



# Non finite axiomatisability of weak bisimulation-based congruences

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## ABSTRACT

We study the axiomatisability of CCS parallel composition operator modulo weak bisimulation-based congruences. Specifically, we prove that all congruences that are coarser than rooted branching bisimilarity, and finer than rooted weak bisimilarity, do not admit a finite equational axiomatisation over the recursion, restriction, and relabelling free fragment of CCS.

## 1. Introduction

The equational characterisation of the *parallel composition operator*  $\parallel$  (also known as *full merge* [13,14]) of the Calculus of Communicating Systems (CCS) [29] is a key problem in the study of the equational logic of process calculi. The “saga” of the axiomatisation of this operator has its roots in the works by Hennessy and Milner [24,25], who studied the *equational theory* of (recursion free) CCS and proposed a *ground-complete axiomatisation* for it modulo *strong bisimilarity* and *observational congruence*, two classic notions of behavioural *congruence* (i.e., an equivalence relation that is compositional with respect to the language operators) that allow one to establish whether two processes have the same *observable behaviour* [36]. (An equational axiomatisation is ground-complete with respect to some congruence over a language if it can prove all the sound equations relating variable-free terms.)

The axiomatisation by Hennessy and Milner included infinitely many axioms, which were instances of the celebrated *expansion law* used to “simulate equationally” the operational semantics of  $\parallel$ . Then, Bergstra and Klop showed, in [13], that a *finite* ground-complete axiomatisation modulo bisimilarity can be obtained by enriching CCS with two auxiliary operators, i.e., the *left merge*  $\ll$  and the *communication merge*  $|$ . Their result was strengthened by Aceto et al. in [7], where it is proved that, over the fragment of CCS without recursion, restriction and relabelling, the auxiliary operators  $\ll$  and  $|$  allow for finitely axiomatising  $\parallel$  modulo bisimilarity. Moreover, in [10] that result is extended to the fragment of CCS with relabelling and restriction, but without communication.

From those studies, we can infer that  $\ll$  and  $|$  are *sufficient* to finitely axiomatise  $\parallel$  over CCS modulo bisimilarity. Moller showed, in [32,33], that the use of auxiliary operators is also *necessary*. He considered a minimal fragment of CCS, including only the inactive process, action prefixing, nondeterministic choice and interleaving parallel composition, and proved that, even in the presence of a single action, bisimilarity does not afford a finite ground-complete axiomatisation over that language. Moller’s proof technique

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was then used in [6] to show that the same negative result holds if we replace  $\parallel$  and  $|$  with the so called *Hennessey's merge* operator from [23], which denotes an asymmetric interleaving parallel operator with communication, or, more generally, with a single binary auxiliary operator satisfying three assumptions given in [2].

The aforementioned works considered equational characterisations of  $\parallel$  modulo strong bisimilarity. However, a plethora of strong behavioural congruences have been proposed in the literature, corresponding to different levels of abstraction from the information on process execution. Hence, another chapter in the saga consisted in extending the studies recalled above to the behavioural strong congruences in van Glabbeek's linear time-branching time spectrum [17]. In [5] we delineated the *boundary* between finite and non-finite axiomatisability of  $\parallel$  modulo all the strong congruences in the spectrum. *Henceforth, we consider only the recursion, restriction and relabelling free fragment of CCS.*

### Weak congruences

Some information on process behaviour can either be considered irrelevant or be unavailable to an external observer. *Weak behavioural semantics* have been introduced to study the effects of these unobservable (or *silent*) actions, usually denoted by  $\tau$ , on the observable behaviour of processes, each semantics considering a different level of abstraction. A taxonomy of weak semantics is given in [18], and studies on the equational theories of various of these semantics have been carried out over the algebra BCCSP, which consists of the basic operators from CCS and CSP [26] but does not include  $\parallel$  (see, among others, [9,16,21,25,35]). A finite, ground-complete axiomatisation of parallel composition modulo *rooted weak bisimilarity* (also known as *observational congruence* [25]) is provided by Bergstra and Klop in [14] over the algebra  $ACP_\tau$  that includes the auxiliary operators  $\parallel$  and  $|$ . To the best of our knowledge, the only study on the axiomatisability of CCS's  $\parallel$  over open terms modulo weak congruences is the negative result from [1], which shows that a class of weak congruences (including rooted weak bisimilarity) does not afford a finite, complete axiomatisation over the open terms of the minimal fragment of CCS with interleaving.

To our mind, in light of the key role played by weak behavioural congruences in the theory and applications of process algebras, the lack of results pertaining to the non-finite axiomatisability of such congruences over CCS is unsatisfactory. We always believed that Moller's negative results for strong bisimilarity extend to weak bisimulation-based congruences, since the family of process equations underlying Moller's proofs does not involve the internal  $\tau$  action and sound axioms involving  $\tau$  do not seem to help in proving those process equations. However, negative results and their proofs are sensitive to even seemingly minor changes in the collection of operators in a language or in the behavioural semantics and therefore that belief of ours must be supported by formal proofs. Ideally, such proofs will eventually be based on general techniques for lifting theorems proven for strong congruences, specially the negative ones, to their weak counterparts. Indeed, such techniques may lead to elegant proofs and might be used to solve a number of open problems in the field, apart from allowing one to establish negative results without having to go through their direct proofs, which are undeniably long, complex, and technically heavy.

However, the development of such general techniques has thus far proved elusive, since both negative and positive finite-axiomatisability results are very delicate and, to our mind, we still lack a suitably deep understanding of the reasons that make an algebra of processes finitely based. Moreover, the development of widely applicable techniques is often preceded by specific results from which one can glean underlying general principles. Hence, as an intermediate step, in this paper we study the non-finite axiomatisability of the main weak bisimulation-based congruences over CCS.

### Our contribution

We start by considering *branching bisimilarity* [20], which generalises strong bisimilarity to abstract away from  $\tau$ -steps of terms while preserving their *branching structure* [20,21], and its *rooted* version, which is a congruence with respect to CCS operators. As a first main contribution, we show that *rooted branching bisimilarity affords no finite ground-complete axiomatisation over CCS*. To this end, we adapt the proof-theoretic technique used by Moller to prove the corresponding negative result for strong bisimilarity. We remark that, even though the general proof strategy is a natural extension of Moller's, our proof requires a number of original, non-trivial technical results on (rooted) branching bisimilarity. In particular, we observe that equational proofs of  $\tau$ -free equations might involve terms having occurrences of  $\tau$  in some intermediate steps (see, e.g., page 175 of Moller's thesis [32]), and our proof of the negative result for rooted branching bisimilarity will account for those uses of  $\tau$ , thus making our results specific to the considered weak congruence. Moreover, as an intermediate step in our proof, we establish a result of independent interest: we show that *each CCS process has a unique decomposition into indecomposable processes modulo branching bisimilarity*. A similar result was proven in [28], but only for interleaving parallel composition. Here, we extend this result to the full merge operator, including thus the possibility of communication between the parallel components.

Then we switch to rooted weak bisimilarity, i.e., Milner's observational congruence, and we prove our second main result: *rooted weak bisimilarity is not finitely axiomatisable over CCS*. To obtain it, we show how to adapt the proof technique used for rooted branching congruence to account for the subtleties of observational congruence. Moreover, we also extend the unique parallel decomposition result to weak bisimilarity.

For the sake of completeness, we argue that the two negative results do not depend on the possibility of synchronisation between the parallel components, and we show that they still hold if we replace the full merge operator with its purely interleaving variant.

Finally, we extend the non finite axiomatisability result to all the congruences that are finer than rooted weak bisimilarity and coarser than rooted branching bisimilarity, thus including rooted  $\eta$ -bisimilarity [11] and rooted delay bisimilarity [30].

### What's new

A preliminary version of the negative results for rooted branching bisimilarity appeared in [3]. Besides providing the full technical development and proofs for those results, we have enriched our previous contribution as follows:

1. We show that every weak bisimulation equivalence class of CCS processes has a unique parallel decomposition into indecomposables.
2. We prove that rooted weak bisimilarity admits no finite equational axiomatisation over CCS and over CCS with purely interleaving parallel composition.
3. We extend those negative results to all congruences that are coarser than rooted branching bisimilarity, and finer than rooted weak bisimilarity.

### Organisation of contents

In Section 2 we review basic notions on weak bisimulation-based semantics, CCS and equational logic. In Section 3 we give a bird's-eye view on the proof strategy that we will apply to obtain the two negative results, which are presented, respectively, in Sections 8 and 10. Sections 5–7 are devoted to the technical development of all the intermediate steps necessary to obtain the two results. Specifically, Section 4 deals with the decomposition of the behaviour of terms. In Section 5 we discuss the existence of a unique parallel decomposition into indecomposables. Section 6 presents the saturation of axiom systems, modulo the considered congruences. Section 7 contains technical results specific for rooted branching bisimilarity, while those for rooted weak bisimilarity are presented in Section 9. In Section 11 we discuss the extension of the negative results obtained over CCS with full merge, to CCS with pure interleaving parallel composition. The four negative results are then extended to other weak bisimulation-based congruences in Section 12. We conclude the paper in Section 13 with a discussion on future works.

## 2. Background

*Labelled transition systems* As semantic model we consider classic *labelled transition systems* [27]. We assume a set of action names  $\mathcal{A}$ , and we let  $\overline{\mathcal{A}}$  denote the set of action co-names, i.e.,  $\overline{\mathcal{A}} = \{\overline{a} \mid a \in \mathcal{A}\}$ . As usual, we postulate that  $\overline{\overline{a}} = a$  and  $a \neq \overline{a}$  for all  $a \in \mathcal{A}$ . Then, we let  $\mathcal{A}_\tau = \mathcal{A} \cup \overline{\mathcal{A}} \cup \{\tau\}$ , where  $\tau \notin \mathcal{A}$ . Henceforth, we let  $\mu, \nu, \dots$  range over actions in  $\mathcal{A}_\tau$ , and  $\alpha, \beta, \dots$  range over actions in  $\mathcal{A} \cup \overline{\mathcal{A}}$ .

**Definition 1** (*Labelled Transition System*). A *labelled transition system* (LTS) is a triple  $(\mathbf{P}, \mathcal{A}_\tau, \rightarrow)$ , where  $\mathbf{P}$  is a set of *processes* (or *states*),  $\mathcal{A}_\tau$  is a set of *actions*, and  $\rightarrow \subseteq \mathbf{P} \times \mathcal{A}_\tau \times \mathbf{P}$  is a (*labelled*) *transition relation*.

As usual, we use  $p \xrightarrow{\mu} p'$  in lieu of  $(p, \mu, p') \in \rightarrow$ . For each  $p \in \mathbf{P}$  and  $\mu \in \mathcal{A}$ , we write  $p \xrightarrow{\mu}$  if  $p \xrightarrow{\mu} p'$  holds for some  $p'$ , and  $p \not\xrightarrow{\mu}$  otherwise. The *initials* of  $p$  are the actions that label the outgoing transitions of  $p$ , that is,  $\text{init}(p) = \{\mu \in \mathcal{A}_\tau \mid p \xrightarrow{\mu}\}$ . We also introduce the notion of *derivative* of a process.

**Definition 2** (*Derivative*). For a process  $p$ , the set of *derivatives* of  $p$ , notation  $\text{der}(p)$ , is the least set containing  $p$  that is closed under  $\rightarrow$ , i.e., the least set satisfying:

- $p \in \text{der}(p)$ , and
- if  $q \in \text{der}(p)$  and  $q \xrightarrow{\mu} q'$ , for some action  $\mu \in \mathcal{A}_\tau$ , then  $q' \in \text{der}(p)$ .

In particular, we say that  $p' \in \text{der}(p)$  is a  $\mu$ -*derivative* of  $p$ , for some  $\mu \in \mathcal{A}_\tau$ , if  $p \xrightarrow{\mu} p'$ . Moreover, we say that  $p'$  is a *proper derivative* of  $p$  if  $p' \in \text{der}(p) \setminus \{p\}$ .

*The language CCS* We consider the recursion, relabelling and restriction free fragment of Milner's CCS [31], which for simplicity we still call CCS, given by the following grammar:

$$t ::= \mathbf{0} \mid x \mid \mu.t \mid t + t \mid t \parallel t ,$$

where  $x$  is a variable drawn from a countably infinite set  $\mathcal{V}$ , and  $\mu \in \mathcal{A}_\tau$ . We shall use the meta-variables  $t, u, v, w$  to range over terms, and write  $\text{var}(t)$  for the collection of variables occurring in the term  $t$ . We use a *summation*  $\sum_{i \in \{1, \dots, k\}} t_i$  to denote the term  $t = t_1 + \dots + t_k$ , where the empty sum represents  $\mathbf{0}$ . The *size* of a term  $t$ , denoted by  $\text{size}(t)$ , is the number of operator symbols in  $t$ . A term is *closed* if it does not contain any variables. Closed terms, or *processes*, will be denoted by  $p, q, r$ . Moreover, we omit trailing  $\mathbf{0}$ 's from terms.

The SOS rules [37] for the CCS operators given above are reported in Table 1. The operational semantics for our language is then modelled by the LTS whose processes are the closed terms, and whose labelled transitions are those that are provable from the SOS rules. Henceforth, we let  $\mathbf{P}$  denote the set of CCS processes.

**Table 1**  
The SOS rules for CCS operators ( $\mu \in \mathcal{A}_\tau$ ,  $\alpha \in \mathcal{A} \cup \overline{\mathcal{A}}$ ).

$\frac{}{\mu.t \xrightarrow{\mu} t}$	$\frac{t \xrightarrow{\mu} t'}{t + u \xrightarrow{\mu} t'}$	$\frac{u \xrightarrow{\mu} u'}{t + u \xrightarrow{\mu} u'}$
$\frac{t \xrightarrow{\mu} t'}{t \parallel u \xrightarrow{\mu} t' \parallel u}$	$\frac{u \xrightarrow{\mu} u'}{t \parallel u \xrightarrow{\mu} t \parallel u'}$	$\frac{t \xrightarrow{\alpha} t' \quad u \xrightarrow{\overline{\alpha}} u'}{t \parallel u \xrightarrow{\tau} t' \parallel u'}$

We can extend the operational semantics to terms, by considering as transitions the literals  $t \xrightarrow{\mu} t'$  that are provable from the SOS rules. We remark that no transitions can be inferred for variables from the SOS rules in Table 1. We will investigate in Section 4 how to broaden the set of SOS rules in order to equip variables with an operational semantics. We can then extend the notion of set of initial actions from processes to terms in the obvious way:  $\text{init}(t) = \{\mu \in \mathcal{A}_\tau \mid t \xrightarrow{\mu} t'\}$ . We remark that whenever  $t \xrightarrow{\mu} t'$ , for some terms  $t, t'$  and action  $\mu \in \mathcal{A}_\tau$ , then  $\text{size}(t) > \text{size}(t')$ .

A (closed) substitution is a mapping from process variables to (closed) terms. Substitutions are extended from variables to terms, transitions, and rules in the usual way. Note that  $\sigma(t)$  is closed, if so is  $\sigma$ . We let  $\sigma[x \mapsto p]$  denote the substitution that maps the variable  $x$  to process  $p$  and behaves like  $\sigma$  on all other variables. In particular,  $[x \mapsto p]$  denotes the substitution that maps the variable  $x$  to process  $p$  and behaves like the identity on all other variables.

*Weak bisimulation-based semantics Behavioural equivalences* have been introduced as a tool to establish whether the behaviours of two processes are *indistinguishable for their observers*. Roughly, they allow us to check whether the *observable* semantics of two processes is *the same*. In the literature we can find several notions of behavioural equivalence based on the observations that an external observer can make on the process. A classic behavioural equivalence is (strong) *bisimilarity* [29,37].

In this paper, we focus on the behavioural equivalences known as *weak bisimilarity* and *branching bisimilarity*, which we formally introduce below. Briefly, weak bisimulation generalises bisimulation to abstract away from (non observable, irrelevant, hidden, silent)  $\tau$ -labelled computation steps of terms. Branching bisimulation refines it by requiring that the *branching structure* of terms is preserved in the abstraction. This key feature is mainly due to the *stuttering* nature of branching bisimulation (see Definition 4 below), which guarantees that the potential of a term is preserved in the execution of a sequence of silent steps [20,21].

Let  $\xrightarrow{\epsilon}$  denote the reflexive and transitive closure of the transition  $\xrightarrow{\tau}$ . Moreover, let  $p \xRightarrow{\mu} q$  be a shorthand for  $p \xrightarrow{\epsilon} p' \xrightarrow{\mu} p'' \xrightarrow{\epsilon} q$ , for some  $p', p''$ .

**Definition 3** (*Weak and branching bisimilarity*). Let  $(\mathbf{P}, \mathcal{A}_\tau, \rightarrow)$  be a LTS.

*Weak bisimilarity*, denoted by  $\sim_{\text{WB}}$ , is the largest binary relation over  $\mathbf{P}$  such that whenever  $p \sim_{\text{WB}} q$ :

- whenever  $p \xrightarrow{\mu} p'$ , then either:
  - $\mu = \tau$  and  $p' \sim_{\text{WB}} q$ , or
  - there is a process  $q'$  such that  $q \xRightarrow{\mu} q'$  and  $p' \sim_{\text{WB}} q'$ ;
- whenever  $q \xrightarrow{\mu} q'$ , then either:
  - $\mu = \tau$  and  $p \sim_{\text{WB}} q'$ , or
  - there is a process  $p'$  such that  $p \xRightarrow{\mu} p'$  and  $p' \sim_{\text{WB}} q'$ .

*Branching bisimilarity*, denoted by  $\sim_{\text{BB}}$ , is the largest binary relation over  $\mathbf{P}$  such that whenever  $p \sim_{\text{BB}} q$ :

- whenever  $p \xrightarrow{\mu} p'$ , then either:
  - $\mu = \tau$  and  $p' \sim_{\text{BB}} q$ , or
  - there are processes  $q', q''$  such that  $q \xrightarrow{\epsilon} q'' \xrightarrow{\mu} q'$ ,  $p \sim_{\text{BB}} q''$ , and  $p' \sim_{\text{BB}} q'$ ;
- whenever  $q \xrightarrow{\mu} q'$ , then either:
  - $\mu = \tau$  and  $p \sim_{\text{BB}} q'$ , or
  - there are processes  $p', p''$  such that  $p \xrightarrow{\epsilon} p'' \xrightarrow{\mu} p'$ ,  $p'' \sim_{\text{BB}} q$ , and  $p' \sim_{\text{BB}} q'$ .

**Example 1.** We present a classic example, from [18], showing the difference between weak and branching bisimilarity. Consider processes  $p = a + \tau.(b + \tau.c)$  and  $q = a + \tau.(b + \tau.c) + \tau.c$  whose process graphs, obtained from the respective LTSs, are depicted in Fig. 1. We have that  $p \sim_{\text{WB}} q$ , but  $p \not\sim_{\text{BB}} q$ , due to the rightmost (red)  $\tau$ -transition from  $q$ . In fact, in order for  $p$  to match the transition  $q \xrightarrow{\tau} c$  modulo branching bisimilarity, we should have that either  $p \sim_{\text{BB}} c$ , or there are  $p', p''$  such that  $p \xrightarrow{\epsilon} p' \xrightarrow{\tau} p''$ ,  $p' \sim_{\text{BB}} q$  and  $p'' \sim_{\text{BB}} c$ . However, it is immediate to check that  $p \not\sim_{\text{BB}} c$ . Moreover, the only sequence of  $\tau$ -moves from  $p$  reaching a process that is branching bisimilar to  $c$  does not preserve the branching structure: we have  $p \xrightarrow{\tau} \tau.c + b \xrightarrow{\tau} c$  with  $c \sim_{\text{BB}} c$ , but  $\tau.c + b \not\sim_{\text{BB}} c$ . ◀

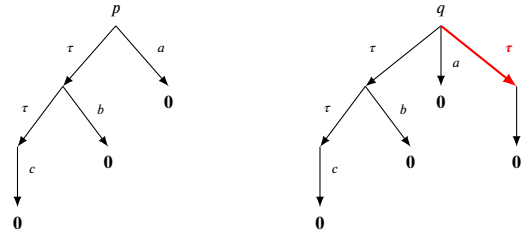


Fig. 1.  $p \sim_{\text{WB}} q$  but  $p \not\sim_{\text{BB}} q$ . (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

Branching bisimilarity satisfies the *stuttering property* (see [21, Lemma 2.5]).

**Definition 4** (*Stuttering property*, [12,21]). A binary relation  $\mathcal{R} \subseteq \mathbf{P} \times \mathbf{P}$  has the *stuttering property* if and only if for all  $n \geq 0$  and processes  $p, p_1, \dots, p_n, q, q_1, \dots, q_n \in \mathbf{P}$  it holds that:

- whenever  $p \mathcal{R} q$ ,  $p \xrightarrow{\tau} p_1 \xrightarrow{\tau} \dots \xrightarrow{\tau} p_n$  and  $p_n \mathcal{R} q$ , then  $p_i \mathcal{R} q$  for all  $i = 1, \dots, n-1$ ;
- whenever  $p \mathcal{R} q$ ,  $q \xrightarrow{\tau} q_1 \xrightarrow{\tau} \dots \xrightarrow{\tau} q_n$  and  $p \mathcal{R} q_n$ , then  $p \mathcal{R} q_i$  for all  $i = 1, \dots, n-1$ .

One of the main concerns in the development of a meta-theory of process languages is to guarantee their *compositionality*, i.e., that the *replacement* of a component of a system with a  $\sim$ -equivalent one, for a chosen behavioural equivalence  $\sim$ , does not affect the behaviour of that system. In algebraic terms, this is known as the *congruence property* of  $\sim$  with respect to all language operators, and consists in verifying whether, for all  $n$ -ary operators  $f$  in the studied language

$$t_i \sim t'_i, \forall i = 1, \dots, n \implies f(t_1, \dots, t_n) \sim f(t'_1, \dots, t'_n).$$

Weak and branching bisimilarity are equivalence relations [12,21,25], and are congruences with respect to the operators that meet, respectively, the *simple WB cool* and the *simple BB cool* rule formats from [15,19]. In our setting, this means that  $\sim_{\text{WB}}$  and  $\sim_{\text{BB}}$  are compositional with respect to prefixing and parallel composition, but not with respect to nondeterministic choice, as shown in the following example.

**Example 2.** Consider processes  $b$  and  $\tau.b$ . It is immediate to verify that  $b \sim_{\text{WB}} \tau.b$  and  $b \sim_{\text{BB}} \tau.b$ . However,  $a + b \not\sim_{\text{WB}} a + \tau.b$  and  $a + b \not\sim_{\text{BB}} a + \tau.b$ : process  $a + b$  can simulate the transition  $a + \tau.b \xrightarrow{\tau} b$  only by performing no action, but  $a + b \sim_{\text{WB}} b$  and  $a + b \sim_{\text{BB}} b$ . ◀

To remedy this shortcoming, the *root condition* is introduced: Initial  $\tau$ -transitions of one process must be matched by some sequence of  $\tau$ -transitions of the other.

**Definition 5** (*Rooted weak and branching bisimilarity*). *Rooted weak bisimilarity*, denoted by  $\sim_{\text{RWB}}$ , is the binary relation over  $\mathbf{P}$  such that whenever  $p \sim_{\text{RWB}} q$ :

- whenever  $p \xrightarrow{\mu} p'$ , then there is a process  $q'$  such that  $q \xrightarrow{\mu} q'$  and  $p' \sim_{\text{WB}} q'$ ;
- whenever  $q \xrightarrow{\mu} q'$ , then there is a process  $p'$  such that  $p \xrightarrow{\mu} p'$  and  $p' \sim_{\text{WB}} q'$ .

*Rooted branching bisimilarity*, denoted by  $\sim_{\text{RBB}}$ , is the binary relation over  $\mathbf{P}$  such that whenever  $p \sim_{\text{RBB}} q$ :

- whenever  $p \xrightarrow{\mu} p'$ , then there is a process  $q'$  such that  $q \xrightarrow{\mu} q'$  and  $p' \sim_{\text{BB}} q'$ ;
- whenever  $q \xrightarrow{\mu} q'$ , then there is a process  $p'$  such that  $p \xrightarrow{\mu} p'$  and  $p' \sim_{\text{BB}} q'$ .

Rooted weak bisimilarity and rooted branching bisimilarity are equivalence relations [12,21,25], and they are congruences over CCS (see, e.g., [19]).

**Example 3.** Consider processes  $p$  and  $q$  from Example 2. As  $p$  cannot perform an initial  $\tau$ -move, we have that  $p \not\sim_{\text{RWB}} q$  and  $p \not\sim_{\text{RBB}} q$ . ◀

**Remark 1.** As the root condition requires an initial  $\tau$ -move by a process to be mimicked by at least one  $\tau$ -move by the other, we get that whenever  $p \sim_{\text{RBB}} q$  (respectively,  $p \sim_{\text{RWB}} q$ ) and  $\tau \notin \text{init}(p)$ , then  $\tau \notin \text{init}(q)$ .

For a sequence of actions  $\varphi = \alpha_1 \cdots \alpha_k \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$  ( $k \geq 0$ ), and processes  $p, p'$ , we write that  $p \xRightarrow{\varphi} p'$  if and only if there exists a sequence of transitions  $p = p_0 \xrightarrow{\alpha_1} p_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} p_k$ . If  $p \xRightarrow{\varphi} p'$  holds for some process  $p'$ , then  $\varphi$  is an *observable trace* of  $p$ . Moreover, we say that  $\varphi$  is a *maximal observable trace* of  $p$  if  $\text{init}(p') = \emptyset$ . By means of observable traces, we associate a classic notion with a process  $p$ , i.e., its (*observable*) *depth*, denoted by  $\text{depth}(p)$ . For a process  $p$  whose set of traces is finite, it expresses the length of a *longest* observable trace. Formally, denoting by  $|\varphi|$  the length of  $\varphi$ ,

$$\text{depth}(p) = \sup\{k \mid p \xRightarrow{\varphi} p' \wedge |\varphi| = k\}.$$

Weak and branching bisimilarity preserve the observable depth of processes:

**Lemma 1.** *Whenever  $p \sim_{\text{BB}} q$  (respectively,  $p \sim_{\text{WB}} q$ ), then  $\text{depth}(p) = \text{depth}(q)$ .*

The cases of rooted equivalences are slightly more complicated, as we need to treat a possible initial  $\tau$ -move as an observable action. To this end, we define the *rooted depth* of a process, denoted by  $\text{Rdepth}(p)$ . For  $p = \mathbf{0}$ , we have that  $\text{Rdepth}(\mathbf{0}) = 0$ . For  $p \neq \mathbf{0}$ ,  $\text{Rdepth}(p)$  is defined as the maximal depth of the proper derivatives of  $p$  augmented by 1:

$$\text{Rdepth}(p) = 1 + \sup_{p' \in \text{der}(p) \setminus \{p\}} \text{depth}(p').$$

Rooted weak bisimilarity and rooted branching bisimilarity preserve the rooted depth of processes:

**Lemma 2.** *Whenever  $p \sim_{\text{RBB}} q$  (respectively,  $p \sim_{\text{RWB}} q$ ), then  $\text{Rdepth}(p) = \text{Rdepth}(q)$ .*

Moreover, it is well known that rooted relations are finer than the corresponding equivalences, in the sense that whenever  $p \sim_{\text{RBB}} q$  (respectively,  $p \sim_{\text{RWB}} q$ ), then also  $p \sim_{\text{BB}} q$  (respectively,  $p \sim_{\text{WB}} q$ ) holds. In general, the converse implication does not hold. However, we present here a simple case in which we can establish that two branching bisimilar (respectively, weak bisimilar) processes are also rooted branching bisimilar (respectively, rooted weak bisimilar):

**Lemma 3.** *Assume that processes  $p, q$  are such that  $p \sim_{\text{BB}} q$  (respectively,  $p \sim_{\text{WB}} q$ ). If  $\tau \notin \text{init}(p) \cup \text{init}(q)$ , then  $p \sim_{\text{RBB}} q$  (respectively,  $p \sim_{\text{RWB}} q$ ).*

*Moreover, if  $\tau \notin \text{init}(p)$ , then  $p \sim_{\text{RBB}} \mathbf{0}$  (respectively,  $p \sim_{\text{RWB}} \mathbf{0}$ ) implies  $p \sim_{\text{BB}} \mathbf{0}$  (respectively,  $p \sim_{\text{WB}} \mathbf{0}$ ).*

Notice that we can give the following inductive characterisation of the (rooted) depth of CCS processes:

- $\text{depth}(\mathbf{0}) = 0$ ;
- $\text{depth}(\tau.p) = \text{depth}(p)$ ;
- $\text{depth}(\alpha.p) = 1 + \text{depth}(p)$ ;
- $\text{depth}(p + q) = \max\{\text{depth}(p), \text{depth}(q)\}$ ;
- $\text{depth}(p \parallel q) = \text{depth}(p) + \text{depth}(q)$ .
- $\text{Rdepth}(\mathbf{0}) = 0$ ;
- $\text{Rdepth}(\mu.p) = 1 + \text{depth}(p)$ ;
- $\text{Rdepth}(p + q) = \max\{\text{Rdepth}(p), \text{Rdepth}(q)\}$ ;
- $\text{Rdepth}(p \parallel q) = \max\left\{\begin{array}{l} \text{Rdepth}(p) + \text{depth}(q) \\ \text{depth}(p) + \text{Rdepth}(q) \end{array}\right\}$ .

**Remark 2.** We remark that the definition of the depth of parallel composition as the sum of the depth of the parallel components is justified by the fact that, in the considered fragment of CCS, no operator can prevent one of its arguments to perform an action. Hence, any process of the form  $p \parallel q$  is always allowed to perform a trace that corresponds to the concatenation of two maximal traces of  $p$  and  $q$  (as they can be interleaved arbitrarily). The case of rooted depth follows by noticing that initial  $\tau$ -moves from a parallel component may not be initial steps in the interleaved semantics. However, we remark that if  $\tau \notin \text{init}(p \parallel q)$  (and thus  $\tau \notin \text{init}(p) \cup \text{init}(q)$ ), then

$$\text{Rdepth}(p \parallel q) = \text{Rdepth}(p) + \text{Rdepth}(q) = \text{depth}(p) + \text{depth}(q) = \text{depth}(p \parallel q).$$

Finally, we also notice that the (rooted) depth of a closed instance of a CCS term is always an upper bound to the (rooted) depths of the closed terms substituted for the variables occurring in it.

**Lemma 4.** *Let  $t$  be a CCS term, and  $\sigma$  be a closed substitution. For each  $x \in \text{var}(t)$ ,  $\text{depth}(\sigma(x)) \leq \text{depth}(\sigma(t))$ , and  $\text{Rdepth}(\sigma(x)) \leq \text{Rdepth}(\sigma(t))$ .*

*Equational logic* An axiom system  $\mathcal{E}$  is a collection of (process) equations  $t \approx u$  over the considered language, thus CCS in this paper. An equation  $t \approx u$  is *derivable* from an axiom system  $\mathcal{E}$ , notation  $\mathcal{E} \vdash t \approx u$ , if there is an *equational proof* for it from  $\mathcal{E}$ , namely if  $t \approx u$  can be inferred from the axioms in  $\mathcal{E}$  using the *rules of equational logic*. The rules over CCS are reported in Table 2.

We typically also assume that  $(e_2)$  is only used when  $t \approx u \in \mathcal{E}$  and so that rule is used at leaves of proofs.

We are interested in equations that are valid modulo some congruence relation  $\sim$  over closed terms. The equation  $t \approx u$  is said to be *sound* modulo  $\sim$  if  $\sigma(t) \sim \sigma(u)$  for all closed substitutions  $\sigma$ . For simplicity, if  $t \approx u$  is sound, then we write  $t \sim u$ . An axiom system is *sound* modulo  $\sim$  if, and only if, all of its equations are sound modulo  $\sim$ . Conversely, we say that  $\mathcal{E}$  is *complete* modulo  $\sim$  if  $t \sim u$

**Table 2**  
The rules of equational logic.

$(e_1)$ $t \approx t$	$(e_2)$ $\frac{t \approx u}{u \approx t}$	$(e_3)$ $\frac{t \approx u \quad u \approx v}{t \approx v}$	$(e_4)$ $\frac{t \approx u}{\sigma(t) \approx \sigma(u)}$
$(e_4)$ $\frac{t \approx u}{\mu.t \approx \mu.u}$	$(e_5)$ $\frac{t \approx u \quad t' \approx u'}{t + t' \approx u + u'}$	$(e_6)$ $\frac{t \approx u \quad t' \approx u'}{t \parallel t' \approx u \parallel u'}$	

**Table 3**  
Some classical axioms for the considered bisimulation-based congruences.

Some axioms for bisimilarity over CCS:	
A0 $x + \mathbf{0} \approx x$	P0 $x \parallel \mathbf{0} \approx x$
A1 $x + y \approx y + x$	P1 $x \parallel y \approx y \parallel x$
A2 $(x + y) + z \approx x + (y + z)$	P2 $(x \parallel y) \parallel z \approx x \parallel (y \parallel z)$
A3 $x + x \approx x$	
EL $\sum_{i \in I} \mu_i x_i \parallel \sum_{j \in J} \nu_j y_j \approx \sum_{i \in I} \mu_i (x_i \parallel \sum_{j \in J} \nu_j y_j) + \sum_{j \in J} \nu_j (\sum_{i \in I} \mu_i x_i \parallel y_j) + \sum_{i,j   \mu_i = \nu_j} \tau(x_i \parallel y_j)$	
Additional axioms for rooted branching bisimilarity over CCS:	
TB $\mu(\tau(x + y) + y) \approx \mu(x + y)$	T1 $\mu \tau x \approx \mu x$
Additional axioms for rooted weak bisimilarity over CCS:	
T2 $\tau x \approx \tau x + x$	T3 $\mu(\tau x + y) \approx \mu(\tau x + y) + \mu x$

implies  $\mathcal{E} \vdash t \approx u$  for all terms  $t, u$ . If we restrict ourselves to consider only equations in which there are no occurrences of variables, then  $\mathcal{E}$  is said to be *ground-complete* modulo  $\sim$ . We say that  $\sim$  has a finite, (ground) complete axiomatisation, if there is a finite axiom system  $\mathcal{E}$  that is sound and (ground) complete for  $\sim$ .

For instance, the set of axioms  $\{A0-A3, P0-P1, EL\}$ , reported in the upper part of Table 3, is a sound and ground-complete axiomatisation for strong bisimilarity.

Throughout the paper, we also exploit the associativity and commutativity of  $+$  and  $\parallel$  modulo the behavioural equivalences we will consider, and define terms modulo them, i.e., we do not distinguish  $t + u$  and  $u + t$ , nor  $(t + u) + v$  and  $t + (u + v)$ , nor  $t \parallel u$  and  $u \parallel t$ , nor  $t \parallel (u \parallel r)$  and  $(t \parallel u) \parallel r$ . The symbol  $=$  will then denote equality modulo the above identifications. Moreover, we recall that, thanks to the axioms in Table 3, we can prove the following classic result for CCS terms, that will be useful in the rest of paper.

**Lemma 5.** *For every CCS term  $t \neq \mathbf{0}$  there are CCS terms  $t_1, \dots, t_n$  ( $n \geq 0$ ), different from  $\mathbf{0}$  and not having  $+$  as head operator, such that  $t$  can be univocally rewritten as  $\sum_{i=1}^n t_i$  modulo A0–A3. The terms  $t_i$  are then called the summands of  $t$ .*

We recall that  $\mathbf{0}$  is sometimes defined as the empty summation.

We conclude this section by recalling a classic result for bisimulation-based equivalences: equivalent terms have the same variables.

**Lemma 6.** *Let  $X \in \{\text{WB}, \text{BB}, \text{RWB}, \text{RBB}\}$ . If  $t \approx u$  is sound modulo  $\sim_X$ , then  $\text{var}(t) = \text{var}(u)$ .*

### 3. Proof strategy

In light of the technicality of the results that we will need to fulfil our objectives, we devote this section to an informal presentation of the proof strategy we will apply. The idea is that the reader can use this section as a road map for their hike through the technical development of our contribution.

#### 3.1. The main theorems

Our principal goal is to prove the following theorems:

**Theorem 1.** *Rooted branching bisimilarity admits no finite equational axiomatisation over CCS.*

**Theorem 2.** *Rooted weak bisimilarity admits no finite equational axiomatisation over CCS.*

To this end, we apply the *proof-theoretic approach to non finite axiomatisability results*, whose application in process algebra stems from [32–34], in which Møller proved that CCS modulo strong bisimilarity is not finitely based. The main idea behind this proof technique is this. Whenever an equation  $t \approx u$  is provable from an axiom system  $\mathcal{E}$ , then there is a *proof* of it, i.e., a sequence of equations  $t_i \approx u_i$ , for  $i = 1, \dots, n$ , such that  $t = t_n$ ,  $u = u_n$ , and each equation  $t_i \approx u_i$  is in turn derivable from  $\mathcal{E} \cup \{t_j \approx u_j \mid j < i\}$ . The

aim in the proof-theoretic approach is to show that no such sequence exists for some sound equation, so that the considered equation cannot be derived from  $\mathcal{E}$ . To this end, we need to identify a *specific property* of terms, say  $\mathbb{P}_n$  for  $n \geq 0$ , that, when  $n$  is *large enough*, is preserved by provability from finite, sound axiom systems. Roughly, in our case, this means that if:

- $\mathcal{E}$  is a finite set of axioms that are sound modulo  $\sim_{\text{RBB}}$  (respectively,  $\sim_{\text{RWB}}$ ),
- the equation  $p \approx q$  is provable from  $\mathcal{E}$ , and
- $n > \text{size}(t)$  for any term  $t$  in the equations in  $\mathcal{E}$ ,

then either both  $p$  and  $q$  satisfy  $\mathbb{P}_n$ , or none of them does. Then, we exhibit a family of infinitely many equations  $\{\varepsilon_n \mid n \geq 0\}$  that are all sound modulo  $\sim_{\text{RBB}}$  (respectively,  $\sim_{\text{RWB}}$ ), but are such that only one side of  $\varepsilon_n$  satisfies  $\mathbb{P}_n$ , for each  $n \geq 0$ . In particular, this implies that whenever  $n$  is larger than the size of any term occurring in the equations in a finite, sound axiom system  $\mathcal{E}$ , then the  $n$ -th equation  $\varepsilon_n$  in the family (as well as any equation  $\varepsilon_m$  with  $m > n$ ) cannot be derived from  $\mathcal{E}$ . Hence, infinitely many equations must be included in the axiomatisation, which yields that the studied behavioural congruence is *not* finitely based.

### 3.2. General strategy

Firstly, we discuss how the aforementioned proof-theoretic approach will be applied to prove Theorems 1 and 2. We remark that although the two theorems follow from the same proof strategy, we will need to develop different intermediate technical results for them, to take into account the different subtleties of the two congruences.

*The choice of  $\mathbb{P}_n$  and the family of equations* The key observation behind Moller’s proof of the non-finite axiomatisability of strong bisimilarity was that no finite, sound axiom system  $\mathcal{E}$  over CCS is powerful enough to expand the initial behaviour of a term of the form  $a \parallel r$  when  $r$  has a large branching degree. Informally, this is mainly due to the fact that parallel composition does not distribute over choice in either of its arguments, modulo strong bisimilarity. Hence, the only way to describe equationally the initial behaviour of a term of the form  $p \parallel q$  is first to express  $p$  as  $\sum_{i \in I} p_i$  and  $q$  as  $\sum_{j \in J} q_j$  and then to use the classic “expansion law” (see EL in Table 3) to “simulate” equationally the rules giving the operational semantics of  $\parallel$ . Note that EL is an axiom schema representing an infinite family of equations.

Since parallel composition does not distribute over choice in either of its arguments, modulo  $\sim_{\text{RBB}}$  and  $\sim_{\text{RWB}}$ , it is natural to exploit a similar “weakness” to prove Theorems 1 and 2. In detail, we will consider, for each  $n \geq 2$ , the process

$$p_n = \sum_{i=2}^n a a^{<i},$$

where  $a^{<i} = \sum_{j=1}^i a^j$  and  $a^j = a a^{j-1}$ , with  $a^0 = \mathbf{0}$ , for each  $j = 1, \dots, i$ ,  $i = 2, \dots, n$ . Then, for each  $n \geq 2$ , the property  $\mathbb{P}_n$  will consist in having a summand rooted branching bisimilar (respectively, rooted weak bisimilar) to the process  $a \parallel p_n$ . We will show that, when  $n$  is *large enough*, property  $\mathbb{P}_n$  is an invariant under provability from finite, sound axiom systems (Theorem 3 in Section 8 for  $\sim_{\text{RBB}}$ , and Theorem 4 in Section 10 for  $\sim_{\text{RWB}}$ ). Then, we consider the infinite family of equations  $\{\varepsilon_n \mid n \geq 2\}$ , where, for each  $n \geq 2$ ,

$$\varepsilon_n : a \parallel p_n \approx a p_n + \sum_{i=2}^n a(a \parallel a^{<i}).$$

Each equation  $\varepsilon_n$  is sound modulo rooted branching bisimilarity, but it does not satisfy property  $\mathbb{P}_n$ . In fact, while the left-hand side of  $\varepsilon_n$  (i.e., process  $a \parallel p_n$ ) satisfies the property of having a summand rooted branching bisimilar to  $a \parallel p_n$  (namely the process itself), the right-hand side of the equation (i.e., process  $a p_n + \sum_{i=2}^n a(a \parallel a^{<i})$ ) does not satisfy such a property, as none of the summands (namely processes  $a p_n$  and  $a(a \parallel a^{<i})$  for each  $i = 2, \dots, n$ ) alone is rooted branching bisimilar to  $a \parallel p_n$ . As a consequence, since for  $n$  large enough the property  $\mathbb{P}_n$  is an invariant under provability from any finite, sound axiom system  $\mathcal{E}$ , the equation  $\varepsilon_n$  cannot be proved from such an axiom system  $\mathcal{E}$ . The infinite family of equations  $\{\varepsilon_n \mid n \geq 2\}$  thus gives the desired negative result.

Although the general proof idea is a natural (if not immediate) extension of Moller’s work on strong bisimilarity, it is based on a number of technical, non-trivial results on the considered congruences that we present in the upcoming sections. In particular, we will need to deal with the possibility that equations like  $\varepsilon_n$  cannot be shown using a finite number of sound axioms that, like TB and T1 in Table 3, allow one to introduce and then remove  $\tau$ ’s in terms. One pivotal ingredient to prove that the property  $\mathbb{P}_n$  is an invariant in equational proofs, will be the fact that processes  $p_n$  and  $a^{<i}$ , for  $i \in \{2, \dots, n\}$ ,  $n \geq 2$ , are *indecomposable*. The existence of a unique parallel decomposition into indecomposable processes modulo *branching bisimilarity* and *weak bisimilarity* over CCS with *interleaving parallel composition* has been studied in [28].

In Section 5, we extend the result from [28] to the full merge operator, thus including communication. In particular, we show that *every branching equivalence class* of CCS processes has a *unique parallel decomposition* into indecomposables (Proposition 3). Then we show that the same holds for *every equivalence class* of CCS processes *modulo weak bisimilarity* (Proposition 4).

Moreover, in order to simplify the equational proofs, we will restrict our attention to axiom systems that do not introduce terms that are rooted branching bisimilar to  $\mathbf{0}$  (and thus are not necessary) in the proofs.

Hence, in Section 6 we discuss the closure property of *saturation* for axiom systems that are sound modulo rooted branching bisimilarity, and for those that are sound modulo rooted weak bisimilarity.

*The choice of  $n$*  As outlined above, the choice of a sufficiently large  $n$  *large enough* plays a crucial role in proving that  $\mathbb{P}_n$  is an invariant under provability from a finite, sound axiom system  $\mathcal{E}$ . This will be evident in the case in which  $p \approx q$  is derived by an application of the substitution rule of equational logic (Proposition 7 in Section 8 for  $\sim_{\text{RBB}}$ , and Proposition 9 in Section 10 for  $\sim_{\text{RWB}}$ ), namely  $p = \sigma(t)$ ,  $q = \sigma(u)$ , and  $t \approx u \in \mathcal{E}$  for some terms  $t, u$  and closed substitution  $\sigma$ .

In our setting,  $n$  being *large enough* will mean that  $n$  is *greater than the size* of each term occurring in the equations in  $\mathcal{E}$ . In fact, under this constraint we can prove that  $p = \sigma(t)$  satisfies  $\mathbb{P}_n$  because of the behaviour of the closed instance of some variable  $x$  occurring in  $t$ .

To this end, in Section 4, we will study the decomposition of the behaviour of closed instances of terms with respect to the behaviour of the closed instances of variables occurring in them, that is establishing how a transition  $\sigma(t) \xrightarrow{\alpha} p$  can be derived knowing that  $\sigma(x) \xrightarrow{\beta} q$ , for some closed substitution  $\sigma$ , term  $t$ , variable  $x \in \text{var}(t)$ , and processes  $p, q$ . We remark that since the results in Section 4 concern the derivation of process behaviours, their validity does not depend on the considered congruence relation.

Moreover, we can also prove that, for  $t \approx u$  to be sound modulo rooted branching bisimilarity, whenever a variable  $x$  occurs in  $t$  then it must also occur in  $u$  (as ensured by Lemma 6 above), with the stronger requirement that if  $x$  is a summand of  $t$ , then it is also a summand of  $u$  (Proposition 6 in Section 7). A similar result can be obtained for rooted weak bisimilarity, but it requires some stronger assumptions on the structure of  $t$  and  $u$  (Proposition 8 in Section 9) Hence, we can infer that  $\sigma(x)$  will trigger in  $\sigma(u)$  the same behaviour that it induced in  $\sigma(t)$ , and thus that also  $q = \sigma(u)$  will satisfy  $\mathbb{P}_n$ .

All the additional properties of process  $a \parallel p_n$ , that are necessary to achieve this conclusion for  $\sim_{\text{RBB}}$ , are presented in Section 7. Those necessary for  $\sim_{\text{RWB}}$  are presented in Section 9.

### 3.3. The cases of interleaving parallel composition

We will prove Theorems 1 and 2 over CCS with the full merge operator (cf. Table 1). For sake of completeness, we also show that the negative results do not depend on the possibility of communication between the parallel components.

In Section 11 we prove that Theorems 1 and 2 remain valid even if we restrict our attention to CCS with purely interleaving parallel composition.

## 4. Decomposing the semantics of terms

In the proofs to follow, we shall sometimes need to establish a correspondence between the behaviour of open terms and that of their closed instances. In detail, we are interested in the derivation of a transition  $\sigma(t) \xrightarrow{\mu} p$ , for some term  $t$ , closed substitution  $\sigma$ , action  $\mu$ , and process  $p$ , from the behaviour of  $t$  and that of  $\sigma(x)$ , for each variable  $x$  occurring in  $t$ . The purpose is to characterise, and bind, the syntactic structure of a term given its behaviour. The simplest case is a direct application of the operational semantics in Table 1.

**Lemma 7.** For all terms  $t, t'$ , action  $\mu \in \mathcal{A}_\tau$ , and substitution  $\sigma$ , if  $t \xrightarrow{\mu} t'$  then  $\sigma(t) \xrightarrow{\mu} \sigma(t')$ .

Let us focus now on the role of variables. A transition  $\sigma(t) \xrightarrow{\mu} p$  may also derive from the initial behaviour of some closed term  $\sigma(x)$ , provided that the collection of initial moves of  $\sigma(t)$  depends, in some formal sense, on that of the closed term substituted for the variable  $x$ . In this case, we say that  $x$  *triggers the behaviour* of  $t$ .

**Example 4.** Consider the CCS term  $x + a$ . Using the SOS rules in Table 1, we can immediately infer that  $a \xrightarrow{a} \mathbf{0}$ , and, thus, that  $x + a \xrightarrow{a} \mathbf{0}$ . However, those rules do not allow us to establish a direct relation between the (potential) behaviour of  $x$  and that of  $x + a$ , although we know that as soon as a term substituted for  $x$  can perform a transition, then  $x + a$  will mimic such a transition. Our aim is then to introduce a set of auxiliary rules and transitions allowing us to indicate that  $x$  triggers the behaviour of  $x + a$ , and how it does it. Informally, we would like to be able to state that if  $x \xrightarrow{\ell} \xi$ , for some label  $\ell$  and syntactic object  $\xi$ , then  $x + a \xrightarrow{\ell} \xi$ . ◀

To establish how a variable  $x$  triggers the behaviour of a term  $t$ , we need to introduce a framework that would allow us to infer the behaviour of  $t$  from that of the variables and subterms occurring in it, including  $x$ . In other words, we need a means to assign a semantics to variables. We start from an extended version of the notion of *configuration over terms*, which stems from [8]: the novelty with respect to the literature is that we will need to add special labels and indexes in order to deal with communications. First of all,

**Table 4**Inference rules for the transition relation  $\xrightarrow{\ell}_\rho$ .

$(a_1) \frac{}{x \xrightarrow{(x)}_\mu x_\mu}$	$(a_2) \frac{t \xrightarrow{\ell}_\rho c}{t + u \xrightarrow{\ell}_\rho c}$	$(a_3) \frac{t \xrightarrow{\ell}_\rho c}{t \parallel u \xrightarrow{\ell}_\rho c \parallel u}$
$(a_4) \frac{t \xrightarrow{\ell}_\alpha c \quad u \xrightarrow{\ell'}_{\bar{\alpha}} c'}{t \parallel u \xrightarrow{(\ell, \ell')}_{\tau} c \parallel c'}$	$(a_5) \frac{t \xrightarrow{\ell}_\alpha c \quad u \xrightarrow{\bar{\alpha}} u'}{t \parallel u \xrightarrow{\ell}_{\alpha, \tau} c \parallel u'}$	$(a_6) \frac{t \xrightarrow{\alpha} t' \quad u \xrightarrow{\ell}_{\bar{\alpha}} c}{t \parallel u \xrightarrow{\ell}_{\bar{\alpha}, \tau} t' \parallel c}$

we introduce a set of action-indexed variables  $\mathcal{V}_{\mathcal{A}_\tau} = \{x_\mu \mid x \in \mathcal{V}, \mu \in \mathcal{A}_\tau\}$ , disjoint from  $\mathcal{V}$ , and CCS terms. Intuitively, the symbol  $x_\mu$  denotes that the closed term substituted for an occurrence of variable  $x$  has begun its execution (expressed in terms of a  $\mu$ -action), and it contributes thus to triggering the behaviour of the term in which  $x$  occurs. We postpone a detailed explanation of why we need to keep track of action  $\mu$  in  $x_\mu$  to Example 5 below.

**Definition 6.** The collection of CCS configurations, denoted by  $C$ , is given by the following grammar:

$$c ::= x_\mu \mid t \mid c \parallel c ,$$

where  $t$  is a CCS term, and  $x_\mu \in \mathcal{V}_{\mathcal{A}_\tau}$ .

We extend the definition of  $\text{var}(c)$  according to the newly introduced set of variables:  $\text{var}(c) = \{x \in \mathcal{V} \mid x \text{ occurs in } c\} \cup \{x_\mu \in \mathcal{V}_{\mathcal{A}_\tau} \mid x_\mu \text{ occurs in } c\}$ .

The next step towards obtaining a semantics for variables consists in introducing an auxiliary transition relation over open terms and configurations. Letting  $\mathcal{T}$  denote the set of CCS terms over  $\mathcal{V}$ , we introduce the transition relation  $\rightarrow \subseteq \mathcal{T} \times \mathbb{L} \times \mathbb{A} \times C$  where  $\mathbb{L} = \mathcal{V} \cup \mathcal{V} \times \mathcal{V}$  is a set of labels, and  $\mathbb{A} = \mathcal{A}_\tau \cup ((\mathcal{A} \cup \bar{\mathcal{A}}) \times \{\tau\})$  is a set of action indexes. Informally, we use an auxiliary transition  $t \xrightarrow{\ell}_\rho c$  to denote that the behaviour of the variable(s) in  $\ell$  trigger(s) a  $\rho$ -labelled transition of term  $t$  resulting in the configuration  $c$ . The reason why we need to allow pairs of variables as labels, and pairs  $\alpha, \tau$ , for  $\alpha \in \mathcal{A} \cup \bar{\mathcal{A}}$ , as action indexes, is indeed to model communication. First of all, we remark that CCS has handshaking communication, which means that only two parallel components can engage into a communication. As a consequence, the behaviour of a term can be triggered by one or two variables occurring in it. Action indexes are then necessary to characterise the syntactic structure of a parallel composition. Given  $t = x \parallel u$ , we want to be able to distinguish:

- an interleaving  $\tau$ -transition triggered by  $x$ ;
- a  $\tau$ -transition triggered by a communication between  $x$  and a variable  $y$  occurring in  $u$ ;
- a  $\tau$ -transition triggered by a communication between  $x$  and a prefixed subterm of  $u$  (i.e., a subterm such that  $u \xrightarrow{\alpha} u'$  is derivable, for some action  $\alpha$  and term  $u'$ , from the SOS rules in Table 1).

Transitions of the first two kinds can be distinguished by labels, as we will have  $t \xrightarrow{(x)}_\tau c_1$  in the case of interleaving, and  $t \xrightarrow{(x,y)}_\tau c_2$  in the case of communication. In order to distinguish the interleaving transition from the communication involving a single variable, we use two actions to index the latter transition: we let  $t \xrightarrow{(x)}_{\alpha, \tau} c_3$  model that by performing an  $\alpha$ -transition (index  $\alpha$ ), variable  $x$  (label  $(x)$ ) triggers a communication in  $t$  (index  $\tau$ ) that results in configuration  $c_3$ .

The auxiliary transitions of the form  $\xrightarrow{\ell}_\rho$  are formally defined via the inference rules in Table 4, where we omitted the symmetric rules to  $(a_2)$ ,  $(a_4)$ ,  $(a_5)$  and  $(a_6)$ . Finally, given a variable  $x$  and a label  $\ell$ , we write  $x \in \ell$  if  $x$  occurs in  $\ell$ .

First of all, we can characterise the structure of target configurations.

**Lemma 8.** Let  $t$  be a CCS term and  $x$  be a variable in  $\mathcal{V}$ . If  $t \xrightarrow{\ell}_\rho c$  for some  $\rho \in \mathcal{A}_\tau \cup ((\mathcal{A} \cup \bar{\mathcal{A}}) \times \{\tau\})$  and label  $\ell$  with  $x \in \ell$ , then  $c$  is of the form  $x_\mu \parallel c'$ , for some configuration  $c'$  (possibly  $\mathbf{0}$ ), and action  $\mu \in \mathcal{A}_\tau$  (modulo axioms P0–P2 in Table 3).

The introduction of the distinguished variables  $x_\mu$  simplifies the technical results to follow. In fact they allow us to keep track of which variable triggers the behaviour of the term and by means of which action this is obtained, and they also allow us to present substitutions in an intuitive fashion. Yet, as explained in the following example, it is precisely because of substitutions (and communication) that we need to make the action  $\mu$  explicit in  $x_\mu$ .

**Example 5.** Let  $x \in \mathcal{V}$  and consider the CCS term  $x \parallel x$ . The term is a well-defined CCS term, and there is no reason why we should exclude a priori a possible synchronisation between  $x$  and itself. In fact, we can apply the auxiliary rules  $(a_1)$  and  $(a_4)$  in Table 4 and obtain that  $x \parallel x \xrightarrow{(x,x)}_\tau x_\alpha \parallel x_{\bar{\alpha}}$  because  $x \xrightarrow{(x)}_\alpha x_\alpha$  and  $x \xrightarrow{(x)}_{\bar{\alpha}} x_{\bar{\alpha}}$ . Hence, given any (closed) substitution  $\sigma$  such that  $\sigma(x) \xrightarrow{\alpha} p_1$  and  $\sigma(x) \xrightarrow{\bar{\alpha}} p_2$ , for some (closed) terms  $p_1, p_2$ , we want to be able to correctly infer that  $\sigma(x) \parallel \sigma(x) \xrightarrow{\tau} p_1 \parallel p_2$ . Since the two

occurrences of  $x$ , namely  $x_\alpha$  and  $x_{\bar{\alpha}}$ , can be distinguished by the subscripts, the substitution  $\sigma[x_\alpha \mapsto p_1, x_{\bar{\alpha}} \mapsto p_2](x_\alpha \parallel x_{\bar{\alpha}}) = p_1 \parallel p_2$  is well-defined. Without the distinction made by the subscripts, it would have not been possible to correctly define the substitution  $\sigma$  on the configuration  $c$  that is the target of  $x \parallel x \xrightarrow{(x,x)}_\tau c$ . ◀

**Lemma 9.** *Let  $t$  be CCS term and  $\sigma$  be a closed substitution. Let  $x, y \in \mathcal{V}$ .*

1. For any  $\mu \in \mathcal{A}_\tau$ , if  $\sigma(x) \xrightarrow{\mu} p$ , for some process  $p$ , and  $t \xrightarrow{(x)}_\mu c$ , for some configuration  $c \in C$ , then  $\sigma(t) \xrightarrow{\mu} \sigma[x_\mu \mapsto p](c)$ .
2. For any  $\alpha \in \mathcal{A} \cup \bar{\mathcal{A}}$ , if  $\sigma(x) \xrightarrow{\alpha} p$ , for some process  $p$ , and  $t \xrightarrow{(x)}_{\alpha, \tau} c$ , for some configuration  $c \in C$ , then  $\sigma(t) \xrightarrow{\tau} \sigma[x_\alpha \mapsto p](c)$ .
3. For any  $\alpha \in \mathcal{A} \cup \bar{\mathcal{A}}$ , if  $\sigma(x) \xrightarrow{\alpha} p_x$ ,  $\sigma(y) \xrightarrow{\bar{\alpha}} p_y$ , for some processes  $p_x, p_y$ , and  $t \xrightarrow{(x,y)}_\tau c \in C$ , for some configuration  $c$ , then  $\sigma(t) \xrightarrow{\tau} \sigma[x_\alpha \mapsto p_x, y_{\bar{\alpha}} \mapsto p_y](c)$ .

**Proof.** The proof of each statement is by induction on the derivation of  $t \xrightarrow{(x)}_\mu c$ ,  $t \xrightarrow{(x)}_{\alpha, \tau} c$ , and  $t \xrightarrow{(x,y)}_\tau c$ , respectively. ◻

In Lemma 9 we showed how, given the behaviour of  $\sigma(x)$ , we can use auxiliary transitions to derive the behaviour of  $\sigma(t)$ . We are now interested in the converse situation: given the behaviour of  $\sigma(t)$ , we show how the behaviour of  $t$ , expressed in terms of open or auxiliary transitions, can be derived from it. For simplicity, we consider the case of visible actions  $\sigma(t) \xrightarrow{\alpha} p$  separately from the case of silent actions  $\sigma(t) \xrightarrow{\tau} p$  (respectively, Lemma 10 and Lemma 11 below), since the latter requires a more detailed analysis.

**Lemma 10.** *Let  $\alpha \in \mathcal{A} \cup \bar{\mathcal{A}}$ ,  $t$  be a CCS term,  $\sigma$  be a closed substitution, and  $p$  be a process. If  $\sigma(t) \xrightarrow{\alpha} p$ , then one of the following holds:*

1. There is a CCS term  $t'$  such that  $t \xrightarrow{\alpha} t'$  and  $\sigma(t') = p$ .
2. There are a variable  $x$ , a process  $q$  and a configuration  $c$  such that  $\sigma(x) \xrightarrow{\alpha} q$ ,  $t \xrightarrow{(x)}_\alpha c$ , and  $\sigma[x_\alpha \mapsto q](c) = p$ .

**Proof.** The proof follows by structural induction over  $t$ . ◻

**Lemma 11.** *Let  $t$  be a CCS term,  $\sigma$  be a closed substitution, and  $p$  be a process. If  $\sigma(t) \xrightarrow{\tau} p$ , then one of the following holds:*

1. There is a CCS term  $t'$  such that  $t \xrightarrow{\tau} t'$  and  $\sigma(t') = p$ .
2. There are a variable  $x$ , a process  $q$ , and a configuration  $c$  such that  $\sigma(x) \xrightarrow{\tau} q$ ,  $t \xrightarrow{(x)}_\tau c$ , and  $\sigma[x_\tau \mapsto q](c) = p$ .
3. There are a variable  $x$ , a process  $q$ , a subterm  $u$  of  $t$  and a configuration  $c$  such that, for some  $\alpha \in \mathcal{A} \cup \bar{\mathcal{A}}$ ,  $\sigma(x) \xrightarrow{\alpha} q$ ,  $u \xrightarrow{\bar{\alpha}} u' t \xrightarrow{(x)}_{\alpha, \tau} c$ , and  $\sigma[x_\alpha \mapsto q](c) = p$ .
4. There are variables  $x, y$ , processes  $q_x, q_y$  and a configuration  $c$  such that, for some  $\alpha \in \mathcal{A} \cup \bar{\mathcal{A}}$ ,  $\sigma(x) \xrightarrow{\alpha} q_x$ ,  $\sigma(y) \xrightarrow{\bar{\alpha}} q_y$ ,  $t \xrightarrow{(x,y)}_\tau c$ , and  $\sigma[x_\alpha \mapsto q_x, y_{\bar{\alpha}} \mapsto q_y](c) = p$ .

**Proof.** The proof follows by induction on the derivation of the transition  $\sigma(t) \xrightarrow{\tau} p$ . The interesting case is the base case corresponding to a communication in the term  $t = t_1 \parallel t_2$ , for some CCS terms  $t_1, t_2$ , that we expand below.

We can assume, without loss of generality, that  $\sigma(t_1) \xrightarrow{\alpha} p_1$  and  $\sigma(t_2) \xrightarrow{\bar{\alpha}} p_2$ , for some processes  $p_1, p_2$  such that  $p_1 \parallel p_2 = p$ . The symmetric case of  $\sigma(t_1) \xrightarrow{\bar{\alpha}}$  and  $\sigma(t_2) \xrightarrow{\alpha}$  follows by applying the same arguments we use below, switching the roles of  $\sigma(t_1)$  and  $\sigma(t_2)$ . By Lemma 10, from  $\sigma(t_1) \xrightarrow{\alpha} p_1$  we can infer that either  $t_1 \xrightarrow{\alpha} t'_1$  for some term  $t'_1$  such that  $\sigma(t'_1) = p_1$ , or there are a variable  $x$ , a process  $q_1$  and a configuration  $c_1$  such that  $\sigma(x) \xrightarrow{\alpha} q_1$ ,  $t_1 \xrightarrow{(x)}_\alpha c_1$ , and  $\sigma[x_\alpha \mapsto q_1](c_1) = p_1$ . Similarly, by Lemma 10, from  $\sigma(t_2) \xrightarrow{\bar{\alpha}} p_2$  we can infer that either  $t_2 \xrightarrow{\bar{\alpha}} t'_2$  for some term  $t'_2$  such that  $\sigma(t'_2) = p_2$ , or there are a variable  $y$ , a process  $q_2$  and a configuration  $c_2$  such that  $\sigma(y) \xrightarrow{\bar{\alpha}} q_2$ ,  $t_2 \xrightarrow{(y)}_{\bar{\alpha}} c_2$ , and  $\sigma[y_{\bar{\alpha}} \mapsto q_2](c_2) = p_2$ . We can then distinguish four cases, depending on how the possible derivations of the transitions of  $t_1$  and  $t_2$  mentioned above are combined in the derivation of the synchronisation.

1. Case  $t_1 \xrightarrow{\alpha} t'_1$  and  $t_2 \xrightarrow{\bar{\alpha}} t'_2$ .

In this case, the operational semantics of  $\parallel$  allows us to directly infer that  $t_1 \parallel t_2 \xrightarrow{\tau} t'_1 \parallel t'_2$ . Hence we have that there is  $t' = t'_1 \parallel t'_2$  such that  $t \xrightarrow{\tau} t'$  and  $\sigma(t') = p_1 \parallel p_2 = p$ .

2. Case  $\sigma(x) \xrightarrow{\alpha} q_1$ ,  $t_1 \xrightarrow{(x)}_\alpha c_1$ ,  $\sigma[x_\alpha \mapsto q_1](c_1) = p_1$ , and  $t_2 \xrightarrow{\bar{\alpha}} t'_2$ .

- As  $t_1 \xrightarrow{\alpha} c_1$  and  $t_2 \xrightarrow{\bar{\alpha}} t'_2$  we can apply the auxiliary rule ( $a_5$ ) and obtain that  $t_1 \parallel t_2 \xrightarrow{\alpha, \tau} c_1 \parallel t'_2$ . Hence we have that there are a variable  $x$ , a subterm  $t_2$  of  $t$ , a process  $q_1$  and a configuration  $c = c_1 \parallel t'_2$  such that  $\sigma(x) \xrightarrow{\alpha} q_1$ ,  $t_2 \xrightarrow{\bar{\alpha}} t'_2$ ,  $t \xrightarrow{\alpha, \tau} c$  and  $\sigma[x_\alpha \mapsto q_1](c) = \sigma[x_\alpha \mapsto q_1](c_1) \parallel \sigma(t'_2) = p_1 \parallel p_2 = p$ .
3. Case  $t_1 \xrightarrow{\alpha} t'_1$ ,  $\sigma(y) \xrightarrow{\bar{\alpha}} q_2$ ,  $t_2 \xrightarrow{\bar{\alpha}} c_2$ , and  $\sigma[y_\alpha \mapsto q_2](c_2) = p_2$ .  
This is the symmetrical of the previous case and it follows by applying the same reasoning used above, using the auxiliary rule ( $a_6$ ) in place of ( $a_5$ ).
4. Case  $\sigma(x) \xrightarrow{\alpha} q_1$ ,  $t_1 \xrightarrow{\alpha} c_1$ ,  $\sigma[x_\alpha \mapsto q_1](c_1) = p_1$ ,  $\sigma(y) \xrightarrow{\bar{\alpha}} q_2$ ,  $t_2 \xrightarrow{\bar{\alpha}} c_2$ , and  $\sigma[y_\alpha \mapsto q_2](c_2) = p_2$ .  
As  $t_1 \xrightarrow{\alpha} c_1$  and  $t_2 \xrightarrow{\bar{\alpha}} c_2$ , we can apply the auxiliary rule ( $a_4$ ) and obtain thus  $t_1 \parallel t_2 \xrightarrow{\alpha, \tau} c_1 \parallel c_2$ . Hence we have obtained that there are variables  $x, y$ , processes  $q_1, q_2$  and a configuration  $c = c_1 \parallel c_2$  such that  $\sigma(x) \xrightarrow{\alpha} q_1$ ,  $\sigma(y) \xrightarrow{\bar{\alpha}} q_2$ ,  $t \xrightarrow{\alpha, \tau} c$ , and  $\sigma[x_\alpha \mapsto q_1, y_\alpha \mapsto q_2](c) = \sigma[x_\alpha \mapsto q_1](c_1) \parallel \sigma[y_\alpha \mapsto q_2](c_2) = p_1 \parallel p_2 = p$ , where we can distribute the substitutions over  $\parallel$  since  $x_\alpha$  and  $y_\alpha$  are unique by construction (even if  $x = y$ , the two subscripts allow us to distinguish them).  $\square$

## 5. Unique parallel decomposition

As explained in Section 3, our approach for establishing that the property  $\mathbb{P}_n$  is preserved by equational proofs relies on processes having a unique parallel decomposition modulo branching bisimilarity, and modulo weak bisimilarity.

**Definition 7** (*Parallel decomposition modulo  $\sim_X$* ). Let  $X \in \{\text{BB}, \text{WB}\}$ . A process  $p$  is *indecomposable* if  $p \sim_X \mathbf{0}$  and  $p \sim_X p_1 \parallel p_2$  implies  $p_1 \sim_X \mathbf{0}$  or  $p_2 \sim_X \mathbf{0}$ , for all processes  $p_1$  and  $p_2$ . A *parallel decomposition* of a process  $p$  is a finite multiset  $\{p_1, \dots, p_k\}$  of indecomposable processes  $p_1, \dots, p_k$  such that  $p \sim_X p_1 \parallel \dots \parallel p_k$ . We say that  $p$  has a *unique parallel decomposition* if  $p$  has a parallel decomposition  $\{p_1, \dots, p_k\}$  and for every other parallel decomposition  $\{p'_1, \dots, p'_\ell\}$  of  $p$  there exists a bijection  $f : \{1, \dots, k\} \rightarrow \{1, \dots, \ell\}$  such that  $p_i \sim_X p'_{f(i)}$  for all  $1 \leq i \leq k$ .

To prove that processes have a unique parallel decomposition we shall exploit a general result stating that a partial commutative monoid has unique decomposition if it can be endowed with a *weak decomposition order* that satisfies *power cancellation* [28, Theorem 34]; we shall define and explain the notions below. Note that, for  $X \in \{\text{BB}, \text{WB}\}$ , in view of axioms P0–P2, which are sound modulo  $\sim_X$ , the set of processes  $\mathbf{P}$  modulo  $\sim_X$  is a commutative monoid with respect to the binary operation naturally induced by  $\parallel$  on  $\sim_X$ -equivalence classes that has the  $\sim_X$ -equivalence class of  $\mathbf{0}$  as identity element. We permit ourselves a minor abuse in notation and use  $\rightarrow$  to (also) denote the binary relation  $\{(p, q) \mid \exists \mu. p \xrightarrow{\mu} q\}$ , and proceed to argue that  $\rightarrow$  induces on the commutative monoid of processes modulo  $\sim_X$  a weak decomposition order satisfying power cancellation. We present in detail only the case of branching bisimilarity. The technical development of the result for weak bisimilarity is similar (if not simpler), and it is therefore omitted.

Given any process  $p$  and  $n \geq 1$ , let  $p^n$  denote the  $n$ -fold parallel composition  $p \parallel p^{n-1}$ , with  $p^0 = \mathbf{0}$ . We first state some properties of the reflexive-transitive closure  $\rightarrow^*$  of  $\rightarrow$ :

**Proposition 1.** *The relation  $\rightarrow^*$  is an inversely well-founded partial order on processes satisfying the following properties:*

1. for every process  $p$  there exists a process  $p'$  such that  $p \rightarrow^* p' \sim_{\text{BB}} \mathbf{0}$ ;
2. for all processes  $p, p'$  and  $q$ , if  $p \rightarrow^* p'$ , then  $p \parallel q \rightarrow^* p' \parallel q$  and  $q \parallel p \rightarrow^* q \parallel p'$ ;
3. for all processes  $p, q$  and  $r$ , if  $p \parallel q \rightarrow^* r$ , then there exist  $p'$  and  $q'$  such that  $p \rightarrow^* p'$ ,  $q \rightarrow^* q'$  and  $r = p' \parallel q'$ ;
4. for all processes  $p$  and  $q$ , if  $p \rightarrow^* q^n$  for all  $n \in \mathbb{N}$ , then  $q \sim_{\text{BB}} \mathbf{0}$ .

**Proof.** Clearly,  $\rightarrow^*$  is reflexive and transitive by definition, and since  $p \xrightarrow{\mu} p'$  implies  $\text{size}(p) > \text{size}(p')$  it follows that  $\rightarrow^*$  is inversely well-founded. Every inversely well-founded reflexive and transitive relation is clearly also anti-symmetric, so  $\rightarrow^*$  is an inversely well-founded partial order.

By inverse well-foundedness, for every process  $p$  there exists a process  $p'$  such that  $p \rightarrow^* p'$  and  $p' \nrightarrow$ ; from  $p' \nrightarrow$  it follows that  $p' \sim_{\text{BB}} \mathbf{0}$ . Hence  $\rightarrow^*$  satisfies property 1.

That  $\rightarrow^*$  satisfies property 2 is an immediate consequence of the two rules at the bottom left of Table 1.

That  $\rightarrow^*$  satisfies property 3 is straightforwardly established with induction on the length of a transition sequence witnessing  $p \parallel q \rightarrow^* r$ , using that the last SOS rule applied in the derivation of each individual transition must be one of the three rules for  $\parallel$ .

To see that  $p \rightarrow^* q^n$  for all  $n \in \mathbb{N}$  implies  $q \sim_{\text{BB}} \mathbf{0}$ , note that  $p \rightarrow^* q^n$  implies  $\text{depth}(p) \geq \text{depth}(q^n)$  and  $\text{depth}(q^n) = n \cdot \text{depth}(q)$ , from which it follows that  $\text{depth}(q) = 0$  and hence  $q \sim_{\text{BB}} \mathbf{0}$ . This proves that  $\rightarrow^*$  satisfies property 4.  $\square$

The following lemma is a straightforward consequence of the definition of branching bisimilarity.

**Lemma 12.** *For all processes  $p, p'$  and  $q$ , if  $p \sim_{\text{BB}} q$  and  $p \rightarrow^* p'$ , then there exists  $q'$  such that  $q \rightarrow^* q'$  such that  $p' \sim_{\text{BB}} q'$ .*

By this lemma we can define a binary relation  $\leq$  on  $\mathbf{P}/\sim_{\text{BB}}$ , the set of  $\sim_{\text{BB}}$ -equivalence classes of processes by stating that  $[p]_{\sim_{\text{BB}}} \leq [q]_{\sim_{\text{BB}}}$  if, and only if, there exists  $p' \in [p]_{\sim_{\text{BB}}}$  such that  $q \rightarrow^* p'$  (here  $[p]_{\sim_{\text{BB}}}$  and  $[q]_{\sim_{\text{BB}}}$  denote the  $\sim_{\text{BB}}$ -equivalence classes of  $p$  and  $q$ , respectively). The following result is then a straightforward corollary of Proposition 1.

**Corollary 1.** *The relation  $\leq$  is a weak decomposition order on  $\mathbf{P}/\sim_{\text{BB}}$ , namely:*

1. *it is well-founded, i.e., every non-empty subset of  $\mathbf{P}/\sim_{\text{BB}}$  has a  $\leq$ -minimal element;*
2. *the identity element  $[\mathbf{0}]_{\sim_{\text{BB}}}$  of  $\mathbf{P}/\sim_{\text{BB}}$  is the least element of  $\mathbf{P}/\sim_{\text{BB}}$  with respect to  $\leq$ , i.e.,  $[\mathbf{0}]_{\sim_{\text{BB}}} \leq [p]_{\sim_{\text{BB}}}$  for all  $p \in \mathbf{P}$ ;*
3. *it is compatible (with respect to  $\parallel$ ), i.e., for all  $p, q, r \in \mathbf{P}$  if  $[p]_{\sim_{\text{BB}}} \leq [q]_{\sim_{\text{BB}}}$ , then  $[p \parallel r]_{\sim_{\text{BB}}} \leq [q \parallel r]_{\sim_{\text{BB}}}$ ;*
4. *it is precompositional, i.e., for all  $p, q, r \in \mathbf{P}$  we have that  $[p]_{\sim_{\text{BB}}} \leq [q \parallel r]_{\sim_{\text{BB}}}$  implies  $[p]_{\sim_{\text{BB}}} = [q' \parallel r']_{\sim_{\text{BB}}}$  for some  $[q']_{\sim_{\text{BB}}} \leq [q]_{\sim_{\text{BB}}}$  and  $[r']_{\sim_{\text{BB}}} \leq [r]_{\sim_{\text{BB}}}$ ; and*
5. *it is Archimedean, i.e., for all  $p, q \in \mathbf{P}$  we have that  $[p^n]_{\sim_{\text{BB}}} \leq [q]_{\sim_{\text{BB}}}$  for all  $n \in \mathbb{N}$  implies that  $[p]_{\sim_{\text{BB}}} = [\mathbf{0}]_{\sim_{\text{BB}}}$ .*

According to the result in [28] it now remains to prove that  $\leq$  satisfies power cancellation. We shall proceed in more detail for this property. The weak decomposition order  $\leq$  on the commutative monoid of processes modulo  $\sim_{\text{BB}}$  satisfies *power cancellation* if for every indecomposable process  $p$  and for all processes  $q$  and  $r$  such that  $[p]_{\sim_{\text{BB}}} \not\leq [q]_{\sim_{\text{BB}}}, [r]_{\sim_{\text{BB}}}$  and for all  $k \in \mathbb{N}$  we have that  $[p^k \parallel q]_{\sim_{\text{BB}}} = [p^k \parallel r]_{\sim_{\text{BB}}}$  implies  $[q]_{\sim_{\text{BB}}} = [r]_{\sim_{\text{BB}}}$ .

**Proposition 2.** *The weak decomposition order  $\leq$  on the commutative monoid of processes modulo  $\sim_{\text{BB}}$  satisfies power cancellation.*

**Proof.** Let  $p$  be an indecomposable process and let  $q$  and  $r$  be processes such that  $[p]_{\sim_{\text{BB}}} \not\leq [q]_{\sim_{\text{BB}}}, [r]_{\sim_{\text{BB}}}$ , and suppose that  $[p^k \parallel q]_{\sim_{\text{BB}}} = [p^k \parallel r]_{\sim_{\text{BB}}}$  for some  $k \in \mathbb{N}$ . We need to prove that  $[q]_{\sim_{\text{BB}}} = [r]_{\sim_{\text{BB}}}$ , and for this it suffices to argue that the relation

$$\mathcal{R} = \{(q, r)\} \cup \sim_{\text{BB}}$$

is a branching bisimulation relation.

The weak decomposition order  $\leq$ , since it is well-founded, induces a well-founded order on triples of  $\sim_{\text{BB}}$ -equivalence classes of processes, defining  $([p_1]_{\sim_{\text{BB}}}, [p_2]_{\sim_{\text{BB}}}, [p_3]_{\sim_{\text{BB}}}) \leq ([q_1]_{\sim_{\text{BB}}}, [q_2]_{\sim_{\text{BB}}}, [q_3]_{\sim_{\text{BB}}})$  if, and only if,  $[p_1]_{\sim_{\text{BB}}} \leq [q_1]_{\sim_{\text{BB}}}$  and whenever  $[p_1]_{\sim_{\text{BB}}} = [q_1]_{\sim_{\text{BB}}}$ , then also  $[p_2]_{\sim_{\text{BB}}} \leq [q_2]_{\sim_{\text{BB}}}$  and  $[p_3]_{\sim_{\text{BB}}} \leq [q_3]_{\sim_{\text{BB}}}$ . Our proof is by induction on the well-founded order  $\leq$  on triples of  $\sim_{\text{BB}}$ -equivalence classes of processes: we assume, by way of induction hypothesis, that for all  $p', q'$  and  $r'$  such that

$$((p')^k \parallel q')_{\sim_{\text{BB}}}, [q']_{\sim_{\text{BB}}}, [r']_{\sim_{\text{BB}}}) < ((p^k \parallel q)_{\sim_{\text{BB}}}, [q]_{\sim_{\text{BB}}}, [r]_{\sim_{\text{BB}}})$$

we have that  $(p')^k \parallel q' \sim_{\text{BB}} (p')^k \parallel r'$  implies  $q' \sim_{\text{BB}} r'$ , and establish the property for  $p, q$  and  $r$ .

Since  $q \rightarrow q'$  implies  $\text{size}(q) > \text{size}(q')$ , and hence  $\rightarrow$  is terminating, we may, without loss of generality, choose  $q$  such that there does not exist  $q'$  such that  $q \rightarrow q'$  and  $q \sim_{\text{BB}} q'$ , and similarly we may, without loss of generality, choose  $r$  such that there does not exist an  $r' \neq r$  such that  $r \rightarrow r'$  and  $r \sim_{\text{BB}} r'$ .

Since  $\sim_{\text{BB}}$  is the largest branching bisimulation, it is immediate that all pairs in  $\sim_{\text{BB}}$  satisfy the conditions of branching bisimulations. We do need to establish that the pair  $(q, r)$  also satisfies these conditions, and to this end suppose that  $q \xrightarrow{\mu} q'$  for some  $q'$ ; we establish that either  $\mu = \tau$  and  $q' \mathcal{R} r$ , or there exist processes  $r'$  and  $r''$  such that  $r \xrightarrow{\epsilon} r' \xrightarrow{\mu} r''$ ,  $q \mathcal{R} r'$  and  $q' \mathcal{R} r''$ . We distinguish two cases:

1. Suppose that  $p^k \parallel q' \sim_{\text{BB}} p^k \parallel q$ . Then, since  $\sim_{\text{BB}}$  preserves depth and the depth of a parallel composition is the sum of the depths of its components, we must have that  $\text{depth}(q') = \text{depth}(q)$  and hence  $\mu = \tau$ . Now, by the choice of  $q$  it follows that  $[q']_{\sim_{\text{BB}}} < [q]_{\sim_{\text{BB}}}$ . Moreover, from  $p^k \parallel q' \sim_{\text{BB}} p^k \parallel q$  it follows that  $[p^k \parallel q']_{\sim_{\text{BB}}} = [p^k \parallel q]_{\sim_{\text{BB}}}$ . Thus we find that

$$([p^k \parallel q']_{\sim_{\text{BB}}}, [q']_{\sim_{\text{BB}}}, [r]_{\sim_{\text{BB}}}) < ([p^k \parallel q]_{\sim_{\text{BB}}}, [q]_{\sim_{\text{BB}}}, [r]_{\sim_{\text{BB}}}) ,$$

so by the induction hypothesis  $q' \sim_{\text{BB}} r$ , and hence  $q' \mathcal{R} r$ .

2. Suppose that  $p^k \parallel q' \not\sim_{\text{BB}} p^k \parallel q$ . Then  $[p^k \parallel q']_{\sim_{\text{BB}}} < [p^k \parallel q]_{\sim_{\text{BB}}}$ , and, by the induction hypothesis, the weak decomposition order  $\leq$  on the partial commutative submonoid  $\{x \mid x \leq [p^k \parallel q']_{\sim_{\text{BB}}}\}$  of  $\mathbf{P}$  modulo  $\sim_{\text{BB}}$  satisfies power cancellation. Hence, by [28, Theorem 34], if  $s$  is any process such that  $p^k \parallel q' \sim_{\text{BB}} ; \rightarrow^* s$  (where  $;$  denotes relation composition), then  $s$  has a unique parallel decomposition.

From  $q \xrightarrow{\mu} q'$  it follows that  $p^k \parallel q \xrightarrow{\mu} p^k \parallel q'$ , and hence, since  $p^k \parallel q \sim_{\text{BB}} p^k \parallel r$  there exist processes  $p', p'', r', r''$  such that

$$p^k \rightarrow^* p' \rightarrow^* p'' ,$$

$$r \rightarrow^* r' \rightarrow^* r'' ,$$

$$p^k \parallel q \sim_{\text{BB}} p' \parallel r' , \text{ and}$$

$$p^k \parallel q' \sim_{\text{BB}} p'' \parallel r'' .$$

Moreover, from  $p'' \parallel r'' \sim_{\text{BB}} p^k \parallel q' \sim_{\text{BB}} p^k \parallel q \sim_{\text{BB}} p^k \parallel r$  it follows that either  $p'' \sim_{\text{BB}} p^k$  or  $r'' \sim_{\text{BB}} r$ ; we distinguish these two subcases:

- (a) Suppose  $r'' \sim_{\text{BB}} r$ . Then, since  $[r'']_{\sim_{\text{BB}}} < [r]_{\sim_{\text{BB}}}$  and  $[p]_{\sim_{\text{BB}}} \not\prec [r]_{\sim_{\text{BB}}}$ , the unique parallel decomposition of  $r''$  cannot have occurrences of a process branching bisimilar to  $p$ . Since the unique decomposition of  $p^k \parallel q'$  has at least  $k$  occurrences of a process branching bisimilar to  $p$ , it follows that  $[p^k]_{\sim_{\text{BB}}} \leq [p'']_{\sim_{\text{BB}}} \leq [p']_{\sim_{\text{BB}}} \leq [p^k]_{\sim_{\text{BB}}}$ , so  $p' \sim_{\text{BB}} p'' \sim_{\text{BB}} p^k$ . From  $p^k \parallel q' \sim_{\text{BB}} p^k \parallel r''$  it then follows by the induction hypothesis that  $q' \sim_{\text{BB}} r''$ , and hence  $q' \mathcal{R} r''$ . It remains to establish that  $q \mathcal{R} r'$ . If  $r' = r$ , then, since  $q \mathcal{R} r$  this is immediate. If  $r' \neq r$ , then by the choice of  $r$  such that there does not exist  $r'$  such that  $r \rightarrow r'$  and  $r \sim_{\text{BB}} r'$ , it follows that  $[r']_{\sim_{\text{BB}}} < [r]_{\sim_{\text{BB}}}$ . Furthermore, since clearly  $p^k \parallel r \xrightarrow{\epsilon} p^k \parallel r' \xrightarrow{\epsilon} p' \parallel r'$  and  $p^k \parallel r \sim_{\text{BB}} p' \parallel r'$ , it follows by the stuttering property that  $p^k \parallel r \sim_{\text{BB}} p^k \parallel r'$ . Then  $q \sim_{\text{BB}} r'$  follows by the induction hypothesis, and hence  $q \mathcal{R} r'$ .
- (b) Suppose  $p'' \sim_{\text{BB}} p^k$ . Then  $[p'']_{\sim_{\text{BB}}} < [p^k]_{\sim_{\text{BB}}}$ , so the multiplicity of  $p$  in the unique decomposition of  $p''$  is at most  $k - 1$ . Hence, since  $p'' \parallel r'' \sim_{\text{BB}} p^k \parallel q'$ , it follows that  $p$  must be an element of the parallel decomposition of  $r''$ . This means that  $[p]_{\sim_{\text{BB}}} \leq [r'']_{\sim_{\text{BB}}} \leq [r']_{\sim_{\text{BB}}} \leq [r]_{\sim_{\text{BB}}}$  while at the same time  $[p]_{\sim_{\text{BB}}} \not\prec [r]_{\sim_{\text{BB}}}$  by assumption. It follows that

$$p \sim_{\text{BB}} r \sim_{\text{BB}} r' \sim_{\text{BB}} r'' .$$

Hence, since  $r''$  can contribute at most 1 to the multiplicity of  $p$  in the unique parallel decomposition of  $p'' \parallel r''$ , while the multiplicity of  $p$  is at least  $k$ , it now follows that the multiplicity of  $p$  in the parallel decomposition of  $p''$  must be  $k - 1$ . So we can assume without loss of generality that there exist processes  $p_1, \dots, p_k$  and  $p'_1$  such that

$$\begin{aligned} p' &= p_1 \parallel p_2 \parallel \dots \parallel p_k , \\ p'' &= p'_1 \parallel p_2 \parallel \dots \parallel p_k , \\ p &\xrightarrow{\epsilon} p_i \quad (1 \leq i \leq k) , \\ p &\sim_{\text{BB}} p_i \quad (2 \leq i \leq k) , \text{ and} \\ p_1 &\xrightarrow{\mu} p'_1 . \end{aligned}$$

We must have that  $p'_1 \sim_{\text{BB}} p$ , for otherwise we would have  $p'' \sim_{\text{BB}} p^k$ , contradicting the assumption in this case. Therefore, since  $p_1 \sim_{\text{BB}} p \sim_{\text{BB}} r$ , there exist  $r'_1$  and  $r''_1$  such that  $r \xrightarrow{\epsilon} r'_1 \xrightarrow{\mu} r''_1$  with  $p \sim_{\text{BB}} r'_1$  and  $p'_1 \sim_{\text{BB}} r''_1$ . By our choice of  $r$  such that there does not exist an  $r'_1 \neq r$  such that  $r \rightarrow r'_1$  and  $r \sim_{\text{BB}} r'_1$ , it is immediate that  $r = r'_1$  and hence  $q \mathcal{R} r'$ . Moreover, from  $p^k \parallel q' \sim_{\text{BB}} r \parallel p_2 \parallel \dots \parallel p_k \parallel p'_1$ ,  $p \sim_{\text{BB}} r$  and  $p \sim_{\text{BB}} p_i$  ( $2 \leq i \leq k$ ) it follows that  $q' \sim_{\text{BB}} p'_1 \sim_{\text{BB}} r''_1$ , so  $q' \mathcal{R} r''_1$ .  $\square$

We have now established that  $\leq$  is a weak decomposition order on the commutative monoid of processes modulo  $\sim_{\text{BB}}$  that satisfies power cancellation. Thus, by using [28, Theorem 34], we obtain the following unique parallel decomposition result.

**Proposition 3.** *Every process in  $\mathbf{P} / \sim_{\text{BB}}$  has a unique parallel decomposition.*

As shown in [28], we can apply the same strategy to prove that  $\leq$  is a weak decomposition order on the commutative monoid of processes modulo  $\sim_{\text{WB}}$  that satisfies power cancellation.

The unique parallel composition result for  $\sim_{\text{WB}}$  then follows from an application of [28, Theorem 34].

**Proposition 4.** *Every process in  $\mathbf{P} / \sim_{\text{WB}}$  has a unique parallel decomposition.*

In what follows, we shall make use of the following immediate consequence of Propositions 3 and 4.

**Corollary 2.** *If  $p \parallel r \sim_{\text{BB}} q \parallel r$ , then  $p \sim_{\text{BB}} q$ .*

*If  $p \parallel r \sim_{\text{WB}} q \parallel r$ , then  $p \sim_{\text{WB}} q$ .*

## 6. Saturated axiom systems

In the technical developments to follow, we can restrict ourselves to considering axiom systems that do not introduce unnecessary terms that are rooted branching bisimilar (respectively, rooted weak bisimilar) to  $\mathbf{0}$  in equational proofs, namely  $\mathbf{0}$  summands and  $\mathbf{0}$  factors. In the remainder of this section, let  $X \in \{\text{RBB}, \text{RWB}\}$ .

**Definition 8.** We say that a CCS term  $t$  has a  $\mathbf{0}$  factor if it contains a subterm of the form  $t' \parallel t''$ , where either  $t' \sim_X \mathbf{0}$  or  $t'' \sim_X \mathbf{0}$ .

An example of a collection of equations over CCS that are sound modulo  $\sim_X$  is given by axioms A0–A3, P0–P2, in Table 3. Interestingly, axioms A0 and P0 in Table 3 (used from left to right) are enough to establish that each CCS term that is  $\sim_X$  equivalent to  $\mathbf{0}$  is also provably equal to  $\mathbf{0}$ .

Before proceeding, we observe the following:

**Remark 3.** Whenever a term  $t$  has neither  $\mathbf{0}$  summands nor factors then we can assume that, for some finite (non-empty) index set  $I$ ,  $t = \sum_{i \in I} t_i$  for some terms  $t_i$  such that none of them has  $+$  as head operator and moreover, none of them has  $\mathbf{0}$  summands nor factors.

**Lemma 13.** Let  $t$  be a CCS term. Then  $t \sim_{\chi} \mathbf{0}$  if, and only if, the equation  $t \approx \mathbf{0}$  is provable using axioms A0 and P0 in Table 3 from left to right.

**Proof.** The “if” implication is an immediate consequence of the soundness of the equations A0 and P0 with respect to  $\sim_{\chi}$ . To prove the “only if” implication, define, first of all, the subset NIL of CCS terms generated by the following grammar:

$$t ::= \mathbf{0} \mid t + t \mid t \parallel t .$$

We claim that, given any CCS term  $t$ ,  $t \sim_{\chi} \mathbf{0}$  iff  $t \in \text{NIL}$ . Using this claim and structural induction on  $t \in \text{NIL}$ , it is a simple matter to show that if  $t \sim_{\chi} \mathbf{0}$ , then  $t \approx \mathbf{0}$  is provable using axioms A0 and P0 from left to right, which was to be shown.

To complete the proof, it therefore suffices to show the above claim. To establish the “if” implication one proves, using structural induction on  $t$  and the congruence properties of  $\sim_{\chi}$ , that if  $t \in \text{NIL}$ , then  $\sigma(t) \sim_{\chi} \mathbf{0}$  for every closed substitution  $\sigma$ . To show the “only if” implication, we establish the contrapositive statement, viz. that if  $t \notin \text{NIL}$ , then  $\sigma(t) \not\sim_{\chi} \mathbf{0}$  for some closed substitution  $\sigma$ . To this end, it suffices only to show, using structural induction on  $t$ , that if  $t \notin \text{NIL}$ , then  $\sigma_a(t) \xrightarrow{\mu}$  for some action  $\mu \in \mathcal{A}_\tau$ , where  $\sigma_a$  is the closed substitution mapping each variable to the closed term  $a.\mathbf{0}$ . The details of this argument are not hard, and are therefore omitted.  $\square$

It is well-known (see, e.g., Section 2 in [22]) that if an equation  $e$  relating two closed terms can be proved from an axiom system  $\mathcal{E}$ , then there is a closed proof for it. Moreover, if  $\mathcal{E}$  satisfies a further closure property, called *saturation* [32], and  $e$  relates two terms containing no occurrences of  $\mathbf{0}$  as a summand or factor, then there is a closed proof for  $e$  in which all of the terms have no occurrences of  $\mathbf{0}$  as a summand or factor [32, Proposition 5.1.5]. Specifically, saturation was introduced in [32] as a property of axiom systems that are sound modulo strong bisimilarity. Here, we rephrase and adjust the definitions and properties from [32] to cover axiom systems that are sound modulo  $\sim_{\chi}$ .

**Definition 9.** For each CCS term  $t$ , we define  $t/\mathbf{0}$  thus:

$$\begin{array}{l} \mathbf{0}/\mathbf{0} = \mathbf{0} \qquad x/\mathbf{0} = x \qquad \mu t/\mathbf{0} = \mu(t/\mathbf{0}) \\ (t + u)/\mathbf{0} = \begin{cases} u/\mathbf{0} & \text{if } t \sim_{\chi} \mathbf{0} \\ t/\mathbf{0} & \text{if } u \sim_{\chi} \mathbf{0} \\ (t/\mathbf{0}) + (u/\mathbf{0}) & \text{otherwise} \end{cases} \qquad t \parallel u/\mathbf{0} = \begin{cases} u/\mathbf{0} & \text{if } t \sim_{\chi} \mathbf{0} \\ t/\mathbf{0} & \text{if } u \sim_{\chi} \mathbf{0} \\ (t/\mathbf{0}) \parallel (u/\mathbf{0}) & \text{otherwise} \end{cases} \end{array}$$

Intuitively,  $t/\mathbf{0}$  is the term that results by removing *all* occurrences of  $\mathbf{0}$  as a summand or factor from  $t$ .

The following lemma, whose simple proof by structural induction on terms is omitted, collects the basic properties of the above construction.

**Lemma 14.** For each CCS term  $t$ , the following statements hold:

1. the equation  $t \approx t/\mathbf{0}$  can be proven using the equations A0–A3, P0–P2, and therefore  $t \sim_{\chi} t/\mathbf{0}$ ;
2. the term  $t/\mathbf{0}$  has no occurrence of  $\mathbf{0}$  as a summand or factor;
3.  $t/\mathbf{0} = t$ , if  $t$  has no occurrence of  $\mathbf{0}$  as a summand or factor;
4.  $\sigma(t/\mathbf{0})/\mathbf{0} = \sigma(t)/\mathbf{0}$ , for each substitution  $\sigma$ .

**Definition 10.** We say that a substitution  $\sigma$  is a  $\mathbf{0}$ -substitution iff  $\sigma(x) \neq x$  implies that  $\sigma(x) = \mathbf{0}$ , for each variable  $x$ .

**Definition 11.** Let  $\mathcal{E}$  be an axiom system. We define the axiom system  $\text{cl}(\mathcal{E})$  thus:

$$\text{cl}(\mathcal{E}) = \mathcal{E} \cup \{ \sigma(t)/\mathbf{0} \approx \sigma(u)/\mathbf{0} \mid (t \approx u) \in \mathcal{E}, \sigma \text{ a } \mathbf{0}\text{-substitution} \} .$$

An axiom system  $\mathcal{E}$  is *saturated* if  $\mathcal{E} = \text{cl}(\mathcal{E})$ .

The following lemma collects some basic sanity properties of the closure operator  $\text{cl}(\cdot)$ . (Note that the application of  $\text{cl}(\cdot)$  to an axiom system preserves closure with respect to symmetry.)

**Lemma 15.** Let  $\mathcal{E}$  be an axiom system. Then the following statements hold.

1.  $\text{cl}(\mathcal{E}) = \text{cl}(\text{cl}(\mathcal{E}))$ .
2.  $\text{cl}(\mathcal{E})$  is finite, if so is  $\mathcal{E}$ .
3.  $\text{cl}(\mathcal{E})$  is sound, if so is  $\mathcal{E}$ .

4.  $\text{cl}(\mathcal{E})$  is closed with respect to symmetry, if so is  $\mathcal{E}$ .
5.  $\text{cl}(\mathcal{E})$  and  $\mathcal{E}$  prove the same equations, if  $\mathcal{E}$  contains the equations A0–A3, P0–P2.

**Proof.** We limit ourselves to sketching the proofs of statements 1 and 5 in the lemma.

In the proof of statement 1, the only non-trivial thing to check is that the equation

$$\sigma(\sigma'(t)/\mathbf{0})/\mathbf{0} \approx \sigma(\sigma'(u)/\mathbf{0})/\mathbf{0}$$

is contained in  $\text{cl}(\mathcal{E})$ , whenever  $(t \approx u) \in \mathcal{E}$  and  $\sigma, \sigma'$  are  $\mathbf{0}$ -substitutions. This follows from Lemma 14(4) because the collection of  $\mathbf{0}$ -substitutions is closed under composition.

To show statement 5, it suffices only to argue that each equation  $t \approx u$  that is provable from  $\text{cl}(\mathcal{E})$  is also provable from  $\mathcal{E}$ , if  $\mathcal{E}$  contains the equations A0–A3, P0–P2. This can be done by induction on the depth of the proof of the equation  $t \approx u$  from  $\text{cl}(\mathcal{E})$ , using Lemma 14(1) for the case in which  $t \approx u$  is a substitution instance of an axiom in  $\text{cl}(\mathcal{E})$ .  $\square$

In light of this result, the saturation of a finite axiom system that includes the equations A0–A3, P0–P2 results in an equivalent, finite collection of equations (Lemma 15(2) and (5)).

We are now ready to state our counterpart of [32, Proposition 5.1.5].

**Proposition 5.** *Assume that  $\mathcal{E}$  is a saturated axiom system. Suppose furthermore that we have a closed proof from  $\mathcal{E}$  of the closed equation  $p \approx q$ . Then replacing each term  $r$  in that proof with  $r/\mathbf{0}$  yields a closed proof of the equation  $p/\mathbf{0} \approx q/\mathbf{0}$ . In particular, the proof from  $\mathcal{E}$  of an equation  $p \approx q$ , where  $p$  and  $q$  are terms not containing occurrences of  $\mathbf{0}$  as a summand or factor, need not use terms containing occurrences of  $\mathbf{0}$  as a summand or factor.*

**Proof.** The proof follows the lines of that of [32, Proposition 5.1.5], and is therefore omitted.  $\square$

In light of Proposition 5, henceforth, we shall limit ourselves to considering saturated axiom systems. This will allow us to reduce considerably the number of case distinctions in the technical proofs in the upcoming sections, as well as the complexity of their analysis.

## 7. Preliminary results for rooted branching bisimilarity

In this section we discuss some general properties of CCS terms modulo (rooted) branching bisimilarity, alongside some properties of the processes  $p_n$ ,  $a^{\leq i}$  and  $a \parallel p_n$ , introduced in Section 3, on which we will build our proof of the negative result for  $\sim_{\text{RBB}}$ , alongside. In particular, we will focus on indecomposability of those processes modulo  $\sim_{\text{BB}}$ .

First of all, we prove that whenever two terms are provably equal modulo rooted branching bisimilarity, then they have the same variables as summands.

**Proposition 6.** *Let  $t, u$  be two CCS terms without  $\mathbf{0}$  summands or factors, and let  $t \approx u$  be sound modulo  $\sim_{\text{RBB}}$ . If  $t$  has a summand  $x$ , then so does  $u$ .*

**Proof.** Observe, first of all, that since  $t$  and  $u$  have no  $\mathbf{0}$  summands or factors, by Remark 3 we can assume that  $t = \sum_{i \in I} t_i$  and  $u = \sum_{j \in J} u_j$  for some finite non-empty index sets  $I, J$ , where none of the  $t_i$  ( $i \in I$ ) and  $u_j$  ( $j \in J$ ) has  $+$  as its head operator, and none of the  $t_i$  ( $i \in I$ ) and  $u_j$  ( $j \in J$ ) have  $\mathbf{0}$  summands or factors. Therefore, if  $t$  has a summand  $x$ , then that there is some index  $i \in I$  such that  $t_i = x$ . We aim to show that  $u$  has a summand  $x$  as well. Consider the substitution  $\sigma_a$  mapping each variable to  $a.\mathbf{0}$ . Pick an integer  $m$  larger than the rooted depth of  $\sigma_a(t)$  and of  $\sigma_a(u)$ . Let  $\sigma$  be the substitution mapping  $x$  to the term  $a\bar{a}^m$  and agreeing with  $\sigma_a$  on all the other variables.

As  $t \approx u$  is sound modulo rooted branching bisimilarity, we have that  $\sigma(t) \sim_{\text{RBB}} \sigma(u)$ . Moreover, the term  $\sigma(t)$  affords the transition  $\sigma(t) \xrightarrow{a} \bar{a}^m$ , for  $t_i = x$  and  $\sigma(x) = a\bar{a}^m \xrightarrow{a} \bar{a}^m$ . Hence, for some closed term  $p$ ,

$$\sigma(u) = \sum_{j \in J} \sigma(u_j) \xrightarrow{a} p \sim_{\text{BB}} \bar{a}^m.$$

This means that there is a  $j \in J$  such that  $\sigma(u_j) \xrightarrow{a} p$ . We claim that this  $u_j$  can only be the variable  $x$ . To see that this claim holds, observe, first of all, that  $x \in \text{var}(u_j)$ . In fact, if  $x$  did not occur in  $u_j$ , then we would reach a contradiction thus:

$$\begin{aligned} m &= \text{depth}(p) && \text{(by } p \sim_{\text{BB}} \bar{a}^m \text{ and Lemma 1)} \\ &< \text{Rdepth}(\sigma(u_j)) && \text{(by } \sigma(u_j) \xrightarrow{a} p \text{ and definition of Rdepth)} \\ &= \text{Rdepth}(\sigma_a(u_j)) && \text{(by } \sigma(u_j) = \sigma_a(u_j) \text{ if } x \notin \text{var}(u_j)) \\ &\leq \text{Rdepth}(\sigma_a(u)) && \text{(by } u_j \text{ summand of } u \text{ and definition of Rdepth)} \end{aligned}$$

<  $m$ (by the choice of  $m$ ).

Using this observation and Lemma 4, it is not hard to show that, for each of the other possible forms  $u_j$  may have,  $\sigma(u_j)$  does not afford an  $a$ -labelled transition leading to a term of depth  $m$ . We may therefore conclude that  $u_j = x$ , which was to be shown.  $\square$

We now proceed to analyse the properties of the following family of processes, which will play a crucial role in the proof of our main result:

$$p_n = \sum_{i=2}^n a \cdot a^{\leq i} \quad (n \geq 2)$$

where  $a^{\leq i} = \sum_{j=1}^i a^j$ , and  $a^j = a \cdot a^{j-1}$ , with  $a^0 = \mathbf{0}$ , for each  $j \in \{1, \dots, i\}$ ,  $i \in \{2, \dots, n\}$ .

**Lemma 16.** *Let  $i \geq 2$ . Process  $a^{\leq i}$  is indecomposable modulo  $\sim_{\text{BB}}$ .*

**Proof.** Assume, towards a contradiction, that there are processes  $p, q \approx_{\text{BB}} \mathbf{0}$  such that  $a^{\leq i} \sim_{\text{BB}} p \parallel q$ . As  $a^{\leq i} \xrightarrow{a} \mathbf{0}$ , we have that  $p \parallel q \xrightarrow{\epsilon} r \xrightarrow{a} r'$  for some processes  $r, r'$  such that  $r \sim_{\text{BB}} a^{\leq i}$  and  $r' \sim_{\text{BB}} \mathbf{0}$ . Notice that  $p \approx_{\text{BB}} \mathbf{0}$  implies that there is at least one  $p' \in \text{der}(p)$  such that  $p' \xrightarrow{\mu} p''$  for some process  $p''$  and action  $\mu \neq \tau$ . A similar property holds for  $q$ . Since, moreover,  $a^{\leq i}$  can only perform sequences of  $a$ -moves,  $p \parallel q \sim_{\text{BB}} a^{\leq i}$  implies that also  $p$  and  $q$  can perform (weak) sequences of  $a$ -moves. In particular, it follows that there is no derivative of  $p$  or  $q$  that can perform action  $\bar{a}$ . As a consequence,  $p$  and  $q$  cannot synchronise. This implies that there are processes  $p', q'$  such that  $p \xrightarrow{\epsilon} p', q \xrightarrow{\epsilon} q'$  and  $r = p' \parallel q'$ . Then  $r \xrightarrow{a} r'$  can follow either from  $p' \xrightarrow{a} p''$ , or  $q' \xrightarrow{a} q''$ , for some  $p'', q''$ . Assume, without loss of generality, that  $p' \xrightarrow{a} p''$  and  $r' = p'' \parallel q'$  (the case  $q' \xrightarrow{a} q''$  is analogous). Now  $p'' \parallel q' \sim_{\text{BB}} \mathbf{0}$  implies  $p'' \sim_{\text{BB}} \mathbf{0}$  and  $q' \sim_{\text{BB}} \mathbf{0}$ . Hence,  $p' \sim_{\text{BB}} p' \parallel q' = r \sim_{\text{BB}} a^{\leq i}$  and, thus,  $\text{depth}(p) \geq i$ . Since  $q \approx_{\text{BB}} \mathbf{0}$  gives  $\text{depth}(q) \geq 1$ , we get that  $\text{depth}(p \parallel q) > i$ . This contradicts our assumption that  $p \parallel q \sim_{\text{BB}} a^{\leq i}$ , since  $\text{depth}(a^{\leq i}) = i \neq \text{depth}(p \parallel q)$  (cf. Lemma 1).  $\square$

**Remark 4.** We could not directly infer that  $a^{\leq i}$  is indecomposable from the fact that the norm of the process is 1 (where the norm of a process is the length of the shortest maximal observable trace that it can perform). This is due to the fact that we are considering the full merge operator. Consider, for instance, the process  $\tau \cdot a + \alpha \cdot (a \cdot \bar{a} + \bar{a} \cdot a) + \bar{a} \cdot \alpha \cdot a$ . This process has norm 1, yet it is not indecomposable since it is branching bisimilar to  $\alpha \cdot a \parallel \bar{a}$ . Similarly, the process  $\tau + \bar{a} \cdot \alpha + \alpha \cdot \bar{a}$  has norm 1, but it is (rooted) branching bisimilar to  $\alpha \parallel \bar{a}$ .

**Lemma 17.** *Let  $n \geq 2$ . Process  $p_n$  is indecomposable modulo  $\sim_{\text{BB}}$ .*

**Proof.** Assume, towards a contradiction, that there are processes  $p, q \approx_{\text{BB}} \mathbf{0}$  such that  $p_n \sim_{\text{BB}} p \parallel q$ . Notice that  $p \approx_{\text{BB}} \mathbf{0}$  implies that there is at least one  $p' \in \text{der}(p)$  such that  $p' \xrightarrow{\mu} p''$  for some process  $p''$  and action  $\mu \neq \tau$ . A similar property holds for  $q$ . Given any  $i \in \{2, \dots, n\}$ , we have that  $p_n \xrightarrow{a} a^{\leq i}$ . Hence, from  $p_n \sim_{\text{BB}} p \parallel q$ , we get that  $p \parallel q \xrightarrow{\epsilon} r \xrightarrow{a} r'$  for some processes  $r, r'$  such that  $r \sim_{\text{BB}} p_n$  and  $r' \sim_{\text{BB}} a^{\leq i}$ . We remark that, since  $p_n$  can only perform sequences of  $a$ -moves,  $p \parallel q \sim_{\text{BB}} p_n$  implies that also  $p$  and  $q$  can perform (weak) sequences of  $a$ -moves. In particular, it follows that there is no derivative of  $p$  or  $q$  that can perform action  $\bar{a}$ . As a consequence,  $p$  and  $q$  cannot synchronise. We can then distinguish four cases, according to how the sequence of transitions  $p \parallel q \xrightarrow{\epsilon} r \xrightarrow{a} r'$  is derived:

- $p \parallel q \xrightarrow{\epsilon} p' \parallel q \xrightarrow{a} p'' \parallel q$ , with  $p' \parallel q \sim_{\text{BB}} p_n$  and  $p'' \parallel q \sim_{\text{BB}} a^{\leq i}$ . Since  $q \approx_{\text{BB}} \mathbf{0}$  and  $a^{\leq i}$  is indecomposable by Lemma 16, we get that  $p'' \sim_{\text{BB}} \mathbf{0}$  and  $q \sim_{\text{BB}} a^{\leq i}$ . Hence, given any  $j \in \{0, \dots, i-1\}$ ,  $a^{\leq i} \xrightarrow{a} a^j$  implies that  $q \xrightarrow{\epsilon} q' \xrightarrow{a} q''$  for some  $q', q''$  such that  $q' \sim_{\text{BB}} a^{\leq i}$  and  $q'' \sim_{\text{BB}} a^j$ . Therefore,  $p \parallel q \xrightarrow{\epsilon} p'' \parallel q' \xrightarrow{a} p'' \parallel q''$  and  $p_n \sim_{\text{BB}} p \parallel q$  imply that  $p_n \xrightarrow{a} r$  for some process  $r$  such that  $r \sim_{\text{BB}} q'' \sim_{\text{BB}} a^j$  ( $p_n$  cannot perform any  $\tau$ -move, hence it must match any initial sequence of  $\tau$ -transitions by doing nothing). However,  $p_n \xrightarrow{a} a^{\leq k}$ , for any  $k \in \{2, \dots, n\}$ , and there are no  $j, k$  such that  $a^j \sim_{\text{BB}} a^{\leq k}$ , thus giving a contradiction with  $p_n \sim_{\text{BB}} p \parallel q$ .
- $p \parallel q \xrightarrow{\epsilon} p \parallel q' \xrightarrow{a} p \parallel q''$ , with  $p \parallel q' \sim_{\text{BB}} p_n$  and  $p \parallel q'' \sim_{\text{BB}} a^{\leq i}$ . This case is symmetric to the previous one and therefore omitted.
- $p \parallel q \xrightarrow{\epsilon} p' \parallel q' \xrightarrow{a} p'' \parallel q'$ , with  $p' \parallel q' \sim_{\text{BB}} p_n$  and  $p'' \parallel q' \sim_{\text{BB}} a^{\leq i}$ . Since  $a^{\leq i}$  is indecomposable (Lemma 16), we can distinguish two cases:
  - $p'' \sim_{\text{BB}} \mathbf{0}$  and  $q' \sim_{\text{BB}} a^{\leq i}$ . Then we can proceed as in the previous case, and show that  $p \parallel q \xrightarrow{\epsilon} p \parallel q' \xrightarrow{a} a^j$ , for some  $j \leq i-1$ , and thus  $p_n \sim_{\text{BB}} p \parallel q$ .
  - $p'' \sim_{\text{BB}} a^{\leq i}$  and  $q' \sim_{\text{BB}} \mathbf{0}$ . In this case we have that  $p \parallel q \xrightarrow{\epsilon} p' \parallel q' \xrightarrow{a} p'' \parallel q'$ . Hence, since  $p_n$  cannot perform any  $\tau$ -move,  $p_n \sim_{\text{BB}} p \parallel q$  requires that  $p_n \xrightarrow{a} a^{\leq k}$  for some  $k \in \{2, \dots, n\}$  such that  $a^{\leq k} \sim_{\text{BB}} a^{\leq i} \parallel q$ . As  $a^{\leq k}$  is indecomposable (Lemma 16),  $a^{\leq i} \approx_{\text{BB}} \mathbf{0}$  for any  $i \in \{2, \dots, n\}$ , and by the proviso of the lemma  $q \approx_{\text{BB}} \mathbf{0}$ , there is no index  $k \in \{2, \dots, n\}$  realising the required equivalence. Hence, also this case gives a contradiction with  $p_n \sim_{\text{BB}} p \parallel q$ .
- $p \parallel q \xrightarrow{\epsilon} p' \parallel q' \xrightarrow{a} p' \parallel q''$ , with  $p' \parallel q' \sim_{\text{BB}} p_n$  and  $p' \parallel q'' \sim_{\text{BB}} a^{\leq i}$ . This case is symmetric to the previous one and therefore omitted.  $\square$

The next lemma characterises the closed parallel components of a process that is branching bisimilar to the process  $a \parallel p_n$ , for some  $n \geq 2$ .

**Lemma 18.** *Let  $n \geq 2$ . Assume that  $p \parallel q \sim_{\text{BB}} a \parallel p_n$  for some processes  $p, q \sim_{\text{BB}} \mathbf{0}$ . Then either  $p \sim_{\text{BB}} a$  and  $q \sim_{\text{BB}} p_n$ , or  $p \sim_{\text{BB}} p_n$  and  $q \sim_{\text{BB}} a$ .*

**Proof.** First of all, we notice that we can apply the same reasoning used in the proof of Lemma 16 to argue that no  $\tau$ -move performed by any derivative of  $p \parallel q$  can follow from a communication between the two parallel components, for otherwise we get a contradiction with  $p \parallel q \sim_{\text{BB}} a \parallel p_n$ .

We have that  $a \parallel p_n \xrightarrow{a} \mathbf{0} \parallel p_n \sim_{\text{BB}} p_n$ . Hence, as  $p \parallel q \sim_{\text{BB}} a \parallel p_n$ , there are processes  $r, r'$  such that  $p \parallel q \xrightarrow{\epsilon} r \xrightarrow{a} r'$ ,  $r \sim_{\text{BB}} a \parallel p_n$ , and  $r' \sim_{\text{BB}} p_n$ . Since no communication can occur, we have that there are processes  $p', q'$  such that  $p \xrightarrow{\epsilon} p'$ ,  $q \xrightarrow{\epsilon} q'$ , and  $r = p' \parallel q'$ . As  $r \xrightarrow{a} r'$ , assume, without loss of generality, that  $p' \xrightarrow{a} p''$  and  $r' = p'' \parallel q'$ . (Under this assumption we will obtain that  $p \sim_{\text{BB}} a$  and  $q \sim_{\text{BB}} p_n$ . The symmetric case is obtained when  $q' \xrightarrow{a} q''$  and  $r' = p' \parallel q''$ , by applying the same reasoning.) Since  $p'' \parallel q' \sim_{\text{BB}} p_n$  and  $p_n$  is indecomposable by Lemma 17, we can distinguish the following two cases:

- $p'' \sim_{\text{BB}} p_n$  and  $q' \sim_{\text{BB}} \mathbf{0}$ . As  $p' \xrightarrow{a} p''$ , this implies that  $\text{depth}(p) \geq n + 2$ . At the same time,  $q \sim_{\text{BB}} \mathbf{0}$  gives  $\text{depth}(q) \geq 1$ . Consequently, we have that  $\text{depth}(p \parallel q) = \text{depth}(p) + \text{depth}(q) \geq n + 3 > n + 2 = \text{depth}(a \parallel p_n)$ , giving thus a contradiction with  $p \parallel q \sim_{\text{BB}} a \parallel p_n$ .
- $p'' \sim_{\text{BB}} \mathbf{0}$  and  $q' \sim_{\text{BB}} p_n$ . In this case, we have that  $r = p' \parallel q' \sim_{\text{BB}} p' \parallel p_n$ . Moreover,  $q \xrightarrow{\epsilon} q'$  and  $\text{depth}(q') = n + 1$  give that  $\text{depth}(q) \geq n + 1$ . Now, as  $p \sim_{\text{BB}} \mathbf{0}$  gives  $\text{depth}(p) \geq 1$ , from  $\text{depth}(p \parallel q) = \text{depth}(a \parallel p_n) = n + 2$  we infer that  $\text{depth}(q) = n + 1$  and  $\text{depth}(p) = 1$ . As  $a \parallel p_n$  can perform only sequences of  $a$ -moves, and  $\text{depth}(p) = 1$ , we obtain that whenever  $p \xrightarrow{\alpha} p'$  then  $\alpha = a$  and  $p' \sim_{\text{BB}} \mathbf{0}$ .

We can also show that there is no  $p'$  such that  $p \xrightarrow{\epsilon} p'$  and  $p' \sim_{\text{BB}} \mathbf{0}$ . Assume, towards a contradiction, that such a process  $p'$  exists. Then  $p \parallel q \xrightarrow{\epsilon} p' \parallel q \sim_{\text{BB}} q$ . As  $a \parallel p_n$  cannot perform any  $\tau$ -transition, the branching bisimulation relation requires that  $a \parallel p_n \sim_{\text{BB}} q$ . However, this is not possible since  $\text{depth}(a \parallel p_n) \neq \text{depth}(q)$ .

We have therefore obtained that:

- whenever  $p \xrightarrow{\alpha} p'$  then  $\alpha = a$  and  $p' \sim_{\text{BB}} \mathbf{0}$ , and
- there is no  $p'$  such that  $p \xrightarrow{\epsilon} p'$  and  $p' \sim_{\text{BB}} \mathbf{0}$ .

This is enough to infer that  $p \sim_{\text{BB}} a$  (as the second condition guarantees that whenever  $p \xrightarrow{\epsilon} p' \xrightarrow{a} p''$  then  $p' \sim_{\text{BB}} a$  also holds.)

Let  $p'$  be any process such that  $p \xrightarrow{\epsilon} p' \xrightarrow{a} p''$  for some  $p'' \sim_{\text{BB}} \mathbf{0}$ . Then  $p \parallel q \xrightarrow{\epsilon} p' \parallel q \xrightarrow{a} p'' \parallel q \sim_{\text{BB}} q$ . As  $p \parallel q \sim_{\text{BB}} a \parallel p_n$ , then  $a \parallel p_n \xrightarrow{\epsilon} r \xrightarrow{a} r'$  for some processes  $r, r'$  such that  $r \sim_{\text{BB}} p' \parallel q$  and  $r' \sim_{\text{BB}} q$ . Since, however,  $a \parallel p_n$  cannot perform any  $\tau$ -transition, we get that  $a \parallel p_n \xrightarrow{a} r'$  for some  $r' \sim_{\text{BB}} q$ . Moreover, as  $\text{depth}(q) = n + 1$ , we have that either  $r' = p_n$  or  $r' = a \parallel a^{\leq n}$ , as not other  $a$ -derivative of  $a \parallel p_n$  has observable depth  $n + 1$ . Hence, to conclude, it is enough to show that the case  $r' = a \parallel a^{\leq n}$  yields a contradiction with  $p \parallel q \sim_{\text{BB}} a \parallel p_n$ . Assume that  $q \sim_{\text{BB}} a \parallel a^{\leq n}$ . Then, as  $p \sim_{\text{BB}} a$ , we get that  $p \parallel q \sim_{\text{BB}} a \parallel (a \parallel a^{\leq n})$ . Now  $a \parallel (a \parallel a^{\leq n}) \xrightarrow{a} a \parallel (a \parallel a^{n-1}) \sim_{\text{BB}} a^{n+1}$ . However, there is no process  $r$  such that  $a \parallel p_n \xrightarrow{a} r$  and  $r \sim_{\text{BB}} a^{n+1}$ . Hence  $a \parallel p_n \not\sim_{\text{BB}} a \parallel (a \parallel a^{\leq n})$ , giving thus a contradiction with  $a \parallel p_n \sim_{\text{BB}} p \parallel q$ .

We have therefore obtained that  $p \sim_{\text{BB}} a$  and  $q \sim_{\text{BB}} p_n$ , thus concluding the proof.  $\square$

We also need Lemma 18 to hold modulo rooted branching bisimilarity.

**Lemma 19.** *Let  $n \geq 2$ . Assume that  $p \parallel q \sim_{\text{RBB}} a \parallel p_n$  for some processes  $p, q \sim_{\text{RBB}} \mathbf{0}$ . Then either  $p \sim_{\text{BB}} a$  and  $q \sim_{\text{RBB}} p_n$ , or  $p \sim_{\text{RBB}} p_n$  and  $q \sim_{\text{RBB}} a$ .*

**Proof.** First of all, we notice that, by the root condition,  $\tau \notin \text{init}(a \parallel p_n)$  implies  $\tau \notin \text{init}(p \parallel q)$ . In particular, this guarantees that  $\tau \notin \text{init}(p)$  and  $\tau \notin \text{init}(q)$ , and thus that  $p, q \sim_{\text{RBB}} \mathbf{0}$  gives  $p, q \sim_{\text{BB}} \mathbf{0}$  (Lemma 3).

We have that  $a \parallel p_n \xrightarrow{a} \mathbf{0} \parallel p_n \sim_{\text{BB}} p_n$ . Hence, as  $p \parallel q \sim_{\text{RBB}} a \parallel p_n$ , there is a process  $r$  such that  $p \parallel q \xrightarrow{a} r$  and  $r \sim_{\text{BB}} p_n$ . We can assume, without loss of generality, that  $p \xrightarrow{a} p'$  and  $r = p' \parallel q$ , for some process  $p'$ . (Under this assumption we will obtain that  $p \sim_{\text{RBB}} a$  and  $q \sim_{\text{RBB}} p_n$ . The symmetric case is obtained when  $q \xrightarrow{a} q'$  and  $r = p \parallel q'$ , by applying the same reasoning.) Since  $p' \parallel q \sim_{\text{BB}} p_n$ ,  $q \sim_{\text{BB}} \mathbf{0}$ , and  $p_n$  is indecomposable by Lemma 17, it follows that  $p' \sim_{\text{BB}} \mathbf{0}$  and  $q \sim_{\text{BB}} p_n$ . Moreover, as  $\text{init}(q) = \text{init}(p_n)$  and  $\tau \notin \text{init}(p_n)$ , by Lemma 3 we get  $q \sim_{\text{RBB}} p_n$ .

Hence,  $p \parallel q \sim_{\text{RBB}} p \parallel p_n$ . Assume now that  $p \xrightarrow{\mu} p'$  for some action  $\mu$  and process  $p'$ . From  $p \parallel p_n \sim_{\text{RBB}} a \parallel p_n$  we infer  $\mu = a$ , and we can distinguish two cases, according to  $a$ -move is performed by  $a \parallel p_n$  to match the  $a$ -transition by  $p \parallel p_n$ :

- $p' \parallel p_n \sim_{\text{BB}} p_n$ . As  $p_n$  is indecomposable (Lemma 17), we can directly conclude that  $p' \sim_{\text{BB}} \mathbf{0}$ .
- $p' \parallel p_n \sim_{\text{BB}} a \parallel a^{\leq i}$  for some  $i \in \{2, \dots, n\}$ . Due to the form of  $p_n$ , this case is not possible.

Thus, whenever  $p \xrightarrow{a} p'$  we have that  $p' \sim_{\text{BB}} \mathbf{0}$ , which, together with  $\text{init}(p) = \{a\}$ , gives  $p \sim_{\text{RBB}} a$ .  $\square$

We conclude this section with the following result, showing a particular case in which we can establish that not only a variable  $x$  triggers the behaviour of a term  $t$ , but that  $t$  actually coincides with  $x$ .

**Lemma 20.** *Let  $t$  be a CCS term that does not have  $+$  as head operator, and let  $\sigma$  be a closed substitution. Assume that  $\sigma(t)$  has neither  $\mathbf{0}$  summands or factors, and that  $\sigma(t) \sim_{\text{RBB}} \sum_{k=1}^m a.a^{\leq i_k}$ , for some  $m > 1$ , and  $2 \leq i_1 < \dots < i_m$ . Then  $t = x$  for some variable  $x$ .*

**Proof.** Assume towards a contradiction that  $t$  is not a variable. We proceed by a case analysis on the other possible forms  $t$  may have:

- $t = \mu.t'$  for some  $\mu \in \mathcal{A}_\tau$  and term  $t'$ . Then  $\mu = a$  and  $a^{\leq i_1} \sim_{\text{BB}} \sigma(t') \sim_{\text{BB}} a^{\leq i_m}$ . However, this is a contradiction because, as  $i_1 < i_m$ , the terms  $a^{\leq i_1}$  and  $a^{\leq i_m}$  have different observable depths, and therefore they cannot be branching bisimilar.
- $t = t' \parallel t''$ . Since  $\sigma(t)$  has no  $\mathbf{0}$  factors we have that  $\sigma(t'), \sigma(t'') \not\sim_{\text{RBB}} \mathbf{0}$ . In particular, notice that since  $\tau \notin \text{init}(\sigma(t')) \cup \text{init}(\sigma(t''))$  (as  $\tau \notin \text{init}(\sum_{k=1}^m a.a^{\leq i_k})$ ), it follows that  $\sigma(t'), \sigma(t'') \sim_{\text{BB}} \mathbf{0}$  as well (Lemma 3). Moreover, observe that  $\sum_{k=1}^m a.a^{\leq i_k} \xrightarrow{a} a^{\leq i_m}$ . As  $\sigma(t) = \sigma(t') \parallel \sigma(t'') \sim_{\text{RBB}} \sum_{k=1}^m a.a^{\leq i_k}$ , there is a process  $r$  such that  $\sigma(t') \parallel \sigma(t'') \xrightarrow{a} r$  and  $r \sim_{\text{BB}} a^{\leq i_m}$ . Assume, without loss of generality, that  $\sigma(t') \xrightarrow{a} p$  and  $r = p \parallel \sigma(t'')$ , for some process  $p$ . As  $\sigma(t'') \sim_{\text{BB}} \mathbf{0}$  and  $a^{\leq i_m}$  is indecomposable (Lemma 16), it follows that  $p \sim_{\text{BB}} \mathbf{0}$  and  $\sigma(t') \sim_{\text{BB}} a^{\leq i_m}$ . Now notice that  $\sum_{k=1}^m a.a^{\leq i_k} \xrightarrow{a} a^{\leq i_1}$ . We can distinguish two cases according to how this transition is matched by  $\sigma(t') \parallel \sigma(t'')$  in the rooted branching bisimulation game:
  - $\sigma(t') \xrightarrow{a} p'$ , for some process  $p'$ , and  $p' \parallel \sigma(t'') \sim_{\text{BB}} a^{\leq i_1}$ . Then we can proceed as above to show that  $\sigma(t'') \sim_{\text{BB}} a^{\leq i_1}$ . As  $a^{\leq i_1} \not\sim_{\text{BB}} a^{\leq i_m}$ , this gives a contradiction.
  - $\sigma(t'') \xrightarrow{a} q$ , for some process  $q$ , and  $\sigma(t') \parallel q \sim_{\text{BB}} a^{\leq i_1}$ . Since  $\sigma(t') \sim_{\text{BB}} \mathbf{0}$  and  $a^{\leq i_1}$  is indecomposable (Lemma 16), it follows that  $q \sim_{\text{BB}} \mathbf{0}$  and  $\sigma(t') \sim_{\text{BB}} a^{\leq i_1}$ . This implies that  $\sigma(t') \parallel \sigma(t'') \sim_{\text{BB}} a^{\leq i_1} \parallel a^{\leq i_m}$ , and thus  $\text{depth}(\sigma(t') \parallel \sigma(t'')) = i_1 + i_m > i_m + 1$  (since  $i_1 \geq 2$ ). Since  $\tau \notin \text{init}(\sigma(t') \parallel \sigma(t'')) \cup \text{init}(a^{\leq i_1} \parallel a^{\leq i_m})$ , we can infer that  $\text{depth}(\sigma(t') \parallel \sigma(t'')) = \text{Rdepth}(\sigma(t') \parallel \sigma(t''))$  (Remark 2). Hence, we get a contradiction with  $\sigma(t') \parallel \sigma(t'') \sim_{\text{RBB}} \sum_{k=1}^m a.a^{\leq i_k}$ , as  $\text{Rdepth}(\sum_{k=1}^m a.a^{\leq i_k}) = i_m + 1$ .  $\square$

## 8. The negative result for rooted branching bisimilarity

We devote this section to the proof of our first main result, namely:

**Theorem 1.** *Rooted branching bisimilarity admits no finite equational axiomatisation over CCS.*

Following the proof strategy sketched in Section 3, we introduce a particular family of equations on which we will build our negative result. Recall the special processes

$$p_n = \sum_{i=2}^n a.a^{\leq i} \quad (n \geq 2)$$

introduced in Sections 3 and 7. Then, we define:

$$\varepsilon_n : \quad a \parallel p_n \approx a p_n + \sum_{i=2}^n a(a \parallel a^{\leq i}) \quad (n \geq 2) .$$

It is easy to check that each equation  $\varepsilon_n$ , for  $n \geq 2$ , is sound modulo rooted branching bisimilarity (as, in particular, it is sound modulo strong bisimilarity).

In order to prove Theorem 1, we proceed to show that no finite collection of equations over CCS that are sound modulo rooted branching bisimilarity can prove all of the equations  $\varepsilon_n$  ( $n \geq 2$ ) from the family given above.

Formally, this is obtained by proving the following theorem:

**Theorem 3.** *Let  $\mathcal{E}$  be a finite axiom system over CCS that is sound modulo  $\sim_{\text{RBB}}$ ,  $n$  be larger than the size of each term in the equations in  $\mathcal{E}$ , and  $p, q$  be closed terms such that  $p, q \sim_{\text{RBB}} a \parallel p_n$ . If  $\mathcal{E} \vdash p \approx q$  and  $p$  has a summand rooted branching bisimilar to  $a \parallel p_n$ , then so does  $q$ .*

Briefly, Theorem 3 proves that the property of having a summand that is rooted branching bisimilar to  $a \parallel p_n$ , for some  $n \geq 2$ , is an invariant under provability from finite, sound axiom systems, namely, if one side of a sound equation provable from a finite axiom system satisfies the property, then so does the other side. Thus, as none of the equations  $\varepsilon_n$  in the family above satisfies this property, we can conclude that no finite, sound axiom system can prove all of them. In fact, for all  $n \geq 2$ , we have that only the left-hand side of equation  $\varepsilon_n$  satisfies the property (cf. Section 3).

### 8.1. Proving Theorem 3

The proof of Theorem 3 is strongly based on the proposition below, which ensures that the property of having a summand rooted branching bisimilar to  $a \parallel p_n$  ( $n \geq 2$ ) is preserved by the closure under substitutions of equations in a finite, sound axiom system. Proposition 7 is proved by means of the technical results provided so far.

**Proposition 7.** *Let  $t \approx u$  be an equation over CCS that is sound modulo  $\sim_{\text{RBB}}$ . Let  $\sigma$  be a closed substitution with  $p = \sigma(t)$  and  $q = \sigma(u)$ . Suppose that  $p$  and  $q$  have neither  $\mathbf{0}$  summands nor factors, and  $p, q \sim_{\text{RBB}} a \parallel p_n$  for some  $n$  larger than the sizes of  $t$  and  $u$ . If  $p$  has a summand that is rooted branching bisimilar to  $a \parallel p_n$ , then so does  $q$ .*

**Proof.** Observe, first of all, that since  $\sigma(t) = p$  and  $\sigma(u) = q$  have no  $\mathbf{0}$  summands or factors, then neither do  $t$  and  $u$ . Hence, by Remark 3, we have that for some finite non-empty index sets  $I, J$ ,  $t = \sum_{i \in I} t_i$  and  $u = \sum_{j \in J} u_j$ , where none of the  $t_i$  ( $i \in I$ ) and  $u_j$  ( $j \in J$ ) is  $\mathbf{0}$ , has  $+$  as its head operator, has  $\mathbf{0}$  summands and factors.

As  $p = \sigma(t)$  has a summand rooted branching bisimilar to  $a \parallel p_n$ , there is an index  $i \in I$  such that  $\sigma(t_i) \sim_{\text{RBB}} a \parallel p_n$ .

Our aim is now to show that there is an index  $j \in J$  such that  $\sigma(u_j) \sim_{\text{RBB}} a \parallel p_n$ , proving that  $q = \sigma(u)$  also has a summand rooted branching bisimilar to  $a \parallel p_n$ .

We proceed by a case analysis on the form  $t_i$  may have.

1. CASE  $t_i = x$  FOR SOME VARIABLE  $x$ . In this case, we have  $\sigma(x) \sim_{\text{RBB}} a \parallel p_n$ , and  $t$  has  $x$  as a summand. As  $t \approx u$  is sound modulo rooted branching bisimilarity and neither  $t$  nor  $u$  have  $\mathbf{0}$  summands or factors, it follows that  $u$  also has  $x$  as a summand (Proposition 6). Thus there is an index  $j \in J$  such that  $u_j = x$ , and, modulo rooted branching bisimulation,  $\sigma(u)$  has  $a \parallel p_n$  as a summand, which was to be shown.
2. CASE  $t_i = \mu t'$  FOR SOME TERM  $t'$ . This case is vacuous because, since, by the root condition,  $\tau \notin \text{init}(a \parallel p_n)$  implies  $\mu = a$ . Hence, since  $a\sigma(t') \xrightarrow{a} \sigma(t')$  is the only transition afforded by  $\sigma(t_i)$ , this term cannot be rooted branching bisimilar to  $a \parallel p_n$ . Indeed  $a \parallel p_n$  can perform both,  $a \parallel p_n \xrightarrow{a} \mathbf{0} \parallel p_n \sim_{\text{BB}} p_n$ , and  $a \parallel p_n \xrightarrow{a} a \parallel a^{\leq i}$ , for some  $i \in \{2, \dots, n\}$ . Clearly,  $p_n \not\sim_{\text{BB}} a \parallel a^{\leq i}$  for any  $i \in \{2, \dots, n\}$ , and thus  $\sigma(t')$  cannot be branching bisimilar to both of them.
3. CASE  $t_i = t' \parallel t''$  FOR SOME TERMS  $t', t''$ . In this case, we have  $\sigma(t') \parallel \sigma(t'') \sim_{\text{RBB}} a \parallel p_n$ . As  $\sigma(t_i)$  has no  $\mathbf{0}$  factors, it follows that  $\sigma(t') \not\sim_{\text{RBB}} \mathbf{0}$  and  $\sigma(t'') \not\sim_{\text{RBB}} \mathbf{0}$ . Thus by Lemma 19 (and without loss of generality)  $\sigma(t') \sim_{\text{RBB}} a$  and  $\sigma(t'') \sim_{\text{RBB}} p_n$ . Now,  $t''$  can be written as  $t'' = v_1 + \dots + v_M$ , for some  $M > 0$ , where none of the summands  $v_i$  is  $\mathbf{0}$  or a sum. Observe that, since  $n$  is larger than the size of  $t$ , we have that  $M < n$ . Hence, since  $\sigma(t'') \sim_{\text{RBB}} p_n$ , there must be some  $h \in \{1, \dots, M\}$  such that  $\sigma(v_h) \sim_{\text{RBB}} \sum_{k=1}^m aa^{\leq i_k}$  for some  $m > 1$  and  $2 \leq i_1 < \dots < i_m \leq n$ . The term  $\sigma(v_h)$  has no  $\mathbf{0}$  summands or factors, or else so would  $\sigma(t'')$ , and thus  $p = \sigma(t)$ . By Lemma 20, it follows that  $v_h$  can only be a variable  $x$  and thus that

$$\sigma(x) \sim_{\text{RBB}} \sum_{k=1}^m aa^{\leq i_k} . \quad (1)$$

Observe, for later use, that, since  $t'$  has no  $\mathbf{0}$  factors, the above equation yields that  $x \notin \text{var}(t')$ , or else  $\sigma(t') \not\sim_{\text{RBB}} a$  (Lemma 4). So, modulo rooted branching bisimilarity,  $t_i$  has the form  $t' \parallel (x + t''')$ , for some term  $t'''$ , with  $x \notin \text{var}(t')$  and  $\sigma(t') \sim_{\text{RBB}} a$ . Our order of business will now be to use the information collected so far to argue that  $\sigma(u)$  has a summand rooted branching bisimilar to  $a \parallel p_n$ . To this end, consider the substitution

$$\sigma' = \sigma[x \mapsto \bar{a}(a \parallel p_n)] .$$

We have that

$$\begin{aligned} \sigma'(t_i) &= \sigma'(t') \parallel \sigma'(t''') \\ &= \sigma(t') \parallel \sigma'(t''') && \text{(As } x \notin \text{var}(t')\text{)} \\ &\sim_{\text{RBB}} a \parallel (\bar{a}(a \parallel p_n) + \sigma'(t''')) && \text{(As } t'' = x + t'''\text{)}. \end{aligned}$$

Thus, there is some process  $p'$  such that  $\sigma'(t_i) \xrightarrow{\tau} p' \sim_{\text{BB}} a \parallel p_n$ , so that

$$\sigma'(t) \xrightarrow{\tau} p' \sim_{\text{BB}} a \parallel p_n$$

also holds. Since  $t \approx u$  is sound modulo  $\sim_{\text{RBB}}$ , it follows that

$$\sigma'(t) \sim_{\text{RBB}} \sigma'(u) .$$

Hence, we can infer that there are a  $j \in J$  and a  $q'$  such that

$$\sigma'(u_j) \xrightarrow{\tau} q' \sim_{\text{BB}} a \parallel p_n . \quad (2)$$

Recall that, by one of the assumptions of the proposition,  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ , and thus  $\sigma(u)$  has rooted depth  $n + 2$ . On the other hand, by Equation (2),

$$\text{Rdepth}(\sigma'(u_j)) \geq n + 3 .$$

Since  $\sigma$  and  $\sigma'$  differ only in the closed term to which they map variable  $x$ , it follows that

$$x \in \text{var}(u_j) . \quad (3)$$

We now proceed to show that  $\sigma(u_j) \sim_{\text{RBB}} a \parallel p_n$  by a further case analysis on the form a term  $u_j$  satisfying Equations (2) and (3) may have.

- (a) CASE  $u_j = x$ . This case is vacuous because  $\sigma'(x) = \bar{a}(a \parallel p_n) \not\rightarrow$ , and thus this possible form for  $u_j$  does not meet Equation (2).
- (b) CASE  $u_j = \mu u'$  FOR SOME TERM  $u'$ . In light of Equation (2), we have that  $\mu = \tau$  and  $q' = \sigma'(u') \sim_{\text{BB}} a \parallel p_n$ . Using Equation (3), we get  $x \in \text{var}(u')$  and, thus,  $\text{depth}(\sigma'(u')) \geq n + 3$  (Lemma 4). Since  $a \parallel p_n$  has depth  $n + 2$ , this contradicts  $q' \sim_{\text{BB}} a \parallel p_n$ .
- (c) CASE  $u_j = u' \parallel u''$  FOR SOME TERMS  $u', u''$ . Our assumption that  $\sigma(u)$  has no  $\mathbf{0}$  factors yields that none of the terms  $u', u'', \sigma(u')$  and  $\sigma(u'')$  is rooted branching bisimilar to  $\mathbf{0}$ . In addition,  $\text{init}(\sigma(u)) = \{a\}$ , gives  $\tau \notin \text{init}(\sigma(u')) \cup \text{init}(\sigma(u''))$ , so that, by Lemma 3,  $\sigma(u'), \sigma(u'') \sim_{\text{BB}} \mathbf{0}$  holds. Moreover, by Equation (3), either  $x \in \text{var}(u')$  or  $x \in \text{var}(u'')$  (and a simple observation on the depth of terms guarantees that it cannot occur in both).

By Equation (2), we have that  $\sigma(u') \parallel \sigma(u'') \xrightarrow{\tau} q' \sim_{\text{BB}} a \parallel p_n$ . Hence, we can distinguish three cases, according to the possible origins for such a transition according to Lemma 11:

- i. There is a term  $w$  such that  $u' \parallel u'' \xrightarrow{\tau} w$ . This case is vacuous, since  $u' \parallel u'' \xrightarrow{\tau} w$  implies  $u \xrightarrow{\tau} w$ , and, thus,  $\sigma(u) \xrightarrow{\tau} \sigma(w)$ , giving a contradiction with  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ .
- ii.  $u' \xrightarrow{\alpha} w$  and there is a variable  $y \in \text{var}(u'')$  such that  $\sigma'(y) \xrightarrow{\bar{\alpha}} r_y$ ,  $u' \parallel u'' \xrightarrow{\bar{\alpha}, \tau} c$ , and  $\sigma'[y \mapsto r_y](c) = q'$ . (The symmetric case with  $u'' \xrightarrow{\alpha} w$  and  $y \in \text{var}(u')$  can be treated in the same way and it is therefore omitted.)

Notice that if  $\alpha \neq a$  then  $\sigma(u') \xrightarrow{\alpha}$  implies  $\sigma(u) \xrightarrow{\alpha}$ , and thus  $\text{init}(\sigma(u)) \neq \text{init}(a \parallel p_n)$ , giving a contradiction with  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ . Hence,  $\alpha = a$  and, consequently,  $\bar{\alpha} = \bar{a}$ .

Moreover, if  $y \neq x$  then  $\sigma'(y) = \sigma(y)$  giving  $\sigma(y) \xrightarrow{\bar{a}}$  and thus  $\sigma(u) \xrightarrow{\bar{a}}$ . Also in this case we obtain a contradiction with  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ . Hence  $y = x$ .

Since  $\sigma'(x) \xrightarrow{\bar{a}} a \parallel p_n$ , we have  $\sigma'(u') \parallel \sigma'(u'') \xrightarrow{\tau} \sigma'(w) \parallel (a \parallel p_n) \parallel \sigma'(w')$  (where the general term  $w'$  is included, as from  $x \in \text{var}(u'')$ , and  $u'' \xrightarrow{(x)} \mu$ , we can infer that  $u''$  can be written in the general form  $((x + w_1) \parallel w') + w_2$ ). From  $\sigma'(w) \parallel (a \parallel p_n) \parallel \sigma'(w') \sim_{\text{BB}} a \parallel p_n$  it follows (reasoning on the depth of terms) that  $\sigma'(w) \sim_{\text{BB}} \mathbf{0} \sim_{\text{BB}} \sigma'(w')$  and, in particular,  $\sigma'(w) = \sigma(w)$  and  $\sigma'(w') = \sigma(w')$ . Moreover, as  $x \notin \text{var}(u')$ , we have that  $\sigma'(u') = \sigma(u')$ .

Summarising, so far we have obtained that there is a process  $r$  such that  $\sigma(u') \xrightarrow{a} r$  and  $r \sim_{\text{BB}} \mathbf{0}$ , and that  $x \in \text{var}(u'')$  with  $u \xrightarrow{(x)} \mu$ . Next, we notice that  $\sigma(u') \xrightarrow{a} r$  implies that  $\sigma(u_j) \xrightarrow{a} r \parallel \sigma(u'')$ , so that  $\sigma(u) \xrightarrow{a} r \parallel \sigma(u'') \sim_{\text{BB}} \sigma(u'')$ . From  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ , it then follows that  $a \parallel p_n \xrightarrow{a} r'$  for some process  $r'$  such that  $r' \sim_{\text{BB}} \sigma(u'')$ . Given the structure of  $a \parallel p_n$ , we can distinguish two major cases, according to the possible form of  $r'$ :

- $r' = a \parallel a^{\leq i}$  for some  $i \in \{2, \dots, n\}$ . We proceed to show that in this case it is not possible to have  $r' \sim_{\text{BB}} \sigma(u'')$ . Recall that, by Equation (1),  $\sigma(x) \sim_{\text{RBB}} \sum_{k=1}^m a a^{\leq i_k}$  for  $m > 1$  and  $2 \leq i_1 < \dots < i_m \leq n$ . As  $u'' \xrightarrow{(x)} a$ , by Lemma 9 we have that  $\sigma(u'') \xrightarrow{a} a^{\leq i_1}$  and  $\sigma(u'') \xrightarrow{a} a^{\leq i_m}$  (where no parallel composition is considered because we have seen in the analysis above that any potential parallel component would be branching bisimilar to  $\mathbf{0}$ ). In particular,  $i_1 < i_m$  gives  $a^{\leq i_1} \sim_{\text{BB}} a^{\leq i_m}$  as the two processes have different depths (besides different branching structures). On the one hand, we have that  $r' \xrightarrow{a} a \parallel a^j$  for some  $j \in \{0, \dots, i-1\}$ , and  $a \parallel a^j \sim_{\text{BB}} a^{j+1} \sim_{\text{BB}} a^{\leq i_1}, a^{\leq i_m}$  for any  $j \in \{0, \dots, i-1\}$  and  $i_1, i_m \geq 2$ . On the other hand,  $r' \xrightarrow{a} \mathbf{0} \parallel a^{\leq i} \sim_{\text{BB}} a^{\leq i}$ , but  $a^{\leq i}$  cannot be branching bisimilar to both  $a^{\leq i_1}$  and  $a^{\leq i_m}$ . Hence,  $r' \not\sim_{\text{BB}} \sigma(u'')$  in this case.

- $r' = \mathbf{0} \parallel p_n$ . Then  $\sigma(u'') \sim_{\text{BB}} p_n$ . Notice that  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$  implies  $\text{init}(\sigma(u'')) = \{a\}$ . Hence, by Lemma 3,  $\tau \notin \text{init}(\sigma(u'')) \cup \text{init}(p_n)$  and  $\sigma(u'') \sim_{\text{BB}} p_n$  imply  $\sigma(u'') \sim_{\text{RBB}} p_n$ . In particular, it follows that  $\text{Rdepth}(\sigma(u'')) = n + 1$ . Since  $\text{Rdepth}(\sigma(u)) = n + 2 \geq \text{Rdepth}(\sigma(u_j)) = \text{Rdepth}(\sigma(u')) + \text{Rdepth}(\sigma(u''))$ , we obtain that  $\text{Rdepth}(\sigma(u')) = 1$ . In addition,  $\text{init}(\sigma(u')) = \{a\}$ .

It follows that whenever  $\sigma(u') \xrightarrow{\mu} q'$ , for some process  $q'$ , then  $\mu = a$  and  $q' \sim_{\text{BB}} \mathbf{0}$ . Hence,  $\sigma(u') \sim_{\text{RBB}} a$ .

Summarising, we have obtained that  $\sigma(u_j) = \sigma(u') \parallel \sigma(u'') \sim_{\text{RBB}} a \parallel p_n$ , and it is thus the summand we were looking for.

- iii. There are variables  $y, z$  such that  $\sigma'(y) \xrightarrow{\alpha} r_y$ ,  $\sigma'(z) \xrightarrow{\bar{\alpha}} r_z$ ,  $u' \parallel u'' \xrightarrow{(y,z)} c$ , and  $\sigma'[y \mapsto r_y, z \mapsto r_z](c) = q'$ . We can proceed as in the previous case to argue that if both  $y$  and  $z$  are different from  $x$ , then we obtain a contradiction with  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ . Similarly, it cannot be the case that both  $y$  and  $z$  are equal to  $x$ , for otherwise they would not be able to synchronise as  $\text{init}(\sigma'(x)) = \{\bar{a}\}$ . Hence, we can assume, without loss of generality, that  $y \neq x$  and  $z = x$ . Thus,  $\sigma'(y) = \sigma(y)$  and  $\alpha = a$ . As we can also assume, without loss of generality, that  $y \in \text{var}(u')$  and  $x \in \text{var}(u'')$  (having both in one term gives a contradiction either with  $q' \sim_{\text{BB}} a \parallel p_n$  or  $u', u'' \sim_{\text{RBB}} \mathbf{0}$ ). We can then proceed exactly as in the previous case to show that  $\sigma(u') \sim_{\text{RBB}} a$  and  $\sigma(u'') \sim_{\text{RBB}} p_n$ , giving that  $u_j$  is the desired summand.

This completes the proof for the case  $u_j = u' \parallel u''$  for some terms  $u', u''$ .

This completes the analysis for the case  $t_i = t' \parallel t''$  for some terms  $t', t''$ .

The proof of Proposition 7 is now complete.  $\square$

We now have all the ingredients necessary to prove Theorem 3, and thus that the property of having a summand rooted branching bisimilar to  $a \parallel p_n$  is indeed an invariant under provability from finite sound axiom systems.

**Theorem 3.** *Let  $\mathcal{E}$  be a finite axiom system over CCS that is sound modulo  $\sim_{\text{RBB}}$ ,  $n$  be larger than the size of each term in the equations in  $\mathcal{E}$ , and  $p, q$  be closed terms such that  $p, q \sim_{\text{RBB}} a \parallel p_n$ . If  $\mathcal{E} \vdash p \approx q$  and  $p$  has a summand rooted branching bisimilar to  $a \parallel p_n$ , then so does  $q$ .*

**Proof.** Assume that  $\mathcal{E}$  is a finite axiom system over the language CCS that is sound modulo rooted branching bisimilarity. Recall that, without loss of generality, we may assume that the closed terms involved in the proof of an equation  $p \approx q$  have no  $\mathbf{0}$  summands or factors (by Proposition 5, as  $\mathcal{E}$  may be assumed to be saturated), and that applications of symmetry happen first in equational proofs (that is,  $\mathcal{E}$  is closed with respect to symmetry). Hence, by the proviso of the theorem, the following hold, for some closed terms  $p$  and  $q$  and positive integer  $n$  larger than the size of each term in the equations in  $\mathcal{E}$ :

1.  $\mathcal{E} \vdash p \approx q$ ,
2.  $p \sim_{\text{RBB}} q \sim_{\text{RBB}} a \parallel p_n$ ,
3.  $p$  and  $q$  contain no occurrences of  $\mathbf{0}$  as a summand or factor, and
4.  $p$  has a summand rooted branching bisimilar to  $a \parallel p_n$ .

We prove that  $q$  also has a summand rooted branching bisimilar to  $a \parallel p_n$  by induction on the depth of the closed proof of the equation  $p \approx q$  from  $\mathcal{E}$ .

We proceed by a case analysis on the last rule used in the proof of  $p \approx q$  from  $\mathcal{E}$ . The case of reflexivity is trivial, and that of transitivity follows immediately by using the inductive hypothesis twice. Below we only consider the other possibilities.

- CASE  $\mathcal{E} \vdash p \approx q$ , BECAUSE  $\sigma(t) = p$  AND  $\sigma(u) = q$  FOR SOME EQUATION  $(t \approx u) \in \mathcal{E}$  AND CLOSED SUBSTITUTION  $\sigma$ . Since  $\sigma(t) = p$  and  $\sigma(u) = q$  have no  $\mathbf{0}$  summands or factors, and  $n$  is larger than the size of each term mentioned in equations in  $\mathcal{E}$ , the claim follows by Proposition 7.
- CASE  $\mathcal{E} \vdash p \approx q$ , BECAUSE  $p = \mu p'$  AND  $q = \mu q'$  FOR SOME  $p', q'$  SUCH THAT  $\mathcal{E} \vdash p' \approx q'$ . This case is vacuous because  $p = \mu p' \sim_{\text{RBB}} a \parallel p_n$ , and thus  $p$  does not have a summand rooted branching bisimilar to  $a \parallel p_n$ .
- CASE  $\mathcal{E} \vdash p \approx q$ , BECAUSE  $p = p' + p''$  AND  $q = q' + q''$  FOR SOME  $p', q', p'', q''$  SUCH THAT  $\mathcal{E} \vdash p' \approx q'$  AND  $\mathcal{E} \vdash p'' \approx q''$ . Since  $p$  has a summand rooted branching bisimilar to  $a \parallel p_n$ , we have that so does either  $p'$  or  $p''$ . Assume, without loss of generality, that  $p'$  has a summand rooted branching bisimilar to  $a \parallel p_n$ . Since  $p$  is bisimilar to  $a \parallel p_n$ , so is  $p'$ . Using the soundness of  $\mathcal{E}$  modulo  $\sim_{\text{RBB}}$ , it follows that  $q' \sim_{\text{RBB}} a \parallel p_n$ . The inductive hypothesis now yields that  $q'$  has a summand rooted branching bisimilar to  $a \parallel p_n$ . Hence,  $q$  has a summand rooted branching bisimilar to  $a \parallel p_n$ , which was to be shown.
- CASE  $\mathcal{E} \vdash p \approx q$ , BECAUSE  $p = p' \parallel p''$  AND  $q = q' \parallel q''$  FOR SOME  $p', q', p'', q''$  SUCH THAT  $\mathcal{E} \vdash p' \approx q'$  AND  $\mathcal{E} \vdash p'' \approx q''$ . Since the proof involves no uses of  $\mathbf{0}$  as a summand or a factor, we have that  $p', p'' \sim_{\text{RBB}} \mathbf{0}$  and  $q', q'' \sim_{\text{RBB}} \mathbf{0}$ . It follows that  $q$  is a summand of itself. By our assumptions,  $a \parallel p_n \sim_{\text{RBB}} q$ . Therefore we have that  $q$  has a summand bisimilar to  $a \parallel p_n$ , and we are done.

This completes the proof of Theorem 3.  $\square$

As the left-hand side of equation  $\varepsilon_n$ , i.e., the term  $a \parallel p_n$ , has a summand rooted branching bisimilar to  $a \parallel p_n$ , whilst the right-hand side, i.e., the term  $ap_n + \sum_{i=2}^n a(a \parallel a^{\leq i})$ , does not, we can conclude that the collection of infinitely many equations  $\varepsilon_n$  ( $n \geq 2$ ) is the desired infinite family of equations. Hence, the proof of Theorem 1 is now complete.

Having shown that  $\sim_{\text{RBB}}$  has no finite equational (ground-) complete axiomatisation we will now proceed to prove that neither does  $\sim_{\text{RWB}}$ .

## 9. Preliminary results for rooted weak bisimilarity

We devote this section to a presentation of the properties of CCS terms and specific processes that are necessary to prove the negative result for rooted weak bisimilarity.

We start by discussing the properties of processes  $p_n$  (for  $n \geq 2$ ), and of their subprocesses, modulo (rooted) weak bisimilarity.

**Lemma 21.** *Let  $i \geq 2$ . Process  $a^{\leq i}$  is indecomposable modulo  $\sim_{\text{WB}}$ .*

**Proof.** Assume, towards a contradiction, that there are processes  $p, q \sim_{\text{WB}} \mathbf{0}$  such that  $a^{\leq i} \sim_{\text{WB}} p \parallel q$ .

As  $a^{\leq i} \xrightarrow{a} \mathbf{0}$ , we have that  $p \parallel q \xrightarrow{a} r' \xrightarrow{a} r'' \xrightarrow{\varepsilon} r$  for some processes  $r', r'', r$  with  $r \sim_{\text{WB}} \mathbf{0}$ . Notice that  $p \sim_{\text{WB}} \mathbf{0}$  implies that there is at least one  $p' \in \text{der}(p)$  such that  $p' \xrightarrow{\mu} p''$  for some process  $p''$  and action  $\mu \neq \tau$ . A similar property holds for  $q$ . Since, moreover,  $a^{\leq i}$  can only perform sequences of  $a$ -moves,  $p \parallel q \sim_{\text{WB}} a^{\leq i}$  implies that also  $p$  and  $q$  can perform (weak) sequences of  $a$ -moves. In particular, it follows that there is no derivative of  $p$  or  $q$  that can perform action  $\bar{a}$ . As a consequence,  $p, q$ , and their derivatives,

cannot synchronise. This implies that there are processes  $p', q'$  such that  $p \xrightarrow{\varepsilon} p', q \xrightarrow{\varepsilon} q'$  and  $r' = p' \parallel q'$ . Then  $r' \xrightarrow{a} r''$  can follow either from  $p' \xrightarrow{a} p''$ , or  $q' \xrightarrow{a} q''$ , for some  $p'', q''$ . Assume, without loss of generality, that  $p' \xrightarrow{a} p''$  and  $r'' = p'' \parallel q'$  (the case  $q' \xrightarrow{a} q''$  is analogous). Also in this case, as no synchronisation can occur between the derivatives of  $p$  and  $q$ , we have that there are processes  $p_1, q_1$  such that  $p'' \xrightarrow{\varepsilon} p_1, q' \xrightarrow{\varepsilon} q_1$  and  $r = p_1 \parallel q_1$ . Now  $p_1 \parallel q_1 \sim_{\text{WB}} \mathbf{0}$  implies  $p_1 \sim_{\text{WB}} \mathbf{0}$  and  $q_1 \sim_{\text{WB}} \mathbf{0}$ .

In particular, we have obtained that  $q \xrightarrow{\varepsilon} q_1$  for some  $q_1 \sim_{\text{WB}} \mathbf{0}$ , so that  $p \parallel q \xrightarrow{\varepsilon} p \parallel q_1 \sim_{\text{WB}} p$ . From  $p \parallel q \sim_{\text{WB}} a^{\leq i}$ , this implies that  $a^{\leq i} \xrightarrow{\varepsilon} r_1$  for some  $r_1 \sim_{\text{WB}} p$ . However, as  $\tau \notin \text{init}(a^{\leq i})$ , the only possibility is to have  $a^{\leq i} \sim_{\text{WB}} p$ , so that  $\text{depth}(p) = i$ . Since  $q \sim_{\text{WB}} \mathbf{0}$  gives  $\text{depth}(q) \geq 1$ , we get that  $\text{depth}(p \parallel q) = \text{depth}(p) + \text{depth}(q) \geq i + 1 > i = \text{depth}(a^{\leq i})$ , giving thus a contradiction with  $p \parallel q \sim_{\text{WB}} a^{\leq i}$  (cf. Lemma 1).  $\square$

**Lemma 22.** *Let  $n \geq 2$ . Process  $p_n$  is indecomposable modulo  $\sim_{\text{WB}}$ .*

**Proof.** Assume, towards a contradiction, that there are processes  $p, q \sim_{\text{WB}} \mathbf{0}$  such that  $p_n \sim_{\text{WB}} p \parallel q$ . Notice that  $p \sim_{\text{WB}} \mathbf{0}$  implies that there is at least one  $p' \in \text{der}(p)$  such that  $p' \xrightarrow{\mu} p''$  for some process  $p''$  and action  $\mu \neq \tau$ . A similar property holds for  $q$ . Given any  $i \in \{2, \dots, n\}$ , we have that  $p_n \xrightarrow{a} a^{\leq i}$ . Hence, from  $p_n \sim_{\text{WB}} p \parallel q$ , we get that  $p \parallel q \xrightarrow{\varepsilon} r' \xrightarrow{a} r'' \xrightarrow{\varepsilon} r$  for some processes  $r', r'', r$  with  $r \sim_{\text{WB}} a^{\leq i}$ . We remark that, since  $p_n$  can only perform sequences of  $a$ -moves,  $p \parallel q \sim_{\text{WB}} p_n$  implies that also  $p$  and  $q$  can perform (weak) sequences of  $a$ -moves. In particular, it follows that there is no derivative of  $p$  or  $q$  that can perform action  $\bar{a}$ . As a consequence,  $p$  and  $q$  cannot synchronise. Hence,  $p \xrightarrow{\varepsilon} p', q \xrightarrow{\varepsilon} q'$  for some processes  $p', q'$  such that  $r' = p' \parallel q'$ . Then  $r' \xrightarrow{a} r''$  can follow either from  $p' \xrightarrow{a} p''$ , or  $q' \xrightarrow{a} q''$ , for some  $p'', q''$ . Assume, without loss of generality, that  $p' \xrightarrow{a} p''$  and  $r'' = p'' \parallel q'$  (the case  $q' \xrightarrow{a} q''$  is analogous). Also in this case, as no synchronisation can occur between the derivatives of  $p$  and  $q$ , we have that there are processes  $p_1, q_1$  such that  $p'' \xrightarrow{\varepsilon} p_1, q' \xrightarrow{\varepsilon} q_1$  and  $r = p_1 \parallel q_1 \sim_{\text{WB}} a^{\leq i}$ . As  $a^{\leq i}$  is indecomposable (Lemma 21), we can distinguish the following two cases:

- $p_1 \sim_{\text{WB}} a^{\leq i}$  and  $q_1 \sim_{\text{WB}} \mathbf{0}$ . Then  $q \xrightarrow{\varepsilon} q_1$  for some  $q_1 \sim_{\text{WB}} \mathbf{0}$ , so that  $p \parallel q \xrightarrow{\varepsilon} p \parallel q_1 \sim_{\text{WB}} p$ . We can then proceed as in the proof of Lemma 21 above to prove that this would imply  $\text{depth}(p \parallel q) > \text{depth}(p_n)$ , giving thus a contradiction with  $p \parallel q \sim_{\text{WB}} p_n$ .
- $p_1 \sim_{\text{WB}} \mathbf{0}$  and  $q_1 \sim_{\text{WB}} a^{\leq i}$ . Then, as  $a^{\leq i} \xrightarrow{a} a^j$  for any  $j \in \{0, \dots, i-1\}$ , we have that  $q_1 \xrightarrow{a} q_1^j$  for some process  $q_1^j \sim_{\text{WB}} a^j$ . This gives that  $p \parallel q \xrightarrow{a} p \parallel q_1^j \sim_{\text{WB}} p \parallel a^j$ . From  $p_n \sim_{\text{WB}} p \parallel q$  and  $p_n \xrightarrow{\tau}$ , the only way in which  $p_n$  can mimic the (weak)  $a$ -move from  $p \parallel q$  is via an  $a$ -move of the form  $p_n \xrightarrow{a} a^{\leq k}$  for some  $k \in \{2, \dots, n\}$ , with  $a^{\leq k} \sim_{\text{WB}} p \parallel a^j$ . As  $a^{\leq k}$  is indecomposable, we can distinguish two sub-cases:
  - $p \sim_{\text{WB}} \mathbf{0}$  and  $a^j \sim_{\text{WB}} a^{\leq k}$ . This case gives a contradiction with the assumption  $p \sim_{\text{WB}} \mathbf{0}$ , and moreover there is no pair  $j, k$  such that  $a^j \sim_{\text{WB}} a^{\leq k}$  holds.
  - $p \sim_{\text{WB}} a^{\leq k}$  and  $a^j \sim_{\text{WB}} \mathbf{0}$ . This, together with  $q \xrightarrow{\varepsilon} q_1$ , gives that  $p \parallel q \xrightarrow{\varepsilon} p \parallel q_1 \sim_{\text{WB}} a^{\leq k} \parallel a^{\leq i}$ . As  $p_n \xrightarrow{\tau}$ , then  $p \parallel q \sim_{\text{WB}} p_n$  implies  $p_n \sim_{\text{WB}} a^{\leq k} \parallel a^{\leq i}$ . However, it is immediate to verify that there is no pair of indexes  $i, k \in \{2, \dots, n\}$  such that the last equivalence holds. This gives a contradiction with  $p_n \sim_{\text{WB}} p \parallel q$ .  $\square$

The characterisation of the closed parallel components of a process that is (rooted) weak bisimilar to  $a \parallel p_n$ , for  $n \geq 2$ , is the same obtained for (rooted) branching bisimilarity.

**Lemma 23.** *Let  $n \geq 2$ . Assume that  $p \parallel q \sim_{\text{WB}} a \parallel p_n$  for some processes  $p, q \sim_{\text{WB}} \mathbf{0}$ . Then either  $p \sim_{\text{WB}} a$  and  $q \sim_{\text{WB}} p_n$ , or  $p \sim_{\text{WB}} p_n$  and  $q \sim_{\text{WB}} a$ .*

**Proof.** Straightforward adaptation of the proof of Lemma 18.  $\square$

**Lemma 24.** *Let  $n \geq 2$ . Assume that  $p \parallel q \sim_{\text{RWB}} a \parallel p_n$  for some processes  $p, q \sim_{\text{RWB}} \mathbf{0}$ . Then either  $p \sim_{\text{RWB}} a$  and  $q \sim_{\text{RWB}} p_n$ , or  $p \sim_{\text{RWB}} p_n$  and  $q \sim_{\text{RWB}} a$ .*

**Proof.** Straightforward adaptation of the proof of Lemma 19.  $\square$

Let us now investigate some properties of open CCS terms modulo rooted weak bisimilarity. Firstly, we identify a special case in which we can infer that a CCS term showing a specific behaviour must be a variable.

**Lemma 25.** *Let  $t$  be a CCS term that does not have  $+$  as head operator, and let  $\sigma$  be a closed substitution. Assume that  $\sigma(t)$  has neither  $\mathbf{0}$  summands or factors, and that  $\sigma(t) \sim_{\text{RWB}} \sum_{k=1}^m aa^{\leq i_k}$ , for some  $m > 1$ , and  $2 \leq i_1 < \dots < i_m$ . Then  $t = x$  for some variable  $x$ .*

**Proof.** Also in this case we can easily adapt the proof of the corresponding result for rooted branching bisimilarity (Lemma 20). Specifically, for the case  $t = t' \parallel t''$  it is enough to notice that since  $\tau \notin \text{init}(\sigma(t''))$  and no communication can occur between  $\sigma(t')$  (and its derivatives) and  $\sigma(t'')$ , the only possible  $\tau$ -moves in the sequence  $\sigma(t') \parallel \sigma(t'') \xrightarrow{a} p' \parallel \sigma(t'') \xrightarrow{\varepsilon} r$ , mimicking the transition

$\sum_{k=1}^m aa^{\leq i_k} \xrightarrow{a} a^{\leq i_m}$ , can be performed by process  $p'$ . Hence  $r = p'' \parallel \sigma(t'')$  for some  $p''$  such that  $\sigma(t') \xrightarrow{a} p' \xrightarrow{\varepsilon} p''$ , and the rest of the proof then follows as in the case of  $\sim_{\text{RBB}}$ .  $\square$

In the proof of Proposition 7 we used the general property that if two terms  $t$  and  $u$  are rooted branching bisimilar, and  $t$  as a variable  $x$  as a summand, then so does  $u$ . This property does not hold, in general, for rooted weak bisimilar terms, as clearly demonstrated by axiom T2 ( $\tau x \approx \tau x + x$ ) in Table 3. However, as shown in the following lemma, we can take advantage of the particular structure of the terms on which we build our negative result to obtain the desired property.

**Proposition 8.** *Let  $t \approx u$  be an equation over CCS that is sound modulo  $\sim_{\text{RWB}}$ . Let  $\sigma$  be a closed substitution such that  $\sigma(t), \sigma(u)$  have neither  $\mathbf{0}$  summands nor factors, and  $\sigma(t), \sigma(u) \sim_{\text{RWB}} a \parallel p_n$  for some  $n$  larger than the sizes of  $t$  and  $u$ . If  $t$  has a summand  $x$  such that  $\sigma(x)$  is rooted weak bisimilar to  $a \parallel p_n$ , then so does  $u$ .*

**Proof.** Observe, first of all, that since  $\sigma(t)$  and  $\sigma(u)$  have no  $\mathbf{0}$  summands or factors so do  $t$  and  $u$  so that, by Remark 3 we can assume that  $t = \sum_{i \in I} t_i$  and  $u = \sum_{j \in J} u_j$  for some finite non-empty index sets  $I, J$ , where none of the  $t_i$  ( $i \in I$ ) and  $u_j$  ( $j \in J$ ) has  $+$  as its head operator, and none of the  $t_i$  ( $i \in I$ ) and  $u_j$  ( $j \in J$ ) have  $\mathbf{0}$  summands or factors. Therefore, if  $t$  has a summand  $x$ , then that there is some index  $i \in I$  such that  $t_i = x$ . We aim to show that  $u$  has a summand  $x$  as well.

First of all, we remark that  $\sigma(t), \sigma(u) \sim_{\text{RWB}} a \parallel p_n$  and  $\tau \notin \text{init}(a \parallel p_n)$ , imply  $\tau \notin \text{init}(\sigma(t)) \cup \text{init}(\sigma(u))$  (Lemma 3). Let  $\bar{\sigma}$  be the closed substitution such that  $\bar{\sigma}(x) = \sigma(x)$  and  $\bar{\sigma}(y) = \mathbf{0}$  for all  $y \neq x$ . As  $t \approx u$  is sound modulo  $\sim_{\text{RWB}}$ , we have that  $\bar{\sigma}(t) \sim_{\text{RWB}} \bar{\sigma}(u)$ . From  $\bar{\sigma}(x) = \sigma(x) \sim_{\text{RWB}} a \parallel p_n, a \parallel p_n \xrightarrow{a} \mathbf{0} \parallel p_n \sim_{\text{WB}} p_n$ , and  $\tau \notin \text{init}(\sigma(x))$ , we infer that there is a process  $q$  such that  $\sigma(x) \xrightarrow{a} \xrightarrow{\varepsilon} q$  and  $q \sim_{\text{WB}} p_n$ . Hence, from  $\bar{\sigma}(t) \sim_{\text{RWB}} \bar{\sigma}(u)$ , and  $\tau \notin \text{init}(\bar{\sigma}(u))$ , we obtain that there is a summand  $u_j$  of  $u$  such that  $\bar{\sigma}(u_j) \xrightarrow{a} \xrightarrow{\varepsilon} r$  for a process  $r$  with  $r \sim_{\text{WB}} q \sim_{\text{WB}} p_n$ . We proceed to show that  $u_j = x$  by arguing that any other form  $u_j$  might have leads to a contradiction.

- $u_j = a.u'$ . Hence,  $\bar{\sigma}(u_j) \xrightarrow{a} \bar{\sigma}(u') \xrightarrow{\varepsilon} r$ . From  $r \sim_{\text{WB}} p_n$  we get  $\text{depth}(r) = \text{depth}(p_n) = n + 1$ , so that  $\text{depth}(\bar{\sigma}(u')) \geq n + 1$ . We can now distinguish two cases, according to whether or not  $x$  occurs in  $u'$ :
  - $x \notin \text{var}(u')$ . Then, as  $\bar{\sigma}$  maps every variable  $y \neq x$  to  $\mathbf{0}$ , we have that  $n + 1 \leq \text{depth}(\bar{\sigma}(u')) < \text{size}(u) < n$ , giving thus a contradiction.
  - $x \in \text{var}(u')$ . As  $\bar{\sigma}(x) = \sigma(x)$  and  $\text{depth}(\sigma(x)) = n + 2$ , by Lemma 4 we get that

$$\text{depth}(\sigma(u_j)) = \text{depth}(\sigma(u_j)) + 1 \geq \text{depth}(\sigma(x)) + 1 = n + 3,$$

which implies

$$n + 2 = \text{depth}(a \parallel p_n) = \text{depth}(\sigma(u)) \geq \text{depth}(\sigma(u_j)) \geq n + 3,$$

giving thus a contradiction with  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$

- $u_j = u' \parallel u''$ . Assume, without loss of generality, that  $\bar{\sigma}(u') \xrightarrow{a} r'$ , for some process  $r'$ , and that  $r' \xrightarrow{\varepsilon} r'', \bar{\sigma}(u'') \xrightarrow{\varepsilon} r'''$  so that  $\bar{\sigma}(u') \parallel \bar{\sigma}(u'') \xrightarrow{a} \xrightarrow{\varepsilon} r'' \parallel r''' \sim_{\text{WB}} p_n$ . Since  $\tau \notin \text{init}(\bar{\sigma}(u''))$ , we have that  $r''' = \bar{\sigma}(u'')$ . Since, moreover,  $p_n$  is indecomposable, we get that either  $r'' \sim_{\text{WB}} \mathbf{0}$  and  $\bar{\sigma}(u'') \sim_{\text{WB}} p_n$ , or  $r'' \sim_{\text{WB}} p_n$  and  $\bar{\sigma}(u'') \sim_{\text{WB}} \mathbf{0}$ . In the former case, we can proceed as in the case of prefixing above to show that we reach a contradiction regardless of whether or not  $x \in \text{var}(u'')$ . Hence, let us consider the latter case in which  $r'' \sim_{\text{WB}} p_n$  and  $\bar{\sigma}(u'') \sim_{\text{WB}} \mathbf{0}$ . Since  $u$  has no  $\mathbf{0}$  summands or factors, we get that  $u'' \not\sim_{\text{WB}} \mathbf{0}$ , so that  $\bar{\sigma}(u'') \sim_{\text{WB}} \mathbf{0}$  implies  $\bar{\sigma}(u'') = y$  for some variable  $y \neq x$ . Then, a simple argument on the depth of the considered processes gives that  $x \in \text{var}(u')$ . We also notice that  $\sigma(u)$  has no  $\mathbf{0}$  factors,  $\sigma(u'') \not\sim_{\text{WB}} \mathbf{0}$ , and thus  $\text{depth}(\sigma(u'')) \geq 1$ . Hence, we obtain that

$$n + 2 = \text{depth}(a \parallel p_n) = \text{depth}(\sigma(u)) \geq \text{depth}(\sigma(u')) + \text{depth}(\sigma(u'')) \geq \text{depth}(\sigma(x)) + 1 = n + 3,$$

giving thus a contradiction with  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$ .  $\square$

## 10. The negative result for rooted weak bisimilarity

We can now prove the second main result of our paper, namely:

**Theorem 2.** *Rooted weak bisimilarity admits no finite equational axiomatisation over CCS.*

As outlined in Sections 3 and 8, proving Theorem 2 boils down to show the validity of the following:

**Theorem 4.** *Let  $\mathcal{E}$  be a finite axiom system over CCS that is sound modulo  $\sim_{\text{RWB}}$ ,  $n$  be larger than the size of each term in the equations in  $\mathcal{E}$ , and  $p, q$  be closed terms such that  $p, q \sim_{\text{RWB}} a \parallel p_n$ . If  $\mathcal{E} \vdash p \approx q$  and  $p$  has a summand rooted weak bisimilar to  $a \parallel p_n$ , then so does  $q$ .*

Since Theorem 4 shows that the property of having a summand that is rooted weak bisimilar to  $a \parallel p_n$ , for some  $n \geq 2$ , is an invariant under provability from finite, sound axiom systems, we can conclude that no finite, sound axiom system can prove all the equations  $\varepsilon_n$  presented in Section 8.

The following proposition is the crux in the proof of Theorem 4.

**Proposition 9.** *Let  $t \approx u$  be an equation over CCS that is sound modulo  $\sim_{\text{RWB}}$ . Let  $\sigma$  be a closed substitution with  $p = \sigma(t)$  and  $q = \sigma(u)$ . Suppose that  $p$  and  $q$  have neither  $\mathbf{0}$  summands or factors, and  $p, q \sim_{\text{RWB}} a \parallel p_n$  for some  $n$  larger than the sizes of  $t$  and  $u$ . If  $p$  has a summand that is rooted weak bisimilar to  $a \parallel p_n$ , then so does  $q$ .*

**Proof.** Observe, first of all, that since  $\sigma(t) = p$  and  $\sigma(u) = q$  have no  $\mathbf{0}$  summands or factors, then neither do  $t$  and  $u$ . Hence, by Remark 3, we have that for some finite non-empty index sets  $I, J$ ,  $t = \sum_{i \in I} t_i$  and  $u = \sum_{j \in J} u_j$ , where none of the  $t_i$  ( $i \in I$ ) and  $u_j$  ( $j \in J$ ) is  $\mathbf{0}$ , has  $+$  as its head operator, has  $\mathbf{0}$  summands and factors.

As  $p = \sigma(t)$  has a summand rooted weak bisimilar to  $a \parallel p_n$ , there is an index  $i \in I$  such that  $\sigma(t_i) \sim_{\text{RWB}} a \parallel p_n$ .

Our aim is now to show that there is an index  $j \in J$  such that  $\sigma(u_j) \sim_{\text{RWB}} a \parallel p_n$ , proving that  $q = \sigma(u)$  also has a summand rooted weak bisimilar to  $a \parallel p_n$ .

We proceed by a case analysis on the form  $t_i$  may have.

1. CASE  $t_i = x$  FOR SOME VARIABLE  $x$ . In this case, we have that  $t$  has a summand  $x$  such that  $\sigma(x) \sim_{\text{RWB}} a \parallel p_n$ . As  $t \approx u$  is sound modulo rooted weak bisimilarity, neither  $t$  nor  $u$  have  $\mathbf{0}$  summands or factors, and  $\sigma(t), \sigma(u) \sim_{\text{RWB}} a \parallel p_n$ , by Proposition 8 it follows that also  $u$  has  $x$  as a summand. Thus there is an index  $j \in J$  such that  $u_j = x$ , and, modulo rooted weak bisimulation,  $\sigma(u)$  has  $a \parallel p_n$  as a summand, which was to be shown.
2. CASE  $t_i = \mu t'$  FOR SOME TERM  $t'$ . This case is vacuous because, since, by the root condition,  $\tau \notin \text{init}(a \parallel p_n)$  implies  $\mu = a$ . Hence, since  $a\sigma(t') \xrightarrow{a} \sigma(t')$  is the only transition afforded by  $\sigma(t_i)$ , this term cannot be rooted weak bisimilar to  $a \parallel p_n$ . Indeed  $a \parallel p_n$  can perform both,  $a \parallel p_n \xrightarrow{a} \mathbf{0} \parallel p_n \sim_{\text{WB}} p_n$ , and  $a \parallel p_n \xrightarrow{a} a \parallel a^{\leq i}$ , for some  $i \in \{2, \dots, n\}$ . Clearly,  $p_n \not\sim_{\text{WB}} a \parallel a^{\leq i}$  for any  $i \in \{2, \dots, n\}$ , and thus  $\sigma(t')$  cannot be weak bisimilar to both of them.
3. CASE  $t_i = t' \parallel t''$  FOR SOME TERMS  $t', t''$ . In this case, we have  $\sigma(t') \parallel \sigma(t'') \sim_{\text{RWB}} a \parallel p_n$ . As  $\sigma(t_i)$  has no  $\mathbf{0}$  factors, it follows that  $\sigma(t') \not\sim_{\text{RWB}} \mathbf{0}$  and  $\sigma(t'') \not\sim_{\text{RWB}} \mathbf{0}$ . Thus by Lemma 24 (and without loss of generality)  $\sigma(t') \sim_{\text{RWB}} a$  and  $\sigma(t'') \sim_{\text{RWB}} p_n$ . Now,  $t''$  can be written as  $t'' = v_1 + \dots + v_M$ , for some  $M > 0$ , where none of the summands  $v_i$  is  $\mathbf{0}$  or a sum. Observe that, since  $n$  is larger than the size of  $t$ , we have that  $M < n$ . Hence, since  $\sigma(t'') \sim_{\text{RWB}} p_n$ , there must be some  $h \in \{1, \dots, M\}$  such that  $\sigma(v_h) \sim_{\text{RWB}} \sum_{k=1}^m a a^{\leq i_k}$  for some  $m > 1$  and  $2 \leq i_1 < \dots < i_m \leq n$ . The term  $\sigma(v_h)$  has no  $\mathbf{0}$  summands or factors, or else so would  $\sigma(t'')$ , and thus  $p = \sigma(t)$ . By Lemma 25, it follows that  $v_h$  can only be a variable  $x$  and thus that

$$\sigma(x) \sim_{\text{RWB}} \sum_{k=1}^m a a^{\leq i_k} . \quad (4)$$

Observe, for later use, that, since  $t'$  has no  $\mathbf{0}$  factors, the above equation yields that  $x \notin \text{var}(t')$ , or else  $\sigma(t') \not\sim_{\text{RWB}} a$  (Lemma 4). So, modulo rooted weak bisimilarity,  $t_i$  has the form  $t' \parallel (x + t''')$ , for some term  $t'''$ , with  $x \notin \text{var}(t')$  and  $\sigma(t') \sim_{\text{RWB}} a$ .

Our order of business will now be to use the information collected so far to argue that  $\sigma(u)$  has a summand rooted weak bisimilar to  $a \parallel p_n$ . To this end, consider the substitution

$$\sigma' = \sigma[x \mapsto \bar{a}(a \parallel p_n)] .$$

We have that

$$\begin{aligned} \sigma'(t_i) &= \sigma'(t') \parallel \sigma'(t'') \\ &= \sigma'(t') \parallel \sigma'(t'') && \text{(As } x \notin \text{var}(t')\text{)} \\ &\sim_{\text{RWB}} a \parallel (\bar{a}(a \parallel p_n) + \sigma'(t''')) && \text{(As } t'' = x + t'''\text{)}. \end{aligned}$$

Thus, there is some process  $p'$  such that  $\sigma'(t_i) \xrightarrow{\tau} p' \sim_{\text{WB}} a \parallel p_n$ , so that

$$\sigma'(t) \xrightarrow{\tau} p' \sim_{\text{WB}} a \parallel p_n$$

also holds. Since  $t \approx u$  is sound modulo  $\sim_{\text{RWB}}$ , it follows that

$$\sigma'(t) \sim_{\text{RWB}} \sigma'(u) .$$

Hence, we can infer that there are a  $j \in J$  and a  $q'$  such that

$$\sigma'(u_j) \xrightarrow{\tau} q' \sim_{\text{WB}} a \parallel p_n . \quad (5)$$

Recall that, by one of the assumptions of the proposition,  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$ , and thus  $\sigma(u)$  has rooted depth  $n + 2$ . On the other hand, by Equation (2),

$$\text{Rdepth}(\sigma'(u_j)) \geq n + 3 .$$

Since  $\sigma$  and  $\sigma'$  differ only in the closed term to which they map variable  $x$ , it follows that

$$x \in \text{var}(u_j) . \quad (6)$$

We now proceed to show that  $\sigma(u_j) \sim_{\text{RWB}} a \parallel p_n$  by a further case analysis on the form a term  $u_j$  satisfying Equations (5) and (6) may have.

- (a) CASE  $u_j = x$ . This case is vacuous because  $\sigma'(x) = \bar{a}(a \parallel p_n) \not\rightarrow$ , and thus this possible form for  $u_j$  does not meet Equation (5).
- (b) CASE  $u_j = \mu u'$  FOR SOME TERM  $u'$ . In light of Equation (5), we have that  $\mu = \tau$  and  $\sigma'(u') \xrightarrow{\varepsilon} q' \sim_{\text{WB}} a \parallel p_n$ . Now,  $\mu u = \tau$  gives  $\sigma(u_j) \xrightarrow{\tau} \sigma(u')$ , and thus that  $\tau \in \text{init}(\sigma(u))$ . Since  $\tau \notin \text{init}(a \parallel p_n)$ , this gives a contradiction with  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$ .
- (c) CASE  $u_j = u' \parallel u''$  FOR SOME TERMS  $u', u''$ . Our assumption that  $\sigma(u)$  has no  $\mathbf{0}$  factors yields that none of the terms  $u', u'', \sigma(u')$  and  $\sigma(u'')$  is rooted weak bisimilar to  $\mathbf{0}$ . In addition,  $\text{init}(\sigma(u)) = \{a\}$ , gives  $\tau \notin \text{init}(\sigma(u')) \cup \text{init}(\sigma(u''))$ , so that, by Lemma 3,  $\sigma(u'), \sigma(u'') \sim_{\text{WB}} \mathbf{0}$  holds. Moreover, by Equation (6), either  $x \in \text{var}(u')$  or  $x \in \text{var}(u'')$  (and a simple observation on the depth of terms guarantees that it cannot occur in both).

By Equation (5), we have that  $\sigma(u') \parallel \sigma(u'') \xrightarrow{\tau} q' \sim_{\text{WB}} a \parallel p_n$ . Hence, we can distinguish three cases, according to the possible origins for such a transition according to Lemma 11:

- i. There is a term  $w$  such that  $u' \parallel u'' \xrightarrow{\tau} w$ . This case is vacuous, since  $u' \parallel u'' \xrightarrow{\tau} w$  implies  $u \xrightarrow{\tau} w$ , and, thus,  $\sigma(u) \xrightarrow{\tau} \sigma(w)$ , giving a contradiction with  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$ .
- ii.  $u' \xrightarrow{\alpha} w$  and there is a variable  $y \in \text{var}(u'')$  such that  $\sigma'(y) \xrightarrow{\bar{\alpha}} r_y, u' \parallel u'' \xrightarrow{\bar{\alpha}, \tau} c$ , and  $\sigma'[y \mapsto r_y](c) = q'$ . (The symmetric case with  $u'' \xrightarrow{\alpha} w$  and  $y \in \text{var}(u')$  can be treated in the same way and it is therefore omitted.)

Notice that if  $\alpha \neq a$  then  $\sigma(u') \xrightarrow{\alpha}$  implies  $\sigma(u) \xrightarrow{\alpha}$ , and thus  $\text{init}(\sigma(u)) \neq \text{init}(a \parallel p_n)$ , giving a contradiction with  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$ . Hence,  $\alpha = a$  and, consequently,  $\bar{\alpha} = \bar{a}$ .

Moreover, if  $y \neq x$  then  $\sigma'(y) = \sigma(y)$  giving  $\sigma(y) \xrightarrow{\bar{a}}$  and thus  $\sigma(u) \xrightarrow{\bar{a}}$ . Also in this case we obtain a contradiction with  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$ . Hence  $y = x$ .

Since  $\sigma'(x) \xrightarrow{\bar{a}} a \parallel p_n$ , we have  $\sigma'(u') \parallel \sigma'(u'') \xrightarrow{\tau} \sigma'(w) \parallel (a \parallel p_n) \parallel \sigma'(w') \xrightarrow{\varepsilon} q'$  (where the general term  $w'$  is included, as from  $x \in \text{var}(u'')$ , and  $u'' \xrightarrow{(x)} \mu$ , we can infer that  $u''$  can be written in the general form  $((x + w_1) \parallel u') + w_2$ ). Let us now focus on the possible sequence of  $\tau$ -moves through which we reach  $q'$ . Reasoning on the depth of terms, we get that  $x \notin \text{var}(w) \cup \text{var}(w')$ , giving  $\sigma'(w) = \sigma(w)$  and  $\sigma'(w') = \sigma(w')$ . From the structure of  $u''$ , and that of  $u_j$ , we can then infer that  $\tau \notin \text{init}(\sigma'(w'))$ , for otherwise we would get  $\tau \in \text{init}(\sigma(u))$  contradicting thus  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$ . Moreover,  $\sigma'(w) = \sigma(w)$  and  $\sigma'(w') = \sigma(w')$  also guarantee that none of the potential  $\tau$ -moves can be due to a synchronisation. Indeed, any  $\alpha$ -action from  $\sigma(w)$  or  $\sigma(w')$  with  $\alpha \neq a$  would give a contradiction with  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$ . Hence, the potential sequence of  $\tau$ -moves, can only be due to  $\sigma(w) \xrightarrow{\varepsilon} r''$  for some process  $r''$  such that  $r'' \parallel (a \parallel p_n) \parallel \sigma(w') \sim_{\text{WB}} a \parallel p_n$ . Reasoning on the depth of terms, it then follows that  $r'' \sim_{\text{WB}} \mathbf{0} \sim_{\text{WB}} \sigma(w')$ .

Summarising, we have obtained that there are processes  $r, r''$  such that  $\sigma(u') \xrightarrow{a} r \xrightarrow{\varepsilon} r''$  and  $r'' \sim_{\text{WB}} \mathbf{0}$ , and that  $x \in \text{var}(u'')$  with  $u \xrightarrow{(x)} \mu$ . Next, we notice that  $\sigma(u') \xrightarrow{a} r \xrightarrow{\varepsilon} r''$  implies that  $\sigma(u) \xrightarrow{a} r \parallel \sigma(u'') \xrightarrow{\varepsilon} r'' \parallel \sigma(u'') \sim_{\text{WB}} \sigma(u'')$ . As  $a \parallel p_n$  and its derivatives cannot perform any  $\tau$ -move,  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$  implies that  $a \parallel p_n \xrightarrow{a} r'$  for some process  $r'$  such that  $r' \sim_{\text{WB}} \sigma(u'')$ . Given the structure of  $a \parallel p_n$ , we can distinguish two major cases, according to the possible form of  $r'$ :

- $r' = a \parallel a^{\leq i}$  for some  $i \in \{2, \dots, n\}$ . We proceed to show that in this case it is not possible to have  $r' \sim_{\text{WB}} \sigma(u'')$ . Recall that, by Equation (4),  $\sigma(x) \sim_{\text{RWB}} \sum_{k=1}^m aa^{\leq i_k}$  for  $m > 1$  and  $2 \leq i_1 < \dots < i_m \leq n$ . As  $u'' \xrightarrow{(x)} a$ , by Lemma 9 we have that  $\sigma(u'') \xrightarrow{a} a^{\leq i_1}$  and  $\sigma(u'') \xrightarrow{a} a^{\leq i_m}$  (where no parallel composition is considered because we have seen in the analysis above that any potential parallel component would be weak bisimilar to  $\mathbf{0}$ ). In particular,  $i_1 < i_m$  gives  $a^{\leq i_1} \sim_{\text{WB}} a^{\leq i_m}$  as the two processes have different depths (besides different branching structures). On the one hand, we have that  $r' \xrightarrow{a} a \parallel a^j$  for some  $j \in \{0, \dots, i-1\}$ , and  $a \parallel a^j \sim_{\text{WB}} a^{j+1} \sim_{\text{WB}} a^{\leq i_1}, a^{\leq i_m}$  for any  $j \in \{0, \dots, i-1\}$  and  $i_1, i_m \geq 2$ . On the other hand,  $r' \xrightarrow{a} \mathbf{0} \parallel a^{\leq i} \sim_{\text{WB}} a^{\leq i}$ , but  $a^{\leq i}$  cannot be weak bisimilar to both  $a^{\leq i_1}$  and  $a^{\leq i_m}$ . Hence,  $r' \not\sim_{\text{WB}} \sigma(u'')$  in this case.
- $r' = \mathbf{0} \parallel p_n$ . Then  $\sigma(u'') \sim_{\text{WB}} p_n$ . Notice that  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$  implies  $\text{init}(\sigma(u'')) = \{a\}$ . Hence, by Lemma 3,  $\tau \notin \text{init}(\sigma(u'')) \cup \text{init}(p_n)$  and  $\sigma(u'') \sim_{\text{WB}} p_n$  imply  $\sigma(u'') \sim_{\text{RWB}} p_n$ . In particular, it follows that  $\text{Rdepth}(\sigma(u'')) = n + 1$ . Since  $\text{Rdepth}(\sigma(u)) = n + 2 \geq \text{Rdepth}(\sigma(u_j)) = \text{Rdepth}(\sigma(u')) + \text{Rdepth}(\sigma(u''))$ , and  $\text{Rdepth}(\sigma(u')) \neq 0$ , we obtain that  $\text{Rdepth}(\sigma(u')) = 1$ . In addition,  $\text{init}(\sigma(u')) = \{a\}$ . It follows that whenever  $\sigma(u') \xrightarrow{\mu} q'$ , for some process  $q'$ , then  $\mu = a$  and  $q' \sim_{\text{WB}} \mathbf{0}$ . Hence,  $\sigma(u') \sim_{\text{RWB}} a$ .

Summarising, we have obtained that  $\sigma(u_j) = \sigma(u') \parallel \sigma(u'') \sim_{\text{RWB}} a \parallel p_n$ , and it is thus the summand we were looking for.

- iii. There are variables  $y, z$  such that  $\sigma'(y) \xrightarrow{\alpha} r_y, \sigma'(z) \xrightarrow{\bar{\alpha}} r_z, u' \parallel u'' \xrightarrow{(y,z)} \tau c$ , and  $\sigma'[y \mapsto r_y, z \mapsto r_z](c) = q'$ . We can proceed as in the previous case to argue that if both  $y$  and  $z$  are different from  $x$ , then we obtain a contradiction with  $\sigma(u) \sim_{\text{RWB}} a \parallel p_n$ . Similarly, it cannot be the case that both  $y$  and  $z$  are equal to  $x$ , for otherwise they would not be able to synchronise as  $\text{init}(\sigma'(x)) = \{\bar{a}\}$ . Hence, we can assume, without loss of generality, that  $y \neq x$  and  $z = x$ . Thus,  $\sigma'(y) = \sigma(y)$  and  $\alpha = a$ . As we can also assume, without loss of generality, that  $y \in \text{var}(u')$  and  $x \in \text{var}(u'')$  (having both in one term gives a

**Table 5**  
Inference rules for the transition  $\xrightarrow{x}$ .

$(i_1) \frac{}{x \xrightarrow{x} x_d}$	$(i_2) \frac{t \xrightarrow{x} c}{t + u \xrightarrow{x} c}$	$(i_3) \frac{t \xrightarrow{x} c}{t \parallel u \xrightarrow{x} c \parallel u}$
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contradiction either with  $q' \sim_{\text{WB}} a \parallel p_n$  or  $u', u'' \sim_{\text{RWB}} \mathbf{0}$ . We can then proceed exactly as in the previous case to show that  $\sigma(u') \sim_{\text{RWB}} a$  and  $\sigma(u'') \sim_{\text{RWB}} p_n$ , giving that  $u_j$  is the desired summand.

This completes the proof for the case  $u_j = u' \parallel u''$  for some terms  $u', u''$ .

This completes the analysis for the case  $t_i = t' \parallel t''$  for some terms  $t', t''$ .

The proof of Proposition 9 is now complete.  $\square$

One can now proceed to prove Theorem 4. Since the proof is a mere rewriting of that of Theorem 3 (using Proposition 9 in place of Proposition 7), we omit it.

Theorem 2 then follows by noticing that the term  $a \parallel p_n$ , in the left-hand side of equation  $\varepsilon_n$ , has a summand rooted weak bisimilar to  $a \parallel p_n$ , whilst the term  $ap_n + \sum_{i=2}^n a(a \parallel a^{\leq i})$ , i.e., the right-hand side, does not. Hence, the collection of infinitely many equations  $\varepsilon_n$  ( $n \geq 2$ ) is the desired infinite family of equations showing the negative result.

## 11. The case of pure interleaving parallel composition

The proof method applied above to obtain the negative result stated in Theorems 1 and 2 can be employed, with minor modifications, to prove that the negative result holds even if we substitute the full merge operator with *pure interleaving parallel composition* (i.e., parallel composition with no communication). Specifically, we can rephrase all the technical results presented in Sections 4, and 6–8 in terms of interleaving parallel composition (recall that the unique parallel decomposition presented in Section 5 was originally proven with respect to interleaving in [28]), and use them to prove:

**Theorem 5.** *Rooted branching bisimilarity admits no finite equational axiomatisation over CCS with interleaving.*

**Theorem 6.** *Rooted weak bisimilarity admits no finite equational axiomatisation over CCS with interleaving.*

The rephrasing is a simple technical exercise that we leave to the interested reader. However, for the sake of completeness and readability, we report below the decomposition of the semantics of open terms in CCS with interleaving (Section 11.1), the proof of the result corresponding to Proposition 7 (Proposition 10 in Section 11.2), since it is the crux of the proof method and it requires a careful crafting of the substitution that allows us to reason by contradiction, and we also show how the absence of communication allows us to lift the requirements on the structure of terms in Proposition 8, leading to a more general result on provably equal terms having the same variables as summands (Proposition 11 in Section 11.3).

In what follows, let  $\text{CCS}_I$  denote CCS with interleaving parallel composition.

### 11.1. Configurations over CCS with interleaving

$\text{CCS}_I$  configurations are terms defined over a set of variables  $\mathcal{V}_d = \{x_d \mid x \in \mathcal{V}\}$ , disjoint from  $\mathcal{V}$ , and  $\text{CCS}_I$  terms. We use the symbol  $x_d$  to denote that the closed term substituted for an occurrence of variable  $x$  has begun its execution, and it contributes thus to triggering the behaviour of the term in which  $x$  occurs.

**Definition 12** ( $\text{CCS}_I$  configuration). The collection of  $\text{CCS}_I$  configurations, denoted by  $C_I$ , is given by:

$$c ::= x_d \mid t \mid c \parallel c ,$$

where  $t$  is a  $\text{CCS}_I$  term, and  $x_d \in \mathcal{V}_d$ .

Since configurations can have occurrences of variables from  $\mathcal{V} \cup \mathcal{V}_d$ , we extend the definition of  $\text{var}(c)$  accordingly:  $\text{var}(c) = \{x \in \mathcal{V} \mid x \text{ occurs in } c\} \cup \{x_d \in \mathcal{V}_d \mid x_d \text{ occurs in } c\}$ .

We introduce auxiliary transitions of the form  $t \xrightarrow{x} c$ , to denote that variable  $x$  triggers a transition of term  $t$  that results in configuration  $c$ . The auxiliary transitions  $\rightarrow_{\subseteq} \mathcal{T}_I \times \mathcal{V} \times C_I$ , with  $\mathcal{T}_I$  the set of  $\text{CCS}_I$  terms, are formally defined via the inference rules in Table 5, where we omit the symmetric rules to  $(i_2)$  and  $(i_3)$ .

The following lemmas follow from a simple inspection of the rules in Table 5.

**Lemma 26.** *Let  $t$  be a  $\text{CCS}_I$  term and  $x$  be a variable in  $\mathcal{V}$ . If  $t \xrightarrow{x} c$ , then  $c$  is of the form  $x_d \parallel c'$ , for some configuration  $c'$  (possibly  $\mathbf{0}$ ) (modulo axioms P0–P2 in Table 3).*

**Lemma 27.** Let  $t$  be a  $\text{CCS}_1$  term,  $\sigma$  be a closed substitution, and  $x \in \mathcal{V}$ . If  $\sigma(x) \xrightarrow{\mu} p$ , for some  $\mu \in \mathcal{A}_\tau$  and process  $p$ , and  $t \xrightarrow{x} c$ , for some configuration  $c \in C_j$ , then  $\sigma(t) \xrightarrow{\mu} \sigma[x_d \mapsto p](c)$ .

We can now characterise the cases in which a variable triggers the behaviour of a  $\text{CCS}_1$  term.

**Lemma 28.** Let  $t$  be a  $\text{CCS}_1$  term,  $\sigma$  be a closed substitution. If  $\sigma(t) \xrightarrow{\mu} p$ , for some  $\mu \in \mathcal{A}_\tau$  and process  $p$ , then one of the following holds:

1. There is a  $\text{CCS}_1$  term  $t'$  such that  $t \xrightarrow{\mu} t'$  and  $\sigma(t') = p$ .
2. There are a variable  $x$ , a process  $q$ , and a configuration  $c$  such that  $\sigma(x) \xrightarrow{\mu} q$ ,  $t \xrightarrow{x} c$ , and  $\sigma[x_d \mapsto q](c) = p$ .

### 11.2. The crux in the proof of Theorem 5

In order to prove Theorem 5 we can use the same family of equations  $\{\varepsilon_n \mid n \geq 2\}$  that we introduced in Section 8 to prove Theorem 1. In particular, we recall that, for each  $n \geq 2$ , we let  $p_n = \sum_{i=2}^n aa^{\leq i}$ .

We now proceed to prove the result corresponding to Proposition 7. Please notice that whenever in the proof we refer to a result provided in Section 7, we actually mean the version of that result over  $\text{CCS}_1$ .

**Proposition 10.** Let  $t \approx u$  be an equation over  $\text{CCS}$  with interleaving that is sound modulo  $\sim_{\text{RBB}}$ . Let  $\sigma$  be a closed substitution with  $p = \sigma(t)$  and  $q = \sigma(u)$ . Suppose that  $p$  and  $q$  have neither  $\mathbf{0}$  summands nor factors, and  $p, q \sim_{\text{RBB}} a \parallel p_n$  for some  $n$  larger than the sizes of  $t$  and  $u$ . If  $p$  has a summand that is rooted branching bisimilar to  $a \parallel p_n$ , then so does  $q$ .

**Proof.** Observe, first of all, that since  $\sigma(t) = p$  and  $\sigma(u) = q$  have no  $\mathbf{0}$  summands or factors, then neither do  $t$  and  $u$ . Hence, by Remark 3, we have that for some finite non-empty index sets  $I, J$ ,  $t = \sum_{i \in I} t_i$  and  $u = \sum_{j \in J} u_j$ , where none of the  $t_i$  ( $i \in I$ ) and  $u_j$  ( $j \in J$ ) is  $\mathbf{0}$ , has  $+$  as its head operator, has  $\mathbf{0}$  summands and factors.

As  $p = \sigma(t)$  has a summand rooted branching bisimilar to  $a \parallel p_n$ , there is an index  $i \in I$  such that  $\sigma(t_i) \sim_{\text{RBB}} a \parallel p_n$ .

Our aim is now to show that there is an index  $j \in J$  such that  $\sigma(u_j) \sim_{\text{RBB}} a \parallel p_n$ , proving that  $q = \sigma(u)$  also has a summand rooted branching bisimilar to  $a \parallel p_n$ .

We proceed by a case analysis on the form  $t_i$  may have.

1. CASE  $t_i = x$  FOR SOME VARIABLE  $x$ . In this case, we have  $\sigma(x) \sim_{\text{RBB}} a \parallel p_n$ , and  $t$  has  $x$  as a summand. As  $t \approx u$  is sound modulo rooted branching bisimilarity and neither  $t$  nor  $u$  have  $\mathbf{0}$  summands or factors, it follows that  $u$  also has  $x$  as a summand (Proposition 6). Thus there is an index  $j \in J$  such that  $u_j = x$ , and, modulo rooted branching bisimulation,  $\sigma(u)$  has  $a \parallel p_n$  as a summand.
2. CASE  $t_i = \mu t'$  FOR SOME TERM  $t'$ . This case is vacuous because, since the root condition implies  $\mu = a$ , and  $a\sigma(t') \xrightarrow{a} \sigma(t')$  is the only transition afforded by  $\sigma(t_i)$ , this term cannot be branching bisimilar to  $a \parallel p_n$ . Indeed  $a \parallel p_n$  can perform both,  $a \parallel p_n \xrightarrow{a} \mathbf{0} \parallel p_n \sim_{\text{BB}} p_n$ , and  $a \parallel p_n \xrightarrow{a} a \parallel a^{\leq i}$ , for some  $i \in \{2, \dots, n\}$ . Clearly,  $p_n \sim_{\text{BB}} a \parallel a^{\leq i}$  for any  $i \in \{2, \dots, n\}$ , and thus  $\sigma(t')$  cannot be branching bisimilar to both of them.
3. CASE  $t_i = t''$  FOR SOME TERMS  $t', t''$ . In this case, we have  $\sigma(t') \parallel \sigma(t'') \sim_{\text{RBB}} a \parallel p_n$ . As  $\sigma(t_i)$  has no  $\mathbf{0}$  factors, it follows that  $\sigma(t') \sim_{\text{RBB}} \mathbf{0}$  and  $\sigma(t'') \sim_{\text{RBB}} \mathbf{0}$ . Thus by Lemma 19 (and without loss of generality)  $\sigma(t') \sim_{\text{RBB}} a$  and  $\sigma(t'') \sim_{\text{RBB}} p_n$ . Now,  $t''$  can be written as  $t'' = v_1 + \dots + v_M$ , ( $M > 0$ ), where none of the summands  $v_i$  is  $\mathbf{0}$  or a sum. Observe that, since  $n$  is larger than the size of  $t$ , we have that  $M < n$ . Hence, since  $\sigma(t'') \sim_{\text{RBB}} p_n$ , there must be some  $h \in \{1, \dots, M\}$  such that  $\sigma(v_h) \sim_{\text{RBB}} \sum_{k=1}^m aa^{\leq i_k}$  for some  $m > 1$  and  $2 \leq i_1 < \dots < i_m \leq n$ . The term  $\sigma(v_h)$  has no  $\mathbf{0}$  summands or factors, or else so would  $\sigma(t'')$ , and thus  $p = \sigma(t)$ . By Lemma 20, it follows that  $v_h$  can only be a variable  $x$  and thus that

$$\sigma(x) \sim_{\text{RBB}} \sum_{k=1}^m aa^{\leq i_k} . \quad (7)$$

Observe, for later use, that, since  $t'$  has no  $\mathbf{0}$  factors, the above equation yields that  $x \notin \text{var}(t')$ , or else  $\sigma(t') \sim_{\text{RBB}} a$  (Lemma 4). So, modulo rooted branching bisimilarity,  $t_i$  has the form  $t' \parallel (x + t'')$ , for some term  $t''$ , with  $x \notin \text{var}(t')$  and  $\sigma(t') \sim_{\text{RBB}} a$ .

Our order of business will now be to use the information collected so far to argue that  $\sigma(u)$  has a summand rooted branching bisimilar to  $a \parallel p_n$ . To this end, consider the substitution

$$\sigma' = \sigma[x \mapsto \tau.p_n] .$$

We have that

$$\begin{aligned} \sigma'(t_i) &= \sigma'(t') \parallel \sigma'(t'') \\ &= \sigma(t') \parallel \sigma'(t'') && \text{(As } x \notin \text{var}(t')\text{)} \\ &\sim_{\text{RBB}} a \parallel (\tau.p_n + \sigma'(t'')) && \text{(As } t'' = x + t'''\text{)}. \end{aligned}$$

Thus, there is some process  $p'$  such that  $\sigma'(t_i) \xrightarrow{\tau} p' \sim_{\text{BB}} a \parallel p_n$ , so that

$$\sigma'(t) \xrightarrow{\tau} p' \sim_{\text{BB}} a \parallel p_n$$

also holds. Since  $t \approx u$  is sound modulo  $\sim_{\text{RBB}}$ , it follows that

$$\sigma'(t) \sim_{\text{RBB}} \sigma'(u) .$$

Hence, we can infer that there are a  $j \in J$  and a  $q'$  such that

$$\sigma'(u_j) \xrightarrow{\tau} q' \sim_{\text{BB}} a \parallel p_n . \quad (8)$$

Recall that, by one of the assumptions of the proposition,  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ , and thus  $\sigma(u)$  has rooted depth  $n + 2$ . On the other hand, by Equation (8),

$$\text{Rdepth}(\sigma'(u_j)) \geq n + 3 .$$

Since  $\sigma$  and  $\sigma'$  differ only in the closed term they map variable  $x$  to, it follows that

$$x \in \text{var}(u_j) . \quad (9)$$

We now proceed to show that  $\sigma(u_j) \sim_{\text{RBB}} a \parallel p_n$  by a further case analysis on the form a term  $u_j$  satisfying Equation (8) and (9) may have.

- (a) CASE  $u_j = x$ . This case is vacuous because  $\sigma'(x) = \tau.p_n \xrightarrow{\tau} p_n \sim_{\text{BB}} a \parallel p_n$ , and thus this possible form for  $u_j$  does not meet Equation (8).
- (b) CASE  $u_j = \mu u'$  FOR SOME TERM  $u'$ . In light of Equation (8), we have that  $\mu = \tau$  and  $q' = \sigma'(u') \sim_{\text{BB}} a \parallel p_n$ . However,  $\mu = \tau$  gives an immediate contradiction with  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ . In fact,  $\sigma'(u_j) = \tau.\sigma'(u')$  gives  $\sigma(u_j) = \tau.\sigma(u')$  (since it is a prefix, the  $\tau$ -move is not triggered by  $x$ ) and, thus,  $\sigma(u) \xrightarrow{\tau} \sigma(u')$  while  $a \parallel p_n \xrightarrow{\tau} \cdot$ .
- (c) CASE  $u_j = u' \parallel u''$  FOR SOME TERMS  $u', u''$ . Our assumption that  $\sigma(u)$  has no  $\mathbf{0}$  factors yields that none of the terms  $u', u'', \sigma(u')$  and  $\sigma(u'')$  is rooted branching bisimilar to  $\mathbf{0}$ . In addition,  $\text{init}(\sigma(u)) = \{a\}$ , gives  $\tau \notin \text{init}(\sigma(u')) \cup \text{init}(\sigma(u''))$ , so that, by Lemma 3,  $\sigma(u'), \sigma(u'') \sim_{\text{BB}} \mathbf{0}$  holds. Moreover, by Equation (9), either  $x \in \text{var}(u')$  or  $x \in \text{var}(u'')$  (and a simple observation on the depth of terms guarantees that it cannot occur in both).

By Equation (8), we have that  $\sigma(u') \parallel \sigma(u'') \xrightarrow{\tau} q' \sim_{\text{BB}} a \parallel p_n$ . Hence, we can distinguish two cases, according to the possible origins for such a transition according to Lemma 28:

- i. There is a term  $w$  such that  $u' \parallel u'' \xrightarrow{\tau} w$ . This case is vacuous, since  $u' \parallel u'' \xrightarrow{\tau} w$  implies  $u \xrightarrow{\tau} w$ , and, thus,  $\sigma(u) \xrightarrow{\tau} \sigma(w)$ , giving a contradiction with  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ .
- ii. There is a variable  $y \in \text{var}(u')$  such that  $\sigma'(y) \xrightarrow{\tau} r_y$ ,  $u' \parallel u'' \xrightarrow{y} c$ , and  $\sigma'[y \mapsto r_y](c) = q'$ . (The symmetric case with  $y \in \text{var}(u'')$  can be treated in a similar way and it is therefore omitted.)

Notice that if  $y \neq x$  then  $\sigma'(y) = \sigma(y)$  giving  $\sigma(y) \xrightarrow{\tau}$  and thus  $\sigma(u) \xrightarrow{\tau}$  (Lemma 7). As  $a \parallel p_n \xrightarrow{\tau} \cdot$ , we obtain a contradiction with  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ . Hence  $y = x$ .

Since  $\sigma'(x) \xrightarrow{\tau} p_n$ , we have that  $\sigma'(u') \parallel \sigma'(u'') \xrightarrow{\tau} p_n \parallel \sigma'(u'') \parallel \sigma'(w)$  (where the general term  $w$  is included, as from  $x \in \text{var}(u')$ , and  $u' \xrightarrow{x}$ , we can infer that  $u'$  can be written in the general form  $((x + w_1) \parallel w) + w_2$ ). From  $p_n \parallel \sigma'(u'') \parallel \sigma'(w) \sim_{\text{BB}} a \parallel p_n$  it follows (reasoning on the depth of terms) that  $\sigma'(u) \sim_{\text{BB}} \mathbf{0}$  and, in particular,  $\sigma'(w) = \sigma(w)$ . Moreover, as  $x \notin \text{var}(u'')$ , we have that  $\sigma'(u'') = \sigma(u'')$ .

Summarising, we have obtained that  $\sigma'(u') \parallel \sigma'(u'') = \sigma'(u') \parallel \sigma'(u'') \xrightarrow{\tau} p_n \parallel \sigma(u'') \sim_{\text{BB}} a \parallel p_n$ . By Lemma 18, we get that  $\sigma(u'') \sim_{\text{BB}} a$ . Moreover, as  $\tau \notin \text{init}(\sigma(u''))$  (for otherwise  $\sigma(u) \xrightarrow{\tau}$  contradicting thus  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ ), we can directly infer that  $\sigma(u'') \sim_{\text{RBB}} a$ . Hence there is a process  $r$  such that  $\sigma(u'') \xrightarrow{a} r$  and  $r \sim_{\text{BB}} \mathbf{0}$ . This implies that  $\sigma(u_j) = \sigma(u') \parallel \sigma(u'') \xrightarrow{a} \sigma(u') \parallel r \sim_{\text{BB}} \sigma(u')$ . From  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$ , it then follows that there is a process  $r'$  such that  $a \parallel p_n \xrightarrow{a} r'$  and  $r' \sim_{\text{BB}} \sigma(u')$ . Given the structure of  $a \parallel p_n$ , we can distinguish two major cases, according to the possible form of  $r'$ :

- $r' = a \parallel a^{\leq i}$  for some  $i \in \{2, \dots, n\}$ . We proceed to show that in this case it is not possible to have  $r' \sim_{\text{BB}} \sigma(u')$ . Recall that, by Equation (7),  $\sigma(x) \sim_{\text{RBB}} \sum_{k=1}^m a a^{\leq i k}$  for  $m > 1$  and  $2 \leq i_1 < \dots < i_m \leq n$ . As  $u' \xrightarrow{x}$ , by Lemma 9 we have that  $\sigma(u') \xrightarrow{a} a^{\leq i_1}$  and  $\sigma(u') \xrightarrow{a} a^{\leq i_m}$  (where no parallel composition is considered because we have seen in the analysis above that any potential parallel component would be branching bisimilar to  $\mathbf{0}$ ). In particular,  $i_1 < i_m$  gives  $a^{\leq i_1} \sim_{\text{BB}} a^{\leq i_m}$  as the two processes have different depths (besides different branching structures). On the one hand, we have that  $r' \xrightarrow{a} a \parallel a^j$  for some  $j \in \{0, \dots, i-1\}$ , and  $a \parallel a^j \sim_{\text{BB}} a^{j+1} \sim_{\text{BB}} a^{\leq i_1}, a^{\leq i_m}$  for any  $j \in \{0, \dots, i-1\}$  and  $i_1, i_m \geq 2$ . On the other hand,  $r' \xrightarrow{a} \mathbf{0} \parallel a^{\leq i} \sim_{\text{BB}} a^{\leq i}$ , but  $a^{\leq i}$  cannot be branching bisimilar to both  $a^{\leq i_1}$  and  $a^{\leq i_m}$ . Hence,  $r' \not\sim_{\text{BB}} \sigma(u')$  in this case.
- $r' = \mathbf{0} \parallel p_n$ . Then  $\sigma(u') \sim_{\text{BB}} p_n$ . Notice that  $\sigma(u) \sim_{\text{RBB}} a \parallel p_n$  implies  $\text{init}(\sigma(u')) = \{a\}$ . Hence, by Lemma 3,  $\tau \notin \text{init}(\sigma(u')) \cup \text{init}(p_n)$  and  $\sigma(u') \sim_{\text{BB}} p_n$  imply  $\sigma(u') \sim_{\text{RBB}} p_n$ .

Summarising, we have obtained that  $\sigma(u_j) = \sigma(u') \parallel \sigma(u'') \sim_{\text{RBB}} a \parallel p_n$ , and it is thus the summand we were looking for. This completes the proof for the case  $u_j = u' \parallel u''$  for some terms  $u', u''$ .

This completes the analysis for the case  $t_i = t' \parallel t''$  for some terms  $t', t''$ .

The proof of Proposition 10 is now complete.  $\square$

### 11.3. Proving Theorem 6

In the proof of the negative result for rooted weak bisimilarity over CCS, we made use of Proposition 8 to ensure that two rooted weak bisimilar terms had the same variable as a summand. That result is based on quite strict assumptions on the form of closed substitution instances of the two terms and of the variable. We now show that those constraints can be relaxed when dealing with  $\text{CCS}_1$  terms: to guarantee that two rooted weak bisimilar terms have the same variable as a summand, it is enough to require that one of them does not have  $\tau$  as an initial move.

**Proposition 11.** *Let  $t, u$  be two  $\text{CCS}_1$  terms without  $\mathbf{0}$  summands and  $\mathbf{0}$  factors. Let  $t \approx u$  be sound modulo rooted weak bisimilarity. Assume that  $\tau \notin \text{init}(t)$ . If  $t$  has a summand  $x$ , then so does  $u$ .*

**Proof.** Observe, first of all, that since  $t$  and  $u$  have no  $\mathbf{0}$  summands or factors, by Remark 3 we can assume that  $t = \sum_{i \in I} t_i$  and  $u = \sum_{j \in J} u_j$  for some finite non-empty index sets  $I, J$ , where none of the  $t_i$  ( $i \in I$ ) and  $u_j$  ( $j \in J$ ) has  $+$  as its head operator, and none of the  $t_i$  ( $i \in I$ ) and  $u_j$  ( $j \in J$ ) have  $\mathbf{0}$  summands or factors. Therefore, if  $t$  has a summand  $x$ , then that there is some index  $i \in I$  such that  $t_i = x$ . We aim to show that  $u$  has a summand  $x$  as well. Consider the substitution  $\sigma_a$  mapping each variable to  $a.\mathbf{0}$ . Pick an integer  $m$  larger than the rooted depths of  $\sigma_a(t)$  and of  $\sigma_a(u)$ . Let  $\sigma$  be the substitution mapping  $x$  to the term  $\tau.a^m$  and agreeing with  $\sigma_a$  on all the other variables.

As  $t \approx u$  is sound modulo rooted weak bisimilarity, we have that  $\sigma(t) \sim_{\text{RWB}} \sigma(u)$ . Moreover, the term  $\sigma(t)$  affords the transition  $\sigma(t) \xrightarrow{\tau} a^m$ , for  $t_i = x$  and  $\sigma(x) = \tau a^m \xrightarrow{\tau} a^m$ . Hence, for some closed term  $p$ ,

$$\sigma(u) = \sum_{j \in J} \sigma(u_j) \xrightarrow{\tau} p \sim_{\text{WB}} a^m.$$

This means that there is a  $j \in J$  such that  $\sigma(u_j) \xrightarrow{\tau} p$ . We claim that this  $u_j$  can only be the variable  $x$ . To see that this claim holds, observe, first of all, that  $x \in \text{var}(u_j)$ . In fact, if  $x$  did not occur in  $u_j$ , then we would reach a contradiction thus:

$$m = \text{depth}(p) < \text{Rdepth}(\sigma(u_j)) = \text{Rdepth}(\sigma_a(u_j)) \leq \text{Rdepth}(\sigma_a(u)) < m.$$

Moreover, we remark that  $\sigma(u_j) \xrightarrow{\tau} p$  implies that  $\sigma(u_j)$  must perform at least one  $\tau$ -step in the sequence of weak transitions leading to  $p$ . Let us now proceed by a case analysis on the other forms that  $u_j$  might have, and show, in each case, that we reach some contradictory conclusion.

- $u_j = \mu.u'$  for some  $u'$  with  $x \in \text{var}(u')$ . As  $\sigma(u_j) \xrightarrow{\tau}$ , we get that  $\mu = \tau$ . However, this together with  $\tau \notin \text{init}(t)$  gives a contradiction with  $t \approx u$  being sound modulo  $\sim_{\text{RWB}}$ , as, for instance,  $\sigma_a(t) \sim_{\text{RWB}} \sigma_a(u)$  since  $\sigma_a(u) \xrightarrow{\tau}$  and  $\sigma_a(t) \not\xrightarrow{\tau}$ .
  - $u_j = u' \parallel u''$ . As  $\sigma(u_j) \xrightarrow{\tau}$ , we can distinguish two cases, according to Lemma 28:
    - $u' \xrightarrow{\tau}$ , or  $u'' \xrightarrow{\tau}$ . We can then proceed as in the case of prefixing discussed above to see that this case gives a contradiction with the soundness of  $t \approx u$  modulo  $\sim_{\text{RWB}}$ .
    - There are a variable  $y$ , a process  $r_y$ , and a configuration  $c$  such that  $\sigma(y) \xrightarrow{\tau} r_y$ ,  $u_j \xrightarrow{y} c$ , and  $\sigma[y_d \mapsto r_y](c) \xrightarrow{\epsilon} p$ . By the choice of  $\sigma$ , it is clear that the only possibility is to have  $y = x$  and  $r_y = a^m$ . Assume now, without loss of generality, that  $x \in \text{var}(u')$ , so that  $u' \xrightarrow{x} c'$  and  $u_j \xrightarrow{x} c' \parallel u''$ . In particular, by Lemma 26, we have that  $c'$  is of the form  $x_d \parallel c_1$  for some configuration  $c_1$ , giving that  $\sigma(u_j) \xrightarrow{\tau} a^m \parallel \sigma(c_1) \parallel \sigma(u'')$ . We can also infer that none of the potential  $\tau$ -moves in the sequence leading to  $p$  can be due to  $\sigma(u'') \xrightarrow{\tau}$ . In fact, by the choice of  $\sigma$ ,  $\sigma(u'')$  would have only two ways to perform a  $\tau$ -move:
      - \* Either because  $u'' \xrightarrow{\tau}$ , but we have already ruled out this possibility.
      - \* Or because  $u'' \xrightarrow{x} c''$  for some configuration  $c''$ . In this case, Lemma 26 would give  $c'' = x_d \parallel c_2$ , so that  $\sigma(u_j) \xrightarrow{\tau} a^m \parallel \sigma(c_1) \parallel \sigma(u'') \xrightarrow{\tau} a^m \parallel \sigma(c_1) \parallel a^m \parallel \sigma(c_2)$ . This would give  $a^m \parallel \sigma(c_1) \parallel a^m \parallel \sigma(c_2) \xrightarrow{\epsilon} a^m \parallel r_1 \parallel a^m \parallel r_2 = p$ , for some processes  $r_1, r_2$ . Hence  $\text{depth}(p) = m + m + \text{depth}(r_1) + \text{depth}(r_2) > m = \text{depth}(a^m)$ , giving thus a contradiction with  $p \sim_{\text{WB}} a^m$ .
- Hence  $a^m \parallel \sigma(c_1) \parallel \sigma(u'') \xrightarrow{\epsilon} p$  can only follow from  $\sigma(c_1) \xrightarrow{\epsilon} r_1$  for some process  $r_1$ , and  $p = a^m \parallel r_1 \parallel \sigma(u'')$ . Notice now that as  $u$  has no  $\mathbf{0}$  factors, and  $\sigma(y) \neq \mathbf{0}$  for all  $y \in \mathcal{V}$ , we have that  $\text{depth}(\sigma(u'')) \geq 1$ . We can therefore conclude that  $\text{depth}(p) = m + 1 + \text{depth}(r_1) \geq m + 1 > m = \text{depth}(a^m)$ , giving thus a contradiction with  $p \sim_{\text{WB}} a^m$ .

We may therefore conclude that  $u_j = x$ , which was to be shown.  $\square$

The proof of the negative result for rooted weak bisimilarity over  $\text{CCS}_1$  (Theorem 6) then proceeds as in Section 10, and it is therefore omitted. For the sake of completeness, we remark that to prove the analogue to Proposition 9, we use Proposition 11 in place of Proposition 8, and the same substitution  $\sigma'$  employed in the proof of Proposition 10.

## 12. Extending the negative results to other congruences in the spectrum

The equations  $\varepsilon_n$  ( $n \geq 1$ ) in the family that we used to show the four negative results are, clearly, all sound modulo rooted branching bisimilarity and rooted weak bisimilarity. Consequently, they are also sound modulo any congruence  $\mathcal{R}$  that is coarser than the former, and finer than the latter, i.e., any congruence  $\mathcal{R}$  such that  $\sim_{\text{RBB}} \subseteq \mathcal{R} \subseteq \sim_{\text{RWB}}$ . Hence, the negative result for all these congruences can be derived from that for  $\sim_{\text{RWB}}$ , by exploiting this fact and that any finite axiom system that is sound modulo  $\mathcal{R}$  is also sound modulo  $\sim_{\text{RWB}}$ .

**Theorem 7.** *Let  $\mathcal{R}$  be a congruence such that  $\sim_{\text{RBB}} \subseteq \mathcal{R} \subseteq \sim_{\text{RWB}}$ . Then  $\mathcal{R}$  has no finite, ground-complete, equational axiomatisation over CCS, and over  $\text{CCS}_1$ .*

**Proof.** We prove the result for CCS. The one for  $\text{CCS}_1$  is obtained from a straightforward adaptation. Let  $\mathcal{E}$  be a finite equational axiomatisation for CCS that is sound modulo  $\mathcal{R}$ . Since  $\mathcal{R}$  is included in  $\sim_{\text{RWB}}$ , we have that the axiom system  $\mathcal{E}$  is sound modulo  $\sim_{\text{RWB}}$ . Let  $N \geq 2$  and larger than the size of each term in the equations in  $\mathcal{E}$ . Theorem 4 implies that the equation

$$a \parallel p_N \approx a p_N + \sum_{i=1}^N a(a \parallel a^{\leq i})$$

cannot be derived from  $\mathcal{E}$ . Since this equation is sound modulo  $\mathcal{R}$ , namely

$$a \parallel p_N \mathcal{R} a p_N + \sum_{i=1}^N a(a \parallel a^{\leq i})$$

it follows that  $\mathcal{E}$  is not complete modulo  $\mathcal{R}$ .  $\square$

Theorem 7 can be applied to establish, in particular, that the rooted versions of  $\eta$ -bisimilarity [11] and delay bisimilarity [30] have no finite, ground-complete equational axiomatisation over CCS, and over  $\text{CCS}_1$ .

**Definition 13** ( $\eta$  and delay bisimulations). Let  $(\mathbf{P}, \mathcal{A}_\tau, \rightarrow)$  be a LTS.

$\eta$ -bisimilarity, denoted by  $\sim_{\eta\text{B}}$ , is the largest binary relation over  $\mathbf{P}$  such that whenever  $p \sim_{\eta\text{B}} q$ :

- whenever  $p \xrightarrow{\mu} p'$ , then either:
  - $\mu = \tau$  and  $p' \mathcal{R} q$ , or
  - there are processes  $q_1, q_2, q'$  such that  $q \xrightarrow{\varepsilon} q_1 \xrightarrow{\mu} q_2 \xrightarrow{\varepsilon} q'$ ,  $p \mathcal{R} q_1$  and  $p' \mathcal{R} q'$ ;
- whenever  $q \xrightarrow{\mu} q'$ , then either:
  - $\mu = \tau$  and  $p \mathcal{R} q'$ , or
  - there are processes  $p_1, p_2, p'$  such that  $p \xrightarrow{\varepsilon} p_1 \xrightarrow{\mu} p_2 \xrightarrow{\varepsilon} p'$ ,  $q \mathcal{R} p_1$  and  $p' \mathcal{R} q'$ .

Then, we say that  $p$  and  $q$  are *rooted  $\eta$  bisimilar*, denoted by  $p \sim_{\text{R}\eta\text{B}} q$ , iff:

- if  $p \xrightarrow{\mu} p'$  then  $q \xrightarrow{\mu} q_1 \xrightarrow{\varepsilon} q'$  for some  $q_1, q'$  such that  $p' \sim_{\eta\text{B}} q'$ ;
- if  $q \xrightarrow{\mu} q'$  then  $p \xrightarrow{\mu} p_1 \xrightarrow{\varepsilon} p'$  for some  $p_1, p'$  such that  $p' \sim_{\eta\text{B}} q'$ .

*Delay bisimilarity*, denoted by  $\sim_{\text{DB}}$ , is the largest binary relation over  $\mathbf{P}$  such that whenever  $p \sim_{\text{DB}} q$ :

- whenever  $p \xrightarrow{\mu} p'$ , then either:
  - $\mu = \tau$  and  $p' \mathcal{R} q$ , or
  - there are processes  $q_1, q'$  such that  $q \xrightarrow{\varepsilon} q_1 \xrightarrow{\mu} q'$  and  $p' \mathcal{R} q'$ ;
- whenever  $q \xrightarrow{\mu} q'$ , then either:
  - $\mu = \tau$  and  $p \mathcal{R} q'$ , or
  - there are processes  $p_1, p'$  such that  $p \xrightarrow{\varepsilon} p_1 \xrightarrow{\mu} p'$  and  $p' \mathcal{R} q'$ .

Then we say that  $p$  and  $q$  are *rooted delay bisimilar*, denoted by  $p \sim_{\text{RDB}} q$ , iff:

- if  $p \xrightarrow{\mu} p'$  then  $q \xrightarrow{\varepsilon} q_1 \xrightarrow{\mu} q'$  for some  $q_1, q'$  such that  $p' \sim_{\text{DB}} q'$ ;

- if  $q \xrightarrow{\mu} q'$  then  $p \xrightarrow{\varepsilon} p_1 \xrightarrow{\xi} p'$  for some  $p_1, p'$  such that  $p' \sim_{\text{DB}} q'$ .

The following theorem is a corollary of Theorems 4 and 7.

**Theorem 8.** *Rooted  $\eta$ -bisimilarity and rooted delay bisimilarity admit no finite, ground-complete, equational axiomatisation over the languages CCS and CCS<sub>τ</sub>.*

### 13. Concluding remarks

In this paper, we have shown that rooted branching bisimilarity, rooted weak bisimilarity, and all the congruences included between them (in particular, the rooted versions of  $\eta$  and delay bisimilarities), do not admit a finite, ground-complete axiomatisation over (the recursion, restriction, and relabelling free fragment of) CCS. The validity of these results does not depend on whether the synchronisation between parallel components is allowed. We remark that our negative results on *ground-completeness* directly entail the non existence of finite, *complete* axiomatisations for the considered congruences over CCS.

Now that we have established the non finite axiomatisability results, a natural question is whether the use of the auxiliary operators from [13] can help us to obtain finite equational bases for the considered congruences.

A positive answer to this question is the contribution of the companion paper [4] currently submitted for publication to this journal. There, we show that if we add the left merge operator  $\mathbb{L}$  to CCS<sub>τ</sub>, obtaining the language CCS<sub>τ</sub> $\mathbb{L}$ , then we can provide a finite, complete axiomatisation for each of the four weak bisimulation-based congruences considered in this work (i.e., branching,  $\eta$ , delay, and weak) over CCS<sub>τ</sub> $\mathbb{L}$ . Those axiomatisations are obtained by extending the complete axiom system for strong bisimilarity over CCS<sub>τ</sub> $\mathbb{L}$  from [7] with axioms expressing the behaviour of  $\mathbb{L}$  in the presence of  $\tau$ -actions (from [14]) and with the suitable  $\tau$ -laws (from [21,25]) necessary to deal with each of the considered semantics. Similarly, we show that if we add  $\mathbb{L}$  and the communication merge operator  $|$  to CCS, obtaining the language denoted CCS<sub>τ</sub> $\mathbb{L}$  $|$ , then we can provide an equational basis for each one of them over CCS<sub>τ</sub> $\mathbb{L}$  $|$ .

The development of general techniques for lifting positive and negative results on the existence of finite equational axiomatisations from concrete semantics, which consider  $\tau$ -steps as observable, and their  $\tau$ -abstracting weak versions is a natural, tantalising and very challenging topic for future research.

### CRediT authorship contribution statement

**Luca Aceto:** Funding acquisition, Writing – original draft, Writing – review & editing. **Valentina Castiglioni:** Funding acquisition, Writing – original draft, Writing – review & editing. **Anna Ingólfssdóttir:** Project administration. **Bas Luttik:** Funding acquisition, Writing – original draft, Writing – review & editing.

### Declaration of competing interest

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### Data availability

No data was used for the research described in the article.

### References

- [1] L. Aceto, E. Anastasiadi, V. Castiglioni, A. Ingólfssdóttir, B. Luttik, In search of lost time: axiomatising parallel composition in process algebras, in: Proceedings of LICS 2021, IEEE, 2021, pp. 1–14.
- [2] L. Aceto, V. Castiglioni, W.J. Fokink, A. Ingólfssdóttir, B. Luttik, Are two binary operators necessary to finitely axiomatise parallel composition?, in: Proceedings of CSL 2021, in: LIPIcs, vol. 183, 2021, pp. 8:1–8:17.
- [3] L. Aceto, V. Castiglioni, A. Ingólfssdóttir, B. Luttik, On the axiomatisation of branching bisimulation congruence over CCS, in: Proceedings of CONCUR 2022, in: LIPIcs, vol. 243, 2022, pp. 6:1–6:18.
- [4] L. Aceto, V. Castiglioni, A. Ingólfssdóttir, B. Luttik, Axiomatising weak bisimulation congruences over ccs with left merge and communication merge, <https://drive.google.com/file/d/1rHUnTP9ensKk57IcJnL8d7llkqp8r7a4/view?usp=sharing>, 2023, Companion paper, currently under review.
- [5] L. Aceto, V. Castiglioni, A. Ingólfssdóttir, B. Luttik, M.R. Pedersen, On the axiomatisability of parallel composition: a journey in the spectrum, in: Proceedings of CONCUR 2020, in: LIPIcs, vol. 171, 2020, pp. 18:1–18:22.
- [6] L. Aceto, W. Fokink, A. Ingólfssdóttir, B. Luttik, CCS with Hennessy’s merge has no finite-equational axiomatization, Theor. Comput. Sci. 330 (2005) 377–405, <https://doi.org/10.1016/j.tcs.2004.10.003>.

- [7] L. Aceto, W. Fokkink, A. Ingólfssdóttir, B. Luttik, A finite equational base for CCS with left merge and communication merge, *ACM Trans. Comput. Log.* 10 (2009) 6:1–6:26, <https://doi.org/10.1145/1459010.1459016>.
- [8] L. Aceto, W. Fokkink, A. Ingólfssdóttir, S. Nain, Bisimilarity is not finitely based over BPA with interrupt, *Theor. Comput. Sci.* 366 (2006) 60–81, <https://doi.org/10.1016/j.tcs.2006.07.003>.
- [9] L. Aceto, D. de Frutos-Escrig, C. Gregorio-Rodríguez, A. Ingólfssdóttir, Axiomatizing weak simulation semantics over BCCSP, *Theor. Comput. Sci.* 537 (2014) 42–71, <https://doi.org/10.1016/j.tcs.2013.03.013>.
- [10] L. Aceto, A. Ingólfssdóttir, B. Luttik, P. van Tilburg, Finite equational bases for fragments of CCS with restriction and relabelling, in: *Proceedings of IFIP TCS 2008*, in: *IFIP*, vol. 273, 2008, pp. 317–332.
- [11] J.C.M. Baeten, R.J. van Glabbeek, Another look at abstraction in process algebra (extended abstract), in: *Proceedings of ICALP87*, in: *Lecture Notes in Computer Science*, vol. 267, 1987, pp. 84–94.
- [12] B. Basten, Branching bisimilarity is an equivalence indeed!, *Inf. Process. Lett.* 58 (1996) 141–147, [https://doi.org/10.1016/0020-0190\(96\)00034-8](https://doi.org/10.1016/0020-0190(96)00034-8).
- [13] J.A. Bergstra, J.W. Klop, Process algebra for synchronous communication, *Inf. Control* 60 (1984) 109–137, [https://doi.org/10.1016/S0019-9958\(84\)80025-X](https://doi.org/10.1016/S0019-9958(84)80025-X).
- [14] J.A. Bergstra, J.W. Klop, Algebra of communicating processes with abstraction, *Theor. Comput. Sci.* 37 (1985) 77–121, [https://doi.org/10.1016/0304-3975\(85\)90088-X](https://doi.org/10.1016/0304-3975(85)90088-X).
- [15] B. Bloom, Structural operational semantics for weak bisimulations, *Theor. Comput. Sci.* 146 (1995) 25–68, [https://doi.org/10.1016/0304-3975\(94\)00152-9](https://doi.org/10.1016/0304-3975(94)00152-9).
- [16] T. Chen, W.J. Fokkink, R.J. van Glabbeek, Ready to preorder: the case of weak process semantics, *Inf. Process. Lett.* 109 (2008) 104–111, <https://doi.org/10.1016/j.ipl.2008.09.003>.
- [17] R.J. van Glabbeek, The linear time-branching time spectrum (extended abstract), in: *Proceedings of CONCUR '90*, in: *Lecture Notes in Computer Science*, vol. 458, 1990, pp. 278–297.
- [18] R.J. van Glabbeek, The linear time - branching time spectrum II, in: *Proceedings of CONCUR'93*, in: *Lecture Notes in Computer Science*, vol. 715, 1993, pp. 66–81.
- [19] R.J. van Glabbeek, On cool congruence formats for weak bisimulations, *Theor. Comput. Sci.* 412 (2011) 3283–3302, <https://doi.org/10.1016/j.tcs.2011.02.036>.
- [20] R.J. van Glabbeek, W.P. Weijland, Branching time and abstraction in bisimulation semantics (extended abstract), in: *IFIP Congress*, 1989, pp. 613–618.
- [21] R.J. van Glabbeek, W.P. Weijland, Branching time and abstraction in bisimulation semantics, *J. ACM* 43 (1996) 555–600, <https://doi.org/10.1145/233551.233556>.
- [22] J.F. Groote, A new strategy for proving omega-completeness applied to process algebra, in: *Proceedings of CONCUR '90*, in: *Lecture Notes in Computer Science*, vol. 458, 1990, pp. 314–331.
- [23] M. Hennessy, Axiomatising finite concurrent processes, *SIAM J. Comput.* 17 (1988) 997–1017, <https://doi.org/10.1137/0217063>.
- [24] M. Hennessy, R. Milner, On observing nondeterminism and concurrency, in: *Proceedings of ICALP 1980*, in: *Lecture Notes in Computer Science*, vol. 85, 1980, pp. 299–309.
- [25] M. Hennessy, R. Milner, Algebraic laws for nondeterminism and concurrency, *J. ACM* 32 (1985) 137–161, <https://doi.org/10.1145/2455.2460>.
- [26] C.A.R. Hoare, *Communicating Sequential Processes*, Prentice-Hall, 1985.
- [27] R.M. Keller, Formal verification of parallel programs, *Commun. ACM* 19 (1976) 371–384, <https://doi.org/10.1145/360248.360251>.
- [28] B. Luttik, Unique parallel decomposition in branching and weak bisimulation semantics, *Theor. Comput. Sci.* 612 (2016) 29–44, <https://doi.org/10.1016/j.tcs.2015.10.013>.
- [29] R. Milner, *A Calculus of Communicating Systems*, *Lecture Notes in Computer Science*, vol. 92, Springer, 1980.
- [30] R. Milner, A modal characterisation of observable machine-behaviour, in: *Proceedings of CAAP '81*, in: *Lecture Notes in Computer Science*, vol. 112, 1981, pp. 25–34.
- [31] R. Milner, *Communication and Concurrency*, *PHI Series in Computer Science*, Prentice Hall, 1989.
- [32] F. Moller, *Axioms for Concurrency*, Ph.D. thesis, Department of Computer Science, University of Edinburgh, 1989, Report CST-59-89. Also published as ECS-LFCS-89-84.
- [33] F. Moller, The importance of the left merge operator in process algebras, in: *Proceedings of ICALP '90*, in: *Lecture Notes in Computer Science*, vol. 443, 1990, pp. 752–764.
- [34] F. Moller, The nonexistence of finite axiomatisations for CCS congruences, in: *Proceedings of LICS '90*, 1990, pp. 142–153.
- [35] R.D. Nicola, M. Hennessy, Testing equivalence for processes, in: *Proceedings of ICALP 1983*, in: *Lecture Notes in Computer Science*, vol. 154, 1983, pp. 548–560.
- [36] D.M.R. Park, Concurrency and automata on infinite sequences, in: *Proceedings of GI-Conference*, in: *Lecture Notes in Computer Science*, vol. 104, 1981, pp. 167–183.
- [37] G.D. Plotkin, A structural approach to operational semantics, Report DAIMI FN-19, Computer Science Department, Aarhus University, 1981.