states from Example 2.8 does not automatically satisfy this requirement, since it does not automatically lie in a single band space.

The analysis of Schrödinger operators in regimes where the band gap shrinks in the scaling limit at rate $\varepsilon^{1/2}$ is subtle, which is the case in the weak-coupling regime. Using semiclassical tools similar to those presented here, transitions known as Landau–Zener transitions have been shown in some toy models, for instance in [CG02]. See [PST03] for the situation where the authors analyse band crossings in the semiclassical limit for a finite number of bands that, for fixed λ, remain uniformly in-ε separated from all other bands.

In the article [MMP94], the authors also prove convergence of the charge density, current density and energy density under further assumptions. Such results could potentially also be derived in the weak coupling regime using the ideas presented here, but we do not pursue this here.

A different perspective of the results of [MMP94] can be found in [GMMP98], where certain other potentials are also treated, using Wigner measures and microlocal analysis.

4.2 Proof of Theorem 4.10

Although the Wigner series need not be positive, its limits always are, as stated by the following

Lemma 4.13. *Let $w_{I,m}$, w_m be accumulation points of $w_{I,m}^{\varepsilon,\lambda}$ and $w_m^{\varepsilon,\lambda}$ in the $\mathscr{B}^*$ and $L^\infty((0,\infty);\mathscr{B}^*)$ topologies respectively. Then*

$$w_{I,m} \geqslant 0 \text{ on } \mathbb{R}^d \times B, \qquad w_m \geqslant 0 \text{ on } \mathbb{R}^d \times B \times (0,\infty),$$

in the sense of measures.

The proofs are the content of Lemmas 4.2, 4.3 and 4.4 in [MMP94].

Consider the time evolution of the band Wigner functions given by the following

Lemma 4.14. *The m^{th}-rescaled band Wigner function $w_m^{\varepsilon,\lambda}$ solves*

$$\partial_t w_m^{\varepsilon,\lambda}(t,x,\theta) + i\sum_{n\in\mathbb{Z}^d} \mathscr{E}_m^\lambda(n)\left[\frac{w_m^{\varepsilon,\lambda}(t,x+\frac{\varepsilon}{2}n,\theta) - w_m^{\varepsilon,\lambda}(t,x-\frac{\varepsilon}{2}n,\theta)}{\varepsilon}\right]e^{2\pi i\theta\cdot n} = 0 \tag{4.16}$$

Proof. We compute

$$\partial_t w_m^{\varepsilon,\lambda}(t,x,\theta) = \sum_{\mu\in\mathbb{Z}^d} \partial_t\Big(\varphi_m^{\varepsilon,\lambda,*}\Big(t,x+\frac{\varepsilon}{2}\mu\Big)\varphi_m^{\varepsilon,\lambda}\Big(t,x-\frac{\varepsilon}{2}\mu\Big)\Big)e^{2\pi i\theta\cdot\mu}.$$

By using the chain rule and equation (4.13), this is

$$\begin{aligned}
&= \sum_{\mu\in\mathbb{Z}^d} \partial_t\varphi_m^{\varepsilon,\lambda,*}\Big(t,x+\frac{\varepsilon}{2}\mu\Big)\varphi_m^{\alpha,\lambda}\Big(t,x-\frac{\varepsilon}{2}\mu\Big)e^{2\pi i\theta\cdot\mu}\\
&+ \sum_{\mu\in\mathbb{Z}^d} \varphi_m^{\varepsilon,\lambda,*}\Big(t,x+\frac{\varepsilon}{2}\mu\Big)\partial_t\varphi_m^{\varepsilon,\lambda}\Big(t,x-\frac{\varepsilon}{2}\mu\Big)e^{2\pi i\theta\cdot\mu}\\
&= \frac{i}{\varepsilon}\sum_{\mu\in\mathbb{Z}^d}\left(\sum_{n\in\mathbb{Z}^d}\mathscr{E}_m^\lambda(n)\varphi_m^{\varepsilon,\lambda,*}\Big(t,x+\frac{\varepsilon}{2}\mu+\varepsilon n\Big)\right)\varphi_m^{\varepsilon,\lambda}\Big(t,x-\frac{\varepsilon}{2}\mu\Big)e^{2\pi i\theta\cdot\mu}\\
&- \frac{i}{\varepsilon}\sum_{\mu\in\mathbb{Z}^d}\varphi_m^{\varepsilon,*}\Big(t,x+\frac{\varepsilon}{2}\mu\Big)\left(\sum_{n\in\mathbb{Z}^d}\mathscr{E}_m^\lambda(n)\varphi_m^{\varepsilon,\lambda}\Big(t,x-\frac{\varepsilon}{2}\mu+\varepsilon n\Big)\right)e^{2\pi i\theta\cdot\mu}.
\end{aligned}$$

Exchanging sums and changing variables to $\mu = \mu' - n$ in the first sum and $\mu = \mu' + n$ in the second sum respectively, this is

$$\begin{aligned}
&= \frac{i}{\varepsilon}\sum_{n\in\mathbb{Z}^d}\mathscr{E}_m^\lambda(n)\sum_{\mu'\in\mathbb{Z}^d}\varphi_m^{\varepsilon,\lambda,*}\Big(t,x+\frac{\varepsilon}{2}\mu'+\frac{\varepsilon}{2}n\Big)\varphi_m^{\varepsilon,\lambda}\Big(t,x-\frac{\varepsilon}{2}\mu'+\frac{\varepsilon}{2}n\Big)e^{2\pi i\theta\cdot(\mu'-n)}\\
&- \frac{i}{\varepsilon}\sum_{n\in\mathbb{Z}^d}\mathscr{E}_m^\lambda(n)\sum_{\mu'\in\mathbb{Z}^d}\varphi_m^{\varepsilon,\lambda,*}\Big(t,x+\frac{\varepsilon}{2}\mu'+\frac{\varepsilon}{2}n\Big)\varphi_m^{\varepsilon,\lambda}\Big(t,x-\frac{\varepsilon}{2}\mu'+\frac{\varepsilon}{2}n\Big)e^{2\pi i\theta\cdot(\mu'+n)}.
\end{aligned}$$

Using the definition of the Wigner series and regrouping, this is

$$\begin{aligned}
&= \frac{i}{\varepsilon}\sum_{n\in\mathbb{Z}^d}\mathscr{E}_m^\lambda(n)w_m^{\varepsilon,\lambda}\Big(t,x+\frac{\varepsilon}{2}n,\theta\Big)e^{-2\pi i\theta\cdot n} - \frac{i}{\varepsilon}\sum_{n\in\mathbb{Z}^d}\mathscr{E}_m^\lambda(n)w_m^{\varepsilon,\lambda}\Big(t,x+\frac{\varepsilon}{2}n,\theta\Big)e^{2\pi i\theta\cdot n}\\
&= \frac{i}{\varepsilon}\sum_{n\in\mathbb{Z}^d}\Big[\mathscr{E}_m^\lambda(-n)w_m^{\varepsilon,\lambda}\Big(t,x-\frac{\varepsilon}{2}n,\theta\Big) - \mathscr{E}_m^\lambda(n)w_m^{\varepsilon,\lambda}\Big(t,x+\frac{\varepsilon}{2}n,\theta\Big)\Big]e^{2\pi i\theta\cdot n}.
\end{aligned}$$

Since we are working with real potentials, one has that $\mathscr{E}_m^{\lambda}(-n) = \mathscr{E}_m^{\lambda}(n)$ so this is

$$= i\sum_{n\in\mathbb{Z}^d} \mathscr{E}_m^{\lambda}(n)\left[\frac{w_m^{\varepsilon,\lambda}(t, x-\frac{\varepsilon}{2}n, \theta) - w_m^{\varepsilon,\lambda}(t, x+\frac{\varepsilon}{2}n, \theta)}{\varepsilon}\right]e^{2\pi i\theta\cdot n}.$$

Hence we arrive at equation (4.16). □

Before we prove Theorem 4.10, we need one more ingredient, which is a continuity property of the energy bands with respect to the potential.

Lemma 4.15. *One has the following continuity property for the energy bands. For two potentials V, V' and $m \in \mathbb{N}$*

$$\sup_{\theta\in B} |E_m^V(\theta) - E_m^{V'}(\theta)| \leq \|V - V'\|_{L^\infty(\mathbb{R}^d)} \tag{4.17}$$

and

$$\sup_{n\in\mathbb{Z}^d} |\mathscr{E}_m^V(n) - \mathscr{E}_m^{V'}(n)| \leq \|V - V'\|_{L^\infty(\mathbb{R}^d)} \tag{4.18}$$

Proof. Denote by $H^V(\theta)$ and $H^{V'}(\theta)$ the fibered operators associated to V and V' respectively. The proof relies on the min-max theorem, which states that

$$E_m^V(\theta) = \min_{S\subset L^2(C):\dim S = m}\ \max_{\psi\in S:\|\psi\|_{L^2}=1} \langle\psi, H^V(\theta)\psi\rangle.$$

Fix S any m-dimensional subspace of $L^2(C)$. One has that for any $\psi\in S$, since $H^V(\theta) - H^{V'}(\theta) = V - V'$, that

$$\langle\psi, H^V(\theta)\psi\rangle = \langle\psi, H^{V'}(\theta)\psi\rangle + \langle\psi, (V - V')\psi\rangle.$$

Maximizing over unit norm functions in S and minimizing over all possible subspaces S,

$$E_m^V(\theta) \leq E_m^{V'}(\theta) + \|V - V'\|_{L^\infty(\mathbb{R}^d)}.$$

This proves (4.17). Reversing the roles of V and V', we conclude, since the bound is uniform in k. Taking the Fourier series, one has that

$$\mathscr{E}_m^V(n) - \mathscr{E}_m^{V'}(n) = \int_B \mathrm{d}\theta e^{-2\pi i n\cdot\theta}(E_m^V(\theta) - E_m^{V'}(\theta)).$$

Using (4.17) one has (4.18). □

Corollary 4.16. *Let χ be a smooth function bounded by 1, and periodic in B. One has that*

$$\sup_{\theta\in B} |E_m^V(\theta)\chi_m(\theta) - E_m^{V'}(\theta)\chi_m(\theta)| \leq \|V - V'\|_{L^\infty(\mathbb{R}^d)}, \tag{4.19}$$

and for $\widetilde{\mathscr{E}_m^V}, \widetilde{\mathscr{E}_m^{V'}}$ the Fourier coefficients associated to $E_m^V\chi$ and $E_m^{V'}\chi$ respectively,

$$\sup_{n\in\mathbb{Z}^d} |\widetilde{\mathscr{E}_m^V}(n) - \widetilde{\mathscr{E}_m^{V'}}(n)| \leq \|V - V'\|_{L^\infty(\mathbb{R}^d)}. \tag{4.20}$$

Proof. This is trivial using Lemma 4.15 since

$$\sup_{\theta\in B}|E_m^V(\theta)\chi_m(\theta)-E_m^{V'}(\theta)\chi_m(\theta)|\leqslant\sup_{k\in B}|E_m^V(\theta)-E_m^{V'}(\theta)|\sup_{k\in B}|\chi(k)|,$$

and likewise, since

$$\widetilde{\mathscr{E}_m^V}(n)-\widetilde{\mathscr{E}_m^{V'}}(n)=\int_B \mathrm{d}\theta e^{-2\pi i n\cdot\theta}(E_m^V(\theta)-E_m^{V'}(\theta))\chi(\theta),$$

one has by Lemma 4.15,

$$\sup_{n\in\mathbb{Z}^d}|\widetilde{\mathscr{E}_m^V}(n)-\widetilde{\mathscr{E}_m^{V'}}(n)|\leqslant\|V-V'\|_{L^\infty(\mathbb{R}^d)}. \qquad \square$$

Proof. (of Theorem 4.10) For any subsequence $\varepsilon_j\longrightarrow 0$, as a consequence of Lemma 4.8 one has a further subsequence that converges. As a consequence of Lemma 4.13, one has that the limit is positive. If we can show the limit always satisfies equation (4.15), then as a consequence of uniqueness of distributional solutions to this equation, we have that the entire sequence converges to a solution of (4.15), thus proving the Theorem. We are thus left with showing that any limit must satisfy equation (4.15).

Multiplying equation (4.16) by a real test function $\varphi(t,x,k)$ that is $\mathbb{Z}^d$-periodic in k and whose Fourier transform is in $C_c^\infty([0,\infty)\times\mathbb{R}_x^d\times(\bar{B}-F_0))$ we get that

$$\int_{[0,\infty)\times\mathbb{R}_x^d\times(\bar{B}-F_0^0)}\mathrm{d}t\mathrm{d}x\mathrm{d}\theta\partial_t w_m^{\varepsilon,\lambda}(x,k,t)\varphi(x,k,t)$$

$$+i\int_{[0,\infty)\times\mathbb{R}_x^d\times(\bar{B}-F_0^0)}\mathrm{d}t\mathrm{d}x\mathrm{d}k\theta$$

$$\sum_{n\in\mathbb{Z}^d}\mathscr{E}_m^\lambda(n)\left[\frac{w_m^{\varepsilon,\lambda}(t,x+\frac{\varepsilon}{2}n,\theta)-w_m^{\varepsilon,\lambda}(t,x-\frac{\varepsilon}{2}n,\theta)}{\varepsilon}\right]e^{2\pi i\theta\cdot n}\varphi(t,x,\theta)=0.$$

Integrating the first term by parts in time, one gets that

$$0=\int_{[0,\infty)\times\mathbb{R}_x^d\times(\bar{B}-F_0^0)}\mathrm{d}t\mathrm{d}x\mathrm{d}\theta w_m^{\varepsilon,\lambda}(t,x,\theta)\partial_t\varphi(t,x,\theta)$$

$$+\int_{\mathbb{R}_x^d\times(\bar{B}-F_0^0)}\mathrm{d}x\mathrm{d}\theta w_{I,m}^{\varepsilon,\lambda}(x,\theta)\varphi(0,x,\theta)$$

$$-i\int_{[0,\infty)\times\mathbb{R}_x^d\times(\bar{B}-F_0^0)}\mathrm{d}t\mathrm{d}x\mathrm{d}\theta\varphi(t,x,\theta)$$

$$\sum_{n\in\mathbb{Z}^d}\mathscr{E}_m^\lambda(n)\left[\frac{w_m^{\varepsilon,\lambda}(t,x+\frac{\varepsilon}{2}n,\theta)-w_m^{\varepsilon,\lambda}(t,x-\frac{\varepsilon}{2}n,\theta)}{\varepsilon}\right]e^{2\pi i\theta\cdot n}.$$

Now, Plancherel's formula says that for φ real

$$\int_{\mathbb{R}_x^d}f\left(x-\frac{\varepsilon}{2}n\right)\varphi(x)\mathrm{d}x=\int_{\mathbb{R}^d}\tau_{\frac{\varepsilon n}{2}}f(x)\varphi^*(x)\mathrm{d}x$$

$$=\int_{\mathbb{R}^d}\mathscr{F}_x\tau_{\frac{\varepsilon n}{2}}f(\xi)(\mathscr{F}_x\varphi)^*(\xi)\mathrm{d}\xi=\int_{\mathbb{R}^d}e^{-2\pi i\frac{\varepsilon n}{2}\cdot\xi}\mathscr{F}_xf(\xi)(\mathscr{F}_x\varphi)^*(\xi)\mathrm{d}\xi.$$

The third expression is thus

$$
\begin{aligned}
&= i\int_{[0,\infty)\times\mathbb{R}_x^d\times(\bar{B}-F_0^0)} \mathrm{d}t\mathrm{d}x\mathrm{d}\theta\mathscr{F}_x w_m^{\varepsilon,\lambda}(t,\xi,\theta)(\mathscr{F}_x\varphi)^*(t,\xi,\theta)\\
&\qquad \sum_{n\in\mathbb{Z}^d} \mathscr{E}_m^\lambda(n)\frac{\left(e^{2\pi i\left(\theta-\frac{\varepsilon\xi}{2}\right)\cdot n}-e^{2\pi i\left(\theta+\frac{\varepsilon\xi}{2}\right)\cdot n}\right)}{\varepsilon}\\
&= -i\int_{[0,\infty)\times\mathbb{R}_x^d\times(\bar{B}-F_0^0)} \mathrm{d}t\mathrm{d}x\mathrm{d}k\theta\mathscr{F}_x w_m^{\varepsilon,\lambda}(t,\xi,\theta)(\mathscr{F}_x\varphi)^*(t,\xi,\theta)\\
&\qquad \frac{\left(E_m^\lambda\left(\theta+\frac{\varepsilon\xi}{2}\right)-E_m^\lambda\left(\theta-\frac{\varepsilon\xi}{2}\right)\right)}{\varepsilon}.
\end{aligned}
$$

Using the fact that φ has compact support, we know there exists a φ dependent open set G_0 on which φ is zero such that G_0 contains the zero measure set F_0^0 where the band E_m^0 is not analytic. Let H_0 be an open set such that $G_0 \Supset H_0 \Supset F_0^0$ and let χ be a smooth function such that

$$
\chi = \begin{cases} 1 & \text{on}\,\bar{B}\backslash G_0,\\ 0 & \text{in}\,H_0, \end{cases}
$$

and set

$$
\widetilde{E_m^\lambda}(\theta) := E_m^\lambda(\theta)\chi(\theta).
$$

Then, since modifying the energy band on the set where the test function is zero does not change the expression, the above term is

$$
= \int_{[0,\infty)\times\mathbb{R}_x^d\times(\bar{B}-F_0^0)} \mathrm{d}t\mathrm{d}\xi\mathrm{d}\theta\mathscr{F}_x w_m^{\varepsilon,\lambda}(t,\xi,\theta)(\mathscr{F}_x\Theta_m^{\varepsilon,\lambda})^*(t,\xi,\theta),
$$

where

$$
(\mathscr{F}_x\Theta_m^{\varepsilon,\lambda})(t,\xi,\theta) = i\frac{\left(\widetilde{E_m^\lambda}\left(\theta+\frac{\varepsilon\xi}{2}\right)-\widetilde{E_m^\lambda}\left(\theta-\frac{\varepsilon\xi}{2}\right)\right)}{\varepsilon}\mathscr{F}_x\varphi(t,\xi,\theta).
$$

We need to show that

$$
\Theta_m^{\varepsilon,\lambda} \xrightarrow{\lambda=\varepsilon^{1/2},\varepsilon\longrightarrow 0} \frac{1}{2\pi}\nabla_\theta\widetilde{E_m^0}\cdot\nabla_x\varphi\,\text{in}\,L^1((0,\infty);\mathscr{B})
$$

where $\widetilde{E_m^0}(\theta) := E_m^0(\theta)\chi(\theta)$. Hence, we estimate

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}t\left\|\Theta_m^{\varepsilon,\lambda}-\frac{1}{2\pi}\nabla_\theta\widetilde{E_m^0}\cdot\nabla_x\varphi\right\|_{\mathscr{B}}\\
&= \int_0^\infty \mathrm{d}t\sum_{\mu\in\mathbb{Z}^d}\left\|\mathscr{F}_\theta\Theta_m^{\varepsilon,\lambda}(t,\cdot,\mu)-\frac{1}{2\pi}\mathscr{F}_\theta(\nabla_\theta\widetilde{E_m^0}\cdot\nabla_x\varphi)(t,\cdot,\mu)\right\|_{L^\infty(\mathbb{R}_x^d)}\\
&\leqslant \int_0^\infty \mathrm{d}t\sum_{\mu\in\mathbb{Z}^d}\left\|\mathscr{F}_x^{-1}\mathscr{F}_\theta\Theta_m^{\varepsilon,\lambda}(t,\cdot,\mu)-\frac{1}{2\pi}\mathscr{F}_x^{-1}\mathscr{F}_\theta(\nabla_\theta\widetilde{E_m^0}\cdot\nabla_x\varphi)(t,\cdot,\mu)\right\|_{L^1(\mathbb{R}_\xi^d)}.
\end{aligned}
$$

We first compute for $\xi\in\mathbb{R}^d, \mu\in\mathbb{Z}^d$

$$
\begin{aligned}
\frac{1}{2\pi}\mathscr{F}_x^{-1}\mathscr{F}_\theta(\nabla_\theta\widetilde{E_m^0}\cdot\nabla_x\varphi)(t,\xi,\mu) &= \frac{1}{2\pi}\sum_{j=1}^d \mathscr{F}_x^{-1}\mathscr{F}_\theta(\partial_{\theta_j}\widetilde{E_m^0}\partial_{x_j}\varphi)(t,\xi,\mu)\\
&= \frac{1}{2\pi}\sum_{j=1}^d (\mathscr{F}_\theta(\partial_{\theta_j}\widetilde{E_m^0}) * \mathscr{F}_x^{-1}\mathscr{F}_\theta(\partial_{x_j}\varphi))(t,\xi,\mu)
\end{aligned}
$$

we have that $\mathscr{F}_\theta(\partial_{\theta_j}\widetilde{E^0_m})(\mu) = \int_B \mathrm{d}\theta e^{-2\pi i\theta\cdot\mu}\partial_{\theta_j}\widetilde{E^0_m}(\theta) = 2\pi i\mu_j\widetilde{\mathscr{E}^0_m}(\mu)$ so this is

$$= \sum_{j=1}^{d}\sum_{\mu'\in\mathbb{Z}^d} i\mu'_j\widetilde{\mathscr{E}^0_m}(\mu')\mathscr{F}_x^{-1}\mathscr{F}_\theta(\partial_{x_j}\varphi)(t,\xi,\mu-\mu')$$

Now $\mathscr{F}_x^{-1}\mathscr{F}_\theta(\partial_{x_j}\varphi)(\xi,\mu) = \int_{\mathbb{R}^d}\mathrm{d}x e^{2\pi i\xi\cdot x}\partial_{x_j}\mathscr{F}_\theta\varphi(x,\mu) = -2\pi i\xi_j\mathscr{F}_x\mathscr{F}_\theta\varphi(-\xi,\mu)$ so this is

$$= 2\pi\sum_{j=1}^{d}\sum_{\mu'\in\mathbb{Z}^d}\xi_j\mu'_j\widetilde{\mathscr{E}^0_m}(\mu')\mathscr{F}_x\mathscr{F}_\theta\varphi(t,-\xi,\mu-\mu')$$

$$= 2\pi\sum_{\mu'\in\mathbb{Z}^d}\xi\cdot\mu'\widetilde{\mathscr{E}^0_m}(\mu')\mathscr{F}_x\mathscr{F}_\theta\varphi(t,-\xi,\mu-\mu').$$

On the other hand

$$\mathscr{F}_x^{-1}\mathscr{F}_\theta\Theta^{\varepsilon,\lambda}_m(t,\xi,\mu) = \mathscr{F}_\theta\mathscr{F}_x\Theta^{\varepsilon,\lambda}_m(t,-\xi,\mu)$$

$$= \mathscr{F}_\theta\left(i\frac{\widetilde{E^\lambda_m}\left(\theta-\frac{\varepsilon\xi}{2}\right)-\left(\widetilde{E^\lambda_m}\left(\theta+\frac{\varepsilon\xi}{2}\right)\right)}{\varepsilon}\hat{\varphi}(t,-\xi,\theta)\right)$$

$$= \frac{i}{\varepsilon}\left(\mathscr{F}_\theta\left(\left(\tau_{\frac{\varepsilon\xi}{2}}\widetilde{E^\lambda_m}\right)\hat{\varphi}\right)-\mathscr{F}_\theta\left(\left(\tau_{-\frac{\varepsilon\xi}{2}}\widetilde{E^\lambda_m}\right)\hat{\varphi}\right)\right)(t,-\xi,\mu)$$

$$= i\left(\sum_{\mu'\in\mathbb{Z}^d}\frac{\left(e^{-2\pi i\frac{\varepsilon\xi}{2}\cdot\mu'}-e^{2\pi i\frac{\varepsilon\xi}{2}\cdot\mu'}\right)}{\varepsilon}\widetilde{\mathscr{E}^\lambda_m}(\mu')\mathscr{F}_\theta\hat{\varphi}(t,-\xi,\mu-\mu')\right)$$

$$= -i\sum_{\mu'\in\mathbb{Z}^d}\widetilde{\mathscr{E}^\lambda_m}(\mu')\mathscr{F}_\theta\hat{\varphi}(-\xi,\mu-\mu')\left(\frac{e^{\pi i\varepsilon\xi\cdot\mu'}-e^{-\pi i\varepsilon\xi\cdot\mu'}}{\varepsilon}\right).$$

Hence

$$\int_0^\infty \mathrm{d}t\|\Theta^{\varepsilon,\lambda}_m-\nabla_\theta\widetilde{E^0_m}\cdot\nabla_x\varphi\|_{\mathscr{B}}$$

$$\leq\int_0^\infty\mathrm{d}t\sum_{\mu\in\mathbb{Z}^d}\sum_{\mu'\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\mathrm{d}\xi|\mathscr{F}_k\hat{\varphi}(-\xi,\mu-\mu',t)|$$

$$\left|-i\widetilde{\mathscr{E}^\lambda_m}(\mu')\left(\frac{e^{\pi i\varepsilon\xi\cdot\mu'}-e^{-\pi i\varepsilon\xi\cdot\mu'}}{\varepsilon}\right)-2\pi\widetilde{\mathscr{E}^0_m}(\mu')\xi\cdot\mu'\right|.$$

We have that

$$\left|-i\widetilde{\mathscr{E}^\lambda_m}(\mu')\left(\frac{e^{\pi i\varepsilon\xi\cdot\mu'}-e^{-\pi i\varepsilon\xi\cdot\mu'}}{\varepsilon}\right)-2\pi\widetilde{\mathscr{E}^0_m}(\mu')\xi\cdot\mu'\right|$$

$$= \left|2\pi i\widetilde{\mathscr{E}^0_m}(\mu')\xi\cdot\mu'-\widetilde{\mathscr{E}^\lambda_m}(\mu')\left(\frac{e^{\pi i\varepsilon\xi\cdot\mu'}-e^{-\pi i\varepsilon\xi\cdot\mu'}}{\varepsilon}\right)\right|.$$

Adding and subtracting $2\pi i\widetilde{\mathscr{E}^\lambda_m}(\mu')\xi\cdot\mu'$ inside the absolute value, and using the triangle inequality, this is

$$\leq|\widetilde{\mathscr{E}^\lambda_m}(\mu')|\left|\left(2\pi i\xi\cdot\mu'-\left(\frac{e^{\pi i\varepsilon\xi\cdot\mu'}-e^{-\pi i\varepsilon\xi\cdot\mu'}}{\varepsilon}\right)\right)\right|+2\pi|\widetilde{\mathscr{E}^\lambda_m}(\mu')-\widetilde{\mathscr{E}^0_m}(\mu')||\xi\cdot\mu'|.$$

Since we set $\lambda = \varepsilon^{1/2}$, $|\widetilde{\mathcal{E}^\lambda_m}(\mu') - \widetilde{\mathcal{E}^0_m}(\mu')| \xrightarrow{\varepsilon \to 0} 0$ by Corollary 4.16 and

$$\left| \left(2\pi i \xi \cdot \mu' - \left(\frac{e^{\pi i \varepsilon \xi \cdot \mu'} - e^{-\pi i \varepsilon \xi \cdot \mu'}}{\varepsilon} \right) \right) \right| = o(\varepsilon),$$

we can conclude if we can find a way to justify using the dominated convergence theorem. We first consider

$$\int_0^\infty \mathrm{d}t \sum_{\mu \in \mathbb{Z}^d} \sum_{\mu' \in \mathbb{Z}^d} \int_{\mathbb{R}^3} \mathrm{d}\xi |\mathcal{F}_\theta \hat{\varphi}(-\xi, \mu - \mu', t)| |\widetilde{\mathcal{E}^\lambda_m}(\mu')|$$
$$\left| \left(2\pi i \xi \cdot \mu' - \left(\frac{e^{\pi i \varepsilon \xi \cdot \mu'} - e^{-\pi i \varepsilon \xi \cdot \mu'}}{\varepsilon} \right) \right) \right|.$$

Now, since $\left(\frac{e^{\pi i \varepsilon \xi \cdot \mu'} - e^{-\pi i \varepsilon \xi \cdot \mu'}}{\varepsilon} \right) = \frac{2i \sin(\pi \varepsilon \xi \cdot \mu')}{\varepsilon}$ and $\sin(x) \leq |x|$, this is

$$\begin{aligned}
&\leq 4\pi \sum_{\mu' \in \mathbb{Z}^d} |\widetilde{\mathcal{E}^\lambda_m}(\mu')| |\mu'| \int_0^\infty \mathrm{d}t \sum_{\mu \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \mathrm{d}\xi |\mathcal{F}_\theta \hat{\varphi}(-\xi, \mu - \mu', t)| |\xi| \\
&= 4\pi \sum_{\mu' \in \mathbb{Z}^d} |\widetilde{\mathcal{E}^\lambda_m}(\mu')| |\mu'| \left(\int_0^\infty \mathrm{d}t \sum_{\mu \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \mathrm{d}\xi |\mathcal{F}_\theta \hat{\varphi}(-\xi, \mu, t)| |\xi| \right) \\
&\lesssim \sum_{\mu' \in \mathbb{Z}^d} |\widetilde{\mathcal{E}^0_m}(\mu')| |\mu'| \left(\int_0^\infty \mathrm{d}t \sum_{\mu \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \mathrm{d}\xi |\mathcal{F}_\theta \hat{\varphi}(-\xi, \mu, t)| |\xi| \right) < \infty,
\end{aligned}$$

for $\lambda = \varepsilon^{1/2}$ small enough, due to the smoothness of the modified energy bands and the test function. The final bound is uniform in ε. Hence dominated convergence can be used. Similarly, for the second term, one has for λ small enough that by Corollary 4.16

$$\begin{aligned}
&\int_0^\infty \mathrm{d}t \sum_{\mu \in \mathbb{Z}^d} \sum_{\mu' \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \mathrm{d}\xi |\mathcal{F}_\theta \hat{\varphi}(-\xi, \mu - \mu', t)| 2\pi |\widetilde{\mathcal{E}^\lambda_m}(\mu') - \widetilde{\mathcal{E}^0_m}(\mu')| |\xi \cdot \mu'| \\
&\lesssim \sum_{\mu' \in \mathbb{Z}^d} |\widetilde{\mathcal{E}^0_m}(\mu')| |\mu'| \left(\int_0^\infty \mathrm{d}t \sum_{\mu \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \mathrm{d}\xi |\mathcal{F}_k \hat{\varphi}(-\xi, \mu, t)| |\xi| \right) < \infty,
\end{aligned}$$

and also for this term dominated convergence can be used. This concludes the proof. □

Remark 4.17. If one had worked with $w^{\varepsilon,\lambda}_{s,ml}(t,x,\theta) = w^\varepsilon_s(\varphi^{\varepsilon,\lambda}_m, \varphi^{\varepsilon,\lambda}_l)(t,x,\theta)$ as introduced in Remark 4.7, the evolution equation one would need to use would be different, and one would have

$$\begin{aligned}
\partial_t w^{\varepsilon,\lambda}_{s,ml}(t,x,\theta) &= i \sum_{n \in \mathbb{Z}^d} \frac{[\mathcal{E}^\gamma_m(n) w^{\varepsilon,\lambda}_{s,ml}(t, x - \frac{\varepsilon}{2}n, \theta) - \mathcal{E}^\gamma_l(n) w^{\varepsilon,\lambda}_{s,ml}(t, x + \frac{\varepsilon}{2}n, \theta)]}{\varepsilon} e^{2\pi i \theta \cdot n} \\
&= i \sum_{n \in \mathbb{Z}^d} \mathcal{E}^\gamma_m(n) \frac{[w^{\varepsilon,\lambda}_{s,ml}(t, x - \frac{\varepsilon}{2}n, \theta) - w^{\varepsilon,\lambda}_{s,ml}(t, x + \frac{\varepsilon}{2}n, \theta)]}{\varepsilon} e^{2\pi i \theta \cdot n} \\
&\quad + i \sum_{n \in \mathbb{Z}^d} w^{\varepsilon,\lambda}_{s,ml}\left(t, x + \frac{\varepsilon}{2}n, \theta\right) \frac{[\mathcal{E}^\gamma_m(n) - \mathcal{E}^\gamma_l(n)]}{\varepsilon} e^{2\pi i \theta \cdot n}.
\end{aligned}$$

The second term can be approximated as

$$\approx i w^{\varepsilon,\lambda}_{s,ml}(t,x,\theta) \sum_{n\in\mathbb{Z}^d} \frac{[\mathscr{E}^{\gamma}_m(n) - \mathscr{E}^{\gamma}_l(n)]}{\varepsilon} e^{2\pi i\theta\cdot n} = i w^{\varepsilon,\lambda}_{s,ml}(t,x,\theta) \frac{[E_m(\theta) - E_l(\theta)]}{\varepsilon}.$$

Hence, overall one has the evolution

$$\partial_t w^{\varepsilon,\lambda}_{s,ml}(t,x,\theta) \approx i w^{\varepsilon,\lambda}_{s,ml}(t,x,\theta) \frac{[E_m(\theta) - E_l(\theta)]}{\varepsilon} + Q^{\varepsilon}_{s,ml} w^{\varepsilon,\lambda}_{s,ml}(t,x,\theta),$$

where

$$Q^{\varepsilon}_{s,ml} w^{\varepsilon,\lambda}_{s,ml}(t,x,\theta) = i \sum_{n\in\mathbb{Z}^d} \mathscr{E}^{\gamma}_m(n) \frac{[w^{\varepsilon,\lambda}_{s,ml}(t, x - \frac{\varepsilon}{2}n, \theta) - w^{\varepsilon,\lambda}_{s,ml}(t, x + \frac{\varepsilon}{2}n, \theta)]}{\varepsilon} e^{2\pi i\theta\cdot n}.$$

Treating the $Q^{\varepsilon}_{s,ml} w^{\varepsilon,\lambda}_{s,ml}(t,x,\theta)$, one can use the Duhamel principle to write

$$w^{\varepsilon,\lambda}_{s,ml}(t,x,\theta) = e^{i\frac{[E_m(\theta)-E_l(\theta)]}{\varepsilon}t} w^{\varepsilon,\lambda}_{s,ml}(0,x,\theta) + \int_0^t e^{i\frac{[E_m(\theta)-E_l(\theta)]}{\varepsilon}(t-s)} Q^{\varepsilon}_{s,ml} w^{\varepsilon,\lambda}_{s,ml}(s,x,\theta).$$

Away from band crossings, the term $e^{i\frac{[E_m(\theta)-E_l(\theta)]}{\varepsilon}t}$ is like $e^{i\frac{c}{\varepsilon}t}$ for some c, and produces oscillations that average to 0, yielding trivial limits for $w^{\varepsilon,\lambda}_{s,ml}$ tested against some nice test function φ. This can be shown for the first term for instance using the Riemann–Lebesgue lemma. We omit the details here, and refer the reader to [GMMP98], in particular to Lemma 4.13 for the details of an equivalent approach.

4.3 Proof of minor lemmas

Proof. (of Lemma 4.2) We compute

$$\begin{aligned} H^{\varepsilon,\lambda}\Psi^{\varepsilon,\lambda}_m(x,\theta) &= \varepsilon^{-d/2} H^{\varepsilon,\lambda}\Psi^{\lambda}_m\Big(\frac{x}{\varepsilon},\theta\Big) = \varepsilon^{-d/2}\Big(-\varepsilon^2\Delta_x + \lambda V\Big(\frac{x}{\varepsilon}\Big)\Big)\Psi^{\lambda}_m\Big(\frac{x}{\varepsilon},\theta\Big) \\ &= \varepsilon^{-d/2}\Big(-\Delta_x\Psi^{\lambda}_m\Big(\frac{x}{\varepsilon},\theta\Big) + \lambda V\Big(\frac{x}{\varepsilon}\Big)\Psi^{\lambda}_m\Big(\frac{x}{\varepsilon},\theta\Big)\Big) \\ &= \varepsilon^{-d/2} E^{\lambda}_m(\theta)\Psi^{\lambda}_m\Big(\frac{x}{\varepsilon},\theta\Big) = E^{\lambda}_m(\theta)\Psi^{\varepsilon,\lambda}_m(x,\theta). \end{aligned}$$

Next, we check that $\Psi^{\varepsilon,\lambda}_m$ satisfy the periodicity conditions:

$$\Psi^{\varepsilon,\lambda}_m(x+\varepsilon n,\theta) = \varepsilon^{-d/2}\Psi^{\lambda}_m\Big(\frac{x}{\varepsilon}+n,\theta\Big) = e^{2\pi i n\cdot\theta}\varepsilon^{-d/2}\Psi^{\lambda}_m\Big(\frac{x}{\varepsilon},\theta\Big) = e^{2\pi i n\cdot\theta}\Psi^{\varepsilon,\lambda}_m(x,\theta),$$

and

$$\frac{\partial\Psi^{\varepsilon,\lambda}_m}{\partial x_l}(x+\varepsilon n,\theta) = \frac{\partial}{\partial x_l}(\Psi^{\varepsilon,\lambda}_m(x+\varepsilon n,\theta)) = \varepsilon^{-d/2}\frac{\partial}{\partial x_l}(\Psi^{\lambda}_m(y+n,\theta)),$$

with $y = \frac{x}{\varepsilon}$. The chain rule says this is

$$= \varepsilon^{-(d+2)/2}\frac{\partial\Psi^{\lambda}_m}{\partial y_l}(y+n,\theta) = \varepsilon^{-(d+2)/2} e^{2\pi i n\cdot\theta}\frac{\partial\Psi^{\lambda}_m}{\partial y_l}(y,\theta) = e^{2\pi i n\cdot\theta}\frac{\partial\Psi^{\varepsilon,\lambda}_m}{\partial x_l}(x,\theta).$$ □

To prove Lemma 4.4, it will be useful to have some machinery.

Definition 4.18. *For $\psi\in\mathcal{S}(\mathbb{R}^d;\mathbb{C})$, $\varepsilon\in(0,1]$, $\lambda\in[0,1]$, $\theta\in B$, $m\in\mathbb{N}$ define the Bloch coefficients, for*

$$\widetilde{\psi^{\varepsilon,\lambda}}(\theta,m) := \int_{\mathbb{R}^d}\psi(x)\Psi_m^{\varepsilon,\lambda,*}(x,\theta)\mathrm{d}x$$

Remark 4.19. The above definition extends unitarily to $L^2(\mathbb{R}^d;\mathbb{C})$.

Proposition 4.20. *One has that for $\varepsilon\in(0,1]$, $\lambda\in[0,1]$,*

1. *(Reconstruction)* $\psi(x)=\sum_{m=1}^{\infty}\int_B \mathrm{d}\theta\,\widetilde{\psi^{\varepsilon,\lambda}}(\theta,m)\Psi_m^{\varepsilon,\lambda}(x,\theta)$
2. *(norm preservation)* $\|\psi\|_{L^2(\mathbb{R}^d)}^2=\sum_{m=1}^{\infty}\int_B|\widetilde{\psi^{\varepsilon,\lambda}}(\theta,m)|^2\mathrm{d}\theta$
3. *(eigenvalue property) For $\theta\in B, m\in\mathbb{N}$,* $\widetilde{H^{\varepsilon,\lambda}\psi}(\theta,m)=E_m^{\lambda}(\theta)\,\widetilde{\psi^{\varepsilon,\lambda}}(\theta,m)$
4. *(Plancherel) For $\psi_1,\psi_2\in L^2(\mathbb{R}^d)$ one has that*

$$\int_{\mathbb{R}^d}\mathrm{d}x\,\psi_1(x)\,\psi_2^*(x)=\sum_{m=1}^{\infty}\int_B \mathrm{d}\theta\,\widetilde{\psi_1^{\varepsilon,\lambda}}(\theta,m)\,\widetilde{\psi_2^{\varepsilon,\lambda}}^*(\theta,m)$$

Proof. The proof reduces to using properties of the classical Bloch–Floquet Zak transform that were proved in Section 2.1.

1. We begin by considering $\mathcal{U}_{cl}\psi(\theta,x)$ introduced in Definition 2.9. Since $\Psi_m^{\varepsilon,\lambda}(x,\theta)$ is a basis of $L^2(C;\mathbb{C})$, we can write

$$\mathcal{U}_{cl}\psi(\theta,x)=\sum_{m\in\mathbb{N}}c_m(\theta)\Psi_m^{\varepsilon,\lambda}(x,\theta)$$

where

$$c_m(\theta)=\int_C\mathcal{U}_{cl}\psi(\theta,x)\Psi_m^{\varepsilon,\lambda,*}(x,\theta)\mathrm{d}x$$
$$=\int_C\sum_{n\in\mathbb{Z}^d}e^{2\pi i\theta\cdot n}\psi(x-n)\Psi_m^{\varepsilon,\lambda,*}(x,\theta)\mathrm{d}x.$$

Changing variables, this is

$$=\sum_{n\in\mathbb{Z}^d}\int_{C-n}\mathrm{d}y e^{2\pi i\theta\cdot n}\psi(y)\Psi_m^{\varepsilon,\lambda,*}(y+n,\theta)\mathrm{d}y$$
$$=\sum_{n\in\mathbb{Z}^d}\int_{C-n}\mathrm{d}y\psi(y)\Psi_m^{\varepsilon,\lambda,*}(y,\theta)\mathrm{d}y$$
$$=\int_{\mathbb{R}^d}\mathrm{d}y\psi(y)\Psi_m^{\varepsilon,\lambda,*}(y,\theta)=\widetilde{\psi^{\varepsilon,\lambda}}(\theta,m).$$

The claim follows from the inversion formula for $\mathcal{U}_{cl}\psi$:

$$\psi(x)=\int_B\mathrm{d}\theta\mathcal{U}_{cl}\psi(\theta,x)=\int_B\mathrm{d}\theta\sum_{m\in\mathbb{N}}c_m(\theta)\Psi_m^{\varepsilon,\lambda}(x,\theta)$$
$$=\int_B\mathrm{d}\theta\sum_{m\in\mathbb{N}}\widetilde{\psi^{\varepsilon,\lambda}}(\theta,m)\Psi_m^{\varepsilon,\lambda}(x,\theta)=\sum_{m\in\mathbb{N}}\int_B\mathrm{d}\theta\widetilde{\psi^{\varepsilon,\lambda}}(\theta,m)\Psi_m^{\varepsilon,\lambda}(x,\theta).$$

2. One has that
$$\sum_{m=1}^{\infty}\int_B |\widetilde{\psi^{\varepsilon,\lambda}}(\theta,m)|^2 \mathrm{d}k=\sum_{m=1}^{\infty}\int_B |c_m(\theta)|^2 \mathrm{d}k.$$
On the other hand by the unitarity of $\mathscr{U}_{cl}$,
$$\|\psi\|^2_{L^2(\mathbb{R}^d)}=\int_B \mathrm{d}\theta\int_C \mathrm{d}x\mathscr{U}_{cl}\psi^*(\theta,x)\mathscr{U}_{cl}\psi(\theta,x)$$
$$=\int_B \mathrm{d}\theta\int_C \mathrm{d}x\sum_{m,m'\in\mathbb{N}} c_m(\theta)c_{m'}(\theta)\Psi_m^{\varepsilon,\lambda,*}(x,\theta)\Psi_{m'}^{\varepsilon,\lambda}(x,\theta).$$
We conclude using Fubini's theorem and the orthonormality of the family $\{\Psi_m^{\varepsilon,\lambda}\}_{m\in\mathbb{N}}$, since this is
$$=\int_B \mathrm{d}\theta\sum_{m\in\mathbb{N}} |c_m(\theta)|^2.$$

3. This follows from the self-adjointness of $H^{\varepsilon,\lambda}$ and Lemma 4.2:
$$\widetilde{H^{\varepsilon,\lambda}\psi}(\theta,m)=\langle\Psi_m^{\varepsilon,\lambda}(\theta),H^{\varepsilon,\lambda}\psi\rangle=\langle H^{\varepsilon,\lambda}\Psi_m^{\varepsilon,\lambda}(\theta),\psi\rangle=E_m^{\lambda}(\theta)\langle\Psi_m^{\varepsilon,\lambda}(\theta),\psi\rangle$$
$$=E_m^{\lambda}(\theta)\widetilde{\psi^{\varepsilon,\lambda}}(\theta,m).$$

4. Let $\mathscr{U}_{cl}\psi_i(\theta,x)=\sum_{m\in\mathbb{N}} c_{i,m}(\theta)\Psi_m^{\varepsilon,\lambda}(x,\theta)$ for $i\in\{1,2\}$. One has by the unitarity of the classical Bloch–Floquet transform that
$$\langle\psi_1,\psi_2\rangle_{L^2(\mathbb{R}^d)}=\int_B \mathrm{d}\theta\langle\mathscr{U}_{cl}\psi_1(\theta,\cdot),\mathscr{U}_{cl}\psi_2(\theta,\cdot)\rangle_{L^2(C;\mathbb{C})}.$$
We can conclude since
$$\langle\mathscr{U}_{cl}\psi_1(\theta,\cdot),\mathscr{U}_{cl}\psi_2(\theta,\cdot)\rangle_{L^2(C;\mathbb{C})}=\sum_{m,n\in\mathbb{N}} c_{1,m}(\theta)c^*_{2,n}(\theta)\int_C\Psi_m^{\varepsilon,\lambda}(x,\theta)\Psi_n^{\varepsilon,\lambda}(x,\theta)\mathrm{d}x$$
$$=\sum_{m\in\mathbb{N}} c_{1,m}(\theta)c^*_{2,m}(\theta)=\sum_{m=1}^{\infty}\widetilde{\psi_1^{\varepsilon,\lambda}}(\theta,m)\widetilde{\psi_2^{\varepsilon,\lambda}}^*(\theta,m)$$
□

Lemma 4.21. *Consider $S_m^{\varepsilon,\lambda}$ defined in equation (4.11). For $m,l\in\mathbb{N}$, $f_m\in S_m^{\varepsilon,\lambda}$, $f_l\in S_l^{\varepsilon,\lambda}$ one has that*
$$\int_{\mathbb{R}^d}\mathrm{d}xf_m(x)f_l^*(x)=\delta_{ml}.$$

Proof. One has that $f_m(x)=\int_B \mathrm{d}\theta\sigma_m(\theta)\Psi_m^{\varepsilon,\lambda}(x,\theta)$. As a consequence of the first part of the previous Proposition $f_m(x)=\sum_{n=1}^{\infty}\int_B \mathrm{d}\theta\tilde{f}_m(\theta,n)\Psi_n^{\varepsilon,\lambda}(x,\theta)$. Hence $\tilde{f}_m(\theta,n)=\sigma_n(\theta)\delta_{nk}$. Similarly, $f_l(x)=\sum_{n=1}^{\infty}\int_B \mathrm{d}\theta\tilde{f}_l(\theta,n)\Psi_n^{\varepsilon,\lambda}(x,\theta)=\int_B \mathrm{d}\theta\sigma_l(\theta)\Psi_l^{\varepsilon,\lambda}(x,\theta)$. Hence $\tilde{f}_l(\theta,n)=\sigma_n(\theta)\delta_{ln}$. One can conclude using part 4. of Proposition 4.20. □

Proof. (of Lemma 4.4) Since $\psi\in S_m^{\varepsilon,\lambda}$ one has that
$$\psi(x)=\frac{1}{|B|}\int_B \mathrm{d}\theta\widetilde{\psi^{\varepsilon,\lambda}}(\theta,m)\Psi_m^{\varepsilon,\lambda}(x,\theta)$$

Then by the eigenvalue property 3 of Proposition 4.20, one has that

$$\begin{aligned} H^{\varepsilon,\lambda}\psi(x) &= \int_B \mathrm{d}\theta\, \widetilde{\psi^{\varepsilon,\lambda}}(\theta,m) H^{\varepsilon,\lambda}\Psi_m^{\varepsilon,\lambda}(x,\theta) \\ &= \int_B \mathrm{d}k\, \widetilde{\psi^{\varepsilon,\lambda}}(\theta,m) E_m^{\lambda}(\theta)\Psi_m^{\varepsilon,\lambda}(x,\theta). \end{aligned}$$

Using the periodicity of the Bloch bands, one uses a Fourier series representation, to rewrite this as

$$= \int_B \mathrm{d}\theta \sum_{n\in\mathbb{Z}^d} \mathscr{E}_m^{\lambda}(n)\, e^{2\pi i n\cdot\theta}\widetilde{\psi^{\varepsilon,\lambda}}(\theta,m)\Psi_m^{\varepsilon,\lambda}(x,\theta).$$

By using Fubini's theorem and the quasiperiodicity of the eigenfunctions, this is

$$\begin{aligned} &= \sum_{n\in\mathbb{Z}^d} \mathscr{E}_m^{\lambda}(n) \int_B \mathrm{d}\theta\, \widetilde{\psi^{\varepsilon,\lambda}}(\theta,m)\Psi_m^{\varepsilon,\lambda}(x+\varepsilon n,\theta) \\ &= \sum_{n\in\mathbb{Z}^d} \mathscr{E}_m^{\lambda}(n)\, \psi(x+\varepsilon n). \end{aligned}$$

□

Lemma 4.22. *For* $f,g\in\mathcal{S}(\mathbb{R}^d), x\in\mathbb{R}^d, \theta\in B$

$$w_s^{\varepsilon}(f,g)(x,\theta) = \sum_{\gamma\in\mathbb{Z}^d} W^{\varepsilon}(f,g)(x,\theta+\gamma)$$

Proof. Defining $\Phi_x^{\varepsilon}(\mu) := f(x+\frac{\varepsilon}{2}\mu)g(x-\frac{\varepsilon}{2}\mu)$, one has that for any $x\in\mathbb{R}^d, \Phi_x^{\varepsilon}\in\mathcal{S}(\mathbb{R}_{\mu}^d)$. Hence, since $W^{\varepsilon}(f,g)(x,k) = \widehat{\Phi_x^{\varepsilon}}(-k)$, one has that $W^{\varepsilon}(f,g)(x,\cdot)\in\mathcal{S}(\mathbb{R}_k^d)$ for any $x\in\mathbb{R}^d$. Hence for any $\theta\in\left[-\frac{1}{2},\frac{1}{2}\right]^d, \phi(x,\theta) := \sum_{\mu'\in\mathbb{Z}^d} W^{\varepsilon}(f,g)(x,\theta+\mu')$ is well-defined, and is periodic in the k-variable. Taking the Fourier series, one has for $\xi\in\mathbb{Z}^d$

$$\begin{aligned} \int_B \phi(x,\theta)e^{-2\pi i\xi\cdot\theta}\mathrm{d}\theta &= \int_B \mathrm{d}\theta \sum_{\mu'\in\mathbb{Z}^d} W^{\varepsilon}(f,g)(x,\theta+\mu')e^{-2\pi i\xi\cdot\theta} \\ &= \int_B \mathrm{d}\theta \sum_{\mu'\in\mathbb{Z}^d} W^{\varepsilon}(f,g)(x,\theta+\mu')e^{-2\pi i\xi\cdot(\theta+\mu')} \\ &= \int_{\mathbb{R}^d} W^{\varepsilon}(f,g)(x,k)e^{-2\pi i\xi\cdot k}\mathrm{d}k \\ &= \int_{\mathbb{R}^d} \mathrm{d}k\widehat{\Phi_x^{\varepsilon}}(-k)e^{-2\pi i k\cdot\xi} = \int_{\mathbb{R}^d} \mathrm{d}k\widehat{\Phi_x^{\varepsilon}}(k)e^{2\pi i k\cdot\xi} = \Phi_x^{\varepsilon}(\xi). \end{aligned}$$

On the other hand, taking the Fourier series of $w_m^{\varepsilon}(f,g)(x,\theta)$ gives

$$\begin{aligned} \int_B w_m^{\varepsilon}(f,g)(x,\theta)e^{-2\pi i\xi\cdot\theta}\mathrm{d}\theta &= \int_B \mathrm{d}\theta \sum_{\mu\in\mathbb{Z}^d} \Phi_x^{\varepsilon}(\mu)e^{2\pi i\theta\cdot(\mu-\xi)} \\ &= \sum_{\mu\in\mathbb{Z}^d} \Phi_x^{\varepsilon}(\mu)\int_B \mathrm{d}\theta e^{2\pi i\theta\cdot(\mu-\xi)} = \Phi_x^{\varepsilon}(\xi). \end{aligned}$$

This identity can be extended to less regular f,g, see Remark 4.4 in [GMMP98]. □

Proof. (of Lemma 4.8) One has that for $\varphi \in \mathscr{B}$ that

$$\int_B \mathrm{d}\theta \int_{\mathbb{R}^d} \mathrm{d}x w_m^{\varepsilon,\lambda}(t,x,\theta)\varphi(x,\theta) = \int_B \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}x \sum_{\mu\in\mathbb{Z}^d} z_m^{\varepsilon,\lambda}\Big(t, x+\frac{\varepsilon}{2}\mu, x-\frac{\varepsilon}{2}\mu\Big)\varphi(x,k)e^{2\pi i\theta\cdot\mu},$$

where

$$z_m^{\varepsilon,\lambda}\Big(t, x+\frac{\varepsilon}{2}\mu, x-\frac{\varepsilon}{2}\mu\Big) = \psi_m^{\varepsilon,\lambda,*}\Big(t, x+\frac{\varepsilon}{2}\mu\Big)\psi_m^{\varepsilon,\lambda}\Big(t, x-\frac{\varepsilon}{2}\mu\Big).$$

Hence

$$\int_B \mathrm{d}\theta \int_{\mathbb{R}^d} \mathrm{d}x w_m^{\varepsilon,\lambda}(t,x,\theta)\varphi(x,\theta) = \int_{\mathbb{R}^d} \mathrm{d}x \sum_{\mu\in\mathbb{Z}^d} z_m^{\varepsilon,\lambda}\Big(t, x+\frac{\varepsilon}{2}\mu, x-\frac{\varepsilon}{2}\mu\Big)\mathscr{F}_\theta\varphi(x,-\mu).$$

Taking absolute values

$$\Big|\int_B \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}x w_m^{\varepsilon,\lambda}(t,x,\theta)\varphi(x,\theta)\Big| \leq \|\varphi\|_{\mathscr{B}} \sup_{\mu\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathrm{d}x \Big|z_m^{\varepsilon,\lambda}\Big(t, x+\frac{\varepsilon}{2}\mu, x-\frac{\varepsilon}{2}\mu\Big)\Big|.$$

Now

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{d}x \Big|z_m^{\varepsilon,\lambda}\Big(t, x+\frac{\varepsilon}{2}\mu, x-\frac{\varepsilon}{2}\mu\Big)\Big| &= \Big|\int_{\mathbb{R}^d} \mathrm{d}x \psi_m^{\varepsilon,\lambda,*}\Big(t, x+\frac{\varepsilon}{2}\mu\Big)\psi_m^{\varepsilon,\lambda}\Big(t, x-\frac{\varepsilon}{2}\mu\Big)\Big| \\ &\leq \int_{\mathbb{R}^3} \mathrm{d}x \Big|\psi_m^{\varepsilon,\lambda,*}\Big(t, x+\frac{\varepsilon}{2}\mu\Big)\psi_m^{\varepsilon,\lambda}\Big(t, x-\frac{\varepsilon}{2}\mu\Big)\Big| \\ &\leq \|\tau_{-\frac{\varepsilon\mu}{2}}\psi_m^{\varepsilon,\lambda,*}(t)\|_{L^2(\mathbb{R}^d)} \|\tau_{\frac{\varepsilon\mu}{2}}\psi_m^{\varepsilon,\lambda}(t)\|_{L^2(\mathbb{R}^d)} \\ &= \|\psi_m^{\varepsilon,\lambda,*}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} = \|\psi_m^{\varepsilon,\lambda,*}(0,\cdot)\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

and this concludes the proof. □

4.4 A remark on long time behavior

We end this chapter with a brief remark on the long time behavior of Schrödinger equations with periodic potentials. We begin with equation (4.2). Similar to expression (4.6), one can obtain a direct integral decomposition of $H^{1,\lambda}$ using $\mathscr{U}_{\mathrm{BFZ}}$ instead of $\mathscr{U}_{cl}$:

$$\mathscr{U}_{\mathrm{BFZ}} H^{1,\lambda} \mathscr{U}_{\mathrm{BFZ}}^{-1} = \int_B \mathrm{d}\theta H^{1,\lambda}_{\mathrm{BFZ}}(\theta)$$

One has that $H^{1,\lambda}_{\mathrm{BFZ}}(\theta) = (-i\nabla_x + 2\pi\theta)\cdot(-i\nabla_x + 2\pi\theta) + V$ on $L^2(\mathbb{T}^d)$. $H^{1,\lambda}_{\mathrm{BFZ}}(\theta)$ is unitarily related to $H^{1,\lambda}_{cl}(\theta)$ from (4.6) (the details can be found in Remark 1 of [MP14]). Hence $H^{1,\lambda}_{\mathrm{BFZ}}(\theta)$ has a discrete spectrum, and we can write

$$H^{1,\lambda}_{\mathrm{BFZ}}(\theta) = \sum_{n=1}^{\infty} E_n^\lambda(\theta) P_n^\lambda(\theta)$$

where $E_n^\lambda(\theta)$ are the eigenvalues in non-decreasing order, as introduced at the start of this section and $P_n^\lambda(\theta)$ are the corresponding eigenprojections. Using the notation

$$x^m\psi(x) = (x_1^m\psi(x), \ldots, x_d^m\psi(x)), \qquad (\nabla_\theta f)^m := ((\partial_{\theta_1} f)^m, \ldots, (\partial_{\theta_d} f)^m)$$

for $m \in \mathbb{N}$, we state the following

Theorem 4.23. *(Theorem 3.1 of [dMS23]) Assume we have an initial condition* $\psi \in H^{2m}(\mathbb{R}^d)$, $\psi \neq 0$ *and such that*

$$\|x^m \psi\|_{L^2(\mathbb{R}^d)} < \infty$$

Then

$$\lim_{t \longrightarrow +\infty} \frac{\|x^m e^{itH^{1,\lambda}} \psi\|^2_{L^2(\mathbb{R}^d)}}{t^{2m}} = \int_{\mathbb{T}^d} \mathrm{d}\theta \sum_{n=1}^{\infty} |(\nabla_\theta E_n^\lambda(\theta))^m|^2 \|P_n^\lambda(\theta) U_{\mathrm{BFZ}} \psi(\theta, \cdot)\|^2_{L^2(\mathbb{T}^d)} > 0$$

This behavior is called ballistic transport, which is also exhibited by the free Schrödinger equation. The case $m = 1$ was proven earlier, see Theorem 2.3 in [AK98].

The proof reduces to showing that

$$\lim_{t \longrightarrow \infty} \int_B \mathrm{d}\theta \int_C \mathrm{d}x \left| \frac{(\mathscr{U}_{\mathrm{BFZ}} x^m(t) \psi)(\theta, x)}{t^m} - \left(\int_B \mathrm{d}k \sum_{n=1}^{\infty} (\nabla_k E_n^\lambda(k))^m P_n^\lambda(k) \right) \mathscr{U}_{\mathrm{BFZ}} \psi(\theta, x) \right|^2 = 0$$

where $x^m(t)$ is the operator $e^{itH} x^m e^{-itH}$, as is common within the Heisenberg picture of quantum mechanics. One can compute for ψ satisfying $\mathscr{U}_{\mathrm{BFZ}} \psi(\theta, x) = \sum_{n=1}^{N} P_n(\theta) \mathscr{U}_{\mathrm{BFZ}} \psi(\theta, x)$ that

$$\frac{(\mathscr{U}_{\mathrm{BFZ}} x^m(t) \psi)(\theta, x)}{t^m} = \int_B \mathrm{d}\theta \sum_{n=1}^{\infty} (\nabla_\theta E_n^\lambda(\theta))^m P_n^\lambda(\theta) \mathscr{U}_{\mathrm{BFZ}} \psi(\theta, x) + O_{n,\theta}\left(\frac{1}{t}\right).$$

It remains unclear what implications the above information on the moments of the wavefunction has for the weak-coupling scaling limit of the observables.

In the case of a Gaussian random potential V on the other hand, the limit

$$\lim_{\varepsilon \longrightarrow 0} e^{i\varepsilon^{-1}\tau H^{1,\varepsilon^{1/2}}} A$$

is shown to exist, for A dependent only on momentum, in [Spo77]. Furthermore, the limit operator is related to the semigroup of a linear Boltzmann equation. However, understanding the long time behavior remains a major open goal. For instance, for $d = 3$, if one shows that for all $\psi \in L^2(\mathbb{R}^3; \mathbb{C})$ one has that

$$\lim_{t \longrightarrow \infty} \frac{\|x(t) \psi\|_{L^2}}{t^\alpha} > 0$$

for some real $\alpha > 0$ and for a Gaussian random field with fixed coupling constant, this would settle the extended states conjecture for the Anderson model. Roughly speaking, the extended states conjecture says that there are no localized states for the Anderson operator in $d = 3$. The lecture notes [Erd10] explain one possible approach towards the resolution of this conjecture, where the rigorous derivation of the weak coupling limit is an important first step. One can currently go beyond kinetic time scales and get diffusive behavior, but this is currently possible only for weakly coupled potentials. We refer the reader to [ESY08] and [ESY07] for the original proof. More recent advances can be found in [Her24], [BDH25] and [BDH].

Chapter 5

Kinetic limit via the sewing lemma

We quickly recall the setting introduced in Chapter 1. Consider the linear Schrödinger equation with periodic potential: for $x \in \mathbb{R}^d, t \in \mathbb{R}$,

$$i\partial_t \varphi(t,x) = -\Delta_x \varphi(t,x) + \varepsilon^{1/2} V(x) \varphi(t,x), \qquad \varphi(0,x) = \varphi_0(x). \tag{5.1}$$

Here $0 < \varepsilon^{1/2} \ll 1$ is a small coupling constant. The time-independent potential V is assumed to be $\mathbb{Z}^d$-periodic, i.e., $V(x+n) = V(x), \forall x \in \mathbb{R}^d, \forall n \in \mathbb{Z}^d$. We also assume throughout this chapter that

$$V \in \cap_{m \in \mathbb{N}_0} H^m(\mathbb{T}^d).$$

We will not actually need so much regularity. In practice, we need only that $V \in \cap_{m \leq M} H^m(\mathbb{T}^d)$ for some sufficiently large $M \in \mathbb{N}_0$. Details of the initial data φ_0 will be mentioned in the next section, once we have introduced some more terminology.

In Section 2.2, we introduced the Wigner function as a tool to study the problem in phase space. We recall its definition here: for $\varphi \in L^2(\mathbb{R}^d; \mathbb{C}), x, k \in \mathbb{R}^d$:

$$W_\varphi(x,k) := \int_{\mathbb{R}^d} \mathrm{d}y \, e^{2\pi i k \cdot y} \varphi\left(x - \frac{y}{2}\right) \varphi^*\left(x + \frac{y}{2}\right). \tag{5.2}$$

To study the weak coupling regime, it is common to work with the rescaled Wigner function:

$$W^\varepsilon(t,x,k) := W_{\varphi(\frac{t}{\varepsilon})}\left(\frac{x}{\varepsilon}, k\right) = W_\varphi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, k\right), \tag{5.3}$$

with initial conditions $W_0^\varepsilon(x,k) = W_{\varphi_{0,\varepsilon}}(x,k)$, where φ is the solution to (5.1) with initial data $\varphi_{0,\varepsilon}$.

We are interested in the limit of time evolution of rescaled observables, which correspond to quantities of the form

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}x \, W^\varepsilon(t,x,k) F(x,k) \tag{5.4}$$

for some $F \in \mathcal{S}(\mathbb{R}^{2d})$. We will not answer this question completely, but present some partial progress towards this goal.

In Theorem 4.10, we showed that for certain observables, the limiting behavior is given by the transport equation. This was shown for observables supported away from a zero measure set of resonant momenta. In this chapter, we introduce a generalized phase space object via a representation formula of the Wigner transform, and use it to study observables that are also supported on the resonant momenta. We state the main result of this Chapter, Theorem 5.7 in Section 5.1. This result states once again that one can obtain free transport in the limit for non-resonant momenta. Importantly, the proof does not use the Wigner series, and provides a glimpse of the structure we plan to exploit in the study of the rescaled observables. Section 5.2 contains the proof of Theorem 5.7. Section 5.3 is devoted to the study of the rescaled observables. Finally in Section 5.4 we present the proofs of minor results used in the previous sections.
The material presented in this chapter is the content of the work [GS].

5.1 Summary of the chapter

We quickly recall some well-posedness theory for the linear Schrödinger equation (5.1).

Definition 5.1. *A mild solution to equation (5.1) is defined to be a function*

$$\varphi \in C([0,+\infty);L^2(\mathbb{R}^d;\mathbb{C})) \cap C^1((0,+\infty);L^2(\mathbb{R}^d;\mathbb{C})) \cap C((0,+\infty);H^2(\mathbb{R}^d;\mathbb{C}))$$

such that $\varphi(t) = \varphi_0$ *and* $\partial_t\varphi(t) = H\varphi(t)$ *for all* $t \in \mathbb{R}_+^*$.

By Theorem X.15 of [RS75] we know that for V satisfying the above assumptions that $H := -\Delta + \varepsilon^{1/2}V$ is self-adjoint on $D(-\Delta) = H^2(\mathbb{R}^d;\mathbb{C})$. Stone's theorem (Theorem 3.24 of [EN00]) then says that H is the infinitesimal generator of a $\mathscr{C}_0$-group of unitary operators, $S(t)$. By using standard results from the theory of semigroups, one has that for any $\varphi_0 \in H^2(\mathbb{R}^d;\mathbb{C})$, that $\varphi(t) = S(t)\varphi_0 \in C^1(\mathbb{R}_+;L^2(\mathbb{R}^d;\mathbb{C})) \cap C(\mathbb{R}_+;H^2(\mathbb{R}^d;\mathbb{C}))$ is the unique mild solution to equation (5.1).
Recall the definition of the Bloch–Floquet–Zak decomposition introduced in Section 2.1: For $\varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C}), \theta \in \mathbb{R}^d, x \in \mathbb{R}^d$,

$$(\mathscr{U}_{\mathrm{BFZ}}\varphi)_\theta(x) := \tilde{\varphi}(\theta,x) := \sum_{m \in \mathbb{Z}^d} e^{2\pi i\theta\cdot(x-m)}\varphi(x-m).$$

In Section 5.4.1 we use these properties to prove the following representation formula for the Wigner function:

Lemma 5.2. *Let* $\varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$. *Then for for* $W_\varphi(x,k)$ *defined in (5.2), decomposing the momentum* $k = \kappa - \eta$, *with* $\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d$ *and* $\eta \in \left[-\frac{1}{4},\frac{1}{4}\right)^d$, *we have the following representation:*

$$W_\varphi(x,k) = \int_{\mathbb{T}^d}\mathrm{d}y\int_{\mathbb{T}^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot x}e^{-4\pi i\kappa\cdot y}\tilde{\varphi}(\eta+\theta,x+y)\tilde{\varphi}^*(\eta-\theta,x-y). \tag{5.5}$$

This can be extended by a density argument to $L^2(\mathbb{R}^d;\mathbb{C})$. Noting that the variable x is not just periodic in the overall integrand but individually as an argument of $\tilde{\varphi}$ and $\tilde{\varphi}^*$, due to the properties of the BFZ transform, we found it useful to introduce the following generalization.

Definition 5.3. *Let* $z \in \mathbb{T}^d, p \in \mathbb{R}^d$ *play the role of position variables and let* $\eta \in \left[-\frac{1}{4}, \frac{1}{4}\right)^d$, $\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d$ *play the role of momentum variables. Then for* $\varphi \in L^2(\mathbb{R}^d;\mathbb{C})$ *with associated BFZ decomposition* $\tilde{\varphi}$ *the Bloch–Wigner function is defined as*

$$\tilde{W}_\varphi(z,p,\eta,\kappa) := \int_{\mathbb{T}^d} \mathrm{d}y \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \mathrm{d}\theta\, e^{-4\pi i\theta\cdot p} e^{-4\pi i\kappa\cdot y}$$

$$\tilde{\varphi}(\eta+\theta, z+y)\, \tilde{\varphi}^*(\eta-\theta, z-y). \tag{5.6}$$

Note now that we break the periodicity of the integrand in the θ-variable. This expression is well-defined as a consequence of the unitarity of the BFZ transform. We see that for $k = \kappa - \eta$, with $\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d, \eta \in \left[-\frac{1}{4}, \frac{1}{4}\right)^d$ that

$$W_\varphi(x,k) = \tilde{W}_\varphi(x - \lfloor x \rfloor, x, \eta, \kappa). \tag{5.7}$$

We study the time evolution of the Bloch–Wigner transform associated to a solution of the Schrödinger equation. Denote

$$L^\infty_{\eta,p,\kappa,z} := L^\infty_{\eta,p} l^\infty_\kappa L^\infty_z \left(\left[-\frac{1}{4}, \frac{1}{4}\right)^d \times \mathbb{R}^d \times \left(\frac{\mathbb{Z}}{2}\right)^d \times \mathbb{T}^d \right), \tag{5.8}$$

and

$$L^\infty_{\eta,p,\kappa} L^2_z := L^\infty_{\eta,p} l^\infty_\kappa \left(\left(\left[-\frac{1}{4}, \frac{1}{4}\right)^d \times \mathbb{R}^d \times \left(\frac{\mathbb{Z}}{2}\right)^d \right); L^2_z(\mathbb{T}^d) \right). \tag{5.9}$$

We prove in Section 5.4.2 the following

Proposition 5.4. *For* $\varphi \in C(\mathbb{R}_{\geqslant 0}; H^2(\mathbb{R}^d;\mathbb{C})) \cap C^1(\mathbb{R}^*_+; L^2(\mathbb{R}^d;\mathbb{C}))$ *a mild solution of (5.1) with initial data* $\varphi_0 \in H^2(\mathbb{R}^d;\mathbb{C})$, *one has that* $\tilde{W}_\varphi \in C(\mathbb{R}_{\geqslant 0}; L^\infty_{\eta,p,\kappa,z}) \cap C^1(\mathbb{R}_+; L^\infty_{\eta,p,\kappa,z})$. *Furthermore,* $(\kappa-\eta)\cdot(\nabla_p + \nabla_z)\,\tilde{W}_\varphi \in C(\mathbb{R}_+; L^\infty_{\eta,p,\kappa,z})$ *and* $\tilde{W}_\varphi$ *satisfies*

$$\partial_t \tilde{W}_\varphi = -4\pi(\kappa-\eta)\cdot\nabla_p \tilde{W}_\varphi - 4\pi(\kappa-\eta)\cdot\nabla_z \tilde{W}_\varphi + i\varepsilon^{1/2} Q \tilde{W}_\varphi, \tag{5.10}$$

with initial data $\tilde{W}_{\varphi_0}$, *where the collision operator* Q *is defined via*

$$Qf(z,p,\eta,\kappa) := \sum_{n \in \mathbb{Z}^d} e^{2\pi i n\cdot z} \hat{V}(n) \left[f\left(z,p,\eta,\kappa+\frac{n}{2}\right) - f\left(z,p,\eta,\kappa-\frac{n}{2}\right) \right]. \tag{5.11}$$

In order to study the scaling limit, we define the rescaled Bloch–Wigner transform to be

$$\tilde{W}^\varepsilon_\varphi(t,z,p,\eta,\kappa) := \tilde{W}_\varphi\left(\frac{t}{\varepsilon}, z, \frac{p}{\varepsilon}, \eta, \kappa\right), \tag{5.12}$$

with initial data

$$\tilde{W}^{\varepsilon}_{\varphi_{0,\varepsilon}}(z,p,\eta,\kappa) = \tilde{W}_{\varphi_{0,\varepsilon}}\Big(z,\frac{p}{\varepsilon},\eta,\kappa\Big). \tag{5.13}$$

Remark 5.5. In this chapter we will focus on initial data $\varphi_{0,\varepsilon}$ for (5.1) that are uniformly bounded in $L^2(\mathbb{R}^d;\mathbb{C})$ so that by Proposition 5.4, expression (5.13) will be uniformly bounded in $L^\infty_{\eta,p,\kappa,z}$. For instance, one can pick $\varphi_{0,\varepsilon} = \varphi_0$, in which case the limit in $L^\infty_{\eta,p,\kappa,z}$ is trivial. However, the equation for the evolution remains the same, and we will see that even in the setting of initial data converging to something trivial, there are challenges of proving convergence of the rescaled Bloch–Wigner functions, in the topology relevant to the study of the observables. We will shortly make another related remark after stating our main theorem.

One can compute the time evolution of the rescaled Bloch–Wigner transform to be

$$\partial_t \tilde{W}^{\varepsilon}_{\varphi} = -4\pi\varepsilon^{-1}(\kappa-\eta)\cdot\nabla_z \tilde{W}^{\varepsilon}_{\varphi} - 4\pi(\kappa-\eta)\cdot\nabla_p \tilde{W}^{\varepsilon}_{\varphi} + i\varepsilon^{-1/2} Q \tilde{W}^{\varepsilon}_{\varphi} \tag{5.14}$$

We work in a moving coordinate frame, and pullback the fast transport in the z-variable via

$$U^{\varepsilon}(t,z,p,\eta,\kappa) := \tilde{W}^{\varepsilon}_{\varphi}(t,z+4\pi\varepsilon^{-1}(\kappa-\eta)t,p,\eta,\kappa). \tag{5.15}$$

We study the Fourier transform of U^{ε} in the periodic variable z. Define

$$T^{\varepsilon}(t,\xi,p,\eta,\kappa) := \mathscr{F}_z U^{\varepsilon}(t,\xi,p,\eta,\kappa). \tag{5.16}$$

One therefore has that

$$T^{\varepsilon,\eta}_0(\xi,p,\kappa) = T^{\varepsilon}_0(\xi,p,\eta,\kappa) = \mathscr{F}_z \tilde{W}^{\varepsilon}_{\varphi_{0,\varepsilon}}(z,p,\eta,\kappa). \tag{5.17}$$

The time evolution of T^{ε} can be computed using equations (5.14)-(5.16). One has that

$$\partial_t T^{\varepsilon}(t,\xi,p,\eta,\kappa) = A^{\kappa-\eta} T^{\varepsilon}(t,\xi,p,\eta,\kappa) + Q^{\varepsilon}_t T^{\varepsilon}(t,\xi,p,\eta,\kappa), \tag{5.18}$$

where

$$A^{\kappa-\eta} G(\xi,p,\eta,\kappa) := -4\pi(\kappa-\eta)\cdot\nabla_p G(\xi,p,\eta,\kappa)$$

and

$$Q^{\varepsilon}_t G(\xi,p,\eta,\kappa) := i\varepsilon^{-1/2} \sum_{n\in\mathbb{Z}^d} e^{4\pi^2 i\varepsilon^{-1} n\cdot(2\kappa-2\eta+n-\xi)t} \hat{V}(n) G\Big(\xi-n,p,\eta,\kappa+\frac{n}{2}\Big)$$

$$-i\varepsilon^{-1/2} \sum_{n\in\mathbb{Z}^d} e^{4\pi^2 i\varepsilon^{-1} n\cdot(2\kappa-2\eta-n+\xi)t} \hat{V}(n) G\Big(\xi-n,p,\eta,\kappa-\frac{n}{2}\Big).$$

To make sense of the limit of equation (5.18), as $\varepsilon \longrightarrow 0$, we will use some of the machinery introduced in [BG17]. To this end, we introduce the following scale: for $m \geq 0$ define the Banach space

$$E_m := \left\{ \psi \in L^1_p l^1_\kappa l^2_\xi \colon \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \langle \kappa \rangle^m \| D^\beta_p \psi(\xi, p, \kappa) \|_{l^2_\xi} < \infty \right\}. \tag{5.19}$$

with corresponding dual spaces $E_{-m} := E^*_m$. In particular

$$E_0 = L^1_p l^1_\kappa l^2_\xi \left(\mathbb{R}^d \times \left(\frac{\mathbb{Z}}{2} \right)^d \times \mathbb{Z}^d \right), \qquad E_{-0} = L^\infty_p l^\infty_\kappa l^2_\xi \left(\mathbb{R}^d \times \left(\frac{\mathbb{Z}}{2} \right)^d \times \mathbb{Z}^d \right).$$

This is motivated by the structure of the unbounded operator $A^{\kappa-\eta}$ (which worsens regularity in p and summability in k), and by the following a-priori bound on $T^{\varepsilon,\eta}$

$$\operatorname{esssup}_{\eta \in \left[-\frac{1}{4}, \frac{1}{4}\right)^d} \| T^{\varepsilon,\eta} \|_{L^\infty([0,T];E_{-0})} \leq C \tag{5.20}$$

This follows from Proposition 5.10, and equations (5.12), (5.15) and (5.16). Next, we fix

$$\delta \in \left(0, \frac{1}{d+3} \right), \tag{5.21}$$

and define

$$\mathscr{A}_\eta := \left\{ \eta \in \left[\frac{1}{4}, \frac{1}{4} \right)^d \colon |2n \cdot \eta - a|^{-1} \leq c_\delta(\eta)^{\frac{1}{1-\delta}} \langle n \rangle^{d+2} \forall a \in \mathbb{Z}, n \in \mathbb{Z}^d \backslash \{0\} \right\}, \tag{5.22}$$

where $c_\delta \in L^1_\eta\left(\left[-\frac{1}{4}, \frac{1}{4} \right)^d \right)$ is a certain integrable function that will be constructed explicitly in Lemma 5.10. Lemma 5.10 also asserts that $\mathscr{A}_\eta$ is a full measure subset of $\left[\frac{1}{4}, \frac{1}{4} \right)^d$. Let

$$Y^{\eta,*} \psi(\xi, p, \kappa) := \frac{i}{4\pi^2} \sum_{n \in \mathbb{Z}^d \backslash \{0\}} |\hat{V}(n)|^2 \left(\frac{\psi(\xi, p, \kappa) - \psi(\xi, p, \kappa - n) \mathbb{I}_{n \perp \xi}}{n \cdot (2\kappa - 2\eta - \xi - n)} \right) \tag{5.23}$$

$$- \frac{i}{4\pi^2} \sum_{n \in \mathbb{Z}^d \backslash \{0\}} |\hat{V}(n)|^2 \left(\frac{\psi(\xi, p, \kappa + n) \mathbb{I}_{n \perp \xi} - \psi(\xi, p, \kappa)}{n \cdot (2\kappa - 2\eta + \xi + n)} \right). \tag{5.24}$$

Since the dynamics of (5.18) do not affect η, and $\mathscr{A}_\eta$ is a full-measure set, we will make a statement for almost every $\eta \in \left[-\frac{1}{4}, \frac{1}{4} \right)^d$.

Definition 5.6. *Let $\tau > 0, \eta \in \mathscr{A}_\eta$. A weak solution in E_{-0} of the linear Boltzmann equation*

$$\partial_t f(t, \xi, p, \kappa) = A^{\kappa-\eta} f(t, \xi, p, \kappa) + Y^\eta f(t, \xi, p, \kappa) \tag{5.25}$$

with initial data f_0, in the time interval $[0, \tau]$, is defined to be a function $f \in C_{w^*}([0, \tau]; E_{-0})$ such that

$$\langle f(\tau), \varphi(\tau) \rangle - \langle f_0, \varphi(0) \rangle = \int_0^\tau \mathrm{d}t \langle f(t), (\partial_t + A^{\kappa-\eta,*} + Y^{\eta,*}) \varphi(t) \rangle$$

for all $\varphi \in C^1((0,\tau);E_0) \cap C((0,\tau);E_1) \cap C([0,\tau];E_0)$.

We are now able to state our main result, which is the following

Theorem 5.7. *Assume that we have initial data* $\varphi_{0,\varepsilon}$ *for the Schrödinger equation (5.1) such that* $T_0^{\varepsilon,\eta}(\xi,p,\kappa)$ *defined in (5.17) converges weakly-*$\overset{*}{\rightharpoonup}$ *in* E_{-0} *to a limit* $T_0^{\eta}(\xi,p,\kappa)$*. Then, for any* $\tau > 0$*, a.e.* $\eta \in \left[-\frac{1}{4},\frac{1}{4}\right)^d$ *one has that* $T^{\varepsilon,\eta}(t,\xi,p,\kappa)$ *converges weakly-*$\overset{*}{\rightharpoonup}$ *in* $L^\infty([0,\tau];E_{-0})$ *to a limit* $T^{\eta}(t,\xi,p,\kappa)$ *and* T^{η} *is the unique weak solution in* E_{-0} *of equation (5.25) in* $[0,\tau]$ *with initial data* T_0^{η}*.*

The proof of this Theorem is contained in Section 5.2. This passage from rescaled Wigner type dynamics to linear Boltzmann dynamics avoids estimates and combinatorics of Duhamel iterates of equation (5.1), and uses the sewing lemma (Lemma 2.26) instead. This is one of our main contributions.

Remark 5.8. Recalling equation (5.3), we were interested in

$$W^{\varepsilon}(t,p,k) = W\left(\frac{t}{\varepsilon},\frac{p}{\varepsilon},k\right) = \tilde{W}_{\varphi}\left(\frac{t}{\varepsilon},\left[\!\left[\frac{p}{\varepsilon}\right]\!\right],\frac{p}{\varepsilon},\eta,\kappa\right) = \tilde{W}^{\varepsilon}\left(t,\left[\!\left[\frac{p}{\varepsilon}\right]\!\right],p,\eta,\kappa\right),$$

i.e., the rescaled Bloch–Wigner transform with $z = \frac{p}{\varepsilon}$. Since z is a variable in the torus, we had the intuition that setting $z = \frac{p}{\varepsilon}$ would cause a homogenization effect in this variable, leading to an average over z in the limit. We chose as a first step to not rescale z in equation (5.12), and to just examine the zero Fourier mode after taking the limit. This will be remarked on further at the beginning of Section 5.3, where we will be more precise about this intuition, when we heuristically use a stationary phase argument to rewrite the expression for the observables in terms of T^{ε}, evaluated at $\xi = 0$.
As Theorem 5.7 suggests, we work with the limit of T^{ε}, instead of $\tilde{W}^{\varepsilon}$. Furthermore at this stage, unlike in the expression for the observables, we do not integrate in η, and consider fixed $\eta \in \mathscr{A}_{\eta}$.

We now mention two important shortcomings of this result:

1. Theorem 5.7 does not say something conclusive about the evolution of observables, which involve integrating over η. It leaves open the possibility that when we integrate in η and then pass to the limit, we could have concentration at certain η's for the collision term. We report partial progress on this in Section 5.3. We will make a heuristic stationary phase argument and restrict to studying the case $\xi = 0$, and will see that the term involving second order resonant term $\mathbb{Y}_{st}^{\varepsilon,\eta,*}$ requires continuity in η of T^{ε} on the zero-measure set of momenta related to energy band crossings of the free Laplacian. Since we do not currently have such information, this remains an obstruction to taking the limit $\varepsilon \longrightarrow 0$. This set of problematic momenta is the same as the one encountered in the previous chapters.

2. Furthermore, we stress once more that we have results in E_{-0}, since we have an *a priori* bound and uniform estimates on scales E_m. However, since we do not expect to have a family of initial conditions that converge to a nontrivial limit in E_{-0}, this is a weak statement at the moment. However we believe that the method of proof we will demonstrate here can be adapted to other scales of Banach spaces in the future, once an appropriate a-priori bound has been identified. We show that even in our simpler case, there is an issue trying to prove a convergence for the observables.

5.2 Proof of Theorem 5.7

This section contains the proof of Theorem 5.7. In Subsection 5.2.1 we use the Duhamel principle to derive a difference equation (5.28) for the time increments of T^ε, and our aim will then be to show that it has the structure of a rough difference equation. Subsection 5.2.2 introduces some useful lemmas that will then be used in Subsection 5.2.3 to prove uniform in ε-estimates for the operators arising in equation (5.28). This will then imply a naive ε-dependent bound for the remainder term in the equation, using the sewing lemma from the theory of rough paths. Applying the sewing lemma in this context will require certain ideas from the theory of unbounded rough drivers introduced in [BG17]. Finally in Subsection 5.2.4, we will estimate also the remainder uniformly in ε, and this will allow us to pass to and characterize the limit equation, thus proving the theorem.

5.2.1 Deriving a rough difference equation

By using Proposition 5.4 and testing equation (5.18) against functions in E_2, we have that for every $F \in E_2$ that

$$\langle \partial_t T_t^{\varepsilon,\eta}, F\rangle = \langle A^{\kappa-\eta} T_t^{\varepsilon,\eta}, F\rangle + \langle Q_t^{\varepsilon,\eta} T_t^{\varepsilon,\eta}, F\rangle,$$

where $\langle\cdot,\cdot\rangle$ means we are integrating and summing in p, κ and ξ. We write $T^{\varepsilon,\eta}$ to indicate that we fix η, noting that it is just a parameter in the equation and does not get changed by the dynamics. In terms of the adjoint operators $A^{\kappa-\eta,*}$ and $Q_t^{\varepsilon,\eta,*}$ this is

$$\langle \partial_t T_t^{\varepsilon,\eta}, F\rangle = \langle T_t^{\varepsilon,\eta}, A^{\kappa-\eta,*} F\rangle + \langle T_t^{\varepsilon,\eta}, Q_t^{\varepsilon,\eta,*} F\rangle, \tag{5.26}$$

where $A^{\kappa-\eta,*} = -A^{\kappa-\eta}$ and

$$\begin{aligned} Q_u^{\varepsilon,\eta,*} F(\xi,p,\kappa) = i\varepsilon^{-1/2} &\sum_{n\in\mathbb{Z}^d} e^{4\pi^2 i\varepsilon^{-1} n\cdot(2\kappa-2\eta-n-\xi)u}\hat{V}(n) F\Big(\xi+n,p,\kappa-\frac{n}{2}\Big) \\ -i\varepsilon^{-1/2} &\sum_{n\in\mathbb{Z}^d} e^{4\pi^2 i\varepsilon^{-1} n\cdot(2\kappa-2\eta+n+\xi)u}\hat{V}(n) F\Big(\xi+n,p,\kappa+\frac{n}{2}\Big). \end{aligned}$$

Integrating in time, one has that

$$\int_s^t \mathrm{d}u \langle \partial_t T_u^{\varepsilon,\eta}, F\rangle = \langle \delta_{st} T^{\varepsilon,\eta}, F\rangle. \tag{5.27}$$

Using this and equation (5.26) once more, one has that the first term on the right hand side is

$$\int_s^t \mathrm{d}u \langle T_u^{\varepsilon,\eta}, A^{\kappa-\eta,*} F\rangle = \int_s^t \mathrm{d}u \langle T_s^{\varepsilon,\eta}, A^{\kappa-\eta,*} F\rangle + \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \langle T_v^{\varepsilon,\eta}, A^{\kappa-\eta,*} A^{\kappa-\eta,*} F\rangle$$

$$+ \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \langle T_v^{\varepsilon,\eta}, Q_v^{\varepsilon,\eta,*} A^{\kappa-\eta,*} F\rangle.$$

Finally, by using (5.27) and equation (5.26) twice more, the second term on the right hand side is

$$\int_s^t \mathrm{d}u \langle T_u^{\varepsilon}, Q_u^{\varepsilon,\eta,*} F\rangle = \int_s^t \mathrm{d}u \langle T_s^{\varepsilon}, Q_u^{\varepsilon,\eta,*} F\rangle + \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \langle T_v^{\varepsilon}, A^{\kappa-\eta,*} Q_u^{\varepsilon,\eta,*} F\rangle$$

$$+ \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \langle T_v^{\varepsilon}, Q_v^{\varepsilon,\eta,*} Q_u^{\varepsilon,\eta,*} F\rangle$$

$$= \int_s^t \mathrm{d}u \langle T_s^{\varepsilon}, Q_u^{\varepsilon,\eta,*} F\rangle + \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \langle T_v^{\varepsilon}, A^{\kappa-\eta,*} Q_u^{\varepsilon,\eta,*} F\rangle$$

$$+ \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \langle T_s^{\varepsilon}, Q_v^{\varepsilon,\eta,*} Q_u^{\varepsilon,\eta,*} F\rangle + \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \int_s^v \mathrm{d}w \langle T_w^{\varepsilon}, A^{\kappa-\eta,*} Q_v^{\varepsilon,\eta,*} Q_u^{\varepsilon,\eta,*} F\rangle$$

$$+ \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \int_s^v \mathrm{d}w \langle T_w^{\varepsilon}, Q_w^{\varepsilon,\eta,*} Q_v^{\varepsilon,\eta,*} Q_u^{\varepsilon,\eta,*} F\rangle.$$

Hence, overall, one has that

$$\langle \delta_{st} T^{\varepsilon,\eta}, F\rangle = \langle T_s^{\varepsilon,\eta}, (\mathbb{A}_{st}^{\kappa-\eta,*} + \mathbb{X}_{st}^{1,\varepsilon,\eta,*} + \mathbb{X}_{st}^{2,\varepsilon,\eta,*}) F\rangle + \langle T_{st}^{\varepsilon,\eta,\natural}, F\rangle \tag{5.28}$$

where

$$\mathbb{A}_{st}^{\kappa-\eta,*} F(\xi,p,\kappa) = 4\pi(t-s)(\kappa-\eta)\cdot\nabla_p F(\xi,p,\kappa),$$

$$\mathbb{X}_{st}^{1,\varepsilon,*} F(\xi,p,\kappa) = \int_s^t \mathrm{d}u Q_u^{\varepsilon,\eta,*} F(\xi,p,\kappa),$$

and

$$\mathbb{X}_{st}^{2,\varepsilon,\eta,*} F(\xi,p,\kappa) = \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v Q_v^{\varepsilon,\eta,*} Q_u^{\varepsilon,\eta,*} F(\xi,p,\kappa).$$

$T_{st}^{\varepsilon,\eta,\natural}$ is, for any fixed s, t, the unique linear functional in E_{-2} such that for all $F \in E_2$ we have

$$\langle T_{st}^{\varepsilon,\eta,\natural}, F\rangle = \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \langle T_v^{\varepsilon,\eta}, A^{\kappa-\eta,*} A^{\kappa-\eta,*} F\rangle \tag{5.29}$$

$$+ \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \langle T_v^{\varepsilon,\eta}, (Q_v^{\varepsilon,\eta,*} A^{\kappa-\eta,*} + A^{\kappa-\eta,*} Q_u^{\varepsilon,\eta,*}) F\rangle \tag{5.30}$$

$$+ \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \int_s^v \mathrm{d}w \langle T_w^{\varepsilon}, (A^{\kappa-\eta,*} + Q_w^{\varepsilon,\eta,*}) Q_v^{\varepsilon,\eta,*} Q_u^{\varepsilon,\eta,*} F\rangle \tag{5.31}$$

Lemma 5.19 below proves that the expression on the right hand side indeed defines a linear functional on E_2. Our goal is to use the structure of equation (5.28) and the idea of Lemma 2.29 to obtain the limit as $\varepsilon \longrightarrow 0$.

We now compute $\mathbb{X}_{st}^{1,\varepsilon,*}$ and $\mathbb{X}_{st}^{2,\varepsilon,\eta,*}$ explicitly.

$$\mathbb{X}_{st}^{1,\varepsilon,*} F(\xi,p,\kappa) = \int_s^t \mathrm{d}u Q_u^{\varepsilon,\eta,*} F(\xi,p,\kappa)$$

$$= i\varepsilon^{-1/2} \int_s^t \mathrm{d}u \sum_{n\in\mathbb{Z}^d} e^{4\pi^2 i\varepsilon^{-1} n\cdot(2\kappa-2\eta-n-\xi)u} \hat{V}(n) F\Big(\xi+n,p,\kappa-\frac{n}{2}\Big)$$

$$- i\varepsilon^{-1/2} \int_s^t \mathrm{d}u \sum_{n\in\mathbb{Z}^d} e^{4\pi^2 i\varepsilon^{-1} n\cdot(2\kappa-2\eta+n+\xi)u} \hat{V}(n) F\Big(\xi+n,p,\kappa+\frac{n}{2}\Big)$$

$$= i\varepsilon^{-1/2} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \hat{V}(n) \int_s^t \mathrm{d}u \Big[e^{ia_1' u} F\Big(\xi+n,p,\kappa-\frac{n}{2}\Big) - e^{ia_2' u} F\Big(\xi+n,p,\kappa+\frac{n}{2}\Big)\Big],$$

where

$$a_1' = 4\pi^2 \varepsilon^{-1} n\cdot(2\kappa-2\eta-n-\xi), \tag{5.32}$$

$$a_2' = 4\pi^2 \varepsilon^{-1} n\cdot(2\kappa-2\eta+n+\xi). \tag{5.33}$$

Next, consider

$$\mathbb{X}_{st}^{2,\varepsilon,\eta,*} F(\xi,p,\kappa) = \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v Q_v^{\varepsilon,\eta,*} Q_u^{\varepsilon,\eta,*} F(\xi,p,\kappa)$$

$$= \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v\, i\varepsilon^{-1/2} \sum_{n\in\mathbb{Z}^d} e^{ia_1' v} \hat{V}(n) Q_u^{\varepsilon,\eta,*} F\Big(\xi+n,p,\kappa-\frac{n}{2}\Big)$$

$$- i\varepsilon^{-1/2} \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \sum_{n\in\mathbb{Z}^d} e^{ia_2' v} \hat{V}(n) Q_u^{\varepsilon,\eta,*} F\Big(\xi+n,p,\kappa+\frac{n}{2}\Big).$$

We have that

$$Q_u^{\varepsilon,\eta,*} F\Big(\xi+n,p,\kappa-\frac{n}{2}\Big) =$$

$$i\varepsilon^{-1/2} \sum_{n'\in\mathbb{Z}^d} e^{4\pi^2 i\varepsilon^{-1} n'\cdot(2\kappa-2n-2\eta-n'-\xi)u} \hat{V}(n') F\bigg(\xi+n+n',p,\kappa-\frac{n}{2}-\frac{n'}{2}\bigg)$$

$$- i\varepsilon^{-1/2} \sum_{n'\in\mathbb{Z}^d} e^{4\pi^2 i\varepsilon^{-1} n'\cdot(2\kappa-2\eta+n'+\xi)u} \hat{V}(n') F\bigg(\xi+n+n',p,\kappa-\frac{n}{2}+\frac{n'}{2}\bigg),$$

and

$$Q_u^{\varepsilon,\eta,*} F\Big(\xi+n,p,\kappa+\frac{n}{2}\Big) =$$

$$i\varepsilon^{-1/2} \sum_{n'\in\mathbb{Z}^d} e^{4\pi^2 i\varepsilon^{-1} n'\cdot(2\kappa-2\eta-n'-\xi)u} \hat{V}(n') F\bigg(\xi+n+n',p,\kappa+\frac{n}{2}-\frac{n'}{2}\bigg)$$

$$- i\varepsilon^{-1/2} \sum_{n'\in\mathbb{Z}^d} e^{4\pi^2 i\varepsilon^{-1} n'\cdot(2\kappa+2n-2\eta+n'+\xi)u} \hat{V}(n') F\bigg(\xi+n+n',p,\kappa+\frac{n}{2}+\frac{n'}{2}\bigg).$$

Hence

$$\mathbb{X}_{st}^{2,\varepsilon,\eta,*}F(\xi,p,\kappa) = -\varepsilon^{-1} \sum_{n,n' \in \mathbb{Z}^d \setminus \{0\}} \hat{V}(n)\hat{V}(n')\int_s^t \mathrm{d}u \int_s^u \mathrm{d}v$$

$$\left[e^{ia_1'v}e^{ib_1'u}F\left(\xi+n+n',p,\kappa-\frac{n}{2}-\frac{n'}{2}\right) - e^{ia_1'v}e^{ib_2'u}F\left(\xi+n+n',p,\kappa-\frac{n}{2}+\frac{n'}{2}\right)\right]$$

$$+\varepsilon^{-1} \sum_{n,n' \in \mathbb{Z}^d \setminus \{0\}} \hat{V}(n)\hat{V}(n')\int_s^t \mathrm{d}u \int_s^u \mathrm{d}v$$

$$\left[e^{ia_2'v}e^{ib_3'u}F\left(\xi+n+n',p,\kappa+\frac{n}{2}-\frac{n'}{2}\right) - e^{ia_2'v}e^{ib_4'u}F\left(\xi+n+n',p,\kappa+\frac{n}{2}+\frac{n'}{2}\right)\right],$$

where

$$b_1' = 4\pi^2\varepsilon^{-1}n'\cdot(2\kappa-2n-2\eta-n'-\xi), \tag{5.34}$$

$$b_2' = 4\pi^2\varepsilon^{-1}n'\cdot(2\kappa-2\eta+n'+\xi), \tag{5.35}$$

$$b_3' = 4\pi^2\varepsilon^{-1}n'\cdot(2\kappa-2\eta-n'-\xi), \tag{5.36}$$

$$b_4' = 4\pi^2\varepsilon^{-1}n'\cdot(2\kappa+2n-2\eta+n'+\xi). \tag{5.37}$$

We now split the four terms of $\mathbb{X}_{st}^{2,\varepsilon,\eta,*}F(\xi,p,\kappa)$ into their "resonant" and "nonresonant" parts, where by resonant we mean that a and b in the double exponential $e^{iau}e^{ibv}$ are of the form $a+b=0$, for almost every $\eta \in \left[-\frac{1}{4},\frac{1}{4}\right)^d$. Note that

$$\begin{aligned} a_1'+b_1'=0 &\Longleftrightarrow n'\cdot(2\kappa-2n-2\eta-n'-\xi)+n\cdot(2\kappa-2\eta-n-\xi)=0 \\ &\Longleftrightarrow (n+n')\cdot(2\kappa-2\eta-n-\xi)-n'\cdot(n+n')=0 \\ &\Longleftrightarrow (n+n')\cdot(2\kappa-2\eta-n-n'-\xi)=0 \end{aligned}$$

For this to be 0 for almost every $\eta \in \left[-\frac{1}{4},\frac{1}{4}\right)^d$, we need $n+n'=0$. Next, consider that

$$\begin{aligned} a_1'+b_2'=0 &\Longleftrightarrow n\cdot(2\kappa-2\eta-n-\xi)+n'\cdot(2\kappa-2\eta+n'+\xi)=0 \\ &\Longleftrightarrow (n+n')\cdot(2\kappa-2\eta)+(n'-n)\cdot\xi+|n'|^2-|n|^2 \end{aligned}$$

For this to be 0 for almost every $\eta \in \left[-\frac{1}{4},\frac{1}{4}\right)^d$, we need $n+n'=0$ and $n \perp \xi$. We argue similarly for (a_2',b_3') and (a_2',b_4').

We now introduce the notation: $a,b \in \mathbb{R}\setminus\{0\}$,

$$\varphi_{st}(a,b) := \int_s^t \mathrm{d}u \int_s^u \mathrm{d}r e^{iau}e^{ibr} \tag{5.38}$$

Putting together what we have above, we decompose $\mathbb{X}_{st}^{2,\varepsilon,\eta,*} = \mathbb{Y}_{st}^{\varepsilon,\eta,*} + \mathbb{Z}_{st}^{\varepsilon,\eta,*}$ where the resonant term

$$\mathbb{Y}_{st}^{\varepsilon,\eta,*}F(\xi,p,\kappa) :=$$

$$-\varepsilon^{-1} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |\hat{V}(n)|^2 \varphi_{st}(-a_1',a_1')[F(\xi,p,\kappa)-F(\xi,p,\kappa-n)\mathbb{I}_{n\perp\xi}]$$

$$+\varepsilon^{-1} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |\hat{V}(n)|^2 \varphi_{st}(-a_2',a_2')[F(\xi,p,\kappa+n)\mathbb{I}_{n\perp\xi}-F(\xi,p,\kappa)],$$

and the nonresonant term

$$\mathbb{Z}_{st}^{\varepsilon,\eta,*}F(\xi,p,\kappa):=$$

$$-\varepsilon^{-1}\sum_{n,n'\in\mathbb{Z}^d\backslash\{0\}}\hat{V}(n)\hat{V}(n')\varphi_{st}(b_1',a_1')F\left(\xi+n+n',p,\kappa-\frac{n}{2}-\frac{n'}{2}\right)\mathbb{I}_{a_1'\neq b_1'}$$

$$+\varepsilon^{-1}\sum_{n,n'\in\mathbb{Z}^d\backslash\{0\}}\hat{V}(n)\hat{V}(n')\varphi_{st}(b_2',a_1')F\left(\xi+n+n',p,\kappa-\frac{n}{2}+\frac{n'}{2}\right)\mathbb{I}_{a_1'\neq b_2'}$$

$$+\varepsilon^{-1}\sum_{n,n'\in\mathbb{Z}^d\backslash\{0\}}\hat{V}(n)\hat{V}(n')\varphi_{st}(b_3',a_2')F\left(\xi+n+n',p,\kappa+\frac{n}{2}-\frac{n'}{2}\right)\mathbb{I}_{a_2'\neq b_3'}$$

$$-\varepsilon^{-1}\sum_{n,n'\in\mathbb{Z}^d\backslash\{0\}}\hat{V}(n)\hat{V}(n')\varphi_{st}(b_4',a_2')F\left(\xi+n+n',p,\kappa+\frac{n}{2}+\frac{n'}{2}\right)\mathbb{I}_{a_2'\neq b_4'}.$$

5.2.2 Some useful lemmas

We will collect here some lemmas that will be useful in estimating terms that show up when attempting to bound the rough drivers on a scale. The first lemma deals with integrals involving two exponentials. We have

Lemma 5.9. *Let $\gamma\in(0,1)$. For $\varphi_{st}(a,b)$ as defined in expression (5.38) one has that*

1. *For $a+b\neq 0$,*

$$|\varphi_{st}(a,b)|\lesssim\frac{|t-s|^{2\gamma}}{|b|}\left(\frac{1}{|a|^{1-2\gamma}}+\frac{1}{|a+b|^{1-2\gamma}}\right).$$

2. *For a bound that is symmetric in a and b one has for $a+b\neq 0$*

$$|\varphi_{st}(a,b)|\lesssim|t-s|^{2\gamma}\left(\frac{1}{|a|^{1-\gamma}|b|^{1-\gamma}}+\frac{1}{|b|^{1/2}|a|^{1-\gamma}|a+b|^{\frac{1-2\gamma}{2}}}\right)$$

$$+(t-s)^{2\gamma}\left(\frac{1}{|a|^{1/2}|b|^{1-\gamma}|a+b|^{\frac{1-2\gamma}{2}}}+\frac{1}{|b|^{1/2}|a|^{1/2}|a+b|^{1-2\gamma}}\right).$$

3. *When $a+b=0$, one has*

$$|\varphi_{st}(a,b)|\lesssim\frac{|t-s|}{|a|}.$$

Proof. First notice that for $k\in\mathbb{R}\backslash\{0\}$ we have that

$$\left|\int_s^t e^{iku}\mathrm{d}u\right|\leqslant|t-s|,\qquad\left|\int_s^t e^{iku}\mathrm{d}u\right|=\left|\frac{e^{ikt}-e^{iks}}{ik}\right|\lesssim\frac{1}{|k|},$$

which by interpolation gives

$$\left|\int_s^t e^{iku}\mathrm{d}u\right|\lesssim\frac{|t-s|^{2\gamma}}{|k|^{1-2\gamma}}.$$

Now, in the first case we have that

$$\varphi_{st}(a,b) = \int_s^t \mathrm{d}u \int_s^u \mathrm{d}r e^{iau} e^{ibr} = \int_s^t \mathrm{d}u \left(e^{iau} \frac{e^{ibu} - e^{ibs}}{ib} \right)$$

$$= \frac{1}{ib} \int_s^t \mathrm{d}u e^{i(a+b)u} - \frac{e^{ibs}}{ib} \int_s^t \mathrm{d}u e^{iau}.$$

Applying the triangle inequality and the above interpolated estimate to each term, we get

$$|\varphi_{st}(a,b)| \leq \frac{1}{|b|} \left(\left| \int_s^t e^{i(a+b)u} \mathrm{d}u \right| + \left| \int_s^t e^{iau} \mathrm{d}u \right| \right) \lesssim \frac{|t-s|^{2\gamma}}{|b|} \left(\frac{1}{|a|^{1-2\gamma}} + \frac{1}{|a+b|^{1-2\gamma}} \right).$$

Now note that

$$\varphi_{st}(a,b) = \int_s^t \mathrm{d}u \int_s^u \mathrm{d}r e^{iau} e^{ibr} = \int_s^t \mathrm{d}r \int_r^t \mathrm{d}u e^{iau} e^{ibr} = \frac{e^{iat}}{ia} \int_s^t \mathrm{d}r e^{ibr} - \frac{1}{ia} \int_s^t \mathrm{d}r e^{i(a+b)r}.$$

Using the triangle inequality and the interpolated estimate as before, one has

$$|\varphi_{st}(a,b)| \lesssim \frac{|t-s|^{2\gamma}}{|a|} \left(\frac{1}{|b|^{1-2\gamma}} + \frac{1}{|a+b|^{1-2\gamma}} \right).$$

Interpolating between the two bounds obtained above, one has

$$|\varphi_{st}(a,b)| \lesssim |t-s|^{2\gamma} \left(\frac{1}{|a|^{1-\gamma}|b|^{1-\gamma}} + \frac{1}{|b|^{1/2}|a|^{1-\gamma}|a+b|^{\frac{1-2\gamma}{2}}} \right)$$

$$+ |t-s|^{2\gamma} \left(\frac{1}{|a|^{1/2}|b|^{1-\gamma}|a+b|^{\frac{1-2\gamma}{2}}} + \frac{1}{|b|^{1/2}|a|^{1/2}|a+b|^{1-2\gamma}} \right).$$

When $a+b=0$, we only have that

$$\varphi_{st}(a,b) = \int_s^t \mathrm{d}u \int_s^u \mathrm{d}r e^{ia(u-r)} = \int_s^t \mathrm{d}u e^{iau} \int_s^u \mathrm{d}r e^{-iar} = -\int_s^t \frac{1 - e^{ia(u-s)}}{ia} \mathrm{d}u,$$

from which we obtain the claim. □

This lemma is the reason why the resonant and nonresonant terms have different effects. Next we have a lemma that allows us to deal with small divisors. Within our problem, after applying the lemma above, we will encounter terms where a and b are of the form $cn\cdot(\kappa' - \eta')$ for $n \in \mathbb{Z}^d \backslash \{0\}, \kappa' \in \mathbb{Z}^d, \eta' \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d$. So in order to bound $\frac{1}{|a|}$, one will need to bound $|a|$ away from 0.

Lemma 5.10. *Let* $\delta_0 = \frac{1}{d+3}$. *For any* $\delta \in (0, \delta_0)$ *there exists* $c_\delta \in L^1\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^d\right)$ *such that for almost every* $\eta \in \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ *one has*

$$|n\cdot\eta - a|^{-1} \leq c_\delta(\eta)^{\frac{1}{1-\delta}} \langle n \rangle^{d+2} < \infty, \qquad \forall a \in \mathbb{Z}, n \in \mathbb{Z}^d \backslash \{0\}.$$

Proof. Fix $\delta \in (0, \delta_0)$ and define

$$c'_\delta(\eta) = \sum_{n \in \mathbb{Z}^d \backslash \{0\}} \sum_{a : |a| \leq 2\sqrt{d}|n|} \langle n \rangle^{-d-1-\delta} |n\cdot\eta - a|^{\delta - 1}$$

We claim

$$\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} |n\cdot\eta-a|^{\delta-1}\mathrm{d}\eta \leq C_{d,\delta} < \infty$$

uniformly in n and $a\colon |a|\leq 2\sqrt{d}|n|$. To see this, consider

$$\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} |n\cdot\eta-a|^{\delta-1}\mathrm{d}\eta \leq \int_{B_{\sqrt{d}}(0)} \mathrm{d}\eta |n\cdot\eta-a|^{\delta-1}\mathrm{d}\eta.$$

For any rotation matrix R, one can define $\eta' := R^{-1}\eta$ to get that this is

$$= \int_{B_{\sqrt{d}}(0)} |n\cdot R\eta'-a|^{\delta-1}\mathrm{d}\eta' = \int_{B_{\sqrt{d}}(0)} |R^T n\cdot\eta'-a|^{\delta-1}\mathrm{d}\eta'.$$

Now, one can choose $R\equiv R(n)$ such that $R^T n=(|n|,0,\ldots,0)$. This then becomes

$$= \int_{B_{\sqrt{d}}(0)} ||n|\eta_1'-a|^{\delta-1}\mathrm{d}\eta' \leq \left(2\sqrt{d}\right)^{d-1}\int_{-\sqrt{d}}^{\sqrt{d}} \mathrm{d}\eta' ||n|\eta'-a|^{\delta-1}.$$

By a change of variables this is

$$= \frac{\left(2\sqrt{d}\right)^{d-1}}{|n|}\int_{-|n|\sqrt{d}-a}^{|n|\sqrt{d}-a} \mathrm{d}\eta' |\eta'|^{\delta-1}.$$

Now, since $|a|\leq 2\sqrt{d}|n|$ in the definition of c_δ, one can estimate

$$\max\left\{\left||n|\sqrt{d}-a\right|, \left||n|\sqrt{d}+a\right|\right\} \leq 3|n|\sqrt{d}.$$

So

$$\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} |n\cdot\eta-a|^{\delta-1}\mathrm{d}\eta \leq \frac{2\left(2\sqrt{d}\right)^{d-1}}{|n|}\int_0^{3|n|\sqrt{d}} \mathrm{d}\eta' |\eta_1'|^{\delta-1}$$

$$\leq \frac{2\left(2\sqrt{d}\right)^{d-1}}{|n|}\left[\int_0^1 \eta^{\delta-1}\mathrm{d}\eta + \int_1^{3|n|\sqrt{d}} \mathrm{d}\eta\right] \leq \frac{C_1(\delta)}{|n|} + C_2(d) \leq C_{\delta,d},$$

for $n\in\mathbb{Z}^d\backslash\{0\}$. Hence we have that

$$\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} c_\delta'(\eta)\mathrm{d}\eta = \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \sum_{n\in\mathbb{Z}^d\backslash\{0\}} \sum_{a\in\mathbb{Z}:|a|\leq 2\sqrt{d}|n|} \langle n\rangle^{-d-1-\delta} |n\cdot\eta-a|^{\delta-1}\mathrm{d}\eta$$

By Tonelli's theorem and the dominated convergence this is

$$= \sum_{n\in\mathbb{Z}^d\backslash\{0\}} \sum_{a\in\mathbb{Z}:|a|\leq 2\sqrt{d}|n|} \langle n\rangle^{-d-1-\delta} \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} |n\cdot\eta-a|^{\delta-1}\mathrm{d}\eta$$

$$\leq C_{d,\delta} \sum_{n\in\mathbb{Z}^d\backslash\{0\}} \langle n\rangle^{-d-1-\delta} \sum_{a\in\mathbb{Z}:|a|\leq 2\sqrt{d}|n|} 1$$

$$\leq C_{d,\delta} \sum_{n\in\mathbb{Z}^d\backslash\{0\}} \langle n\rangle^{-d-1-\delta}\langle n\rangle \leq C_{d,\delta} \sum_{n\in\mathbb{Z}^d\backslash\{0\}} \langle n\rangle^{-d-\delta} < \infty$$

Hence we have that $c_\delta'(\eta)$ is finite a.e., and hence, for almost every η, by the fact that $c_\delta'(\eta)$ consists of positive terms one has

$$\langle n\rangle^{-d-1-\delta}|n\cdot\eta-a|^{\delta-1}\leq c_\delta'(\eta)$$

$$\Longrightarrow\quad \langle n\rangle^{d+1+\delta}|n\cdot\eta-a|^{1-\delta}\geq c_\delta'(\eta)^{-1}$$

$$\Longrightarrow\quad |n\cdot\eta-a|\geq c_\delta'(\eta)^{-1/1-\delta}\langle n\rangle^{-(d+1+\delta)/1-\delta}$$

$$\Longrightarrow\quad |n\cdot\eta-a|^{-1}\leq c_\delta'(\eta)^{1/1-\delta}\langle n\rangle^{(d+1+\delta)/1-\delta},$$

which for $0<\delta<\frac{1}{d+3}$

$$\Longrightarrow |n\cdot\eta-a|^{-1}\leq c_\delta'(\eta)^{\frac{1}{1-\delta}}\langle n\rangle^{d+2}.$$

In the case that $|a|>2\sqrt{d}|n|$ we have by the reverse triangle inequality that

$$|a-n\cdot\eta|\geq|a|-|n\cdot\eta|\geq|a|-\sqrt{d}|n|\geq\sqrt{d}|n|\geq 1$$

$$\Longrightarrow |n\cdot\eta-a|^{-1}\leq 1.$$

Defining $c_\delta(\eta)=\max\{c_\delta'(\eta),1\}$ we have the claim. □

Remark 5.11. Notice that $\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}c_\delta(\eta)\mathrm{d}\eta\xrightarrow{\delta\to 0}\infty$. On the right hand side of the expression in the statement of the lemma, one has $c_\delta(\eta)^{1/1-\delta}$, which is not in $L^1_\eta\left(\left[-\frac{1}{2},\frac{1}{2}\right]^d\right)$. Clearly, there can never be an L^1 function on the right hand side, because one knows that η^{-1} is not integrable between 0 and 1.

From this point on, we fix some $\delta\in\left(1,\frac{1}{d+3}\right)$ and drop δ from the notation of c_δ. We write

$$c(\eta)=c_\delta(\eta) \tag{5.39}$$

through the rest of the section.
The above lemma says that a.e $\eta\in\left[-\frac{1}{4},\frac{1}{4}\right]^d$ belongs to $\mathscr{A}_\eta$. (Recall Definition 5.22). We next have the following estimates on the double exponentials that show up in the operators of the rough difference equation:

Lemma 5.12. *It holds that* $\forall n\in\mathbb{Z}^d\backslash\{0\}, l\in\mathbb{Z}^d, 0\leq s<t, \eta\in\mathscr{A}_\eta$ *and any* $\gamma\in(0,1)$

$$\left|\varepsilon^{-1/2}\int_s^t\mathrm{d}u e^{4\pi^2 i\varepsilon^{-1}n\cdot(l-2\eta)u}\right|\lesssim c(\eta)^{\frac{1-\gamma}{1-\delta}}\varepsilon^{1/2-\gamma}|t-s|^\gamma\langle n\rangle^{(d+2)(1-\gamma)},$$

where $c(\eta)$ *is the one defined in (5.39).*

Proof. We have by the same proof techniques as in Lemma 5.9 that

$$\left|\int_s^t\mathrm{d}u e^{4\pi^2 i\varepsilon^{-1}n\cdot(l-2\eta)u}\right|\lesssim\frac{|t-s|^\gamma}{\varepsilon^{-1+\gamma}|n\cdot(l-2\eta)|^{1-\gamma}},$$

and by Lemma 5.10 and the fact that $\eta\in\mathscr{A}_\eta$, this is

$$\leq\varepsilon^{1-\gamma}|t-s|^\gamma c(\eta)^{\frac{1-\gamma}{1-\delta}}\langle n\rangle^{(d+2)(1-\gamma)}.$$

for some $c\in L^1\left(\left[-\frac{1}{4},\frac{1}{4}\right]^d\right)$. Hence

$$\left|\varepsilon^{-1/2}\int_s^t \mathrm{d}u e^{4\pi^2 i\varepsilon^{-1} n\cdot(l-2\eta)u}\right| \lesssim c(\eta)^{\frac{1-\gamma}{1-\delta}}\varepsilon^{1/2-\gamma}|t-s|^{\gamma}\langle n\rangle^{(d+2)(1-\gamma)}.$$

□

Note that when $\gamma>\delta$ one can integrate the expression on the right hand side over η when $\frac{1-\gamma}{1-\delta}<1$, since $c\in L^1$.

Lemma 5.13. *It holds that* $\forall n\in\mathbb{Z}^d\backslash\{0\}, l\in\mathbb{Z}^d, 0\leq s<u<t, \eta\in\mathscr{A}_\eta$

$$\left|\varepsilon^{-1}\int_s^t \mathrm{d}u\int_s^u \mathrm{d}v e^{4\pi^2 i\varepsilon^{-1} n\cdot(l-2\eta)(u-v)}\right| \lesssim (t-s)\,c(\eta)^{\frac{1}{1-\delta}}\langle n\rangle^{d+2},$$

where $c(\eta)$ *is the one defined in (5.39).*

Proof. As in case 2 of Lemma 5.9 we set $a=4\pi^2\varepsilon^{-1}\,n\cdot(l-2\eta)$ and see that

$$\left|\varepsilon^{-1}\int_s^t\int_s^u e^{ia(u-r)}\mathrm{d}r\mathrm{d}u\right| \lesssim \frac{\varepsilon^{-1}(t-s)}{|a|}$$

and by plugging in the value of a and using Lemma 5.10 we get that

$$\left|\varepsilon^{-1}\int_s^t\int_s^u e^{ia(u-r)}\mathrm{d}r\mathrm{d}u\right| \lesssim (t-s)\,c(\eta)^{\frac{1}{1-\delta}}\langle n\rangle^{d+2}.$$

□

Lemma 5.14. *It holds that* $\forall n,n'\in\mathbb{Z}^d\backslash\{0\}, l,l'\in\mathbb{Z}^d, \eta\in\mathscr{A}_\eta$ *satisfying* $n\cdot(l-2\eta)+n'\cdot(l'-2\eta)\neq 0$, *and* $0\leq s<u<t, \gamma\in(0,1)$ *one has*

$$\begin{aligned}&\left|\varepsilon^{-1}\int_s^t\int_s^u e^{4\pi^2 i\varepsilon^{-1} n\cdot(l-2\eta)u}e^{4\pi^2 i\varepsilon^{-1} n'\cdot(l'-2\eta)v}\mathrm{d}v\mathrm{d}u\right|\\ &\lesssim \varepsilon^{1-2\gamma}|t-s|^{2\gamma}c(\eta)^{\frac{2-2\gamma}{1-\delta}}(\langle n\rangle\langle n'\rangle)^{2(d+2)},\end{aligned}$$

where $c(\eta)$ *is the one defined in (5.39).*

Proof. Considering the $a+b\neq 0$ case of Lemma 5.9 we set $a=4\pi^2\varepsilon^{-1}\,n\cdot(l-2\eta)$ and $b=4\pi^2\varepsilon^{-1}\,n'\cdot(l'-2\eta)$ and see that when $a+b\neq 0$,

$$\begin{aligned}\left|\varepsilon^{-1}\int_s^t\int_s^u e^{iau}e^{ibv}\mathrm{d}v\mathrm{d}u\right| &\lesssim \varepsilon^{-1}|t-s|^{2\gamma}\left(\frac{1}{|a|^{1-\gamma}|b|^{1-\gamma}}+\frac{1}{|b|^{1/2}|a|^{1-\gamma}|a+b|^{\frac{1-2\gamma}{2}}}\right)\\ &+\varepsilon^{-1}|t-s|^{2\gamma}\left(\frac{1}{|a|^{1/2}|b|^{1-\gamma}|a+b|^{\frac{1-2\gamma}{2}}}+\frac{1}{|b|^{1/2}|a|^{1/2}|a+b|^{1-2\gamma}}\right).\end{aligned}$$

By Lemma 5.10 this is

$$\begin{aligned}&\lesssim \varepsilon^{1-2\gamma}|t-s|^{2\gamma}c(\eta)^{\frac{2-2\gamma}{1-\delta}}\langle n\rangle^{(d+2)(1-\gamma)}\langle n'\rangle^{(d+2)(1-\gamma)}\\ &+\varepsilon^{1-2\gamma}|t-s|^{2\gamma}c(\eta)^{\frac{2-2\gamma}{1-\delta}}\langle n'\rangle^{(d+2)/2}\langle n\rangle^{(d+2)(1-\gamma)}\langle n+n'\rangle^{(d+2)(1-2\gamma)/2}\\ &+\varepsilon^{1-2\gamma}|t-s|^{2\gamma}c(\eta)^{\frac{2-2\gamma}{1-\delta}}\langle n\rangle^{(d+2)/2}\langle n'\rangle^{(d+2)(1-\gamma)}\langle n+n'\rangle^{(d+2)(1-2\gamma)/2}\\ &+\varepsilon^{1-2\gamma}|t-s|^{2\gamma}c(\eta)^{\frac{2-2\gamma}{1-\delta}}\langle n\rangle^{(d+2)/2}\langle n'\rangle^{(d+2)/2}\langle n+n'\rangle^{(d+2)(1-2\gamma)}.\end{aligned}$$

Combining these, one has that this is

$$\begin{aligned}&\lesssim \varepsilon^{1-2\gamma}|t-s|^{2\gamma}c(\eta)^{\frac{2-2\gamma}{1-\delta}}\langle n\rangle^{d+2}\langle n'\rangle^{d+2}\langle n+n'\rangle^{d+2}\\&\lesssim \varepsilon^{1-2\gamma}|t-s|^{2\gamma}c(\eta)^{\frac{2-2\gamma}{1-\delta}}\langle n\rangle^{2(d+2)}\langle n'\rangle^{2(d+2)}.\end{aligned}$$

□

5.2.3 Uniform operator norm and naive remainder estimates

This Subsection will be devoted to proving the following: For any $m\in\mathbb{N}_0, \eta\in\mathscr{A}_\eta$, $\gamma\in\left(\frac{1}{3},\frac{1}{2}\right), \kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d, \eta\in\left[-\frac{1}{4},\frac{1}{4}\right)^d, 0\leqslant s<t, \delta$ as in (5.21), we have for any $\psi\in E_m$,

$$\|\mathbb{X}_{st}^{1,\varepsilon,\eta,*}\psi\|_{E_m}\lesssim \varepsilon^{1/2-\gamma}|t-s|^{\gamma}\|\psi\|_{E_m},$$

$$\|\mathbb{Y}_{st}^{\varepsilon,\eta,*}\psi\|_{E_m}\lesssim (t-s)\|\psi\|_{E_m},$$

$$\|\mathbb{Z}_{st}^{\varepsilon,\eta,*}\psi\|_{E_m}\lesssim \varepsilon^{1-2\gamma}|t-s|^{2\gamma}\|\psi\|_{E_m},$$

and $\forall\psi\in E_{m+1}$,

$$\|\mathbb{A}_{st}^{\kappa-\eta,*}\psi\|_{E_m}\lesssim |t-s|\|\psi\|_{E_{m+1}}.$$

In the above bounds, the constants implicit in $\lesssim$ could depend on $d, V, \gamma, m, \delta, \eta$ but not on ε.
We will also prove the following naive bound on the remainder

$$\|T_{st}^{\varepsilon,\eta,\natural}\|_{E_{-2}}\lesssim \varepsilon^{-3/2}|t-s|^2.$$

Lemma 5.15. *For any $m\in\mathbb{N}_0, \eta\in\mathscr{A}_\eta, \gamma\in\left(\frac{1}{3},\frac{1}{2}\right), \psi\in E_m, 0\leqslant s<t, \delta$ as in (5.21), we have*

$$\|\mathbb{X}_{st}^{1,\varepsilon,\eta,*}\psi\|_{E_m}\lesssim c(\eta)^{\frac{1-\gamma}{1-\delta}}\varepsilon^{1/2-\gamma}|t-s|^{\gamma}\|\psi\|_{E_m} \tag{5.40}$$

where $c(\eta)$ is defined in (5.39).

Proof. Recall that

$$\mathbb{X}_{st,\eta}^{1,\varepsilon,\eta,*}\psi(\xi,p,\kappa)=$$

$$i\varepsilon^{-1/2}\sum_{n\in\mathbb{Z}^d\backslash\{0\}}\hat{V}(n)\int_s^t \mathrm{d}u\left[e^{ia_1'u}\psi\left(\xi+n,p,\kappa-\frac{n}{2}\right)-e^{ia_2'u}\psi\left(\xi+n,p,\kappa+\frac{n}{2}\right)\right],$$

where a_1' and a_2' were defined in (5.32) and (5.33). Since we want to see where $\mathbb{X}_{st}^{1,\varepsilon,\eta,*}$ maps $\psi\in E_m$ we first consider for $\beta:|\beta|\leqslant m$

$$\|D_p^\beta\mathbb{X}_{st}^{1,\varepsilon,\eta,*}\psi(\xi,p,\kappa)\|_{l_\xi^2}=\left(\sum_{\xi\in\mathbb{Z}^d}|D_p^\beta\mathbb{X}_{st}^{1,\varepsilon,\eta,*}\psi(\xi,p,\kappa)|^2\right)^{1/2}.$$

To this end, we use Young's inequality to get

$$|a-b|^2=|a|^2+|b|^2-2a\cdot b\leqslant|a|^2+|b|^2+2\left(\frac{|a|^2}{2}+\frac{|b|^2}{2}\right)=2|a|^2+2|b|^2.$$

This allows us to estimate

$$\sum_{\xi\in\mathbb{Z}^d} |D_p^\beta \mathbb{X}_{st}^{1,\varepsilon,\eta,*}\psi(\xi,p,\kappa)|^2$$

$$\leq 2\sum_{\xi\in\mathbb{Z}^d}\left|\sum_{n\in\mathbb{Z}^d\backslash\{0\}} \hat{V}(n)\left(\varepsilon^{-1/2}\int_s^t e^{ia_1'u}\mathrm{d}u\right)D_p^\beta\psi\left(\xi+n,p,\kappa-\frac{n}{2}\right)\right|^2$$

$$+2\sum_{\xi\in\mathbb{Z}^d}\left|\sum_{n\in\mathbb{Z}^d\backslash\{0\}} \hat{V}(n)\left(\varepsilon^{-1/2}\int_s^t e^{ia_2'u}\mathrm{d}u\right)D_p^\beta\psi\left(\xi+n,p,\kappa+\frac{n}{2}\right)\right|^2$$

$$= A + B.$$

We shall estimate just the first term, and the second term can be estimated the same way. By Cauchy–Schwarz

$$A \lesssim \sum_{\xi\in\mathbb{Z}^d}\left(\sum_{n\in\mathbb{Z}^d\backslash\{0\}} |\hat{V}(n)|^2\langle n\rangle^{2M}\right)$$

$$\left(\sum_{n\in\mathbb{Z}^d\backslash\{0\}} \langle n\rangle^{-2M}\left|\varepsilon^{-1/2}\int_s^t e^{ia_1'u}\mathrm{d}u\right|^2\left|D_p^\beta\psi\left(\xi+n,p,\kappa-\frac{n}{2}\right)\right|^2\right),$$

which by Lemma 5.12 and the expression for a_1' is, for $\eta\in\mathscr{A}_\eta$

$$\lesssim c(\eta)^{\frac{2(1-\gamma)}{1-\delta}}\varepsilon^{1-2\gamma}|t-s|^{2\gamma}$$

$$\sum_{\xi\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d\backslash\{0\}} \langle n\rangle^{-2M}\langle n\rangle^{2(d+2)(1-\gamma)}|D_p^\beta\psi(p,\xi+n,k-n)|^2.$$

By using Tonelli's theorem to change the order of summation, this is

$$\lesssim c(\eta)^{\frac{2(1-\gamma)}{1-\delta}}\varepsilon^{1-2\gamma}|t-s|^{2\gamma}\sum_{n\in\mathbb{Z}^d\backslash\{0\}} \langle n\rangle^{-2M}\langle n\rangle^{2(d+2)}\left\|D_p^\beta\psi\left(\cdot,p,\kappa-\frac{n}{2}\right)\right\|_{l_\xi^2}^2.$$

So, we have that

$$\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\|D_p^\beta\mathbb{X}_{st}^{1,\varepsilon,\eta,*}\psi(p,\xi,\kappa)\|_{l_\xi^2} \lesssim \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m A^{1/2} + \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m B^{1/2}.$$

We estimate

$$\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m A^{1/2} \lesssim c(\eta)^{\frac{1-\gamma}{1-\delta}}\varepsilon^{1/2-\gamma}|t-s|^\gamma$$

$$\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\left(\sum_{n\in\mathbb{Z}^d\backslash\{0\}}\langle n\rangle^{-2M}\langle n\rangle^{2(d+2)}\left\|D_p^\beta\psi\left(\cdot,p,\kappa-\frac{n}{2}\right)\right\|_{l_\xi^2}^2\right)^{1/2}.$$

The idea is to now show that for large M,

$$\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\left(\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\langle n\rangle^{-2M}f^2\left(\kappa-\frac{n}{2}\right)\right)^{1/2}=\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\left(\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\langle 2\kappa-n\rangle^{-2M}f^2\left(\frac{n}{2}\right)\right)^{1/2}$$

$$\approx\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m(f^2(\kappa))^{1/2}=\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m f(\kappa).$$

Since the l^2-norm of a sequence is upper bounded by its l^1-norm, we get that

$$A\lesssim c(\eta)^{\frac{1-\gamma}{1-\delta}}\varepsilon^{1/2-\gamma}|t-s|^{\gamma}$$

$$\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\langle\kappa\rangle^m\langle n\rangle^{-M}\langle n\rangle^{(d+2)/2}\left\|D_p^{\beta}\psi\left(\xi,p,\kappa-\frac{n}{2}\right)\right\|_{l_\xi^2}.$$

By using Tonelli's theorem twice more and changing variables, this

$$\lesssim c(\eta)^{\frac{1-\gamma}{1-\delta}}\varepsilon^{1/2-\gamma}|t-s|^{\gamma}\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\|D_p^{\beta}\psi(\cdot,p,\kappa)\|_{l_\xi^2}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\left\langle\kappa+\frac{n}{2}\right\rangle^m\langle n\rangle^{-M}\langle n\rangle^{(d+2)/2}$$

Let $M'=M-\frac{d+2}{2}$. Then

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\left\langle\kappa+\frac{n}{2}\right\rangle^m\langle n\rangle^{-M}\langle n\rangle^{(d+2)/2}=\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\left\langle\kappa+\frac{n}{2}\right\rangle^m\langle n\rangle^{-M'},$$

and for M' large enough, this is

$$\leqslant C_m\sum_{n\in\mathbb{Z}^d\setminus\{0\}}(\langle\kappa\rangle^m+\langle n\rangle^m)\langle n\rangle^{-M'}\leqslant C_{m,M}\langle\kappa\rangle^m.$$

So we have

$$\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m A^{1/2}\lesssim c(\eta)^{\frac{1-\gamma}{1-\delta}}\varepsilon^{1/2-\gamma}|t-s|^{\gamma}\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\|D_p^{\beta}\psi(\cdot,p,\kappa)\|_{l_\xi^2}\langle\kappa\rangle^m,$$

and similarly for $\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m B^{1/2}$. Hence

$$\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\|D_p^{\beta}\mathbb{X}_{st}^{1,\varepsilon,\eta,*}\psi(\xi,p,\kappa)\|_{l_\xi^2}\lesssim c(\eta)^{\frac{1-\gamma}{1-\delta}}\varepsilon^{1/2-\gamma}|t-s|^{\gamma}\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\|D_p^{\beta}\psi(\xi,p,\kappa)\|_{l_\xi^2}\langle\kappa\rangle^m.$$

Integrating in p and summing over $\beta\colon|\beta|\leqslant m$, we have

$$\|\mathbb{X}_{st}^{1,\varepsilon,*}\psi\|_{E_m}\lesssim c(\eta)^{\frac{1-\gamma}{1-\delta}}\varepsilon^{1/2-\gamma}|t-s|^{\gamma}\|\psi\|_{E_m}.$$ □

Lemma 5.16. *For* $\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d$, $\eta\in\left[-\frac{1}{4},\frac{1}{4}\right)^d$, $m\in\mathbb{N}_0$, $\psi\in E_{m+1}$, $0\leqslant s<t$ *we have*

$$\|\mathbb{A}_{st}^{\kappa-\eta,*}\psi\|_{E_m}\lesssim|t-s|\|\psi\|_{E_{m+1}}\tag{5.41}$$

Proof. We bound the operator $\mathbb{A}_{st}^{\kappa-\eta,*}\psi(\xi,p,\kappa)=4\pi(t-s)(\kappa-\eta)\cdot\nabla_p\psi(\xi,p,\kappa)$ in the scale as follows, by first considering the quantity

$$\begin{aligned}
\sum_{\xi}|D_p^{\beta}\mathbb{A}_{st}^{\kappa-\eta,*}\psi|^2 &= 16\pi^2(t-s)^2\sum_{\xi}|D_p^{\beta}(\kappa-\eta)\cdot\nabla_p\psi|^2\\
&=16\pi^2|t-s|^2\sum_{\xi}|D_p^{\beta}(\kappa-\eta)\cdot\nabla_p\psi(\xi,p,\kappa)|^2\\
&=16\pi^2|t-s|^2\sum_{\xi}\left|D_p^{\beta}\sum_{j=1}^{d}(\kappa-\eta)_jD_p^{e_j}\psi(\xi,p,\kappa)\right|^2\\
&=16\pi^2|t-s|^2\sum_{\xi}\left|\sum_{j=1}^{d}(\kappa-\eta)_jD_p^{\beta+e_j}\psi(\xi,p,\kappa)\right|^2\\
&\lesssim_d|t-s|^2\langle\kappa\rangle^2\sum_{\xi}\sum_{j=1}^{d}|D_p^{\beta+e_j}\psi(\xi,p,\kappa)|^2\\
&=|t-s|^2\langle\kappa\rangle^2\sum_{j=1}^{d}\sum_{\xi}|D_p^{\beta+e_j}\psi(\xi,p,\kappa)|^2.
\end{aligned}$$

By subadditivity of the square root function

$$\begin{aligned}
&4\pi(t-s)\sum_{\kappa}\langle\kappa\rangle^m\left(\sum_{\xi}|D_p^{\beta}(\kappa-\eta)\cdot\nabla_p\psi(\xi,p,\kappa)|^2\right)^{1/2}\\
&\lesssim|t-s|\sum_{j=1}^{d}\sum_{\kappa}\langle\kappa\rangle^{m+1}\left(\sum_{\xi}|D_p^{\beta+e_j}\psi(\xi,p,\kappa)|^2\right)^{1/2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\mathbb{A}_{st}^{\kappa-\eta,*}\psi\|_{E_m}&=\sum_{|\beta|\le m}\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa}\langle\kappa\rangle^m\left(\sum_{\xi}|D_p^{\beta}\mathbb{A}_{st}^{\kappa-\eta,*}\psi(\xi,p,\kappa)|^2\right)^{1/2}\\
&\lesssim|t-s|\sum_{|\beta|\le m}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa}\langle\kappa\rangle^{m+1}\left(\sum_{\xi}|D_p^{\beta+e_j}\psi(\xi,p,\kappa)|^2\right)^{1/2}\\
&\lesssim|t-s|\sum_{|\beta|\le m+1}\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa}\langle\kappa\rangle^{m+1}\|D_p^{\beta}\psi(\cdot,p,\kappa)\|_{l_\xi^2}\lesssim 4\pi|t-s|\|\psi\|_{E_{m+1}}.
\end{aligned}$$

So

$$\|\mathbb{A}_{st}^{\kappa-\eta,*}\psi\|_{E_m}\lesssim|t-s|\|\psi\|_{E_{m+1}}.$$

□

Recall that we had

$$\mathbb{X}_{st}^{2,\varepsilon,\eta,*}=\mathbb{Y}_{st}^{\varepsilon,\eta,*}+\mathbb{Z}_{st}^{\varepsilon,\eta,*}$$

We now estimate the two operators, and will see that the resonant term $\mathbb{Y}_{st}^{\varepsilon,\eta,*}$ will have a contribution in the limit $\varepsilon\longrightarrow 0$ while the nonresonant term $\mathbb{Z}_{st}^{\varepsilon,\eta,*}$ will vanish.

Lemma 5.17. *For all* $m\in\mathbb{N}_0,\eta\in\mathscr{A}_\eta,\psi\in E_m,0\le s<t,\delta$ *as in* (5.21) *we have*

$$\|\mathbb{Y}_{st}^{\varepsilon,\eta,*}\psi\|_{E_m}\lesssim(t-s)c(\eta)^{\frac{1}{1-\delta}}\|\psi\|_{E_m},\tag{5.42}$$

where $c(\eta)$ is defined in (5.39).

Proof. We recall

$$\mathbb{Y}^{\varepsilon,\eta,*}_{st}F(\xi,p,\kappa):=$$

$$-\varepsilon^{-1}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}|\hat{V}(n)|^2\int_s^t\mathrm{d}u\int_s^u\mathrm{d}v e^{ia_1'(v-u)}[F(\xi,p,\kappa)-F(\xi,p,\kappa-n)\mathbb{I}_{n\perp\xi}]$$

$$+\varepsilon^{-1}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}|\hat{V}(n)|^2\int_s^t\mathrm{d}u\int_s^u\mathrm{d}v e^{ia_2'(v-u)}[F(\xi,p,\kappa+n)\mathbb{I}_{n\perp\xi}-F(\xi,p,\kappa)].$$

where a_1' and a_2' were defined in (5.32) and (5.33). We would like to estimate how $\mathbb{Y}^{\varepsilon,\eta,*}_{st}$ maps on the scale. To this end we first compute

$$\|D^\beta_p\mathbb{Y}^{\varepsilon,\eta,*}_{st}\psi(\cdot,p,\kappa)\|_{l^2_\xi}=\left(\sum_{\xi\in\mathbb{Z}^d}|D^\beta_p\mathbb{Y}^{\varepsilon,\eta,*}_{st}\psi(\xi,p,\kappa)|^2\right)^{1/2}.$$

We have that

$$\sum_{\xi\in\mathbb{Z}^d}|D^\beta_p\mathbb{Y}^{\varepsilon,*}_{st}\psi(\xi,p,\kappa)|^2$$

$$\lesssim\sum_{\xi\in\mathbb{Z}^d}\left|\sum_{n\in\mathbb{Z}^d\setminus\{0\}}|\hat{V}(n)|^2\Big(\varepsilon^{-1}\int_s^t\int_s^u e^{ia_1'(v-u)}\mathrm{d}v\mathrm{d}u\Big)D^\beta_p\psi(\xi,p,\kappa)\right|^2$$

$$+\sum_{\xi\in\mathbb{Z}^d}\left|\sum_{n\in\mathbb{Z}^d\setminus\{0\}}|\hat{V}(n)|^2\Big(\varepsilon^{-1}\int_s^t\int_s^u e^{ia_1'(v-u)}\mathrm{d}v\mathrm{d}u\Big)D^\beta_p\psi(\xi,p,\kappa-n)\mathbb{I}_{n\perp\xi}\right|^2$$

$$+\sum_{\xi\in\mathbb{Z}^d}\left|\sum_{n\in\mathbb{Z}^d\setminus\{0\}}|\hat{V}(n)|^2\Big(\varepsilon^{-1}\int_s^t\int_s^u e^{ia_2'(v-u)}\mathrm{d}v\mathrm{d}u\Big)D^\beta_p\psi(\xi,p,\kappa+n)\mathbb{I}_{n\perp\xi}\right|^2$$

$$+\sum_{\xi\in\mathbb{Z}^d}\left|\sum_{n\in\mathbb{Z}^d\setminus\{0\}}|\hat{V}(n)|^2\Big(\varepsilon^{-1}\int_s^t\int_s^u e^{ia_2'(v-u)}\mathrm{d}v\mathrm{d}u\Big)D^\beta_p\psi(\xi,p,\kappa)\right|^2$$

$$=A_{\mathbb{Y}}+B_{\mathbb{Y}}+C_{\mathbb{Y}}+D_{\mathbb{Y}}.$$

We have

$$\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\|D^\beta_p\mathbb{Y}^{\varepsilon,\eta,*}_{st}\psi(\cdot,p,\kappa)\|_{l^2_\xi}\lesssim\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m(A^{1/2}_{\mathbb{Y}}+B^{1/2}_{\mathbb{Y}}+C^{1/2}_{\mathbb{Y}}+D^{1/2}_{\mathbb{Y}}).$$

We will see how to estimate the first two terms, and the last two are estimated analogously. Beginning with the $A_{\mathbb{Y}}$ term we have by Cauchy Schwartz

$$A_{\mathbb{Y}}=\sum_{\xi\in\mathbb{Z}^d}\left|\sum_{n\in\mathbb{Z}^d\setminus\{0\}}|\hat{V}(n)|^2\Big(\varepsilon^{-1}\int_s^t\int_s^u e^{ia_1'(v-u)}\mathrm{d}v\mathrm{d}u\Big)D^\beta_p\psi(\xi,p,\kappa)\right|^2$$

$$\lesssim\sum_{\xi\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}|\hat{V}(n)|^4\langle n\rangle^{2M}$$

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\langle n\rangle^{-2M}\Big(\varepsilon^{-1}\int_s^t\int_s^u e^{ia_1'(v-u)}\mathrm{d}v\mathrm{d}u\Big)^2|D^\beta_p\psi(\xi,p,\kappa)|^2.$$

Now, since $V \in \cap_{M \geqslant 0} H^M(\mathbb{T}^d)$ we have that

$$\sum_{n \in \mathbb{Z}^d \setminus \{0\}} |\hat{V}(n)|^4 \langle n \rangle^{2M} = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |\hat{V}(n)|^2 \langle n \rangle^{2M} |\hat{V}(n)|^2$$

$$\leqslant \|\hat{V}\|^2_{l^\infty(\mathbb{Z}^d)} \|V\|^2_{H^M(\mathbb{T}^d)} < \infty.$$

So by Lemma 5.13

$$A_{\mathbb{Y}} \lesssim (t-s)^2 c(\eta)^{\frac{2}{1-\delta}} \sum_{\xi \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \langle n \rangle^{-2M} \langle n \rangle^{2d+4} |D_p^\beta \psi(\xi, p, \kappa)|^2,$$

which, for M large enough is

$$\lesssim (t-s)^2 c(\eta)^{\frac{2}{1-\delta}} \|D_p^\beta \psi(\cdot, p, \kappa)\|^2_{l^2_\xi}.$$

Hence

$$\sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \langle \kappa \rangle^m A_{\mathbb{Y}}^{1/2} \lesssim (t-s) c(\eta)^{\frac{1}{1-\delta}} \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \langle \kappa \rangle^m \|D_p^\beta \psi(\cdot, p, \kappa)\|_{l^2_\xi}.$$

Similarly, for the $B_{\mathbb{Y}}$ term, we have

$$B_{\mathbb{Y}} = \sum_{\xi \in \mathbb{Z}^d} \left| \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |\hat{V}(n)|^2 \Big(\varepsilon^{-1} \int_s^t \int_s^u e^{i a_1'(v-u)} \mathrm{d}v \mathrm{d}u \Big) D_p^\beta \psi(\xi, p, \kappa - n) \mathbb{I}_{n \perp \xi} \right|^2$$

$$\lesssim \sum_{\xi \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |\hat{V}(n)|^4 \langle n \rangle^{2M}$$

$$\sum_{n \in \mathbb{Z}^d \setminus \{0\}} \langle n \rangle^{-2M} \Big(\varepsilon^{-1} \int_s^t \int_s^u e^{i a_1'(v-u)} \mathrm{d}v \mathrm{d}u \Big)^2 |D_p^\beta \psi(\xi, p, \kappa - n)|^2 \mathbb{I}_{n \perp \xi}.$$

Using Lemma 5.13 and the fact that $V \in \cap_{M \geqslant 0} H^M(\mathbb{T}^d)$ this is

$$\lesssim (t-s)^2 c(\eta)^{\frac{2}{1-\delta}} \sum_{\xi \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \langle n \rangle^{-2M} \langle n \rangle^{2d+4} |D_p^\beta \psi(p, \xi, \kappa - n)|^2 \mathbb{I}_{n \perp \xi}.$$

Hence

$$\sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \langle \kappa \rangle^m B_{\mathbb{Y}}^{1/2} \lesssim (t-s) c(\eta)^{\frac{1}{1-\delta}} \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \langle n \rangle^{-M} \langle n \rangle^{d+2}$$

$$\langle \kappa \rangle^m \|D_p^\beta \psi(\cdot, p, \kappa - n)\|_{l^2_\xi}.$$

Using Tonelli's theorem twice the RHS is

$$= (t-s) c(\eta)^{\frac{1}{1-\delta}} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \langle n \rangle^{-M} \langle n \rangle^{d+2} \langle \kappa + n \rangle^m \|D_p^\beta \psi(\cdot, p, \kappa)\|_{l^2_\xi}$$

$$= (t-s) c(\eta)^{\frac{1}{1-\delta}} \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \|D_p^\beta \psi(\cdot, p, \kappa)\|_{l^2_\xi} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \langle n \rangle^{-M} \langle n \rangle^{d+2} \langle \kappa + n \rangle^m.$$

As we did in the estimate for $\mathbb{X}_{st}^{1,\varepsilon,*}$ this can be bounded by

$$\lesssim (t-s) c(\eta)^{\frac{1}{1-\delta}} \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \|D_p^\beta \psi(\cdot, p, \kappa)\|_{l^2_\xi} \langle \kappa \rangle^m.$$

Estimating the C and D terms analogously and integrating in p and summing over $\beta\colon |\beta|\leq m$ we have that

$$\|\mathbb{Y}_{st}^{\varepsilon,\eta,*}\psi\|_{E_m}\lesssim (t-s)c(\eta)^{\frac{1}{1-\delta}}\|\psi\|_{E_m}. \qquad \square$$

Lemma 5.18. *For all* $m\in\mathbb{N}_0, \eta\in\mathscr{A}_\eta, \gamma\in\left(\frac{1}{3},\frac{1}{2}\right), \psi\in E_m, 0\leq s<t, \delta$ *as in* (5.21) *one has*

$$\|\mathbb{Z}_{st}^{\varepsilon,\eta,*}\psi\|_{E_m}\lesssim \varepsilon^{1-2\gamma}|t-s|^{2\gamma}c(\eta)^{\frac{(2-2\gamma)}{1-\delta}}\|\psi\|_{E_m} \tag{5.43}$$

Proof. Recall

$$\mathbb{Z}_{st}^{\varepsilon,\eta,*}F(\xi,p,\kappa):=$$

$$-\varepsilon^{-1}\sum_{n,n'\in\mathbb{Z}^d\setminus\{0\}}\hat{V}(n)\hat{V}(n')\int_s^t\mathrm{d}u\int_s^u\mathrm{d}v e^{ia_1'v}e^{ib_1'u}F\left(\xi+n+n',p,\kappa-\frac{n}{2}-\frac{n'}{2}\right)\mathbb{I}_{a_1'\neq b_1'}$$

$$+\varepsilon^{-1}\sum_{n,n'\in\mathbb{Z}^d\setminus\{0\}}\hat{V}(n)\hat{V}(n')\int_s^t\mathrm{d}u\int_s^u\mathrm{d}v e^{ia_1'v}e^{ib_2'u}F\left(\xi+n+n',p,\kappa-\frac{n}{2}+\frac{n'}{2}\right)\mathbb{I}_{a_1'\neq b_2'}$$

$$+\varepsilon^{-1}\sum_{n,n'\in\mathbb{Z}^d\setminus\{0\}}\hat{V}(n)\hat{V}(n')\int_s^t\mathrm{d}u\int_s^u\mathrm{d}v e^{ia_2'v}e^{ib_3'u}F\left(\xi+n+n',p,\kappa+\frac{n}{2}-\frac{n'}{2}\right)\mathbb{I}_{a_2'\neq b_3'}$$

$$-\varepsilon^{-1}\sum_{n,n'\in\mathbb{Z}^d\setminus\{0\}}\hat{V}(n)\hat{V}(n')\int_s^t\mathrm{d}u\int_s^u\mathrm{d}v e^{ia_2'v}e^{ib_4'u}F\left(\xi+n+n',p,\kappa+\frac{n}{2}+\frac{n'}{2}\right)\mathbb{I}_{a_2'\neq b_4'}$$

where a_1' and a_2' were defined in (5.32) and (5.33), and b_1', b_2', b_3' and b_4' were defined in (5.34)-(5.37). As usual, we first compute

$$\|D_p^\beta\mathbb{Z}_{st}^{\varepsilon,\eta,*}\psi(\cdot,p,\kappa)\|_{l_\xi^2}=\left(\sum_{\xi\in\mathbb{Z}^d}|D_p^\beta\mathbb{Z}_{st}^{\varepsilon,\eta,*}\psi(\xi,p,\kappa)|^2\right)^{1/2}$$

We have that

$$\sum_{\xi\in\mathbb{Z}^d}|D_p^\beta\mathbb{Z}_{st}^{\varepsilon,\eta,*}\psi(\xi,p,\kappa)|^2\lesssim\sum_{\xi\in\mathbb{Z}^d}|D_p^\beta A_Z|^2+|D_p^\beta B_Z|^2+|D_p^\beta C_Z|^2+|D_p^\beta D_Z|^2$$

Hence

$$\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\|D_p^\beta\mathbb{Z}_{st}^{\varepsilon,*}\psi(\cdot,p,\kappa)\|_{l_\xi^2}$$

$$\lesssim\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\left(\|D_p^\beta A_Z\|_{l_\xi^2}+\|D_p^\beta B_Z\|_{l_\xi^2}+\|D_p^\beta C_Z\|_{l_\xi^2}+\|D_p^\beta D_Z\|_{l_\xi^2}\right).$$

We will estimate the first term, and the rest are estimated analogously. We have by using the Cauchy–Schwartz inequality that

$$\|D_p^\beta A_Z\|_{l_\xi^2}^2=\sum_{\xi\in\mathbb{Z}^d}|D_p^\beta A_Z|^2\lesssim\sum_{\xi\in\mathbb{Z}^d}\left(\sum_{n,n'\in\mathbb{Z}^d\setminus\{0\}}|\hat{V}(n)\hat{V}(n')|^2\langle n\rangle^{2M}\langle n'\rangle^{2M}\right)$$

$$\sum_{n,n'\in\mathbb{Z}^d\setminus\{0\}}(\langle n\rangle\langle n'\rangle)^{-2M}\left(\varepsilon^{-1}\int_s^t\int_s^u e^{ia_1'v}e^{ib_1'u}\mathrm{d}v\mathrm{d}u\right)^2$$

$$\left|D_p^\beta\psi\left(\xi+n+n_1,p,\kappa-\frac{n}{2}-\frac{n'}{2}\right)\right|^2\mathbb{I}_{n+n'\neq0}.$$

Since we have $V \in \cap_{M \geq 0} H^M(\mathbb{T}^d)$, one has that

$$\sum_{n,n' \in \mathbb{Z}^d \backslash \{0\}} |\hat{V}(n)\hat{V}(n')|^2 \langle n \rangle^{2M} \langle n' \rangle^{2M} = \|V\|^4_{H^M(\mathbb{T}^d)} < \infty.$$

So

$$\|D^\beta_p A_Z\|^2_{l^2_\xi} \lesssim \sum_{\xi \in \mathbb{Z}^d} \sum_{n,n' \in \mathbb{Z}^d \backslash \{0\}} (\langle n \rangle \langle n' \rangle)^{-2M} \left(\varepsilon^{-1} \int_s^t \int_s^u e^{ia_1' v} e^{ib_1' u} \mathrm{d}v \mathrm{d}u \right)^2$$

$$\left| D^\beta_p \psi \left(\xi + n + n', p, \kappa - \frac{n}{2} - \frac{n'}{2} \right) \right|^2 \mathbb{I}_{n+n' \neq 0}.$$

Due to the presence of $\mathbb{I}_{n+n' \neq 0}$, by Lemma 5.14 we have

$$\|D^\beta_p A_Z\|^2_{l^2_\xi} \lesssim \varepsilon^{2-4\gamma} |t-s|^{4\gamma} c(\eta)^{\frac{2 \cdot (2-2\gamma)}{1-\delta}}$$

$$\sum_{\xi \in \mathbb{Z}^d} \sum_{n,n' \in \mathbb{Z}^d \backslash \{0\}} (\langle n \rangle \langle n' \rangle)^{-2M+2(d+2)} \left| D^\beta_p \psi \left(\xi + n + n', p, \kappa - \frac{n}{2} - \frac{n'}{2} \right) \right|^2 \mathbb{I}_{n+n' \neq 0}$$

$$\lesssim \varepsilon^{2-4\gamma} |t-s|^{4\gamma} c(\eta)^{\frac{2 \cdot (2-2\gamma)}{1-\delta}}$$

$$\sum_{n,n' \in \mathbb{Z}^d \backslash \{0\}} (\langle n \rangle \langle n' \rangle)^{-2M+2(d+2)} \left\| D^\beta_p \psi \left(\cdot, p, \kappa - \frac{n}{2} - \frac{n'}{2} \right) \right\|^2_{l^2_\xi} \mathbb{I}_{n+n' \neq 0}.$$

This implies that

$$\sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \langle \kappa \rangle^m \|D^\beta_p A_Z\|_{l^2_\xi} \lesssim \varepsilon^{1-2\gamma} |t-s|^{2\gamma} c(\eta)^{\frac{(2-2\gamma)}{1-\delta}}$$

$$\sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \langle \kappa \rangle^m \sum_{n,n' \in \mathbb{Z}^d \backslash \{0\}} (\langle n \rangle \langle n' \rangle)^{-M+(d+2)} \left\| D^\beta_p \psi \left(\cdot, p, \kappa - \frac{n}{2} - \frac{n'}{2} \right) \right\|_{l^2_\xi} \mathbb{I}_{n+n' \neq 0}.$$

By using Tonelli's theorem twice we have that this is

$$\lesssim \varepsilon^{1-2\gamma} |t-s|^{2\gamma} c(\eta)^{\frac{(2-2\gamma)}{1-\delta}} \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \|D^\beta_p \psi(p, \xi, \kappa)\|_{l^2_\xi}$$

$$\sum_{n,n' \in \mathbb{Z}^d \backslash \{0\}} (\langle n \rangle \langle n' \rangle)^{-M+(d+2)} \left\langle \kappa + \frac{n}{2} + \frac{n'}{2} \right\rangle^m \mathbb{I}_{n+n' \neq 0},$$

which for M large enough is bounded by

$$\lesssim \varepsilon^{1-2\gamma} |t-s|^{2\gamma} c(\eta)^{\frac{(2-2\gamma)}{1-\delta}} \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \|D^\beta_p \psi(\cdot, p, \kappa)\|_{l^2_\xi} \langle \kappa \rangle^m.$$

By arguing similarly for the other terms B_Z, C_Z and D_Z one has that

$$\|\mathbb{Z}^{\varepsilon,\eta,*}_{st} \psi\|_{E_m} \lesssim \varepsilon^{1-2\gamma} |t-s|^{2\gamma} c(\eta)^{\frac{(2-2\gamma)}{1-\delta}} \|\psi\|_{E_m}.$$

□

Lemma 5.19. *$T^{\varepsilon,\eta,\natural}_{st}$ defined by equations (5.29), (5.30) and (5.31) is a bounded linear functional on E_2, and we have the naive bound*

$$\|T^{\varepsilon,\eta,\natural}_{st}\|_{E_{-2}} \lesssim \varepsilon^{-3/2} |t-s|^2. \tag{5.44}$$

Proof. The proof reduces to showing that for $m\in\{0,1\}$,

$$\|A^{\kappa-\eta,*}\|_{\mathscr{L}(E_{m+1}\to E_m)}\lesssim 1$$

and

$$\|Q_t^{\varepsilon,\eta,*}\|_{\mathscr{L}(E_m\to E_m)}\lesssim\varepsilon^{-1/2}$$

As an example, take the first term, for $F\in E_2$, one has

$$|\langle T_t^{\varepsilon,\eta},A^{\kappa-\eta,*}A^{\kappa-\eta,*}F\rangle|\leqslant\|T_t^{\varepsilon,\eta}\|_{E_{-0}}\|A^{\kappa-\eta,*}A^{\kappa-\eta,*}F\|_{E_0}$$

and

$$\|A^{\kappa-\eta,*}F\|_{E_0}\leqslant\|A^{\kappa-\eta,*}\|_{\mathscr{L}(E_2\to E_0)}\|A^{\kappa-\eta,*}\|_{\mathscr{L}(E_1\to E_0)}\|F\|_{E_1}\lesssim\|F\|_{E_1}\lesssim\|F\|_{E_2}<\infty$$

The bound for $A^{\kappa-\eta,*}$ follows from Lemma 5.41. We now compute for $Q_t^{\varepsilon,\eta,*}$: Let $m\in\mathbb{N}_0$, we then have

$$\|Q_t^{\varepsilon,\eta,*}F\|_{E_m}=\sum_{|\beta|\leqslant m}\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\left(\sum_{\xi\in\mathbb{Z}^d}|D_p^\beta Q_t^{\varepsilon,\eta,*}F(\xi,p,\kappa)|^2\right)^{1/2}.$$

$\sum_{\xi\in\mathbb{Z}^d}|D_p^\beta Q_t^{\varepsilon,\eta,*}F(\xi,p,\kappa)|^2$ can be naively estimated as

$$\begin{aligned}&\lesssim\varepsilon^{-1}\sum_{\xi\in\mathbb{Z}^d}\left|\sum_{n\in\mathbb{Z}^d}e^{4\pi^2i\varepsilon^{-1}n\cdot(2\kappa-2\eta-n-\xi)t}\hat{V}(n)D_p^\beta F\left(\xi+n,p,\kappa-\frac{n}{2}\right)\right|^2\\&+\varepsilon^{-1}\sum_{\xi\in\mathbb{Z}^d}\left|\sum_{n\in\mathbb{Z}^d}e^{4\pi^2i\varepsilon^{-1}n\cdot(2\kappa-2\eta+n+\xi)t}\hat{V}(n)D_p^\beta F\left(\xi+n,p,\kappa+\frac{n}{2}\right)\right|^2=c_1+c_2.\end{aligned}$$

We estimate just the first term by Cauchy–Schwartz, the second is estimated the same way.

$$\begin{aligned}c_1&\lesssim\varepsilon^{-1}\sum_{\xi\in\mathbb{Z}^d}\left(\sum_{n\in\mathbb{Z}^d}|\hat{V}(n)|^2\langle n\rangle^{2M}\right)\left(\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{-2M}(D_p^\beta F)^2\left(\xi+n,p,\kappa-\frac{n}{2}\right)\right)\\&\lesssim\varepsilon^{-1}\|V\|_{H^M(\mathbb{T}^d)}^2\sum_{\xi\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{-2M}(D_p^\beta F)^2\left(\xi+n,p,\kappa-\frac{n}{2}\right).\end{aligned}$$

By Tonelli's theorem this is

$$\begin{aligned}&\lesssim\varepsilon^{-1}\|V\|_{H^M(\mathbb{T}^d)}^2\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{-2M}\sum_{\xi\in\mathbb{Z}^d}(D_p^\beta F)^2\left(\xi+n,p,\kappa-\frac{n}{2}\right)\\&=\varepsilon^{-1}\|V\|_{H^M(\mathbb{T}^d)}^2\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{-2M}\left\|D_p^\beta F\left(\cdot,p,\kappa-\frac{n}{2}\right)\right\|_{l_\xi^2}^2.\end{aligned}$$

Using that the l^2-norm of a sequence is upper bounded by its l^1-norm,

$$\begin{aligned}&\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\left(\sum_{\xi\in\mathbb{Z}^d}|D_p^\beta Q_t^{\varepsilon,\eta,*}F(\xi,p,\kappa)|^2\right)^{1/2}\lesssim\varepsilon^{-1/2}\|V\|_{H^M(\mathbb{T}^d)}\\&\qquad\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{-M}\left\|D_p^\beta F\left(\cdot,p,\kappa-\frac{n}{2}\right)\right\|_{l_\xi^2}.\end{aligned}$$

Tonelli's theorem says this is

$$\lesssim \varepsilon^{-1/2}\|V\|_{H^M(\mathbb{T}^d)} \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} \|D_p^\beta F(\cdot,p,\kappa)\|_{l_\xi^2} \sum_{n\in\mathbb{Z}^d} \left\langle \kappa+\frac{n}{2}\right\rangle^m \langle n\rangle^{-M}$$

$$\lesssim \varepsilon^{-1/2}\|V\|_{H^M(\mathbb{T}^d)} \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} \langle\kappa\rangle^m \|D_p^\beta F(\cdot,p,\kappa)\|_{l_\xi^2}.$$

Hence

$$\|Q_t^{\varepsilon,\eta,*}F\|_{E_m} \lesssim \varepsilon^{-1/2}\|V\|_{H^M(\mathbb{T}^d)}\|F\|_{E_m}.$$

So we have the claim that

$$\|Q_t^{\varepsilon,\eta,*}\|_{\mathscr{L}(E_m\to E_m)} \lesssim \varepsilon^{-1/2}$$

Hence by chaining the above estimates together with the fact that each time integral gives a $|t-s|$ in the bound, we have proved the claim. □

5.2.4 Passing to the limit

At this stage, we have established that for almost every $\eta\in\left[-\frac{1}{4},\frac{1}{4}\right)^d$, there exists a family of solutions $\{T^{\varepsilon,\eta}\}_{\varepsilon\in(0,1)}$ uniformly bounded in $L^\infty([0,T];E_{-0})$ which satisfy for all $\psi\in E_2$:

$$\langle\delta T_{st}^{\varepsilon,\eta},\psi\rangle = \langle T_s^{\varepsilon,\eta},\mathbb{A}_{st}^{\kappa-\eta,*}\psi\rangle + \langle T_s^{\varepsilon,\eta},\mathbb{X}_{st}^{1,\varepsilon,\eta,*}\psi\rangle + \langle T_s^{\varepsilon,\eta},\mathbb{X}_{st}^{2,\varepsilon,\eta,*}\psi\rangle + \langle T_{st}^{\varepsilon,\eta,\natural},\psi\rangle,$$

where $T_{st}^{\varepsilon,\eta,\natural}$ is an E_{-2} valued 3γ-Hölder map which for any $\psi:\|\psi\|_{E_2}\leqslant 1$ has the property that $\langle T_{st}^{\varepsilon,\eta,\natural},\psi\rangle \leqslant \varepsilon^{-3/2}|t-s|^2$.

We now use the machinery of unbounded rough drivers introduced in [BG17] and [DGHT18] to obtain a bound on this remainder term that is uniform in ε, which will allow us to pass to the limit. This will use the uniform bounds on the drivers. Recall that from estimates (5.40)-(5.43) we have for $\eta\in\mathscr{A}_\eta$, uniformly in ε, and for any $m\in\mathbb{N}_0$ that

$$\begin{aligned}
\|\mathbb{X}_{st}^{1,\varepsilon,\eta,*}\psi\|_{E_m} &\lesssim |t-s|^\gamma\|\psi\|_{E_m},\\
\|\mathbb{A}_{st}^{\kappa-\eta,*}\psi\|_{E_m} &\lesssim |t-s|\|\psi\|_{E_{m+1}},\\
\|\mathbb{Z}_{st}^{\varepsilon,\eta,*}\psi\|_{E_m} &\lesssim |t-s|^{2\gamma}\|\psi\|_{E_m},\\
\|\mathbb{Y}_{st}^{\varepsilon,\eta,*}\psi\|_{E_m} &\lesssim |t-s|\|\psi\|_{E_m}.
\end{aligned}$$

In these bounds, the constant c in $\lesssim$ depends on η,δ,γ,m,V,d but not on ε. We then define for $\psi\in E_1$,

$$\langle T_{st}^{\varepsilon,\eta,\#},\psi\rangle := \langle\delta T_{st}^{\varepsilon,\eta},\psi\rangle - \langle T_s^{\varepsilon,\eta},\mathbb{X}_{st}^{1,\varepsilon,\eta,*}\psi\rangle.$$

For $\psi\in E_2$ we have that

$$\langle T_{st}^{\varepsilon,\eta,\#},\psi\rangle = \langle T_s^{\varepsilon,\eta},(\mathbb{A}_{st}^{\kappa-\eta,*}+\mathbb{X}_{st}^{2,\varepsilon,\eta,*})\psi\rangle + \langle T_{st}^{\varepsilon,\eta,\natural},\psi\rangle.$$

Next, define the smoothing operators, for $\nu\in(0,1)$:

$$\mathcal{J}_\nu\varphi(p,\kappa) := e^{-\nu^{1/2}\langle\kappa\rangle}(\varphi *_p \phi_\nu), \tag{5.45}$$

where

$$\phi_\nu(p) = \nu^{-d/2}\phi\left(\frac{p}{\nu^{1/2}}\right), \tag{5.46}$$

for some mollifier $\phi \in C_c^\infty(\mathbb{R}^d)$: $\int \mathrm{d}x \phi(x) = 1$. We have that for $(m,n) \in \{(1,1),(1,2),(2,2)\}$

$$\|\mathfrak{J}_\nu\|_{\mathscr{L}(E_m \to E_n)} \lesssim \nu^{-(n-m)}, \tag{5.47}$$

and

$$\|\mathfrak{J}_\nu - \mathrm{Id}\|_{\mathscr{L}(E_2 \to E_1)} \lesssim \nu^{1/2}. \tag{5.48}$$

See Lemmas 5.29 and 5.30 for proofs of these claims. Now, we can prove the following

Lemma 5.20. *Fix $T>0$. One has that for any $\varepsilon>0$, $\gamma \in \left(\frac{1}{3},\frac{1}{2}\right)$ a.e. $\eta \in \left[-\frac{1}{4},\frac{1}{4}\right)^d$ and $0 \leqslant s < t \leqslant T$ that*

$$\|T_{st}^{\varepsilon,\eta,\natural}\|_{E_{-2}} \lesssim C_{3\gamma} M |t-s|^{3\gamma} \tag{5.49}$$

uniformly in ε, where $C_{3\gamma}$ is the constant appearing in the sewing lemma, and $M := \|T^{\varepsilon,\eta}\|_{L^\infty([0,T];E_{-0})}$.

Remark 5.21. The core idea of the proof is contained in Lemma 2.29. Here we deal with the additional difficulty of working with an unbounded operator that moves us from one space in the scale E_m to another.

Proof. Assume $\|\psi\|_{E_2} \leqslant 1$. Let $0 < \frac{\min\{1,L\}}{2} \leqslant |I| \leqslant \min\{1,L\}$ where $L>0$ will be defined below in the argument. We compute the increments for the remainder term : for $0 \leqslant s < u < t \leqslant |I|$, we have

$$\langle \delta T_{sut}^{\varepsilon,\eta,\natural}, \psi \rangle = \langle T_{st}^{\varepsilon,\eta,\natural}, \psi \rangle - \langle T_{su}^{\varepsilon,\eta,\natural}, \psi \rangle - \langle T_{ut}^{\varepsilon,\eta,\natural}, \psi \rangle.$$

Using equation (5.28), this is

$$\begin{aligned}
=&\langle \delta T_{st}^{\varepsilon,\eta}, \psi \rangle - \langle T_s^{\varepsilon,\eta}, (\mathbb{A}_{st}^{\kappa-\eta,*} + \mathbb{X}_{st}^{1,\varepsilon,\eta,*} + \mathbb{X}_{st}^{2,\varepsilon,\eta,*})\psi \rangle \\
-&\langle \delta T_{su}^{\varepsilon,\eta}, \psi \rangle + \langle T_s^{\varepsilon,\eta}, (\mathbb{A}_{su}^{\kappa-\eta,*} + \mathbb{X}_{su}^{1,\varepsilon,\eta,*} + \mathbb{X}_{su}^{2,\varepsilon,\eta,*})\psi \rangle \\
-&\langle \delta T_{ut}^{\varepsilon,\eta}, \psi \rangle + \langle T_u^{\varepsilon,\eta}, \mathbb{A}_{ut}^{\kappa-\eta,*} + \mathbb{X}_{ut}^{1,\varepsilon,\eta,*} + \mathbb{X}_{ut}^{2,\varepsilon,\eta,*}\psi \rangle.
\end{aligned}$$

By adding and subtracting $\langle T_s^{\varepsilon,\eta}, (\mathbb{A}_{ut}^{\kappa-\eta,*} + \mathbb{X}_{ut}^{1,\varepsilon,\eta,*} + \mathbb{X}_{ut}^{2,\varepsilon,\eta,*})\psi \rangle$ and using the Chen relations, this is

$$= \langle \delta T_{su}^{\varepsilon}, (\mathbb{A}_{ut}^{\kappa-\eta,*} + \mathbb{X}_{ut}^{1,\varepsilon,\eta,*} + \mathbb{X}_{ut}^{2,\varepsilon,\eta,*})\psi \rangle - \langle T_s^{\varepsilon,\eta}, \mathbb{X}_{su}^{1,\eta,*}\mathbb{X}_{ut}^{1,\eta,*}\psi \rangle.$$

Adding and subtracting the smoothing operators from (5.45) to the transport term, this is

$$= \langle T_{su}^{\varepsilon,\eta,\sharp}, \mathbb{X}_{ut}^{1,\varepsilon,\eta,*}\psi \rangle + \langle \delta T_{su}^{\varepsilon,\eta}, (\mathbb{I} \pm \mathfrak{J}_\nu)\mathbb{A}_{ut}^{\kappa-\eta,*}\psi \rangle + \langle \delta T_{su}^{\varepsilon,\eta}, \mathbb{X}_{ut}^{2,\varepsilon,\eta,*}\psi \rangle.$$

Since $\mathbb{X}_{ut}^{1,\varepsilon,\eta,*}\psi \in E_2$ this is

$$\begin{aligned}
&= \langle T_s^{\varepsilon,\eta}, \mathbb{A}_{su}^{\kappa-\eta,*}\mathbb{X}_{ut}^{1,\varepsilon,\eta,*}\psi \rangle + \langle T_s^{\varepsilon,\eta}, \mathbb{X}_{su}^{2,\varepsilon,\eta,*}\mathbb{X}_{ut}^{1,\varepsilon,\eta,*}\psi \rangle + \langle T_{su}^{\varepsilon,\eta,\natural}, \mathbb{X}_{ut}^{1,\varepsilon,\eta,*}\psi \rangle \\
&+ \langle T_s^{\varepsilon,\eta}, (\mathbb{A}_{su}^{\kappa-\eta,*} + \mathbb{X}_{su}^{1,\varepsilon,\eta,*} + \mathbb{X}_{su}^{2,\varepsilon,\eta,*})\mathfrak{J}_\nu \mathbb{A}_{ut}^{\kappa-\eta,*}\psi \rangle + \langle T_{su}^{\varepsilon,\eta,\natural}, \mathfrak{J}_\nu \mathbb{A}_{ut}^{\kappa-\eta,*}\psi \rangle \\
&+ \langle \delta T_{su}^{\varepsilon,\eta}, (\mathbb{I} - \mathfrak{J}_\nu)\mathbb{A}_{ut}^{\kappa-\eta,*}\psi \rangle \\
&+ \langle T_s^{\varepsilon,\eta}, (\mathbb{A}_{su}^{\kappa-\eta,*} + \mathbb{X}_{su}^{1,\varepsilon,\eta,*} + \mathbb{X}_{su}^{2,\varepsilon,\eta,*})\mathbb{X}_{ut}^{2,\varepsilon,\eta,*}\psi \rangle + \langle T_{su}^{\varepsilon,\eta,\natural}, \mathbb{X}_{ut}^{2,\varepsilon,\eta,*}\psi \rangle \\
&= I_1 + \cdots + I_{12}.
\end{aligned}$$

We have that

$$|I_1|+|I_2|\leqslant M(\|\mathbb{A}^{\kappa-\eta,*}_{su}\mathbb{X}^{1,\varepsilon,\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_1)}+\|\mathbb{X}^{2,\varepsilon,\eta,*}_{su}\mathbb{X}^{1,\varepsilon,\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_2)}),$$

$$|I_4|+|I_5|\lesssim M(\|\mathbb{A}^{\kappa-\eta,*}_{su}\mathcal{J}_\nu\mathbb{A}^{\kappa-\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_0)}+\|\mathbb{X}^{1,\varepsilon,\eta,*}_{su}\mathcal{J}_\nu\mathbb{A}^{\kappa-\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_1)}),$$

$$|I_6|+|I_9|\lesssim M(\|\mathbb{X}^{2,\varepsilon,\eta,*}_{su}\mathcal{J}_\nu\mathbb{A}^{\kappa-\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_1)}+\|\mathbb{A}^{\kappa-\eta,*}_{su}\mathbb{X}^{2,\varepsilon,\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_1)}),$$

$$|I_{10}|+|I_{11}|\lesssim M(\|\mathbb{X}^{1,\varepsilon,\eta,*}_{su}\mathbb{X}^{2,\varepsilon,\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_2)}+\|\mathbb{X}^{2,\varepsilon,\eta,*}_{su}\mathbb{X}^{2,\varepsilon,\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_2)}),$$

$$|I_8|\lesssim M\|(\mathbb{I}-\mathcal{J}_\nu)\mathbb{A}^{\kappa-\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_0)},$$

$$|I_3|+|I_{12}|\lesssim\|T^{\varepsilon,\eta,\natural}_{su}\|_{E_{-2}}(\|\mathbb{X}^{1,\varepsilon,\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_2)}+\|\mathbb{X}^{2,\varepsilon,\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_2)}),$$

$$|I_7|\lesssim\|T^{\varepsilon,\eta,\natural}_{su}\|_{E_{-2}}\|\mathcal{J}_\nu\mathbb{A}^{\kappa-\eta,*}_{ut}\|_{\mathscr{L}(E_2\to E_2)}.$$

Using the ε-independent bounds on the drivers, and properties (5.47) and (5.48) of the smoothing operator $\mathcal{J}_\nu$, we have

$$|I_1|+|I_2|+|I_4|+|I_5|\lesssim M(|t-s|^{1+\gamma}+|t-s|^{3\gamma}+|t-s|^2+|t-s|^{1+\gamma}),$$

$$|I_6|+|I_9|+|I_{10}|+|I_{11}|\lesssim M(|t-s|^{1+2\gamma}+|t-s|^{1+2\gamma}+|t-s|^{3\gamma}+|t-s|^{4\gamma}),$$

$$|I_3|+|I_7|+|I_{12}|\lesssim\|T^{\varepsilon,\eta,\natural}_{su}\|_{E_{-2}}(|t-s|^{\gamma}+\nu^{-1}|t-s|+|t-s|^{2\gamma}),$$

$$|I_8|\lesssim M\nu^{1/2}|t-s|.$$

Pick $\nu=\frac{|t-s|^{2\gamma}}{|I|^{2\gamma}}\in(0,1)$, so $\nu^{-1}|t-s|=|I|^{2\gamma}|t-s|^{1-2\gamma}\leqslant|I|$, to get that

$$|I_1|+|I_2|+|I_4|+|I_5|+|I_6|+|I_8|+|I_9|+|I_{10}|+|I_{11}|\lesssim M|t-s|^{3\gamma},$$

$$|I_3|+|I_7|+|I_{12}|\lesssim\|T^{\varepsilon,\eta,\natural}_{su}\|_{E_{-2}}|I|^{\gamma}.$$

Hence

$$|\langle\delta T^{\varepsilon,\eta,\natural}_{sut},\psi\rangle|\lesssim M|t-s|^{3\gamma}+\|T^{\varepsilon,\eta,\natural}_{su}\|_{E_{-2}}|I|^{\gamma}.$$

Consider the germ $G^{\varepsilon,\eta,\psi}_{st}:=\langle T^{\varepsilon,\eta}_s,(\mathbb{A}^{\kappa-\eta,*}_{st}+\mathbb{X}^{1,\varepsilon,\eta,*}_{st}+\mathbb{X}^{2,\varepsilon,\eta,*}_{st})\psi\rangle$. One has that

$$\delta G^{\varepsilon,\eta,\psi}_{sut}=\langle\delta T^{\varepsilon,\eta}_{sut},\psi\rangle-\langle\delta T^{\varepsilon,\eta,\natural}_{sut},\psi\rangle=-\langle\delta T^{\varepsilon,\eta,\natural}_{sut},\psi\rangle$$

By what we have shown above, we have that for each $\psi\in E_2$, $G^{\varepsilon,\eta,\psi}_{sut}$ is a germ to which the sewing lemma can be applied. By the sewing lemma (see Lemma 2.26 and Remark 2.27), we have that

$$\|T^{\varepsilon,\eta,\natural}_{st}\|_{E_{-2}}\lesssim C_{3\gamma}(M|t-s|^{3\gamma}+\|T^{\varepsilon,\eta,\natural}_{su}\|_{E_{-2}}|I|^{\gamma}),$$

where $C_{3\gamma}$ is the constant appearing in Lemma 2.26 with $\beta=\gamma$. Now the $\lesssim$ means there is a constant $c_{\eta,\delta,\gamma,d,V}$ depending on V,γ,d,η but not on ε such that

$$\|T^{\varepsilon,\eta,\natural}_{st}\|_{E_{-2}}\leqslant c_{\eta,\delta,\gamma,d,V}C_{3\gamma}(M|t-s|^{3\gamma}+\|T^{\varepsilon,\eta,\natural}_{su}\|_{E_{-2}}|I|^{\gamma})$$

Choosing L such that $c_{\eta,\delta,\gamma,d,V}C_\gamma L^\gamma<\frac{1}{2}$ and recalling that I is such that $0<|I|\leqslant\min\{1,L\}$ one has

$$\|T^{\varepsilon,\eta,\natural}_{st}\|_{E_{-2}}\leqslant c_{\eta,\delta,\gamma,d,V}C_\gamma C_T|t-s|^{3\gamma}$$

Covering the interval $[0,T]$ with small intervals of the above size, we have the uniform bound for the remainder on $[0,T]$. This concludes the proof. □

Next, recall equations (5.23) and (5.24). We have the following

Lemma 5.22. *Let* $T>0$. *For* $\eta\in\mathscr{A}_\eta$, $\psi\in E_0$ *and* $0\leqslant s<t\leqslant T$ *it holds that*

$$\lim_{\varepsilon\longrightarrow 0}\|(\mathbb{Y}^{\varepsilon,\eta,*}_{st}-\mathbb{Y}^{\eta,*}_{st})\psi\|_{E_0}=0.$$

Proof. Recall that

$$\mathbb{Y}^{\varepsilon,\eta,*}_{st}\psi(\xi,p,\kappa):=$$

$$-\varepsilon^{-1}\sum_{n\in\mathbb{Z}^d\backslash\{0\}}|\hat{V}(n)|^2\int_s^t\mathrm{d}u\int_s^u\mathrm{d}v e^{ia_1'(v-u)}[\psi(\xi,p,\kappa)-\psi(\xi,p,\kappa-n)\mathbb{I}_{n\perp\xi}]$$

$$+\varepsilon^{-1}\sum_{n\in\mathbb{Z}^d\backslash\{0\}}|\hat{V}(n)|^2\int_s^t\mathrm{d}u\int_s^u\mathrm{d}v e^{ia_2'(v-u)}[\psi(\xi,p,\kappa+n)\mathbb{I}_{n\perp\xi}-\psi(\xi,p,\kappa)].$$

where $a_1'=4\pi^2\varepsilon^{-1}n\cdot(2\kappa-2\eta-n-\xi)$ and $a_2'=4\pi^2\varepsilon^{-1}n\cdot(2\kappa-2\eta+n+\xi)$. We have that for

$$\varepsilon^{-1}\int_s^t\mathrm{d}u\int_s^u\mathrm{d}v e^{ia_1'(v-u)}=\varepsilon^{-1}\int_s^t\mathrm{d}u\frac{1-e^{ia_1'(s-u)}}{ia_1'}=\varepsilon^{-1}\left(\frac{(t-s)}{ia_1'}+\frac{e^{ia_1'(s-t)}}{(ia_1')^2}-\frac{1}{(ia_1')^2}\right)$$

$$=\frac{(t-s)}{4\pi^2 in\cdot(2\kappa-2\eta-n-\xi)}+o(\varepsilon)$$

Hence in the limit as $\varepsilon\longrightarrow 0$ we have that the first term is

$$=\frac{i(t-s)}{4\pi^2}\sum_{n\in\mathbb{Z}^d\backslash\{0\}}|\hat{V}(n)|^2\frac{\psi(\xi,p,\kappa)-\psi(\xi,p,\kappa-n)\mathbb{I}_{n\perp\xi}}{n\cdot(2\kappa-2\eta-\xi-n)}$$

Computing the other terms similarly, we have (5.23) and (5.24). □

Finally, we are ready to prove Theorem 5.7.

Proof. (of Theorem 5.7) Recall the weak formulation of the rough equation, for $\psi\in E_2\colon\|\psi\|_{E_2}\leqslant 1$

$$\langle\delta T^{\varepsilon,\eta}_{st},\psi\rangle=\langle T^{\varepsilon,\eta}_s,\mathbb{A}^{\kappa-\eta,*}_{st}\psi\rangle+\langle T^{\varepsilon,\eta}_s,\mathbb{X}^{1,\varepsilon,\eta,*}_{st}\psi\rangle+\langle T^{\varepsilon,\eta}_s,\mathbb{X}^{2,\varepsilon,\eta,*}_{st}\psi\rangle+\langle T^{\varepsilon,\eta,\natural}_{st},\psi\rangle$$

one has for a.e. $\eta\in\left[-\frac{1}{4},\frac{1}{4}\right)^d$ that for $M:=\|T^{\varepsilon,\eta}\|_{L^\infty([0,T];E_{-0})}$

$$|\langle\delta T^{\varepsilon,\eta}_{st},\psi\rangle|\lesssim M(|t-s|+|t-s|^\gamma+|t-s|^{2\gamma})+\|T^{\varepsilon,\eta,\natural}_{st}\|_{E_{-2}}$$

which by the uniform a-priori bounds is

$$|\langle\delta T^{\varepsilon,\eta}_{st},\psi\rangle|\lesssim|t-s|^\gamma M+\|T^{\varepsilon,\eta,\natural}_{st}\|_{E_{-2}}\lesssim|t-s|^\gamma M+M|t-s|^{3\gamma}\lesssim M|t-s|^\gamma$$

uniformly in ε. This argument works for any subinterval of $[0,T]$ that is shorter than 1, and so it works for the entire interval $[0,T]$ by a covering argument. Hence, for any $\psi\in E_2\colon\|\psi\|_{E_2}\leqslant 1$, the family $\langle T^{\varepsilon,\eta}(t),\psi\rangle$ which form a uniformly bounded sequence of $\mathbb{R}$-valued γ-Hölder paths, are uniformly equicontinuous in time, which means by the theorem of Arzela–Ascoli and standard analysis arguments that there exists a subsequence which converges uniformly to some γ-Hölder real valued function on $[0,T]$.

Next, we note that by the uniform bound $\|T^{\varepsilon,\eta}(t)\|_{E_{-0}} \lesssim C_T^\eta = \|T^{\varepsilon,\eta}\|_{L^\infty([0,T];E_{-0})}$ and by the Banach–Alaoglu theorem that there exists a subsequence T^{ε_j} converging weakly-* to a function $T^\eta \in L^\infty([0,T];E_{-0})$. In particular, for any $\psi \in E_2, \langle T^{\varepsilon_j,\eta}, \psi\rangle \longrightarrow \langle T^\eta, \psi\rangle \in L^\infty([0,T];\mathbb{R})$ and by the fact that there exists a uniformly converging subsequence from the previous paragraph, the limit $\langle T^\eta(t), \psi\rangle$ is also a γ-Hölder path.
Now for $\psi \in E_2$, this allows passing to the limit (along the subsequence) in the term on the left hand side of the rough equation, and the first three terms on the right hand side, and this defines the term $\langle T_{st}^{\eta,\natural}, \psi\rangle$. We have

$$\langle T_s^{\varepsilon_j,\eta}, \mathbb{A}_{st}^{\kappa-\eta,*}\psi\rangle \longrightarrow \langle T_s^\eta, \mathbb{A}_{st}^*\psi\rangle,$$

$$|\langle T_s^{\varepsilon_j,\eta}, \mathbb{X}_{st}^{1,\varepsilon_j,\eta,*}\psi\rangle| \leqslant \|T_s^{\varepsilon_j,\eta}\|_{E_{-0}}\|\mathbb{X}_{st}^{1,\varepsilon_j,\eta,*}\psi\|_{E_0} \lesssim \varepsilon_j^{1/2-\gamma} \longrightarrow 0.$$

For the term

$$\langle T_s^{\varepsilon_j,\eta}, \mathbb{X}_{st}^{2,\varepsilon_j,*}\psi\rangle = \langle T_s^{\varepsilon_j,\eta}, \mathbb{Y}_{st}^{\varepsilon_j,\eta,*}\psi\rangle + \langle T_s^{\varepsilon_j,\eta}, \mathbb{Z}_{st}^{\varepsilon_j,\eta,*}\psi\rangle,$$

one has that

$$|\langle T_s^{\varepsilon_j,\eta}, \mathbb{Z}_{st}^{\varepsilon_j,\eta,*}\psi\rangle| \leqslant \|T_s^{\varepsilon_j,\eta}\|_{E_{-0}}\|\mathbb{Z}_{st}^{\varepsilon_j,\eta,*}\psi\|_{E_0} \lesssim \varepsilon_j^{1-2\gamma} \longrightarrow 0$$

Finally, consider

$$\langle T_s^{\varepsilon_j,\eta}, \mathbb{Y}_{st}^{\varepsilon_j,\eta,*}\psi\rangle = \langle T_s^{\varepsilon_j,\eta}, \mathbb{Y}_{st}^{\eta,*}\psi\rangle + \langle T_s^{\varepsilon_j,\eta}, (\mathbb{Y}_{st}^{\varepsilon_j,\eta,*} - \mathbb{Y}_{st}^{\eta,*})\psi\rangle.$$

We see that the first term

$$T_s^{\varepsilon_j,\eta}(\mathbb{Y}_{st}^{\eta,*}\psi) \longrightarrow T_s^\eta(\mathbb{Y}_{st}^{\eta,*}\psi).$$

The second term goes to 0, as a consequence of Lemma 5.22.
Hence one has

$$\langle T_{st}^{\eta,\natural}, \psi\rangle = \langle \delta T_{st}^\eta, \psi\rangle - \langle T_s^\eta, \mathbb{A}_{st}^{\kappa-\eta,*}\psi\rangle - \langle T_s^\eta, \mathbb{Y}_{st}^{\eta,*}\psi\rangle = \lim_{j\to\infty} T_{st}^{\varepsilon_j,\eta,\natural}(\psi),$$

and we have

$$|\langle T_{st}^{\eta,\natural}, \psi\rangle| \lesssim M\|\psi\|_{E_2}|t-s|^{3\gamma},$$

as a consequence of the fact that the a-priori bounds are uniform in ε. Hence we have the existence of an E_{-0} valued path $T(t)$ satisfying, for all $\psi \in E_2$, the rough difference equation

$$\langle \delta T_{st}^\eta, \psi\rangle = \langle T_s^\eta, \mathbb{A}_{st}^{\kappa-\eta,*}\psi\rangle + \langle T_s^\eta, \mathbb{Y}_{st}^{\eta,*}\psi\rangle + \langle T_{st}^{\eta,\natural}, \psi\rangle.$$

Now, since $\mathbb{A}_{st}^{\kappa-\eta,*}$ and $\mathbb{Y}_{st}^{\eta,*}$ are of order $|t-s|$, we deduce that

$$\langle T_T^\eta, \psi\rangle - \langle T_0^\eta, \psi\rangle = \int_0^T \mathrm{d}s\langle T_s^\eta, (A^{\kappa-\eta,*} + Y_{st}^{\eta,*})\psi\rangle,$$

where $Y^{\eta,*} = \frac{\mathbb{Y}_{st}^{\eta,*}}{(t-s)}$. In fact, for a.e. $t \in [0,T]$ it holds that

$$\partial_t T_t^\eta(\psi) = T_t^\eta(A^{\kappa-\eta,*}\psi) + T_t^\eta(Y^{\eta,*}\psi), \tag{5.50}$$

Finally, by the fact that every subsequence of equation (5.28) has a further subsequence converging to a limit that also satisfies equation (5.50), and by the uniqueness of solutions to this equation in E_{-0} (see Section 2.3), we have that the entire sequence $T^{\varepsilon,\eta}$ converges to T^η. This concludes the proof. □

5.3 Observables

This section contains estimates for the observables, after making a heuristic stationary phase argument to restrict to the term $\xi = 0$. When attempting to use the same strategy using the sewing lemma as in Section 5.2, we show in Subsection 5.3.1 that one can prove uniform in ε-estimates for the terms involving the non-resonant drivers from (5.28). In Subsection 5.3.2, we characterize the obstruction to the convergence of the resonant term, and make the connection to energy band crossings of the free Laplacian, which we encountered in the previous chapters.

Recall the problem of understanding the weak coupling limit of the observables: For $F \in \mathcal{S}(\mathbb{R}^{2d})$, consider

$$\int_{\mathbb{R}^{2d}} \mathrm{d}x \mathrm{d}k\, W^{\varepsilon}(t,x,k) F(x,k) = \int_{\mathbb{R}^{2d}} \mathrm{d}x \mathrm{d}k\, W\Big(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, k\Big) F(x,k).$$

Using equations (5.7) and (5.12), this is

$$= \int_{\mathbb{R}^d} \mathrm{d}x \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \tilde{W}_{\varphi}\Big(\frac{t}{\varepsilon}, \Big[\!\Big[\frac{x}{\varepsilon}\Big]\!\Big], \frac{x}{\varepsilon}, \eta, \kappa\Big) F(x, \kappa-\eta)$$

$$= \int_{\mathbb{R}^d} \mathrm{d}p \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \tilde{W}^{\varepsilon}_{\varphi}\Big(t, \frac{p}{\varepsilon}, p, \eta, \kappa\Big) F(p, \kappa-\eta).$$

Now using equations (5.15) and (5.16), this can be rewritten as

$$= \int_{\mathbb{R}^d} \mathrm{d}p \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} U^{\varepsilon}(t, \varepsilon^{-1}p - 4\pi\varepsilon^{-1}(\kappa-\eta)t, p, \eta, \kappa) F(p, \kappa-\eta)$$

$$= \int_{\mathbb{R}^d} \mathrm{d}p \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \sum_{\xi \in \mathbb{Z}^d} e^{2\pi i \varepsilon^{-1}\xi\cdot(p-4\pi(\kappa-\eta)t)} T^{\varepsilon}(t, \xi, p, \eta, \kappa) F(p, \kappa-\eta).$$

Consider the mode $\xi = 0$, which we expect to be the only one contributing to the limit by a stationary phase argument (we are being formal here). Hence we restrict to considering

$$O^{\varepsilon}_t = \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \langle T^{\varepsilon}_t, \mathbb{I}_{\xi=0} F\rangle = \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} T^{\varepsilon}(t, 0, p, \eta, \kappa) F(p, \kappa-\eta).$$

Then, equation (5.28) and the fact that $\mathbb{X}^{2,\varepsilon,\eta,*}_{st} = \mathbb{Y}^{\varepsilon,\eta,*}_{st} + \mathbb{Z}^{\varepsilon,\eta,*}_{st}$ gives that

$$\delta O^{\varepsilon}_{st} = \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \langle \delta T^{\varepsilon,\eta}_{st}, \mathbb{I}_{\xi=0} F\rangle$$

$$= \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \langle T^{\varepsilon,\eta}_{s}, A^{\kappa-\eta,*}_{st} \mathbb{I}_{\xi=0} F\rangle + \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \langle T^{\varepsilon,\eta}_{s}, \mathbb{X}^{1,\varepsilon,\eta,*}_{st} \mathbb{I}_{\xi=0} F\rangle$$

$$+ \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \langle T^{\varepsilon,\eta}_{s}, \mathbb{Y}^{\varepsilon,\eta,*}_{st} \mathbb{I}_{\xi=0} F\rangle + \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \langle T^{\varepsilon,\eta}_{s}, \mathbb{Z}^{\varepsilon,\eta,*}_{st} \mathbb{I}_{\xi=0} F\rangle$$

$$+ \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \langle T^{\varepsilon,\eta,\natural}_{st}, \mathbb{I}_{\xi=0} F\rangle.$$

We will now attempt to use the same strategy as the previous section - prove uniform in ε-estimates for the leading order terms, and a naive bound on the remainder. We will show that

$$\left|\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \langle T_s^{\varepsilon,\eta}, \mathbb{X}_{st}^{1,\varepsilon,\eta,*}\mathbb{I}_{\xi=0}F\rangle\right| \lesssim \varepsilon^{1/2-\gamma}|t-s|^{\gamma}\|T_s^{\varepsilon}\|_{L^{\infty}_{\eta,p,\kappa}l^2_{\xi}}\|F\|_{W^{1,1}_{p,k}(\mathbb{R}^{2d})}, \tag{5.51}$$

and

$$\left|\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \langle T_s^{\varepsilon,\eta}, \mathbb{Z}_{st}^{\varepsilon,\eta,*}\mathbb{I}_{\xi=0}F\rangle\right| \lesssim |t-s|^{2\gamma}\varepsilon^{1-2\gamma}\|T_s^{\varepsilon}\|_{L^{\infty}_{\eta,p,\kappa}l^2_{\xi}}\|F\|_{W^{1,1}_{p,k}(\mathbb{R}^{2d})}, \tag{5.52}$$

and that the terms with $A_{st}^{\kappa-\eta,*}$ and $\mathbb{Y}_{st}^{\varepsilon,\eta,*}$are uniformly bounded in ε, and have time regularity $|t-s|$, but do not decay as $\varepsilon\longrightarrow 0$. This is similar to what we did in the case of fixed η. However, we will see that the limit $\varepsilon\longrightarrow 0$ cannot be taken for the resonant term without having continuity in the η-variable for T^{ε} and any potential limit T. The transport term is the easiest to handle. We have that

$$\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \langle T_s^{\varepsilon}, A_{st}^{*}\mathbb{I}_{\xi=0}F\rangle = \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} \int_{\mathbb{R}^d} \mathrm{d}p\, T_s^{\varepsilon}(0,p,\eta,\kappa) A_{st}^{*}F(p,\kappa-\eta)$$

$$=4\pi(t-s)\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} \int_{\mathbb{R}^d} \mathrm{d}p\, T_s^{\varepsilon}(0,p,\eta,\kappa)(\kappa-\eta)\cdot\nabla_p F(p,\kappa-\eta).$$

Hence

$$\left|\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \langle T_s^{\varepsilon}, A_{st}^{*}\mathbb{I}_{\xi=0}F\rangle\right|$$

$$\lesssim |t-s|\|T_s^{\varepsilon}\|_{L^{\infty}_{\eta,p,\kappa}l^2_{\xi}} \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} \int_{\mathbb{R}^d} \mathrm{d}p\, |(\kappa-\eta)\cdot\nabla_p F(p,\kappa-\eta)|.$$

5.3.1 Uniform bounds on the non-resonant terms

In this section we prove the bounds in equations (5.51) and (5.52). First consider the term with $\mathbb{X}_{st}^{1,\varepsilon,*}$.

$$\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \langle T_s^{\varepsilon}, \mathbb{X}_{st}^{1,\varepsilon,*}\mathbb{I}_{\xi=0}F\rangle.$$

We will now see that this can be bounded uniformly in ε. This is

$$=\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\xi\in\mathbb{Z}^d} T_s^{\varepsilon}(\xi,p,\eta,\kappa)\mathbb{X}_{st}^{1,\varepsilon,*}(\mathbb{I}_{\xi=0}F(p,\kappa-\eta))$$

Since

$$=i\varepsilon^{-1/2}\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\xi\in\mathbb{Z}^d} T_s^{\varepsilon}(\xi,p,\eta,\kappa)$$

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}} \hat{V}(n)\int_s^t \mathrm{d}u\left[e^{ia_1'u}F\left(p,\kappa-\eta-\frac{n}{2}\right)-e^{ia_2'u}F\left(p,\kappa-\eta+\frac{n}{2}\right)\right]\mathbb{I}_{\xi+n=0},$$

where we recall that a_1' and a_2' are given by expressions (5.32) and (5.33). Simplifying, this is

$$= i\varepsilon^{-1/2}\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d}\mathrm{d}\eta \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\int_{\mathbb{R}^d}\mathrm{d}p \sum_{n\in\mathbb{Z}^d\setminus\{0\}}\hat{V}(n)\,T_s^\varepsilon(-n,p,\eta,\kappa)$$

$$\left[F\left(p,\kappa-\eta-\frac{n}{2}\right)-F\left(p,\kappa-\eta+\frac{n}{2}\right)\right]\int_s^t \mathrm{d}u e^{8\pi^2 i\varepsilon^{-1}n\cdot(\kappa-\eta)u}$$

Splitting terms, and taking absolute values, the first term can be bounded by

$$\leqslant \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d}\mathrm{d}\eta \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\int_{\mathbb{R}^d}\mathrm{d}p$$

$$\left|\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\hat{V}(n)\,T_s^\varepsilon(-n,p,\eta,\kappa)F\left(p,\kappa-\eta-\frac{n}{2}\right)\left(\varepsilon^{-1/2}\int_s^t \mathrm{d}u e^{8\pi^2 i\varepsilon^{-1}n\cdot(\kappa-\eta)u}\right)\right|.$$

Using the Hölder's inequality, this is

$$\leqslant \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d}\mathrm{d}\eta \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\int_{\mathbb{R}^d}\mathrm{d}p\left(\sup_{n\in\mathbb{Z}^d\setminus\{0\}}|\hat{V}(n)|\langle n\rangle^M\right)$$

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\langle n\rangle^{-M}|T_s^\varepsilon(-n,p,\eta,\kappa)|F\left(p,\kappa-\eta-\frac{n}{2}\right)\left(\varepsilon^{-1/2}\int_s^t \mathrm{d}u e^{8\pi^2 i\varepsilon^{-1}n\cdot(\kappa-\eta)u}\right)$$

$$\lesssim_V \|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa}l^\infty_\xi}\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d}\mathrm{d}\eta \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\int_{\mathbb{R}^d}\mathrm{d}p$$

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}}F\left(p,\kappa-\eta-\frac{n}{2}\right)\langle n\rangle^{-M}\left|\varepsilon^{-1/2}\int_s^t \mathrm{d}u e^{8\pi^2 i\varepsilon^{-1}n\cdot(\kappa-\eta)u}\right|.$$

By Lemma 5.12, this is

$$\lesssim_V \|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa}l^2_\xi}\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d}\mathrm{d}\eta \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\int_{\mathbb{R}^d}\mathrm{d}p$$

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\langle n\rangle^{-M}F\left(p,\kappa-\eta-\frac{n}{2}\right)c(\eta)^{\frac{1-\gamma}{1-\delta}}\varepsilon^{\frac{1}{2}-\gamma}|t-s|^\gamma\langle n\rangle^{(d+2)(1-\gamma)}.$$

By Tonelli's theorem, this is

$$\lesssim_V \varepsilon^{1/2-\gamma}|t-s|^\gamma\|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa}l^2_\xi}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\langle n\rangle^{-M}\langle n\rangle^{(d+2)(1-\gamma)}$$

$$\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\sup_{\eta\in\left[-\frac{1}{4},\frac{1}{4}\right]^d}F\left(p,\kappa-\eta-\frac{n}{2}\right)\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d}\mathrm{d}\eta\, c(\eta)^{\frac{1-\gamma}{1-\delta}}.$$

Now, choosing $\delta<\min\left\{\gamma,\frac{1}{d+3}\right\}$, the integral in η is finite, since $c(\eta)\in L^1$. The sum over n is finite choosing M large enough and the regularity of F says that the above expression is uniformly bounded in ε, i.e.,

$$\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d}\mathrm{d}\eta\langle T_s^\varepsilon,\mathbb{X}_{st}^{1,\varepsilon,*}\mathbb{I}_{\xi=0}F\rangle\lesssim_V \varepsilon^{1/2-\gamma}|t-s|^\gamma\|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa}l^2_\xi}\|F\|_{W^{1,1}_{p,k}(\mathbb{R}^{2d})}$$

This is equation (5.51). Next we consider the term with $\mathbb{Z}_{st}^{\varepsilon,\eta,*}$

$$\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \langle T_s^{\varepsilon,\eta}, \mathbb{Z}_{st}^{\varepsilon,\eta,*} \mathbb{I}_{\xi=0} F \rangle$$

$$= \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} T_s^{\varepsilon}(0,p,\eta,\kappa) \mathbb{Z}_{st}^{\varepsilon,\eta,*}(\mathbb{I}_{\xi=0} F(p,\kappa-\eta)),$$

where

$$\mathbb{Z}_{st}^{\varepsilon,\eta,*}(\mathbb{I}_{\xi=0} F(p,\kappa-\eta)) =$$

$$= -\varepsilon^{-1} \sum_{n,n' \in \mathbb{Z}^d \setminus \{0\}} \hat{V}(n) \hat{V}(n') \varphi_{st}(b_1', a_1') F\left(p, \kappa - \eta - \frac{n}{2} - \frac{n'}{2}\right) \mathbb{I}_{a_1' \neq b_1'} \mathbb{I}_{\xi+n+n'=0}$$

$$+ \varepsilon^{-1} \sum_{n,n' \in \mathbb{Z}^d \setminus \{0\}} \hat{V}(n) \hat{V}(n') \varphi_{st}(b_2', a_1') F\left(p, \kappa - \eta - \frac{n}{2} + \frac{n'}{2}\right) \mathbb{I}_{a_1' \neq b_2'} \mathbb{I}_{\xi+n+n'=0}$$

$$+ \varepsilon^{-1} \sum_{n,n' \in \mathbb{Z}^d \setminus \{0\}} \hat{V}(n) \hat{V}(n') \varphi_{st}(b_3', a_2') F\left(\kappa - \eta + \frac{n}{2} - \frac{n'}{2}\right) \mathbb{I}_{a_2' \neq b_3'} \mathbb{I}_{\xi+n+n'=0}$$

$$- \varepsilon^{-1} \sum_{n,n' \in \mathbb{Z}^d \setminus \{0\}} \hat{V}(n) \hat{V}(n') \varphi_{st}(b_4', a_2') F\left(\kappa - \eta + \frac{n}{2} + \frac{n'}{2}\right) \mathbb{I}_{a_2' \neq b_4'} \mathbb{I}_{\xi+n+n'=0}.$$

Recalling expressions (5.32)-(5.38), for

$$\alpha_1 = 4\pi^2 \varepsilon^{-1} n \cdot (2\kappa - 2\eta + n'), \qquad \alpha_2 = 4\pi^2 \varepsilon^{-1} n \cdot (2\kappa - 2\eta - n'),$$

$$\beta_1 = 4\pi^2 \varepsilon^{-1} n' \cdot (2\kappa - 2\eta - n), \qquad \beta_2 = 4\pi^2 \varepsilon^{-1} n' \cdot (2\kappa - 2\eta + n),$$

and

$$G_1(p,\kappa,\eta,n,n') = F\left(p, \kappa - \eta - \frac{n}{2} - \frac{n'}{2}\right) - F\left(p, \kappa - \eta - \frac{n}{2} + \frac{n'}{2}\right),$$

$$G_2(p,\kappa,\eta,n,n') = F\left(p, \kappa - \eta + \frac{n}{2} - \frac{n'}{2}\right) - F\left(p, \kappa - \eta + \frac{n}{2} - \frac{n'}{2}\right),$$

one has that

$$\mathbb{Z}_{st}^{\varepsilon,\eta,*}(\mathbb{I}_{\xi=0} F(p,\kappa-\eta)) =$$

$$= -\varepsilon^{-1} \sum_{n,n' \in \mathbb{Z}^d \setminus \{0\}} \hat{V}(n) \hat{V}(n') \varphi_{st}(\beta_1, \alpha_1) G_1(p,\kappa,\eta,n,n') \mathbb{I}_{\alpha_1 \neq \beta_1}$$

$$+ \varepsilon^{-1} \sum_{n,n' \in \mathbb{Z}^d \setminus \{0\}} \hat{V}(n) \hat{V}(n') \varphi_{st}(\beta_2, \alpha_2) G_2(p,\kappa,\eta,n,n') \mathbb{I}_{\alpha_2 \neq \beta_2}$$

$$= O_Z^1 + O_Z^2.$$

Both terms can be estimated the same way. We will show how to bound O_Z^1. Consider now

$$O_Z^1 = -\varepsilon^{-1} \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} T_s^{\varepsilon}(0,p,\eta,\kappa) \sum_{n,n' \in \mathbb{Z}^d \setminus \{0\}} \hat{V}(n) \hat{V}(n')$$

$$\varphi_{st}(\beta_1, \alpha_1) G_1(p,\kappa,\eta,n,n') \mathbb{I}_{\alpha_1 \neq \beta_1}.$$

So

$$|O_Z^1| \leqslant \varepsilon^{-1} \|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa} l^\infty_\xi} \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d}$$

$$\left| \sum_{n,n' \in \mathbb{Z}^d \backslash \{0\}} \hat{V}(n) \hat{V}(n') \varphi_{st}(\beta_1, \alpha_1) G_1(p, \kappa, \eta, n, n') \mathbb{I}_{\alpha_1 \neq \beta_1} \right|.$$

By Hölder's inequality, this is

$$\leqslant \varepsilon^{-1} \|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa} l^2_\xi} \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \left(\sup_n |\hat{V}(n)| \langle n \rangle^M \right)^2$$

$$\sum_{n,n' \in \mathbb{Z}^d \backslash \{0\}} \langle n \rangle^{-M} \langle n' \rangle^{-M} \varphi_{st}(\beta_1, \alpha_1) |G_1(p, \kappa, \eta, n, n')| \mathbb{I}_{\alpha_1 \neq \beta_1}$$

$$\lesssim \varepsilon^{-1} \|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa} l^2_\xi} \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \left(\sup_n |\hat{V}(n)| \langle n \rangle^M \right)^2$$

$$\sum_{n,n' \in \mathbb{Z}^d \backslash \{0\}} \langle n \rangle^{-M} \langle n' \rangle^{-M} \sup_\eta |G_1(p, \kappa, \eta, n, n')| \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \varphi_{st}(\beta_1, \alpha_1) \mathbb{I}_{\alpha_1 \neq \beta_1}$$

$$\lesssim_V \varepsilon^{-1} \|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa} l^2_\xi} \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \sup_{\eta \in \left[-\frac{1}{4},\frac{1}{4}\right]^d} F(p, \kappa - \eta)$$

$$\sum_{n,n' \in \mathbb{Z}^d \backslash \{0\}} \langle n \rangle^{-M} \langle n' \rangle^{-M} \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \varphi_{st}(\beta_1, \alpha_1) \mathbb{I}_{\alpha_1 \neq \beta_1}.$$

By Lemma 5.9 we have that

$$\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta |\varphi_{st}(\beta_1, \alpha_1)| \mathbb{I}_{\alpha_1 \neq \beta_1}$$

$$\lesssim \frac{|t-s|^{2\gamma}}{\varepsilon^{2\gamma-2}} \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \left(\frac{1}{|\beta_1|^{1-\gamma} |\alpha_1|^{1-\gamma}} + \frac{1}{|\alpha_1|^{1/2} |\beta_1|^{1-\gamma} |\beta_1 + \alpha_1|^{\frac{1-2\gamma}{2}}} \right) \mathbb{I}_{\alpha_1 \neq \beta_1}$$

$$+ \frac{|t-s|^{2\gamma}}{\varepsilon^{2\gamma-2}} \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \left(\frac{\mathbb{I}_{\alpha_1 \neq \beta_1}}{|\beta_1|^{1/2} |\alpha_1|^{1-\gamma} |\beta_1 + \alpha_1|^{\frac{1-2\gamma}{2}}} + \frac{\mathbb{I}_{\alpha_1 \neq \beta_1}}{|\alpha_1|^{1/2} |\beta_1|^{1/2} |\beta_1 + \alpha_1|^{1-2\gamma}} \right).$$

If we can show that each of the integrals in η can be bounded by $C(d,\gamma) \langle n \rangle^4 \langle n' \rangle^4$, we will have that

$$|O_Z^1| \lesssim |t-s|^{2\gamma} \varepsilon^{1-2\gamma} \|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa} l^2_\xi} \|F\|_{W^{1,1}_{p,k}(\mathbb{R}^{2d})},$$

which is uniformly bounded in ε, and vanishes as $\varepsilon \longrightarrow 0$. To show the bound on the integrals in η, we need to modify the argument used in Lemma 5.10, by noticing that each singular term is integrable when isolated from the others. This will be the content of the following

Lemma 5.23. *For $i\in\{1,2\}$, we have that for $\alpha+\beta+\sigma=2-2\gamma, 0<\alpha,\beta<1, 0\leqslant\sigma<1$ that*

$$\int_{[-\frac{1}{4},\frac{1}{4}]^d}\mathrm{d}\eta\frac{1}{|\beta_i|^{\alpha}|\alpha_i|^{\beta}|\beta_i+\alpha_i|^{\sigma}}\mathbb{I}_{\alpha_i\neq\beta_i}\lesssim_{d,\gamma}\langle n\rangle^4\langle n'\rangle^4$$

Proof. We will prove it for the case $i=1$, the case $i=2$ is the done the same way. Note that $\sigma<\alpha$ and $\sigma<\beta$ in each expression. Using the form of α_1 and β_1, this can be rewritten as

$$\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\eta\frac{1}{|n\cdot\eta-k_1|^{\alpha}|n'\cdot\eta-k_2|^{\beta}|n\cdot\eta-k_1+n'\cdot\eta-k_2|^{\sigma}}\mathbb{I}_{n\cdot\eta-k_1+n'\cdot\eta-k_2\neq0},$$

for some $k_1,k_2\in\mathbb{Z}$. We break this down into two cases, one where n,n' are collinear and one where they aren't. For the collinear case, assume without loss of generality that $n'=cn, c\in\mathbb{Z}\backslash\{0\}$. Hence $|c|\geqslant1$. Note also that in the collinear case, we cannot then have $k_2=ck_1$, since if we use the actual form of k_1 and k_2 from α_1,β_1 we would have that $k_2=ck_1$ would mean that

$$n'\cdot2\kappa-n'\cdot n=k_2\overset{!}{=}ck_1=c(2n\cdot\kappa+n\cdot n')\Longleftrightarrow2cn\cdot\kappa-cn\cdot n=2cn\cdot\kappa+c^2n\cdot n$$

$$\Longleftrightarrow c^2=-c\Longleftrightarrow c=-1.$$

But in this case $n\cdot\eta-k_1+n'\cdot\eta-k_2=n\cdot\eta-k_1-n'\cdot\eta+k_1=0$ which violates the non-resonance condition. Hence we have that $k_2\neq ck_1$ in the collinear case where $n'=cn$. Then writing $k_2=ck_1+l$ for some $l\in\mathbb{Z}\backslash\{0\}$ the above expression becomes

$$\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\eta\frac{1}{|n\cdot\eta-k_1|^{\alpha}|cn\cdot\eta-ck_1-l|^{\beta}|n\cdot\eta-k_1+cn\cdot\eta-ck_1-l|^{\sigma}}$$

$$=\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\eta\frac{1}{|n\cdot\eta-k_1|^{\alpha}|c|^{\beta}\left|n\cdot\eta-k_1-\frac{l}{c}\right|^{\beta}|n\cdot\eta-k_1+cn\cdot\eta-ck_1-l|^{\sigma}}$$

If $c=-1$ then $|n\cdot\eta-k_1+cn\cdot\eta-ck_1-l|^{\sigma}=|l|^{\sigma}$ and one adapts the steps we show below in the case $c\neq-1$. For $c\neq-1$, this is

$$=\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\eta\frac{1}{|n\cdot\eta-k_1|^{\alpha}|c|^{\beta}\left|n\cdot\eta-k_1-\frac{l}{c}\right|^{\beta}|1+c|^{\sigma}\left|n\cdot\eta-k_1-\frac{l}{1+c}\right|^{\sigma}}.$$

Let R be a rotation matrix such that $Rn=|n|e_1$. Then changing variables via $\eta'=R^T\eta$ this is

$$\leqslant\int_{B_{\sqrt{d}}(0)}\mathrm{d}\eta'\frac{1}{||n|\eta_1'-k_1|^{\alpha}|c|^{\beta}\left||n|\eta_1'-k_1-\frac{l}{c}\right|^{\beta}|1+c|^{\sigma}\left||n|\eta_1'-k_1-\frac{l}{1+c}\right|^{\sigma}}$$

$$\lesssim\int_{-\sqrt{d}}^{\sqrt{d}}\mathrm{d}\eta'\frac{1}{||n|\eta_1'-k_1|^{\alpha}|c|^{\beta}\left||n|\eta_1'-k_1-\frac{l}{c}\right|^{\beta}|1+c|^{\sigma}\left||n|\eta_1'-k_1-\frac{l}{1+c}\right|^{\sigma}}.$$

Changing variables once more $\eta = |n|\eta_1'$ and since $|c| \geqslant 1, |1+c| \geqslant 1$ this is

$$\lesssim \frac{1}{|n|}\int_{-|n|\sqrt{d}}^{|n|\sqrt{d}} \mathrm{d}\eta \frac{1}{|\eta - k_1|^{\alpha} \left|\eta - k_1 - \frac{l}{c}\right|^{\beta} \left|\eta - k_1 - \frac{l}{1+c}\right|^{\sigma}}.$$

We have singularities therefore at $\eta = k_1, \eta = k_1 + \frac{l}{c}, \eta = k_1 + \frac{l}{1+c}$. (In the case $c = -1$ this important point would still be true for the two denominator terms). Since $l \in \mathbb{Z} \backslash \{0\}$ the singularities never occur at the same point. We can therefore rewrite this as

$$\frac{1}{|n||c|^{\beta}|1+c|^{\sigma}}\int_{-|n|\sqrt{d}}^{|n|\sqrt{d}} \mathrm{d}\eta \frac{1}{|\eta - k_1|^{\alpha}| \eta - k_2|^{\beta}|\eta - k_3|^{\sigma}},$$

for $k_1 < k_2 < k_3$. We assume that k_1, k_2, k_3 lie in $\left[-|n|\sqrt{d}, |n|\sqrt{d}\right]$, if not we would have a constant which the term lying outside would be upper bounded by, and we could estimate the remaining integral as we will do below. Now let $r = \min\left\{1, \frac{k_2 - k_1}{2}, \frac{k_3 - k_2}{2}\right\}$ and let $S_{k_j} = [k_j - l, k_j + l] \cap \left[-|n|\sqrt{d}, |n|\sqrt{d}\right]$ for $j \in \{1, 2, 3\}$ and let $N = \left[-|n|\sqrt{d}, |n|\sqrt{d}\right] \backslash \cup_{j=1}^{3} S_{k_j}$. On N, each of the denominators is greater than $\frac{r}{2}$, so

$$\int_N \mathrm{d}\eta \frac{\mathrm{d}\eta}{|\eta - k_1|^{\alpha}| \eta - k_2|^{\beta}|\eta - k_3|^{\sigma}} \leqslant \frac{|N|}{r^{2-2\gamma}} \lesssim_{d,\gamma} \frac{|n|}{r^{2-2\gamma}}.$$

On any of the S_{k_j} for instance, S_{k_1} one has that

$$\int_{S_{k_1}} \mathrm{d}\eta \frac{\mathrm{d}\eta}{|\eta - k_1|^{\alpha}| \eta - k_2|^{\beta}|\eta - k_3|^{\sigma}} \lesssim \frac{1}{r^{\beta+\sigma}}\int_{S_{k_1}} \mathrm{d}\eta \frac{\mathrm{d}\eta}{|\eta - k_1|^{\alpha}} \lesssim \frac{1}{r^{2-2\gamma}}\int_{-1}^{1} \mathrm{d}\eta \frac{\mathrm{d}\eta}{|\eta|^{\alpha}} \lesssim \frac{1}{r^{2-2\gamma}}.$$

Hence overall we have that

$$\frac{1}{|n|}\int_{-|n|\sqrt{d}}^{|n|\sqrt{d}} \mathrm{d}\eta \frac{1}{|\eta - k_1|^{\alpha}| \eta - k_2|^{\beta}|\eta - k_3|^{\sigma}} \lesssim \frac{1}{r^{2-2\gamma}}.$$

Now plugging in that $k_2 - k_1 = \frac{l}{1+c}$ and $k_3 - k_2 = \frac{l}{c} - \frac{l}{1+c} = \frac{l}{c(1+c)}$ we have that for $c \notin \{0, 1\}$ that $r \geqslant \frac{1}{|c||1+c|}$ hence,

$$\frac{1}{|n|}\int_{-|n|\sqrt{d}}^{|n|\sqrt{d}} \mathrm{d}\eta \frac{1}{|\eta - k_1|^{\alpha}| \eta - k_2|^{\beta}|\eta - k_3|^{\sigma}} \lesssim |c|^{4-4\gamma}.$$

Finally using that $|c| = \frac{|n'|}{|n|} \leqslant |n'| \leqslant \langle n' \rangle$ we have that this is

$$\lesssim \langle n' \rangle^{4-4\gamma} \lesssim \langle n' \rangle^{4} \langle n \rangle^{4},$$

as claimed. We next consider the non-collinear case. Here, since $n' \neq cn$, we have that n, n' span a two dimensional plane $P_{n,n'}$. We define two orthogonal vectors on this plane via the Gram–Schmidt procedure

$$u_1 := \frac{n}{\|n\|}, \qquad u_2 := \frac{\tilde{n}}{\|\tilde{n}\|}, \tilde{n} = n' - (n' \cdot u_1) u_1.$$

We then create an orthogonal matrix R of the form

$$R = \begin{pmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_d^T \end{pmatrix}.$$

Defining $s = R\eta \iff \eta = R^T s = s_1 u_1 + s_2 u_2 + \sum_{j=3}^{d} s_j u_j$ we have that $n \cdot \eta = \|n\| u_1 \cdot \eta = \|n\| s_1$ and since $n' \in \operatorname{span}\{u_1, u_2\}, n' \cdot \eta = (n' \cdot u_1) s_1 + (n' \cdot u_2) s_2$. Then

$$\begin{aligned}
&\int_{[-\frac{1}{2},\frac{1}{2}]^d} \mathrm{d}\eta \frac{1}{|n\cdot\eta - k_1|^\alpha |n'\cdot\eta - k_2|^\beta |n\cdot\eta - k_1 + n'\cdot\eta - k_2|^\sigma} \\
&\leq \int_{B_{\sqrt{d}}(0)} \mathrm{d}s \frac{1}{|\|n\| s_1 - k_1|^\alpha |(n'\cdot u_1) s_1 + (n'\cdot u_2) s_2 - k_2|^\beta} \\
&\qquad \frac{1}{|\|n\| s_1 - k_1 + (n'\cdot u_1) s_1 + (n'\cdot u_2) s_2 - k_2|^\sigma} \\
&\lesssim \int_{B_{\sqrt{d}}(0)} \mathrm{d}s_2 \mathrm{d}s_1 \frac{1}{|\|n\| s_1 - k_1|^\alpha |(n'\cdot u_1) s_1 + (n'\cdot u_2) s_2 - k_2|^\beta} \\
&\qquad \frac{1}{|\|n\| s_1 - k_1 + (n'\cdot u_1) s_1 + (n'\cdot u_2) s_2 - k_2|^\sigma},
\end{aligned}$$

where now we are in a two dimensional ball. Let us define

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = C \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \qquad C = \begin{pmatrix} \|n\| & 0 \\ (n'\cdot u_1) & (n'\cdot u_2) \end{pmatrix}.$$

Then $\det C = \|n\| (n' \cdot u_2)$.

$$\begin{aligned}
n'\cdot u_2 &= n' \cdot \frac{\tilde{n}}{\|\tilde{n}\|} = n' \cdot \frac{n'}{\|\tilde{n}\|} - (n'\cdot u_1) \frac{(n'\cdot u_1)}{\|\tilde{n}\|} = \frac{\|n'\|^2}{\|\tilde{n}\|} - \frac{\|n'\cdot u_1\|^2}{\|\tilde{n}\|} \\
&= \frac{\|n'\|^2}{\|\tilde{n}\|} - \frac{\|n'\cdot n\|^2}{\|n\|^2 \|\tilde{n}\|} = \frac{\|n'\|^2}{\|\tilde{n}\|} - \frac{\|n'\|^2}{\|\tilde{n}\|} \cos^2(\theta) = \frac{\|n'\|^2}{\|\tilde{n}\|} \sin^2(\theta) = \|n'\| |\sin\theta|,
\end{aligned}$$

since by construction $\tilde{n} = n' \sin\theta \Longrightarrow \|\tilde{n}\| = \|n'\| |\sin\theta|$. Here θ is the angle between n and n' in P. Hence

$$\det C = \|n\| \|n'\| |\sin\theta|$$

Hence

$$\begin{aligned}
&\int_{B_{\sqrt{d}}(0)} \mathrm{d}s_2 \mathrm{d}s_1 \frac{1}{|\|n\| s_1 - k_1|^\alpha |(n'\cdot u_1) s_1 + (n'\cdot u_2) s_2 - k_2|^\beta} \\
&\qquad \frac{1}{|\|n\| s_1 - k_1 + (n'\cdot u_1) s_1 + (n'\cdot u_2) s_2 - k_2|^\sigma} \\
&\lesssim \frac{1}{\|n\| \|n'\| |\sin\theta|} \int_{C(B_{\sqrt{d}}(0))} \mathrm{d}v_2 \mathrm{d}v_1 \frac{1}{|v_1 - k_1|^\alpha |v_2 - k_2|^\beta |v_1 - k_1 + v_2 - k_2|^\sigma} \\
&\lesssim \frac{1}{\|n\| \|n'\| |\sin\theta|} \int_{B_{\sqrt{d}(|n| + |n'|)}(0)} \mathrm{d}v_2 \mathrm{d}v_1 \frac{1}{|v_1 - k_1|^\alpha |v_2 - k_2|^\beta |v_1 - k_1 + v_2 - k_2|^\sigma}.
\end{aligned}$$

We see that there are singularities at $v_1 = k_1, v_2 = k_2$, but if these singularities lie outside the ball they are harmless, since we then just need to estimate the volume of the ball in two dimensions and we are done. Hence we can bound the above by

$$\lesssim \frac{1}{\|n\|\|n'\||\sin\theta|} \int_{B_{\sqrt{d}(|n|+|n'|)}(0)} \mathrm{d}v_2 \mathrm{d}v_1 \frac{1}{|v_1|^{\alpha}|v_2|^{\beta}|v_1+v_2|^{\sigma}}.$$

Changing to polar coordinates and writing $R = \sqrt{d}(|n| + |n'|)$, this is

$$\lesssim \frac{1}{\|n\|\|n'\||\sin\theta|} \int_0^R \frac{r}{r^{2-2\gamma}} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \frac{1}{|\cos\theta|^{\alpha}|\sin\theta|^{\beta}|\cos\theta+\sin\theta|^{\sigma}}$$

$$= \frac{1}{\|n\|\|n'\||\sin\theta|} \frac{R^{2\gamma}}{2\gamma} \int_0^{2\pi} \frac{\mathrm{d}\theta}{|\cos\theta|^{1/2}|\sin\theta|^{1-\gamma}|\cos\theta+\sin\theta|^{\frac{1-2\gamma}{2}}}.$$

Now the problematic points in θ are when one of the denominators becomes 0. The important point is they cannot all be 0 at the same θ. $\cos\theta = 0$ at $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \frac{3\pi}{2}$. $\sin\theta = 0$ at $\theta_3 = 0, \theta_4 = \pi, \theta_5 = 2\pi$ and $\sin\theta + \cos\theta = 0$ at $\theta_6 = \frac{3\pi}{4}, \theta_7 = \frac{7\pi}{4}$. By picking $0 < \delta_\theta \ll 1$ and defining $S_{\theta_j} = (\theta_j + [-\delta, \delta]) \cap [0, 2\pi]$ we can decompose the integral onto each S_{θ_j} and $C \backslash \cup_{j=1}^{7} S_{\theta_j}$. On $C \backslash \cup_{j=1}^{7} S_{\theta_j}$ each of the denominators is bounded away from 0 and then the integral can be easily estimated. On each S_{θ_j}, there is only singular contribution. For instance, for S_{θ_1} we have that there exists a constant c such that $\frac{1}{|\sin\theta|}, \frac{1}{|\cos\theta+\sin\theta|} \leqslant c$. Then

$$\int_{S_{\theta_j}} \frac{\mathrm{d}\theta}{|\cos\theta|^{1/2}|\sin\theta|^{1-\gamma}|\cos\theta+\sin\theta|^{\frac{1-2\gamma}{2}}} \lesssim c^{\frac{3}{2}-2\gamma} \int_{\frac{\pi}{2}-\delta_\theta}^{\frac{\pi}{2}+\delta_\theta} \frac{\mathrm{d}\theta}{|\cos\theta|^{1/2}}$$

$$\lesssim c^{\frac{3}{2}-2\gamma} \int_0^{\delta_\theta} \frac{\mathrm{d}\theta}{\theta^{1/2}} \lesssim c^{\frac{3}{2}-2\gamma} \delta_\theta^{1-1/2} \lesssim c^{2-2\gamma} \delta_\theta^{1-\min\left\{\frac{1}{2}, 1-\gamma, \frac{1}{2}-\gamma\right\}},$$

and this constant c only depends on δ_θ. Hence, overall the term is bounded by

$$\lesssim \frac{R^{2\gamma}}{\|n\|\|n'\||\sin\theta|} \lesssim \frac{\|n\|^{2\gamma-1} + \|n'\|^{2\gamma-1}}{|\sin\theta|} \lesssim \|n\|\|n'\|(\|n\|^{2\gamma-1} + \|n'\|^{2\gamma-1})$$

$$\lesssim \|n\|^{2\gamma}\|n'\| + \|n\|\|n'\|^{2\gamma} \lesssim \|n\|\|n'\| \lesssim \langle n \rangle^4 \langle n' \rangle^4.$$

Hence we can conclude. □

5.3.2 The resonant term

Next consider

$$\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \langle T_s^{\varepsilon,\eta}, \mathbb{Y}_{st}^{\varepsilon,\eta,*} \mathbb{I}_{\xi=0} F \rangle$$

$$= \int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} T_s^{\varepsilon}(0, p, \eta, \kappa) \mathbb{Y}_{st}^{\varepsilon,\eta,*}(\mathbb{I}_{\xi=0} F(p, \kappa - \eta)).$$

We recall that

$$\mathbb{Y}_{st}^{\varepsilon,\eta,*} f(\xi,p,\kappa) =$$

$$-\varepsilon^{-1} \sum_{n\in\mathbb{Z}^d\backslash\{0\}} |\hat{V}(n)|^2 \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v e^{ia_1'(v-u)} [f(\xi,p,\kappa) - f(\xi,p,\kappa-n)\mathbb{I}_{n\perp\xi}]$$

$$+\varepsilon^{-1} \sum_{n\in\mathbb{Z}^d\backslash\{0\}} |\hat{V}(n)|^2 \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v e^{ia_2'(v-u)} [f(\xi,p,\kappa+n)\mathbb{I}_{n\perp\xi} - f(\xi,p,\kappa)].$$

So, the expression concerned becomes

$$-\varepsilon^{-1} \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} T_s^\varepsilon(0,p,\eta,\kappa) \sum_{n\in\mathbb{Z}^d\backslash\{0\}} |\hat{V}(n)|^2 \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v e^{ic_1(v-u)}$$

$$[F(p,\kappa-\eta) - F(p,\kappa-\eta-n)]$$

$$+\varepsilon^{-1} \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} T_s^\varepsilon(0,p,\eta,\kappa) \sum_{n\in\mathbb{Z}^d\backslash\{0\}} |\hat{V}(n)|^2 \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v e^{ic_2(v-u)}$$

$$[F(p,\kappa-\eta+n) - F(p,\kappa-\eta)],$$

where now

$$c_1 = 4\pi^2\varepsilon^{-1} n\cdot(2\kappa-2\eta-n), \qquad c_2 = 4\pi^2\varepsilon^{-1} n\cdot(2\kappa-2\eta+n).$$

Changing variables in the second expression, this is

$$=-\varepsilon^{-1} \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} T_s^\varepsilon(0,p,\eta,\kappa) \sum_{n\in\mathbb{Z}^d\backslash\{0\}} |\hat{V}(n)|^2 \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v e^{ic_1(v-u)}$$

$$[F(p,\kappa-\eta) - F(p,\kappa-\eta-n)]$$

$$-\varepsilon^{-1} \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} T_s^\varepsilon(0,p,\eta,\kappa) \sum_{n\in\mathbb{Z}^d\backslash\{0\}} |\hat{V}(-n)|^2 \int_s^t \mathrm{d}u \int_s^u \mathrm{d}v e^{-ic_1(v-u)}$$

$$[F(p,\kappa-\eta) - F(p,\kappa-\eta-n)].$$

Recalling that $|\hat{V}(n)|^2 = |\hat{V}(-n)|^2$, and using that

$$e^{ic_1(v-u)} + e^{-ic_1(v-u)} = 2\cos(c_1(v-u)),$$

this simplifies to

$$=2 \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} T_s^\varepsilon(0,p,\eta,\kappa) \sum_{n\in\mathbb{Z}^d\backslash\{0\}} |\hat{V}(n)|^2$$

$$\int_s^t \mathrm{d}u \int_s^u \mathrm{d}v (\varepsilon^{-1}\cos(c_1(v-u))) [F(p,\kappa-\eta-n) - F(p,\kappa-\eta)].$$

We next perform the time integration:

$$\varepsilon^{-1}\int_s^t \mathrm{d}u \int_s^u \mathrm{d}v \cos(c_1(v-u)) = -\frac{\varepsilon^{-1}}{c_1}\int_s^t \mathrm{d}u \sin(c_1(s-u)) = \frac{\varepsilon^{-1}}{c_1}\int_s^t \mathrm{d}u \sin(c_1(u-s))$$

$$= -\frac{\varepsilon^{-1}}{c_1^2}(\cos(c_1(t-s)) - 1)$$

$$= \frac{\varepsilon^{-1}}{|4\pi^2\varepsilon^{-1}n\cdot(2\kappa-2\eta-n)|^2}(1-\cos(4\pi^2\varepsilon^{-1}n\cdot(2\kappa-2\eta-n)(t-s))).$$

A computation shows that this is an approximation of the delta function on the hyperplane $n\cdot(2\kappa-2\eta-n)=0$, up to a constant factor. We can show the concentration on this set as follows: Let $H_{n,k,\eta}(r)=\left\{(n,k,\eta)\in\mathbb{Z}^d\backslash\{0\}\times\left(\frac{\mathbb{Z}}{2}\right)^d\times\left(-\frac{1}{4},\frac{1}{4}\right)^d : |n\cdot(2\kappa-2\eta-n)|\leq r\right\}$. Fix $r>0$. Let $c_1'=\varepsilon c_1$. From the above computations we have that

$$\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d}\mathrm{d}\eta\langle T_s^{\varepsilon,\eta}, \Upsilon_{st}^{\varepsilon,\eta,*}\mathbb{I}_{\xi=0}F\rangle$$

$$=2(t-s)\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d}\mathrm{d}\eta\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}T_s^{\varepsilon}(0,p,\eta,\kappa)\sum_{n\in\mathbb{Z}^d\backslash\{0\}}|\hat{V}(n)|^2$$

$$\left(\varepsilon^{-1}(t-s)\frac{1-\cos(\varepsilon^{-1}c_1'(t-s))}{|\varepsilon^{-1}c_1'(t-s)|^2}\right)[F(p,\kappa-\eta-n)-F(p,\kappa-\eta)]$$

$$(\mathbb{I}_{H_{n,k,\eta}(r)}+\mathbb{I}_{H_{n,k,\eta}^c(r)}).$$

For the term containing $\mathbb{I}_{H_{n,k,\eta}^c(r)}$, we have that $c_1'>\frac{r}{4\pi^2}$. Hence, we can bound that term in absolute value by

$$\lesssim\frac{\varepsilon}{r}\|T_s^{\varepsilon}\|_{L^\infty_{\eta,p,\kappa,z}}\sum_{n\in\mathbb{Z}^d\backslash\{0\}}|\hat{V}(n)|^2\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d}\mathrm{d}\eta\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}|F(p,\kappa-\eta)-F(p,\kappa-\eta-n)|$$

$$\lesssim\frac{\varepsilon}{r}\|T_s^{\varepsilon}\|_{L^\infty_{\eta,p,\kappa,z}}\|F\|_{L^1(\mathbb{R}^{2d})},$$

which for any positive r, in the limit $\varepsilon\longrightarrow 0$ is 0.

However, the difficulty in concluding is that on the zero measure set, we do not know continuity of T^{ε} in η, hence we do not know yet if the limit exists.

Remark 5.24. Note that using the notation from Chapter 3

$$4\pi^2 n\cdot(2\kappa-2\eta-n)=4\pi^2(2n\cdot(\kappa-\eta)-|n|^2)=4\pi^2|\kappa-\eta|^2-4\pi^2|\kappa-\eta-n|^2$$

$$=E_{\kappa}(\eta)-E_{\kappa-n}(\eta).$$

So we see how the crossing of energy bands appears naturally in the resonant term. When working with the Bloch–Wigner transform, the problem is no longer in the definition of the derivative of the energy bands, as was the case when working with the Wigner series, it is rather a matter of knowing that there is sufficient regularity at certain momenta of the Bloch–Wigner in order to pass to the limit.

We have the following image in $d=2$, of what the problematic zero-measure set looks like

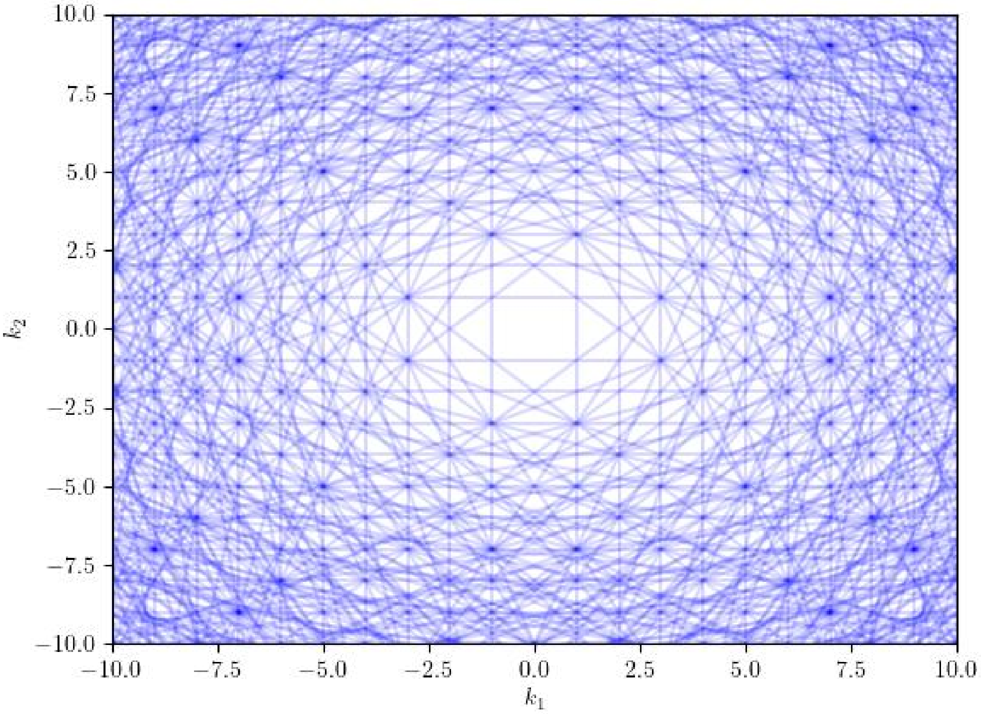


Figure 5.1. The set of resonant momenta in $\mathbb{R}^2$ for a generic smooth periodic potential.

The image is generated purely with straight lines as follows: Since $2\kappa - 2\eta$ are representations for a momenta $k \in \mathbb{R}^d$, we are looking for momenta $k \in \mathbb{R}^d$ such that there exists a non-zero lattice vector $n \in \mathbb{Z}^d \backslash \{0\}$ for which $n \cdot (\kappa - n) = 0$. The diagram is generated by looping over lattice vectors n and drawing a line perpendicular to the line segment $[0, n]$, passing through n. Any vector k on this line has the property that $n \cdot k = |n|^2$.

Example 5.25. (Single mode potential) Let's demonstrate what the resonant term looks like when the potential has only one mode. Assume and $\hat{V}(n) = 0$ unless $n = \binom{\sigma}{0}$

for $\sigma \in \{\pm 1\}$. Then we have an expression of the form

$$\int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \sum_{\sigma \in \{\pm 1\}} T_s^\varepsilon(0,p,\eta,\kappa)\,\delta_\varepsilon(2\kappa_1 - 2\eta_1 - \sigma)$$

$$\left[F\left(p, \kappa - \eta - \begin{pmatrix} \sigma \\ 0 \end{pmatrix}\right) - F(p, \kappa - \eta)\right].$$

If we assume $\operatorname{supp} F \subset S_1 \times \mathbb{R}^{d-1}$, where $S_1 = \left(\left[-\frac{1}{2} - \rho, -\frac{1}{2} + \rho\right] \cup \left[\frac{1}{2} - \rho, \frac{1}{2} + \rho\right]\right)^c$, for some $\rho \ll 1$, then the systems

$$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} \sigma \\ d \end{pmatrix}, \quad \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} \sigma \\ d + \sigma \end{pmatrix}$$

have no solution for any $d \in S_1$. One has that for any $\varepsilon' > 0, \sigma \in \{\pm 1\}$, there exists $\varepsilon > 0$ such $\int_{[\frac{\sigma}{2} - \rho, \frac{\sigma}{2} + \rho]^c} \delta_\varepsilon(2\kappa_1 - 2\eta_1 - \sigma) < \varepsilon'$. Then the above term is bounded in absolute value by

$$\lesssim \|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa} L^2_\xi} \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \sum_{\sigma \in \{\pm 1\}} \delta_\varepsilon(2\kappa_1 - 2\eta_1 - \sigma)$$

$$\left|F\left(p, \kappa - \eta - \begin{pmatrix} \sigma \\ 0 \end{pmatrix}\right) - F(p, \kappa - \eta)\right| \left(\mathbb{I}_{\kappa_1 - \eta_1 \in [\frac{\sigma}{2} - \rho, \frac{\sigma}{2} + \rho]} + \mathbb{I}_{\kappa_1 - \eta_1 \notin [\frac{\sigma}{2} - \rho, \frac{\sigma}{2} + \rho]}\right).$$

When $\kappa_1 - \eta_1 \notin [\frac{\sigma}{2} - \rho, \frac{\sigma}{2} + \rho]$, one can bound this by

$$\lesssim \|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa} L^2_\xi} \|F\|_{L^1_p L^\infty_k} \int_{[\frac{\sigma}{2} - \rho, \frac{\sigma}{2} + \rho]^c} \delta_\varepsilon(2\kappa_1 - 2\eta_1 - \sigma)$$

$$\lesssim \varepsilon' \|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa} L^2_\xi} \|F\|_{L^1_p L^\infty_k}.$$

Since ε' was arbitrary, this part is 0. On the other hand, for the term

$$\|T_s^\varepsilon\|_{L^\infty_{\eta,p,\kappa} L^2_\xi} \int_{[-\frac{1}{4},\frac{1}{4}]^d} \mathrm{d}\eta \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \sum_{\sigma \in \{\pm 1\}} \delta_\varepsilon(2\kappa_1 - 2\eta_1 - \sigma)$$

$$\left|F\left(p, \kappa - \eta - \begin{pmatrix} \sigma \\ 0 \end{pmatrix}\right) - F(p, \kappa - \eta)\right| \mathbb{I}_{\kappa_1 - \eta_1 \in [\frac{\sigma}{2} - \rho, \frac{\sigma}{2} + \rho]},$$

one has that $\kappa_1 - \eta_1 \in [\frac{\sigma}{2} - \rho, \frac{\sigma}{2} + \rho]$ hence both $\kappa_1 - \eta_1 \in S_1^c$ and $\kappa_1 - \eta_1 - \sigma \in S_1^c$. Hence $F\left(p, \kappa - \eta - \begin{pmatrix} \sigma \\ 0 \end{pmatrix}\right) = F(p, \kappa - \eta) = 0$ and also this part is 0. Hence, for any $F \in C_c^\infty(S_1 \times \mathbb{R}^{d-1})$, the entire term is zero, and we expect only trivial transport in the limit for its corresponding observable. For this potential, in the picture below, the problematic zero-measure set reduces to the two red vertical lines passing through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ respectively.

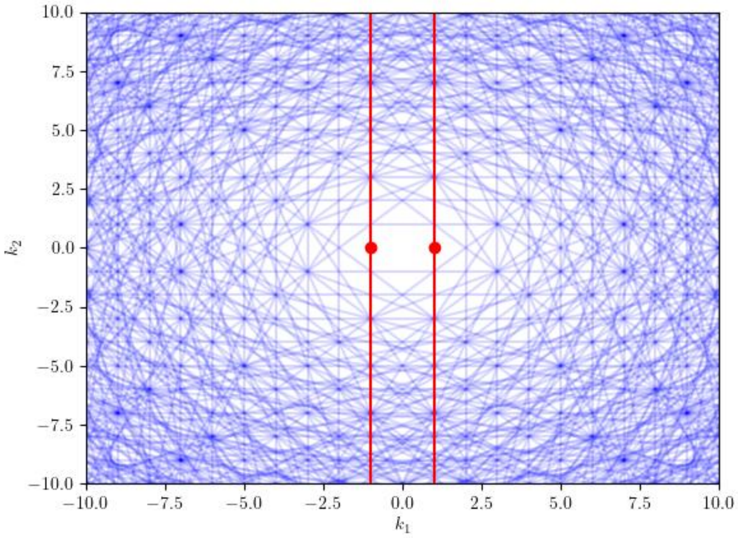


Figure 5.2. The resonant set of momenta for a single mode potential (in red).

5.4 Proofs of auxiliary results

5.4.1 Proof of Lemma 5.2

In the proof of Lemma 5.2 below, we will use several properties of the BFZ transform that we have listed in Appendix 2.1.

Proof. For $x, k \in \mathbb{R}^d, \varphi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$,

$$W_\varphi(x,k) = \int_{\mathbb{R}^d} \mathrm{d}y e^{2\pi i k\cdot y} \varphi\Big(x - \frac{y}{2}\Big) \varphi^*\Big(x + \frac{y}{2}\Big).$$

We now regularize this expression. This is

$$= \int_{\mathbb{R}^d} \mathrm{d}y \lim_{\varepsilon \to 0} e^{2\pi i k\cdot y} e^{-\pi\varepsilon|y|^2} \varphi\Big(x - \frac{y}{2}\Big) \varphi^*\Big(x + \frac{y}{2}\Big).$$

By the dominated convergence theorem, this is

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \mathrm{d}y e^{2\pi i k\cdot y} e^{-\pi\varepsilon|y|^2} \varphi\Big(x - \frac{y}{2}\Big) \varphi^*\Big(x + \frac{y}{2}\Big).$$

Plugging in expression (2.3) and its complex conjugate into this equation, this is

$$
\begin{gathered}
= \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{T}^d} \mathrm{d}\theta \int_{\mathbb{T}^d} \mathrm{d}\theta' e^{2\pi i k \cdot y} e^{-\pi\varepsilon|y|^2} \\
e^{-2\pi i\theta \cdot (x - \frac{y}{2})} \tilde{\varphi}\Big(\theta, x - \frac{y}{2}\Big) e^{2\pi i \theta' \cdot (x + \frac{y}{2})} \tilde{\varphi}^*\Big(\theta', x + \frac{y}{2}\Big) \\
= \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{T}^d} \mathrm{d}\theta \int_{\mathbb{T}^d} \mathrm{d}\theta' e^{2\pi i \left(k + \frac{\theta}{2} + \frac{\theta'}{2}\right) \cdot y} e^{-\pi\varepsilon|y|^2} e^{-2\pi i(\theta - \theta') \cdot x} \tilde{\varphi}\Big(\theta, x - \frac{y}{2}\Big) \tilde{\varphi}^*\Big(\theta', x + \frac{y}{2}\Big).
\end{gathered}
$$

We decompose $\mathbb{R}^d \ni y = y + m \in [-1, 1]^d + (2\mathbb{Z})^d$ to get that this is

$$
\begin{gathered}
= \lim_{\varepsilon \longrightarrow 0} \int_{[-1,1]^d} \mathrm{d}y \sum_{m \in (2\mathbb{Z})^d} \int_{\mathbb{T}^d} \mathrm{d}\theta \int_{\mathbb{T}^d} \mathrm{d}\theta' e^{2\pi i \left(k + \frac{\theta}{2} + \frac{\theta'}{2}\right) \cdot (y + m)} e^{-\pi\varepsilon|y + m|^2} e^{-2\pi i(\theta - \theta') \cdot x} \\
\tilde{\varphi}\Big(\theta, x - \frac{y}{2} - \frac{m}{2}\Big) \tilde{\varphi}^*\Big(\theta', x + \frac{y}{2} + \frac{m}{2}\Big),
\end{gathered}
$$

which, by the $\mathbb{Z}^d$-periodicity of the BFZ transform in the second variable, and a change of variables, is

$$
\begin{gathered}
= \lim_{\varepsilon \longrightarrow 0} \int_{\left[-\frac{1}{2}, \frac{1}{2}\right]^d} \mathrm{d}y \sum_{m \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \mathrm{d}\theta \int_{\mathbb{T}^d} \mathrm{d}\theta' e^{2\pi i \left(k + \frac{\theta}{2} + \frac{\theta'}{2}\right) \cdot (2y + 2m)} e^{-\pi\varepsilon|2y + 2m|^2} e^{-2\pi i(\theta - \theta') \cdot x} \\
\tilde{\varphi}(\theta, x - y) \tilde{\varphi}^*(\theta', x + y).
\end{gathered}
$$

Since we introduced the regularization, Fubini's theorem says that this is

$$
\begin{gathered}
= \lim_{\varepsilon \longrightarrow 0} \int_{\left[-\frac{1}{2}, \frac{1}{2}\right]^d} \mathrm{d}y \int_{\mathbb{T}^d} \mathrm{d}\theta \int_{\mathbb{T}^d} \mathrm{d}\theta' \sum_{m \in \mathbb{Z}^d} e^{2\pi i(2k + \theta + \theta') \cdot (y + m)} e^{-4\pi\varepsilon|y + m|^2} e^{-2\pi i(\theta - \theta') \cdot x} \\
\tilde{\varphi}(\theta, x - y) \tilde{\varphi}^*(\theta', x + y).
\end{gathered}
$$

We let $g(z) = e^{-4\pi\varepsilon|z|^2}$, $h(m) = e^{2\pi i(2k + \theta + \theta') \cdot z} g(z)$. Let $f(z) = \tau_{-y} h(z)$, where $\tau_{-y} F(x) := F(x + y)$. Then $\hat{f}(\xi) = e^{2\pi i y \cdot \xi} \hat{h}(\xi)$ and

$$
\begin{aligned}
\hat{h}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot z} e^{2\pi i(2k + \theta + \theta') \cdot z} g(z) \mathrm{d}z &= \hat{g}(\xi - 2k - \theta - \theta') \\
&= \frac{1}{(4\varepsilon)^{\frac{d}{2}}} e^{-\frac{\pi|\xi - 2k - \theta - \theta'|2}{4\varepsilon}}.
\end{aligned}
$$

Hence

$$
\hat{f}(\xi) = \frac{1}{(4\varepsilon)^{\frac{d}{2}}} e^{-\frac{\pi|\xi - 2k - \theta - \theta'|2}{4\varepsilon}} e^{2\pi i y \cdot \xi}.
$$

Using the Poisson summation formula, we get that

$$
\begin{gathered}
W_\varphi(x, k) = \lim_{\varepsilon \longrightarrow 0} \int_{\left[-\frac{1}{2}, \frac{1}{2}\right]^d} \mathrm{d}y \int_{\mathbb{T}^d} \mathrm{d}\theta \int_{\mathbb{T}^d} \mathrm{d}\theta' \sum_{m \in \mathbb{Z}^d} e^{2\pi i y \cdot m} \frac{e^{-\frac{\pi|m - 2k - \theta - \theta'|2}{4\varepsilon}}}{(4\varepsilon)^{d/2}} \\
e^{-2\pi i(\theta - \theta') \cdot x} \tilde{\varphi}(\theta, x - y) \tilde{\varphi}^*(\theta', x + y).
\end{gathered}
$$

Notice that as $\varepsilon \longrightarrow 0$, the Gaussian better approximates the delta and we expect only certain terms to survive in the limit. To see this rigorously, we split the sum over m as

$$
\begin{aligned}
=&\lim_{\varepsilon\longrightarrow 0}\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta'\,\tilde{\varphi}(\theta,x-y)\tilde{\varphi}^*(\theta',x+y)\\
&\sum_{m\in\mathbb{Z}^d}e^{2\pi i y\cdot m}\frac{e^{-\frac{\pi|m-2k-\theta-\theta'|2}{4\varepsilon}}}{(4\varepsilon)^{d/2}}e^{-2\pi i(\theta-\theta')\cdot x}\mathbb{I}_{[-1,1]^d+2k+\theta}(m)\\
+&\lim_{\varepsilon\longrightarrow 0}\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta'\,\tilde{\varphi}(\theta,x-y)\tilde{\varphi}^*(\theta',x+y)\\
&\sum_{m\in\mathbb{Z}^d}e^{2\pi i y\cdot m}\frac{e^{-\frac{\pi|m-2k-\theta-\theta'|2}{4\varepsilon}}}{(4\varepsilon)^{d/2}}e^{-2\pi i(\theta-\theta')\cdot x}\left(1-\mathbb{I}_{[-1,1]^d+2k+\theta}(m)\right).
\end{aligned}
$$

Since $\sum_{m\in\mathbb{Z}^d}\frac{e^{-\frac{\pi|m-2k-\theta-\theta'|2}{4\varepsilon}}}{(4\varepsilon)^{d/2}}\left(1-\mathbb{I}_{[-1,1]^d+2k+\theta}(m)\right)\leqslant C$ uniformly in $\varepsilon\in(0,1)$ and $\theta,\theta'\in\left[-\frac{1}{2},\frac{1}{2}\right]^d$, the dominated convergence says the second sum is

$$
\begin{aligned}
=&\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta'\,\tilde{\varphi}(\theta,x-y)\tilde{\varphi}^*(\theta',x+y)\\
&\sum_{m\in\mathbb{Z}^d}e^{2\pi i y\cdot m}\lim_{\varepsilon\longrightarrow 0}\frac{e^{-\frac{\pi|m-2k-\theta-\theta'|2}{4\varepsilon}}}{(4\varepsilon)^{d/2}}e^{-2\pi i(\theta-\theta')\cdot x}\left(1-\mathbb{I}_{[-1,1]^d+2k+\theta}(m)\right).
\end{aligned}
$$

Let $c=\operatorname{dist}\left(\left\{\left[-\frac{1}{2},\frac{1}{2}\right]^d,([-1,1]^d)^c\right\}\right)$, so $c>0$, and the above term is bounded in absolute value by

$$
\begin{aligned}
\leqslant&\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta'\,|\tilde{\varphi}(\theta,x-y)\tilde{\varphi}^*(\theta',x+y)|\\
&\sum_{m\in\mathbb{Z}^d}\underbrace{\lim_{\varepsilon\longrightarrow 0}\frac{e^{-c\pi/4\varepsilon}}{(4\varepsilon)^{d/2}}}_{=0}\left(1-\mathbb{I}_{[-1,1]^d+2k+\theta}(m)\right)=0.
\end{aligned}
$$

For a fixed k,θ the sum in the first term becomes a finite sum and hence we can use the linearity of the Lebesgue integral to write it as

$$
\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta\tilde{\varphi}(\theta,x-y)\sum_{m\in\mathbb{Z}^d}e^{2\pi i y\cdot m}\mathbb{I}_{[-1,1]^d+2k+\theta}(m)\\
&\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta'\frac{e^{-\frac{\pi|m-2k-\theta-\theta'|2}{4\varepsilon}}}{(4\varepsilon)^{d/2}}e^{-2\pi i(\theta-\theta')\cdot x}\tilde{\varphi}^*(\theta',x+y)\\
=&\lim_{\varepsilon\longrightarrow 0}\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta\tilde{\varphi}(\theta,x-y)\sum_{m\in\mathbb{Z}^d}e^{2\pi i y\cdot m}\mathbb{I}_{[-1,1]^d+2k+\theta}(m)\\
&\int_{\mathbb{R}^d}\mathrm{d}\theta'\frac{e^{-\frac{\pi|m-2k-\theta-\theta'|2}{4\varepsilon}}}{(4\varepsilon)^{d/2}}e^{-2\pi i(\theta-\theta')\cdot x}\tilde{\varphi}^*(\theta',x+y)\mathbb{I}_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}(\theta').
\end{aligned}
$$

We make the following change of variables

$$\bar{\theta} = \frac{(m-2k-\theta-\theta')}{\sqrt{4\varepsilon\pi^{-1}}} \Longrightarrow \theta' = m-2k-\theta-\sqrt{4\varepsilon\pi^{-1}}\,\bar{\theta}, \qquad \mathrm{d}\theta' = -(4\varepsilon\pi^{-1})^{d/2}\mathrm{d}\bar{\theta}.$$

Hence the above expression is

$$= -\lim_{\varepsilon\longrightarrow 0}\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta\tilde{\varphi}(\theta,x-y)\sum_{m\in\mathbb{Z}^d} e^{2\pi i y\cdot m}\mathbb{I}_{[-1,1]^d+2k+\theta}(m)\times$$

$$\times\int_{\mathbb{R}^d}\frac{\mathrm{d}\bar{\theta}}{(\pi)^{d/2}}e^{-|\bar{\theta}|^2}e^{-2\pi i\left(2\theta-m+2k+\sqrt{4\varepsilon\pi^{-1}}\bar{\theta}\right)\cdot x}$$

$$\tilde{\varphi}^*\left(m-2k-\theta-\sqrt{4\varepsilon\pi^{-1}}\,\bar{\theta},x+y\right)\mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]^d+2k+\theta-m}\left(m-\sqrt{4\varepsilon\pi^{-1}}\,\bar{\theta}\right).$$

The continuity of $\tilde{\varphi}^*$ implies that $\sup_{\theta,x\in[-\frac{1}{2},\frac{1}{2}]^d}\tilde{\varphi}^*(\theta,x)\leqslant C$ and also allows us to use dominated convergence once more, to get that

$$W_\varphi(x,k) = -\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta\tilde{\varphi}(\theta,x-y)\sum_{m\in\mathbb{Z}^d} e^{2\pi i y\cdot m}\mathbb{I}_{[-1,1]^d+2k+\theta}(m)$$

$$\int_{\mathbb{R}^d}\frac{\mathrm{d}\bar{\theta}}{(\pi)^{d/2}}e^{-|\bar{\theta}|^2}e^{-2\pi i(2\theta-m+2k)\cdot x}\tilde{\varphi}^*(m-2k-\theta,x+y)\mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]^d+2k+\theta}(m).$$

Taking the product of the indicator functions, and using the fact that $\int_{\mathbb{R}^d}\mathrm{d}x e^{-|x|^2} = (\pi)^{d/2}$, this is

$$= -\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta\tilde{\varphi}(\theta,x-y)\sum_{m\in\mathbb{Z}^d} e^{2\pi i y\cdot m}\mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]^d+2k+\theta}(m)$$

$$e^{-2\pi i(2\theta-m+2k)\cdot x}\tilde{\varphi}^*(m-2k-\theta,x+y).$$

By $\mathbb{Z}^d$-quasiperiodicity of φ^* in the first variable, this is

$$= -\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta\tilde{\varphi}(\theta,x-y)\sum_{m\in\mathbb{Z}^d} e^{2\pi i y\cdot m}\mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]^d+2k+\theta}(m)$$

$$e^{-2\pi i(2\theta-m+2k)\cdot x}\tilde{\varphi}^*(-2k-\theta,x+y)e^{-2\pi i m\cdot(x+y)}$$

$$= -\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta\tilde{\varphi}(\theta,x-y)e^{-4\pi i(\theta+k)\cdot x}\tilde{\varphi}^*(-2k-\theta,x+y)$$

$$\sum_{m\in\mathbb{Z}^d}\mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]^d+2k+\theta}(m).$$

Since $\sum_{m\in\mathbb{Z}^d}\mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]^d+2k+\theta}(m) = 1, \theta a.s.$ this is

$$= -\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta\tilde{\varphi}(\theta,x-y)e^{-4\pi i(\theta+k)\cdot x}\tilde{\varphi}^*(-2k-\theta,x+y).$$

By the $\mathbb{Z}^d$-periodicity of the integrands in y and θ, this is

$$= -\int_{\mathbb{T}^d}\mathrm{d}y\int_{\mathbb{T}^d}\mathrm{d}\theta\tilde{\varphi}(\theta,x-y)e^{-4\pi i(\theta+k)\cdot x}\tilde{\varphi}^*(-2k-\theta,x+y).$$

Now by splitting $k = \kappa - \eta \in \left(\frac{\mathbb{Z}}{2}\right)^d + \left[-\frac{1}{4}, \frac{1}{4}\right)^d$ we have

$$W_\varphi(x,k) = -\int_{\mathbb{T}^d} \mathrm{d}y \int_{\mathbb{T}^d} \mathrm{d}\theta e^{-4\pi i(\theta+\kappa-\eta)\cdot x} \tilde{\varphi}(\theta, x-y)\tilde{\varphi}^*(-2\kappa+2\eta-\theta, x+y).$$

Using $\mathbb{Z}^d$-quasiperiodicity in the first variable of φ^* once more

$$= -\int_{\mathbb{T}^d} \mathrm{d}y \int_{\mathbb{T}^d} \mathrm{d}\theta e^{-4\pi i(\theta-\eta)\cdot x} e^{4\pi i\kappa\cdot y} \tilde{\varphi}(\theta, x-y)\tilde{\varphi}^*(2\eta-\theta, x+y).$$

Finally by shifting by η in the θ-variable

$$= -\int_{\mathbb{T}^d} \mathrm{d}y \int_{\mathbb{T}^d} \mathrm{d}\theta e^{-4\pi i\theta\cdot x} e^{4\pi i\kappa\cdot y} \tilde{\varphi}(\eta+\theta, x-y)\tilde{\varphi}^*(\eta-\theta, x+y).$$

We conclude by a change of variables that

$$W_\varphi(x,k) = \int_{\mathbb{T}^d} \mathrm{d}y \int_{\mathbb{T}^d} \mathrm{d}\theta e^{-4\pi i\theta\cdot x} e^{-4\pi i\kappa\cdot y} \tilde{\varphi}(\eta+\theta, x+y)\tilde{\varphi}^*(\eta-\theta, x-y). \qquad \square$$

Remark 5.26. By another change of variables, we see that

$$W_\varphi(x,k) = 2^{2d} \int_{\mathbb{T}^d} \mathrm{d}y \int_{\mathbb{T}^d} \mathrm{d}\theta e^{2\pi i\theta\cdot x} e^{2\pi i\kappa\cdot y} \tilde{\varphi}\left(\eta - \frac{\theta}{2}, x - \frac{y}{2}\right)\tilde{\varphi}^*\left(\eta + \frac{\theta}{2}, x + \frac{y}{2}\right).$$

In this form, the similarity to the usual Wigner transform is even more apparent.

5.4.2 Proof of Proposition 5.4

Similar to the L^∞ estimate on the Wigner function (see Lemma 2.6) we have an a-priori bound on the Bloch–Wigner function. Denote

$$L^\infty_{\eta,p,\kappa,z} := L^\infty_{\eta,p} l^\infty_\kappa L^\infty_z \left(\left[-\frac{1}{4}, \frac{1}{4}\right)^d \times \mathbb{R}^d \times \left(\frac{\mathbb{Z}}{2}\right)^d \times \mathbb{T}^d\right),$$

and

$$L^\infty_{\eta,p,\kappa} L^2_z := L^\infty_{\eta,p} l^\infty_\kappa \left(\left(\left[-\frac{1}{4}, \frac{1}{4}\right)^d \times \mathbb{R}^d \times \left(\frac{\mathbb{Z}}{2}\right)^d\right); L^2_z(\mathbb{T}^d)\right).$$

Lemma 5.27. *For $\varphi \in L^2(\mathbb{R}^d;\mathbb{C})$ one has that $\tilde{W}_\varphi \in L^\infty_{\eta,p,\kappa,z}$*

Proof. We compute

$$|\tilde{W}_\varphi(z,p,\eta,\kappa)| = \left|\int_{\mathbb{T}^d} \mathrm{d}y \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \mathrm{d}\theta e^{-4\pi i\theta\cdot p} e^{-4\pi i\kappa\cdot y} \tilde{\varphi}(\eta+\theta, z+y)\tilde{\varphi}^*(\eta-\theta, z-y)\right|$$

$$\leq \int_{\mathbb{T}^d} \mathrm{d}y \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \mathrm{d}\theta |\tilde{\varphi}(\eta+\theta, z+y)\tilde{\varphi}^*(\eta-\theta, z-y)|.$$

By using Cauchy–Schwartz inequality, this is

$$\leq \left(\int_{\mathbb{T}^d} \mathrm{d}y \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \mathrm{d}\theta |\tilde{\varphi}(\eta+\theta, z+y)|^2\right)^{1/2} \left(\int_{\mathbb{T}^d} \mathrm{d}y \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \mathrm{d}\theta |\tilde{\varphi}^*(\eta-\theta, z-y)|^2\right)^{1/2}$$

Using equation (2.2), this is

$$=\Big(\int_{\mathbb{T}^d}\mathrm{d}y\int_{\mathbb{T}^d}\mathrm{d}\theta|\tilde{\varphi}(\eta+\theta,z+y)|^2\Big)^{1/2}\Big(\int_{\mathbb{T}^d}\mathrm{d}y\int_{\mathbb{T}^d}\mathrm{d}\theta|\tilde{\varphi}^*(\eta-\theta,z-y)|^2\Big)^{1/2}.$$

Hence we can shift in the θ and y variables and use the fact that $\mathscr{U}_{\mathrm{BFZ}}$ is a unitary transformation, to have that this is

$$=\Big(\int_{\mathbb{T}^d}\mathrm{d}y\int_{\mathbb{T}^d}\mathrm{d}\theta|\tilde{\varphi}(\theta,y)|^2\Big)^{1/2}\Big(\int_{\mathbb{T}^d}\mathrm{d}y\int_{\mathbb{T}^d}\mathrm{d}\theta|\tilde{\varphi}^*(\theta,y)|^2\Big)^{1/2}=\|\tilde{\varphi}\|^2_{\mathscr{H}_\gamma}=\|\varphi\|^2_{L^2(\mathbb{R}^d)}$$

□

Hence

$$\tilde{W}_\varphi(z,p,\eta,\kappa)\in L^\infty_{\eta,p,\kappa,z}\subset L^\infty_{\eta,p,\kappa}L^2_z. \tag{5.53}$$

We will now consider now $\tilde{W}_\varphi(t,z,p,\eta,\kappa)$ associated to $\varphi(t,x)$, the solution of the Schrödinger equation (5.1).

First, we define a Hermitian form

$$F[\tilde{\varphi},\tilde{\psi}](z,p,\eta,\kappa):=\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}\tilde{\varphi}(\eta+\theta,x+y)\tilde{\psi}^*(\eta-\theta,x-y),$$

and

$$G[\tilde{\varphi}](z,p,\eta,\kappa):=F[\tilde{\varphi},\tilde{\varphi}](z,p,\eta,\kappa).$$

Lemma 5.28. *$F:\mathscr{H}_\gamma\times\mathscr{H}_\gamma\longrightarrow L^\infty_{\eta,p,\kappa,z}$ and $G:\mathscr{H}_\gamma\longrightarrow L^\infty_{\eta,p,\kappa,z}$ are continuous maps.*

Proof. By picking $\tilde{\varphi},\tilde{\psi},\tilde{\varphi}_1,\tilde{\psi}_1\in\mathscr{H}_f$ one has that

$$\begin{aligned}&F[\tilde{\varphi},\tilde{\psi}](z,p,\eta,\kappa)-F[\tilde{\varphi}_1,\tilde{\psi}_1](z,p,\eta,\kappa)\\
&=\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}\tilde{\varphi}(\eta+\theta,z+y)\tilde{\psi}^*(\eta-\theta,z-y)\\
&-\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}\tilde{\varphi}_1(\eta+\theta,z+y)\tilde{\psi}_1^*(\eta-\theta,z-y)\\
&\pm\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}\tilde{\varphi}_1(\eta+\theta,z+y)\tilde{\psi}^*(\eta-\theta,z-y).\end{aligned}$$

By rearranging the terms, this is

$$\begin{aligned}&=\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}(\tilde{\varphi}-\tilde{\varphi}_1)(\eta+\theta,z+y)\tilde{\psi}^*(\eta-\theta,z-y)\\
&+\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}\tilde{\varphi}_1(\eta+\theta,z+y)(\tilde{\psi}^*-\tilde{\psi}_1^*)(\eta-\theta,z-y).\end{aligned}$$

Hence

$$\begin{aligned}&|F[\tilde{\varphi},\tilde{\psi}](z,p,\eta,\kappa)-F[\tilde{\varphi}_1,\tilde{\psi}_1](z,p,\eta,\kappa)|\\
&\leq\|(\tilde{\varphi}-\tilde{\varphi}_1)(\eta+\cdot,z+\cdot)\tilde{\psi}^*(\eta-\cdot,z-\cdot)\|_{L^1([-\frac{1}{2},\frac{1}{2}]^d\times\mathbb{T}^d)}\\
&+\|\tilde{\varphi}_1(\eta+\cdot,z+\cdot)(\tilde{\psi}^*-\tilde{\psi}_1^*)(\eta-\cdot,z-\cdot)\|_{L^1([-\frac{1}{2},\frac{1}{2}]^d\times\mathbb{T}^d)}\end{aligned}$$

By the Cauchy–Schwartz inequality and (2.2) this is

$$\begin{aligned}&\leqslant \|(\tilde{\varphi}-\tilde{\varphi}_1)(\eta+\cdot,z+\cdot)\|_{L^2(\mathbb{T}^d\times\mathbb{T}^d)}\|\tilde{\psi}^*(\eta-\cdot,z-\cdot)\|_{L^2(\mathbb{T}^d\times\mathbb{T}^d)}\\&+\|\tilde{\varphi}_1(\eta+\cdot,z+\cdot)\|_{L^2(\mathbb{T}^d\times\mathbb{T}^d)}\|(\tilde{\psi}^*-\tilde{\psi}_1^*)(\eta-\cdot,z-\cdot)\|_{L^2(\mathbb{T}^d\times\mathbb{T}^d)}.\end{aligned}$$

By the shift invariance of the L^2 norm, this is

$$\leqslant \|\tilde{\varphi}-\tilde{\varphi}_1\|_{\mathscr{H}_\gamma}\|\tilde{\psi}^*\|_{\mathscr{H}_\gamma}+\|\tilde{\varphi}_1\|_{\mathscr{H}_\gamma}\|\tilde{\psi}^*-\tilde{\psi}_1^*\|_{\mathscr{H}_f}.$$

One can deduce continuity of the map from here, and also for $G[\tilde{\varphi}]$, by replacing $\tilde{\psi}$ and $\tilde{\psi}_1$ by $\tilde{\varphi}$ and $\tilde{\varphi}_1$ respectively in the above computations. □

Proof. (of Proposition 5.4) One can compute for $\varphi\in C(\mathbb{R}_{\geqslant 0};H^2(\mathbb{R}^d;\mathbb{C}))\cap C^1(\mathbb{R}_+;L^2(\mathbb{R}^d;\mathbb{C}))$:

$$\frac{\tilde{W}_\varphi(t+h)-\tilde{W}_\varphi(t)}{h}=\frac{F[\tilde{\varphi}(t+h),\tilde{\varphi}(t+h)]-F[\tilde{\varphi}(t),\tilde{\varphi}(t)]}{h}\pm\frac{F[\tilde{\varphi}(t),\tilde{\varphi}(t+h)]}{h}.$$

Since F is a Hermitian form by Lemma 5.28, this is

$$=F\left[\frac{(\tilde{\varphi}(t+h)-\tilde{\varphi}(t))}{h},\tilde{\varphi}(t+h)\right]+F\left[\tilde{\varphi}(t),\frac{\tilde{\varphi}(t+h)-\tilde{\varphi}(t)}{h}\right].$$

By the continuity of F one has

$$\begin{aligned}\partial_t\tilde{W}_\varphi(t)&=\lim_{h\to 0}\frac{\tilde{W}_\varphi(t+h)-\tilde{W}_\varphi(t)}{h}\\&=F[\partial_t\tilde{\varphi}(t),\tilde{\varphi}(t)]+F[\tilde{\varphi}(t),\partial_t\tilde{\varphi}(t)].\end{aligned}$$

Thus, we have

$$\partial_t\tilde{W}_\varphi(t)=F[\partial_t\tilde{\varphi}(t),\tilde{\varphi}(t)]+F[\tilde{\varphi}(t),\partial_t\tilde{\varphi}(t)] \tag{5.54}$$

Similarly, by using Lemma 2.6, one has that,

$$\partial_{z_j}\tilde{W}_\varphi(t)=F[\partial_{z_j}\tilde{\varphi}(t),\tilde{\varphi}(t)]+F[\tilde{\varphi}(t),\partial_{z_j}\tilde{\varphi}(t)],$$

and

$$\begin{aligned}\partial_{z_iz_j}\tilde{W}_\varphi(t)&=F[\partial_{z_iz_j}\tilde{\varphi}(t),\tilde{\varphi}(t)]+F[\tilde{\varphi}(t),\partial_{z_iz_j}\tilde{\varphi}(t)]\\&+F[\partial_{z_i}\tilde{\varphi}(t),\partial_{z_j}\tilde{\varphi}(t)]+F[\partial_{z_j}\tilde{\varphi}(t),\partial_{z_i}\tilde{\varphi}(t)].\end{aligned}$$

Since $\tilde{\varphi}\in C^1(\mathbb{R}_+;\mathscr{H}_f)\cap C(\mathbb{R}_{\geqslant 0};L^2_{\mathrm{loc}}(\mathbb{R}^d;H^2(\mathbb{T}^d;\mathbb{C})))$, one has using Lemma 5.28 that for $i,j\in\{1,\ldots,d\}$ and $t\in\mathbb{R}$,

$$\partial_t\tilde{W}_\varphi,\partial_{z_j}\tilde{W}_\varphi,\partial_{z_iz_j}\tilde{W}_\varphi\in C(\mathbb{R}_{\geqslant 0};L^\infty_{z,p,\eta,\kappa}).$$

Furthermore

$$\begin{aligned}&|F[V\tilde{\varphi}(t),\tilde{\varphi}(t)](z,p,\eta,\kappa)|\\&=\left|\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}V(z+y)\tilde{\varphi}(\eta+\theta,z+y)\tilde{\varphi}^*(\eta-\theta,z-y)\right|\\&\leqslant\|V\|_{L^\infty(\mathbb{R}^d)}\|\tilde{W}_\varphi\|_{L^\infty_{z,p,\eta,\kappa}}\leqslant\|V\|_{L^\infty}\|\tilde{\varphi}\|^2_{\mathscr{H}_\gamma}.\end{aligned}$$

A similar bound holds for $|F[\tilde{\varphi}(t), V\tilde{\varphi}(t)](z,p,\eta,\kappa)|$. Hence also

$$F[V\tilde{\varphi}, \tilde{\varphi}], F[\tilde{\varphi}, V\tilde{\varphi}] \in C(\mathbb{R}_{\geqslant 0}; L^{\infty}_{z,p,\eta,\kappa}).$$

Now, assume that $\varphi \in C(\mathbb{R}_{\geqslant 0}; H^2(\mathbb{R}^d;\mathbb{C})) \cap C^1(\mathbb{R}_+; L^2(\mathbb{R}^d;\mathbb{C}))$ satisfies the Schrödinger equation (5.1). We have that by applying the BFZ transform to both sides of the Schrödinger equation that

$$i\widetilde{\partial_t \varphi} = -\widetilde{\Delta\varphi} + \varepsilon^{1/2}\widetilde{V\varphi}.$$

Multiplying by $-i$ on both sides

$$\widetilde{\partial_t \varphi} = i\widetilde{\Delta\varphi} - i\varepsilon^{1/2}\widetilde{V\varphi}.$$

Using Lemma 2.6 and

$$\widetilde{V\varphi}(\theta, x) = \sum_{m\in\mathbb{Z}^d} e^{2\pi i\theta\cdot(x-m)} V(x-m)\varphi(t, x-m)$$

$$= V(x)\sum_{m\in\mathbb{Z}^d} e^{2\pi i\theta\cdot(x-m)}\varphi(t, x-m) = V(x)\tilde{\varphi}(\theta, x),$$

we get that

$$\partial_t\tilde{\varphi}(t,\theta,x) = i\widetilde{\Delta\varphi}(t,\theta,x) - i\varepsilon^{1/2}\, V(x)\tilde{\varphi}(t,\theta,x).$$

Plugging this into equation (5.54) one has

$$\partial_t \tilde{W}_{\varphi}(t) = F[\partial_t\tilde{\varphi}(t), \tilde{\varphi}(t)] + F[\tilde{\varphi}(t), \partial_t\tilde{\varphi}(t)]$$

$$= F[i\widetilde{\Delta\varphi}(t), \tilde{\varphi}(t)] + F[-i\varepsilon^{1/2}\, V\tilde{\varphi}(t), \tilde{\varphi}(t)] + F[\tilde{\varphi}(t), i\widetilde{\Delta\varphi}(t)] + F[\tilde{\varphi}(t), -i\varepsilon^{1/2} V\tilde{\varphi}(t)].$$

Since F is a Hermitian form, this is

$$= i\left[F[\widetilde{\Delta\varphi}(t), \tilde{\varphi}(t)] - F[\tilde{\varphi}(t), \widetilde{\Delta\varphi}(t)]\right] + i\varepsilon^{1/2}\left[F[\tilde{\varphi}(t), V\tilde{\varphi}(t)] - F[V\tilde{\varphi}(t), \tilde{\varphi}(t)]\right].$$

The terms with the potential can be computed first:

$$i\varepsilon^{1/2}\left[F[\tilde{\varphi}(t), V\tilde{\varphi}(t)] - F[V\tilde{\varphi}(t), \tilde{\varphi}(t)]\right](z,p,\eta,\kappa)$$

$$= i\varepsilon^{1/2}\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p} e^{-4\pi i\kappa\cdot y}\tilde{\varphi}(\eta+\theta, z+y)\tilde{\varphi}^*(\eta-\theta, z-y)$$

$$[V(z-y) - V(z+y)]$$

$$= i\varepsilon^{1/2}\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p} e^{-4\pi i\kappa\cdot y}\tilde{\varphi}(\eta+\theta, z+y)\tilde{\varphi}^*(\eta-\theta, z-y)$$

$$\left[\sum_{n\in\mathbb{Z}^d} e^{2\pi i n\cdot(z-y)}\hat{V}(n) - e^{2\pi i n\cdot(z+y)}\hat{V}(n)\right].$$

By using Fubini's theorem to interchange the sum and integrals, this is

$$= i\varepsilon^{1/2}\sum_{n\in\mathbb{Z}^d} e^{2\pi i n\cdot z}\hat{V}(n)\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}\tilde{\varphi}(\eta+\theta, z+y)\tilde{\varphi}^*(\eta-\theta, z-y)$$

$$\left[e^{-4\pi i\left(\kappa+\frac{n}{2}\right)\cdot y} - e^{-4\pi i\left(\kappa-\frac{n}{2}\right)\cdot y}\right]$$

$$= i\varepsilon^{1/2}\sum_{n\in\mathbb{Z}^d} e^{2\pi i n\cdot z}\hat{V}(n)\left[\tilde{W}_{\varphi}\left(t,z,p,\eta,\kappa+\frac{n}{2}\right) - \tilde{W}_{\varphi}\left(t,z,p,\eta,\kappa-\frac{n}{2}\right)\right].$$

Now, consider the terms with the Laplacian,

$$i\left[F[\widetilde{\Delta\varphi}(t),\tilde{\varphi}(t)]-F[\tilde{\varphi}(t),\widetilde{\Delta\varphi}(t)]\right](z,p,\eta,\kappa).$$

We temporarily leave t out of the notation, and shorten $\tilde{\varphi}(\eta+\theta,z+y)$ to $\tilde{\varphi}$ and $\tilde{\varphi}^*(\eta-\theta,z-y)$ to $\tilde{\varphi}^*$ to improve legibility in the computations below. The above expression is then

$$\begin{aligned}&=i\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}\widetilde{\Delta\varphi}(\eta+\theta,z+y)\tilde{\varphi}^*\\&-i\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}\tilde{\varphi}\widetilde{\Delta\varphi}^*(\eta-\theta,z-y).\end{aligned}$$

Using equations (2.5) and (2.6) this is

$$\begin{aligned}&=i\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}\tilde{\varphi}^*[(\Delta_z-4\pi^2|\eta+\theta|^2-4\pi i(\eta+\theta)\cdot\nabla_z)\tilde{\varphi}]\\&-i\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}\tilde{\varphi}[(\Delta_z-4\pi^2|\eta-\theta|^2+4\pi i(\eta-\theta)\cdot\nabla_z)\tilde{\varphi}^*].\end{aligned}$$

Splitting this into three parts, one has

$$i\left[F[\widetilde{\Delta\varphi},\tilde{\varphi}]-F[\tilde{\varphi},\widetilde{\Delta\varphi}]\right](z,p,\eta,\kappa)=A_1+A_2+A_3,$$

where

$$\begin{aligned}A_1&=-4\pi^2 i\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[|\eta+\theta|^2-|\eta-\theta|^2]\tilde{\varphi}\tilde{\varphi}^*,\\A_2&=i\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[\Delta_z\tilde{\varphi}\tilde{\varphi}^*-\tilde{\varphi}\Delta_z\tilde{\varphi}^*],\\A_3&=4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}(((\eta+\theta)\cdot\nabla_z\tilde{\varphi})\tilde{\varphi}^*+((\eta-\theta)\cdot\nabla_z\tilde{\varphi}^*)\tilde{\varphi}).\end{aligned}$$

One can compute that

$$\begin{aligned}A_1&=-4\pi^2 i\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[|\eta|^2+|\theta|^2+2\eta\cdot\theta-|\eta|^2-|\theta|^2+2\eta\cdot\theta]\tilde{\varphi}\tilde{\varphi}^*\\&=4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}(-4\pi i\eta\cdot\theta)\tilde{\varphi}\tilde{\varphi}^*\\&=4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta\eta\cdot\nabla_p(e^{-4\pi i\theta\cdot p})e^{-4\pi i\kappa\cdot y}\tilde{\varphi}\tilde{\varphi}^*\\&=4\pi\eta\cdot\nabla_p\tilde{W}_\varphi(t,z,p,\eta,\kappa).\end{aligned}$$

Next, consider

$$A_2=i\int_{\mathbb{T}^d}\mathrm{d}y\int_{[-\frac{1}{2},\frac{1}{2}]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[\Delta_z\tilde{\varphi}\tilde{\varphi}^*-\tilde{\varphi}\Delta_z\tilde{\varphi}^*].$$

We note that

$$\nabla_z\tilde{\varphi}=\nabla_z\tilde{\varphi}(\eta+\theta,z+y)=\nabla_y\tilde{\varphi}(\eta+\theta,z+y),$$

and similarly

$$\nabla_z\tilde{\varphi}^*=-\nabla_y\tilde{\varphi}^*,\qquad\Delta_z\tilde{\varphi}=\Delta_y\tilde{\varphi},\qquad\Delta_z\tilde{\varphi}^*=\Delta_y\tilde{\varphi}^*.$$

So

$$A_2 = i\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[\Delta_y\tilde{\varphi}\tilde{\varphi}^* - \tilde{\varphi}\Delta_y\tilde{\varphi}^*]$$

$$= i\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[(\mathrm{div}\nabla_y\tilde{\varphi})\tilde{\varphi}^* - \tilde{\varphi}(\mathrm{div}\nabla_y\tilde{\varphi}^*)]$$

Since $a\,\mathrm{div}\,(v) = \mathrm{div}\,(av) - v\cdot\nabla a$, we have that

$$a^*\,\mathrm{div}\,(v) - a\,\mathrm{div}\,(v^*) = \mathrm{div}\,(a^*v - av^*) - v\cdot\nabla a^* + v^*\cdot\nabla a,$$

and since in our case $a = \tilde{\varphi}$, $v = \nabla_y\tilde{\varphi}$, the last two terms cancel since

$$-v\cdot\nabla a^* + v^*\cdot\nabla a = -\nabla_y\tilde{\varphi}\cdot\nabla_y\tilde{\varphi}^* + \nabla_y\tilde{\varphi}^*\cdot\nabla_y\tilde{\varphi} = 0.$$

Hence

$$A_2 = i\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}\mathrm{div}\,[(\nabla_y\tilde{\varphi})\tilde{\varphi}^* - \tilde{\varphi}(\nabla_y\tilde{\varphi}^*)].$$

Integrating by parts in y, this is

$$= -i\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}\nabla_y(e^{-4\pi i\kappa\cdot y})[(\nabla_y\tilde{\varphi})\tilde{\varphi}^* - \tilde{\varphi}(\nabla_y\tilde{\varphi}^*)]$$

$$= -4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[(\kappa\cdot\nabla_y\tilde{\varphi})\tilde{\varphi}^* - \tilde{\varphi}(\kappa\cdot\nabla_y\tilde{\varphi}^*)]$$

$$= -4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[(\kappa\cdot\nabla_z\tilde{\varphi})\tilde{\varphi}^* + \tilde{\varphi}(\kappa\cdot\nabla_z\tilde{\varphi}^*)]$$

$$= -4\pi\kappa\cdot\nabla_z\tilde{W}_\varphi(t, z, p, \eta, \kappa).$$

Finally, consider the term

$$A_3 = 4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[((\eta+\theta)\cdot\nabla_z\tilde{\varphi})\tilde{\varphi}^* + ((\eta-\theta)\cdot\nabla_z\tilde{\varphi}^*)\tilde{\varphi}]$$

$$= 4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[((\eta+\theta)\cdot\nabla_y\tilde{\varphi})\tilde{\varphi}^* + ((\theta-\eta)\cdot\nabla_y\tilde{\varphi}^*)\tilde{\varphi}].$$

Using integration by parts for the first term in the sum, this is

$$= 4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[\tilde{\varphi}((-\eta-\theta)\cdot\nabla_y\tilde{\varphi}^*) + ((\theta-\eta)\cdot\nabla_y\tilde{\varphi}^*)\tilde{\varphi}]$$

$$-4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}(\eta+\theta)\cdot\nabla_y(e^{-4\pi i\kappa\cdot y})\tilde{\varphi}\tilde{\varphi}^*.$$

The terms with θ in the first expression cancel. Hence

$$A_3 = -4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[\tilde{\varphi}(\eta\cdot\nabla_y\tilde{\varphi}^*)]$$

$$-4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[\tilde{\varphi}(\eta\cdot\nabla_y\tilde{\varphi}^*)]$$

$$+16\pi^2 i\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}(\eta+\theta)\cdot\kappa e^{-4\pi i\kappa\cdot y}\tilde{\varphi}\tilde{\varphi}^*.$$

Using integration by parts once more, this is

$$\begin{aligned}
&=4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[(\eta\cdot\nabla_y\tilde{\varphi})\tilde{\varphi}^*-\tilde{\varphi}(\eta\cdot\nabla_y\tilde{\varphi}^*)]\\
&\quad+16\pi^2 i\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}(\eta+\theta)\cdot\kappa e^{-4\pi i\kappa\cdot y}\tilde{\varphi}\tilde{\varphi}^*\\
&\quad+4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}\eta\cdot\nabla_y(e^{-4\pi i\kappa\cdot y})\tilde{\varphi}\tilde{\varphi}^*\\
&=4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[(\eta\cdot\nabla_y\tilde{\varphi})\tilde{\varphi}^*-\tilde{\varphi}(\eta\cdot\nabla_y\tilde{\varphi}^*)]\\
&\quad+16\pi^2 i\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}(\eta+\theta)\cdot\kappa e^{-4\pi i\kappa\cdot y}\tilde{\varphi}\tilde{\varphi}^*\\
&\quad-16\pi^2 i\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}\eta\cdot\kappa e^{-4\pi i\kappa\cdot y}\tilde{\varphi}\tilde{\varphi}^*.
\end{aligned}$$

The η terms in the last two expressions cancel to give

$$\begin{aligned}
A_3=&\,4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}e^{-4\pi i\kappa\cdot y}[(\eta\cdot\nabla_z\tilde{\varphi})\tilde{\varphi}^*+\tilde{\varphi}(\eta\cdot\nabla_z\tilde{\varphi}^*)]\\
&+16\pi^2 i\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}\theta\cdot\kappa e^{-4\pi i\kappa\cdot y}\tilde{\varphi}\tilde{\varphi}^*\\
=&\,4\pi\eta\cdot\nabla_z\tilde{W}_\varphi(t,z,p,\eta,\kappa)-4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta e^{-4\pi i\theta\cdot p}(-4\pi i\theta\cdot\kappa)\,e^{-4\pi i\kappa\cdot y}\tilde{\varphi}\tilde{\varphi}^*\\
=&\,4\pi\eta\cdot\nabla_z\tilde{W}_\varphi(t,z,p,\eta,\kappa)-4\pi\int_{\mathbb{T}^d}\mathrm{d}y\int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\mathrm{d}\theta\kappa\cdot\nabla_p\,(e^{-4\pi i\theta\cdot p})\,e^{-4\pi i\kappa\cdot y}\tilde{\varphi}\tilde{\varphi}^*\\
=&\,4\pi\eta\cdot\nabla_z\tilde{W}_\varphi(t,z,p,\eta,\kappa)-4\pi\kappa\cdot\nabla_p\,\tilde{W}_\varphi(t,z,p,\eta,\kappa).
\end{aligned}$$

Overall, one has

$$\begin{aligned}
\partial_t\tilde{W}_\varphi(t,z,p,\eta,\kappa)=&\,4\pi\eta\cdot\nabla_p\tilde{W}_\varphi(t,z,p,\eta,\kappa)-4\pi\kappa\cdot\nabla_z\tilde{W}_\varphi(t,z,p,\eta,\kappa)\\
&+4\pi\eta\cdot\nabla_z\tilde{W}_\varphi(t,z,p,\eta,\kappa)-4\pi\kappa\cdot\nabla_p\,\tilde{W}_\varphi(t,z,p,\eta,\kappa)\\
&+i\varepsilon^{1/2}\sum_{n\in\mathbb{Z}^d}e^{2\pi in\cdot z}\hat{V}(n)\left[\tilde{W}_\varphi\left(t,z,p,\eta,\kappa+\frac{n}{2}\right)-\tilde{W}_\varphi\left(t,z,p,\eta,\kappa-\frac{n}{2}\right)\right],
\end{aligned}$$

and this proves the claim. □

5.4.3 Estimates on smoothing operators

Lemma 5.29. *For $\mathcal{J}_\nu$ defined in expression (5.45), we have that estimate (5.47) holds.*

Proof. Let $(m,n)\in\{(1,1),(1,2),(2,2)\}$. Consider

$$\|\mathcal{J}_\nu\varphi\|_{E_n}=\sum_{|\beta|\leq n}\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^n\|D_p^\beta\mathcal{J}_\nu\varphi(\cdot,p,\kappa)\|_{l_\xi^2}.$$

We have that

$$\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^n\|D_p^{\beta}\mathfrak{J}_\nu\varphi(\cdot,p,\kappa)\|_{l_\xi^2}=\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^n\left(\sum_{\xi\in\mathbb{Z}^d}|D_p^{\beta}\mathfrak{J}_\nu\varphi(\xi,p,\kappa)|^2\right)^{1/2}$$

$$=\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^n\left(\sum_{\xi\in\mathbb{Z}^d}|D_p^{\beta}(e^{-\nu^{1/2}\langle\kappa\rangle}(\varphi(\xi,\cdot,\kappa)*_p\phi_\nu(\cdot))(p))|^2\right)^{1/2}$$

$$=\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}e^{-\nu^{1/2}\langle\kappa\rangle}\langle\kappa\rangle^n\|D_p^{\beta}((\varphi(\xi,\cdot,\kappa)*_p\phi_\nu(\cdot))(p))\|_{l_\xi^2}.$$

Distributing the derivatives between the terms, we split $\beta=\alpha_1(\beta)+\alpha_2(\beta)=\alpha_1+\alpha_2$, such that $|\alpha_1(\beta)|=\min(|\beta|,m)$. So

$$D_p^{\beta}((\varphi(\xi,\cdot,\kappa)*_p\phi_\nu(\cdot))(p))=(D_p^{\alpha_1}\varphi(\xi,\cdot,\kappa)*_pD_p^{\alpha_2}\phi_\nu(\cdot))(p).$$

Hence by the triangle inequality for the l^2-norm

$$\|D_p^{\beta}((\varphi(\xi,\cdot,\kappa)*_p\phi_\nu(\cdot))(p))\|_{l_\xi^2}\leqslant\|(D_p^{\alpha_1}\varphi(\xi,\cdot,\kappa)*_pD_p^{\alpha_2}\phi_\nu(\cdot))(p)\|_{l_\xi^2}.$$

By Minkowski's inequality, this is

$$\leqslant\int_{\mathbb{R}^d}\mathrm{d}q\|D_p^{\alpha_1}\varphi(\cdot,q,\kappa)\|_{l_\xi^2}|D_p^{\alpha_2}\phi_\nu(p-q)|.$$

Hence

$$\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^n\|D_p^{\beta}\mathfrak{J}_\nu\varphi(\cdot,p,\kappa)\|_{l_\xi^2}=\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\langle\kappa\rangle^{n-m}\|D_p^{\beta}\mathfrak{J}_\nu\varphi(\cdot,p,\kappa)\|_{l_\xi^2}.$$

By Tonelli's theorem, this is

$$\leqslant\int_{\mathbb{R}^d}\mathrm{d}q\sup_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}e^{-\nu^{1/2}\langle\kappa\rangle}\langle\kappa\rangle^{n-m}\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\|D_p^{\alpha_1}\varphi(\cdot,q,\kappa)\|_{l_\xi^2}|D_p^{\alpha_2}\phi_\nu(p-q)|.$$

Integrating in p, and using Tonelli's theorem once more, we have that this is

$$\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^n\|D_p^{\beta}\mathfrak{J}_\nu\varphi(\cdot,p,\kappa)\|_{l_\xi^2}\leqslant\sup_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}e^{-\nu^{1/2}\langle\kappa\rangle}\langle\kappa\rangle^{n-m}$$

$$\int_{\mathbb{R}^d}\mathrm{d}q\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\|D_p^{\alpha_1}\varphi(\cdot,q,\kappa)\|_{l_\xi^2}\int_{\mathbb{R}^d}\mathrm{d}p|D_p^{\alpha_2}\phi_\nu(p-q)|.$$

Next we estimate

$$\int_{\mathbb{R}^d}\mathrm{d}p|D_p^{\alpha_2}\phi_\nu(p-q)|=\int_{\mathbb{R}^d}\mathrm{d}r|D_p^{\alpha_2}\phi_\nu(r)|=\nu^{-\frac{d}{2}}\int_{\mathbb{R}^d}\mathrm{d}r\left|D_p^{\alpha_2}\left(\phi\left(\frac{r}{\nu^{1/2}}\right)\right)\right|$$

$$=\nu^{-\frac{d+|\alpha_2|}{2}}\int_{\mathbb{R}^d}\mathrm{d}r\left|D_p^{\alpha_2}\phi\left(\frac{r}{\nu^{1/2}}\right)\right|=\nu^{-\frac{|\alpha_2|}{2}}\int_{\mathbb{R}^d}\mathrm{d}r'|D_p^{\alpha_2}\phi(r')|\lesssim\nu^{-\frac{|\alpha_2|}{2}}\lesssim\nu^{-\frac{n-m}{2}}.$$

Since $|\alpha_2| = |\beta| - |\alpha_1|$ due to how we choose to split the derivative and since we also have that

$$\sup_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d} e^{-\nu^{1/2}\langle\kappa\rangle}\langle\kappa\rangle^{n-m} = \sup_x e^{-x}\langle x\nu^{-1/2}\rangle^{n-m} \lesssim \nu^{-\frac{n-m}{2}}\sup_x e^{-x}\langle x\rangle^{n-m} \lesssim \nu^{-\frac{n-m}{2}},$$

$$\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^n\|D_p^\beta\mathcal{J}_\nu\varphi(\cdot,p,\kappa)\|_{l_\xi^2} \lesssim \nu^{-(n-m)}\int_{\mathbb{R}^d}\mathrm{d}q\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle\|D_p^{\alpha_1}\varphi(\cdot,q,\kappa)\|_{l_\xi^2}.$$

Hence summing over β, one has that

$$\|\mathcal{J}_\nu\varphi\|_{E_n} \lesssim \nu^{-(n-m)}\sum_{|\beta|\leq n}\int_{\mathbb{R}^d}\mathrm{d}q\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\|D_p^{\alpha_1(\beta)}\varphi(\cdot,q,\kappa)\|_{l_\xi^2}.$$

since for every β we chose $\alpha_1 : |\alpha_1| \leq 1$, at the price of worsening the constant, this is

$$\lesssim \nu^{-(n-m)}\sum_{|\alpha_1|\leq m}\int_{\mathbb{R}^d}\mathrm{d}q\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\langle\kappa\rangle^m\|D_p^{\alpha_1}\varphi(\cdot,q,\kappa)\|_{l_\xi^2}$$

$$\lesssim \nu^{-(n-m)}\|\varphi\|_{E_1}.$$ ☐

Lemma 5.30. *For $\mathcal{J}_\nu$ defined in expression (5.45), we have that estimate (5.48) holds.*

Proof. Consider

$$\|(\mathcal{J}_\nu - \mathrm{Id})\varphi\|_{E_0} = \int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\|(\mathcal{J}_\nu - \mathrm{Id})\varphi(\cdot,p,\kappa)\|_{l_\xi^2}.$$

Now

$$(\mathcal{J}_\nu - \mathrm{Id})\varphi(\xi,p,\kappa) = e^{-\nu^{1/2}\langle\kappa\rangle}(\varphi(\xi,\cdot,\kappa) *_p \phi_\nu(\cdot))(p) - \varphi(\xi,p,\kappa)$$

$$= (e^{-\nu^{1/2}\langle\kappa\rangle} - 1)(\varphi(\xi,\cdot,\kappa) *_p \phi_\nu(\cdot))(p) - (\varphi(\xi,p,\kappa) - (\varphi(\xi,\cdot,\kappa) *_p \phi_\nu(\cdot))(p)).$$

We handle the two terms separately, beginning with

$$\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\|(e^{-\nu^{1/2}\langle\kappa\rangle} - 1)(\varphi(\xi,\cdot,\kappa) *_p \phi_\nu(\cdot))(p)\|_{l_\xi^2}$$

$$= \int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}|e^{-\nu^{1/2}\langle\kappa\rangle} - 1|\|(\varphi(\xi,\cdot,\kappa) *_p \phi_\nu(\cdot))(p)\|_{l_\xi^2}.$$

Now using that $|e^{-x} - 1| \leq x$ for $x > 0$, this is

$$\leq \nu^{1/2}\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\|(\varphi(\xi,\cdot,\kappa) *_p \phi_\nu(\cdot))(p)\|_{l_\xi^2}.$$

By Minkowski's inequality, this is

$$\leq \nu^{1/2}\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}\int_{\mathbb{R}^d}\mathrm{d}q\|\varphi(\cdot,q,\kappa)\|_{l_\xi^2}\phi_\nu(p-q).$$

By using Tonelli's theorem, this is

$$\begin{aligned}
&\leqslant \nu^{1/2} \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \int_{\mathbb{R}^d} \mathrm{d}q \|\varphi(\cdot, q, \kappa)\|_{l^2_\xi} \int_{\mathbb{R}^d} \mathrm{d}p \phi_\nu(p-q) \\
&\leqslant \nu^{1/2} \int_{\mathbb{R}^d} \mathrm{d}q \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \|\varphi(\cdot, q, \kappa)\|_{l^2_\xi} = \nu^{1/2} \|\varphi\|_{E_0} \\
&\leqslant \nu^{1/2} \|\varphi\|_{E_1}.
\end{aligned}$$

For the second term, we consider

$$\int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \|\varphi(\cdot, p, \kappa) - (\varphi(\cdot, \cdot, \kappa) *_p \phi_\nu(\cdot))(p)\|_{l^2_\xi}.$$

Now

$$\begin{aligned}
\varphi(\xi, p, \kappa) - (\varphi(\xi, \cdot, \kappa) *_p \phi_\nu(\cdot))(p) &= \int_{\mathbb{R}^d} \mathrm{d}q (\varphi(\xi, p-q, \kappa) - \varphi(\xi, p, \kappa)) \phi_\nu(q) \\
&= \int_{\mathbb{R}^d} \mathrm{d}r (\varphi(\xi, p - \nu^{1/2} r, \kappa) - \varphi(\xi, p, \kappa)) \phi(r).
\end{aligned}$$

Hence

$$\begin{aligned}
\|\varphi(\cdot, p, \kappa) - (\varphi(\cdot, \cdot, \kappa) *_p \phi_\nu(\cdot))(p)\|_{l^2_\xi} &\leqslant \int_{\mathbb{R}^d} \mathrm{d}r \|\varphi(\cdot, p - \nu^{1/2} r, \kappa) - \varphi(\cdot, p, \kappa)\|_{l^2_\xi} \phi(r) \\
&\leqslant \int_{\mathbb{R}^d} \mathrm{d}r \left\| \int_0^{\nu^{1/2}} \mathrm{d}t r \cdot \nabla \varphi(\cdot, p - tr, \kappa) \right\|_{l^2_\xi} \phi(r) \leqslant \int_{\mathbb{R}^d} \mathrm{d}r \int_0^{\nu^{1/2}} \mathrm{d}t \| r \cdot \nabla \varphi(\cdot, p - tr, \kappa)\|_{l^2_\xi} \\
&\leqslant \int_{\mathbb{R}^d} \mathrm{d}r |r| \phi(r) \int_0^{\nu^{1/2}} \mathrm{d}t \| \nabla \varphi(\cdot, p - tr, \kappa)\|_{l^2_\xi}.
\end{aligned}$$

Hence by Tonelli's theorem

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \|\varphi(\cdot, p, \kappa) - (\varphi(\cdot, \cdot, \kappa) *_p \phi_\nu(\cdot))(p)\|_{l^2_\xi} \\
&\leqslant \int_{\mathbb{R}^d} \mathrm{d}r |r| \phi(r) \int_0^{\nu^{1/2}} \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}p \sum_{\kappa \in \left(\frac{\mathbb{Z}}{2}\right)^d} \| \nabla \varphi(\cdot, p - tr, \kappa)\|_{l^2_\xi} \\
&\leqslant \int_{\mathbb{R}^d} \mathrm{d}r |r| \phi(r) \int_0^{\nu^{1/2}} \mathrm{d}t \|\varphi\|_{E_1} \leqslant \nu^{1/2} \|\varphi\|_{E_1}.
\end{aligned}$$

This concludes the proof. □

Chapter 6

Perspectives

To summarize, our main results have been the following:

1. In Chapter 3 we used an asymptotic expansion that heuristically gives the correct limit equation in the case of the random quantum Lorentz gas. In the case of the periodic quantum Lorentz gas, this method yields trivial transport equations in the limit for certain observables (similar to the case $V = 0$), and hints at the obstructions one encounters when studying other observables.

2. In Chapter 4, we used Wigner series tools to make precise the notion that for observables supported away from energy band crossings of the Laplacian, the weak coupling limit behavior is free transport. In particular, no collisional effects remain. These tools were insufficient to study observables supported on problematic momenta related to the band crossings, except for a restricted class of initial data.

3. In order to study what happens for observables supported on energy band crossings in Chapter 5 we used the Bloch–Floquet–Zak transform to derive a new representation formula for the Wigner transform. Generalizing this representation, we define a rescaled Bloch–Wigner transform and demonstrate once more that for certain observables, there is no collisional effect in the limit. For certain other observables, we provide evidence that the existence of a weak coupling limit depends on certain regularity properties of the Bloch–Wigner transform introduced. The existence (and characterization) of a limit remains open.

More precisely, to connect the first and third point above, in the first result, using the homogenization Ansatz, we arrived at expression (3.19)

$$\partial_t \sigma_\kappa(t,p,\eta) + \frac{1}{2\pi}\nabla_\eta E_j(\eta)\cdot\nabla_p \sigma_\kappa(t,p,\eta)$$

$$= \lim_{\theta\longrightarrow 0} 2\sum_{n\in\mathbb{Z}^d} \hat{R}(n)\left[\sigma_\kappa(t,p,\eta) - \sigma_{\kappa+n}(t,p,\eta)\right]\left[\frac{\theta}{(E_{\kappa+n}(\eta) - E_\kappa(\eta))^2 + \theta^2}\right]$$

Then, working with the Wigner series, we proved in Theorem 4.10 that for observables supported away from band crossings one has trivial transport in the weak-coupling limit. Working with the Bloch–Wigner transform, we proved in Theorem 5.7 a (weak) version of the same statement once more, without needing to work within energy bands. Working with the observables using the Bloch–Wigner transform, we arrived at an expression involving the resonant operator $\mathbb{Y}_{st}^{\varepsilon,\eta,*}$ that simplified to an expression very reminiscent of the right hand side of the equality in the above expression from the asymptotic analysis:

$$=2\int_{\left[-\frac{1}{4},\frac{1}{4}\right]^d}\mathrm{d}\eta\int_{\mathbb{R}^d}\mathrm{d}p\sum_{\kappa\in\left(\frac{\mathbb{Z}}{2}\right)^d}T_s^{\varepsilon}(0,p,\eta,\kappa)\sum_{n\in\mathbb{Z}^d\backslash\backslash\{0\}}|\hat{V}(n)|^2$$

$$\int_s^t\mathrm{d}u\int_s^u\mathrm{d}v(\varepsilon^{-1}\cos(c_1(v-u)))[F(p,\kappa-\eta-n)-F(p,\kappa-\eta)]$$

We see that $T_s^{\varepsilon}(0,p,\eta,\kappa)$ plays a role analogous to $\sigma_\kappa(s,p,\eta)$. Instead of the Fourier transform of the covariance $\hat{R}(n)$ (which we got from taking expectations), we have $|\hat{V}(n)|^2$. We have different approximants to the delta in each expression, and the gain term minus the loss term. Hence we see the extent to which the asymptotic expansion captures the difficulties in the problem. Furthermore, we see that although the final expression we get in the asymptotic expression does not make sense pointwise, it could make sense when integrating over the η's, as is the case when working with the observables. Therefore, if a limit exists, it is not expected to be the solution of a linear Boltzmann equation, and requires a different interpretation.

We end this thesis by highlighting some promising future directions for further research:

1. Staring at the expression above for the problematic term in the observables, one realizes that if the sum over n is a Riemann sum, one formally obtains a linear Boltzmann equation. One could then ask what happens if one considers the period of the potential going to infinity depending on ε. This relates to the first motivation we wrote in the introduction, i.e., to find a different proof of the result of [EY99]. One way to do this this would be to introduce an additional parameter L to our Bloch–Wigner transform, and find a way to ensure uniform estimates in both ε and L on the non-resonant terms. Then, one could repeat the argument with the sewing lemma to pass to the limit. This could be a way to move towards our first goal of providing an alternate proof to the derivation of the linear Boltzmann equation. However, one will need to find a way to relate the potentials that are periodic with different periods, and proving the uniform estimates will probably require additional ideas. Working directly with the evolution equation of the rescaled Wigner transform instead of the rescaled Bloch–Wigner transform was challenging, since it was not clear how to prove the uniform in ε estimates for the terms with low time regularity. Our view is that the ideas in this thesis are a first step towards this larger goal.

2. As mentioned after the statement of the main theorem, working in the L^∞-based Banach space E_{-0} is not the correct framework for most families of initial data that go from the microscopic to macroscopic scale. Eventually, if one manages to make the point above work, one would need to identify the analogue for the rescaled Bloch–Wigner transform, of the Banach algebra in whose dual the usual rescaled Wigner transform is uniformly bounded.

3. As mentioned in the literature review, the limiting behavior of the periodic quantum Lorentz gas has been identified in the low-density scaling regime, assuming that a certain generalized Berry-Tabor conjecture holds, see [GM19], [GM21]. This complements the result of [Cas01], [Cas02], [Gri23] and [PR04], where the authors obtain linear Boltzmann equations by introducing damping terms in various manners. It would be interesting to see if the introduction of damping leads to a linear Boltzmann equation also in the weak coupling regime, and if one can prove such a result using the techniques introduced here. Another potential direction of research is to extend these techniques to the case of the low-density scaling.

4. The results presented in this thesis were all done in the Wigner function picture, and should have a dual version in the Weyl formalism when working with time evolution of operators in the interaction picture of quantum mechanics, as was done for example in [Spo77] and [GM19]. It would be interesting to prove the convergence in operator norm of the series defining the time evolution of the observables. This might require some of the ideas that were developed in this thesis, since one cannot use anymore Gaussian cancellations that were used in [Spo77]. It would be intriguing to see how one encounters the problem of energy-band crossings in this framework.

5. Recently, another approach was developed in the works [Her24], [BDH25] and [BDH] in order to derive the linear Boltzmann equation from the random quantum Lorentz gas in the kinetic regime, and the heat equation from the Schrödinger equation on time scales beyond the kinetic time scale. It would be interesting to establish how this connects to the proof strategy in this thesis.

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