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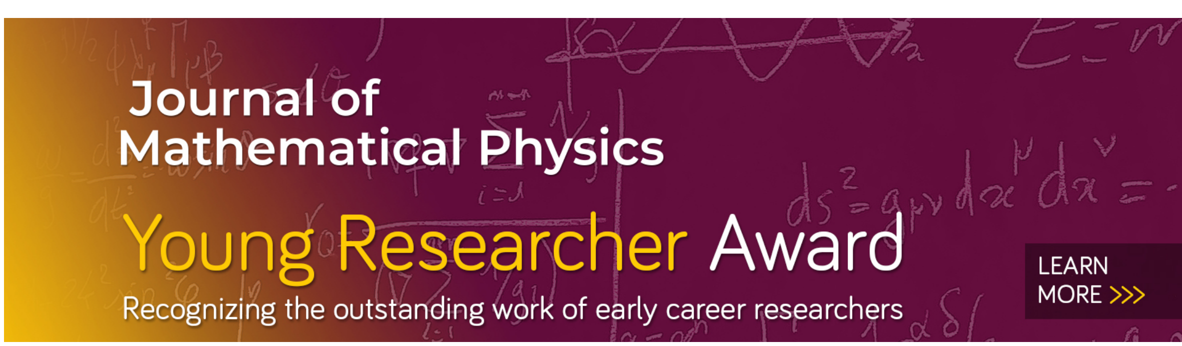

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ABSTRACT

We consider a system of N bosons in a unitary box in the grand-canonical setting interacting through a potential with the scattering length scaling as $N^{-1+\kappa}$, $\kappa \in (0, 2/3)$. This regimes interpolate between the Gross–Pitaevskii regime ($\kappa = 0$) and the thermodynamic limit ($\kappa = 2/3$). In the work of Basti *et al.* [Forum Math., Sigma **9**, E74 (2021)], as an intermediate step to prove an upper bound in agreement with the Lee–Huang–Yang formula in the thermodynamic limit, a second order upper bound on the ground state energy for $\kappa < 5/9$ was obtained. In this paper, thanks to a more careful analysis of the error terms, we extend the mentioned result to $\kappa < 7/12$.

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I. INTRODUCTION AND MAIN RESULT

It was predicted by Lee, Huang, and Yang in Ref. 1 (see also Ref. 2) that the ground state energy per unit volume of a dilute Bose gas satisfies

$$e(\rho) = 4\pi\mathfrak{a}\rho^2\left[1 + \frac{128}{15\sqrt{\pi}}(\rho\mathfrak{a}^3)^{1/2} + o((\rho\mathfrak{a}^3)^{1/2})\right], \tag{1.1}$$

where ρ denotes the particle density of the gas, $\mathfrak{a}$ is the scattering length of the interaction potential, and dilute refers to the fact that the mean interparticle distance is much larger than the scattering length, i.e., $\rho\mathfrak{a}^3 \ll 1$. The expansion (1.1) is known as the Lee–Huang–Yang formula, and its rigorous proof has been an open problem for a long time. In fact, while the leading term was already derived in Ref. 3 as an upper bound for hard sphere interactions, the matching lower bound was obtained 40 years later in Ref. 4. On the other hand, it was only with Ref. 5 (we also mention[6] where the correct constant is recovered in the weak coupling limit) that the next to leading order was proved to be correct as an upper bound for regular potentials and later with Ref. 7 for all potentials in L^3. Finally, in Ref. 8, the Lee–Huang–Yang correction was established as a lower bound for all L^1 potentials and in a recent paper[9] for a larger class of potentials, including the hard sphere case. Note that in the latter case, the matching upper bound is still missing.

We mention that the fermionic analog of the expansion (1.1) predicted in Ref. 10 has not yet been proved; see Refs. 11 and 12 where the first two orders are derived (due to the Pauli principle, an extra term of order $\rho^{5/3}$ appears).

In this paper, we will discuss and complement the result obtained in Ref. 7. There, as in Ref. 5, the core of the proof is to build a grand canonical trial state in a box with periodic boundary conditions whose size changes with ρ; in particular, the side length is assumed to be $\rho^{-\gamma}$ for some $\gamma > 1$. Indeed, following a well-known localization procedure (see, e.g., Ref. 13), this trial state can then be easily modified to provide a trial state with the correct energy on a lager box with Dirichlet boundary conditions. The latter can finally be replicated to recover the thermodynamic box in the limit. Note that in the limit, a grand canonical trial state can be proved to give an upper bound on the canonical ground state energy.

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Due to the strategy just described, Ref. 7 produced a side result of some interest on its own. Namely, it provided an upper bound correct up to the second order on the energy of a Hamiltonian acting on the Fock space built on a box whose side length is of the form $\rho^{-\gamma}$ for some $\gamma > 1$. By scaling, the described setting is equivalent to consider N bosons in the unitary box $\Lambda = [-1/2, 1/2]^3 \subset \mathbb{R}^3$ interacting through the Hamiltonian $\mathcal{H}_N$ acting on the bosonic Fock space $\mathcal{F}(\Lambda)$ whose action on the n-particle sector is given by

$$\mathcal{H}_N^{(n)} = \sum_{j=1}^{n} -\Delta_{x_j} + \sum_{1\le i,j\le n} N^{2-2\kappa} V(N^{1-\kappa}(x_i - x_j)), \tag{1.2}$$

with $\kappa \in (1/2, 2/3)$ [note that the request $k > 1/2$ comes from the assumption $\gamma > 1$ needed to use the localization technique, but it is never used in the proof of (1.3) below, which remains valid for $0 < \kappa < 1/2$]. In Ref. 7, it has been shown that, under suitable assumptions on the potential V, the ground state energy E_N of the Hamiltonian $\mathcal{H}_N$ satisfies

$$E_N \le 4\pi\mathfrak{a}N^{1+\kappa}\left(1 + \frac{128}{15\sqrt{\pi}}(\mathfrak{a}^3 N^{3\kappa-2})^{1/2}\right) + CN^{5\kappa/2}\max\{N^{-\varepsilon}, N^{9\kappa-5+6\varepsilon}, N^{21\kappa/4-3+3\varepsilon}\} \tag{1.3}$$

for all $\kappa \in (1/2, 2/3)$ and ε such that $3\kappa - 2 + 4\varepsilon < 0$. Let us stress that Eq. (1.3), whenever $\kappa < 5/9$, is just the equivalent of Eq. (1.1) written for the rescaled Hamiltonian (1.2) (note that the scattering length of the rescaled potential is given by $\mathfrak{a}/N^{1-\kappa}$ with $\mathfrak{a}$ the scattering length of the original potential).

The first step to construct the trial state leading to (1.3) is to generate the condensate since Bose Einstein condensation is expected to hold in the ground state of (1.2). Then, to add correlations, we act with a Bogoliubov transformation. However, it is known (see Refs. 6 and 14) that a quadratic operator is not enough to capture the correct second order of the energy. In Ref. 5, the exponential of the sum of a quadratic and a cubic operator was taken into account to better describe correlations. On the other hand, in Ref. 7, the exponential of a quadratic and of a cubic operator acts separately, inspired by the methods developed in recent years in Refs. 15 and 16 to study the Gross–Pitaevskii regime [corresponding to $\kappa = 0$ in (1.2)]. The drawback in considering the exponential of a cubic operator is the lack of explicit formulas for its action that makes computations harder. In particular, to handle the desired more singular regimes $\kappa > \frac{1}{2}$, new ideas are needed w.r.t. those used in Refs. 15 and 16. In fact, in Ref. 7, the cubic operator is implemented as a non-unitary operator acting directly on the vacuum with some crucial restrictions on the allowed momenta.

In Ref. 7, it was stated as a remark that the same method could have been pushed to cover all $\kappa < 7/12$, but such extension was out of the scope of that paper. Our aim here is to prove this statement based on a more careful analysis of some error terms. More precisely, we need to prove the following theorem.

Theorem 1.1. *Let $0 < \kappa < 7/12$ and $\varepsilon > 0$ small enough. Let $V \in L^3(\mathbb{R}^3)$ be non-negative and radially symmetric, with $\text{supp}(V) \subset B_R(0)$ and scattering length $\mathfrak{a}$. Then, for all N large enough,*

$$E_N \le 4\pi\mathfrak{a}N^{1+\kappa}\left(1 + \frac{128}{15\sqrt{\pi}}(\mathfrak{a}^3 N^{3\kappa-2})^{1/2}\right) + CN^{5\kappa/2}N^{-\varepsilon}. \tag{1.4}$$

We conclude this section with some comments about the scaling in (1.2). As we already mentioned, for $\kappa = 0$, one recovers the well-known Gross–Pitaevskii regime. In this setting, the expansion of the ground state energy has been established to first order in Refs. 4, 17, and 18, while the second order was proved in Ref. 16 (where also the low energy spectrum is derived) for all potentials in L^3 (and can be extended to all L^1 interactions as discussed in Ref. 19). We also mention[20] where a simplified approach is described. Recently, in Ref. 21 (see also Ref. 22), the second order correction has been obtained as an upper bound in the hard core case.

On the other hand, for $\kappa > 0$, the first order of the ground state energy was derived in Ref. 17, while the second order (and the low-energy excitation spectrum) was established in Ref. 23 for sufficiently small κ, making use of the analysis carried out in Ref. 24. We also mention that, for $\kappa < 1/10$, Bose Einstein condensation can be proved extending the analysis in Refs. 25–27, and in a recent paper,[28] condensation was shown for all $\kappa < 2/5$ (see also Ref. 29 where a similar but simpler regime is considered). Proving condensation for $\kappa = 2/3$, i.e., directly in the thermodynamic limit, is a challenging and widely open problem so far; see Refs. 30 and 31 for preliminary results. Note that all the mentioned results are valid in the canonical setting, while, on the contrary, the grand canonical setting is considered here.

II. DEFINITION OF THE TRIAL STATE

To obtain an upper bound on the ground state energy of the operator $\mathcal{H}_N$ defined in (1.2), we have to show a trial state whose energy is bounded by the rhs of (1.4). We first rewrite the Hamiltonian using the bosonic creation and annihilation operators a_p^*, a_p, and $p \in \Lambda^* = 2\pi\mathbb{Z}^3$ as follows:

$$\mathcal{H}_N = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{1}{2N^{1-\kappa}} \sum_{p,q,r\in\Lambda^*} \widehat{V}(r/N^{1-\kappa})\, a_{p+r}^* a_q^* a_{q+r} a_p. \tag{2.1}$$

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Our trial state is defined as in Ref. 7, and we recall here the definition referring the reader to Ref. 7 for more details. First, we introduce the Weyl operator

$$W_{N_0} = \exp\big[\sqrt{N_0}a_0^* - \sqrt{N_0}a_0\big], \tag{2.2}$$

where $N_0 > 0$ is a parameter to be fixed. The role of W_{N_0} is to generate the condensate; indeed, we expect most of the particle to be in the condensate wave function [i.e., $\varphi(x) = 1$, which is the ground state of the non-interacting problem]. Next, we have to take into account correlations among particles due to the presence of interaction. To this end, we consider the solution to the Neumann problem on the ball $|x| < N^{1-\kappa}\ell$,

$$\left[-\Delta + \frac{1}{2}V\right]f_\ell = \lambda_\ell f_\ell, \tag{2.3}$$

with $0 < \ell < 1/2$ and with the boundary condition $f(x) = 1$ if $|x| = N^{1-\kappa}\ell$. Furthermore, we define $f_{N,\ell}(x) = f_\ell(N^{1-\kappa}x)$ and set $\eta_p = -N\widehat{\omega}_{N,\ell}(p)$, where $w_{N,\ell} = 1 - f_{N,\ell}$ and the hat denotes the Fourier transform. Note that $|\eta_p| \le CN^\kappa p^{-2}$. To define the quadratic transformation, we also have to introduce two sets of momenta: the set of low momenta $P_L = \{p \in \Lambda^* : |p| \le N^{\kappa/2+\varepsilon}\} \subset \Lambda^*$ and its complement P_L^C. Then, to obtain the desired upper bound, we will consider a Bogoliubov transform whose kernel coincides with η on P_L^C. On the other hand, on the set of low momenta, we consider the kernel τ defined by

$$\tanh(2\tau_p) = -\frac{8\pi\mathfrak{a}N^\kappa}{p^2 + 8\pi\mathfrak{a}N^\kappa}.$$

We are now ready to introduce the Bogoliubov transformation

$$T_\nu = \exp\left(\frac{1}{2}\sum_{p\in\Lambda_+^*}\nu_p\big(a_p^*a_{-p}^* - \text{h.c.}\big)\right),$$

where the coefficients ν_p are defined as follows: $\nu_p = \eta_p$ for $p \in P_L^C$ and $\nu_p = \tau_p$ for $p \in P_L$. However, T_ν is still not enough to obtain the energy correct up to the second order, and to give a more precise description of correlations, we consider a cubic operator. To do so, we first introduce the notations $\gamma_p = \cosh(\nu_p)$ and $\sigma_p = \sinh(\nu_p)$. We also need two new sets of momenta: $P_H = \{p \in \Lambda^* : |p| > N^{1-\kappa-\varepsilon}\}$ and $P_S = \{p \in \Lambda^* : N^{\kappa/2-\varepsilon} \le |p| \le N^{\kappa/2+\varepsilon}\} \subset P_L$. Let us mention that considering the restriction of η to P_H (denoted by η_H) and the restriction of γ and σ to P_S (denoted by γ_S and σ_S, respectively), we have

$$\|\eta_H\|^2 \le CN^{3\kappa-1+\varepsilon}, \quad \|\eta_H\|_{H^1}^2 \le CN^{1+\kappa}, \quad \|\eta_H\|_\infty \le CN^{3\kappa-2+2\varepsilon} \tag{2.4}$$

and

$$\begin{aligned} &\|\sigma_S\|^2 \le CN^{3\kappa/2}, && \|\sigma_S\|_{H^1}^2 \le CN^{5\kappa/2+\varepsilon},\\ &\|\gamma_S\sigma_S\|_1 \le CN^{3\kappa/2+\varepsilon}, && \|\gamma_S\|_\infty^2, \|\sigma_S\|_\infty^2 \le CN^\varepsilon. \end{aligned} \tag{2.5}$$

With this notation at hand, we can finally define the desired cubic operator. Specifically,

$$A_\nu = \frac{1}{\sqrt{N}}\sum_{\substack{r\in P_H, v\in P_S:\\ r+v\in P_H}} \eta_r\sigma_v\, a_{r+v}^*a_{-r}^*a_{-v}^*\Theta_{r,v}, \tag{2.6}$$

where the operator $\Theta_{r,v}$, for $r \in P_H, v \in P_S$ with $r + v \in P_S$, is defined by

$$\Theta_{r,v} = \prod_{s\in P_H}[1 - \chi(\mathcal{N}_s > 0)\chi(\mathcal{N}_{-s+v} > 0)] \prod_{w\in P_S}[1 - \chi(\mathcal{N}_w > 0)\chi(\mathcal{N}_{r-w} + \mathcal{N}_{-r-v-w} > 0)]. \tag{2.7}$$

In (2.7), $\mathcal{N}_p = a_p^*a_p$ counts the number of particles with momentum p and χ is the characteristic function. The trial state we are going to consider to prove (1.4) is obtained acting on the vacuum vector $\Omega \in \mathcal{F}$ first with the operator e^{A_ν} and then with T_ν followed by W_{N_0}; finally, since e^{A_ν} is not a unitary operator, we have to normalize

$$\Psi_N = \frac{W_{N_0}T_\nu e^{A_\nu}\Omega}{\|e^{A_\nu}\Omega\|^2}.$$

Here, we fixed $N_0 = N - \|\sigma_L\|^2$.

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Let us stress that the role of the operator $\Theta_{r,v}$ in the definition of A_ν in (2.6) is to avoid certain relations among momenta created by the action of e^{A_ν} on the vacuum, and this results in a drastic simplification of the computations. As discussed in Ref. 7 (Sec. 2), one can write

$$A_\nu^m \Omega = \frac{1}{N^{m/2}} \sum_{\substack{r_1 \in P_H, v_1 \in P_S: \\ r_1+v_1 \in P_H}} \cdots \sum_{\substack{r_m \in P_H, v_m \in P_S: \\ r_m+v_m \in P_H}} \prod_{i=1}^m \eta_{r_i} \sigma_{v_i} \times \theta(\{r_j, v_j\}_{j=1}^m) a^*_{r_m+v_m} a^*_{-r_m} a^*_{-v_m} \ldots a^*_{r_1+v_1} a^*_{-r_1} a^*_{v_1} \Omega,$$

where θ encodes all the restrictions mentioned above,

$$\theta(\{r_j, v_j\}_{j=1}^m) = \prod_{\substack{i,j,\ k=1 \\ j \neq k}}^m \prod_{\substack{p_i \in \{-r_i, r_i+v_i\} \\ p_k \in \{-r_k, r_k+v_k\}}} \delta_{-p_i+v_j \neq p_k}. \tag{2.8}$$

Then, setting $\xi_\nu = e^{A_\nu}\Omega$, one has

$$\|\xi_\nu\|^2 = \sum_{m \geq 0} \frac{1}{2^m m!} \frac{1}{N^m} \sum_{\substack{v_1 \in P_S, r_1 \in P_H: \\ r_1+v_1 \in P_H}} \cdots \sum_{\substack{v_m \in P_S, r_m \in P_H: \\ r_m+v_m \in P_H}} \theta(\{r_j, v_j\}_{j=1}^m) \prod_{i=1}^m (\eta_{r_i} + \eta_{r_i+v_i})^2 \sigma_{v_i}^2. \tag{2.9}$$

III. PROOF OF THE MAIN THEOREM

In this section, we discuss the modifications that are needed to extend the result obtained in Ref. 7 to a larger set of choices of κ, proving Theorem 1.1.

Let

$$\mathcal{G}_N = T_\nu^* W_{N_0}^* \mathcal{H}_N W_{N_0} T_\nu$$

so that $\langle \Psi_N, \mathcal{H}_N \Psi_N \rangle = \langle \xi_N, \mathcal{G}_N \xi_N \rangle / \|\xi_N\|^2$. Moreover, let us introduce the kinetic energy operator $\mathcal{K} = \sum_{p \in \Lambda^*} p^2 a_p^* a_p$ and the operators

$$\mathcal{V}_N^{(H)} = \frac{1}{2\ N} \sum_{\substack{r \in \Lambda^*,\ p,q \in P_H: \\ p+r, q+r \in P_H}} N^\kappa \widehat{V}(r/N^{1-\kappa}) a^*_{p+r} a^*_q a_p a_{q+r} \tag{3.1}$$

and

$$\mathcal{C}_N = \frac{\sqrt{N_0}}{N} \sum_{\substack{p, r \in P_H \\ p+r \in P_S}} N^\kappa \widehat{V}(r/N^{1-\kappa})\ \sigma_{p+r} \gamma_p \gamma_r\ (a^*_{p+r} a^*_{-p} a^*_{-r} + \text{h.c.}). \tag{3.2}$$

Finally, let

$$\begin{aligned} C_{\mathcal{G}_N} = {} & \frac{N^{1+\kappa}}{2} \widehat{V}(0) + \sum_{p \in \Lambda_+^*} p^2 \sigma_p^2 + \sum_{p \in \Lambda_+^*} N^\kappa \widehat{V}(p/N^{1-\kappa}) \sigma_p \gamma_p \\ & + \sum_{p \in P_L} N^\kappa \widehat{V}(p/N^{1-\kappa}) \sigma_p^2 + \frac{1}{2N} \sum_{\substack{p, r \in \Lambda_+^* \\ r \neq p}} N^\kappa \widehat{V}(r/N^{1-\kappa}) \sigma_p \sigma_{p-r} \gamma_p \gamma_{p-r} \\ & - \frac{1}{N} \sum_{v \in P_L} \sigma_v^2 \sum_{p \in P_L^c} N^\kappa \widehat{V}(p/N^{1-\kappa}) \eta_p. \end{aligned} \tag{3.3}$$

It has been shown in Ref. 7 (Proposition 3.1) that, for any $0 < \kappa < 2/3$ and $\varepsilon > 0$ such that $3\kappa - 2 + 4\varepsilon < 0$,

$$\langle \Psi_N, \mathcal{H}_N \Psi_N \rangle = \frac{\langle \xi_\nu, \mathcal{G}_N \xi_\nu \rangle}{\|\xi_\nu\|^2} \leq C_{\mathcal{G}_N} + \frac{\langle \xi_\nu, (\mathcal{K} + \mathcal{V}_N^{(H)} + \mathcal{C}_N) \xi_\nu \rangle}{\|\xi_\nu\|^2} + \frac{\langle \xi_\nu, \mathcal{E} \xi_\nu \rangle}{\|\xi_\nu\|^2}, \tag{3.4}$$

with

$$\frac{\langle \xi_\nu, \mathcal{E} \xi_\nu \rangle}{\|\xi_\nu\|^2} \leq C N^{5\kappa/2} \cdot \max\{N^{-\varepsilon}, N^{9\kappa-5+6\varepsilon}, N^{21\kappa/4-3+3\varepsilon}\}. \tag{3.5}$$

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Moreover, $C_{\mathcal{G}_N}$ and the expectation on ξ_ν of the operators $\mathcal{K}, \mathcal{C}, \mathcal{V}_N^{(H)}$ provide the correct energy up to an error that is small under the previous assumptions (see Ref. 7, Secs. 3 and 5),

$$C_{\mathcal{G}_N} + \frac{\langle \xi_\nu, (\mathcal{K} + \mathcal{V}_N^{(H)} + \mathcal{C}_N)\xi_\nu\rangle}{\|\xi_\nu\|^2} \le 4\pi\mathfrak{a}N^{1+\kappa}\left(1 + \frac{128}{15\sqrt{\pi}}(\mathfrak{a}^3 N^{3\kappa-2})^{1/2}\right) + CN^{5\kappa/2}\max\{N^{-\varepsilon}, N^{12\kappa-7+5\varepsilon}\}. \tag{3.6}$$

The bound (3.5) was obtained in Ref. 7 using suitable bounds on the expectation over ξ_ν of products of the kinetic energy $\mathcal{K}$ and powers of the particle number operator $\mathcal{N} = \sum_{p\in\Lambda^*} a_p^* a_p$. However, it is clearly not compatible with the claim that (1.4) is satisfied for all $\kappa < 7/12$ and is indeed responsible for the request $\kappa < 5/9$ in Ref. 7. Therefore, in order to prove Theorem 1.1, we have to obtain an improved estimate on the expectation of the error term $\mathcal{E}$ coming from the quadratic transformation.

Remark. Note that the error of the form $N^{5\kappa/2}N^{12\kappa-7+5\varepsilon}$ appearing in (3.6) and coming from the action of the cubic operator e^{A_ν}, in particular from the restrictions on the allowed momenta encoded in the operator Θ, cannot be improved with the methods presented here. Therefore, to treat $k > 7/12$, new ideas are needed.

To improve the estimate (3.5), we first identify, with a careful reading of the proof of Proposition 3.1 in Ref. 7, those terms in $\mathcal{E}$, giving the worst rate. Using the notation of Ref. 7, Sec. 4 they are F_2, F_3, the first two terms in G_2, the first term in G_3 and $G_1 - \mathcal{V}_N^{(H)}$ (note that they all come from the conjufation of the cubic and quartic term). Therefore, we rewrite

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \tag{3.7}$$

with

$$\frac{\langle \xi_\nu \mathcal{E}_1 \xi_\nu\rangle}{\|\xi_\nu\|^2} \le CN^{5\kappa/2-\varepsilon} \tag{3.8}$$

for all $\kappa < 2/3$ choosing ε small enough. Thus, we can focus on $\mathcal{E}_2$ that can be split as

$$\mathcal{E}_2 = \mathcal{E}_C + \mathcal{E}_S + \mathcal{E}_H + \mathcal{E}_M. \tag{3.9}$$

Here, $\mathcal{E}_C$ is the cubic operator defined by

$$\mathcal{E}_C = \frac{\sqrt{N_0}}{N}\sum_{\substack{p\in P_H, r\in P_S:\\ p+r\in P_H}} \alpha(p,r)(a_{p+r}^* a_{-p}^* a_{-r}^* + \text{h.c.}), \tag{3.10}$$

where we introduced the notation

$$\alpha(p,r) = N^\kappa\left(\widehat{V}\left(\frac{r}{N^{1-\kappa}}\right) + \widehat{V}\left(\frac{p}{N^{1-\kappa}}\right)\right)(\gamma_r\gamma_p\sigma_{p+r} + \sigma_r\sigma_p\gamma_{p+r}).$$

On the other hand, $\mathcal{E}_S, \mathcal{E}_H$, and $\mathcal{E}_M$ are quartic operators. In particular, in $\mathcal{E}_H$, only operators with high momenta appear, and it is defined by

$$\begin{aligned}\mathcal{E}_H &= \mathcal{E}_{H,1} + \mathcal{E}_{H,2}\\ &= \frac{1}{2N}\sum_{r\in\Lambda^*}\sum_{\substack{p,q\in P_H:\\ p+r,q+r\in P_H}} \beta_1(p,q,r)a_p^* a_{q+r}^* a_q a_{p+r} + \frac{1}{N}\sum_{r\in\Lambda_+^*}\sum_{\substack{p,q\in P_H:\\ p+r,q+r\in P_H}} \beta_2(p,q,r)a_{p+r}^* a_{-p}^* a_{q+r} a_{-q},\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}\beta_1(p,q,r) &= N^\kappa\widehat{V}\left(\frac{r}{N^{1-\kappa}}\right)[(\gamma_p\gamma_q\gamma_{p+r}\gamma_{q+r} - 1) + \sigma_p\sigma_{p+r}\sigma_q\sigma_{q+r} + \gamma_p\gamma_{p+r}\sigma_q\sigma_{q+r}],\\ \beta_2(p,q,r) &= N^\kappa\widehat{V}\left(\frac{r}{N^{1-\kappa}}\right)\gamma_{p+r}\gamma_{q+r}\sigma_p\sigma_q.\end{aligned}$$

Conversely, in $\mathcal{E}_S$, only momenta in P_S are involved. In fact, setting

$$\begin{aligned}\zeta_1(p,q,r) &= N^\kappa\widehat{V}\left(\frac{r}{N^{1-\kappa}}\right)(\sigma_p\sigma_{p+r}\sigma_q\sigma_{q+r} + \gamma_p\gamma_q\gamma_{p+r}\gamma_{q+r} + \gamma_p\gamma_{p+r}\sigma_q\sigma_{q+r}),\\ \zeta_2(p,q,r) &= N^\kappa\widehat{V}\left(\frac{r}{N^{1-\kappa}}\right)\gamma_{p+r}\gamma_{q+r}\sigma_p\sigma_q,\end{aligned}$$

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we have

$$\mathcal{E}_S = \mathcal{E}_{S,1} + \mathcal{E}_{S,2} = \frac{1}{2N}\sum_{r\in\Lambda^*}\sum_{\substack{p,q\in P_S:\\ p+r,q+r\in P_S}} \zeta_1(p,q,r)a_p^* a_{q+r}^* a_q a_{p+r} + \frac{1}{N}\sum_{r\in\Lambda_+^*}\sum_{\substack{p,q\in P_S:\\ p+r,q+r\in P_S}} \zeta_2(p,q,r)a_{p+r}^* a_{-p}^* a_{q+r} a_{-q}. \tag{3.12}$$

Finally, $\mathcal{E}_M$ contains terms where two operators have momenta in P_H and two operators have momenta in P_S. More precisely,

$$\begin{aligned}\mathcal{E}_M &= \mathcal{E}_{M,1} + \mathcal{E}_{M,2} + \mathcal{E}_{M,3}\\ &= \frac{1}{N}\sum_{r\in\Lambda^*}\sum_{\substack{p,q\in P_H:\\ p+r,q+r\in P_S}} \varphi_1(p,q,r)a_p^* a_{q+r}^* a_q a_{p+r}\\ &\quad + \frac{1}{N}\sum_{r\in\Lambda_+^*}\sum_{\substack{p,q\in P_H:\\ p+r,q+r\in P_S}} \varphi_2(p,q,r)a_{p+r}^* a_{-p}^* a_{q+r} a_{-q}\\ &\quad + \frac{1}{N}\sum_{r\in\Lambda^*}\sum_{\substack{p\in P_S,q\in P_H:\\ p+r\in P_S,q+r\in P_H}} \varphi_3(p,q,r)a_p^* a_{q+r}^* a_q a_{p+r},\end{aligned} \tag{3.13}$$

with

$$\begin{aligned}\varphi_1(p,q,r) &= N^\kappa \widehat{V}\left(\frac{r}{N^{1-\kappa}}\right)(\sigma_p\sigma_q\sigma_{p+r}\sigma_{q+r} + \gamma_p\gamma_q\gamma_{p+r}\gamma_{q+r} + 2\gamma_p\gamma_{p+r}\sigma_q\sigma_{q+r}),\\ \varphi_2(p,q,r) &= N^\kappa \widehat{V}\left(\frac{r}{N^{1-\kappa}}\right)(\gamma_{p+r}\gamma_{q+r}\sigma_p\sigma_q + \gamma_p\gamma_{q+r}\sigma_{p+r}\sigma_q + \gamma_{p+r}\gamma_q\sigma_p\sigma_{q+r} + \gamma_p\gamma_q\sigma_{p+r}\sigma_{q+r}),\\ \varphi_3(p,q,r) &= N^\kappa \widehat{V}\left(\frac{r}{N^{1-\kappa}}\right)(\sigma_p\sigma_{p+r}\sigma_q\sigma_{q+r} + \gamma_p\gamma_q\gamma_{p+r}\gamma_{q+r} + \gamma_p\gamma_{p+r}\sigma_q\sigma_{q+r} + \sigma_q\sigma_{p+r}\gamma_p\gamma_{q+r}).\end{aligned}$$

Our goal is to prove

$$\frac{\langle \xi_\nu, \mathcal{E}_2\xi_\nu\rangle}{\|\xi_\nu\|^2} \le CN^{5\kappa/2-\varepsilon} \tag{3.14}$$

for all $\kappa < 2/3$ and ε small enough.

Then, (3.14) together with (3.4), (3.6)–(3.8) immediately yield (1.4)

To obtain the improved estimate (3.14), the idea is to compute the expectation of each operator appearing on the right hand side of (3.9) using the definition of ξ_ν. This is done in the rest of this section.

Remark. Note that (3.7), (3.8), and (3.14) imply that the quadratic conjugation only produces errors that remain small up to the thermodynamic limit, i.e., for all $\kappa < 2/3$. Indeed, the restriction to $\kappa < 7/12$ comes from the action of the exponential of the cubic operator [see (3.6)].

A. Bound of the expectation of $\mathcal{E}_C$ on ξ_ν

We start noting that the cubic error term $\mathcal{E}_C$ has the same structure as the cubic term $\mathcal{C}$, giving a large contribution to the energy analyzed in Ref. 7 (Sec. 5.2).

Recalling the definition of A_ν given in (2.6), we easily obtain

$$\begin{aligned}\langle \xi_\nu, \mathcal{E}_C\xi_\nu\rangle &= 2\frac{\sqrt{N_0}}{N}\sum_{m\ge 1}\frac{1}{m!(m-1)!}\sum_{\substack{p\in P_H,r\in P_S:\\ p+r\in P_H}} \alpha(p,r)\langle A_\nu^m\Omega, a_{p+r}^* a_{-p}^* a_{-r}^* A_\nu^{m-1}\Omega\rangle\\ &= 2\sqrt{\frac{N_0}{N}}\sum_{m\ge 1}\frac{1}{m!(m-1)!N^m}\sum_{\substack{v_1,\tilde{v}_1\in P_S,r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}}\cdots\sum_{\substack{v_m\in P_S,r_m\in P_H:\\ r_m+v_m\in P_H}} \theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^{m-1})\\ &\quad\times\prod_{i=1}^m \eta_{r_j}\sigma_{v_j}\prod_{j=1}^{m-1}\eta_{\tilde{r}_j}\sigma_{\tilde{v}_j}\sum_{\substack{p\in P_H,r\in P_S:\\ p+r\in P_H}} \alpha(p,r)\langle\Omega, a_{r_m+v_m}\dots a_{-v_1}a_{p+r}^* a_{-p}^* a_{-r}^* a_{\tilde{r}_{m-1}+\tilde{v}_{m-1}}^*\dots a_{-\tilde{v}_1}^*\Omega\rangle.\end{aligned}$$

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Noting that the expectation in the last line vanishes unless there exists an index $i \in \{1, \ldots, m\}$ such that $r = v_i$ and pairing the remaining momenta in P_S (by symmetry, we can assume $r = -v_m$ and $\tilde{v}_j = v_j$ for all $j = 1, \ldots, m-1$), we find

$$\begin{aligned}\langle \xi_\nu, \mathcal{E}_C \xi_\nu \rangle = 2\sqrt{\frac{N_0}{N}} \sum_{m\geq 1} \frac{1}{(m-1)! N^m} \sum_{\substack{v_1 \in P_S, r_1, \tilde{r}_1 \in P_H: \\ r_1+v_1, \tilde{r}_1+v_1 \in P_H}} \cdots \sum_{\substack{v_m \in P_S, r_m \in P_H: \\ r_m+v_m \in P_H}} \theta(\{r_j, v_j\}_{j=1}^m) \theta(\{\tilde{r}_j, v_j\}_{j=1}^{m-1}) \\ \times \prod_{j=1}^{m-1} \eta_{r_j} \eta_{\tilde{r}_j} \sigma_{v_j}^2 \eta_{r_m} \sigma_{v_m} \sum_{\substack{p \in P_H: \\ p+v_m \in P_H}} \alpha(p, v_m) \langle \Omega, A_{r_m, v_m} \ldots A_{r_1, v_1} A^*_{p, v_m} \ldots A^*_{-\tilde{r}_1, v_1} \Omega \rangle,\end{aligned} \tag{3.15}$$

where we introduced the notation $A_{r,v} = a_{r+v} a_{-r}$ for any $v \in P_S$ and $r \in P_H$ such that $r + v \in P_H$ (and $A^*_{r,v}$ to denote the adjoint). Using now the fact that, due to the presence of $\theta(\{r_j, v_j\}_{j=1}^m)\theta(\{\tilde{r}_j, v_j\}_{j=1}^{m-1})$, each A_{r_j,v_j} has to be contracted with $A^*_{\tilde{r}_j, v_j}$ for any $j = 1, \ldots, m-1$ and, therefore, A_{r_m, v_m} is contracted with A^*_{p, v_m}, we obtain

$$\begin{aligned}\langle \xi_\nu, \mathcal{E}_C \xi_\nu \rangle = 2\sqrt{\frac{N_0}{N}} \sum_{m\geq 1} \frac{1}{(m-1)! N^m} \sum_{\substack{v_1 \in P_S, r_1, \tilde{r}_1 \in P_H: \\ r_1+v_1, \tilde{r}_1+v_1 \in P_H}} \cdots \sum_{\substack{v_m \in P_S, r_m \in P_H: \\ r_m+v_m \in P_H}} \theta(\{r_j, v_j\}_{j=1}^m) \theta(\{\tilde{r}_j, v_j\}_{j=1}^{m-1}) \\ \times \prod_{j=1}^{m-1} \eta_{r_j} \eta_{\tilde{r}_j} (\delta_{\tilde{r}_j, r_j} + \delta_{-\tilde{r}_j, r_j+v_j}) \sigma_{v_j}^2 \eta_{r_m} \sigma_{v_m} \sum_{\substack{p \in P_H: \\ p+v_m \in P_H}} \alpha(p, v_m) \sum_{p_m \in \{-r_m, r_m+v_m\}} \delta_{-p, p_m}.\end{aligned} \tag{3.16}$$

We now note that

$$|\alpha(p, v_m) \delta_{p, -p_m}| \leq C N^\kappa (\gamma_{v_m} |\sigma_{-p_m+v_m}| + |\sigma_{v_m}||\sigma_{p_m}|).$$

Taking the absolute value in (3.16) and using the fact that when all terms in the sum are positive, we can replace $\theta(\{r_j, v_j\}_{j=1}^m)$ with $\theta(\{r_j, v_j\}_{j=1}^{m-1})$ obtaining an upper bound. Therefore, using (2.4), (2.5), and (2.9) we obtain

$$\frac{|\langle \xi_\nu, \mathcal{E}_C \xi_\nu \rangle|}{\|\xi_\nu\|^2} \leq C N^{\kappa-1} \|\eta_H\|^2 (\|\gamma_S \sigma_S\|_1 + \|\sigma_S\|^2) \leq C N^{5\kappa/2 - \varepsilon} \tag{3.17}$$

for all $\kappa < 2/3$ and ε small enough.

B. Bound of the expectation of $\mathcal{E}_H$ on ξ_ν

To bound the quartic error term $\mathcal{E}_H$, we start considering $\mathcal{E}_{H,1}$ that has the same form as the large quartic term $\mathcal{V}_N^{(H)}$ (see Ref. 7, Sec. 5.3). We write

$$\begin{aligned}\langle \xi_\nu, \mathcal{E}_{H,1} \xi_\nu \rangle = \sum_{m\geq 1} \frac{1}{m!} \frac{1}{2N^{m+1}} \sum_{\substack{v_1 \in P_S, r_1, \tilde{r}_1 \in P_H: \\ r_1+v_1, \tilde{r}_1+v_1 \in P_H}} \cdots \sum_{\substack{v_m \in P_S, r_m, \tilde{r}_m \in P_H: \\ r_m+v_m, \tilde{r}_m+v_m \in P_H}} \theta(\{r_j, v_j\}_{j=1}^m) \theta(\{\tilde{r}_j, v_j\}_{j=1}^m) \prod_{j=1}^m \eta_{r_j} \eta_{\tilde{r}_j} \sigma_{v_j}^2 \\ \times \sum_{r \in \Lambda^*} \sum_{\substack{p,q \in P_H: \\ p+r, q+r \in P_H}} \beta_1(p, q, r) \langle \Omega, A_{r_1, v_1} \ldots A_{r_m, v_m} a^*_p a^*_{q+r} a_q a_{p+r} A^*_{\tilde{r}_1, v_1} \ldots A^*_{\tilde{r}_m, v_m} \Omega \rangle,\end{aligned}$$

where we paired all momenta in P_S.

We now distinguish two contributions: the first one corresponds to the situation in which a_q and a_{p+r} are annihilated with $A^*_{\tilde{r}_i, v_i}$ for some $i = 1, \ldots, m$ [this also implies, taking into account the presence of $\theta(\{r_j, v_j\}_{j=1}^m)\theta(\{\tilde{r}_j, v_j\}_{j=1}^m)$, that a^*_p, a^*_{q+r} are annihilated with A_{r_i, v_i}]. The second case on the other hand arises when a_q and a_{p+r} are annihilated with $a^*_{\tilde{p}_i}, a^*_{\tilde{p}_j}$ for $\tilde{p}_\ell \in \{-\tilde{r}_\ell, \tilde{r}_\ell + v_\ell\}$, $\ell = i$ and j with $i \neq j$ (then, a^*_p, a^*_{q+r} are annihilated with a_{p_i}, a_{p_j} for $p_\ell \in \{-r_\ell, r_\ell + v_\ell\}$, $\ell = i$ and j, and $a^*_{-\tilde{p}_i + v_i}, a^*_{-\tilde{p}_j + v_j}$ are annihilated with $a_{-p_i + v_i}, a_{-p_j + v_j}$).

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We denote the two contributions described in the previous paragraph by A and B, respectively, so that

$$\langle \xi_\nu, \mathcal{E}_{H,1}\xi_\nu\rangle = A + B, \tag{3.18}$$

with

$$\begin{aligned} A &= \sum_{m\geq 1}\frac{1}{(m-1)!}\frac{1}{2N^{m+1}}\sum_{\substack{v_1\in P_S, r_1,\tilde r_1\in P_H:\\ r_1+v_1,\tilde r_1+v_1\in P_H}}\cdots\sum_{\substack{v_m\in P_S, r_m,\tilde r_m\in P_H:\\ r_m+v_m,\tilde r_m+v_m\in P_H}}\theta\big(\{r_j,v_j\}_{j=1}^m\big)\theta\big(\{\tilde r_j,v_j\}_{j=1}^m\big)\\ &\quad\times\prod_{j=1}^m\eta_{r_j}\eta_{\tilde r_j}\sigma_{v_j}^2\prod_{j=1}^{m-1}\big(\delta_{r_j,\tilde r_j}+\delta_{-r_j,\tilde r_j+v_j}\big)\sum_{r\in\Lambda^*}\sum_{p\in P_H}\beta_1(p,v_m-p-r,r)\sum_{\substack{p_m\in\{-r_m,r_m+v_m\}\\ \tilde p_m\in\{-\tilde r_m,\tilde r_m+v_m\}}}\delta_{p,p_m}\delta_{p+r,\tilde p_m},\\ B &= \sum_{m\geq 2}\frac{1}{(m-2)!}\frac{1}{2N^{m+1}}\sum_{\substack{v_1\in P_S, r_1,\tilde r_1\in P_H:\\ r_1+v_1,\ \tilde r_1+v_1\in P_H}}\cdots\sum_{\substack{v_m\in P_S, r_m,\tilde r_m\in P_H:\\ r_m+v_m,\ \tilde r_m+v_m\in P_H}}\theta\big(\{r_j,v_j\}_{j=1}^m\big)\theta\big(\{\tilde r_j,v_j\}_{j=1}^m\big)\\ &\quad\times\prod_{i=1}^m\eta_{r_i}\eta_{\tilde r_i}\sigma_{v_i}^2\prod_{j=1}^{m-2}\big(\delta_{\tilde r_i,r_i}+\delta_{-\tilde r_i,r_i+v_i}\big)\sum_{r\in\Lambda^*}\sum_{p,q\in P_H}\beta_1(p,q,r)\sum_{\substack{p_\ell\in\{-r_\ell,r_\ell+v_\ell\}\\ \ell=m-1,m}}\delta_{p,p_m}\delta_{q+r,p_{m-1}}\\ &\quad\times\sum_{\substack{\tilde p_\ell\in\{-\tilde r_\ell,\tilde r_\ell+v_\ell\}\\ \ell=m-1,m}}\big(\delta_{q,\tilde p_m}\delta_{p+r,\tilde p_{m-1}}+\delta_{q,\tilde p_{m-1}}\delta_{p+r,\tilde p_m}\big)\big(\delta_{\tilde p_m,p_m}+\delta_{-\tilde p_m+v_m,-p_{m-1}+v_{m-1}}\big). \end{aligned}$$

Thus, the bound

$$\begin{aligned} |\beta_1(p,v_m-p-r,r)\delta_{p,p_m}\delta_{p+r,\tilde p_m}| &\leq CN^\kappa\widehat V\Big(\frac{r}{N^{1-\kappa}}\Big)\big(|\eta_p|^3+|\eta_{p+r}|^3+|\eta_{p-v_m}|^3+|\eta_{p+r-v_m}|^3\big)\\ &\quad + CN^\kappa\big(\|\eta_H\|_\infty^2|\eta_p\|\eta_{p+r}|+|\eta_{p-v_m}\|\eta_{p+r-v_m}|\big) \end{aligned}$$

yields

$$\frac{A}{\|\xi_\nu\|^2}\leq CN^{\kappa-2}\|\sigma_S\|^2\big(N\|\eta_H\|_\infty^2\|\eta_H\|^2+\|\eta_H\|_\infty^2\|\eta_H\|^4+\|\eta_H\|^4\big), \tag{3.19}$$

where we used the bound $\sup_{r\in\Lambda^*}\sum_{p\in P_H}N^\kappa\widehat V(r/N^{1-\kappa})/|p-r|^2\leq N$ and the fact that $|\eta_p|\leq CN^\kappa|p|^{-2}$.

On the other hand, since $\sup_{\substack{r\in\Lambda^*,p,q\in P_H:\\ p+r,q+r\in P_H}}|\beta_1(p,q,r)|\leq CN^\kappa\|\eta_H\|_\infty^2$, we obtain

$$\frac{|B|}{\|\xi_\nu\|^2}\leq CN^{\kappa-3}\|\sigma_S\|^4\|\eta_H\|^4\|\eta_H\|_\infty^2. \tag{3.20}$$

We now consider $\mathcal{E}_{H,2}$. First, pairing all momenta in P_S, we rewrite

$$\begin{aligned} \langle \xi_\nu,\mathcal{E}_{H,2}\xi_\nu\rangle &= \sum_{m\geq 1}\frac{1}{m!}\frac{1}{N^{m+1}}\sum_{\substack{v_1\in P_S, r_1,\tilde r_1\in P_H:\\ r_1+v_1,\tilde r_1+v_1\in P_H}}\cdots\sum_{\substack{v_m\in P_S, r_m,\tilde r_m\in P_H:\\ r_m+v_m,\tilde r_m+v_m\in P_H}}\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde r_j,v_j\}_{j=1}^m)\prod_{j=1}^m\eta_{r_j}\eta_{\tilde r_j}\sigma_{v_j}^2\\ &\quad\times\sum_{r\in\Lambda^*}\sum_{\substack{p,q\in P_H:\\ p+r,q+r\in P_H}}\beta_2(p,q,r)\langle\Omega, A_{r_1,v_1}\dots A_{r_m,v_m}a_{p+r}^*a_{-p}^*a_{q+r}a_{-q}A_{\tilde r_1,v_1}^*\dots A_{\tilde r_m,v_m}^*\Omega\rangle. \end{aligned}$$

We note that also in this case, we can distinguish two contributions depending on whether the operators $a_{q+r}a_{-q}$ are annihilated with $A_{\tilde r_j,v_j}^*$ or with $a_{\tilde p_j}^*, a_{\tilde p_k}^*$, with $\tilde p_\ell\in\{-\tilde r_\ell,\tilde r_\ell+v_\ell\}$, $\ell=j,k$ with $j\neq k$.

Hence, we split

$$\langle\xi_\nu,\mathcal{E}_{H,2}\xi_\nu\rangle = C+D, \tag{3.21}$$

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with C and D defined by

$$C = \sum_{m\geq 1} \frac{1}{(m-1)!} \frac{1}{N^{m+1}} \sum_{\substack{v_1\in P_S, r_1, \tilde{r}_1 \in P_H: \\ r_1+v_1, \tilde{r}_1+v_1\in P_H}} \cdots \sum_{\substack{v_m\in P_S, r_m, \tilde{r}_m \in P_H: \\ r_m+v_m, \tilde{r}_m+v_m\in P_H}} \theta(\{r_j, v_j\}_{j=1}^m)\theta(\{\tilde{r}_j, v_j\}_{j=1}^m)\eta_{r_m}\eta_{\tilde{r}_m}\sigma_{v_m}^2$$
$$\times \prod_{j=1}^{m-1} \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}^2 (\delta_{\tilde{r}_j, r_j} + \delta_{-\tilde{r}_j, r_j+v_j}) \sum_{p,q\in P_H} \beta_2(p,q,v_m) \sum_{\substack{p_m\in\{-r_m, r_m+v_m\} \\ \tilde{p}_m\in\{-\tilde{r}_m, \tilde{r}_m+v_m\}}} \delta_{-p,p_m}\delta_{-q,\tilde{p}_m},$$

$$D = \sum_{m\geq 2} \frac{1}{(m-2)!} \frac{1}{N^{m+1}} \sum_{\substack{v_1\in P_S, r_1, \tilde{r}_1 \in P_H: \\ r_1+v_1, \tilde{r}_1+v_1\in P_H}} \cdots \sum_{\substack{v_m\in P_S, r_m, \tilde{r}_m \in P_H: \\ r_m+v_m, \tilde{r}_m+v_m\in P_H}} \theta(\{r_j, v_j\}_{j=1}^m)\theta(\{\tilde{r}_j, v_j\}_{j=1}^m)$$
$$\times \prod_{j=1}^{m} \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}^2 \prod_{i=1}^{m-2} (\delta_{\tilde{r}_j, r_j} + \delta_{-\tilde{r}_j, r_j+v_j}) \sum_{r\in\Lambda^*}\sum_{p,q\in P_H} \beta_2(p,q,r) \sum_{\substack{p_\ell\in\{-r_\ell, r_\ell+v_\ell\} \\ \tilde{p}_\ell\in\{-\tilde{r}_\ell, \tilde{r}_\ell+v_\ell\} \\ \ell=m-1,m}} \delta_{-q,\tilde{p}_m}\delta_{q+r,\tilde{p}_{m-1}}$$
$$\times (\delta_{-p,p_m}\delta_{p+r,p_{m-1}} + \delta_{-p,p_{m-1}}\delta_{p+r,p_m})(\delta_{p_m,\tilde{p}_m} + \delta_{-\tilde{p}_m+v_m, -p_{m-1}+v_{m-1}}).$$

Taking the absolute value, which allows us to forget $\theta(\{\tilde{r}_j, v_j\}_{j=1}^m)$ and replace $\theta(\{r_j, v_j\}_{j=1}^m)$ with $\theta(\{r_j, v_j\}_{j=1}^{m-1})$, and noting that $|\beta_2(p,q,r)| \leq CN^\kappa|\eta_p||\eta_q|$ for any $r\in\Lambda^*, p,q\in P_H$ with $p+r, q+r\in P_H$, we obtain

$$\frac{|C|}{\|\xi_\nu\|^2} \leq CN^{\kappa-2}\|\sigma_S\|^2\|\eta_H\|^4 \tag{3.22}$$

and

$$\frac{|D|}{\|\xi_\nu\|^2} \leq CN^{\kappa-3}\|\sigma_S\|^4\|\eta_H\|^4\|\eta_H\|_\infty^2. \tag{3.23}$$

With (3.18) and (3.21), using the bounds (3.19), (3.20), (3.22), and (3.23) and recalling (2.4) and (2.5), we obtain

$$\frac{|\langle\xi_\nu, \mathcal{E}_H\xi_\nu\rangle|}{\|\xi_\nu\|^2} \leq CN^{5\kappa/2-\varepsilon} \tag{3.24}$$

for any $\kappa < 2/3$ and ε small enough.

C. Bound of the expectation of $\mathcal{E}_S$ on ξ_ν

In this subsection, we focus on $\mathcal{E}_S$. Let us first consider the contribution coming from $\mathcal{E}_{S,1}$. Recalling the definition of $\mathcal{E}_{S,1}$, see (3.12), we can write

$$\langle\xi_\nu, \mathcal{E}_{S,1}\xi_\nu\rangle = \sum_{m\geq 2} \frac{1}{(m!)^2} \frac{1}{2N^{m+1}} \sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H: \\ r_1+v_1, \tilde{r}_1+\tilde{v}_1\in P_H}} \cdots \sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H: \\ r_m+v_m, \tilde{r}_m+\tilde{v}_m\in P_H}}$$
$$\times \theta(\{r_j, v_j\}_{j=1}^m)\theta(\{\tilde{r}_j, \tilde{v}_j\}_{j=1}^m)\prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j} \sum_{r\in\Lambda^*} \sum_{\substack{p,q\in P_S: \\ p+r, q+r\in P_S}} \zeta_1(p,q,r)$$
$$\times \langle\Omega, a_{r_m+v_m}a_{-r_m}a_{-v_m}\ldots a_{-v_1}a_{p+r}^*a_q^*a_pa_{q+r}a_{\tilde{r}_m+\tilde{v}_m}^*a_{-\tilde{r}_m}^*a_{-\tilde{v}_m}^*\ldots a_{-\tilde{v}_1}^*\Omega\rangle. \tag{3.25}$$

We note that the scalar product in the last line of (3.25) does not vanish only if there exist $i, j, k,$ and ℓ such that $q = -\tilde{v}_i, p+r = -\tilde{v}_j, p = -v_k, q+r = -v_\ell$, which immediately implies $r = v_k - \tilde{v}_j$ and $\tilde{v}_i = v_k + v_\ell - \tilde{v}_j$. By symmetry, we can assume $i = k = m$ and $j = \ell = m-1$, obtaining a factor $m^2(m-1)^2$ in front. Pairing also the remaining $m-2$ momenta in P_S, we obtain

$$\langle\xi_\nu, \mathcal{E}_{S,1}\xi_\nu\rangle = \sum_{m\geq 2} \frac{1}{(m-2)!} \frac{1}{2N^{m+1}} \sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H: \\ r_1+v_1, \tilde{r}_1+\tilde{v}_1\in P_H}} \cdots \sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H: \\ r_m+v_m, \tilde{r}_m+\tilde{v}_m\in P_H}} \zeta_1(-v_m, -\tilde{v}_m, v_m - \tilde{v}_{m-1})$$
$$\times \theta(\{r_j, v_j\}_{j=1}^m)\theta(\{\tilde{r}_j, \tilde{v}_j\}_{j=1}^m)\prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j}\delta_{\tilde{v}_m, v_m+v_{m-1}-\tilde{v}_{m-1}}\prod_{i=1}^{m-2}\delta_{\tilde{v}_i, v_i}$$
$$\times \langle\Omega, A_{r_m,v_m}\ldots A_{r_1,v_1}A_{\tilde{r}_m, v_m+v_{m-1}-\tilde{v}_{m-1}}^* A_{\tilde{r}_{m-1},\tilde{v}_{m-1}}^* A_{\tilde{r}_{m-2},v_{m-2}}^*\ldots A_{\tilde{r}_1,v_1}^*\Omega\rangle.$$

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We now distinguish three contributions. The first contribution, which we will denote by I, corresponds to the situation in which the operators in A_{r_m,v_m} and $A_{r_{m-1},v_{m-1}}$ are annihilated with the operators in $A^*_{\tilde{r}_m,v_m+v_{m-1}-\tilde{v}_{m-1}}$ and $A^*_{\tilde{r}_{m-1},\tilde{v}_{m-1}}$, respectively (note that this immediately implies $v_{m-1} = \tilde{v}_{m-1}$). The second contribution, denoted by II, is on the contrary obtained when A_{r_m,v_m} is annihilated with $A^*_{\tilde{r}_{m-1},\tilde{v}_{m-1}}$ and $A_{r_{m-1},v_{m-1}}$ is annihilated with $A^*_{\tilde{r}_m,v_m+v_{m-1}-\tilde{v}_{m-1}}$ (then, $\tilde{v}_{m-1} = v_m$). Finally, there is a third term, denoted by III, arising when one operator in A_{r_m,v_m} is annihilated with an operator in $A^*_{\tilde{r}_{m-1},\tilde{v}_{m-1}}$ and the other with an operator in $A^*_{\tilde{r}_m,v_m+v_{m-1}-\tilde{v}_{m-1}}$ (and analogously, an operator in $A_{r_{m-1},v_{m-1}}$ is annihilated with an operator in $A^*_{\tilde{r}_{m-1},\tilde{v}_{m-1}}$ and the other with an operator in $A^*_{\tilde{r}_m,v_m+v_{m-1}-\tilde{v}_{m-1}}$). Let us stress that in all these cases, the operators A_{r_j,v_j} are annihilated with the operators $A^*_{\tilde{r}_j,v_j}$ for any $j = 1,\dots,m-2$ due to the presence of the restrictions encoded in $\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)$.

Summarizing, we have $\langle \xi_\nu, \mathcal{E}_{S,1}\xi_\nu\rangle = I + II + III$, with

$$\begin{aligned}
I &= \sum_{m\geq 2}\frac{1}{(m-2)!}\frac{1}{2N^{m+1}}\sum_{\substack{v_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+v_1\in P_H}}\cdots\sum_{\substack{v_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+v_m\in P_H}}\zeta_1(-v_m,-v_m,v_m-v_{m-1})\\
&\quad\times\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,v_j\}_{j=1}^m)\prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}^2(\delta_{\tilde{r}_j,r_j}+\delta_{-\tilde{r}_j,r_j+v_j}),\\
II &= \sum_{m\geq 2}\frac{1}{(m-2)!}\frac{1}{2N^{m+1}}\sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}}\cdots\sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}}\zeta_1(-v_m,-\tilde{v}_m,v_m-\tilde{v}_{m-1})\\
&\quad\times\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)\prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j}\delta_{\tilde{v}_m,v_m+v_{m-1}-\tilde{v}_{m-1}}\\
&\quad\times\prod_{i=1}^{m-2}\delta_{\tilde{v}_i,v_i}(\delta_{\tilde{r}_i,r_i}+\delta_{-\tilde{r}_i,r_i+v_i})\delta_{\tilde{v}_{m-1},v_m}\sum_{\substack{p_\ell\in\{-r_\ell,r_\ell+v_\ell\}\\ \tilde{p}_\ell\in\{-\tilde{r}_\ell,\tilde{r}_\ell+\tilde{v}_\ell\}\\ \ell=m-1,m}}\delta_{\tilde{p}_{m-1},p_m}\delta_{\tilde{p}_m,p_{m-1}}.\\
III &= \sum_{m\geq 2}\frac{1}{(m-2)!}\frac{1}{2N^{m+1}}\sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}}\cdots\sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}}\zeta_1(-v_m,-\tilde{v}_m,v_m-\tilde{v}_{m-1})\\
&\quad\times\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)\prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j}\delta_{\tilde{v}_m,v_m+v_{m-1}-\tilde{v}_{m-1}}\\
&\quad\times\prod_{i=1}^{m-2}\delta_{\tilde{v}_i,v_i}(\delta_{\tilde{r}_i,r_i}+\delta_{-\tilde{r}_i,r_i+v_i})\sum_{\substack{p_\ell\in\{-r_\ell,r_\ell+v_\ell\}\\ \tilde{p}_\ell\in\{-\tilde{r}_\ell,\tilde{r}_\ell+\tilde{v}_\ell\}\\ \ell=m-1,m}}\delta_{\tilde{p}_{m-1},p_{m-1}}\delta_{\tilde{p}_m,p_m}\delta_{-\tilde{p}_{m-1}+\tilde{v}_{m-1},-p_m+v_m}.
\end{aligned}$$

Recalling the definition of ζ_1 and using (2.5), we find $|\zeta_1(p,q,r)| \leq CN^{\kappa+2\varepsilon}$ for any $r\in\Lambda^*$, $p,q\in P_S$ such that $p+r, q+r\in P_S$. Thus,

$$\frac{|I|+|II|}{\|\xi_\nu\|^2}\leq CN^{\kappa-3+2\varepsilon}\|\sigma_S\|^4\|\eta_H\|^4. \tag{3.26}$$

To bound III, we note that

$$\begin{aligned}
&|\delta_{\tilde{v}_m,v_m+v_{m-1}-\tilde{v}_{m-1}}\zeta_1(-v_m,-\tilde{v}_m,v_m-\tilde{v}_{m-1})|\leq CN^\kappa\\
&\times(|\sigma_{v_m}\|\sigma_{\tilde{v}_{m-1}}\|\sigma_{v_{m-1}}|\|\sigma_S\|_\infty+|\gamma_{v_m}\|\gamma_{\tilde{v}_{m-1}}\|\gamma_{v_{m-1}}|\|\gamma_S\|_\infty+|\gamma_{v_m}\|\gamma_{\tilde{v}_{m-1}}\|\sigma_{v_{m-1}}|\|\sigma_S\|_\infty).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{|III|}{\|\xi_\nu\|^2}&\leq CN^{\kappa-3}\|\eta_H\|^2\|\eta_H\|_\infty^2\|\sigma_S\|_\infty\\
&\quad\times(\|\sigma_S\|^6\|\sigma_S\|_\infty+\|\gamma_S\sigma_S\|_1^3\|\gamma_S\|_\infty+\|\sigma_S\|^2\|\gamma_S\sigma_S\|_1^2\|\sigma_S\|_\infty).
\end{aligned}\tag{3.27}$$

From (3.26) and (3.27), using (2.4) and (2.5), we conclude

$$\frac{|\langle\xi_\nu,\mathcal{E}_{S,1}\xi_\nu\rangle|}{\|\xi_\nu\|^2}\leq\frac{|I|+|II|+|III|}{\|\xi_\nu\|^2}\leq CN^{5\kappa/2-\varepsilon} \tag{3.28}$$

for any $\kappa < 2/3$ and ε small enough.

The error term $\mathcal{E}_{S,2}$ can be bounded analogously. Indeed, reasoning as before, we split

$$\langle\xi_\nu,\mathcal{E}_{S,2}\xi_\nu\rangle = \widetilde{I}+\widetilde{II}+\widetilde{III}, \tag{3.29}$$

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where

$$\widetilde{I} = \sum_{m\geq 2} \frac{1}{(m-2)!} \frac{1}{N^{m+1}} \sum_{\substack{v_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+v_1\in P_H}} \cdots \sum_{\substack{v_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+v_m\in P_H}} \zeta_2(v_m, v_m, -v_m - v_{m-1})$$
$$\times \theta(\{r_j, v_j\}_{j=1}^m)\theta(\{\tilde{r}_j, \tilde{v}_j\}_{j=1}^m) \prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}^2(\delta_{\tilde{r}_j,r_j} + \delta_{-\tilde{r}_j,r_j+v_j}),$$

$$\widetilde{II} = \sum_{m\geq 2} \frac{1}{(m-2)!} \frac{1}{N^{m+1}} \sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}} \cdots \sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}} \zeta_2(v_m, \tilde{v}_m, -v_m - v_{m-1})$$
$$\times \theta(\{r_j, v_j\}_{j=1}^m)\theta(\{\tilde{r}_j, \tilde{v}_j\}_{j=1}^m) \prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j}\delta_{\tilde{v}_m, v_m+v_{m-1}-\tilde{v}_{m-1}}$$
$$\times \prod_{i=1}^{m-2} \delta_{\tilde{v}_i,v_i}(\delta_{\tilde{r}_i,r_i} + \delta_{-\tilde{r}_i,r_i+v_i})\delta_{\tilde{v}_{m-1},v_m} \sum_{\substack{p_\ell\in\{-r_\ell, r_\ell+v_\ell\}\\ \tilde{p}_\ell\in\{-\tilde{r}_\ell,\tilde{r}_\ell+\tilde{v}_\ell\}\\ \ell=m-1,m}} \delta_{\tilde{p}_{m-1},p_m}\delta_{\tilde{p}_m,p_{m-1}},$$

$$\widetilde{III} = \sum_{m\geq 2} \frac{1}{(m-2)!} \frac{1}{N^{m+1}} \sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}} \cdots \sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}} \zeta_2(v_m, \tilde{v}_m, -v_m - v_{m-1})$$
$$\times \theta(\{r_j, v_j\}_{j=1}^m)\theta(\{\tilde{r}_j, \tilde{v}_j\}_{j=1}^m) \prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j}\delta_{\tilde{v}_m, v_m+v_{m-1}-\tilde{v}_{m-1}}$$
$$\times \prod_{i=1}^{m-2} \delta_{\tilde{v}_i,v_i}(\delta_{\tilde{r}_i,r_i} + \delta_{-\tilde{r}_i,r_i+v_i}) \sum_{\substack{p_\ell\in\{-r_\ell, r_\ell+v_\ell\}\\ \tilde{p}_\ell\in\{-\tilde{r}_\ell,\tilde{r}_\ell+\tilde{v}_\ell\}\\ \ell=m-1,m}} \delta_{\tilde{p}_{m-1},p_{m-1}}\delta_{\tilde{p}_m,p_m}\delta_{-\tilde{p}_{m-1}+\tilde{v}_{m-1},-p_m+v_m}.$$

Hence, noting that $|\zeta_2(p,q,r)| \leq CN^\kappa|\gamma_{p+r}|\gamma_{q+r}\|\sigma_p\|\sigma_q|$, we find

$$\frac{|\widetilde{I}| + |\widetilde{II}|}{\|\xi_\nu\|^2} \leq CN^{\kappa-3+2\varepsilon}\|\sigma_S\|^4\|\eta_H\|^4 \tag{3.30}$$

and

$$\frac{|\widetilde{III}|}{\|\xi_\nu\|^2} \leq CN^{\kappa-3}\|\eta_H\|^2\|\eta_H\|_\infty^2\|\sigma_S\|_\infty^2\|\gamma_S\sigma_S\|_1^2\|\sigma_S\|^2. \tag{3.31}$$

Using (2.4) and (2.5), we obtain from (3.28), (3.30), and (3.31) that

$$\frac{|\langle\xi_\nu, \mathcal{E}_S\xi_\nu\rangle|}{\|\xi_\nu\|^2} \leq CN^{5\kappa/2-\varepsilon} \tag{3.32}$$

for any $\kappa < 2/3$ and ε small enough.

D. Bound of the expectation of $\mathcal{E}_M$ on ξ_ν

To conclude the proof of (3.14), $\mathcal{E}_M$remains to be studied. We start focusing on $\mathcal{E}_{M,1}$, and we rewrite

$$\langle\xi_\nu, \mathcal{E}_{M,1}\xi_\nu\rangle = \sum_{m\geq 1} \frac{1}{(m!)^2} \frac{1}{N^{m+1}} \sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}} \cdots \sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}}$$
$$\times \theta(\{r_j, v_j\}_{j=1}^m)\theta(\{\tilde{r}_j, \tilde{v}_j\}_{j=1}^m) \prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j} \sum_{r\in\Lambda^*} \sum_{\substack{p,q\in P_H:\\ p+r,q+r\in P_S}} \varphi_1(p,q,r)$$
$$\times \langle\Omega, a_{r_m+v_m}\dots a_{-v_1}a^*_{p+r}a^*_q a_p a_{q+r}a^*_{\tilde{r}_m+\tilde{v}_m}\dots a^*_{-\tilde{v}_1}\Omega\rangle.$$

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We now have to assume the existence of i and $j = 1,\ldots,m$ such that $p + r = -v_i$ and $q + r = -\tilde{v}_j$; otherwise, the expectation on the last line would vanish. In particular, we assume $i = j = m$ since the m^2 cases are all equivalent. Pairing the remaining momenta in P_S, we obtain

$$\begin{aligned}\langle \xi_v, \mathcal{E}_{M,1}\xi_v\rangle = \sum_{m\geq 1}\frac{1}{(m-1)!}\frac{1}{N^{m+1}} &\sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}}\cdots\sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}}\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)\\ &\times\prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j}\prod_{i=1}^{m-1}\delta_{v_i,\tilde{v}_i}\sum_{\substack{r\in\Lambda^*:\\ r+v_m, r+\tilde{v}_m\in P_H}}\varphi_1(-r-v_m,-r-\tilde{v}_m,r)\\ &\times\langle\Omega, A_{r_m,v_m}\ldots A_{r_1,v_1}a^*_{-r-\tilde{v}_m}a_{-r-v_m}A^*_{\tilde{r}_m,\tilde{v}_m}\ldots A^*_{\tilde{r}_1,v_1}\Omega\rangle.\end{aligned}$$

At this point, we recognize that, due to the presence of the restrictions encoded in $\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)$, if the operator a_{-r-v_m} is annihilated with an operator in $A^*_{\tilde{r}_j,v_j}$, then the operator $a^*_{-r-\tilde{v}_m}$ has to be annihilated with an operator in A_{r_j,v_j}. In particular, if $j = m$, then the remaining operator in $A^*_{\tilde{r}_m,\tilde{v}_m}$ has to be contracted with the remaining operator in A_{r_m,v_m}, and we obtain a contribution denoted by M_1. On the other hand, if $j \neq m$ (by symmetry, we assume $j = m-1$), we distinguish two cases: either the remaining operator in $A^*_{\tilde{r}_{m-1},v_{m-1}}$ is annihilated with the remaining operator in $A_{r_{m-1},v_{m-1}}$ and the operators $A^*_{\tilde{r}_m,\tilde{v}_m}$ and A_{r_m,v_m} are contracted among themselves (imposing $v_m = \tilde{v}_m$) or the remaining operator in $A^*_{\tilde{r}_{m-1},v_{m-1}}$ is contracted with an operator in A_{r_m,v_m} and the remaining operator in $A_{r_{m-1},v_{m-1}}$ is contracted with an operator in $A^*_{\tilde{r}_m,\tilde{v}_m}$ (then, we are left with one operator in $A^*_{\tilde{r}_m,\tilde{v}_m}$ and one in A_{r_m,v_m} that are necessarily contracted with each other). We denote these contributions with M_2 and M_3, respectively.

Explicitly,

$$\langle\xi_v,\mathcal{E}_{M,1}\xi_v\rangle = M_1 + M_2 + M_3, \tag{3.33}$$

with

$$\begin{aligned}M_1 = \sum_{m\geq 1}\frac{1}{(m-1)!}\frac{1}{N^{m+1}}&\sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}}\cdots\sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}}\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)\\ &\times\prod_{j=1}^m\eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j}\prod_{i=1}^{m-1}\delta_{v_i,\tilde{v}_i}(\delta_{\tilde{r}_i}+\delta_{\tilde{r}_i+v_i})\sum_{r\in\Lambda^*}\varphi_1(-r-v_m,-r-\tilde{v}_m,r)\\ &\times\sum_{\substack{p_m\in\{-r_m,r_m+v_m\}\\ \tilde{p}_m\in\{\tilde{r}_m,\tilde{r}_m+\tilde{v}_m\}}}\delta_{r,-p_m-\tilde{v}_m}\delta_{\tilde{p}_m,p_m+\tilde{v}_m-v_m},\\ M_2 = \sum_{m\geq 2}\frac{1}{(m-2)!}\frac{1}{N^{m+1}}&\sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}}\cdots\sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}}\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)\\ &\times\prod_{j=1}^m\eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j}\prod_{\substack{i=1\\ i\neq m-1}}^{m}\delta_{v_i,\tilde{v}_i}(\delta_{r_i,\tilde{r}_i}+\delta_{-r_i,\tilde{r}_i+v_i})\sum_{r\in\Lambda^*}\varphi_1(-r-v_m,-r-v_m,r)\\ &\times\delta_{\tilde{v}_{m-1},v_{m-1}}\sum_{\substack{p_{m-1}\in\{-r_{m-1},r_{m-1}+v_{m-1}\}\\ \tilde{p}_{m-1}\in\{-\tilde{r}_{m-1},\tilde{r}_{m-1}+v_{m-1}\}}}\delta_{r,-p_{m-1}-v_m}\delta_{\tilde{p}_{m-1},p_{m-1}},\\ M_3 = \sum_{m\geq 2}\frac{1}{(m-2)!}\frac{1}{N^{m+1}}&\sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}}\cdots\sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}}\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)\\ &\times\prod_{j=1}^m\eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j}\prod_{i=1}^{m-2}\delta_{v_i,\tilde{v}_i}(\delta_{\tilde{r}_i}+\delta_{\tilde{r}_i+v_i})\delta_{\tilde{v}_{m-1},v_{m-1}}\sum_{r\in\Lambda^*}\varphi_1(-r-v_m,-r-\tilde{v}_m,r)\\ &\times\sum_{\substack{p_\ell\in\{-r_\ell,r_\ell+v_\ell\}\\ \tilde{p}_\ell\in\{\tilde{r}_\ell,\tilde{r}_\ell+\tilde{v}_\ell\}\\ \ell=m-1,m}}\delta_{r,-p_{m-1}-\tilde{v}_m}\delta_{\tilde{p}_{m-1},p_{m-1}+\tilde{v}_m-v_m}\delta_{\tilde{p}_m,p_m-v_m+\tilde{v}_m}\delta_{p_{m-1},-p_m+v_m-\tilde{v}_m+v_{m-1}}.\end{aligned}$$

Since

$$\begin{aligned}&|\varphi_1(-r-v_m,-r-\tilde{v}_m,r)\delta_{r,-p_m-\tilde{v}_m}\delta_{\tilde{p}_m,p_m+\tilde{v}_m-v_m}|\\ &\qquad\leq CN^\kappa(\|\eta_H\|_\infty^2|\sigma_{v_m}\|\sigma_{\tilde{v}_m}|+|\gamma_{v_m}\|\gamma_{\tilde{v}_m}|+\|\eta_H\|_\infty|\gamma_{v_m}\|\sigma_{\tilde{v}_m}|),\end{aligned}$$

we have

$$\frac{|M_1|}{\|\xi_v\|^2}\leq CN^{\kappa-2}\|\eta_H\|^2(\|\sigma_S\|^4\|\eta_H\|_\infty^2+\|\sigma_S\gamma_S\|_1^2+\|\sigma_S\|^2\|\gamma_S\sigma_S\|_1\|\eta_H\|_\infty). \tag{3.34}$$

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On the other hand, using the fact that

$$|\varphi_1(-r-v_m,-r-v_m,r)|\delta_{r,-p_{m-1}-v_m}\delta_{\tilde{p}_{m-1},p_{m-1}}| \le CN^{\kappa}(\|\eta_H\|_\infty^2\|\sigma_S\|_\infty^2+\|\gamma_S\|_\infty^2+\|\gamma_S\|_\infty\|\sigma_S\|_\infty\|\eta_H\|_\infty),$$

we obtain

$$\frac{|M_2|}{\|\xi_\nu\|^2} \le CN^{\kappa-3}\|\eta_H\|^4\|\sigma_S\|^4(\|\sigma_S\|_\infty^2\|\eta_H\|_\infty^2+\|\gamma_S\|_\infty^2+\|\gamma_S\|_\infty\|\sigma_S\|_\infty\|\eta_H\|_\infty). \tag{3.35}$$

Finally, we note that

$$\begin{aligned}|\varphi_1(-r-v_m,-r-\tilde{v}_m,r)\delta_{r,-p_{m-1}-\tilde{v}_m}\delta_{\tilde{p}_{m-1},p_{m-1}+\tilde{v}_m-v_m}|\\ \le CN^{\kappa}(\|\eta_H\|_\infty^2|\sigma_{v_m}\|\sigma_{\tilde{v}_m}|+|\gamma_{v_m}\|\gamma_{\tilde{v}_m}|+\|\eta_H\|_\infty|\gamma_{v_m}\|\sigma_{\tilde{v}_m}|).\end{aligned}$$

Therefore,

$$\frac{|M_3|}{\|\xi_\nu\|^2} \le N^{\kappa-3}\|\eta_H\|^2\|\eta_H\|_\infty^2\|\sigma_S\|^2(\|\sigma_S\|^4\|\eta_H\|_\infty^2+\|\gamma_S\sigma_S\|_1^2+\|\sigma_S\|^2\|\gamma_S\sigma_S\|_1\|\eta_H\|_\infty). \tag{3.36}$$

With (3.33)–(3.36), we conclude, using (2.4) and (2.5), that for any $\kappa < 2/3$ and ε small enough,

$$\frac{\langle\xi_\nu,\mathcal{E}_{M,1}\xi_\nu\rangle}{\|\xi_\nu\|^2} \le CN^{5\kappa/2-\varepsilon}. \tag{3.37}$$

We now consider the expectation on ξ_ν of $\mathcal{E}_{M,2}$. We have by definition

$$\begin{aligned}\langle\xi_\nu,\mathcal{E}_{M,2}\xi_\nu\rangle &= \sum_{m\ge1}\frac{1}{(m!)^2}\frac{1}{N^{m+1}}\sum_{\substack{v_1,\tilde{v}_1\in P_S,r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}}\cdots\sum_{\substack{v_m,\tilde{v}_m\in P_S,r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}}\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)\\ &\times\prod_{j=1}^m\eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j}\sum_{r\in\Lambda^*}\sum_{\substack{p,q\in P_H:\\ p+r,q+r\in P_S}}\varphi_2(p,q,r)\\ &\times\langle\Omega,a_{r_m+v_m}\ldots a_{-v_1}a_{p+r}^*a_{-p}^*a_{q+r}a_{-q}a_{\tilde{r}_m+\tilde{v}_m}^*\ldots a_{-\tilde{v}_1}^*\Omega\rangle.\end{aligned}$$

We note that there are necessarily i and j such that $p+r=-v_i$ and $q+r=-\tilde{v}_j$. Assuming by symmetry $i=j=m$ and pairing the remaining $m-1$ momenta in P_S, we obtain

$$\begin{aligned}\langle\xi_\nu,\mathcal{E}_{M,2}\xi_\nu\rangle &= \sum_{m\ge1}\frac{1}{(m-1)!}\frac{1}{N^{m+1}}\sum_{\substack{v_1,\tilde{v}_1\in P_S,r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}}\cdots\sum_{\substack{v_m,\tilde{v}_m\in P_S,r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}}\theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)\\ &\times\prod_{j=1}^m\eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j}\prod_{i=1}^{m-1}\delta_{\tilde{v}_i,v_i}\sum_{\substack{r\in\Lambda^*:\\ r+v_m,r+\tilde{v}_m\in P_H}}\varphi_2(-r-v_m,-r-\tilde{v}_m,r)\\ &\times\langle\Omega,A_{r_m,v_m}\ldots A_{r_1,v_1}a_{r+v_m}^*a_{r+\tilde{v}_m}A_{\tilde{r}_m,\tilde{v}_m}^*A_{\tilde{r}_{m-1},v_{m-1}}^*\ldots A_{\tilde{r}_1,v_1}^*\Omega\rangle\\ &= \widetilde{M}_1+\widetilde{M}_2+\widetilde{M}_3,\end{aligned}$$

where

$$\widetilde{M}_1 = \sum_{m\geq 1} \frac{1}{(m-1)!} \frac{1}{N^{m+1}} \sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}} \cdots \sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}} \theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)$$
$$\times \prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j} \prod_{i=1}^{m-1} \delta_{v_i,\tilde{v}_i}(\delta_{\tilde{r}_i} + \delta_{\tilde{r}_i+v_i}) \sum_{r\in\Lambda^*} \varphi_2(-r-v_m, -r-\tilde{v}_m, r)$$
$$\times \sum_{\substack{p_m\in\{-r_m, r_m+v_m\}\\ \tilde{p}_m\in\{\tilde{r}_m,\tilde{r}_m+\tilde{v}_m\}}} \delta_{r,p_m-v_m}\delta_{\tilde{p}_m,p_m+\tilde{v}_m-v_m},$$
$$\widetilde{M}_2 = \sum_{m\geq 2} \frac{1}{(m-2)!} \frac{1}{N^{m+1}} \sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}} \cdots \sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}} \theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)$$
$$\times \prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j} \prod_{\substack{i=1\\ i\neq m-1}}^{m} \delta_{v_i,\tilde{v}_i}(\delta_{r_i,\tilde{r}_i} + \delta_{-r_i,\tilde{r}_i+v_i}) \sum_{r\in\Lambda^*} \varphi_2(-r-v_m, -r-v_m, r)$$
$$\times \delta_{\tilde{v}_{m-1},v_{m-1}} \sum_{\substack{p_{m-1}\in\{-r_{m-1}, r_{m-1}+v_{m-1}\}\\ \tilde{p}_{m-1}\in\{-\tilde{r}_{m-1},\tilde{r}_{m-1}+v_{m-1}\}}} \delta_{r,-p_{m-1}-v_m}\delta_{\tilde{p}_{m-1},p_{m-1}},$$
$$\widetilde{M}_3 = \sum_{m\geq 2} \frac{1}{(m-2)!} \frac{1}{N^{m+1}} \sum_{\substack{v_1,\tilde{v}_1\in P_S, r_1,\tilde{r}_1\in P_H:\\ r_1+v_1,\tilde{r}_1+\tilde{v}_1\in P_H}} \cdots \sum_{\substack{v_m,\tilde{v}_m\in P_S, r_m,\tilde{r}_m\in P_H:\\ r_m+v_m,\tilde{r}_m+\tilde{v}_m\in P_H}} \theta(\{r_j,v_j\}_{j=1}^m)\theta(\{\tilde{r}_j,\tilde{v}_j\}_{j=1}^m)$$
$$\times \prod_{j=1}^m \eta_{r_j}\eta_{\tilde{r}_j}\sigma_{v_j}\sigma_{\tilde{v}_j} \prod_{i=1}^{m-2} \delta_{v_i,\tilde{v}_i}(\delta_{\tilde{r}_i} + \delta_{\tilde{r}_i+v_i})\delta_{\tilde{v}_{m-1},v_{m-1}} \sum_{r\in\Lambda^*} \varphi_2(-r-v_m, -r-\tilde{v}_m, r)$$
$$\times \sum_{\substack{p_\ell\in\{-r_\ell, r_\ell+v_\ell\}\\ \tilde{p}_\ell\in\{\tilde{r}_\ell,\tilde{r}_\ell+\tilde{v}_\ell\}\\ \ell=m-1,m}} \delta_{r,p_{m-1}-v_m}\delta_{\tilde{p}_{m-1},p_{m-1}+\tilde{v}_m-v_m}\delta_{\tilde{p}_m,p_m-v_m+\tilde{v}_m}\delta_{p_{m-1},-p_m+v_m-\tilde{v}_m+v_{m-1}}.$$

Thus, the bound

$$|\varphi_2(-r-v_m, -r-\tilde{v}_m, r)\delta_{r,p_m-v_m}\delta_{\tilde{p}_m,p_m+\tilde{v}_m-v_m}|$$
$$\leq CN^\kappa(\|\eta_H\|_\infty^2|\gamma_{v_m}\|\gamma_{\tilde{v}_m}| + \|\eta_H\|_\infty|\gamma_{\tilde{v}_m}\|\sigma_{v_m}| + \|\eta_H\|_\infty|\gamma_{v_m}\|\sigma_{\tilde{v}_m}| + |\sigma_{v_m}\|\sigma_{\tilde{v}_m}|)$$

implies

$$\frac{|\widetilde{M}_1|}{\|\xi_\nu\|^2} \leq CN^{\kappa-2}\|\eta_H\|^2(\|\gamma_S\sigma_S\|_1^2\|\eta_H\|_\infty^2 + \|\gamma_S\sigma_S\|_1\|\sigma_S\|^2\|\eta_H\|_\infty + \|\sigma_S\|^4). \tag{3.38}$$

On the other hand, noting that

$$|\varphi_2(-r-v_m, -r-v_m, r)\delta_{\tilde{v}_{m-1},v_{m-1}}\delta_{r,-p_{m-1}-v_m}\delta_{\tilde{p}_{m-1},p_{m-1}}|$$
$$\leq CN^\kappa(\|\gamma_S\|_\infty^2\|\eta_H\|_\infty^2 + \|\gamma_S\|_\infty\|\sigma_S\|_\infty\|\eta_H\|_\infty + \|\sigma_S\|_\infty^2),$$

we obtain

$$\frac{|\widetilde{M}_2|}{\|\xi_\nu\|^2} \leq CN^{\kappa-3}\|\eta_H\|^4\|\sigma_S\|^4(\|\gamma_S\|_\infty^2\|\eta_H\|_\infty^2 + \|\eta_H\|_\infty\|\sigma_S\|_\infty\|\gamma_S\|_\infty + \|\sigma_S\|_\infty^2). \tag{3.39}$$

Finally, using

$$|\varphi_2(-r-v_m, -r-\tilde{v}_m, r)\delta_{r,p_{m-1}-v_m}\delta_{\tilde{p}_{m-1},p_{m-1}+\tilde{v}_m-v_m}|$$
$$\leq CN^\kappa(\|\eta_H\|_\infty^2|\gamma_{v_m}\|\gamma_{\tilde{v}_m}| + \|\eta_H\|_\infty|\gamma_{\tilde{v}_m}\|\sigma_{v_m}| + \|\eta_H\|_\infty|\gamma_{v_m}\|\sigma_{\tilde{v}_m}| + |\sigma_{v_m}\|\sigma_{\tilde{v}_m}|),$$

we obtain

$$\frac{|\widetilde{M}_3|}{\|\xi_\nu\|^2} \leq CN^{\kappa-3}\|\eta_H\|^2\|\eta_H\|_\infty^2\|\sigma_S\|^2(\|\gamma_S\sigma_S\|_1^2\|\eta_H\|_\infty^2 + \|\gamma_S\sigma_S\|_1\|\sigma_S\|^2\|\eta_H\|_\infty + \|\sigma_S\|^4). \tag{3.40}$$

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From (3.38)–(3.40) and recalling (2.4) and (2.5), we obtain for $\kappa < 2/3$ and ε small enough

$$\frac{\langle \xi_v, \mathcal{E}_{M,2}\xi_v\rangle}{\|\xi_v\|^2} \le \frac{|\widetilde{M}_1| + |\widetilde{M}_2| + |\widetilde{M}_3|}{\|\xi_v\|^2} \le CN^{5\kappa/2-\varepsilon}. \tag{3.41}$$

To conclude, we still have to bound the expectation of $\mathcal{E}_{M,3}$ on the state ξ_v. Proceeding similarly as before, we write

$$\langle \xi_v, \mathcal{E}_{M,3}\xi_v\rangle = M_1' + M_2' + M_3',$$

where we introduced the notations

$$\begin{aligned} M_1' =& \sum_{m\ge1}\frac{1}{(m-1)!}\frac{1}{N^{m+1}}\sum_{\substack{v_1,\tilde v_1\in P_S, r_1,\tilde r_1\in P_H:\\ r_1+v_1,\tilde r_1+\tilde v_1\in P_H}}\cdots\sum_{\substack{v_m,\tilde v_m\in P_S, r_m,\tilde r_m\in P_H:\\ r_m+v_m,\tilde r_m+\tilde v_m\in P_H}}\theta\big(\{r_j,v_j\}_{j=1}^m\big)\theta\big(\{\tilde r_j,\tilde v_j\}_{j=1}^m\big)\\ &\times\prod_{j=1}^m \eta_{r_j}\eta_{\tilde r_j}\sigma_{v_j}\sigma_{\tilde v_j}\prod_{i=1}^{m-1}\delta_{v_i,\tilde v_i}(\delta_{\tilde r_i,r_i}+\delta_{-\tilde r_i,r_i+v_i})\sum_{q\in P_H}\varphi_3(-v_m,q,v_m-\tilde v_m)\\ &\times\sum_{\substack{p_m\in\{-r_m,r_m+v_m\}\\ \tilde p_m\in\{-\tilde r_m,\tilde r_m+\tilde v_m\}}}\delta_{\tilde p_m,q}\delta_{q,p_m+\tilde v_m-v_m},\\ M_2' =& \sum_{m\ge2}\frac{1}{(m-2)!}\frac{1}{N^{m+1}}\sum_{\substack{v_1,\tilde v_1\in P_S, r_1,\tilde r_1\in P_H:\\ r_1+v_1,\tilde r_1+\tilde v_1\in P_H}}\cdots\sum_{\substack{v_m,\tilde v_m\in P_S, r_m,\tilde r_m\in P_H:\\ r_m+v_m,\tilde r_m+\tilde v_m\in P_H}}\theta\big(\{r_j,v_j\}_{j=1}^m\big)\theta\big(\{\tilde r_j,\tilde v_j\}_{j=1}^m\big)\\ &\times\prod_{j=1}^m \eta_{r_j}\eta_{\tilde r_j}\sigma_{v_j}\sigma_{\tilde v_j}\prod_{\substack{i=1\\ i\ne m-1}}^{m}\delta_{v_i,\tilde v_i}(\delta_{r_i,\tilde r_i}+\delta_{-r_i,\tilde r_i+v_i})\sum_{q\in P_H}\varphi_3(-v_m,q,v_m-\tilde v_m)\\ &\times\delta_{\tilde v_{m-1},v_{m-1}}\sum_{\substack{p_{m-1}\in\{-r_{m-1},r_{m-1}+v_{m-1}\}\\ \tilde p_{m-1}\in\{-\tilde r_{m-1},\tilde r_{m-1}+v_{m-1}\}}}\delta_{q,p_{m-1}}\delta_{\tilde p_{m-1},p_{m-1}},\end{aligned}$$

$$\begin{aligned} M_3' =& \sum_{m\ge2}\frac{1}{(m-2)!}\frac{1}{N^{m+1}}\sum_{\substack{v_1,\tilde v_1\in P_S, r_1,\tilde r_1\in P_H:\\ r_1+v_1,\tilde r_1+\tilde v_1\in P_H}}\cdots\sum_{\substack{v_m,\tilde v_m\in P_S, r_m,\tilde r_m\in P_H:\\ r_m+v_m,\tilde r_m+\tilde v_m\in P_H}}\theta\big(\{r_j,v_j\}_{j=1}^m\big)\theta\big(\{\tilde r_j,\tilde v_j\}_{j=1}^m\big)\\ &\times\prod_{j=1}^m \eta_{r_j}\eta_{\tilde r_j}\sigma_{v_j}\sigma_{\tilde v_j}\prod_{i=1}^{m-2}\delta_{v_i,\tilde v_i}(\delta_{\tilde r_i}+\delta_{\tilde r_i+v_i})\delta_{\tilde v_{m-1},v_{m-1}}\sum_{q\in P_H}\varphi_3(-v_m,q,v_m-\tilde v_m)\\ &\times\sum_{\substack{p_\ell\in\{-r_\ell,r_\ell+v_\ell\}\\ \tilde p_\ell\in\{\tilde r_\ell,\tilde r_\ell+\tilde v_\ell\}\\ \ell=m-1,m}}\delta_{q,p_{m-1}+\tilde v_m-v_m}\delta_{\tilde p_{m-1},q}\delta_{\tilde p_m,p_m-v_m+\tilde v_m}\delta_{p_{m-1},-p_m+v_m-\tilde v_m+v_{m-1}}.\end{aligned}$$

Hence, we have

$$\frac{|M_1'|}{\|\xi_v\|^2} \le CN^{\kappa-1}\|\eta_H\|^2\big(\|\sigma_S\|^4\|\eta_H\|_\infty^2 + \|\gamma_S\sigma_S\|_1^2 + \|\gamma_S\sigma_S\|_1^2\|\eta_H\|_\infty^2 + \|\sigma_S\|^2\|\gamma_S\sigma_S\|_1\|\eta_H\|_\infty\big), \tag{3.42}$$

where we used the bound

$$\begin{aligned}&|\varphi_3(-v_m,q,v_m-\tilde v_m)\delta_{\tilde p_m,q}\delta_{q,p_m+\tilde v_m-v_m}|\\ &\le CN^\kappa\big(\|\eta_H\|_\infty^2|\sigma_{v_m}\|\sigma_{\tilde v_m}| + |\gamma_{v_m}\|\gamma_{\tilde v_m}| + \|\eta_H\|_\infty^2|\gamma_{v_m}\|\gamma_{\tilde v_m}| + \|\eta_H\|_\infty|\sigma_{\tilde v_m}\|\gamma_{v_m}|\big).\end{aligned}$$

Moreover, since

$$\begin{aligned}&|\varphi_3(-v_m,q,v_m-\tilde v_m)\delta_{q,p_{m-1}}\delta_{\tilde p_{m-1},p_{m-1}}|\\ &\le CN^\kappa\big(\|\eta_H\|_\infty^2\|\sigma_S\|_\infty^2 + \|\gamma_S\|_\infty^2 + \|\gamma_S\|_\infty^2\|\eta_H\|_\infty^2 + \|\sigma_S\|_\infty\|\eta_H\|_\infty\|\gamma_S\|_\infty\big),\end{aligned}$$

we obtain

$$\frac{|M_2'|}{\|\xi_v\|^2} \le CN^{\kappa-3}\|\eta_H\|^4\|\sigma_S\|^4\big(\|\sigma_S\|_\infty^2\|\eta_H\|_\infty^2 + \|\gamma_S\|_\infty^2 + \|\gamma_S\|_\infty^2\|\eta_H\|_\infty^2 + \|\gamma_S\|_\infty\|\sigma_S\|_\infty\|\eta_H\|_\infty\big). \tag{3.43}$$

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Finally, we note that

$$\begin{aligned}&|\varphi_3(-v_m, q, v_m - \tilde{v}_m)\delta_{q,p_{m-1}+\tilde{v}_m-v_m}\delta_{\tilde{p}_{m-1},q}|\\&\leq CN^{\kappa}(\|\eta_H\|_\infty^2\|\sigma_{v_m}\|\sigma_{\tilde{v}_m}| + |\gamma_{v_m}\|\gamma_{\tilde{v}_m}| + \|\eta_H\|_\infty^2|\gamma_{v_m}\|\gamma_{\tilde{v}_m}| + \|\eta_H\|_\infty\|\sigma_{\tilde{v}_m}\|\gamma_{v_m}|),\end{aligned}$$

which implies

$$\frac{|M_3'|}{\|\xi_\nu\|^2} \leq CN^{\kappa-3}\|\sigma_S\|^2\|\eta_H\|^2\|\eta_H\|_\infty^2(\|\sigma_S\|^4\|\eta_H\|_\infty^2 + \|\gamma_S\sigma_S\|^2 + \|\gamma_S\sigma_S\|_1\|\sigma_S\|^2\|\eta_H\|_\infty). \tag{3.44}$$

The bounds (3.42)–(3.44) yield, recalling (2.4) and (2.5),

$$\frac{\langle\xi_\nu, \mathcal{E}_{M,3}\xi_\nu\rangle}{\|\xi_\nu\|^2} \leq \frac{|M_1'| + |M_2'| + |M_3'|}{\|\xi_\nu\|^2} \leq CN^{5\kappa/2-\varepsilon} \tag{3.45}$$

under the assumptions $\kappa < 2/3$ and ε small enough.

From (3.9), (3.17), (3.24), (3.32), (3.37), (3.41), and (3.45), we obtain (3.14).

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

Author Contributions

Giulia Basti: Writing – original draft (lead).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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