Hydrodynamic limit for asymmetric simple exclusion with accelerated boundaries

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Abstract

We consider the asymmetric simple exclusion process (ASEP) on the one-dimensional finite lattice $\{1, 2, ..., N\}$. The particles can be created/annihilated at the boundaries with given rates. These rates are L^{∞} functions of time and are independent of the jump rates in the bulk (cf. [1]). The boundary dynamics is modified by a factor N^{θ} with $\theta > 0$. We study the hydrodynamic limit for the particle density profile under the hyperbolic space-time scale. The macroscopic equation is given by (inviscid) Burgers equation with Dirichlet type boundaries that is characterized by the boundary entropy [21]. A grading scheme is developed to control the formulation of boundary layers on the microscopic level.

Nous considérons le processus d'exclusion simple asymétrique (ASEP) sur le réseau fini unidimensionnel $\{1, 2, ..., N\}$. Les particules peuvent être créées/annihilées sur les points de frontière avec des taux qui sont des fonctions L^{∞} du temps et sont indépendants des taux de saut à l'intérieur du système (cf. [1]). La dynamique des bords est modifiée d'un facteur N^{θ} avec $\theta > 0$. Nous étudions la limite hydrodynamique du profil de densité des particules sous l'échelle espace-temps hyperbolique. L'équation macroscopique est donnée par l'équation (non visqueuse) de Burgers avec des conditions aux limites caractérisées par l'entropie du processus sur les points de frontière [21]. Un schéma adapté est développé pour contrôler la formulation des couches limites au niveau microscopique.

1. Introduction

Exclusion process in contact with reservoirs is a widely studied model in non-equilibrium statistical physics. In its dynamics, each particle performs a random walk on the finite lattice $\{1, 2, ..., N\}^d$ in accordance with the exclusion rule: if the target site is occupied, the corresponding jump will be suppressed. At boundaries, the particles can be removed from or added into the system with given rates. For symmetric dynamics, the hydrodynamic limit for the particle density is given by diffusion equation with Dirichlet boundary conditions imposed by the reservoirs, see [12, 13, 16, 17] and the reference therein. If the bulk dynamics performs weak asymmetry, viscous Burgers equation is obtained in the limit [4, 5]. More recent works succeed in getting varies types of boundary conditions by modifying the strength of reservoirs, for instance [2, 7, 14].

The situation differs drastically when a strong drift exists. Liggett [18] first introduced the asymmetric simple exclusion process (ASEP) with open boundaries as a tool for studying the dynamics of the infinite system. For one-dimensional case, the stationary states are studied in [6, 10, 24]. They are given by uniform bulk density which is determined by the reservoir rates through three phases: the high-density phase, the low-density phase, and the max-current phase. The high-density phase and the low-density phase intersect at a critical line, where the bulk density performs a randomly located shock. These stationary states satisfy the variational property [22]: the current is minimized if $\rho_{-} < \rho_{+}$ (drift up-hill) and is maximized if $\rho_{-} > \rho_{+}$ (drift down-hill), where ρ_{\pm} are the left and right boundary rates, respectively. The

hydrodynamic limit for ASEP on \mathbb{Z}^d is obtained in [23] under the hyperbolic space-time scale. The celebrated result of [1] proves that the hydrodynamic limit for a class of boundary driven asymmetric particle systems is given by the unique entropy solution to the initial-boundary problem of hyperbolic conservation law with boundary conditions introduced in [3]. Different from the Dirichlet conditions in diffusion equation, these boundary conditions do not fix the boundary value of the solution. Instead, they impose a set of possible boundary values depending on the boundary data. The proof in [1] relies on a special choice of reservoir densities that approximates optimally the infinite dynamics (see Remark 2.1 for details), thus allows the application of an effective coupling method. Also note that all these previous results are restricted to reservoir densities that do not depend on time.

Recently, the ASEP connected to general reservoirs whose densities vary with time and are regulated by a factor N^{θ} is studied [8, 26]. When $\theta < 0$, the reservoirs are slowed down and the hydrodynamic limit is given by the entropy solution to Burgers equation subjected to degenerated boundary traces [26]. The proof in [26] relies on the vanishing boundary drift created by the slow reservoirs. When $\theta > 0$ and the reservoir densities are smooth functions of time, [8] obtained the quasi-static limit: under the time scale longer than the hyperbolic one, the particle density is determined by the quasi-stationary solution [19] to Burgers equation, that is, the stationary solution associated to dynamical boundary data. The phase diagram [10] and the variational property [22] are also reproduced, see Section 2.3. The proof in [8] is established upon a fine estimate on the boundary entropy production for the balanced dynamics where the reservoirs impose the same density at left and right boundaries. Then, the results are extended to unbalanced case by using a coupling method, which relies on the fact that the quasi-static dynamics always stays close to stationary states.

In the present article, we consider the ASEP in contact to reversible reservoirs whose densities are general L^{∞} functions of time and are accelerated by N^{θ} with $\theta > 0$. We prove in Theorem 2.2 that, under the hyperbolic time scale, the density profile evolves with the entropy solution to Burgers equation with boundary conditions characterized by boundary entropy flux pairs [21]. Since the strengthened reservoirs dominate the drift, the *formal* boundary values in the macroscopic equation coincide with the reversible densities of the reservoirs, see Remark 2.3.

The main difference from the previous works is that, as stated above, the coupling method used in [1] is not available in this general setting of reservoir dynamics. Furthermore, since the dynamics does not have enough time to relax to the stationary state, the strategy in [8] is not applicable. Our new approach contains two main steps. First, we apply the stochastic compensated compactness developed in [15] to show the existence of a random L^{∞} function as the limit density profile. Then, we introduce a grading scheme, which helps to extend the estimate on the boundary entropy production proved in [8] to general unbalanced dynamics. The key point in this step is the auxiliary functions defined in (7.1), which slices the divergent boundary integral into pieces with an exact cancellation, see Lemma 7.5 and 7.6. The proof is finally closed by the uniqueness of the entropy solution established for scalar conservation law on a bounded domain [20, 21].

The technique developed in this article does not rely on the attractiveness of the dynamics, thus it is expected to be applicable to non-attractive systems, such as non-attractive asymmetric zero-range process. Meanwhile, the hydrodynamic limit remains an open problem for $\theta = 0$ except the regime treated in [1]. We expect the macroscopic boundary conditions to be given by the vanishing viscosity limit obtained in [26, Theorem 2.12].

Keywords. Asymmetric simple exclusion process, Open boundary, Hydrodynamic limit, Entropy solution.

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2. One-dimensional ASEP with open boundaries

The one-dimensional asymmetric simple exclusion process with open boundaries is a Markov process defined on the configuration space

$$\Omega_N := \{ \eta = (\eta_1, \eta_2, \dots, \eta_N), \ \eta_i \in \{0, 1\} \}, \quad N \ge 2.$$
(2.1)

Its infinitesimal generator is given by

$$L_N = pL_{\rm TAS} + \sigma L_{\rm SS} + L_- + L_+, \qquad (2.2)$$

where p and σ are strictly positive constants. Here L_{TAS} and L_{SS} generate respectively the totally asymmetric exclusion and symmetric exclusion in bulk:

$$L_{\text{TAS}}f := \sum_{i=1}^{N-1} \eta_i (1 - \eta_{i+1}) [f(\eta^{i,i+1}) - f(\eta)],$$

$$L_{\text{SS}}f := \sum_{i=1}^{N-1} [f(\eta^{i,i+1}) - f(\eta)],$$
(2.3)

for any function f on Ω_N , where $\eta^{i,i+1}$ is the configuration obtained from η upon exchanging η_i and η_{i+1} . The boundary generators L_{\pm} generate the reservoirs that creates/annihilates particles at i = 1 and i = N with given rates $\alpha, \beta, \gamma, \delta > 0$:

$$L_{-}f := [\alpha(1-\eta_{1})+\gamma\eta_{1}][f(\eta^{1})-f(\eta)],$$

$$L_{+}f := [\delta(1-\eta_{N})+\beta\eta_{N}][f(\eta^{N})-f(\eta)],$$
(2.4)

where η^i is the configuration obtained from η by shifting the status at site *i* from η_i to $1 - \eta_i$. Note that L_{\pm} are respectively reversible for the Bernoulli measure with densities

$$\rho_{-} := \frac{\alpha}{\alpha + \gamma}, \quad \rho_{+} := \frac{\delta}{\beta + \delta}.$$
(2.5)

Remark 2.1 (Liggett's boundaries). In [1, 10, 18] a specific choice of boundary parameters α , β , γ , δ are adopted. Given some $\bar{\rho}_{\pm} \in [0, 1]$, define

$$\alpha := (p+\sigma)\bar{\rho}_{-}, \quad \gamma := \sigma(1-\bar{\rho}_{-}), \beta := (p+\sigma)(1-\bar{\rho}_{+}), \quad \delta := \sigma\bar{\rho}_{+}.$$

$$(2.6)$$

This choice corresponds to the projection on the finite interval $\{1, \ldots, N\}$ of the infinite ASEP dynamics with jump rate $p + \sigma$ to the right and σ to the left, where particles are distributed according to Bernoulli measure with density $\bar{\rho}_{-}$ on $\{0, -1, \ldots\}$ and density $\bar{\rho}_{+}$ on $\{N+1, N+2, \ldots\}$. If furthermore $\bar{\rho}_{-} = \bar{\rho}_{+} = \bar{\rho}$, the homogeneous Bernoulli measure with density $\bar{\rho}$ is stationary with respect to the dynamics. Observe that due to the non-vanishing current, the stationary densities differ from the densities of the reservoirs.

2.1. Parameters. We want to treat boundary rates that vary non-smoothly with time. Consider positive functions α , β , γ , $\delta \in L^{\infty}(\mathbb{R}_+)$ and define operators $L_{\pm,t}$ by (2.4) with $(\alpha, \beta, \gamma, \delta)(t)$. Also pick two sequences σ_N and $\tilde{\sigma}_N$ satisfying that

$$\lim_{N \to \infty} N^{\frac{9}{7}} \sigma_N^{-1} = \lim_{N \to \infty} N^{-1} \sigma_N = 0, \quad \lim_{N \to \infty} \widetilde{\sigma}_N = \infty.$$
(2.7)

Define the infinitesimal generator $L_{N,t}$ as

$$L_{N,t} = pL_{\text{TAS}} + \sigma_N L_{\text{SS}} + \widetilde{\sigma}_N (L_{-,t} + L_{+,t}).$$

$$(2.8)$$

In (2.8), the symmetric exclusion is accelerated by σ_N in order to enhance convergence to local equilibrium at a mesoscopic scale, while the reservoirs are strengthened by $\tilde{\sigma}_N$ to fix the densities at 1 and N, see Proposition 7.7. The technical assumption $\sigma_N \gg N^{\frac{5}{7}}$ allows us to choose the specific mesoscopic scale in (4.12). Moreover, this dynamic is NOT the so called *weakly asymmetric exclusion* since the asymmetry is strong enough to survive in the hyperbolic space-time scaling limit. From a macroscopic point of view, $\sigma_N L_{\rm SS}$ plays the role of vanishing viscosity in the hyperbolic conservation law.

2.2. Hydrodynamic limit. Let $\eta(t) = (\eta_1(t), \ldots, \eta_N(t)) \in \Omega_N$ be the Markov process generated by $NL_{N,t}$ and some initial distribution $\mu_{N,0}$. Assume that for all $\phi \in \mathcal{C}([0,1])$ and $\varepsilon > 0$,

$$\lim_{N \to \infty} \mu_{N,0} \left\{ \left| \frac{1}{N} \sum_{i=1}^{N} \phi\left(\frac{i}{N}\right) \eta_i - \int_0^1 \phi(x) u_0(x) dx \right| > \varepsilon \right\} = 0$$
(2.9)

with some $u_0 \in L^{\infty}([0,1])$. Let $\chi_{i,N}$ be the indicator function

$$\chi_{i,N}(x) := \mathbf{1}_{\left\{ \left[\frac{i}{N} - \frac{1}{2N}, \frac{i}{N} + \frac{1}{2N}\right] \cap [0,1] \right\}}(x), \quad \forall x \in [0,1].$$
(2.10)

For each N, define the empirical density $\zeta_N = \zeta_N(t, x)$ as

$$\zeta_N(t,x) := \sum_{i=1}^N \eta_i(t)\chi_{i,N}(x), \quad (t,x) \in [0,+\infty) \times [0,1].$$
(2.11)

Our aim is to show that, as $N \to \infty$, ζ_N converges to the L^{∞} weak entropy solution to the one-dimensional inviscid Burgers equation

$$\partial_t u(t,x) + p \partial_x J(u(t,x)) = 0, \quad x \in (0,1), \quad J(u) = u(1-u),$$
(2.12)

with L^{∞} boundary and initial data formally given by

$$u|_{x=0} = \rho_{-}(t) := \frac{\alpha(t)}{\alpha(t) + \gamma(t)}, \quad u|_{x=1} = \rho_{+}(t) := \frac{\delta(t)}{\beta(t) + \delta(t)}, \quad u|_{t=0} = u_{0}.$$
 (2.13)

Notice that (2.13) is not satisfied in the classical sense. Instead, it allows discontinuity at the boundaries, usually called the boundary layers, by imposing a set of possible boundary values relevant to $\rho_{\pm}(t)$. Rigorous definition of the entropy solution to (2.12)–(2.13) is stated in Section 3. Our main result is stated below.

Theorem 2.2 (Hyperbolic hydrodynamic limit). Suppose (2.7), (2.9) and that

$$\operatorname{essinf}_{0 \le t \le T} \left\{ \alpha(t), \beta(t), \gamma(t), \delta(t) \right\} > 0$$
(2.14)

for some T > 0. For any $\psi \in \mathcal{C}(\mathbb{R}^2)$, the following limit holds in probability:

$$\lim_{N \to \infty} \int_0^T \frac{1}{N} \sum_{i=1}^N \psi\left(t, \frac{i}{N}\right) \eta_i(t) dt = \int_0^T \int_0^1 \psi(t, x) \rho(t, x) dx \, dt, \tag{2.15}$$

where $\rho = \rho(t, x)$ is the entropy solution to (2.12)–(2.13).

Remark 2.3 (Boundary values). Without the factor $\tilde{\sigma}_N$ speeding up the reservoirs, we conjecture the boundary values are given by some $\tilde{\rho}_-(t) = \tilde{\rho}_-(p, \alpha, \gamma)$ and $\tilde{\rho}_+(t) = \tilde{\rho}_+(p, \beta, \delta)$. Due to the non-zero drift from 0 to 1, it is expected that $\tilde{\rho}_- < \rho_-$, $\tilde{\rho}_+ > \rho_+$. For the particular case where $(\alpha, \beta, \gamma, \delta)$ are determined through (2.6) with parameters $\bar{\rho}_{\pm}$ that is constant in time, the boundary conditions are given by [1]

$$u(t,0) = \bar{\rho}_{-}, \quad u(t,1) = \bar{\rho}_{+}.$$
 (2.16)

The result for general case remains an open problem.

2.3. A few words on the quasi-stationary profile. The quasi-stationary solution to (2.12)–(2.13) is defined as the limit, when $\epsilon \to 0^+$, of the entropy solution to

$$\epsilon \partial_t u^{\epsilon} + p \partial_x J(u^{\epsilon}) = 0, \quad u^{\epsilon}|_{x=0} = \rho_-, \quad u^{\epsilon}|_{x=1} = \rho_+, \quad u^{\epsilon}|_{t=0} = u_0.$$
(2.17)

Excluding the critical phase given by

$$(\rho_{-}, \rho_{+}) \in \{(a, b) \in (0, 1)^2, \ a < 1/2, \ a + b = 1\},$$

$$(2.18)$$

this limit is constant in x and is explicitly given in [19]:

$$\bar{u}(t) = \begin{cases} \rho_{-}(t), & \text{if } \rho_{-}(t) < 1/2, \ \rho_{-}(t) < 1 - \rho_{+}(t), \\ \rho_{+}(t), & \text{if } \rho_{+}(t) > 1/2, \ \rho_{+}(t) > 1 - \rho_{-}(t), \\ 1/2, & \text{if } \rho_{-}(t) \ge 1/2, \ \rho_{+}(t) \le 1/2. \end{cases}$$
(2.19)

This extends the variational property in [22] to a quasi-static sense:

$$J(\bar{u}(t)) = \begin{cases} \inf\{J(\rho); \rho \in [\rho_{-}(t), \rho_{+}(t)]\}, & \text{if } \rho_{-}(t) \le \rho_{+}(t), \\ \sup\{J(\rho); \rho \in [\rho_{+}(t), \rho_{-}(t)]\}, & \text{if } \rho_{-}(t) > \rho_{+}(t). \end{cases}$$
(2.20)

The quasi-stationary profile can be directly derived from the microscopic dynamics under the quasi-static time scale. Precisely speaking, for a > 0 let $N^{1+a}L_{N,t}$ generate the process $\eta'(t) \in \Omega_N$. Suppose ζ'_N to be the empirical density of η' defined in (2.11). Then ζ'_N converges to \bar{u} , as $N \to \infty$ in the sense of (2.15). This quasi-static limit has been proved in [8, Theorem 3.3] when the boundary data are smooth functions of t and $\rho_- - \rho_+$ keeps its sign within [0, T]. We expect the same result to hold for general non-smooth boundary data.

3. Entropy solution on bounded domain

Fix two measurable functions $\rho_{\pm} : [0, \infty) \to [0, 1]$ and recall the initial-boundary problem in (2.12)–(2.13). We say that $(f, q) \in \mathcal{C}^2([0, 1]; \mathbb{R}^2)$ is a Lax entropy flux pair associated to J if

$$q'(u) = J'(u)f'(u) = (1 - 2u)f'(u), \quad \forall u \in [0, 1].$$
(3.1)

The pair (f,q) is called convex if $f''(u) \ge 0$ for $u \in [0,1]$. To characterise the boundary conditions, Otto [21] introduced the boundary entropy.

Definition 3.1. A boundary entropy flux pair associated to (2.12) is a couple of \mathcal{C}^2 functions $(F, Q) : [0, 1] \times \mathbb{R} \to \mathbb{R}^2$ such that

$$F(w,w) = Q(w,w) = \partial_u F(w,w) = 0, \quad \forall (u,w) \in [0,1] \times \mathbb{R},$$
(3.2)

and $(F,Q)(\cdot,w)$ is a Lax entropy flux pair for each $w \in \mathbb{R}$. Moreover, we say that the pair (F,Q) is convex if F(u,w) is convex in u for all $w \in \mathbb{R}$.

The entropy solution to (2.12)–(2.13) is a function $\rho \in L^{\infty}(\mathbb{R}_+ \times [0,1])$ satisfying

i. the entropy inequality in the weak sense

$$\int_0^\infty \int_0^1 f(\rho) \partial_t \varphi \, dx \, dt + p \int_0^\infty \int_0^1 q(\rho) \partial_x \varphi \, dx \, dt \ge 0, \tag{3.3}$$

for all convex entropy flux pair (f,q) and positive function $\varphi \in \mathcal{C}^{\infty}(\mathbb{R}_+ \times (0,1))$ with compact support;

ii. the initial condition in the sense

$$\operatorname{esslim}_{t \to 0^+} \int_0^1 |\rho(t, x) - u_0(x)| dx = 0;$$
(3.4)

iii. the boundary conditions introduced by Otto [21]

$$\operatorname{esslim}_{r \to 0^{+}} \int_{0}^{\infty} Q(\rho(t, r), \rho_{-}(t)) \phi(t) dt \leq 0,$$

$$\operatorname{esslim}_{r \to 0^{+}} \int_{0}^{\infty} Q(\rho(t, 1 - r), \rho_{+}(t)) \phi(t) dt \geq 0,$$
(3.5)

for all convex boundary entropy flux Q and positive function $\phi \in \mathcal{C}(\mathbb{R}_+)$ with compact support.

Observe that the equality in (3.3) is reached for (f, q) = (u, J(u)), hence entropy solution is in particular a weak solution. It is not hard to see that $\rho(t, x) \in [0, 1]$ almost surely.

Remark 3.2 (Boundary traces). In the case ρ is of bounded variation, (3.5) coincides with the Bardos–Leroux–Nédélec condition [3]. Later, Vasseur [25] proved that ρ has strong boundary traces even in L^{∞} regime, so Bardos–Leroux–Nédélec's condition is generally valid. Nevertheless, we use an alternative expression that does not contain the explicit information of the boundary traces, see (4.11) for detail.

4. Proof of Theorem 2.2

Fix some T > 0 and denote $\Sigma_T = [0, T] \times [0, 1]$. By a Young measure on $\Sigma_T \times [0, 1]$ we mean a positive measure ν on $\Sigma_T \times [0, 1]$ such that

$$\nu(A \times [0,1]) = \mu_{\text{Leb}}(A), \quad \forall \text{ Borel measurable set } A \subseteq \Sigma_T, \tag{4.1}$$

where μ_{Leb} is the Lebesgue measure. Denote by \mathcal{Y} the set of all Young measures, endowed with the weak topology, i.e., a sequence $\nu^n \to \nu$ if and only if

$$\lim_{n \to \infty} \int f \, d\nu^n = \int f \, d\nu, \quad \forall f \in \mathcal{C}(\Sigma_T \times [0, 1]).$$
(4.2)

The condition (4.1) being stable under weak convergence, so \mathcal{Y} is closed. Since $\Sigma_T \times [0, 1]$ is a Polish space, also is \mathcal{Y} . Moreover, Prokhorov's theorem yields that \mathcal{Y} is compact.

By the slicing measure theorem (cf. [11, pp.14, Theorem 1.10]), $\nu \in \mathcal{Y}$ can be interpreted as a family $\{\nu_{t,x}; (t,x) \in \Sigma_T\}$ of probability measures on [0, 1], such that

$$\int f \, d\nu = \iint_{\Sigma_T} dx \, dt \int_0^1 f(t, x, u) \nu_{t, x}(du), \quad \forall f \in \mathcal{C}(\Sigma_T \times [0, 1]).$$
(4.3)

Let \mathcal{Y}_D be the subset of \mathcal{Y} consisting of all Dirac type measures:

$$\mathcal{Y}_D := \left\{ \nu \in \mathcal{Y} : \nu_{t,x} = \delta_{\rho(t,x)} \text{ a.e., for some } \rho \in L^{\infty}(\Sigma_T), 0 \le \rho(t,x) \le 1 \right\}.$$
(4.4)

Recall the empirical density ζ_N in (2.11) and define the Young measures

$$\nu^{N} := \{ \nu_{t,x}^{N} = \delta_{\zeta_{N}(t,x)}; \ (t,x) \in \Sigma_{T} \}.$$
(4.5)

Since $\zeta_N \in [0,1]$ for all N, ν^N generates a probability measure on \mathcal{Y} that concentrates on \mathcal{Y}_D . To prove (2.15), it suffices to show that ν^N converge, in probability, to the Dirac type measure $\{\delta_{\rho(t,x)}; (t,x) \in \Sigma_T\}$ with the entropy solution ρ to (2.12)–(2.13). Proof of Theorem 2.2. Let K = K(N) be a mesoscopic scale such that K = o(N). The choice of K will be specified later in (4.12). Define the uniform average

$$\bar{\eta}_{i,K} := \frac{1}{K} \sum_{i'=0}^{K-1} \eta_{i-i'}, \quad \forall i = K, K+1, \dots, N.$$
(4.6)

For i = K, ..., N - K + 1, define the smoothly weighted average

$$\hat{\eta}_{i,K} := \frac{1}{K} \sum_{i'=0}^{K-1} \bar{\eta}_{i+i',K} = \sum_{i'=-K+1}^{K-1} w_{i'} \eta_{i-i'}, \quad w_{i'} = \frac{K - |i'|}{K^2}.$$
(4.7)

The corresponding empirical density ρ_N is given by

$$\rho_N(t,x) := \sum_{i=K+1}^{N-K} \chi_{i,N}(x)\hat{\eta}_{i,K}(t), \quad (t,x) \in [0,+\infty) \times [0,1],$$
(4.8)

where $\chi_{i,N}$ is the indicator function in (2.10). Observe that for any $\psi \in \mathcal{C}^1(\Sigma_T)$,

$$\lim_{N \to \infty} \left| \iint_{\Sigma_T} \psi(t, x) \big(\zeta_N(t, x) - \rho_N(t, x) \big) dx \, dt \right| \le \lim_{N \to \infty} \frac{C_{\psi} T K}{N} = 0. \tag{4.9}$$

Therefore, ρ_N can be identified with ζ_N in the macroscopic limit. In the following we prove Theorem 2.2 with η_i replaced by $\hat{\eta}_{i,K}$.

Denote by $\hat{\nu}_N$ the Young measure concentrated on ρ_N . Let \mathfrak{Q}_N be the probability distribution of $\hat{\nu}_N$ on \mathcal{Y} . Since \mathcal{Y} is compact, the sequence \mathfrak{Q}_N is tight, thus we can extract a subsequence $\mathfrak{Q}_{N'}$ that converges weakly to some measure \mathfrak{Q} on \mathcal{Y} , which means that for all $f \in \mathcal{C}(\Sigma_T \times [0, 1]),$

$$\begin{split} E^{\mathfrak{Q}}\left[\iint_{\Sigma_{T}}dx\,dt\int_{0}^{1}f(t,x,u)\nu_{t,x}(du)\right] &= \lim_{N'\to\infty}E^{\mathfrak{Q}_{N'}}\left[\iint_{\Sigma_{T}}dx\,dt\int_{0}^{1}f(t,x,u)\nu_{t,x}(du)\right] \\ &= \lim_{N'\to\infty}\mathbb{E}_{N'}\left[\iint_{\Sigma_{T}}f\left(t,x,\rho_{N'}(t,x)\right)dx\,dt\right]. \end{split}$$

The proof is then divided into two main steps. First, in Section 6 we show that there exists $\rho \in L^{\infty}(\Sigma_T)$, which may depend on the choice of subsequence N', such that

$$\mathfrak{Q}\left\{\nu_{t,x} = \delta_{\rho(t,x)}, \ (t,x) - \text{a.s.}\right\} = 1.$$
(4.10)

Second, in Section 7 we prove that $\rho = \rho(t, x)$ satisfies that

$$-\iint_{\Sigma_{T}} F(\rho,h)\partial_{t}\psi \,dx \,dt - p \iint_{\Sigma_{T}} Q(\rho,h)\partial_{x}\psi \,dx \,dt$$

$$\leq \int_{0}^{1} F(u_{0},h)\psi(0,\cdot)dx + p \int_{0}^{T} F(\rho_{-},h)\psi(\cdot,0)dt + p \int_{0}^{T} F(\rho_{+},h)\psi(\cdot,1)dt$$

$$(4.11)$$

for all convex boundary entropy flux pair (F, Q), any $h \in \mathbb{R}$ and smooth, positive function ψ defined on Σ_T . By [20, Theorem 2.7.31], (4.11) is a necessary and sufficient condition of (3.3)–(3.5). Hence, \mathfrak{Q} is concentrated on the Dirac type Young measure associated to the entropy solution to (2.12)–(2.13). Finally, we conclude from the uniqueness of the entropy solution [20, Theorem 2.7.28] that \mathfrak{Q} is independent of the choice of subsequence.

4.1. Mesoscopic scale. Hereafter we fix K to be such that

$$\lim_{N \to \infty} \frac{N\sigma_N}{K^3} = \lim_{N \to \infty} \frac{NK^2}{\sigma_N^3} = \lim_{N \to \infty} \frac{\sigma_N^2}{NK} = 0.$$
(4.12)

This choice of the mesoscopic scale is crucial for the proof, for instance in Lemma 6.3, 6.7 and 7.4. In view of (2.7), a simple example is to choose

$$\sigma_N = N^{\frac{5}{7}+\kappa}, \quad K = \left[N^{\frac{4}{7}+\frac{17\kappa}{14}+\kappa^2}\right], \quad \kappa \in \left(0,\frac{2}{7}\right), \tag{4.13}$$

In particular, $N^{4/7} \ll K \ll \sigma_N \ll N$.

5. Microscopic entropy production

Given an entropy pair (f, q) and a Young measure $\nu \in \mathcal{Y}$, the entropy production of ν is a distribution $X^f(\nu) \in W^{-1,\infty}(\Sigma_T)$ defined by

$$X^{f}(\nu) := \partial_{t} \langle f \rangle_{\nu_{t,x}} + p \partial_{x} \langle q \rangle_{\nu_{t,x}}, \quad (t,x) \in \Sigma_{T}.$$
(5.1)

Applied on a test function $\psi \in \mathcal{C}^1(\Sigma_T)$, this definition reads

$$X^{f}(\nu,\psi) = -\iint_{\Sigma_{T}} \langle f \rangle_{\nu_{t,x}} \partial_{t} \psi \, dx \, dt - p \iint_{\Sigma_{T}} \langle q \rangle_{\nu_{t,x}} \partial_{x} \psi \, dx \, dt.$$
(5.2)

To simplify the notations, for i = 1, ..., N we write

$$\psi_i(t) := \psi\left(t, \frac{i}{N} - \frac{1}{2N}\right), \quad \bar{\psi}_i(t) := N \int_0^1 \psi(t, x) \chi_{i,N}(x) dx.$$
(5.3)

For a sequence $\{a_i\}$, denote $\nabla a_i = a_{i+1} - a_i$, $\nabla^* a_i = a_{i-1} - a_i$ and

$$\Delta a_i = (-\nabla^* \nabla) a_i = a_{i+1} - 2a_i + a_{i-1}.$$
(5.4)

For i = K + 1, ..., N - K, recall $\hat{\eta}_{i,K}$ defined in (4.7). Observe that they are supported on $\{\eta_2, ..., \eta_{N-1}\}$, so $L_{\pm,t}$ do not contribute to their time evolution and

$$L_{N,t}[\hat{\eta}_{i,K}] = p \nabla^* \hat{J}_{i,K} + \sigma \Delta \hat{\eta}_{i,K}, \quad \hat{J}_{i,K} = \sum_{i'=-K+1}^{K-1} w_{i'} J_{i-i',i-i'+1}, \quad (5.5)$$

where $J_{i,i+1} = \eta_i(1 - \eta_{i+1})$ gives the microscopic current. In most of the following contents, we omit the subscript K in $\hat{\eta}_{i,K}$, $\hat{J}_{i,K}$ and write $\hat{\eta}_i$, \hat{J}_i for short.

Lemma 5.1 (Basic decomposition). Recall the empirical density ρ_N in (4.8) and the Dirac type Young measure $\hat{\nu}_N$ concentrated on it. The entropy production decomposes as

$$X^{f}(\hat{\nu}_{N},\psi) = \frac{1}{N} \sum_{i=K+1}^{N-K} f(\hat{\eta}_{i}(0)) \bar{\psi}_{i}(0) + \mathcal{M}_{N}(\psi) + \mathcal{A}_{N}(\psi) + \mathcal{S}_{N}(\psi) + \sum_{\ell=1,2,3} \mathcal{E}_{N,\ell}(\psi),$$
(5.6)

for all $\psi \in \mathcal{C}^1(\Sigma_T)$ such that $\psi(T, \cdot) = 0$. In (5.6),

$$\mathcal{M}_{N}(\psi) := -\int_{0}^{T} \frac{1}{N} \sum_{i=K+1}^{N-K} \bar{\psi}'_{i}(t) M^{f}_{i,K}(t) dt$$
(5.7)

where $M_{i,K}^{f} = M_{i,K}^{f}(t)$ is a family of martingales,

$$\mathcal{A}_{N}(\psi) := p \int_{0}^{T} \sum_{i=K+1}^{N-K} \bar{\psi}_{i}(t) f'(\hat{\eta}_{i}(t)) \nabla^{*} \left[\hat{J}_{i}(t) - J(\hat{\eta}_{i}(t)) \right] dt,$$
(5.8)

$$\mathcal{S}_N(\psi) := \sigma_N \int_0^T \sum_{i=K+1}^{N-K} \bar{\psi}_i(t) f'(\hat{\eta}_i(t)) \Delta \hat{\eta}_i(t) dt, \qquad (5.9)$$

where J(u) = u(1-u) and for $\ell = 1, 2, 3$,

$$\mathcal{E}_{N,1}(\psi) := p \int_0^T \sum_{i=K+1}^{N-K} \left[\bar{\psi}_i \nabla^* q(\hat{\eta}_i) - q(\hat{\eta}_i) \nabla \psi_i \right] dt,$$
(5.10)

$$\mathcal{E}_{N,2}(\psi) := \int_0^T \sum_{i=K+1}^{N-K} \bar{\psi}_i(t) \left(\epsilon_{i,K}^{(1)} + p\epsilon_{i,K}^{(2)}\right) dt, \qquad (5.11)$$

$$\mathcal{E}_{N,3}(\psi) := -pq(0) \int_0^T \left(\int_0^{\frac{2K+1}{2N}} + \int_{1-\frac{2K-1}{2N}}^1 \right) \partial_x \psi \, dx \, dt, \tag{5.12}$$

where the small correction terms are given by

$$\epsilon_{i,K}^{(1)} := L_{N,t} \big[f(\hat{\eta}_i) \big] - f'(\hat{\eta}_i) L_{N,t}[\hat{\eta}_i], \quad \epsilon_{i,K}^{(2)} := f'(\hat{\eta}_i) \nabla^* J(\hat{\eta}_i) - \nabla^* q(\hat{\eta}_i).$$
(5.13)
By the definition in (4.8)

Proof. By the definition in (4.8),

$$-\iint_{\Sigma_T} f(\rho_N) \partial_t \psi \, dx \, dt = -\int_0^T \frac{1}{N} \sum_{i=K+1}^{N-K} f(\hat{\eta}_i) \bar{\psi}'_i \, dt.$$
(5.14)

By Dynkin formula, $M_{i,K}^f = M_{i,K}^f(t)$ is a martingale, where

$$M_{i,K}^{f}(t) := f(\hat{\eta}_{i}(t)) - f(\hat{\eta}_{i}(0)) - N \int_{0}^{t} L_{N,s}[f(\hat{\eta}_{i}(s))] ds.$$
(5.15)

Its quadratic variance is given by

$$\langle M_{i,K}^f \rangle(t) = N \int_0^t \left\{ L_{N,s} \left[f(\hat{\eta}_i)^2 \right] - 2f(\hat{\eta}_i) L_{N,s} \left[f(\hat{\eta}_i) \right] \right\} ds.$$
 (5.16)

Recalling (5.5) and (5.13), we can decompose as

$$L_{N,t}[f(\hat{\eta}_{i})] = \epsilon_{i,K}^{(1)} + f'(\hat{\eta}_{i}) \left(p \nabla^{*} \hat{J}_{i} + \sigma_{N} \Delta \hat{\eta}_{i} \right)$$

= $\epsilon_{i,K}^{(1)} + p \left\{ f'(\hat{\eta}_{i}) \nabla^{*} \left[\hat{J}_{i} - J(\hat{\eta}_{i}) \right] + \epsilon_{i,K}^{(2)} + \nabla^{*} q(\hat{\eta}_{i}) \right\} + \sigma_{N} f'(\hat{\eta}_{i}) \Delta \hat{\eta}_{i}.$ (5.17)

Therefore, (5.14) and integrate-by-parts formula yield that

$$-\iint_{\Sigma_{T}} f(\rho_{N})\partial_{t}\psi \,dx \,dt = \frac{1}{N} \sum_{i=K+1}^{N-K} f(\hat{\eta}_{i,K}(0))\bar{\psi}_{i}(0) + \mathcal{M}_{N}(\psi) + \mathcal{A}_{N}(\psi) + \mathcal{S}_{N}(\psi) + p \int_{0}^{T} \sum_{i=K+1}^{N-K} \bar{\psi}_{i} \nabla^{*} q(\hat{\eta}_{i}) dt + \mathcal{E}_{N,2}(\psi).$$
(5.18)

On the other hand, elementary manipulation shows that

$$-p \iint_{\Sigma_T} q(\rho_N) \partial_x \psi \, dx \, dt = -p \int_0^T \sum_{i=K+1}^{N-K} q(\hat{\eta}_i) \nabla \psi_i \, dt + \mathcal{E}_{N,3}(\psi), \tag{5.19}$$

tion (5.6) then follows from (5.2), (5.18) and (5.19).

The decomposition (5.6) then follows from (5.2), (5.18) and (5.19).

6. Compensated compactness

The aim of this section is to verify (4.10), following the stochastic compensated compactness developed in [15]. We first prepare some useful notations. Denote by $\mathcal{C}_0^{\infty}(\Sigma_T)$ the class of smooth functions on Σ_T with compact support included in the interior of Σ_T . Let $H_0^1(\Sigma_T)$ be the completion of $\mathcal{C}_0^{\infty}(\Sigma_T)$ under the norm

$$\|\varphi\|_{H^1_0(\Sigma_T)}^2 := \iint_{\Sigma_T} \left[\varphi^2 + (\partial_t \varphi)^2 + (\partial_x \varphi)^2\right] dx \, dt.$$
(6.1)

The dual space of $H_0^1(\Sigma_T)$ is denoted by $H^{-1}(\Sigma_T)$, equipped with the norm

$$||X||_{H^{-1}(\Sigma_T)} := \sup\left\{ |X(\varphi)|, \ \varphi \in \mathcal{C}_0^{\infty}(\Sigma_T), \ ||\varphi||_{H^1_0(\Sigma_T)} \le 1 \right\}.$$
(6.2)

Let $\mathcal{M}(\Sigma_T)$ be the set of signed Radon measures on Σ_T . Define

$$\|\mu\|_{\mathcal{M}(\Sigma_T)} := \sup\left\{ |\mu(\varphi)|, \ \varphi \in \mathcal{C}_0^\infty(\Sigma_T), \ \|\varphi\|_{L^\infty(\Sigma_T)} \le 1 \right\}.$$
(6.3)

Recall the empirical density ρ_N in (4.8) and the Young measure $\hat{\nu}_N$ associated to it. Since $\hat{\nu}_N$ is of Dirac type, $\mathfrak{Q}_N(\mathcal{Y}_D) = 1$, where \mathfrak{Q}_N is the distribution of $\hat{\nu}_N$ and \mathcal{Y}_D is defined in (4.4). Recall that \mathfrak{Q} is a weak limit point of \mathfrak{Q}_N . To show $\mathfrak{Q}(\mathcal{Y}_D) = 1$, the main obstacle is that \mathcal{Y}_D is not closed under the weak topology. In the following, we construct a set $\mathcal{G} \subseteq \mathcal{Y}$ such that the closure of $\mathcal{Y}_D \cap \mathcal{G}$ is contained in \mathcal{Y}_D and $\lim_{N\to\infty} \mathfrak{Q}_N(\mathcal{Y}_D \cap \mathcal{G}) = 1$.

For two given Lax entropy flux pairs (f_1, q_1) , (f_2, q_2) and two sequences $\{a_N\}$, $\{b_N\}$ such that $\lim_{N\to\infty} a_N = 0$, $\sup b_N < \infty$, define

$$\mathcal{G}_{N} := \left\{ \nu \in \mathcal{Y}_{D} \middle| \begin{array}{l} \text{for } j = 1, 2, \ X^{f_{j}}(\nu) = Y_{j} + Z_{j}, \text{ s.t.} \\ \|Y_{j}\|_{H^{-1}(\Sigma_{T})} \leq a_{N}, \ \|Z_{j}\|_{\mathcal{M}(\Sigma_{T})} \leq b_{N} \end{array} \right\},$$
(6.4)

where $X^{f_j}(\nu)$ is the entropy production defined in (5.1). Let \mathcal{G} be the closure of $\bigcap_{N\geq 1}\mathcal{G}_N$ under the weak topology. By the div-curl lemma ([11, Theorem 5.B.4, p.54]), \mathcal{G} is a subset of \mathcal{Y}_D . In Proposition 6.1 below we prove that $\lim_{N\to\infty} \mathfrak{Q}_N(\mathcal{G}_N) = 1$ and consequently that $\mathfrak{Q}(\mathcal{Y}_D) \geq \mathfrak{Q}(\mathcal{G}) = 1$ for any weak limit point \mathfrak{Q} .

Proposition 6.1. The entropy production $X^f(\hat{\nu}_N) = Y_N + Z_N$, such that

$$\mathbb{E}_{N}\left[\|Y_{N}\|_{H^{-1}(\Sigma_{T})}\right] \leq a_{N}, \quad \lim_{N \to \infty} a_{N} = 0,$$

$$\mathbb{E}_{N}\left[\|Z_{N}\|_{\mathcal{M}(\Sigma_{T})}\right] \leq b_{N}, \quad \sup b_{N} < \infty.$$
(6.5)

In order to prove Proposition 6.1, we apply (5.6) with $\varphi \in C_0^{\infty}(\Sigma_T)$ and estimate each term in the right-hand side by several lemmas. Recall the definition of φ_i and $\bar{\varphi}_i$ in (5.3).

Lemma 6.2. As $N \to \infty$, $\mathcal{E}_{N,1}$ and $\mathcal{E}_{N,3}$ vanish uniformly in $H^{-1}(\Sigma_T)$, while $\mathcal{E}_{N,2}$ vanishes uniformly in $\mathcal{M}(\Sigma_T)$.

Proof. We first treat $\mathcal{E}_{N,1}$. Applying summation by parts¹,

$$\mathcal{E}_{N,1}(\varphi) = p \int_0^T \varphi_{K+1} q(\hat{\eta}_K) dt - p \int_0^T \varphi_{N-K+1} q(\hat{\eta}_{N-K}) dt + p \int_0^T \sum_{i=K+1}^{N-K} (\bar{\varphi}_i - \varphi_i) \nabla^* q(\hat{\eta}_i) dt.$$
(6.6)

¹Though φ is compactly supported, the boundary terms can not be omitted autonomously, since we need to take supreme over all φ before send N to ∞ .

Since φ vanishes at the boundary of Σ_T , Cauchy–Schwarz inequality yields that

$$\int_0^T \left[\varphi(t,x)^2 + \varphi(t,1-x)^2\right] dt \le \int_0^T x \left[\int_0^x (\partial_y \varphi)^2 dy + \int_{1-x}^1 (\partial_y \varphi)^2 dy\right] dt$$

$$\le 2x \|\varphi\|_{H_0^1(\Sigma_T)}^2, \quad \forall x \in [0,1].$$
(6.7)

Consequently, $\int_0^T \varphi_{K+1}^2 dt + \int_0^T \varphi_{N-K+1}^2 dt \le CKN^{-1} \|\varphi\|_{H_0^1}^2$ and

$$\left| \int_{0}^{T} \varphi_{K+1} q(\hat{\eta}_{K}) dt - \int_{0}^{T} \varphi_{N-K+1} q(\hat{\eta}_{N-K}) dt \right| \\
\leq C \sqrt{\frac{K}{N}} \|\varphi\|_{H_{0}^{1}(\Sigma_{T})} \left\{ \left[\int_{0}^{T} q(\hat{\eta}_{K})^{2} dt \right]^{\frac{1}{2}} + \left[\int_{0}^{T} q(\hat{\eta}_{N-K})^{2} dt \right]^{\frac{1}{2}} \right\}$$

$$\leq C' T \|q\|_{\infty} \sqrt{\frac{K}{N}} \|\varphi\|_{H_{0}^{1}(\Sigma_{T})}.$$
(6.8)

Also noting that $|\nabla^* q(\hat{\eta}_i)| \le |q'|_{\infty} |\hat{\eta}_{i-1} - \hat{\eta}_i| \le |q'|_{\infty} K^{-1}$,

$$\left| \int_{0}^{T} \sum_{i=K+1}^{N-K} (\bar{\varphi}_{i} - \varphi_{i}) \nabla^{*} q(\hat{\eta}_{i}) dt \right|^{2} \leq |q'|_{\infty}^{2} \frac{NT}{K^{2}} \int_{0}^{T} \sum_{i=K+1}^{N-K} (\bar{\varphi}_{i} - \varphi_{i})^{2} dt \\ \leq C |q'|_{\infty}^{2} K^{-2} \|\varphi\|_{H^{1}_{0}(\Sigma_{T})}^{2}.$$
(6.9)

For $\mathcal{E}_{N,3}$, by Cauchy–Schwarz inequality,

$$\left|\mathcal{E}_{N,3}(\varphi)\right|^{2} \leq \frac{2p^{2}|q(0)|^{2}TK}{N} \iint_{\Sigma_{T}} (\partial_{x}\varphi)^{2} dx \, dt \leq \frac{C|q|_{\infty}^{2}TK}{N} \|\varphi\|_{H_{0}^{1}(\Sigma_{T})}^{2}.$$
(6.10)

The conclusion then follows from (2.7) and (4.12).

Now we estimate $\mathcal{E}_{N,2}$ in $\mathcal{M}(\Sigma_T)$. By the definition (2.8) of $L_{N,t}$,

$$\epsilon_{i,K}^{(1)} = \sum_{j=i-K}^{i+K-1} (p\eta_j + \sigma_N) \left[f(\hat{\eta}_i^{j,j+1}) - f(\hat{\eta}_i) - f'(\hat{\eta}_i) \left(\hat{\eta}_i^{j,j+1} - \hat{\eta}_i \right) \right].$$
(6.11)

Recall $\hat{\eta}_i = \hat{\eta}_{i,K}$ in (4.7). Direct computation shows that

$$\hat{\eta}_{i}^{j,j+1} = \hat{\eta}_{i} - \operatorname{sgn}\left(i - j - \frac{1}{2}\right) \frac{\nabla \eta_{j}}{K^{2}}, \quad j - K + 1 \le i \le j + K$$
(6.12)

and otherwise $\hat{\eta}_i^{j,j+1} - \hat{\eta}_i = 0$. Therefore,

$$\left|\epsilon_{i,K}^{(1)}\right| \le \sum_{j=i-K}^{i+K-1} \frac{(p+\sigma_N)|f''|_{\infty}}{2} \left(\hat{\eta}_i^{j,j+1} - \hat{\eta}_i\right)^2 \le \frac{(p+\sigma_N)|f''|_{\infty}}{K^3}.$$
(6.13)

Also notice that f'(u)J'(u) = q'(u) and $|\nabla^*\hat{\eta}_{i,K}| \leq 1/K$, so that

$$\left|\epsilon_{i,K}^{(2)}\right| \le \frac{|f'|_{\infty}|J''|_{\infty} + |q''|_{\infty}}{K^2} \le \frac{4|f'|_{\infty} + |f''|_{\infty}}{K^2}.$$
(6.14)

From (6.13) and (6.14),

$$\left|\mathcal{E}_{N,2}(\varphi)\right| \le \left[\frac{pN\sigma_N |f''|_{\infty}}{K^3} + \frac{p(4|f'|_{\infty} + |f''|_{\infty})N}{K^2}\right] \|\varphi\|_{L^{\infty}(\Sigma_T)}.$$
(6.15)

In view of (4.12), $\mathcal{E}_{N,2}$ is negligible in $\mathcal{M}(\Sigma_T)$.

Lemma 6.3. As $N \to \infty$, $\mathbb{E}_N[\|\mathcal{M}_N\|_{H^{-1}(\Sigma_T)}]$ vanishes.

Proof. From (5.7) we see that

$$\left|\mathcal{M}_{N}(\varphi)\right|^{2} \leq \int_{0}^{T} \frac{1}{N} \sum_{i=K+1}^{N-K} \left|M_{i,K}^{f}(t)\right|^{2} dt \int_{0}^{T} \frac{1}{N} \sum_{i=K+1}^{N-K} \bar{\varphi}'(t)^{2} dt$$

$$\leq \frac{1}{N} \sum_{i=K+1}^{N-K} \int_{0}^{T} \left|M_{i,K}^{f}(t)\right|^{2} dt \times C \|\varphi\|_{H_{0}^{1}(\Sigma_{T})}^{2}.$$
(6.16)

The definition of $\|\cdot\|_{H^{-1}(\Sigma_T)}$ together with Doob's inequality yields that

$$\mathbb{E}_{N}\left[\left\|\mathcal{M}_{N}\right\|_{H^{-1}(\Sigma_{T})}^{2}\right] \leq \frac{C}{N} \sum_{i=K+1}^{N-K} \mathbb{E}_{N}\left[\int_{0}^{T} \left|M_{i,K}^{f}(t)\right|^{2} dt\right]$$
$$\leq \frac{C'}{N} \sum_{i=K+1}^{N-K} \mathbb{E}_{N}\left[\int_{0}^{T} \left\langle M_{i,K}^{f} \right\rangle(t) dt\right].$$
(6.17)

From (5.16), the quadratic variance reads

$$\mathbb{E}_{N}\left[\left\langle M_{i,K}^{f}\right\rangle(t)\right] = N \int_{0}^{t} \sum_{j=1}^{N-1} (p\eta_{j} + \sigma_{N}) \left[f(\hat{\eta}_{i}^{j,j+1}) - f(\hat{\eta}_{i})\right]^{2} ds$$

$$\leq CN\sigma_{N} |f'|_{\infty}^{2} \int_{0}^{t} \sum_{j=1}^{N-1} \left(\hat{\eta}_{i}^{j,j+1} - \hat{\eta}_{i}\right)^{2} dt \leq \frac{C|f'|_{\infty}^{2} N \sigma_{N}}{K^{3}},$$
(6.18)

where the last inequality follows from (6.12). Therefore,

$$\mathbb{E}_{N}\left[\left\|\mathcal{M}_{N}\right\|_{H^{-1}(\Sigma_{T})}^{2}\right] \leq \frac{C'|f'|_{\infty}^{2}N\sigma_{N}}{K^{3}}.$$
(6.19)

The right-hand side vanishes from (4.12).

To decompose \mathcal{A}_N and \mathcal{S}_N , we make use of the following block estimates. Their proofs are postponed to Section 8.4.

Proposition 6.4 (One-block estimate). There is some constant C, such that

$$\mathbb{E}_{N}\left[\int_{0}^{T}\sum_{i=K}^{N-K} \left[\hat{J}_{i} - J(\hat{\eta}_{i})\right]^{2} dt\right] \leq C\left(\frac{K^{2}}{\sigma_{N}} + \frac{N}{K}\right).$$
(6.20)

Proposition 6.5 (H^1 estimate). There is some constant C, such that

$$\mathbb{E}_N\left[\int_0^T \sum_{i=K}^{N-K} (\nabla \hat{\eta}_i)^2 dt\right] \le C\left(\frac{1}{\sigma_N} + \frac{N}{K^3}\right).$$
(6.21)

Lemma 6.6. A_N satisfies the decomposition in (6.5).

Proof. Without loss of generality, we fix p = 1. With $g_i := \hat{J}_i - J(\hat{\eta}_i)$,

$$\mathcal{A}_{N}(\varphi) = \int_{0}^{T} \sum_{i=K+1}^{N-K} f'(\hat{\eta}_{i+1}) g_{i} \nabla \bar{\varphi}_{i} dt + \int_{0}^{T} \sum_{i=K+1}^{N-K} \bar{\varphi}_{i} g_{i} \nabla f'(\hat{\eta}_{i}) dt + \int_{0}^{T} \left[\bar{\varphi}_{K+1} f'(\hat{\eta}_{K+1}) g_{K} - \bar{\varphi}_{N-K+1} f'(\hat{\eta}_{N-K+1}) g_{N-K} \right] dt.$$
(6.22)

Similarly to (6.8), the boundary integrals vanish uniformly in $H^{-1}(\Sigma_T)$ as $|g_i| \leq 1$.

Denote the first two integrals in (6.22) by $\mathcal{A}_{N,1}(\varphi)$ and $\mathcal{A}_{N,2}(\varphi)$, respectively. Applying Cauchy–Schwarz inequality,

$$\begin{aligned} \left|\mathcal{A}_{N,1}(\varphi)\right|^{2} &\leq \left|f'\right|_{\infty}^{2} \int_{0}^{T} \sum_{i=K+1}^{N-K} g_{i}^{2} dt \int_{0}^{T} \sum_{i=K+1}^{N-K} (\nabla \bar{\varphi}_{i})^{2} dt \\ &\leq \frac{C|f'|_{\infty}^{2}}{N} \|\varphi\|_{H_{0}^{1}(\Sigma_{T})}^{2} \int_{0}^{T} \sum_{i=K+1}^{N-K} g_{i}^{2} dt. \end{aligned}$$

$$(6.23)$$

Proposition 6.4 then shows that

$$\mathbb{E}_{N}\left[\left\|\mathcal{A}_{N,1}\right\|_{H^{-1}(\Sigma_{T})}^{2}\right] \leq \frac{C|f'|_{\infty}^{2}}{N} \mathbb{E}_{N}\left[\int_{0}^{T}\sum_{i=K+1}^{N-K}g_{i}^{2}dt\right]$$

$$\leq C'|f'|_{\infty}^{2}\left(\frac{K^{2}}{N\sigma_{N}}+\frac{1}{K}\right).$$
(6.24)

For the term $\mathcal{A}_{N,2}$, again by Cauchy–Schwarz inequality,

$$\begin{aligned} \left|\mathcal{A}_{N,2}(\varphi)\right|^{2} &\leq \left|f''\right|_{\infty}^{2} \left\|\varphi\right\|_{L^{\infty}(\Sigma_{T})}^{2} \left(\int_{0}^{T} \sum_{i=K+1}^{N-K} \left|g_{i} \nabla \hat{\eta}_{i}\right| dt\right)^{2} \\ &\leq \left|f''\right|_{\infty}^{2} \left\|\varphi\right\|_{L^{\infty}(\Sigma_{T})}^{2} \int_{0}^{T} \sum_{i=K+1}^{N-K} g_{i}^{2} dt \int_{0}^{T} \sum_{i=K+1}^{N-K} \left(\nabla \hat{\eta}_{i}\right)^{2} dt. \end{aligned}$$

$$(6.25)$$

Thanks to Proposition 6.4 and 6.5, we obtain the estimate

$$\mathbb{E}_{N}\left[\left\|\mathcal{A}_{N,2}\right\|_{\mathcal{M}(\Sigma_{T})}\right] \leq C|f''|_{\infty}\sqrt{\frac{K^{2}}{\sigma_{N}} + \frac{N}{K}}\sqrt{\frac{1}{\sigma_{N}} + \frac{N}{K^{3}}}$$

$$= C|f''|_{\infty}\left(\frac{K}{\sigma_{N}} + \frac{N}{K^{2}}\right).$$
(6.26)

Finally, we conclude the result from (2.7) and (4.12).

Lemma 6.7. S_N satisfies the decomposition in (6.5).

Proof. Recall that $\Delta = -\nabla^* \nabla$. Similarly to the previous proof,

$$S_{N}(\varphi) = S_{N,1}(\varphi) + S_{N,2}(\varphi) - \sigma_{N} \int_{0}^{T} \bar{\varphi}_{K+1} f'(\hat{\eta}_{K+1}) \nabla \hat{\eta}_{K} dt + \sigma_{N} \int_{0}^{T} \bar{\varphi}_{N-K+1} f'(\hat{\eta}_{N-K+1}) \nabla \hat{\eta}_{N-K} dt,$$
(6.27)

where

$$S_{N,1}(\varphi) = -\sigma_N \int_0^T \sum_{i=K+1}^{N-K} f'(\hat{\eta}_{i+1}) \nabla \hat{\eta}_i \nabla \bar{\varphi}_i \, dt,$$

$$S_{N,2}(\varphi) = -\sigma_N \int_0^T \sum_{i=K+1}^{N-K} \bar{\varphi}_i \nabla \hat{\eta}_i \nabla f'(\hat{\eta}_i) dt.$$
(6.28)

For the boundary integrals, noting that $|\nabla \hat{\eta}_i| \leq K^{-1}$, by (6.7) we have

$$\mathbb{E}_{N}\left[\sigma_{N}\int_{0}^{T}\bar{\varphi}_{K+1}f'(\hat{\eta}_{K+1})\nabla\hat{\eta}_{K}\,dt\right] \leq T|f'|_{\infty}\frac{\sigma_{N}}{\sqrt{NK}}\|\varphi\|_{H^{1}_{0}}.$$
(6.29)

The term with $\bar{\varphi}_{N-K+1}$ is estimated in the same way.

To deal with $\mathcal{S}_{N,1}$, first apply Cauchy–Schwarz inequality to obtain

$$\begin{aligned} \left| \mathcal{S}_{N,1}(\varphi) \right|^2 &\leq \left| f' \right|_{\infty}^2 \sigma_N^2 \int_0^T \sum_{i=K+1}^{N-K} \left(\nabla \hat{\eta}_i \right)^2 dt \int_0^T \sum_{i=K+1}^{N-K} (\nabla \bar{\varphi}_i)^2 dt \\ &\leq \frac{C |f'|_{\infty}^2 \sigma_N^2}{N} \|\varphi\|_{H_0^1(\Sigma_T)}^2 \int_0^T \sum_{i=K+1}^{N-K} \left(\nabla \hat{\eta}_i \right)^2 dt. \end{aligned}$$
(6.30)

Therefore, using Proposition 6.5 we get

$$\mathbb{E}_{N}\left[\left\|\mathcal{S}_{N,1}\right\|_{H^{-1}(\Sigma_{T})}^{2}\right] \leq C|f'|_{\infty}^{2}\left(\frac{\sigma_{N}}{N} + \frac{\sigma_{N}^{2}}{K^{3}}\right).$$
(6.31)

The term $\mathcal{S}_{N,2}$ is bounded in $\mathcal{M}(\Sigma_T)$. Indeed,

$$\left|\mathcal{S}_{N,2}(\varphi)\right| \le |f''|_{\infty} \sigma_N \|\varphi\|_{L^{\infty}(\Sigma_T)} \int_0^T \sum_{i=K+1}^{N-K} (\nabla\hat{\eta}_i)^2 dt.$$
(6.32)

Therefore, Proposition 6.5 gives the estimate

$$\mathbb{E}_{N}\left[\left\|\mathcal{S}_{N,2}\right\|_{\mathcal{M}(\Sigma_{T})}\right] \leq C|f''|_{\infty}\left(1+\frac{N\sigma_{N}}{K^{3}}\right).$$
(6.33)

We again conclude the result from (2.7) and (4.12).

Remark 6.8. It is clear from the above proof that $S_{N,2}$ in Lemma 6.7 is the only term which survives in the limit $N \to \infty$. It is the microscopic origin of the non-zero macroscopic entropy production appeared in (3.3).

7. Boundary entropy production

From Proposition 6.1, any weak-* limit point of ρ_N concentrates on some $\rho \in L^{\infty}(\Sigma_T)$. Note that ρ may depend on the choice of subsequence. In this section we show that ρ satisfies (4.11), thus is the unique L^{∞} entropy solution of (2.12)–(2.13).

Before stating the result of this section, we point out the main difficulty that prevents us from applying the direct approach in Section 6 and [8] here. Taking a test function ψ that is not compactly supported, the estimate (6.7) does not hold any more. Hence, in performing summation by parts in $S_N(\psi)$ like in Lemma 6.7, the boundary integrals are out of control because of the large factor σ_N . In this section, we introduce a grading argument to treat the divergent integrals mentioned above. The following definition of the auxiliary functions plays a central role. For each N, define for $x \in [0, 1]$ that

$$\alpha_N(x) := 1 - \mathbf{1}_{x \in [0,\delta_N]} \left(\frac{\sigma_N}{\sigma_N + 1}\right)^{Nx} - \mathbf{1}_{x \in (\delta_N,1]} \left(\frac{\sigma_N - 1}{\sigma_N}\right)^{N(1-x)},\tag{7.1}$$

where $\delta_N \in (0, 1)$ is chosen as

$$\delta_N = \frac{\log \sigma_N - \log(\sigma_N - 1)}{\log(\sigma_N + 1) - \log(\sigma_N - 1)}.$$
(7.2)

Observe that $\delta_N \approx 2^{-1} + (4\sigma_N)^{-1} + o(\sigma_N^{-2})$ for large N. For sufficiently large N, the function α_N is continuous, piecewise smooth and satisfies that

- (i). $\alpha_N(0) = \alpha_N(1) = 0, \ \alpha_N(x) \in [0, 1)$ for $x \in [0, 1];$
- (ii). $\lim_{N\to\infty} \alpha_N(x) = 1$ for $x \in (0, 1)$, uniformly on any compact subset;
- (iii). $|\alpha'_N(x)| \leq 2N\sigma_N^{-1}$ for $x \in (0, \delta_N) \cup (\delta_N, 1)$ and $\int |\alpha'_N| dx \leq 2$.

By (ii) together with the boundedness of $f(\rho_N)$ and $q(\rho_N)$, to show (4.11) it suffices to prove the next proposition and send $N \to \infty$.

Proposition 7.1. Assume (2.7) and (4.12), then

$$\lim_{N \to \infty} \mathbb{P}_N \left\{ -\iint_{\Sigma_T} F(\rho_N, h) \alpha_N \partial_t \psi \, dx \, dt - p \iint_{\Sigma_T} Q(\rho_N, h) \alpha_N \partial_x \psi \, dx \, dt \le \Theta_{u_0, \rho_{\pm}}^{F, h}(\psi) \right\} = 1,$$
(7.3)

for all boundary entropy flux pair (F, Q), $h \in \mathbb{R}$ and $\psi \in \mathcal{C}^{\infty}(\Sigma_T)$ such that $\psi \ge 0$, $\psi(T, \cdot) = 0$, where u_0 , ρ_{\pm} are the initial and boundary data in (2.13), and

$$\Theta_{u_0,\rho_{\pm}}^{F,h}(\psi) := \int_0^1 f(u_0)\psi(0,\cdot)dx + p \int_0^T \left[f(\rho_-)\psi(\cdot,0) + f(\rho_+)\psi(\cdot,1)\right]dt.$$
(7.4)

In the following proof, we fix p = 1, an arbitrary convex boundary entropy flux pair (F, Q)and $h \in \mathbb{R}$. We also adopt the short notations $\hat{\eta}_i$, \hat{J}_i for $\hat{\eta}_{i,K}$, $\hat{J}_{i,K}$. Denote $(f,q) = (F,Q)(\cdot,h)$, then (f,q) is an entropy flux pair such that $f \ge 0$, f(h) = 0, $f'' \ge 0$ and

$$|q(u)| = \left| \int_{h}^{u} q'(v) dv \right| \le |J'|_{\infty} \int_{h}^{u} f'(v) dv \le f(u), \quad \forall u \in \mathbb{R}.$$

$$(7.5)$$

For ψ in Proposition 7.1, recall ψ_i , $\overline{\psi}_i$ defined in (5.3). Furthermore, denote

$$\psi^{N}(t,x) := \psi(t,x)\alpha_{N}(x), \quad \alpha_{i}^{N} := \alpha_{N}\left(\frac{i}{N} - \frac{1}{2N}\right), \quad \psi_{i}^{N}(t) := \psi_{i}(t)\alpha_{i}^{N},$$
$$\bar{\psi}_{i}^{N}(t) := N\int_{0}^{1}\psi^{N}(t,x)\chi_{i,N}(x)dx = N\int_{\frac{i}{N} - \frac{1}{2N}}^{\frac{i}{N} + \frac{1}{2N}}\psi(t,x)\alpha_{N}(x)dx.$$

Recall the operators ∇ , ∇^* and Δ defined in (5.4). Observe that $|\nabla \alpha_i^N| \leq \sigma_N^{-1}$ and $|\nabla \psi_i^N| \leq C \sigma_N^{-1}$ for all *i*.

Proof of Proposition 7.1. Applying Lemma 5.1 with $\psi = \psi^N$,

$$X^{f}(\hat{\nu}_{N},\psi^{N}) = \frac{1}{N} \sum_{i=K+1}^{N-K} f(\hat{\eta}(0)) \bar{\psi}_{i}^{N}(0) + \mathcal{M}_{N}(\psi^{N}) + \mathcal{A}_{N}(\psi^{N}) + \mathcal{S}_{N}(\psi^{N}) + \sum_{\ell=1,2,3} \mathcal{E}_{N,\ell}(\psi^{N}).$$
(7.6)

Since $\partial_x \psi^N = \alpha_N \partial_x \psi + \psi \alpha'_N$, from (5.12) we have

$$\mathcal{E}_{N,3}(\psi^N) = \mathcal{E}_{N,3}^*(\psi, \alpha_N) - q(0) \int_0^T \left(\int_0^{\frac{2K+1}{2N}} + \int_{1-\frac{2K-1}{2N}}^1 \right) \psi \alpha'_N \, dx \, dt, \tag{7.7}$$

where

$$\mathcal{E}_{N,3}^{*}(\psi,\alpha_{N}) := -q(0) \int_{0}^{T} \left(\int_{0}^{\frac{2K+1}{2N}} + \int_{1-\frac{2K-1}{2N}}^{1} \right) \alpha_{N} \partial_{x} \psi \, dx \, dt.$$
(7.8)

On the other hand, by the definition of $X^{f}(\nu, \psi)$ in (5.2),

$$X^{f}(\hat{\nu}_{N},\psi^{N}) = -\iint_{\Sigma_{T}} \left[f(\rho_{N})\partial_{t}\psi + q(\rho_{N})\partial_{x}\psi \right] \alpha_{N} \, dx \, dt -\iint_{\Sigma_{T}} q(\rho_{N})\psi\alpha_{N}' \, dx \, dt.$$
(7.9)

We furthermore compute the last term above as

$$\iint_{\Sigma_N} q(\rho_N) \psi \alpha'_N \, dx \, dt = \int_0^T \sum_{i=K+1}^{N-K} q(\hat{\eta}_i) \int_{\frac{i}{N} - \frac{1}{2N}}^{\frac{i}{N} + \frac{1}{2N}} \psi \alpha'_N \, dx \, dt + q(0) \int_0^T \left(\int_0^{\frac{2K+1}{2N}} + \int_{1 - \frac{2K-1}{2N}}^1 \right) \psi \alpha'_N \, dx \, dt.$$
(7.10)

Therefore,

$$-\iint_{\Sigma_{T}} \left[f(\rho_{N})\partial_{t}\psi + q(\rho_{N})\partial_{x}\psi \right] \alpha_{N} \, dx \, dt$$

$$= \frac{1}{N} \sum_{i=K+1}^{N-K} f(\hat{\eta}(0)) \bar{\psi}_{i}^{N}(0) + \mathcal{M}_{N}(\psi^{N}) + \mathcal{A}_{N}(\psi^{N}) + \mathcal{S}_{N}(\psi^{N})$$

$$+ \sum_{\ell=1,2} \mathcal{E}_{N,\ell}(\psi^{N}) + \mathcal{E}_{N,3}^{*}(\psi,\alpha_{N}) + \int_{0}^{T} \sum_{i=K+1}^{N-K} q(\hat{\eta}_{i}) \int_{\frac{i}{N} - \frac{1}{2N}}^{\frac{i}{N} + \frac{1}{2N}} \psi \alpha'_{N} \, dx \, dt.$$
(7.11)

Rewrite the last integral above as $\mathcal{B}_N(\psi, \alpha_N) + \mathcal{E}^*_{N,4}(\psi, \alpha_N)$, where

$$\mathcal{B}_N(\psi, \alpha_N) := \int_0^T \sum_{i=K+1}^{N-K} q(\hat{\eta}_i) \psi_i \nabla \alpha_i^N dt, \qquad (7.12)$$

$$\mathcal{E}_{N,4}^{*}(\psi,\alpha_{N}) := \int_{0}^{T} \sum_{i=K+1}^{N-K} q(\hat{\eta}_{i}) \int_{\frac{i}{N} - \frac{1}{2N}}^{\frac{i}{N} + \frac{1}{2N}} (\psi - \psi_{i}) \alpha_{N}' \, dx \, dt.$$
(7.13)

We then obtain the decomposition

$$-\iint_{\Sigma_T} \left[f(\rho_N) \partial_t \psi + q(\rho_N) \partial_x \psi \right] \alpha_N \, dx \, dt$$

= $\frac{1}{N} \sum_{i=K+1}^{N-K} f(\hat{\eta}(0)) \bar{\psi}_i^N(0) + \mathcal{M}_N(\psi^N) + \mathcal{A}_N(\psi^N) + \mathcal{S}_N(\psi^N)$ (7.14)
+ $\mathcal{B}_N(\psi, \alpha_N) + \sum_{\ell=1,2} \mathcal{E}_{N,\ell}(\psi^N) + \sum_{\ell=3,4} \mathcal{E}_{N,\ell}^*(\psi, \alpha_N).$

Taking the limit $N \to \infty$ in (7.14), Proposition 7.1 then follows from (2.9) and Lemma 7.2–7.6 established below.

In the rest of this section, we fix ψ and estimate the limit of each term in the right-hand side of (7.14). We begin with terms that vanish uniformly.

Lemma 7.2. As $N \to \infty$, $|\mathcal{E}_{N,1}(\psi^N)|$, $|\mathcal{E}_{N,2}(\psi^N)|$, $|\mathcal{E}_{N,3}^*(\psi, \alpha_N)|$ and $|\mathcal{E}_{N,4}^*(\psi, \alpha_N)|$ converges to 0 uniformly.

Proof. Recall that the definition of $\mathcal{E}_{N,1}(\psi^N)$ reads

$$\mathcal{E}_{N,1}(\psi^{N}) = \int_{0}^{T} \sum_{i=K+1}^{N-K} \left[\bar{\psi}_{i}^{N} \nabla^{*} q(\hat{\eta}_{i}) - q(\hat{\eta}_{i}) \nabla \psi_{i}^{N} \right] dt$$

$$= \int_{0}^{T} \psi_{K+1}^{N} q(\hat{\eta}_{K}) dt - \int_{0}^{T} \psi_{N-K+1}^{N} q(\hat{\eta}_{N-K}) dt \qquad (7.15)$$

$$+ \int_{0}^{T} \sum_{i=K+1}^{N-K} \left(\bar{\psi}_{i}^{N} - \psi_{i}^{N} \right) \nabla^{*} q(\hat{\eta}_{i}) dt.$$

Since $K \ll \sigma_N \ll N$, for some c > 0 and sufficiently large N,

$$\alpha_{K+1}^N = 1 - \left[1 - (1 + \sigma_N)^{-1}\right]^{K + \frac{1}{2}} \le 1 - e^{-\frac{(1+c)K}{\sigma_N}} \le \frac{(1+c)K}{\sigma_N}.$$
(7.16)

As $\psi_{K+1}^N = \psi_{K+1} \alpha_{K+1}^N$, the first integral in (7.15) is vanishing. The second one follows similarly. To deal with the last term in (7.15), note that for each i,

$$\begin{aligned} \left|\bar{\psi}_{i}^{N}-\psi_{i}^{N}\right| &\leq N \int_{\frac{i}{N}-\frac{1}{2N}}^{\frac{i}{N}+\frac{1}{2N}} \left|\psi(t,x)\alpha_{N}(x)-\psi_{i}\alpha_{i}^{N}\right| dx \\ &\leq \frac{1}{N} \left(|\alpha_{N}|_{\infty}\|\partial_{x}\psi\|_{L^{\infty}(\Sigma_{T})}+\|\psi\|_{L^{\infty}}|\alpha_{N}'|_{\infty}\right) \leq \frac{C}{\sigma_{N}}, \end{aligned}$$

$$\tag{7.17}$$

where the last inequality follows from (iii). As $|\nabla^* q(\hat{\eta}_i)| \leq C' K^{-1}$,

$$\left| \int_0^T \sum_{i=K+1}^{N-K} \left(\bar{\psi}_i^N - \psi_i^N \right) \nabla^* q(\hat{\eta}_i) dt \right| \le \frac{C''N}{\sigma_N K}.$$
(7.18)

In view of (2.7) and (4.12), $\mathcal{E}_{N,1}(\psi^N)$ vanishes when $N \to \infty$. For $|\mathcal{E}_{N,2}(\psi^N)|$ and $|\mathcal{E}_{N,3}^*(\psi, \alpha_N)|$, observe that they are respectively bounded from above by $|\mathcal{E}_{N,2}(\psi)|$ and $|\mathcal{E}_{N,3}(\psi)|$. The conclusion then follows directly from Lemma 6.2. We are left with $\mathcal{E}_{N,4}^*(\psi, \alpha_N)$. The estimate is also straightforward:

$$\left|\mathcal{E}_{N,4}^{*}(\psi,\alpha_{N})\right| \leq |q|_{\infty} \iint_{\Sigma_{T}} \left|(\psi-\psi_{i})\alpha_{N}'\right| dx \leq \frac{C}{N} \int_{0}^{1} |\alpha_{N}'| dx.$$
(7.19)

Noting that $\int_{[0,1]} |\alpha'_N| dx \equiv 2$, $\mathcal{E}^*_{N,4}(\psi, \alpha_N)$ also vanishes.

Lemma 7.3. As $N \to \infty$, $|\mathcal{M}_N(\psi^N)| \to 0$ in $L^2(\mathbb{P}_N)$.

Proof. Recalling (5.7) and (5.16), it is not hard to get

$$\langle \mathcal{M}_{N}(\psi^{N}) \rangle = \int_{0}^{T} \sum_{j=1}^{N-1} \frac{p\eta_{j} + \sigma_{N}}{N} \left[\sum_{i=K+1}^{N-K} \bar{\psi}_{i}^{N} \left(f(\hat{\eta}_{i}^{j,j+1}) - f(\hat{\eta}_{i}) \right) \right]^{2} dt$$

$$\leq \frac{C\sigma_{N}}{N} \int_{0}^{T} \sum_{j=1}^{N-1} \left[\sum_{i=K+1}^{N-K} \left| \bar{\psi}_{i}^{N} \left(\hat{\eta}_{i}^{j,j+1} - \hat{\eta}_{i} \right) \right| \right]^{2} dt.$$

$$(7.20)$$

Doob's inequality and (6.12) then yield that

$$\mathbb{E}_{N}\left[\left|\mathcal{M}_{N}(\varphi)\right|^{2}\right] \leq 4\mathbb{E}_{N}\left[\left\langle\mathcal{M}_{N}(\varphi)\right\rangle\right] \leq C'\sigma_{N}K^{-2}.$$
(7.21)

This term vanishes since (2.7) and (4.12) contain $K^2 \gg N^{8/7} \gg \sigma_N$.

Lemma 7.4. As $N \to \infty$, $|\mathcal{A}_N(\psi^N)| \to 0$ in $L^2(\mathbb{P}_N)$. *Proof.* Recall that $g_i = \hat{J}_i - J(\hat{\eta}_i)$. Applying 6.22,

$$\mathcal{A}_{N}(\psi^{N}) = \mathcal{A}_{N,1}(\psi^{N}) + \mathcal{A}_{N,2}(\psi^{N}) + \int_{0}^{T} \left[\bar{\psi}_{K+1}^{N} f'(\hat{\eta}_{K+1}) g_{K} - \bar{\psi}_{N-K+1}^{N} f'(\hat{\eta}_{N-K+1}) g_{N-K} \right] dt,$$
(7.22)

where

$$\mathcal{A}_{N,1}(\psi^{N}) := \int_{0}^{T} \sum_{i=K+1}^{N-K} f'(\hat{\eta}_{i+1}) g_{i} \nabla \bar{\psi}_{i}^{N} dt,$$

$$\mathcal{A}_{N,2}(\psi^{N}) := \int_{0}^{T} \sum_{i=K+1}^{N-K} \bar{\psi}_{i}^{N} g_{i} \nabla f'(\hat{\eta}_{i}) dt.$$
(7.23)

The boundary integrals are negligible since g_i is bounded and (7.16):

$$\left| \int_{0}^{T} \left[\bar{\psi}_{K+1}^{N} f'(\hat{\eta}_{K+1}) g_{K} - \bar{\psi}_{N-K+1}^{N} f'(\hat{\eta}_{N-K+1}) g_{N-K} \right] dt \right| \leq \frac{CK}{\sigma_{N}}.$$
 (7.24)

The estimate for the remaining two terms is similar to Lemma 6.6. Using Cauchy–Schwarz inequality,

$$\left|\mathcal{A}_{N,1}(\psi^{N})\right|^{2} \leq |f'|_{\infty}^{2} \int_{0}^{T} \sum_{i=K+1}^{N-K} g_{i}^{2} dt \int_{0}^{T} \sum_{i=K+1}^{N-K} \left|\nabla \bar{\psi}_{i}^{N}\right|^{2} dt.$$
(7.25)

Notice that $|\nabla \bar{\psi}_i^N| \leq C \sigma_N^{-1}$. By Proposition 6.4,

$$\mathbb{E}_{N}\left[\left|\mathcal{A}_{N,1}(\psi^{N})\right|^{2}\right] \leq C\left(\frac{K^{2}}{\sigma_{N}} + \frac{N}{K}\right)\frac{N}{\sigma_{N}^{2}} = C\left(1 + \frac{N\sigma_{N}}{K^{3}}\right)\frac{NK^{2}}{\sigma_{N}^{3}}.$$
(7.26)

Hence, we can conclude from (2.7) and (4.12). Similarly for $\mathcal{A}_{N,2}$,

$$\left|\mathcal{A}_{N,2}(\psi^{N})\right|^{2} \leq |f''|_{\infty}^{2} \|\psi\|_{L^{\infty}}^{2} \int_{0}^{T} \sum_{i=K+1}^{N-K} g_{i}^{2} dt \int_{0}^{T} \sum_{i=K+1}^{N-K} \left|\nabla\hat{\eta}_{i}\right|^{2} dt.$$
(7.27)

The estimate follows exactly the same as (6.26).

The limit of $S_N(\psi^N)$ differs essentially from that of $S_N(\varphi)$ in Lemma 6.7. We see below the divergent term mentioned at the beginning of this section.

Lemma 7.5. As $N \to \infty$, $S_N(\psi^N)$ satisfies that

$$\lim_{N \to \infty} \mathbb{P}_N \left\{ \mathcal{S}_N(\psi^N) + \sigma_N \int_0^T \sum_{i=K+1}^{N-K} \nabla \left(\psi_i f(\hat{\eta}_i) \right) \nabla \alpha_i^N dt \le 0 \right\} = 1.$$
(7.28)

Proof. As in the previous lemma, we start from integration by parts. From (6.27),

$$S_{N}(\psi^{N}) = S_{N,1}(\psi^{N}) + S_{N,2}(\psi^{N}) - \sigma_{N} \int_{0}^{T} \bar{\psi}_{K+1}^{N} f'(\hat{\eta}_{K+1}) \nabla \hat{\eta}_{K} dt + \sigma_{N} \int_{0}^{T} \bar{\psi}_{N-K+1}^{N} f'(\hat{\eta}_{N-K+1}) \nabla \hat{\eta}_{N-K} dt,$$
(7.29)

where $S_{N,1}(\psi^N)$, $S_{N,2}(\psi^N)$ are defined by (6.28) with $\bar{\varphi}_i$ replaced by $\bar{\psi}_i^N$. To deal with the boundary terms, observe from (7.16) that

$$\left|\sigma_{N}\int_{0}^{T}\bar{\psi}_{K+1}^{N}f'(\hat{\eta}_{K+1})\nabla\hat{\eta}_{K}\,dt\right| \leq K\|\psi\|_{L^{\infty}}|f'|_{\infty}\int_{0}^{T}\left|\nabla\hat{\eta}_{K}\right|dt.$$
(7.30)

Notice that the elementary estimate $|\nabla \hat{\eta}_K| \leq K^{-1}$ is insufficient here since it leads to an upper bound of constant order. Instead, we use Remark 8.3 to obtain that

$$\mathbb{E}_{N}\left[\left|\sigma_{N}\int_{0}^{T}\bar{\psi}_{K+1}^{N}f'(\hat{\eta}_{K+1})\nabla\hat{\eta}_{K}\,dt\right|^{2}\right] \leq C\left(\frac{K}{\sigma_{N}}+\frac{1}{K}\right),\tag{7.31}$$

which vanishes as $N \to \infty$. The other boundary term is estimated similarly.

For the remaining two terms, we first look at $S_{N,2}(\psi^N)$. The convexity of f assures that $\nabla \hat{\eta}_i \nabla f'(\hat{\eta}_i) \geq 0$ for all i, therefore,

$$\mathcal{S}_{N,2}(\psi^N) = -\sigma_N \int_0^T \sum_{i=K+1}^{N-K} \bar{\psi}_i^N \nabla \hat{\eta}_i \nabla f'(\hat{\eta}_i) dt \le 0.$$
(7.32)

For $\mathcal{S}_{N,1}(\psi^N)$, the following replacement holds as $|\nabla \bar{\psi}_i^N| \leq C \sigma_N^{-1}$:

$$\left|\mathcal{S}_{N,1}(\psi^N) + \sigma_N \int_0^T \sum_{i=K+1}^{N-K} \nabla f(\hat{\eta}_i) \nabla \bar{\psi}_i^N dt \right| \le \frac{CN}{K^2}.$$
(7.33)

Plugging (7.31)–(7.33) into (7.29) and using (2.7), (4.12),

$$\lim_{N \to \infty} \mathbb{P}_N \left\{ \mathcal{S}_N(\psi^N) + \sigma_N \int_0^T \sum_{i=K+1}^{N-K} \nabla f(\hat{\eta}_i) \nabla \bar{\psi}_i^N dt \le 0 \right\} = 1.$$
(7.34)

Now we look closely at the integral appeared in (7.34). Notice that

$$\nabla \bar{\psi}_i^N = N \int_{\frac{i}{N} - \frac{1}{2N}}^{\frac{i}{N} + \frac{1}{2N}} \left[\psi\left(t, x + \frac{1}{N}\right) \alpha_N\left(x + \frac{1}{N}\right) - \psi(t, x)\alpha_N(x) \right] dx.$$
(7.35)

With the notation $\bar{\alpha}_i^N := N \int \alpha_N(x) \chi_{i,N}(x) dx$,

$$\left|\nabla \bar{\psi}_{i}^{N} - \alpha_{i}^{N} \nabla \bar{\psi}_{i} - \psi_{i+1} \nabla \bar{\alpha}_{i}^{N}\right| \leq \frac{2 \|\partial_{x}\psi\|_{L^{\infty}} |\alpha'_{N}|_{\infty}}{N^{2}} \leq \frac{C}{N\sigma_{N}}.$$
(7.36)

Moreover, observe that for large N and $i \leq N/4$,

$$\left|\nabla\bar{\alpha}_{i}^{N} - \nabla\alpha_{i}^{N}\right| \le CN^{-2} \max\left\{\left|\alpha_{N}^{\prime\prime}(x)\right|; 0 < x < \frac{1}{4}\right\} \le C\sigma_{N}^{-2}.$$
(7.37)

As $\delta_N = 1/2 + o_N(1)$, for $N/4 < i \le N\delta_N$,

$$\left|\nabla\alpha_{i}^{N}-\nabla\bar{\alpha}_{i}^{N}\right| \leq \left|\nabla\alpha_{i}^{N}\right|+\left|\nabla\bar{\alpha}_{i}^{N}\right| \leq 2e^{-N(4\sigma_{N})^{-1}} \leq C\sigma_{N}^{-2}.$$
(7.38)

By (7.36)–(7.38) and similar bound for $i > N\delta_N$,

$$\sigma_{N} \left| \int_{0}^{T} \sum_{i=K+1}^{N-K} \nabla f(\hat{\eta}_{i}) \left(\nabla \bar{\psi}_{i}^{N} - \alpha_{i}^{N} \nabla \bar{\psi}_{i} - \psi_{i+1} \nabla \alpha_{i}^{N} \right) dt \right|$$

$$\leq CN \sigma_{N} \cdot |f'|_{\infty} \sup_{i} \left| \nabla \hat{\eta}_{i} \right| \cdot \left(\frac{1}{N \sigma_{N}} + \frac{1}{\sigma_{N}^{2}} \right) \leq \frac{CN}{\sigma_{N} K}.$$

$$(7.39)$$

Therefore, we are allowed to replace $\nabla \bar{\psi}_i^N$ in (7.34) by $\alpha_i^N \nabla \bar{\psi}_i + \psi_{i+1} \nabla \alpha_i^N$. To complete the proof, we need to compute that

$$\sum_{i=K+1}^{N-K} \left(\alpha_i^N \nabla \bar{\psi}_i + \psi_{i+1} \nabla \alpha_i^N \right) \nabla f(\hat{\eta}_i) - \sum_{i=K+1}^{N-K} \nabla \left(\psi_i f(\hat{\eta}_i) \right) \nabla \alpha_i^N$$

$$= \alpha_i^N \nabla \bar{\psi}_i f(\hat{\eta}_{i+1}) \Big|_{i=K}^{N-K} + \sum_{i=K+1}^{N-K} f(\hat{\eta}_i) \left[\nabla^* (\alpha_i^N \nabla \bar{\psi}_i) - \nabla \psi_i \nabla \alpha_i^N \right].$$
(7.40)

By (7.16), the contribution of the boundary terms is negligible:

$$\sigma_N \left| \int_0^T \alpha_i^N \nabla \bar{\psi}_i f(\hat{\eta}_{i+1}) \right|_{i=K}^{N-K} dt \right| \le C \sigma_N \frac{K}{\sigma_N} \frac{1}{N} \le \frac{CK}{N}.$$
(7.41)

Since $|\nabla^*(\alpha_i^N \nabla \bar{\psi}_i)| \le |\alpha_{i-1}^N \Delta \bar{\psi}_i| + |\nabla \alpha_{i-1}^N \nabla \bar{\psi}_i| \le CN^{-2} + |\nabla \alpha_{i-1}^N \nabla \bar{\psi}_i|,$

$$\sigma_{N} \left| \int_{0}^{T} \sum_{i=K+1}^{N-K} f(\hat{\eta}_{i}) \left[\nabla^{*}(\alpha_{i}^{N} \nabla \bar{\psi}_{i}) - \nabla \psi_{i} \nabla \alpha_{i}^{N} \right] dt \right|$$

$$\leq \frac{C\sigma_{N}}{N} + \sigma_{N} \int_{0}^{T} \sum_{i=K+1}^{N-K} f(\hat{\eta}_{i}) \left(|\nabla \alpha_{i-1}^{N} \nabla \bar{\psi}_{i}| + |\nabla \psi_{i} \nabla \alpha_{i}^{N}| \right) dt.$$
(7.42)

Finally, the definition of α_N yields that

$$\sum_{i=K+1}^{N-K} f(\hat{\eta}_i) \left| \nabla \alpha_{i-1}^N \nabla \bar{\psi}_i \right| \le \frac{|f|_{\infty} ||\partial_x \psi||_{L^{\infty}}}{N} \sum_{i=K+1}^{N-K} \left| \nabla \alpha_{i-1}^N \right| \le \frac{C}{N}.$$
(7.43)

As the same bound holds for $\nabla \psi_i \nabla \alpha_i^N$,

$$\sigma_N \int_0^T \sum_{i=K+1}^{N-K} f(\hat{\eta}_i) \left(|\nabla \alpha_{i-1}^N \nabla \bar{\psi}_i| + |\nabla \psi_i \nabla \alpha_i^N| \right) dt \le \frac{C \sigma_N}{N}.$$
(7.44)

The desired conclusion then follows by putting (7.41)-(7.44) together.

We are left with the last term $\mathcal{B}_N(\psi, \alpha_N)$ in (7.14). The next lemma shows that $\mathcal{B}_N(\psi, \alpha_N)$ cancels exactly the divergent term in Lemma 7.5.

Lemma 7.6. As $N \to \infty$, $\mathcal{B}_N(\psi, \alpha_N)$ satisfies that

$$\lim_{N \to \infty} \mathbb{P}_N \left\{ \mathcal{B}_N(\psi, \alpha_N) - \sigma_N \int_0^T \sum_{i=K+1}^{N-K} \nabla\left(\psi_i f(\hat{\eta}_i)\right) \nabla \alpha_i^N dt \\ \leq \int_0^T f(\rho_-) \psi(\cdot, 0) dt + \int_0^T f(\rho_+) \psi(\cdot, 1) dt \right\} = 1.$$
(7.45)

The following boundary block estimate is necessary for Lemma 7.6. It is proved similarly to Proposition 6.4 and 6.5. We postpone it to Section 8.4.

Proposition 7.7 (Boundary one-block estimates). With some constant C,

$$\mathbb{E}_{N}\left[\int_{0}^{T}\left(\left|\hat{\eta}_{K}-\rho_{-}(t)\right|^{2}+\left|\hat{\eta}_{N-K}-\rho_{+}(t)\right|^{2}\right)dt\right] \leq C\left(\frac{K}{\sigma_{N}}+\frac{1}{\widetilde{\sigma}_{N}}+\frac{1}{K}\right),\tag{7.46}$$

where $\rho_{\pm} = \rho_{\pm}(t)$ are defined in (2.13).

Proof of Lemma 7.6. First notice from (7.5) that

$$\mathcal{B}_N(\psi,\alpha_N) = \int_0^T \sum_{i=K+1}^{N-K} q(\hat{\eta}_i)\psi_i \nabla \alpha_i^N dt \le \int_0^T \sum_{i=K+1}^{N-K} f(\hat{\eta}_i)\psi_i \left| \nabla \alpha_i^N \right| dt.$$
(7.47)

Recall that $\delta_N = 1/2 + O(\sigma_N^{-1})$ in (7.1). For large N, choose the integer $j_N := [N\delta_N + 1/2]$, then j_N satisfies that

$$\delta_N \in \left[\frac{j_N}{N} - \frac{1}{2N}, \frac{j_N}{N} + \frac{1}{2N}\right). \tag{7.48}$$

By the definition of α_N we can rewrite (7.47) as²,

$$\mathcal{B}_N(\psi,\alpha_N) \le \int_0^T \sum_{i=K+1}^{j_N-1} f(\hat{\eta}_i) \psi_i \nabla \alpha_i^N dt - \int_0^T \sum_{i=j_N+1}^{N-K} f(\hat{\eta}_i) \psi_i \nabla \alpha_i^N dt.$$
(7.49)

We calculate the two terms in (7.49) respectively. For the first one,

$$\sum_{i=K+1}^{j_N-1} f(\hat{\eta}_i) \psi_i \nabla \alpha_i^N = \sum_{i=K+1}^{j_N-1} f(\hat{\eta}_i) \psi_i \nabla^* (1 - \alpha_{i+1}^N)$$

= $f(\hat{\eta}_{K+1}) \psi_{K+1} (1 - \alpha_{K+1}^N) - f(\hat{\eta}_{j_N}) \psi_{j_N} (1 - \alpha_{j_N}^N) + \sum_{i=K+1}^{j_N-1} \nabla (f(\hat{\eta}_i) \psi_i) (1 - \alpha_{i+1}^N)$
 $\leq f(\hat{\eta}_{K+1}) \psi_{K+1} + \sum_{i=K+1}^{j_N-1} \nabla (f(\hat{\eta}_i) \psi_i) (1 - \alpha_{i+1}^N).$

The second one follows similarly and we obtain that

$$\sum_{i=j_N+1}^{N-K} f(\hat{\eta}_i) \psi_i \nabla \alpha_i^N \ge -f(\hat{\eta}_{N-K+1}) \psi_{N-K+1} + \sum_{i=j_N+1}^{N-K} \nabla \left(f(\hat{\eta}_i) \psi_i \right) \left(1 - \alpha_{i+1}^N \right).$$

From these estimates and (7.49), we obtain that

$$\mathcal{B}_{N}(\psi,\alpha_{N}) \leq \int_{0}^{T} \left[f(\hat{\eta}_{K+1})\psi_{K+1} + f(\hat{\eta}_{N-K+1})\psi_{N-K+1} \right] dt + \int_{0}^{T} \sum_{i=K+1}^{N-K} \operatorname{sgn}(j_{N}-i)\nabla \left(f(\hat{\eta}_{i})\psi_{i} \right) \left(1 - \alpha_{i+1}^{N} \right) dt.$$
(7.50)

The definition of α_N assures that

$$\nabla \alpha_i^N = \begin{cases} \sigma_N^{-1} (1 - \alpha_{i+1}^N), & 1 \le i \le j_N - 1, \\ -\sigma_N^{-1} (1 - \alpha_{i+1}^N), & j_N + 1 \le i \le N - 1. \end{cases}$$
(7.51)

Therefore, (7.50) can be rewritten as

$$\mathcal{B}_{N}(\psi,\alpha_{N}) - \sigma_{N} \int_{0}^{T} \sum_{i=K+1}^{N-K} \nabla (f(\hat{\eta}_{i-1})\psi_{i-1}) \nabla \alpha_{i}^{N} dt$$

$$\leq \int_{0}^{T} [f(\hat{\eta}_{K+1})\psi_{K+1} + f(\hat{\eta}_{N-K+1})\psi_{N-K+1}] dt.$$
(7.52)

Sending $N \to \infty$ and applying Proposition 7.7, we conclude the result. ²The term with $i = j_N$ is omitted since it is vanishing in the limit $N \to \infty$.

8. Estimates

In this section we establish and prove the technical results used in this article.

8.1. A priori estimate on boundary currents. Recall that $\eta(t)$ is the process generated by $NL_{N,t}$ for the generator $L_{N,t}$ in (2.8). Define the microscopic currents $j_{i,i+1}$ associated to $L_{N,t}$ through the conservation law $L_{N,t}[\eta_i] = j_{i-1,i} - j_{i,i+1}$. They are equal to

$$j_{i,i+1} = \begin{cases} \tilde{\sigma}_N [\alpha(t) - (\alpha(t) + \gamma(t))\eta_1], & i = 0, \\ p\eta_i (1 - \eta_{i+1}) + \sigma_N (\eta_i - \eta_{i+1}), & 1 \le i \le N - 1, \\ \tilde{\sigma}_N [(\beta(t) + \delta(t))\eta_N - \delta(t)], & i = N. \end{cases}$$
(8.1)

Following the argument in [9, Section 2], for i = 1, ..., N - 1 define the counting processes associated to the process $\{\eta(t)\}_{t\geq 0}$ by

$$h_{+}(i,t) := \text{number of jumps } i \to i+1 \text{ in } [0,t],$$

$$h_{-}(i,t) := \text{number of jumps } i+1 \to i \text{ in } [0,t].$$
(8.2)

These definitions extend to the boundaries i = 0 and i = N as

 $h_{+}(0,t) := \text{number of particles created at 1 in } [0,t],$ $h_{-}(0,t) := \text{number of particles annihilated at 1 in } [0,t],$ $h_{+}(N,t) := \text{number of particles annihilated at } N \text{ in } [0,t],$ $h_{-}(N,t) := \text{number of particles created at } N \text{ in } [0,t].$ (8.3)

With $h(i,t) := h_+(i,t) - h_-(i,t)$, the conservation law is microscopically given by

$$\eta_i(t) - \eta_i(0) = h(i-1,t) - h(i,t), \quad \forall i = 1, \dots, N.$$
(8.4)

Furthermore, for i = 0, ..., N, there is a martingale $M_i(t)$ such that

$$h(i,t) = N \int_0^t j_{i,i+1}(s) ds + M_i(t).$$
(8.5)

As $\eta_i(t) \in \{0,1\}$, (8.4) yields that $|h(i,t) - h(i',t)| \le |i-i'|$. Therefore,

$$\left|\mathbb{E}_{N}\left[\int_{0}^{t} j_{i,i+1}(s)ds\right] - \mathbb{E}_{N}\left[\int_{0}^{t} j_{i',i'+1}(s)ds\right]\right| \le \frac{|i-i'|}{N} \le 1$$
(8.6)

for all i, i' = 0, ..., N. In particular, for the fixed time T > 0,

$$\left| \mathbb{E}_{N} \left[\int_{0}^{T} j_{N,N+1}(t) dt \right] - \mathbb{E}_{N} \left[\int_{0}^{T} \frac{1}{N-1} \sum_{i=1}^{N-1} j_{i,i+1}(t) dt \right] \right| \le 1.$$
(8.7)

Since $j_{i,i+1} = p\eta_i(1 - \eta_{i+1}) + \sigma_N(\eta_i - \eta_{i+1}),$

$$\frac{1}{N-1} \sum_{i=1}^{N-1} j_{i,i+1}(t) \bigg| = \bigg| \frac{\sigma_N}{N-1} (\eta_1 - \eta_N) + \frac{1}{N-1} \sum_{i=1}^{N-1} p \eta_i (1 - \eta_{i+1}) \bigg|$$

$$\leq \sigma_N (N-1)^{-1} + p = o(1) + p, \quad N \to \infty.$$
(8.8)

Hence, from (8.7) we see that

$$\lim_{N \to \infty} \sup_{N \to \infty} \left| \mathbb{E}_N \left[\int_0^T j_{N,N+1}(t) dt \right] \right| \le 1 + p.$$
(8.9)

Recalling the definitions of $j_{N,N+1}$ in (8.1), we obtain a priori bound:

$$\lim_{N \to \infty} \sup_{N \to \infty} \left| \mathbb{E}_N \left[\int_0^T \left(\beta(t) + \delta(t) \right) \left(\eta_N - \rho_+(t) \right) dt \right] \right| \le \frac{C}{\widetilde{\sigma}_N}, \tag{8.10}$$

with ρ_+ given by (2.13). It is easy to obtain the same bound for $\eta_1 - \rho_-(t)$.

8.2. Dirichlet forms. Recall the process $\eta(\cdot) \in \Omega_N = \{0,1\}^N$ generated by $NL_{N,t}$. Denote by $\mu_{N,t}$ the distribution of $\eta(t)$. Define the Dirichlet form

$$\mathfrak{D}_{\text{exc},N}(t) := \frac{1}{2} \sum_{\eta \in \Omega_N} \sum_{i=1}^{N-1} \left(\sqrt{\mu_{N,t}(\eta^{i,i+1})} - \sqrt{\mu_{N,t}(\eta)} \right)^2.$$
(8.11)

Denote by $\nu_{\pm,t}$ the Bernoulli measure with densities $\rho_{\pm}(t)$ given in (2.13):

$$\nu_{\pm,t}(\eta) = \prod_{i=1}^{N} \rho_{\pm}(t)^{\eta_i} \left[1 - \rho_{\pm}(t) \right]^{1-\eta_i}, \quad \forall \eta \in \Omega_N.$$
(8.12)

The boundary Dirichlet forms are defined as

$$\mathfrak{D}_{-,N}(t) := \frac{1}{2} \sum_{\eta \in \Omega_N} \rho_{-}^{1-\eta_1} (1-\rho_{-})^{\eta_1} \left(\sqrt{f_{N,t}(\eta^1)} - \sqrt{f_{N,t}(\eta)} \right)^2 \nu_{-,t}(\eta),$$

$$\mathfrak{D}_{+,N}(t) := \frac{1}{2} \sum_{\eta \in \Omega_N} \rho_{+}^{1-\eta_N} (1-\rho_{+})^{\eta_N} \left(\sqrt{f_{N,t}^+(\eta^N)} - \sqrt{f_{N,t}^+(\eta)} \right)^2 \nu_{+,t}(\eta),$$
(8.13)

where $f_{N,t}^{\pm} := \mu_{N,t} / \nu_{\pm,t}$ is the probability density function.

Proposition 8.1. There exists C independent of N such that

$$\int_{0}^{T} \mathfrak{D}_{\text{exc},N}(t) dt \le \frac{C}{\sigma_{N}}, \quad \int_{0}^{T} \left[\mathfrak{D}_{-,N}(t) + \mathfrak{D}_{+,N}(t) \right] dt \le \frac{C}{\widetilde{\sigma}_{N}}.$$
(8.14)

Proof. Define the relative entropy $H_{\pm,N}(t)$ by

$$H_{\pm,N}(t) := \sum_{\eta \in \Omega_N} f_{N,t}^{\pm}(\eta) \log f_{N,t}^{\pm}(\eta) \nu_{\pm,t}(\eta).$$
(8.15)

Standard manipulation with Kolmogorov equation gives that

$$\frac{1}{N}\frac{d}{dt}H_{-,N}(t) = \sum_{\eta \in \Omega_N} f_{N,t}^- L_{N,t} \Big[\log f_{N,t}^-\Big]\nu_{-,t} - \frac{1}{N}\sum_{\eta \in \Omega_N} f_{N,t}^- \frac{d}{dt}\nu_{-,t} \\
\leq \sum_{\eta \in \Omega_N} f_{N,t}^- L_{N,t} \Big[\log f_{N,t}^-\Big]\nu_{-,t} + C.$$
(8.16)

We divide the summation in the right-hand side of (8.16) into two parts:

$$\Gamma_{N,t}^{-,(1)} := \sum_{\eta \in \Omega_N} f_{N,t}^{-} (L_{N,t} - \widetilde{\sigma}_N L_{+,t}) \big[\log f_{N,t}^{-} \big] \nu_{-,t},$$
(8.17)

$$\Gamma_{N,t}^{-,(2)} := \tilde{\sigma}_N \sum_{\eta \in \Omega_N} f_{N,t}^- L_{+,t} \big[\log f_{N,t}^- \big] \nu_{-,t}.$$
(8.18)

Exploiting the inequality $x(\log y - \log x) \le 2\sqrt{x}(\sqrt{y} - \sqrt{x})$ for x, y > 0,

$$\Gamma_{N,t}^{-,(1)} = \sum_{\eta \in \Omega_N} f_{N,t}^{-} \left(pL_{\text{TAS}} + \sigma_N L_{\text{SS}} + \widetilde{\sigma}_N L_{-,t} \right) \left[\log f_{N,t}^{-} \right] \nu_{-,t}$$

$$\leq 2 \sum_{\eta \in \Omega_N} g_{N,t}^{-} \left(pL_{\text{TAS}} + \sigma_N L_{\text{SS}} + \widetilde{\sigma}_N L_{-,t} \right) \left[g_{N,t}^{-} \right] \nu_{-,t}, \tag{8.19}$$

where we denote $g_{N,t}^- := (f_{N,t}^-)^{1/2}$. Since the measure $\nu_{-,t}$ is invariant under L_{TAS} , L_{SS} as well as $L_{-,t}$, direct computation shows that

$$\Gamma_{N,t}^{-,(1)} \le -\widetilde{\sigma}_N \big(\alpha(t) + \gamma(t) \big) \mathfrak{D}_{-,N}(t) - \sigma_N \mathfrak{D}_{\text{exc},N}(t) + C.$$
(8.20)

Meanwhile, since $f_{N,t}^- \nu_{-,t} = f_{N,t}^+ \nu_{+,t}$,

$$\Gamma_{N,t}^{-,(2)} = \widetilde{\sigma}_N \sum_{\eta \in \Omega_N} f_{N,t}^- L_{+,t} \left[\log f_{N,t}^+ + \log \left(\frac{\nu_{+,t}}{\nu_{-,t}} \right) \right] \nu_{-,t}$$

$$\leq -\widetilde{\sigma}_N \big(\beta(t) + \delta(t) \big) \mathfrak{D}_{+,N}(t) + \widetilde{\sigma}_N \sum_{\eta \in \Omega_N} f_{N,t}^- L_{+,t} \left[\log \left(\frac{\nu_{+,t}}{\nu_{-,t}} \right) \right] \nu_{-,t}.$$
(8.21)

By the definition of $\nu_{\pm,t}$ and $L_{-,t}$,

$$L_{-,t}\left[\log\left(\frac{\nu_{+,t}}{\nu_{-,t}}\right)\right] = \Re(t)\big(\beta(t) + \delta(t)\big)\big(\eta_N - \rho_+(t)\big),\tag{8.22}$$

where

$$\Re(t) := \log\left[\frac{\rho_{-}(t)}{1 - \rho_{-}(t)}\right] - \log\left[\frac{\rho_{+}(t)}{1 - \rho_{+}(t)}\right].$$
(8.23)

Applying the priori estimate (8.10),

$$\int_0^T \Gamma_{N,t}^{-,(2)} dt \le -\widetilde{\sigma}_N \int_0^T \left(\beta(t) + \delta(t)\right) \mathfrak{D}_{+,N}(t) dt + C.$$
(8.24)

Integrating (8.16) over [0, T] and using the results we proved above,

$$\frac{H_{-,N}(T) - H_{-,N}(0)}{N} \le C - \sigma_N \int_0^T \mathfrak{D}_{\text{exc},N}(t) dt$$

$$- \widetilde{\sigma}_N \int_0^T \left[\left(\alpha(t) + \gamma(t) \right) \mathfrak{D}_{-,N}(t) + \left(\beta(t) + \delta(t) \right) \mathfrak{D}_{+,N}(t) \right] dt.$$
(8.25)

We can conclude (8.14) since α , β , γ , δ are uniformly bounded from 0.

8.3. Logarithmic Sobolev inequalities. For $\rho \in (0, 1)$, let ν_{ρ} be the product Bernoulli measure on $\Omega_k = \{0, 1\}^k$ with homogeneous density ρ . The micro canonical configuration space Ω_{k,ρ_*} is defined for $\rho_* = 0, 1/k, ..., (k-1)/k, 1$ as

$$\Omega_{k,\rho_*} := \left\{ \eta = (\eta_1, \dots, \eta_k) \in \Omega_k \mid \frac{1}{k} \sum_{i=1}^k \eta_i = \rho_* \right\}.$$
(8.26)

Let $\tilde{\nu}^k(\eta|\rho_*)$ be the uniform measure on Ω_{k,ρ_*} . The log-Sobolev inequality for the simple exclusion ([27]) yields that there exists a universal constant $C_{\rm LS}$ such that

$$\sum_{\eta \in \Omega_{k,\rho_*}} f(\eta) \log f(\eta) \nu^k(\eta|\rho_*)$$

$$\leq \frac{C_{\mathrm{LS}}k^2}{2} \sum_{\eta \in \Omega_{k,\rho_*}} \sum_{i=1}^{k-1} \left(\sqrt{f(\eta^{i,i+1})} - \sqrt{f(\eta)}\right)^2 \nu^k(\eta|\rho_*).$$
(8.27)

for any $f \ge 0$ such that $\sum_{\eta \in \Omega_{k,\rho_*}} f(\eta)\nu(\eta|\rho_*) = 1$. In [8, Proposition A.1], (8.27) is extended to a log-Sobolev inequality associated to ν_{ρ} .

Proposition 8.2. There exists a constant C_{ρ} , such that

$$\sum_{\eta \in \Omega_{k}} f(\eta) \log f(\eta) \nu_{\rho}(\eta) \leq \frac{C_{\rho} k^{2}}{2} \sum_{\eta \in \Omega_{k}} \sum_{i=1}^{k-1} \left(\sqrt{f}(\eta^{i,i+1}) - \sqrt{f}(\eta) \right)^{2} \nu_{\rho}(\eta) + \frac{C_{\rho} k}{2} \sum_{\eta \in \Omega_{k}} \rho^{1-\eta_{1}} (1-\rho)^{\eta_{1}} \left(\sqrt{f}(\eta^{1}) - \sqrt{f}(\eta) \right)^{2} \nu_{\rho}(\eta),$$
(8.28)

for any $f \ge 0$ such that $\sum_{\eta \in \Omega_k} f(\eta) \nu_{\rho}(\eta) = 1$.

We refer to [8, Appendix A] for the details of the proof.

8.4. Block estimates. Now we state the proofs of Proposition 6.4, 6.5 and 7.7. They are based on Proposition 8.1, (8.27) and (8.28). Since the size of the block varies in the proofs, we keep the subscription K in $\bar{\eta}_{i,K}$, $\hat{\eta}_{i,K}$ and $\hat{J}_{i,K}$.

Proof of Proposition 6.4. Recall the space Ω_{k,ρ^*} defined in (8.26) and the uniform measure $\nu^k(\cdot|\rho_*)$ on it. Also define

$$\bar{\mu}_{N,t}^{i,k}(\rho_*) := \mu_{N,t} \left\{ (\eta_{i-k+1}, \dots, \eta_i) \in \Omega_{k,\rho_*} \right\}, \\
\mu_{N,t}^{i,k}(\eta_{i-k+1}, \dots, \eta_i | \rho_*) := \frac{\mu_{N,t}(\eta_{i-k+1}, \dots, \eta_i)}{\bar{\mu}_{N,t}^{i,k}(\rho_*)}.$$
(8.29)

Recall the block averages $\bar{\eta}_{i,K}$, $\hat{\eta}_{i,K}$ and $\hat{J}_{i,K}$ defined in (4.6), (4.7) and (5.5). Note that

$$\left[\hat{J}_{i,K} - J(\hat{\eta}_{i,K})\right]^2 \le 2 \left[\hat{J}_{i,K} - J(\bar{\eta}_{i+K,2K})\right]^2 + 2 \left[J(\hat{\eta}_{i,K}) - J(\bar{\eta}_{i+K,2K})\right]^2.$$
(8.30)

We prove the estimate for the two terms respectively. Denote $h_i = \hat{J}_{i,K} - J(\bar{\eta}_{i+K,2K})$. The relative entropy inequality yields that

$$\int_{\Omega_{2K,\rho_*}} h_i^2 d\mu_{N,t}^{i+K,2K}(\cdot|\rho_*) \leq \frac{1}{aK} \bigg[H\left(\mu_{N,t}^{i+K,2K}(\cdot|\rho_*);\nu^{2K}(\cdot|\rho_*)\right) + \log \int e^{aKh_i^2} d\nu^{2K}(\cdot|\rho_*) \bigg], \quad \forall a > 0,$$
(8.31)

where H is the relative entropy for two probability measures μ and ν :

$$H(\mu;\nu) := \int (\log \mu - \log \nu) d\mu.$$
(8.32)

The log-Sobolev inequality (8.27) yields that with some constant C,

$$H\left(\mu_{N,t}^{i+K,2K}(\,\cdot\,|\rho_*);\nu^{2K}(\,\cdot\,|\rho_*)\right) \le CK^2 \mathfrak{D}_{N,\rho_*}^{i+K,2K}(t),\tag{8.33}$$

where the Dirichlet form in the right-hand side is defined as

$$\mathfrak{D}_{N,\rho_*}^{i,k}(t) := \frac{1}{2} \sum_{\eta \in \Omega_{k,\rho_*}} \sum_{i'=1}^{k-1} \left(\sqrt{\mu_{N,t}^{i,k}(\eta^{i',i'+1}|\rho_*)} - \sqrt{\mu_{N,t}^{i,k}(\eta|\rho_*)} \right)^2.$$
(8.34)

For the uniform integral, with standard argument we can show that

$$\log \int e^{aKh_i^2} d\nu^{2K}(\cdot|\rho_*) \le C,\tag{8.35}$$

for some sufficiently small but fixed a. From (8.31), (8.33) and (8.35),

$$\int_{\Omega_{2K,\rho_*}} h_i^2 d\mu_{N,t}^{i+K,2K}(\cdot | \rho_*) \le CK \mathfrak{D}_{N,\rho_*}^{i+K,2K}(t) + \frac{C}{K},$$
(8.36)

for all i = K, ..., N - K and ρ_* . Integrating the estimate above with respect to $\bar{\mu}_{N,t}^{i+K,2K}(\rho_*)$ and summing up in i, we obtain that

$$\sum_{i=K}^{N-K} \int_{\Omega_N} h_i^2 d\mu_{N,t} \le CK \sum_{i=K}^{N-K} \sum_{\rho_*} \bar{\mu}_{N,t}^{i,k}(\rho_*) \mathfrak{D}_{N,\rho_*}^{i+K,2K}(t) + \frac{CN}{K}$$

$$\le C' K^2 \mathfrak{D}_{\text{exc},N}(t) + \frac{CN}{K}.$$
(8.37)

The desired estimate for h_i then follows from Proposition 8.1. The upper bound for $J(\hat{\eta}_{i,K}) - J(\bar{\eta}_{i+K,2K})$ can be obtained by exactly the same argument.

Proof of Proposition 6.5. Observe that for i = K, K + 1, ..., N - K,

$$\nabla \hat{\eta}_{i,K} = \hat{\eta}_{i+1,K} - \hat{\eta}_{i,K} = \frac{\bar{\eta}_{i+K,K} - \bar{\eta}_{i,K}}{K}.$$
(8.38)

Applying the same argument used in Proposition 6.4,

$$\mathbb{E}_{N}\left[\int_{0}^{T}\sum_{i=K}^{N-K} (\bar{\eta}_{i+K,K} - \bar{\eta}_{i,K})^{2} dt\right] \leq C\left(\frac{K^{2}}{\sigma_{N}} + \frac{N}{K}\right).$$
(8.39)
follows directly.

The conclusion then follows directly.

Remark 8.3. With the arguments in the previous proofs, we also get that

$$\mathbb{E}_{N}\left[\int_{0}^{T} \left|\nabla\hat{\eta}_{i,K}\right|^{2} dt\right] \leq C\left(\frac{1}{\sigma_{N}K} + \frac{1}{K^{3}}\right),\tag{8.40}$$

for each i = K, K + 1, ..., N - K.

Proof of Proposition 7.7. Recall that ν_{ρ} is the product Bernoulli measure with rate ρ . To shorten the notation, let $\alpha_{N,t}$ be the marginal distribution of $\{\eta_1, \eta_2, \ldots, \eta_{2K-1}\}$ under $\mu_{N,t}$. By the relative entropy inequality, for any a > 0,

$$\int_{\Omega_{N}} \left| \hat{\eta}_{K,K} - \rho_{-}(t) \right|^{2} d\mu_{N,t} \leq \frac{1}{aK} \left[H(\alpha_{N,t};\nu_{\rho_{-}(t)}) + \log \int e^{aK |\hat{\eta}_{K,K} - \rho_{-}(t)|^{2}} d\nu_{\rho_{-}(t)} \right].$$
(8.41)

Applying Proposition 8.2 on the box $\{\eta_1, \ldots, \eta_{2K-1}\},\$

$$H(\alpha_{N,t};\nu_{\rho_{-}(t)}) \le CK^2 \mathfrak{D}_{\mathrm{exc},N}(t) + CK\mathfrak{D}_{-,N}(t), \qquad (8.42)$$

where $\mathfrak{D}_{\text{exc},N}(t)$, $\mathfrak{D}_{-,N}(t)$ are the Dirichlet forms defined in (8.11) and (8.13). On the other hand, the canonical integral as can be bound similarly to (8.35) as

$$\log \int e^{aK|\hat{\eta}_{K,K}-\rho_{-}(t)|^{2}} d\nu_{\rho_{-}(t)} \leq C, \qquad (8.43)$$

for sufficiently small a. Therefore,

$$\int_{\Omega_N} \left| \hat{\eta}_{K,K} - \rho_-(t) \right|^2 d\mu_{N,t} \le C \left(K \mathfrak{D}_{\text{exc},N}(t) + \mathfrak{D}_{-,N}(t) + \frac{1}{K} \right).$$
(8.44)

The estimate for $\hat{\eta}_{N-K+1,K}$ is proved in the same way. Finally, in order to close the proof we only need to apply Proposition 8.1.

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