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Axiomatising weak bisimulation congruences over CCS with left merge and communication merge

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ABSTRACT

Classic weak bisimulation-based congruences are not finitely axiomatisable over (the recursion, relabelling, and restriction free fragment of) CCS. Motivated by these negative results, this paper studies the role of auxiliary operators in the finite equational characterisation of CCS parallel composition modulo those congruences. Firstly, we consider CCS with interleaving and left merge. We provide finite equational bases for this language modulo branching, η, delay, and weak bisimulation congruence. In particular, the completeness proofs for η, delay, and weak bisimulation congruence are obtained by reduction to the completeness result for branching bisimulation congruence. Then we extend the language with full merge and communication merge. In this case we provide an equational basis modulo branching bisimulation congruence under the assumption that the set of action names is infinite.

1. Introduction

With our recent paper [5] we enriched the saga of the study of the axiomatisation of the *parallel composition operator* $\|$ in Milner's Calculus of Communicating Systems (CCS) [36]. In detail, we focused on the recursion, restriction, and relabelling free fragment of CCS (henceforth simply referred to as CCS), and we investigated its equational theory modulo *rooted branching bisimilarity* [27,28], which is a classic bisimulation-based notion of congruence that abstracts from internal computation steps in process behaviour. In [5], we showed that CCS is not finitely based modulo the considered congruence. In the companion paper [6], we have extended the non-finite axiomatisability result from [5] to *rooted weak bisimilarity* (also known as *observational congruence*) [31], and to all congruences that are included in it and that include rooted branching bisimilarity, such as *rooted η-bisimilarity* [15] and *rooted delay bisimilarity* [37].

In [5], we also claimed that if we enrich CCS with the *left merge* $\mathbin{\|\!\!\!\_}$ and *communication merge* $|$ operators from Bergrstra and Klop's Algebra of Communicating Processes (ACP) [17], then there is a finite equational basis for rooted branching bisimilarity over the extended language. Recently, thanks to one of the reviewers of an earlier version of this paper, we realised that such a claim is fallacious, due to an issue with Theorem 23 in [5].

Hence, the aim of this paper is to rectify the results from [5] in the first place, and then to investigate the role of the auxiliary operators $\mathbin{\|\!\!\!\_}$ and $|$ towards obtaining finite equational characterisations of the other three bisimulation-based congruences (i.e., rooted η, rooted delay, and rooted weak bisimilarity). Although, as we will see, we were not able to fulfil that goal in full generality, we see

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this contribution as a step to enhance our understanding of equational characterisations of weak semantics over process algebras, in order to pave the way to further studies in this line of research, as well as to obtain some insight on what it would take to develop a technique to lift axiomatisability results from strong to weak semantics over process algebras.

1.1. A brief recap of previous chapters in the saga

In the early 1980s, in the papers [30,31], Hennessy and Milner studied the *equational theory* of (recursion free) CCS and proposed a *ground-complete axiomatisation* for it modulo *strong bisimilarity*, a classic notion of behavioural *congruence* (i.e., an equivalence relation that is compositional with respect to the operators in a language) that allows one to establish whether two processes have the same *observable behaviour* [44]. Notably, the axiomatisation in [30,31] included infinitely many axioms, which were instances of the *expansion law* used to 'simulate equationally' the operational semantics of $\|$.

Then, Bergstra and Klop showed in [17] that, when the set of actions processes can perform is finite, a *finite* ground-complete axiomatisation modulo bisimilarity can be obtained by enriching CCS with two auxiliary operators, namely the left merge $\llfloor$ and the communication merge $|$, expressing one step in the pure interleaving and the synchronous behaviour of $\|$, respectively. Their result was strengthened by Aceto et al. in [9], where it is proved that, over the fragment of CCS without recursion, restriction and relabelling, the auxiliary operators $\llfloor$ and $|$ allow for finitely axiomatising $\|$ modulo bisimilarity also when CCS terms with variables are considered. Moreover, in [13] that result is extended to the fragment of CCS with relabelling and restriction, but without communication.

From those studies, we can infer that $\llfloor$ and $|$ are *sufficient* to finitely axiomatise $\|$ modulo bisimilarity. Moller showed, in [40–42], that they are also *necessary*. He considered a minimal fragment of CCS, including only a constant for an inactive process, action prefixing, nondeterministic choice and interleaving, and proved that, even in the presence of a single action, bisimilarity does not afford a finite ground-complete axiomatisation over that language. (*Please notice that, henceforth, whenever we refer to CCS, we only consider its recursion, restriction and relabelling free fragment.*) Adapting Moller's technique, Aceto et al. proved, in [8], that if we replace $\llfloor$ and $|$ with the so called *Hennessy's merge* $|\!/$ [29], which denotes an asymmetric interleaving operator with communication, then the collection of ground equations that hold modulo bisimilarity over CCS enriched with $|\!/$ is not finitely based. Recently, in [4], in joint work with Fokkink, we showed that, under three reasonable assumptions, a binary auxiliary operator alone does not allow us to obtain a finite, ground-complete axiomatisation modulo bisimilarity.

From bisimilarity to the rest of the spectrum

So far, we have considered equational characterisations of $\|$ modulo strong bisimilarity. However, a plethora of behavioural congruences have been proposed in the literature, corresponding to different levels of abstraction from the information on process execution, which can either be considered irrelevant in an application context, or be unavailable to an external observer. So, another chapter in the saga consisted in extending the studies recalled above to the behavioural congruences in van Glabbeek's linear time-branching time spectrum [21]. In particular, in [7] we delineated the *boundary* between finite and non-finite axiomatisability of $\|$ modulo all the congruences in the spectrum.

From strong to weak semantics

As briefly outlined above, some information on process behaviour can either be considered irrelevant or be unavailable to an external observer. *Weak behavioural semantics* have been introduced to study the effects of these unobservable (or *silent*) actions, usually denoted by τ, on the observable behaviour of processes, each semantics considering a different level of abstraction. A taxonomy of weak semantics is given in [23], and studies on the equational theories of various of these semantics have been carried out over the algebra BCCSP, which consists of the basic operators from CCS and CSP [32] but does not include $\|$ (see, among others, [11,19,22,28,31,43]).

A finite, ground-complete axiomatisation of parallel composition modulo *rooted weak bisimilarity* is provided by Bergstra and Klop in [18] over the algebra ACP_τ that includes the auxiliary operators $\llfloor$ and $|$. Then, in [26], van Glabbeek presented a ground-complete axiomatisation of rooted branching bisimilarity (respectively, rooted η-bisimilarity) over augmented RBB (respectively, RHB) cool GSOS languages via an application of the method from [3] to obtain axiomatisations from an inspection of the SOS rules for the operators of a language. Moreover, in the same paper, he also proposed an extension of that method allowing one to obtain a ground-complete axiomatisation of rooted weak bisimilarity (respectively, rooted delay bisimilarity) over augmented RWB (respectively RDB) cool GSOS languages.

To the best of our knowledge, there is only one study on the axiomatisability of the parallel composition operator of CCS over open terms modulo weak congruences. This is the negative result from [2], where we built on Moller's work in [40,42] to show that a class of weak congruences (including rooted weak bisimilarity) does not afford a finite, complete axiomatisation over the *open* terms of a minimal fragment of CCS with interleaving. This result was strengthened by the negative result in [5], as there we proved the non-existence of a finite *ground-complete* axiomatisation modulo rooted branching bisimilarity over CCS.

1.2. Our contribution: completeness results for weak bisimulation-based semantics

In this paper we consider the congruence relations associated with four classic variants of bisimilarity in the weak setting, namely *rooted branching bisimilarity* ($\sim_{\mathtt{RBB}}$), *rooted η-bisimilarity* ($\sim_{\mathtt{R}\eta\mathtt{B}}$), *rooted delay bisimilarity* ($\sim_{\mathtt{RDB}}$), and *rooted weak bisimilarity* ($\sim_{\mathtt{RWB}}$). Rooted branching bisimilarity is the finest of the four relations and generalises bisimilarity to abstract away from τ-steps of terms while preserving their *branching structure* [27,28]. Rooted weak bisimilarity is the coarsest relation, while $\sim_{\mathtt{R}\eta\mathtt{B}}$ and $\sim_{\mathtt{RDB}}$ both include $\sim_{\mathtt{RBB}}$ and are included in $\sim_{\mathtt{RWB}}$, but are incomparable one with the other.

We divide our studies on the axiomatisability of the four weak congruences over two variants of the considered fragment of CCS, namely $\mathrm{CCS}_{\mathrm{L}}$ and $\mathrm{CCS}_{\mathrm{LC}}$. In $\mathrm{CCS}_{\mathrm{L}}$, the operator $\|$ is the pure interleaving parallel composition operator, and we add only left merge to the syntax of CCS. In $\mathrm{CCS}_{\mathrm{LC}}$, the operator $\|$ is the full merge operator [17,18] and we add both left merge and communication merge to the syntax of CCS.

As a first contribution we show that, in both languages, processes have *unique parallel decompositions* [39] modulo the considered weak equivalences. In [34] it is showed, using techniques from [35], that $\mathrm{CCS}_{\mathrm{L}}$ processes have a unique parallel decomposition modulo branching and weak bisimilarity. Here, we present two extensions of that result. Firstly, we show that, on $\mathrm{CCS}_{\mathrm{L}}$, it holds also modulo η and delay bisimilarity. Secondly, we extend it to $\mathrm{CCS}_{\mathrm{LC}}$ modulo all four weak equivalences.

Then, we focus on $\mathrm{CCS}_{\mathrm{L}}$ and we provide equational bases for the four congruences, which are finite if so is the set of actions over which the language is defined. These axiomatisations are obtained by extending the complete axiom system for strong bisimilarity over $\mathrm{CCS}_{\mathrm{L}}$ from [9] with axioms expressing the behaviour of $\lfloor\!\lfloor$ in the presence of τ-actions (from [18]) and with the suitable τ-laws (from [28,31]) necessary to deal with each of the considered semantics. In detail, first we focus on $\sim_{\mathtt{RBB}}$ and we provide a direct proof of the completeness of the axiom system for it. This is obtained by applying the *distinguishing substitution* proof strategy from [9]. Informally, a substitution $*$ is called *distinguishing* if it makes it possible to identify the original syntactic structure of a normal form N by inspecting the behaviour of $*(N)$. Specifically, such substitutions will allow us to recognise the behaviour induced by the instantiations of variables. Then, we give the complete axiomatisations for the congruences $\sim_{\mathtt{RXB}}$ (for $\mathtt{x} \in \{\eta, \mathtt{d}, \mathtt{w}\}$). These are obtained by combining distinguishing substitutions with a proof technique from [12,24] based on the notion of *saturation*, and on the specification of the semantics of open terms based on the notion of *configuration* from [10]. Intuitively, each relation $\sim_{\mathtt{RXB}}$ coincides with $\sim_{\mathtt{RBB}}$ on closed instances, via distinguishing substitutions, of $\mathtt{x}$-saturated normal forms. Since we show that each normal form is provably equal, using the axiom system for $\sim_{\mathtt{RXB}}$, to an $\mathtt{x}$-saturated one, the completeness of the considered axiom system then directly follows from that for $\sim_{\mathtt{RBB}}$.

Then, we focus on $\mathrm{CCS}_{\mathrm{LC}}$. Also in this case we extend the complete axiomatisation for strong bisimilarity over $\mathrm{CCS}_{\mathrm{LC}}$ from [9] with the axioms used in the case of $\mathrm{CCS}_{\mathrm{L}}$ and with new ones expressing the interplay of $|$ with τ-actions. However, we prove the completeness result, via a direct argument, only under the assumption that the set of actions over which the language is defined is *infinite*. As in the case of $\mathrm{CCS}_{\mathrm{L}}$ we base the proof strategy on the notion of distinguishing substitution, but, to account for the extra complexity entailed by the presence of communication, we need to pair it with the novel notion of *potential of a variable in a term*. This uses, for each variable x, a specially labelled action univocally associated to x, to count how many nested instances of x occur in a normal form N, by analysing the behaviour of the closed instance of N via a distinguishing substitution. The uniqueness of this action, and the additional constraint that it cannot be used for communication with any other subterm in the normal form, are crucial in the technical development of the completeness result, and they can be guaranteed only by assuming the existence of an infinite number of actions.

It remains an open problem to provide a (finite) complete axiomatisation for $\sim_{\mathtt{RBB}}$ over $\mathrm{CCS}_{\mathrm{LC}}$ when the set of actions is finite. Our conjecture is that the proposed axiom system is complete modulo $\sim_{\mathtt{RBB}}$ even if the set of actions is finite, although, so far, all our attempts to obtain a proof have failed (see Section 7 for a detailed discussion).

The axiomatisability of $\sim_{\mathtt{R}\eta\mathtt{B}}, \sim_{\mathtt{RDB}}$ and $\sim_{\mathtt{RWB}}$ over $\mathrm{CCS}_{\mathrm{LC}}$ remain instead open problems. Regarding $\sim_{\mathtt{R}\eta\mathtt{B}}$, similar arguments to those that refuted the positive result in [5] prevent us from applying the saturation proof strategy to it. Moreover, we could not overcome some technical issues preventing us from obtaining a direct proof for it even when the set of actions is infinite (see Section 7 for a detailed discussion). The cases of $\sim_{\mathtt{RDB}}$ and $\sim_{\mathtt{RWB}}$ are even more involved, since the root condition is not enough to ensure the congruence property for delay and weak bisimilarity over $\mathrm{CCS}_{\mathrm{LC}}$. This is mainly due to the presence of the communication merge operator (see Section 8 for more insights).

Our contributions can then be summarised as follows:

1. We show that $\mathrm{CCS}_{\mathrm{L}}$ processes, as well as $\mathrm{CCS}_{\mathrm{LC}}$ processes, admit a unique parallel decomposition modulo the considered (rooted) weak equivalences.
2. We provide a (finite) complete axiomatisation for $\sim_{\mathtt{RBB}}$ over $\mathrm{CCS}_{\mathrm{L}}$.
3. We provide (finite) complete axiomatisations for $\sim_{\mathtt{R}\eta\mathtt{B}}, \sim_{\mathtt{RDB}}, \sim_{\mathtt{RWB}}$ over $\mathrm{CCS}_{\mathrm{L}}$. The completeness proofs are derived in a uniform fashion from the corresponding result for $\sim_{\mathtt{RBB}}$.
4. We provide a complete axiomatisation for $\sim_{\mathtt{RBB}}$ over $\mathrm{CCS}_{\mathrm{LC}}$, provided the set of actions is infinite.

1.3. Organisation of contents

In Section 2 we give an overview of background notions on labelled transition systems, the languages we study in this paper, behavioural equivalences, and equational logic. In Section 3 we discuss the existence of unique parallel decomposition for $\mathrm{CCS}_{\mathrm{L}}$ and $\mathrm{CCS}_{\mathrm{LC}}$ processes modulo the four considered equivalences. In Section 4 we present the (finite) equational basis for rooted branching bisimilarity over $\mathrm{CCS}_{\mathrm{L}}$. The completeness result over $\mathrm{CCS}_{\mathrm{L}}$ is then proved for the other three weak congruences in Section 5. Section 6 then deals with the proof of the completeness result for rooted branching bisimilarity over $\mathrm{CCS}_{\mathrm{LC}}$. In Section 7 we give an overview of the technical subtleties that prevented us from obtaining the completeness result for rooted branching bisimilarity over $\mathrm{CCS}_{\mathrm{LC}}$ when the set of actions is finite, as well as the difficulties encountered in studying an axiomatisation for rooted η-bisimilarity over the same language. We conclude by discussing avenues for future work in Section 8.

Table 1
The SOS rules for CCS$_\mathrm{L}$ operators ($\mu \in \mathcal{A}_\tau$).

$\dfrac{}{\mu.t \xrightarrow{\mu} t}$	$\dfrac{t \xrightarrow{\mu} t'}{t+u \xrightarrow{\mu} t'}$	$\dfrac{u \xrightarrow{\mu} u'}{t+u \xrightarrow{\mu} u'}$
$\dfrac{t \xrightarrow{\mu} t'}{t \parallel u \xrightarrow{\mu} t' \parallel u}$	$\dfrac{u \xrightarrow{\mu} u'}{t \parallel u \xrightarrow{\mu} t \parallel u'}$	$\dfrac{t \xrightarrow{\mu} t'}{t \mathbin{\|\!\!\lfloor} u \xrightarrow{\mu} t' \parallel u}$

Table 2
Additional SOS rules for CCS$_\mathrm{LC}$ ($\alpha \in \mathcal{A} \cup \overline{\mathcal{A}}$).

$\dfrac{t \xrightarrow{\alpha} t' \quad u \xrightarrow{\overline{\alpha}} u'}{t \parallel u \xrightarrow{\tau} t' \parallel u'}$	$\dfrac{t \xrightarrow{\alpha} t' \quad u \xrightarrow{\overline{\alpha}} u'}{t \mid u \xrightarrow{\tau} t' \parallel u'}$

2. Background

Labelled transition systems. As semantic model for describing process behaviour we consider classic *labelled transition systems* (LTSs) [33]. We assume a non-empty set of action names $\mathcal{A}$, and we let $\overline{\mathcal{A}}$ denote the set of action co-names, i.e., $\overline{\mathcal{A}} = \{\overline{a} \mid a \in \mathcal{A}\}$. As usual, we postulate that $\overline{\overline{a}} = a$ and $a \neq \overline{a}$ for all $a \in \mathcal{A}$. Then, we let $\mathcal{A}_\tau = \mathcal{A} \cup \overline{\mathcal{A}} \cup \{\tau\}$, where $\tau \notin \mathcal{A}$. Henceforth, we use $\mu, \nu, \dots$ to range over actions in $\mathcal{A}_\tau$, and $\alpha, \beta, \dots$ to range over actions in $\mathcal{A} \cup \overline{\mathcal{A}}$.

Definition 1 *(Labelled transition system). A labelled transition system (LTS) is a triple* $(\mathbf{P}, \mathcal{A}_\tau, \longrightarrow)$*, where* $\mathbf{P}$ *is a set of* processes *(or* states*),* $\mathcal{A}_\tau$ *is a set of actions, and* $\longrightarrow \subseteq \mathbf{P} \times \mathcal{A}_\tau \times \mathbf{P}$ *is a (labelled) transition relation.*

As usual, we use $p \xrightarrow{\mu} p'$ in lieu of $(p, \mu, p') \in \longrightarrow$. For each $p \in \mathbf{P}$ and $\mu \in \mathcal{A}_\tau$, we write $p \xrightarrow{\mu}$ if $p \xrightarrow{\mu} p'$ holds for some p', and $p \overset{\mu}{\not\longrightarrow}$ otherwise. We write $p \longrightarrow p'$ if there exists $\mu \in \mathcal{A}_\tau$ such that $p \xrightarrow{\mu} p'$; if $p \longrightarrow p'$, then p' is called a *residual* of p. The *initials* of p are the actions that label the outgoing transitions of p, that is, $\mathrm{init}(p) = \{\mu \mid p \xrightarrow{\mu}\}$. We also introduce the notion of *derivative* of a process p, notation $\mathrm{der}(p)$, as the least set containing p that is closed under $\longrightarrow$, i.e., the least set satisfying:

- $p \in \mathrm{der}(p)$, and
- if $q \in \mathrm{der}(p)$ and $q \xrightarrow{\mu} q'$, for some action $\mu \in \mathcal{A}_\tau$, then $q' \in \mathrm{der}(p)$.

In particular, we say that $p' \in \mathrm{der}(p)$ is a *μ-derivative* of p, for some $\mu \in \mathcal{A}_\tau$, if $p \xrightarrow{\mu} p'$. Moreover, we say that p' is a *proper* derivative of p if $p' \in \mathrm{der}(p) \setminus \{p\}$.

The languages CCS$_\mathrm{L}$ and CCS$_{LC}$. We consider two variants of Milner's *Calculus of Communicating Systems* (CCS) [38].

The first language we consider is *CCS$_L$*, namely the recursion, restriction and relabelling free fragment of CCS with *interleaving parallel composition* and enriched with the *left merge* from [17]. Terms in CCS$_\mathrm{L}$ are given by the following grammar:

$$t ::= \mathbf{0} \mid x \mid \mu.t \mid t+t \mid t \parallel t \mid t \mathbin{\|\!\!\lfloor} t \ , \tag{CCS$_\mathrm{L}$}$$

where x is a variable drawn from a countably infinite set $\mathcal{V}$, and $\mu \in \mathcal{A}_\tau$. We use the *Structural Operational Semantics* (SOS) framework [45] to equip processes with an operational semantics. The SOS rules [45] for the CCS$_\mathrm{L}$ operators given above are reported in Table 1.

The second language we consider is *CCS$_{LC}$*, namely the extension of CCS$_\mathrm{L}$ with *full merge* in place of interleaving and the *communication merge* operator from [17]. The terms in that language are those that can be built thus:

$$t ::= \mathbf{0} \mid x \mid \mu.t \mid t+t \mid t \parallel t \mid t \mathbin{\|\!\!\lfloor} t \mid t \mid t \ . \tag{CCS$_\mathrm{LC}$}$$

Following [38], the action symbol τ will result from the synchronised occurrence of two complementary actions, such as a and $\overline{a}$. The SOS rules for the CCS$_\mathrm{LC}$ operators are those given in Table 1 plus those reported in Table 2.

We shall use the meta-variables t, u, v, w to range over process terms, and write $\mathrm{var}(t)$ for the collection of variables occurring in the term t. We use a *summation* $\sum_{i \in \{1,\dots,k\}} t_i$ to denote the term $t = t_1 + \dots + t_k$, with the empty sum corresponding to $\mathbf{0}$. The *size* of a term t, denoted by $\mathrm{size}(t)$, is the number of operator symbols occurring in t. A term is *closed* if it does not contain any variables. Closed terms, or *processes*, will be denoted by p, q, r. Moreover, we omit trailing $\mathbf{0}$'s from terms.

A *(closed) substitution* is a mapping from process variables to (closed) terms. Substitutions are extended from variables to terms, transitions, and rules in the usual way. Note that $\sigma(t)$ is closed, if so is σ. We let $\sigma[x \mapsto p]$ denote the substitution that maps the variable x into process p and behaves like σ on all other variables.

The inference rules in Tables 1 and 2 allow us to derive valid transitions between terms. The operational semantics for our languages are then modelled by the LTSs whose processes are the closed terms, and whose labelled transitions are those that are provable from the SOS rules.

Behavioural equivalences. *Behavioural equivalences* have been introduced as a tool to establish whether the behaviours of two processes are *indistinguishable for their observers*. Roughly, they allow us to check whether the *observable* semantics of two processes is *the same*. In the literature we can find several notions of behavioural equivalence based on the observations that an external observer can make on the process (the interested reader is referred to the taxonomies in [21,23,25] for an exhaustive presentation). Specifically, we can distinguish a so-called *strong* approach and a *weak* approach to the definition of equivalences. The difference is that, in the latter approach, the *internal* computation steps performed by processes, like, e.g., communications between the parallel components, are deemed *unobservable* (*silent, hidden*) and, thus, an external observer cannot use their direct observation to distinguish process behaviours. This abstraction is achieved by labelling those computation steps with the symbol τ, and giving τ-labelled transitions a special treatment in the definition of the behavioural equivalence.

A classic behavioural equivalence is (*strong*) *bisimilarity* [36,45].

Definition 2 *(Strong bisimilarity). Let* $(\mathbf{P}, \mathcal{A}_\tau, \longrightarrow)$ *be a LTS. A symmetric binary relation* $\mathcal{R} \subseteq \mathbf{P} \times \mathbf{P}$ *is a* (strong) bisimulation *if, and only if, whenever* $(p, q) \in \mathcal{R}$ *and* $p \xrightarrow{\mu} p'$ *then there exists a process* q' *such that* $q \xrightarrow{\mu} q'$ *and* $(p', q') \in \mathcal{R}$*. We say that* p *and* q *are* (strongly) bisimilar *if there is a bisimulation relation* $\mathcal{R}$ *such that* $(p, q) \in \mathcal{R}$*.*

It is well known that strong bisimilarity is an equivalence relation and is the largest strong bisimulation [38,44].

Strong bisimilarity has been extended to the weak case in various forms, according to how the τ-moves are taken into account. In particular, in this paper we will consider four weak variants of bisimilarity: *branching bisimilarity* [27,28], *η-bisimilarity* [15], *delay bisimilarity* [37], and *weak bisimilarity* [38] (also known as *observational equivalence* [31]).

Let $\xrightarrow{\varepsilon}$ denote the reflexive and transitive closure of the transition $\xrightarrow{\tau}$.

Definition 3 *(Weak, η, delay and branching bisimulation). Let* $(\mathbf{P}, \mathcal{A}_\tau, \longrightarrow)$ *be a LTS. A symmetric binary relation* $\mathcal{R} \subseteq \mathbf{P} \times \mathbf{P}$ *is a* weak bisimulation *iff whenever* $(p, q) \in \mathcal{R}$ *and* $p \xrightarrow{\mu} p'$ *then*

1. *either* $\mu = \tau$ *and* $p' \mathcal{R} q$*,*
2. *or there are some* q', q_1, q_2 *such that* $q \xrightarrow{\varepsilon} q_1 \xrightarrow{\mu} q_2 \xrightarrow{\varepsilon} q'$ *and* $p' \mathcal{R} q'$*.*

A binary relation $\mathcal{R} \subseteq \mathbf{P} \times \mathbf{P}$ *is an* η-bisimulation *if it is a weak bisimulation with the additional requirements that* $p \mathcal{R} q_1$ *in item 2.*

A binary relation $\mathcal{R} \subseteq \mathbf{P} \times \mathbf{P}$ *is a* delay bisimulation *if it a weak bisimulation with the additional requirements that* $p' \mathcal{R} q_2$ *in item 2.*

A binary relation $\mathcal{R} \subseteq \mathbf{P} \times \mathbf{P}$ *is a* branching bisimulation *if it is a weak bisimulation with the additional requirements that* $p \mathcal{R} q_1$ *and* $p' \mathcal{R} q_2$ *in item 2.*

Two processes p *and* q *are* weak *(respectively,* η, delay, branching*)* bisimilar*, denoted by* $p \sim_{\mathtt{WB}} q$ *(respectively* $\sim_{\eta\mathtt{B}}$, $\sim_{\mathtt{DB}}$, $\sim_{\mathtt{BB}}$*), iff there exists a weak bisimulation (respectively, an* η*, a delay, a branching bisimulation)* $\mathcal{R}$ *such that* $p \mathcal{R} q$*.*

Each of the behavioural equivalences defined above are equivalence relations. Moreover, we have that $\sim_{\mathtt{BB}} \subset \sim_{\eta\mathtt{B}} \subset \sim_{\mathtt{WB}}$, $\sim_{\mathtt{BB}} \subset \sim_{\mathtt{DB}} \subset \sim_{\mathtt{WB}}$, and $\sim_{\eta\mathtt{B}}$ and $\sim_{\mathtt{DB}}$ are incomparable, i.e., there are processes p, q, r, s such that $p \sim_{\eta\mathtt{B}} q$ but $p \nsim_{\mathtt{DB}} q$, and $r \nsim_{\eta\mathtt{B}} s$ but $r \sim_{\mathtt{DB}} s$. We also recall that a behavioural equivalence $\sim$ is a *congruence* for an n-ary operator f if, and only if,

$$\text{whenever } t_i \sim t_i' \text{ for all } i = 1, \dots, n, \text{ then } f(t_1, \dots, t_n) \sim f(t_1', \dots, t_n').$$

It is well known that strong bisimilarity is a congruence with respect to all $\mathrm{CCS_L}$ and $\mathrm{CCS_{LC}}$ operators [46]. Weak, η, delay and branching bisimilarity are, however, not congruences with respect to $+$ and $⫴$ (see, e.g., [26]). To obviate this inconvenience, the *root* condition has been introduced in the literature.

Definition 4 *(Rooted weak, η, delay, branching bisimilarity). Let* $(\mathbf{P}, \mathcal{A}_\tau, \longrightarrow)$ *be a LTS. We say that:*

- p *and* q *are* rooted weak bisimilar*, denoted by* $p \sim_{\mathtt{RWB}} q$*, iff:*
  - *if* $p \xrightarrow{\mu} p'$ *then* $q \xrightarrow{\varepsilon} q_1 \xrightarrow{\mu} q_1 \xrightarrow{\varepsilon} q'$ *for some* q_1, q_2, q' *such that* $p' \sim_{\mathtt{WB}} q'$*;*
  - *if* $q \xrightarrow{\mu} q'$ *then* $p \xrightarrow{\varepsilon} p_1 \xrightarrow{\mu} p_2 \xrightarrow{\varepsilon} p'$ *for some* p_1, p_2, p' *such that* $p' \sim_{\mathtt{WB}} q'$*.*
- p *and* q *are* rooted η bisimilar*, denoted by* $p \sim_{\mathtt{R}\eta\mathtt{B}} q$*, iff:*
  - *if* $p \xrightarrow{\mu} p'$ *then* $q \xrightarrow{\mu} q_1 \xrightarrow{\varepsilon} q'$ *for some* q_1, q' *such that* $p' \sim_{\eta\mathtt{B}} q'$*;*
  - *if* $q \xrightarrow{\mu} q'$ *then* $p \xrightarrow{\mu} p_1 \xrightarrow{\varepsilon} p'$ *for some* p_1, p' *such that* $p' \sim_{\eta\mathtt{B}} q'$*.*
- p *and* q *are* rooted delay bisimilar*, denoted by* $p \sim_{\mathtt{RDB}} q$*, iff:*
  - *if* $p \xrightarrow{\mu} p'$ *then* $q \xrightarrow{\varepsilon} q_1 \xrightarrow{\mu} q'$ *for some* q_1, q' *such that* $p' \sim_{\mathtt{DB}} q'$*;*

Table 3
The rules of equational logic.

$$(e_1)\; t \approx t \qquad (e_2)\; \frac{t \approx u}{u \approx t} \qquad (e_3)\; \frac{t \approx u \quad u \approx v}{t \approx v} \qquad (e_4)\; \frac{t \approx u}{\sigma(t) \approx \sigma(u)}$$

$$(e_4)\; \frac{t \approx u}{\mu.t \approx \mu.u} \qquad (e_5)\; \frac{t \approx u \quad t' \approx u'}{t + t' \approx u + u'} \qquad (e_6)\; \frac{t \approx u \quad t' \approx u'}{t \parallel t' \approx u \parallel u'}$$

$$(e_7)\; \frac{t \approx u \quad t' \approx u'}{t \mathbin{⫡} t' \approx u \mathbin{⫡} u'} \qquad (e_8)\; \frac{t \approx u \quad t' \approx u'}{t \mid t' \approx u \mid u'} \;.$$

  - *if* $q \xrightarrow{\mu} q'$ *then* $p \xrightarrow{\varepsilon} p_1 \xrightarrow{\mu} p'$ *for some* p_1, p' *such that* $p' \sim_{\mathtt{DB}} q'$.
- *p and q are* rooted branching bisimilar, *denoted by* $p \sim_{\mathtt{RBB}} q$, *iff:*
  - *if* $p \xrightarrow{\mu} p'$ *then* $q \xrightarrow{\mu} q'$ *for some* q' *such that* $p' \sim_{\mathtt{BB}} q'$;
  - *if* $q \xrightarrow{\mu} q'$ *then* $p \xrightarrow{\mu} p'$ *for some* p' *such that* $p' \sim_{\mathtt{BB}} q'$.

It is well known that the rooted variants of weak, η, delay and branching bisimilarity proposed above are equivalence relations [16, 28], and they are characterised by the same comparability relations that hold over their unrooted counterparts. Moreover, they are all congruences over $\mathrm{CCS_L}$. Rooted branching equivalence and rooted η equivalence are also congruences over $\mathrm{CCS_{LC}}$. We postpone to Section 8 the discussion of the congruence properties of rooted weak and rooted delay bisimulation over $\mathrm{CCS_{LC}}$.

In what follows, we shall write $p \overset{\mu}{\Longrightarrow} p'$ ($\mu \in \mathcal{A}_\tau$) when $p \xrightarrow{\varepsilon} p_1 \xrightarrow{\mu} p_2 \xrightarrow{\varepsilon} p'$ for some p_1, p_2. Moreover, for a sequence of actions $\xi = \alpha_1 \cdots \alpha_k \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ ($k \geq 0$), and processes p, p', we write that $p \overset{\xi}{\Longrightarrow} p'$ if and only if there exists a sequence of transitions $p = p_0 \overset{\alpha_1}{\Longrightarrow} p_1 \overset{\alpha_2}{\Longrightarrow} \cdots \overset{\alpha_k}{\Longrightarrow} p_k$. If $p \overset{\xi}{\Longrightarrow} p'$ holds for some process p', then ξ is an *observable trace* of p. Moreover, we say that ξ is a *maximal* observable trace of p if $\mathrm{init}(p') = \emptyset$. By means of observable traces, we associate a classic notion with a process p, i.e., its (*observable*) *depth*, denoted by $\mathrm{depth}(p)$. For a process p whose set of traces is finite, it expresses the length of a *longest* observable trace. Formally, denoting by $|\xi|$ the length of ξ,

$$\mathrm{depth}(p) = \sup\{k \mid p \overset{\xi}{\Longrightarrow} p' \;\wedge\; |\xi| = k\}.$$

The weak bisimulation-based equivalences presented above preserve the observable depth of processes:

Lemma 1. *Whenever* $p \sim_{\mathtt{BB}} q$ *(respectively,* $\sim_{\eta\mathtt{B}}, \sim_{\mathtt{DB}}, \sim_{\mathtt{WB}}$*), then* $\mathrm{depth}(p) = \mathrm{depth}(q)$.

Equational logic. An *axiom system* $\mathcal{E}$ is a collection of (*process*) *equations* $t \approx u$ over some languages such as $\mathrm{CCS_L}$ and $\mathrm{CCS_{LC}}$. An equation $t \approx u$ is *derivable* from an axiom system $\mathcal{E}$, notation $\mathcal{E} \vdash t \approx u$, if there is an *equational proof* for it from $\mathcal{E}$, namely if $t \approx u$ can be inferred from the axioms in $\mathcal{E}$ using the *rules* of *equational logic* given in Table 3. Moreover, given two axiom systems $\mathcal{E}_1$ and $\mathcal{E}_2$, we write $\mathcal{E}_1 \vdash \mathcal{E}_2$ if, and only if, $\mathcal{E}_1 \vdash t \approx u$ for every equation $t \approx u$ in $\mathcal{E}_2$.

We are interested in equations that are valid modulo some congruence relation $\sim$ over terms. The equation $t \approx u$ is said to be *sound* modulo $\sim$ if $\sigma(t) \sim \sigma(u)$ for all closed substitutions σ. For simplicity, if $t \approx u$ is sound modulo $\sim$, then we write $t \sim u$. An axiom system is *sound* modulo $\sim$ if, and only if, all of its equations are sound modulo $\sim$. Conversely, we say that $\mathcal{E}$ is *complete* modulo $\sim$ if $t \sim u$ implies $\mathcal{E} \vdash t \approx u$ for all terms t, u. If we restrict ourselves to consider only equations in which there are no occurrences of variables, then $\mathcal{E}$ is said to be *ground-complete* modulo $\sim$. We say that $\sim$ has a finite, (ground) complete axiomatisation, if there is a finite axiom system $\mathcal{E}$ that is sound and (ground) complete for $\sim$.

As an example, in [9] it was proved that the axiom system $\mathcal{E}_{\mathrm{B}}^{\mathrm{L}}$, consisting of all the axioms in the upper part of Table 4, is a complete axiomatisation for bisimulation equivalence over $\mathrm{CCS_L}$. In [9] it was also proved that the axiom system $\mathcal{E}_{\mathrm{B}}$, consisting of all the axioms in Table 4 (except PL), is a complete axiomatisation for bisimulation equivalence over $\mathrm{CCS_{LC}}$. Since axioms L1, C3 and C4 are axiom schemata that generate finitely many axioms if the set of actions is finite, the complete axiomatisations in [9] are finite if $\mathcal{A}_\tau$ is finite.

Throughout the paper, we also exploit the associativity and commutativity of $+$ (axioms A1 and A2) and $\parallel$ (axioms D1 and D2) modulo the behavioural equivalences we will consider, and define terms modulo them, i.e., we do not distinguish $t \odot u$ and $u \odot t$, nor $(t \odot u) \odot v$ and $t \odot (u \odot v)$, where $\odot \in \{+, \parallel\}$. The symbol $=$ will then denote equality modulo the above identifications. We refer the reader interested in the derivation proofs of D0–D4 to [9, Lemma 2.10 for D0–D2, Lemma 2.3 for D4, and Lemma 4.3 for D5]. Moreover, axiom D3 can be derived as follows:

$$\begin{aligned} (x \mid y) \mid z &\overset{\text{(H)}}{\approx} \mathbf{0} \\ &\overset{\text{(H)}}{\approx} (y \mid z) \mid x \\ &\overset{\text{(C1)}}{\approx} x \mid (y \mid z). \end{aligned}$$

Finally, we recall that, thanks to the axioms in Table 4, we can prove the following classic result for $\mathrm{CCS_L}$ and $\mathrm{CCS_{LC}}$ terms, which will be useful in the rest of paper.

Table 4
Equational bases modulo strong bisimilarity.

Complete axiom system for bisimilarity over $\mathrm{CCS_L}$: $\mathcal{E}_{\mathrm{B}}^{\mathrm{L}}$			
A0	$x + \mathbf{0} \approx x$	L0	$\mathbf{0} \mathbin{\|\!\!\lfloor} x \approx \mathbf{0}$
A1	$x + y \approx y + x$	L1	$\mu x \mathbin{\|\!\!\lfloor} y \approx \mu(x \parallel y)$
A2	$(x + y) + z \approx x + (y + z)$	L2	$(x \mathbin{\|\!\!\lfloor} y) \mathbin{\|\!\!\lfloor} z \approx x \mathbin{\|\!\!\lfloor} (y \parallel z)$
A3	$x + x \approx x$	L3	$x \mathbin{\|\!\!\lfloor} \mathbf{0} \approx x$
PL	$x \parallel y \approx x \mathbin{\|\!\!\lfloor} y + y \mathbin{\|\!\!\lfloor} x$	L4	$(x + y) \mathbin{\|\!\!\lfloor} z \approx x \mathbin{\|\!\!\lfloor} z + y \mathbin{\|\!\!\lfloor} z$
Equations derivable from $\mathcal{E}_{\mathrm{B}}^{\mathrm{L}}$:			
D0	$x \parallel \mathbf{0} \approx x$		
D1	$x \parallel y \approx y \parallel x$		
D2	$(x \parallel y) \parallel z \approx x \parallel (y \parallel z)$		
Additional axioms for bisimilarity over $\mathrm{CCS_{LC}}$ (P replaces PL): $\mathcal{E}_{\mathrm{B}}$			
C0	$\mathbf{0} \mid x \approx \mathbf{0}$	C3	$\alpha x \mid \beta y \approx \tau(x \parallel y)$ if $\alpha = \overline{\beta}$
C1	$x \mid y \approx y \mid x$	C4	$\alpha x \mid \beta y \approx \mathbf{0}$ if $\alpha \neq \overline{\beta}$
C2	$(x + y) \mid z \approx (x \mid z) + (y \mid z)$	C5	$(x \mathbin{\|\!\!\lfloor} y) \mid z \approx (x \mid z) \mathbin{\|\!\!\lfloor} y$
H	$(x \mid y) \mid z \approx \mathbf{0}$		
P	$x \parallel y \approx x \mathbin{\|\!\!\lfloor} y + y \mathbin{\|\!\!\lfloor} x + x \mid y$		
Additional equation derivable from $\mathcal{E}_{\mathrm{B}}$			
D3	$(x \mid y) \mid z \approx x \mid (y \mid z)$		
D4	$(x \mathbin{\|\!\!\lfloor} y) \mid (z \mathbin{\|\!\!\lfloor} w) \approx (x \mid z) \mathbin{\|\!\!\lfloor} (y \parallel w)$		
D5	$\tau.x \mid y \approx \mathbf{0}$		

Lemma 2. *For every term t there are terms $t_1, \ldots, t_n$ ($n \geq 0$) that do not have + as head operator such that $t \approx \sum_{i=1}^{n} t_i$ is provable from A0–A3.*

The terms t_i are also called the *summands* of t.

3. Unique parallel decomposition

Throughout this section, we let $\mathtt{XB}$ range over $\{\mathtt{BB}, \eta\mathtt{B}, \mathtt{DB}, \mathtt{WB}\}$. We shall prove that $\mathrm{CCS_L}$ and $\mathrm{CCS_{LC}}$ processes have a unique parallel decomposition modulo $\sim_{\mathtt{XB}}$. We will present the details for $\mathrm{CCS_{LC}}$ processes; the details for $\mathrm{CCS_L}$ processes are similar but slightly simpler.

Definition 5 *(Parallel decomposition modulo $\sim_{\mathtt{XB}}$). A process p is* indecomposable *if $p \nsim_{\mathtt{XB}} \mathbf{0}$ and $p \sim_{\mathtt{XB}} p_1 \parallel p_2$ implies $p_1 \sim_{\mathtt{XB}} \mathbf{0}$ or $p_2 \sim_{\mathtt{XB}} \mathbf{0}$, for all processes p_1 and p_2. A* parallel decomposition *of a process p is a finite multiset $\Lbag p_1, \ldots, p_k \Rbag$ of indecomposable processes $p_1, \ldots, p_k$ such that $p \sim_{\mathtt{XB}} p_1 \parallel \cdots \parallel p_k$. We say that p has a* unique parallel decomposition *if p has a parallel decomposition $\Lbag p_1, \ldots, p_k \Rbag$ and for every other parallel decomposition $\Lbag p'_1, \ldots, p'_\ell \Rbag$ of p there exists a bijection $f : \{1, \ldots, k\} \to \{1, \ldots, \ell\}$ such that $p_i \sim_{\mathtt{XB}} p'_{f(i)}$ for all $1 \leq i \leq k$.*

To prove that processes have a unique parallel decomposition we shall exploit a general result stating that a partial commutative monoid has unique decomposition if it can be endowed with a *weak decomposition order* that satisfies *power cancellation* [34]; we shall define and explain the notions below. Note that, in view of axioms D0–D2, which are sound modulo $\sim_{\mathtt{XB}}$, the set of $\mathrm{CCS_{LC}}$ processes $\mathbf{P}$ modulo $\sim_{\mathtt{XB}}$ is a commutative monoid with respect to the binary operation naturally induced by $\parallel$ on $\sim_{\mathtt{XB}}$-equivalence classes and the $\sim_{\mathtt{XB}}$-equivalence class of $\mathbf{0}$ as identity element. We permit ourselves a minor abuse in notation and use → to (also) denote the binary relation $\{(p, q) \mid \exists \mu.\ p \xrightarrow{\mu} q\}$, and proceed to argue that → induces a weak decomposition order satisfying power cancellation on the commutative monoid of processes modulo $\sim_{\mathtt{XB}}$.

Given any process p and $n \geq 1$, let p^n denote the n-fold parallel composition $p \parallel p^{n-1}$, with $p^0 = \mathbf{0}$. We first state some properties of the reflexive-transitive closure $\to^*$ of $\to$:

Proposition 1. *The relation $\to^*$ is an inversely well-founded partial order on processes satisfying the following properties:*

1. *For every process p there exists a process p' such that $p \to^* p' \sim_{\mathtt{XB}} \mathbf{0}$.*
2. *For all processes p, p' and q, if $p \to^* p'$, then $p \parallel q \to^* p' \parallel q$ and $q \parallel p \to^* q \parallel p'$.*
3. *For all processes p, q and r, if $p \parallel q \to^* r$, then there exist p' and q' such that $p \to^* p'$, $q \to^* q'$ and $r = p' \parallel q'$.*
4. *For all processes p and q, if $p \to^* q^n$ for all $n \in \mathbb{N}$, then $q \sim_{\mathtt{BB}} \mathbf{0}$.*

Proof. Clearly, $\to^*$ is reflexive and transitive by definition, and since $p \xrightarrow{\mu} p'$ implies $\mathrm{size}(p) > \mathrm{size}(p')$ it follows that $\to^*$ is inversely well-founded. Every inversely well-founded reflexive and transitive relation is clearly also anti-symmetric, so $\to^*$ is an inversely well-founded partial order.

By inverse well-foundedness, for every process p there exists a process p' such that $p \to^* p'$ and $p' \not\rightarrow$; from $p' \not\rightarrow$ it follows that $p' \sim_{\mathtt{XB}} \mathbf{0}$. Hence $\to^*$ satisfies property 1.

That $\to^*$ satisfies property 2 is an immediate consequence of the two rules at the bottom left of Table 1.

That $\to^*$ satisfies property 3 is straightforwardly established with induction on the length of a transition sequence witnessing $p \parallel q \to^* r$, using that the last rule applied in the derivation of each individual transition must be one of the three rules for $\parallel$.

To see that $p \to^* q^n$ for all $n \in \mathbb{N}$ implies $q \sim_{\mathtt{XB}} \mathbf{0}$, note that $p \to^* q^n$ implies $\mathrm{depth}(p) \geq \mathrm{depth}(q^n)$ and $\mathrm{depth}(q^n) = n \cdot \mathrm{depth}(q)$, from which it follows that $\mathrm{depth}(q) = 0$ and hence $q \sim_{\mathtt{XB}} \mathbf{0}$. This proves that $\to^*$ satisfies property 4. □

The following lemma is a direct consequence of the definitions of branching, η, delay and weak bisimilarity.

Lemma 3. *For all processes p, p' and q, if $p \sim_{\mathtt{XB}} q$ and $p \to^* p'$, then there exists q' such that $q \to^* q'$ and $p' \sim_{\mathtt{XB}} q'$.*

Given a process p and an equivalence relation $\sim$, we denote by $[p]_\sim$ the equivalence class of p with respect to $\sim$, namely $[p]_\sim = \{q \in \mathbf{P} \mid p \sim q\}$. By Lemma 3 we can define a binary relation $\preceq$ on $\mathbf{P}/{\sim_{\mathtt{XB}}}$, the set of $\sim_{\mathtt{XB}}$-equivalence classes of processes, by stating that $[p]_{\sim_{\mathtt{XB}}} \preceq [q]_{\sim_{\mathtt{XB}}}$ if, and only if, there exists $p' \in [p]_{\sim_{\mathtt{XB}}}$ such that $q \to^* p'$. The following result is then a straightforward corollary of Proposition 1.

Corollary 1. *The relation $\preceq$ is a weak decomposition order on $\mathbf{P}/{\sim_{\mathtt{XB}}}$, namely:*

1. *it is well-founded, i.e., every non-empty subset of $\mathbf{P}/{\sim_{\mathtt{XB}}}$ has a $\preceq$-minimal element;*
2. *the identity element $[\mathbf{0}]_{\sim_{\mathtt{XB}}}$ of $\mathbf{P}/{\sim_{\mathtt{XB}}}$ is the least element of $\mathbf{P}/{\sim_{\mathtt{XB}}}$ with respect to $\preceq$, i.e., $[\mathbf{0}]_{\sim_{\mathtt{XB}}} \preceq [p]_{\sim_{\mathtt{XB}}}$ for all $p \in \mathbf{P}$;*
3. *it is compatible, i.e., for all $p, q, r \in \mathbf{P}$ if $[p]_{\sim_{\mathtt{XB}}} \preceq [q]_{\sim_{\mathtt{XB}}}$, then $[p \parallel r]_{\sim_{\mathtt{XB}}} \preceq [q \parallel r]_{\sim_{\mathtt{XB}}}$;*
4. *it is precompositional, i.e., for all $p, q, r \in \mathbf{P}$ we have that $[p]_{\sim_{\mathtt{XB}}} \preceq [q \parallel r]_{\sim_{\mathtt{XB}}}$ implies $[p]_{\sim_{\mathtt{XB}}} = [q' \parallel r']_{\sim_{\mathtt{XB}}}$ for some $[q']_{\sim_{\mathtt{XB}}} \preceq [q]_{\sim_{\mathtt{XB}}}$ and $[r']_{\sim_{\mathtt{XB}}} \preceq [r]_{\sim_{\mathtt{XB}}}$; and*
5. *it is Archimedean, i.e., for all $p, q \in \mathbf{P}$ we have that $[p^n]_{\sim_{\mathtt{XB}}} \preceq [q]_{\sim_{\mathtt{XB}}}$ for all $n \in \mathbb{N}$ implies that $[p]_{\sim_{\mathtt{XB}}} = [\mathbf{0}]_{\sim_{\mathtt{XB}}}$.*

According to [34, Theorem 34], to prove that processes in $\mathbf{P}$ have a unique parallel decomposition, it now remains to prove that $\preceq$ satisfies power cancellation. The weak decomposition order $\preceq$ on the commutative monoid of processes modulo $\sim_{\mathtt{XB}}$ satisfies *power cancellation* if for every indecomposable process p and for all processes q and r such that $[p]_{\sim_{\mathtt{XB}}} \not\preceq [q]_{\sim_{\mathtt{XB}}}, [r]_{\sim_{\mathtt{XB}}}$, for all $k \in \mathbb{N}$, we have that $[p^k \parallel q]_{\sim_{\mathtt{XB}}} = [p^k \parallel r]_{\sim_{\mathtt{XB}}}$ implies $[q]_{\sim_{\mathtt{XB}}} = [r]_{\sim_{\mathtt{XB}}}$.

Lemma 4 *(Stuttering Property). For all processes $p_0, \dots, p_n$ such that $p_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} p_n$, if $p_0 \sim_{\mathtt{XB}} p_n$, then $p_i \sim_{\mathtt{XB}} p_j$ for all $0 \leq i, j \leq n$.*

Proof. See [20, Lemma 4.8]. □

Proposition 2. *The weak decomposition order $\preceq$ on the commutative monoid of processes modulo $\sim_{\mathtt{XB}}$ satisfies power cancellation.*

Proof. Let p be an indecomposable process and let q and r be processes such that $[p]_{\sim_{\mathtt{XB}}} \not\preceq [q]_{\sim_{\mathtt{XB}}}, [r]_{\sim_{\mathtt{XB}}}$, and suppose that $[p^k \parallel q]_{\sim_{\mathtt{XB}}} = [p^k \parallel r]_{\sim_{\mathtt{XB}}}$ for some $k \in \mathbb{N}$. We need to prove that $[q]_{\sim_{\mathtt{XB}}} = [r]_{\sim_{\mathtt{XB}}}$.

The weak decomposition order $\preceq$, since it is well-founded, induces a well-founded order on triples of $\sim_{\mathtt{XB}}$-equivalence classes of processes, defining $([p_1]_{\sim_{\mathtt{XB}}}, [p_2]_{\sim_{\mathtt{XB}}}, [p_3]_{\sim_{\mathtt{XB}}}) \preceq ([q_1]_{\sim_{\mathtt{XB}}}, [q_2]_{\sim_{\mathtt{XB}}}, [q_3]_{\sim_{\mathtt{XB}}})$ if, and only if, $[p_1]_{\sim_{\mathtt{XB}}} \preceq [q_1]_{\sim_{\mathtt{XB}}}$ and whenever $[p_1]_{\sim_{\mathtt{XB}}} = [q_1]_{\sim_{\mathtt{XB}}}$, then also $[p_2]_{\sim_{\mathtt{XB}}} \preceq [q_2]_{\sim_{\mathtt{XB}}}$ and $[p_3]_{\sim_{\mathtt{XB}}} \preceq [q_3]_{\sim_{\mathtt{XB}}}$. Our proof is by induction on the well-founded order $\preceq$ on triples of $\sim_{\mathtt{XB}}$-equivalence classes of processes: we assume, by way of induction hypothesis, that for all p', q' and r' such that

$$([(p')^k \parallel q']_{\sim_{\mathtt{XB}}}, [q']_{\sim_{\mathtt{XB}}}, [r']_{\sim_{\mathtt{XB}}}) \prec ([p^k \parallel q]_{\sim_{\mathtt{XB}}}, [q]_{\sim_{\mathtt{XB}}}, [r]_{\sim_{\mathtt{XB}}})$$

we have that $(p')^k \parallel q' \sim_{\mathtt{XB}} (p')^k \parallel r'$ implies $q' \sim_{\mathtt{XB}} r'$, and establish the property for p, q and r.

To prove that $[q]_{\sim_{\mathtt{XB}}} = [r]_{\sim_{\mathtt{XB}}}$, it suffices to argue that the relation

$$\mathcal{R} = \{(q, r)\} \cup \sim_{\mathtt{XB}}$$

is a branching, η, delay, or weak bisimulation relation, respectively. Note that q and r are just arbitrary elements of the equivalence classes $[q]_{\sim_{\mathtt{XB}}}$ and $[r]_{\sim_{\mathtt{XB}}}$, respectively. Since $q \to q'$ implies $\mathrm{size}(q) > \mathrm{size}(q')$, and hence is $\to$ is terminating, we may, without loss of generality, choose q such that there does not exist q' such that $q \to q'$ and $q \sim_{\mathtt{XB}} q'$, and similarly we may, without loss of generality, choose r such that there does not exist an $r' \neq r$ such that $r \sim_{\mathtt{XB}} r'$.

Since $\sim_{\mathtt{XB}}$ is the $\mathtt{XB}$ bisimulation, it is immediate that all pairs in $\sim_{\mathtt{XB}}$ satisfy the conditions of $\mathtt{XB}$ bisimulations. We do need to establish that the pair (q, r) also satisfies these conditions, and to this end suppose that $q \xrightarrow{\mu} q'$ for some q'; we establish that either $\mu = \tau$ and $q' \mathrel{\mathcal{R}} r$, or there exist processes r_1, r_2 and r' such that $r \xrightarrow{\epsilon} r_1 \xrightarrow{\mu} r_2 \xrightarrow{\epsilon} r$ and:

- $q \mathrel{\mathcal{R}} r_1$, $q' \mathrel{\mathcal{R}} r_2$, and $q' \mathrel{\mathcal{R}} r'$, if $\mathtt{XB} = \mathtt{BB}$;
- $q \mathrel{\mathcal{R}} r_1$ and $q' \mathrel{\mathcal{R}} r'$, if $\mathtt{XB} = \eta\mathtt{B}$;
- $q' \mathrel{\mathcal{R}} r_2$ and $q' \mathrel{\mathcal{R}} r'$, if $\mathtt{XB} = \mathtt{DB}$;

- $q' \mathcal{R} r'$, if $\mathtt{XB} = \mathtt{WB}$.

We distinguish two cases:

1. Suppose that $p^k \parallel q' \sim_{\mathtt{XB}} p^k \parallel q$. Then, since $\sim_{\mathtt{XB}}$ preserves depth and the depth of a parallel composition is the sum of the depths of its components, we must have that $\mathrm{depth}(q') = \mathrm{depth}(q)$ and hence $\mu = \tau$. Now, by the choice of q it follows that $[q']_{\sim_{\mathtt{XB}}} \prec [q]_{\sim_{\mathtt{XB}}}$. Moreover, from $p^k \parallel q' \sim_{\mathtt{XB}} p^k \parallel q$ it follows that $[p^k \parallel q']_{\sim_{\mathtt{XB}}} = [p^k \parallel q]_{\sim_{\mathtt{XB}}}$. Thus we find that
$$([p^k \parallel q']_{\sim_{\mathtt{XB}}}, [q']_{\sim_{\mathtt{XB}}}, [r]_{\sim_{\mathtt{XB}}}) \prec ([p^k \parallel q]_{\sim_{\mathtt{XB}}}, [q]_{\sim_{\mathtt{XB}}}, [r]_{\sim_{\mathtt{XB}}}) \ ,$$
so by the induction hypothesis $q' \sim_{\mathtt{XB}} r$, and hence $q' \mathcal{R} r$.
2. Suppose that $p^k \parallel q' \nsim_{\mathtt{XB}} p^k \parallel q$. Then $[p^k \parallel q']_{\sim_{\mathtt{XB}}} \prec [p^k \parallel q]_{\sim_{\mathtt{XB}}}$, and, by the induction hypothesis, the weak decomposition order $\preceq$ on the partial commutative submonoid $\{x \mid x \preceq [p^k \parallel q']_{\sim_{\mathtt{XB}}}\}$ of $\mathbf{P}$ modulo $\sim_{\mathtt{XB}}$ satisfies power cancellation. Hence, by [34, Theorem 34], if s is any process such that $p^k \parallel q' \sim_{\mathtt{XB}}; \rightarrow^* s$ (where ; denotes relation composition), then s has a unique parallel decomposition.

   From $q \xrightarrow{\mu} q'$ if follows that $p^k \parallel q \xrightarrow{\mu} p^k \parallel q'$, and hence, since $p^k \parallel q \sim_{\mathtt{XB}} p^k \parallel r$ there exist processes $p_1, p_2, p', r_1, r_2, r'$ such that
$$p^k \xrightarrow{\epsilon} p_1 \xrightarrow{(\mu_1)} p_2 \xrightarrow{\epsilon} p' \ ,$$
$$r \xrightarrow{\epsilon} r_1 \xrightarrow{(\mu_2)} r_2 \xrightarrow{\epsilon} r' \ , \text{ and}$$
$$p^k \parallel q' \sim_{\mathtt{XB}} p' \parallel r' \ ,$$
   where either $\mu_1 = \mu$ and $\mu_2 = \tau$, or $\mu_1 = \tau$ and $\mu_2 = \mu$, or $\mu_1 = \overline{\mu_2}$ and $\mu = \tau$. (Furthermore, we write $t \xrightarrow{(\nu)} t'$ if either $t \xrightarrow{\nu} t'$ or $\nu = \tau$ and $t = t'$.)

   Moreover, if $\mathtt{XB} = \eta\mathtt{B}$, then we additionally assume that also $p^k \parallel q \sim_{\mathtt{XB}} p_1 \parallel r_1$; if $\mathtt{XB} = \mathtt{DB}$, then we additionally assume that also $p^k \parallel q' \sim_{\mathtt{XB}} p_2 \parallel r_2$; and if $\mathtt{XB} = \mathtt{BB}$, then we additionally assume that both $p^k \parallel q \sim_{\mathtt{XB}} p_1 \parallel r_1$ and $p^k \parallel q' \sim_{\mathtt{XB}} p_2 \parallel r_2$. Also, from $p' \parallel r' \sim_{\mathtt{XB}} p^k \parallel q' \nsim_{\mathtt{XB}} p^k \parallel q \sim_{\mathtt{XB}} p^k \parallel r$, and the fact that $\sim_{\mathtt{XB}}$ is preserved by $\parallel$, it follows that $p' \nsim_{\mathtt{BB}} p^k$ or $r' \nsim_{\mathtt{BB}} r$; we distinguish these two subcases:
   1. (a) Suppose $r' \nsim_{\mathtt{XB}} r$. Then, since $[r']_{\sim_{\mathtt{XB}}} \prec [r]_{\sim_{\mathtt{XB}}}$ and $[p]_{\sim_{\mathtt{XB}}} \not\prec [r]_{\sim_{\mathtt{XB}}}$, the unique parallel decomposition of r' cannot have occurrences of a process bisimilar to p. Since the unique decomposition of $p^k \parallel q'$ has at least k occurrences of a process that is branching bisimilar to p, it follows that $[p^k]_{\sim_{\mathtt{XB}}} \preceq [p']_{\sim_{\mathtt{XB}}} \preceq [p^k]_{\sim_{\mathtt{XB}}}$, so $p' \sim_{\mathtt{XB}} p^k$. Note that from $p' \sim_{\mathtt{XB}} p^k$ it follows that $\mu_1 = \tau$, and therefore $\mu_2 = \mu$. By Lemma 4 we then get that $p^k \sim_{\mathtt{XB}} p_1 \sim_{\mathtt{XB}} p_2 \sim_{\mathtt{XB}} p'$.

      From $p^k \parallel q' \sim_{\mathtt{XB}} p^k \parallel r'$ it now follows by the induction hypothesis that $q' \sim_{\mathtt{XB}} r'$, and hence $q' \mathcal{R} r'$. Thereby, we have established that (in this particular case) $\mathcal{R}$ is a weak bisimulation.

      To show that $\mathcal{R}$ is an η-bisimulation if $\mathtt{XB} = \eta\mathtt{B}$, we need to establish that $q \mathcal{R} r_1$. If $r_1 = r$, then, since $q \mathcal{R} r$ this is immediate. If $r_1 \neq r$, then by the choice of r such that there does not exist r_1 such that $r \rightarrow r_1$ and $r \sim_{\eta\mathtt{B}} r_1$, it follows that $[r_1]_{\sim_{\eta\mathtt{B}}} \prec [r]_{\sim_{\eta\mathtt{B}}}$. Furthermore, since clearly $p^k \parallel r \xrightarrow{\epsilon} p_1 \parallel r_1$ and $p^k \parallel r \sim_{\eta\mathtt{B}} p_1 \parallel r_1$, and $p^k \sim_{\eta\mathtt{B}} p_1$, it follows that $p^k \parallel r \sim_{\eta\mathtt{B}} p^k \parallel r_1$. Then $q \sim_{\eta\mathtt{B}} r'$ follows by the induction hypothesis, and hence $q \mathcal{R} r'$.

      To show that $\mathcal{R}$ is a delay bisimulation if $\mathtt{XB} = \mathtt{DB}$, we need to establish that $q' \mathcal{R} r_2$. Note that from $p^k \parallel q' \sim_{\mathtt{DB}} p_2 \parallel r_2$ and $p^k \sim_{\mathtt{DB}} p_2$ it follows that $p^k \parallel q' \sim_{\mathtt{DB}} p^k \parallel r_2$. Thus, $q' \sim_{\mathtt{DB}} r_2$ now follows by the induction hypothesis and hence $q' \mathcal{R} r_2$.

      To show that $\mathcal{R}$ is a branching bisimulation if $\mathtt{XB} = \mathtt{BB}$, we need to establish that $q \mathcal{R} r_1$ and $q' \mathcal{R} r_2$; this can be done in exactly the same way as for η- and delay bisimilarity, respectively.
   2. (b) Suppose $p' \nsim_{\mathtt{XB}} p^k$. Then $[p']_{\sim_{\mathtt{XB}}} \prec [p^k]_{\sim_{\mathtt{XB}}}$, so the multiplicity of p in the unique decomposition of p' is at most $k-1$. Hence, since $p' \parallel r' \sim_{\mathtt{XB}} p^k \parallel q'$, it follows that p must be an element of the parallel decomposition of r'. This means that $[p]_{\sim_{\mathtt{XB}}} \preceq [r']_{\sim_{\mathtt{XB}}} \preceq [r]_{\sim_{\mathtt{XB}}}$ while at the same time $[p]_{\sim_{\mathtt{XB}}} \not\prec [r]_{\sim_{\mathtt{XB}}}$ by assumption. It follows that
$$p \sim_{\mathtt{XB}} r' \sim_{\mathtt{XB}} r \ .$$
      Hence, since r' can contribute at most 1 to the multiplicity of p in the unique parallel decomposition of $p' \parallel r'$, while the multiplicity of p is at least k, it now follows that the multiplicity of p in the parallel decomposition of p' must be $k-1$. So we can assume without loss of generality that there exist processes $p_{1,1}, \dots, p_{1,k}$, $p_{2,1}, \dots, p_{2,k}$ and $p'_1, \dots, p'_k$ such that
$$p_1 = p_{1,1} \parallel p_{1,2} \parallel \dots \parallel p_{1,k} \ ,$$
$$p_2 = p_{2,1} \parallel p_{2,2} \parallel \dots \parallel p_{2,k} \ ,$$
$$p' = p'_1 \parallel p'_2 \parallel \dots \parallel p'_k \ ,$$
$$p \xrightarrow{\epsilon} p_{1,i} \quad (1 \leq i \leq k) \ ,$$
$$p_{1,1} \xrightarrow{\mu} p_{2,1} \ , \quad p_{1,i} = p_{2,i} \quad (2 \leq i \leq k) \ ,$$
$$p_{2,i} \xrightarrow{\epsilon} p'_i \quad (1 \leq i \leq k) \ , \text{ and}$$

Table 5
Equational basis modulo rooted branching bisimilarity over $\mathrm{CCS}_{\mathrm{L}}$.

Additional axioms for $\sim_{\mathtt{RBB}}$	
TB	$\mu(\tau(x+y)+y) \approx \mu(x+y)$
TL1	$x \mathbin{\|\!\_}\, \tau y \approx x \mathbin{\|\!\_}\, y$
Axiom system $\mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}}$: $\mathcal{E}^{\mathrm{L}}_{\mathtt{B}} \cup \{\mathrm{TB},\mathrm{TL1}\}$	
Equations derivable from $\mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}}$	
DT1	$\mu\tau x \approx \mu x$
DT2	$x \mathbin{\|\!\_}\, (\tau(y+z)+y) \approx x \mathbin{\|\!\_}\, (y+z)$

$$p \sim_{\mathtt{XB}} p_{1,i} \sim_{\mathtt{XB}} p_{2,i} \sim_{\mathtt{XB}} p'_i \quad (2 \le i \le k) \ .$$

We must have that $p'_1 \not\sim_{\mathtt{XB}} p$, for otherwise we would have $p' \sim_{\mathtt{XB}} p^k$, contradicting the assumption in this case. Therefore, since $p_{1,1} \sim_{\mathtt{XB}} p \sim_{\mathtt{XB}} r$, either $\mu = \tau$ and $r \sim_{\mathtt{XB}} p_{2,1}$, or there exist r''_1, r''_2 and r'' such that $r \xrightarrow{\epsilon} r''_1 \xrightarrow{\mu} r''_2 \xrightarrow{\epsilon} r''$ such that $p'_1 \sim_{\mathtt{XB}} r''$. Moreover, if $\mathtt{XB} = \eta\mathtt{B}$, then we additionally have that $p_{1,1} \sim_{\mathtt{XB}} r''_1$; if $\mathtt{XB} = \mathtt{DB}$, then we additionally have that $p_{2,1} \sim_{\mathtt{XB}} r''_2$; and if $\mathtt{XB} = \mathtt{BB}$, then we additionally have that $p_{1,1} \sim_{\mathtt{XB}} r''_1$ and $p_{2,1} \sim_{\mathtt{XB}} r''_2$. From $p^k \parallel q' \sim_{\mathtt{XB}} r \parallel p'_2 \parallel \cdots \parallel p'_k \parallel p'_1$, $p \sim_{\mathtt{XB}} r$ and $p \sim_{\mathtt{XB}} p'_i$ $(2 \le i \le k)$ it follows by the induction hypothesis that $q' \sim_{\mathtt{XB}} p'_1 \sim_{\mathtt{XB}} r''$, so $q' \mathcal{R} r''$. Thus, it has now been established that (also in this case) $\mathcal{R}$ is a weak bisimulation.
To see that $\mathcal{R}$ is an η-bisimulation if $\mathtt{XB} = \eta\mathtt{B}$, note that $r \sim_{\eta\mathtt{B}} p_{1,1} \sim_{\eta\mathtt{B}} r''_1$ implies, by the choice of r, that $r = r''_1$, and hence $q \mathcal{R} r''_1$.
To see that $\mathcal{R}$ is a delay bisimulation if $\mathtt{XB} = \mathtt{DB}$, note that from $p^k \parallel q' \sim_{\mathtt{DB}} r \sim_{\mathtt{DB}} p_{2,2} \parallel \cdots \parallel p_{2,k} \parallel p_{2,1}$, $p \sim_{\mathtt{DB}} r$ and $p \sim_{\mathtt{DB}} p_{2,i}$ $(2 \le i \le k)$ it follows by the induction hypothesis that $q' \sim_{\mathtt{DB}} p_{2,1} \sim_{\mathtt{DB}} r''_2$ and hence $q' \mathcal{R} r''_2$.
To see that $\mathcal{R}$ is a branching bisimulation if $\mathtt{XB} = \mathtt{BB}$, we need to establish that $q \mathcal{R} r''_1$ and $q' \mathcal{R} r''_2$, and this is done in exactly the same way as for η- and delay bisimilarity, respectively. □

We have now established that $\preceq$ is a weak decomposition order on the commutative monoid $\mathbf{P}$ of processes modulo $\sim_{\mathtt{XB}}$ that satisfies power cancellation. Thus, with an application of [34, Theorem 34] we get the following unique parallel decomposition result.

Proposition 3. *Every process in* $\mathbf{P}$ *has a unique parallel decomposition.*

In what follows, we shall make use of the following direct consequence of Proposition 3.

Corollary 2. *If* $p \parallel r \sim_{\mathtt{XB}} q \parallel r$, *then* $p \sim_{\mathtt{XB}} q$.

4. Completeness for rooted branching bisimilarity over $\mathrm{CCS}_{\mathrm{L}}$

In this section we present a complete axiomatisation for rooted branching equivalence over $\mathrm{CCS}_{\mathrm{L}}$.

In [28] it was proved that if we consider the fragment BCCS of CCS (i.e., the fragment consisting only of $\mathbf{0}$, variables, prefixing, and nondeterministic choice), then a ground-complete axiomatisation of rooted branching bisimilarity over BCCS is given by $\mathcal{E}_0 \cup \{\mathrm{TB}\}$, where $\mathcal{E}_0 = \{\mathrm{A0,A1,A2,A3}\}$ from Table 4, and axiom TB is in Table 5. Informally, TB mimics the fact that the τ-moves that do not modify the branching structure, and thus the equivalence class, of the term performing them are *stuttering*, and they can be safely disregarded. In [9] it was proved that the axiom system $\mathcal{E}^{\mathrm{L}}_{\mathtt{B}}$, given in Table 4, is a complete axiomatisation of strong bisimilarity over $\mathrm{CCS}_{\mathrm{L}}$.

Starting from these works, we provide a complete axiomatisation for rooted branching bisimilarity. Specifically, our aim is to show that the axiom system $\mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}} = \mathcal{E}^{\mathrm{L}}_{\mathtt{B}} \cup \{\mathrm{TB},\mathrm{TL1}\}$ presented in Table 5 is a *complete axiomatisation of rooted branching bisimulation equivalence over* $\mathit{CCS}_{\mathit{L}}$.

The intuitive reason for adding axiom TL1 is the same of TB, the difference being that while the latter deals with the interplay between τ-prefixing and nondeterministic choice, the former considers left merge. Interestingly, by combining TL1 and TB, it is possible to derive, as shown below, equation DT2 in Table 5, which is the equation for left merge corresponding to TB.

$$\begin{aligned}
x \mathbin{\|\!\_}\, (\tau(y+z)+y) &\overset{(\mathrm{TL1})}{\approx} x \mathbin{\|\!\_}\, \tau(\tau(y+z)+y) \\
&\overset{(\mathrm{TB})}{\approx} x \mathbin{\|\!\_}\, \tau(y+z) \\
&\overset{(\mathrm{TL1})}{\approx} x \mathbin{\|\!\_}\, (y+z).
\end{aligned}$$

Table 5 includes also equation DT1, which can be derived from $\mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}}$, and is useful in the technical development of our results. Notice that DT1 corresponds essentially to the substitution instance of TB in which y is mapped to $\mathbf{0}$.

First of all, it is immediate to prove the soundness of $\mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}}$ modulo $\sim_{\mathtt{RBB}}$.

Theorem 1. *The axiom system* $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}}$ *is sound modulo* $\sim_{\mathrm{RBB}}$ *over* CCS_L.

4.1. The proof strategy

To obtain the desired completeness result, we apply the same strategy used in [9] to prove the completeness, modulo strong bisimilarity, of the axiom system $\mathcal{E}^{\mathrm{L}}_{\mathrm{B}}$ presented in Table 4. That strategy consists of the following steps:

1. We identify *normal forms* for $\mathrm{CCS_L}$ terms and show that each term can be proven equal to a normal form using $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}}$ (Proposition 4).
2. We associate with every two normal forms N_1, N_2 a so-called *distinguishing substitution* $*$ such that if $*(N_1) \sim_{\mathrm{BB}} *(N_2)$, then the equation $\mu.N_1 \approx \mu.N_2$ can be proven using $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}}$ (Proposition 5).
3. Using the two results mentioned above, we show that for all terms t, u, if $t \sim_{\mathrm{RBB}} u$, then $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash t \approx u$ (completeness result, Theorem 2).

Informally, distinguishing substitutions will be crafted in such a way that we can identify the original syntactic structure of a normal form N by analysing the behaviour of $*(N)$. Specifically, they will allow us to recognise the behaviour induced by the instantiations of variables. To this end, in the definition of distinguishing substitutions, we will make use of the following family of processes:

$$\varphi_i = \sum_{k=1}^{i} a^k \qquad i \geq 1, \tag{1}$$

where $a \in \mathcal{A}$ and process a^k is inductively defined by $a^0 = \mathbf{0}$ and $a^k = a.a^{k-1}$ for each $k > 0$.

Then, we define the notion of *branching degree* with respect to branching bisimilarity of a process p as follows:

$$\mathrm{bd}_{\mathrm{BB}}(p) = \left| \{ (\mu, [q]_{\sim_{\mathrm{BB}}}) \mid p \xrightarrow{\varepsilon} p' \xrightarrow{\mu} [q]_{\sim_{\mathrm{BB}}} \text{ with } [p]_{\sim_{\mathrm{BB}}} = [p']_{\sim_{\mathrm{BB}}} \text{ and } [q]_{\sim_{\mathrm{BB}}} \neq [p]_{\sim_{\mathrm{BB}}} \} \right|. \tag{2}$$

Lemma 5. *Whenever* $p \sim_{\mathrm{BB}} q$, *then* $\mathrm{bd}_{\mathrm{BB}}(p) = \mathrm{bd}_{\mathrm{BB}}(q)$.

Remark 1. *We remark that Lemma 5 would not hold without the two requirements on the equivalence classes of processes in Equation (2). Consider, for instance, processes* $s = a$ *and* $t = \tau.a$. *Clearly,* $s \sim_{\mathrm{BB}} t$. *If, in the definition of branching degree, we remove the requirement on the equivalence class of* q *being different than that of* p, *we get* $\mathrm{bd}_{\mathrm{BB}}(s) = 1 \neq 2 = |\{(\tau, [a]_{\sim_{\mathrm{BB}}}), (a, [\mathbf{0}]_{\sim_{\mathrm{BB}}})\}| = \mathrm{bd}_{\mathrm{BB}}(t)$. *The problem is that the inert* τ*-transition from* t *is counted twice: once as* $t \xrightarrow{\tau} a$, *and once in* $t \xrightarrow{\varepsilon} a \xrightarrow{a} \mathbf{0}$. *The requirement on the equivalence class of* a *being different than that of* t, *allows us to exclude the pair* $(\tau, [a]_{\sim_{\mathrm{BB}}})$ *from the calculation.*

Consider now, processes $r = \tau$. *We have that* $s + r \sim_{\mathrm{BB}} r + t$. *However, if we remove the requirement on the process* p' *being equivalent to* p *in the definition of branching degree, then we get* $\mathrm{bd}_{\mathrm{BB}}(s + r) = 2 \neq 3 = |\{(\tau, [\mathbf{0}]_{\sim_{\mathrm{BB}}}), (\tau, [a]_{\sim_{\mathrm{BB}}}), (a, [\mathbf{0}]_{\sim_{\mathrm{BB}}})\}| = \mathrm{bd}_{\mathrm{BB}}(r + t)$. *Also in this situation, the* τ*-transition from* t *is counted twice: once as* $r + t \xrightarrow{\tau} a$, *and once as* $r + t \xrightarrow{\varepsilon} a \xrightarrow{a} \mathbf{0}$. *In this case, the requirement on the equivalence class of* a *being the same as that of* $r + t$, *allows us to exclude the pair* $(a, [\mathbf{0}]_{\sim_{\mathrm{BB}}})$ *from the calculation.*

Another interesting property is that the branching degree of a parallel composition is at least the branching degree of the parallel components.

Lemma 6. *For all* CCS_L *processes* p, q, *it holds that* $\mathrm{bd}_{\mathrm{BB}}(p), \mathrm{bd}_{\mathrm{BB}}(q) \leq \mathrm{bd}_{\mathrm{BB}}(p \parallel q)$.

Proof. Without loss of generality, we show it only for p. The proof for q is analogous. We also abuse notation, and use $\mathrm{bd}_{\mathrm{BB}}(\cdot)$ to denote the set whose cardinality corresponds to the branching degree of a process. To prove the claim, it is enough to show that for each pair $(\mu, [r]_{\sim_{\mathrm{BB}}}) \in \mathrm{bd}_{\mathrm{BB}}(p)$ there is a pair $(\mu, [r \parallel q]_{\sim_{\mathrm{BB}}}) \in \mathrm{bd}_{\mathrm{BB}}(p \parallel q)$.

Let $(\mu, [r]_{\sim_{\mathrm{BB}}}) \in \mathrm{bd}_{\mathrm{BB}}(p)$. Then there is a process p' such that $p \xrightarrow{\varepsilon} p' \xrightarrow{\mu} r$, $p \sim_{\mathrm{BB}} p'$ and $p \nsim_{\mathrm{BB}} r$. From $p \xrightarrow{\varepsilon} p' \xrightarrow{\mu} r$, it follows that $p \parallel q \xrightarrow{\varepsilon} p' \parallel q \xrightarrow{\mu} r \parallel q$. Moreover, since $\sim_{\mathrm{BB}}$ is a congruence over $\mathrm{CCS_L}$, $p \sim_{\mathrm{BB}} p'$ implies $p \parallel q \sim_{\mathrm{BB}} p' \parallel q$. Assume now, towards a contradiction, that $p \parallel q \sim_{\mathrm{BB}} r \parallel q$. By Corollary 2, this would imply $p \sim_{\mathrm{BB}} r$ giving thus a contradiction with $(\mu, [r]_{\sim_{\mathrm{BB}}}) \in \mathrm{bd}_{\mathrm{BB}}(p)$. We have therefore obtained that $p \parallel q \xrightarrow{\varepsilon} p' \parallel q \xrightarrow{\mu} r \parallel q$ with $p \parallel q \sim_{\mathrm{BB}} p' \parallel q$ and $p \parallel q \nsim_{\mathrm{BB}} r \parallel q$. Hence, we can conclude that $(\mu, [r \parallel q]_{\sim_{\mathrm{BB}}}) \in \mathrm{bd}_{\mathrm{BB}}(p \parallel q)$. $\square$

Processes φ_i satisfy some interesting properties, proved in [9, Lemma 2.14] in the case of strong bisimilarity, that can be lifted to our setting. Since these properties hold both for $\mathrm{CCS_L}$ and $\mathrm{CCS_{LC}}$, we discuss them here.

Lemma 7.

1. *For all* $i \geq 1$, *the processes* φ_i *are indecomposable modulo* $\sim_{\mathrm{BB}}$.

2. *The processes* φ_i, $i \geq 1$, *are all distinct, namely* $\varphi_j \sim_{\mathrm{BB}} \varphi_i$ *implies that* $j = i$.
3. *For all* $i \geq 1$, $\mathrm{bd}_{\mathrm{BB}}(\varphi_i) = i$.

Proof. We prove each statement in turn.

1. We prove the first statement only over $\mathrm{CCS}_{\mathrm{LC}}$. The proof over $\mathrm{CCS}_{\mathrm{L}}$ is simpler, and can be adapted from the one we give below. Assume, towards a contradiction, that there are processes $p, q \not\sim_{\mathrm{BB}} \mathbf{0}$ such that $\varphi_i \sim_{\mathrm{BB}} p \parallel q$. As $\varphi_i \xrightarrow{a} \mathbf{0}$, we have that $p \parallel q \xrightarrow{\varepsilon} r \xrightarrow{a} r'$ for some processes r, r' such that $r \sim_{\mathrm{BB}} \varphi_i$ and $r' \sim_{\mathrm{BB}} \mathbf{0}$. Notice that $p \not\sim_{\mathrm{BB}} \mathbf{0}$ implies that there is at least one $p' \in \mathrm{der}(p)$ such that $p' \xrightarrow{\mu} p''$ for some process p'' and action $\mu \neq \tau$. A similar property holds for q. Since, moreover, φ_i can only perform sequences of a-moves, $p \parallel q \sim_{\mathrm{BB}} \varphi_i$ implies that also p and q can perform (weak) sequences of a-moves. In particular, it follows that there is no derivative of p or q that can perform action $\overline{a}$. As a consequence, p and q cannot synchronise. This implies that there are processes p', q' such that $p \xrightarrow{\varepsilon} p'$, $q \xrightarrow{\varepsilon} q'$ and $r = p' \parallel q'$. Then $r \xrightarrow{a} r'$ can follow either from $p' \xrightarrow{a} p''$, or $q' \xrightarrow{a} q''$, for some p'', q''. Assume, without loss of generality, that $p' \xrightarrow{a} p''$ and $r' = p'' \parallel q'$ (the case $q' \xrightarrow{a} q''$ is analogous). Now $p'' \parallel q' \sim_{\mathrm{BB}} \mathbf{0}$ implies $p'' \sim_{\mathrm{BB}} \mathbf{0}$ and $q' \sim_{\mathrm{BB}} \mathbf{0}$. Hence, $p' \sim_{\mathrm{BB}} p' \parallel q' = r \sim_{\mathrm{BB}} \varphi_i$ and, thus, $\mathrm{depth}(p) \geq i$. Since $q \not\sim_{\mathrm{BB}} \mathbf{0}$ gives $\mathrm{depth}(q) \geq 1$, we get that $\mathrm{depth}(p \parallel q) > i$. This contradicts our assumption that $p \parallel q \sim_{\mathrm{BB}} \varphi_i$, since $\mathrm{depth}(\varphi_i) = i \neq \mathrm{depth}(p \parallel q)$ (cf. Lemma 1).
2. We have that, for each $i \geq 1$, $\mathrm{depth}(\varphi_i) = i$. Hence, if $\varphi_j \sim_{\mathrm{BB}} \varphi_i$, then $j = \mathrm{depth}(\varphi_j) = \mathrm{depth}(\varphi_i) = i$.
3. Firstly, we notice that $\varphi_i \xrightarrow{a} a^k.\mathbf{0}$ for all $k = 0, \ldots, i-1$. Since, moreover, $a^k.\mathbf{0} = a^h.\mathbf{0}$ implies $k = h$, for all $k, h = 0, \ldots, i-1$, we can infer that $\mathrm{bd}_{\mathrm{BB}}(\varphi_i)$ is at least i.

   Secondly, we notice that whenever $\varphi_i \xrightarrow{\mu} p$, then $\mu = a$ and $p = a^k.\mathbf{0}$ for some $0 \leq k < i$. Therefore, $\mathrm{bd}_{\mathrm{BB}}(\varphi_i)$ cannot be greater than i. ☐

Remark 2. *Statements 1 and 2 in Lemma 7 hold also modulo* $\sim_{\eta\mathrm{B}}, \sim_{\mathrm{DB}}$ *and* $\sim_{\mathrm{WB}}$. *We omit the proofs because they are straightforward adaptations of the one presented above for branching bisimilarity.*

4.2. Normal forms over $\mathrm{CCS}_{\mathrm{L}}$

According to the strategy described in Section 4.1, the first step towards proving the completeness of $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}}$ over $\mathrm{CCS}_{\mathrm{L}}$ consists in identifying normal forms for $\mathrm{CCS}_{\mathrm{L}}$ terms.

Definition 6 *(Normal form over* $\mathrm{CCS}_{\mathrm{L}}$*). The set of normal forms over* $\mathrm{CCS}_{\mathrm{L}}$ *is generated by the following grammar:*

$$\begin{aligned} S &::= \mu.N \quad \mid \quad x \mathbin{\|\!\!\lfloor} N, \text{ and} \\ N &::= \mathbf{0} \quad \mid \quad S \quad \mid \quad N + N, \end{aligned}$$

where $x \in \mathcal{V}$ *and* $\mu \in \mathcal{A}_\tau$. *Normal forms* S*, which do not have* $+$ *as main operator, are also called* simple *normal forms.*

Each term can be proven equal using $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}}$ to a normal form.

Proposition 4. *For every term* t *there is a normal form* N *such that* $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash t \approx N$.

Proof. The proof can be carried out by induction on the size of terms exactly like in the proof of Lemma 3.3 in [9], and therefore we omit it. ☐

As direct consequence of Proposition 4, we can henceforth assume that each $\mathrm{CCS}_{\mathrm{L}}$ term t can be expressed in the general form

$$t \approx \sum_{i \in I} \mu_i N_i + \sum_{j \in J} x_j \mathbin{\|\!\!\lfloor} N_j \ .$$

Moreover, if t is a $\mathrm{CCS}_{\mathrm{L}}$ normal form, then only axioms A1–A4 are needed to prove the equivalence. The terms $\mu_i.N_i$ $(i \in I)$ and $x_j \mathbin{\|\!\!\lfloor} N_j$ $(j \in J)$ will be referred to as the *syntactic summands* of t.

4.3. Distinguishing substitutions over $\mathrm{CCS}_{\mathrm{L}}$

The second step in our proof strategy consists in associating a specific substitution with every pair of normal forms. The aim is to define the substitution, denoted by $*$, associated to normal forms N and M in such a way that $*(N) \sim_{\mathrm{BB}} *(M)$ implies that $\mu.N \approx \mu.M$ can be proven using $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}}$. Such a substitution is called a *distinguishing substitution*, for, by contraposition, if $\mu.N \approx \mu.M$ can not be proven, then $*$ distinguishes N and M modulo branching bisimilarity.

To this end, we need first to introduce the concept of *width* of a normal form. Each distinguishing substitution will then be parameterised with an upper bound on the widths of the normal forms to which it is associated.

Definition 7 *(Width of CCS$_{\mathrm{L}}$ normal forms). We define the* width $w(N)$ *of a CCS$_{\mathrm{L}}$ normal form N as follows:*

1. *if $N = \mathbf{0}$, then $w(N) = 0$;*
2. *if $N = \mu.N'$, then $w(N) = \max\{w(N'), 1\}$;*
3. *if $N = x \mathbin{\|\!\lfloor} N'$, then $w(N) = \max\{w(N'), 1\}$; and*
4. *if $N = N_1 + N_2$, then $w(N) = w(N_1) + w(N_2)$.*

Definition 8 *(Distinguishing substitution over CCS$_{\mathrm{L}}$). Let N, M be two normal forms and let $W \geq w(N), w(M)$. Consider an injective mapping*

$$\lceil\_\rceil : \mathcal{V} \to \{n \in \mathbb{N} : n > W\}$$

that associates with every variable a unique natural number greater than W. We define the distinguishing substitution *as the closed substitution $*$ such that*

$$*(x) = a.\varphi_{\lceil x\rceil} \ ,$$

for all $x \in \mathcal{V}$, where a is some arbitrary action name in $\mathcal{A}$, and $\varphi_{\lceil x\rceil}$ is defined as in (1).

A similar substitution was used in [9], but there τ was used instead of $a \in \mathcal{A}$.

Lemma 8. *For every CCS$_{\mathrm{L}}$ normal form N, the branching degree with respect to $\sim_{\mathrm{BB}}$ of $*(N)$ is at most $w(N)$.*

Proof. The proof proceeds by structural induction over N. We expand only the inductive step related to summation, since the others cases are immediate. In particular, since every CCS$_{\mathrm{L}}$ normal form can be written as a summation of simple normal forms, it is enough to prove the statement for $N = \sum_{i=1}^{k} S_i$, for some simple normal forms S_i. Let

$$BD(*(N)) = \{(\mu, [q]_{\sim_{\mathrm{BB}}}) \mid *(N) \xrightarrow{\varepsilon} p \xrightarrow{\mu} q \text{ with } *(N) \sim_{\mathrm{BB}} p \text{ and } *(N) \not\sim_{\mathrm{BB}} q\}$$

so that $\mathrm{bd}_{\mathrm{BB}}(*(N)) = |BD(*(N))|$. We can then distinguish two cases:

- There exist processes p, q such that $(\mu, [q]_{\sim_{\mathrm{BB}}}) \in BD(*(N))$ for some μ, $*(N) \xrightarrow{\varepsilon} p \xrightarrow{\mu} q$ and $*(N) \xrightarrow{\varepsilon} p$ with at least one τ-transition in the sequence. By construction of the distinguishing substitution $*$, since $*(x) \overset{\tau}{\not\longrightarrow}$ for all $v \in \mathrm{var}$, then each τ-transition in the sequence $*(N) \xrightarrow{\varepsilon} p$ is due to a summand of the form $\tau.N'$ for some normal form N'. In particular, there are a $j \in \{1, \dots, k\}$ and a normal form M such that $*(S_j) \xrightarrow{\varepsilon} *(M)$, $p = *(M)$, and $*(N) \sim_{\mathrm{BB}} *(M)$. Hence, by Lemma 5, we have that $\mathrm{bd}_{\mathrm{BB}}(*(N)) = \mathrm{bd}_{\mathrm{BB}}(*(M))$, and by structural induction over M we obtain that $\mathrm{bd}_{\mathrm{BB}}(*(M)) \leq w(M)$. By definition of width, we also have that $w(M) \leq w(S_j)$. Summarising, we have obtained that

  $$\mathrm{bd}_{\mathrm{BB}}(*(N)) = \mathrm{bd}_{\mathrm{BB}}(*(M)) \leq w(M) \leq w(S_j) \leq \sum_{i=1}^{k} w(S_i) = w(N).$$

- For each $(\mu, [q]_{\sim_{\mathrm{BB}}}) \in BD(*(N))$, we have that $*(N) \xrightarrow{\mu} q$ and $q \not\sim_{\mathrm{BB}} *(N)$. This means that the branching degree of $*(N)$ is at most the number of outgoing transition from $*(N) = \sum_{i=1}^{k} *(S_i)$, i.e., it is at most k. Since, by definition of width, $w(S_i) \geq 1$ for all $i \in \{1, \dots, k\}$, we can conclude that

  $$\mathrm{bd}_{\mathrm{BB}}(*(N)) \leq k \leq \sum_{i=1}^{k} w(S_i) = w(N).$$

We have obtained, in both cases, that $\mathrm{bd}_{\mathrm{BB}}(*(N)) \leq w(N)$. □

Lemma 9. *Let S be a simple CCS$_{\mathrm{L}}$ normal form, let $\mu \in \mathcal{A}_\tau$, and let p be a process such that $*(S) \xrightarrow{\mu} p$.*

1. *If $S = \nu.N$, then $\mu = \nu$ and $p = *(N)$.*
2. *If $S = x \mathbin{\|\!\lfloor} N$, then $\mu = a$ and $p = \varphi_{\lceil x\rceil} \parallel *(N)$.*

We say that a process p is of *type* n ($n \geq 0$) if its unique parallel decomposition contains precisely n parallel components with a branching degree larger than W. As an immediate corollary to Lemmas 8 and 9 we get that the unique residual of $*(\mu.N)$ is of type 0 and the unique residual of $*(x \mathbin{\|\!\lfloor} N)$ is of type 1.

The following proposition is the crux in the proof of the completeness result.

Proposition 5. *Let N and M be CCS$_\mathrm{L}$ normal forms such that $w(N), w(M) \leq W$. Then $*(N) \sim_{\mathrm{BB}} *(M)$ implies $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \mu.N \approx \mu.M$ for every $\mu \in \mathcal{A}_\tau$.*

Proof. Suppose that $*(N) \sim_{\mathrm{BB}} *(M)$. To prove that $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \mu.N \approx \mu.M$ for every $\mu \in \mathcal{A}_\tau$, we proceed by induction on $\mathrm{size}(N) + \mathrm{size}(M)$.

Let I, J, K and L be finite disjoint sets, $\mu_i, \mu_k \in \mathcal{A}_\tau$, $x_j, x_h \in \mathcal{V}$ and N_i, N_j, M_k and M_h be normal forms for all $i \in I$, $j \in J$, $k \in K$ and $h \in H$ such that

$$N \approx \sum_{i\in I} \mu_i.N_i + \sum_{j \in J} x_j \mathbin{\|\!\!\_} N_j \quad \text{and} \quad M \approx \sum_{k \in K} \mu_k.M_k + \sum_{h\in H} x_h \mathbin{\|\!\!\_} M_h \ .$$

We now distinguish three cases:

1. There is no $i \in I$ such that $\mu_i = \tau$ and $*(N_i) \sim_{\mathrm{BB}} *(M)$, and there is no $k \in K$ such that $\mu_k = \tau$ and $*(N) \sim_{\mathrm{BB}} *(M_k)$.
   In this case, as $*(N) \sim_{\mathrm{BB}} *(M)$, for every μ, each transition $*(N) \xrightarrow{\mu} p$ must be matched by a transition $*(M) \xrightarrow{\mu} q$ for some q such that $p \sim_{\mathrm{BB}} q$. We can now distinguish two cases, according to the form of the syntactic summand S of N inducing the transition:
   - $S = \mu_i.N_i$ for some $i \in I$, such that $\mu = \mu_i$. Then $*(N) \xrightarrow{\mu_i} *(N_i) = p$, where $*(N_i)$ is of type 0, i.e., none of its parallel primes has a branching degree that exceeds W. This transition is then matched by a syntactic summand S' of M such that $*(S') \xrightarrow{\mu_i} q$ for some q such that $*(N_i) \sim_{\mathrm{BB}} q$. Given the uniqueness of the parallel decomposition modulo $\sim_{\mathrm{BB}}$, we have that also q is of type 0. Hence, by Lemma 9, $S' = \mu_k.M_k$ for some $k \in K$ such that $\mu_k = \mu_i$, $*(M_k) = q$ and $*(N_i) \sim_{\mathrm{BB}} *(M_k)$. As the sum of the sizes of N_i and M_k is strictly smaller than $\mathrm{size}(N) + \mathrm{size}(M)$, by induction we get that
     $$\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \mu_i.N_i \approx \mu_i.M_k = \mu_k.M_k. \tag{3}$$
     Since the above reasoning does not depend on the choice of the particular index $i \in I$, we can deduce that for each $i \in I$ there is a $k \in K$ such that $\mu_i = \mu_k$ and $*(N_i) \sim_{\mathrm{BB}} *(M_k)$. Symmetrically, for each summand $\mu_k.M_k$ there is a summand $\mu_i.N_i$ such that $\mu_i = \mu_k$, $*(N_i) \sim_{\mathrm{BB}} *(M_k)$, and Equation (3) holds for every such pair. Hence, we can conclude that $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \sum_{i\in I} \mu_i.N_i \approx \sum_{k \in K} \mu_k.M_k$.
   - $S = x_j \mathbin{\|\!\!\_} N_j$ for some $j \in J$. Then, $\mu = a$ and $*(N) \xrightarrow{a} \varphi_{\lceil x_j \rceil} \| *(N_j) = p$, with p of type 1. Since, by Lemma 7.3, $\mathrm{bd}_{\mathrm{BB}}(\varphi_{\lceil x_j \rceil}) > W$ and $\varphi_{\lceil x_j \rceil}$ is indecomposable, we can also infer that $*(N_j)$ is of type 0. This transition is then matched by a syntactic summand S' of M such that $*(S') \xrightarrow{a} q$ for some q such that $q \sim_{\mathrm{BB}} \varphi_{\lceil x_j \rceil} \| *(N_j)$. Given the uniqueness of the parallel decomposition modulo $\sim_{\mathrm{BB}}$, we have that also q is of type 1. Hence, by Lemma 9, $S' = x_h \mathbin{\|\!\!\_} M_h$ for some $h \in H$ such that $\varphi_{\lceil x_j \rceil} \| *(N_j) \sim_{\mathrm{BB}} \varphi_{\lceil x_h \rceil} \| *(M_h)$. Also in this case, from Lemma 7.3 we infer that $*(M_h)$ is of type 0. Hence, since $*(N_j)$ and $*(M_h)$ are of type 0, we have that the unique parallel decomposition of $*(N_j)$ cannot contain $\varphi_{\lceil x_h \rceil}$ and, likewise, the unique parallel decomposition of $*(M_h)$ does not contain $\varphi_{\lceil x_j \rceil}$. Therefore, $\varphi_{\lceil x_j \rceil} \| *(N_j) \sim_{\mathrm{BB}} \varphi_{\lceil x_h \rceil} \| *(M_h)$ implies that $\varphi_{\lceil x_j \rceil} \sim_{\mathrm{BB}} \varphi_{\lceil x_h \rceil}$ and $*(N_j) \sim_{\mathrm{BB}} *(M_h)$. By Lemma 7.2 and the injectivity of $\lceil \cdot \rceil$, we then get that $x_j = x_h$. Moreover, since the sum of the sizes of N_j and M_h is strictly smaller than $\mathrm{size}(N) + \mathrm{size}(M)$, by induction we obtain that
     $$\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \tau.N_j \approx \tau.M_h. \tag{4}$$
     Then, we have
     $$\begin{aligned} x_j \mathbin{\|\!\!\_} N_j &\overset{(\mathrm{TL1})}{\approx} x_j \mathbin{\|\!\!\_} \tau.N_j \\ &\overset{(4)}{\approx} x_j \mathbin{\|\!\!\_} \tau.M_h \\ &\overset{(\mathrm{TL1})}{\approx} x_j \mathbin{\|\!\!\_} M_h \\ &\overset{(x_j = x_h)}{\approx} x_h \mathbin{\|\!\!\_} M_h. \end{aligned}$$
     Summarising, by symmetry, we have obtained that for each $j \in J$ (respectively, $h \in H$) there is a $h \in H$ (respectively, $j \in J$) such that:
     $$\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash x_j \mathbin{\|\!\!\_} N_j \approx x_h \mathbin{\|\!\!\_} M_h. \tag{5}$$

   Equations (3) and (5) taken together give $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash N \approx M$, from which it is immediate to infer $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \mu.N \approx \mu.M$, for each $\mu \in \mathcal{A}_\tau$, and the proof is complete in this case.
2. Assume now that $\mu_i = \tau$ and $*(N_i) \sim_{\mathrm{BB}} *(M)$ for some $i \in I$, and that $\mu_k = \tau$ and $*(N) \sim_{\mathrm{BB}} *(M_k)$ for some $k \in K$. Clearly, we have that $*(N_i) \sim_{\mathrm{BB}} *(M) \sim_{\mathrm{BB}} *(N) \sim_{\mathrm{BB}} *(M_k)$, and $\mathrm{size}(N_i) + \mathrm{size}(M_k), \mathrm{size}(N_i) + \mathrm{size}(M), \mathrm{size}(N) + \mathrm{size}(M_k) < \mathrm{size}(N) + \mathrm{size}(M)$, so that by induction we obtain
   $$\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \mu.N_i \approx \mu.M_k \qquad \mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \mu.N \approx \mu.M_k \qquad \mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \mu.N_i \approx \mu.M$$

from which $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \mu.N \approx \mu.M$ can be inferred, and the proof is complete in this case.

3. Assume that there is an index $i \in I$ such that $\mu_i = \tau$ and $*(N_i) \sim_{\mathrm{BB}} *(M)$, but there is no $k \in K$ such that $\mu_k = \tau$ and $*(N) \sim_{\mathrm{BB}} *(M_k)$. (The symmetric case can be treated similarly and it is therefore omitted.) For every summand $\tau.N_i$ of N with $*(N_i) \sim_{\mathrm{BB}} *(M)$ we have that the sum of the sizes of N_i and M is strictly smaller than $\mathrm{size}(N) + \mathrm{size}(M)$. Hence, by induction we obtain that $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \tau.N_i \approx \tau.M$ for all such summands. Thus, possibly applying axioms A0–A3, we can infer that

$$\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash N \approx \tau.M + N' \tag{6}$$

where

$$N' = \sum_{i \in I_N} \mu_i.N_i + \sum_{j \in J} x_j \mathbin{⫫} N_j$$

with $I_N = \{i \in I \mid \mu_i \neq \tau \vee *(N_i) \nsim_{\mathrm{BB}} *(M)\}$. Given the condition on the indexes in I_N, and considering that there is no $k \in K$ such that $\mu_k = \tau$ and $*(N) \sim_{\mathrm{BB}} *(M_k)$, it is immediate to verify that whenever $*(N') \rightarrow p$ then $*(M) \rightarrow q$ for some q such that $p \sim_{\mathrm{BB}} q$. In particular, by applying the same reasoning used in the analysis of case 1 above, we have:

- for each $i \in I_N$ there is a $k_i \in K$ such that

$$\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \mu_i.N_i \approx \mu_{k_i}.M_{k_i};$$

- for each $j \in J$ there is a $h_j \in H$ such that

$$\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash x_j \mathbin{⫫} N_j \approx x_{h_j} \mathbin{⫫} M_{h_j}.$$

Summarising, we have obtained that

$$\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash M \approx N' + M' \tag{7}$$

where

$$M' = \sum \{\mu_k.M_k \mid k \neq k_i \text{ for all } i\} + \sum \{x_h \mathbin{⫫} M_h \mid h \neq h_j \text{ for all } j\}.$$

Then:

$$\begin{aligned}
\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash \mu.N &\overset{(6)}{\approx} \mu.(\tau.M + N') \\
&\overset{(7)}{\approx} \mu.(\tau.(N' + M') + N') \\
&\overset{(\mathrm{TB})}{\approx} \mu.(N' + M') \\
&\overset{(7)}{\approx} \mu.M
\end{aligned}$$

and the claim follows also in this case. □

4.4. *The completeness result for $\sim_{\mathrm{RBB}}$ over CCS$_L$*

We are now ready to prove the completeness of the axiom system $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}}$, which follows from Proposition 4 and Proposition 5. We notice that, since TB is an axiom schemata that generates finitely many axioms if the set of actions is finite, the axiom system $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}}$ is finite if so is $\mathcal{A}_\tau$.

Theorem 2. *Let t, u be CCS$_L$ terms. If $t \sim_{\mathrm{RBB}} u$, then $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash t \approx u$.*

Proof. Assume that $t \sim_{\mathrm{RBB}} u$. By Proposition 4, there exist normal forms N and M such that $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash t \approx N$ and $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash u \approx M$, with

$$N \approx \sum_{i \in I} \mu_i.N_i + \sum_{j \in J} x_j \mathbin{⫫} N_j \quad \text{and} \quad M \approx \sum_{k \in K} \mu_k.M_k + \sum_{h \in H} x_h \mathbin{⫫} M_h$$

where all the N_i, N_j, M_k, M_h are in normal form. To prove the claim, it is then enough to show that $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash N \approx M$, since, by transitivity, this would imply $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash t \approx u$. To this end, we proceed to show that $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash N \approx N + M$, since, by symmetry, this also gives $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash M \approx N + M$, and thus that $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash N \approx M$.

By the soundness of $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}}$ (Theorem 1), $t \approx N$ and $u \approx M$ imply that $t \sim_{\mathrm{RBB}} N$ and $u \sim_{\mathrm{RBB}} M$, thus giving $N \sim_{\mathrm{RBB}} M$. Let $W \geq w(N), w(M)$. From $N \sim_{\mathrm{RBB}} M$, it follows that $*(N) \sim_{\mathrm{RBB}} *(M)$. We can then apply the same reasoning used in case 1 of the proof of Proposition 5 and obtain that:

- For each index $k \in K$ there is an index $i_k \in I$ such that $\mu_{i_k} = \mu_k$ and $*(N_{i_k}) \sim_{\mathrm{BB}} *(M_k)$. Then, for each $k \in K$, by Proposition 5 we obtain that

Table 6
Equational bases modulo rooted η, delay and weak bisimulation over $\mathrm{CCS_L}$.

τ-laws:	
T1	$\mu\tau x \approx \mu x$
T2	$\tau x \approx \tau x + x$
T3	$\mu(\tau x + y) \approx \mu(\tau x + y) + \mu x$
Additional axioms for $\sim_{\mathtt{RWB}}$: $\mathcal{E}^{\mathrm{L}}_{\mathtt{RWB}} = \mathcal{E}^{\mathrm{L}}_{\mathtt{B}} \cup \{$T1,T2,T3,TL1,TL2$\}$	
TL2	$x \mathbin{\|\!\!\_} (\tau y + z) \approx x \mathbin{\|\!\!\_} (\tau y + z) + x \mathbin{\|\!\!\_} y$
$\mathcal{E}^{\mathrm{L}}_{\mathtt{R\eta B}} = \mathcal{E}^{\mathrm{L}}_{\mathtt{B}} \cup \{$TB,T3,TL1,TL2$\}$ $\mathcal{E}^{\mathrm{L}}_{\mathtt{RDB}} = \mathcal{E}^{\mathrm{L}}_{\mathtt{B}} \cup \{$T1,T2,TL1,TL2$\}$	

$$\mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}} \vdash \mu_{i_k}.N_{i_k} \approx \mu_{i_k}.M_k = \mu_k.M_k. \tag{8}$$

- For each $h \in H$ there is a $j_h \in J$ such that $x_h = x_{j_h}$ and $*(M_h) \sim_{\mathtt{BB}} *(N_{j_h})$. Then, for each $h \in H$, by Proposition 5 and axiom TL1 we obtain that

$$\mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}} \vdash x_{j_h} \mathbin{\|\!\!\_} N_{j_h} \approx x_{j_h} \mathbin{\|\!\!\_} \tau.N_{j_h} \approx x_h \mathbin{\|\!\!\_} \tau.M_h \approx x_h \mathbin{\|\!\!\_} M_h. \tag{9}$$

Equations (8) and (9) allow us to infer that $\mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}} \vdash N \approx N + M$. By symmetry, $\mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}} \vdash M \approx N + M$ and we are done. ☐

5. Extending the completeness result over $\mathrm{CCS_L}$

In this section we consider the relations of rooted η, delay, and weak bisimilarity, and we provide a complete axiomatisation for them over $\mathrm{CCS_L}$.

Let $\mathtt{RXB} \in \{\mathtt{R\eta B}, \mathtt{RDB}, \mathtt{RWB}\}$. Ground-complete axiomatisations for $\sim_{\mathtt{RXB}}$ over BCCS were proposed in [28], based on the three τ-laws from [31] (axioms T1,T2,T3 in Table 6), axiom TB in Table 5, and the axiom system $\mathcal{E}_0 = \{$A0,A1,A2,A3$\}$. In particular, the complete axiomatisation for $\sim_{\mathtt{RWB}}$ over BCCS is $\mathcal{E}_0 \cup \{$T1,T2,T3$\}$; the one for $\sim_{\mathtt{R\eta B}}$ is $\mathcal{E}_0 \cup \{$TB,T3$\}$; the one for $\sim_{\mathtt{RDB}}$ is $\mathcal{E}_0 \cup \{$T1,T2$\}$. Notice that, differently from TB, an application of axiom T3 does not preserve the branching structure of terms. Then, to obtain a complete axiomatisation for $\sim_{\mathtt{RXB}}$ over $\mathrm{CCS_L}$, we follow the same intuition we had for $\sim_{\mathtt{RBB}}$: we need to extend our axiom system with a counterpart of T3 for left merge. This is done by axiom TL2 in Table 6.

Let

$$\begin{aligned}
\mathcal{E}^{\mathrm{L}}_{\mathtt{RWB}} &= \mathcal{E}^{\mathrm{L}}_{\mathtt{B}} \cup \{\text{T1,T2,T3,TL1,TL2}\},\\
\mathcal{E}^{\mathrm{L}}_{\mathtt{RDB}} &= \mathcal{E}^{\mathrm{L}}_{\mathtt{B}} \cup \{\text{T1,T2,TL1,TL2}\},\\
\mathcal{E}^{\mathrm{L}}_{\mathtt{R\eta B}} &= \mathcal{E}^{\mathrm{L}}_{\mathtt{B}} \cup \{\text{TB,T3,TL1,TL2}\} = \mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}} \cup \{\text{T3,TL2}\}.
\end{aligned}$$

Theorem 3. *Let* $\mathtt{RXB} \in \{\mathtt{RWB}, \mathtt{RDB}, \mathtt{R\eta B}\}$. *Then* $\mathcal{E}^{\mathrm{L}}_{\mathtt{RXB}}$ *is sound modulo* $\sim_{\mathtt{RXB}}$ *over* CCS_L.

We can derive the following expressiveness properties for the aforementioned axiom systems:

Lemma 10. $\mathcal{E}^{\mathrm{L}}_{\mathtt{RWB}} \vdash \mathcal{E}^{\mathrm{L}}_{\mathtt{RDB}} \vdash \mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}}$, *and* $\mathcal{E}^{\mathrm{L}}_{\mathtt{RWB}} \vdash \mathcal{E}^{\mathrm{L}}_{\mathtt{R\eta B}} \vdash \mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}}$.

Proof. The proof is similar to the proof of Proposition 3.3 in [12] and it is therefore omitted. ☐

Clearly, Lemma 10 implies that all the equations that are derivable from $\mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}}$ (see Table 5) are also derivable from $\mathcal{E}^{\mathrm{L}}_{\mathtt{RXB}}$. As a direct consequence of this fact, we get that every $\mathrm{CCS_L}$ term can be proven equal, using $\mathcal{E}^{\mathrm{L}}_{\mathtt{RXB}}$ to a $\mathrm{CCS_L}$ normal form (Definition 6).

Corollary 3. *Let* $\mathtt{x} \in \{\eta, \mathtt{d}, \mathtt{w}\}$. *For every* CCS_L *term* t *there is a* CCS_L *normal form* N *such that* $\mathcal{E}^{\mathrm{L}}_{\mathtt{RXB}} \vdash t \approx N$.

One can apply the same reasoning we used in Section 4.4 to obtain a direct proof of the fact that the axiom systems $\mathcal{E}^{\mathrm{L}}_{\mathtt{RXB}}$ are the complete axiomatisations of $\sim_{\mathtt{RXB}}$ we are looking for. However, our main result in this section consists in showing that the completeness of these axioms systems can be derived from the completeness of $\mathcal{E}^{\mathrm{L}}_{\mathtt{RBB}}$. Interestingly, the derivation technique does not depend on the considered semantics; in fact, we will prove the three completeness results at the same time and in a uniform fashion.

We will apply a proof technique based on *saturation* [28]. Briefly, this consists in identifying, for each of the three congruences, a special class of terms over which the considered congruence coincides with rooted branching bisimilarity. To achieve this, we will first need to define the semantics of *open* $\mathrm{CCS_L}$ terms, by using the notion of configuration from [10]. This allows us to establish a correspondence between the behaviour of open terms and that of their closed instances, with special focus on the role of variables.

Table 7
Inference rules for the transition $\xrightarrow{x}$.

$(a_1)\ \dfrac{}{x \xrightarrow{x} x_{\mathrm{d}}}$	$(a_2)\ \dfrac{t \xrightarrow{x} c}{t + u \xrightarrow{x} c}$	$(a_3)\ \dfrac{u \xrightarrow{x} c}{t + u \xrightarrow{x} c}$
$(a_4)\ \dfrac{t \xrightarrow{x} c}{t \lfloor\!\lfloor u \xrightarrow{x} c \parallel u}$	$(a_5)\ \dfrac{t \xrightarrow{x} c}{t \parallel u \xrightarrow{x} c \parallel u}$	$(a_6)\ \dfrac{u \xrightarrow{x} c}{t \parallel u \xrightarrow{x} t \parallel c}$

5.1. Decomposing the semantics of terms

In the technical developments to follow, we need to establish a correspondence between the behaviour of open terms and that of their closed instances. In detail, we are interested in the derivation of a transition $\sigma(t) \xrightarrow{\mu} p$, for some term t, closed substitution σ, action μ, and process p, from the behaviour of t and that of $\sigma(x)$, for each variable x occurring in t. The simplest case is a direct application of the operational semantics in Table 1.

Lemma 11. *For all CCS$_{\mathrm{L}}$ terms t, t', action $\mu \in \mathcal{A}_\tau$, and substitution σ, if $t \xrightarrow{\mu} t'$ then $\sigma(t) \xrightarrow{\mu} \sigma(t')$.*

Let us focus now on the role of variables. A transition $\sigma(t) \xrightarrow{\mu} p$ may also derive from the initial behaviour of some closed term $\sigma(x)$, provided that the collection of initial moves of $\sigma(t)$ depends, in some formal sense, on that of the closed term substituted for the variable x. In this case, we say that x *triggers the behaviour* of t.

To fully describe this situation, we introduce an auxiliary transition relation over open terms. Informally, we use transitions of the form $t \xrightarrow{x} c$ to denote that variable x triggers that behaviour in term t. The target c is a *configuration over terms*, i.e., a term defined over a set of variables $\mathcal{V}_{\mathrm{d}} = \{x_{\mathrm{d}} \mid x \in \mathcal{V}\}$, disjoint from $\mathcal{V}$, and CCS$_{\mathrm{L}}$ terms. We use the special variable x_{d} to denote that the closed term substituted for an occurrence of variable x has begun its execution, and it contributes thus to triggering the behaviour of the term in which x occurs. This is captured by auxiliary transitions of the form $x \xrightarrow{x} x_{\mathrm{d}}$.

Formally, CCS$_{\mathrm{L}}$-*configurations* are defined as follows.

Definition 9 *(CCS$_{\mathrm{L}}$ configuration). The collection of CCS$_{\mathrm{L}}$ configurations, denoted by C_{L}, is given by:*

$$c ::= x_{\mathrm{d}} \quad | \quad t \quad | \quad c \parallel c \ ,$$

where t is a CCS$_{\mathrm{L}}$ term, and $x_{\mathrm{d}} \in \mathcal{V}_{\mathrm{d}}$.

Since we can have occurrences of variables from $\mathcal{V} \cup \mathcal{V}_{\mathrm{d}}$, we extend the definition of $\mathrm{var}(c)$ accordingly: $\mathrm{var}(c) = \{x \in \mathcal{V} \mid x \text{ occurs in } c\} \cup \{x_{\mathrm{d}} \in \mathcal{V}_{\mathrm{d}} \mid x_{\mathrm{d}} \text{ occurs in } c\}$. Substitutions are then extended to configurations in the expected way.

The auxiliary transitions of the form $\xrightarrow{x}$, where $x \in \mathcal{V}$, are formally defined via the inference rules in Table 7.

The following lemma follows from a simple inspection of the rules in Table 7.

Lemma 12. *Let t be a CCS$_{\mathrm{L}}$ term, σ be a closed substitution, and $x \in \mathcal{V}$. If $\sigma(x) \xrightarrow{\mu} p$, for some $\mu \in \mathcal{A}_\tau$ and process p, and $t \xrightarrow{x} c$, for some configuration $c \in C_{\mathrm{L}}$, then $\sigma(t) \xrightarrow{\mu} \sigma[x_{\mathrm{d}} \mapsto p](c)$.*

We can now characterise the cases in which a variable triggers the behaviour of a CCS$_{\mathrm{L}}$ term.

Lemma 13. *Let t be a CCS$_{\mathrm{L}}$ term, σ be a closed substitution. If $\sigma(t) \xrightarrow{\mu} p$, for some $\mu \in \mathcal{A}_\tau$ and process p, then one of the following holds:*

1. *There is a CCS$_{\mathrm{L}}$ term t' such that $t \xrightarrow{\mu} t'$ and $\sigma(t') = p$.*
2. *There are a variable x, a process q, and a configuration c such that $\sigma(x) \xrightarrow{\mu} q$, $t \xrightarrow{x} c$, and $\sigma[x_{\mathrm{d}} \mapsto q](c) = p$.*

5.2. Saturation

The *saturation* technique has been introduced in [28] to derive the ground-completeness of rooted η, delay, and weak bisimilarity over BCCSP from that of rooted branching bisimilarity over the same language. Given the interest in *ground*-completeness, the notion was defined over closed terms. Later on, in [12], saturation was used to derive the completeness of the axiomatisation for the three congruences over BCCS with prefix iteration, and it was defined on open terms, accordingly. Our notion of saturation is also defined on open terms, but over CCS$_{\mathrm{L}}$. We limit our attention to normal forms: our objective is to identify three special classes of normal forms, called $\mathtt{x}$-*saturated* for $\mathtt{x} \in \{\eta, \mathtt{d}, \mathtt{w}\}$, such that distinguishing substitution instances of $\sim_{\mathtt{RXB}}$ equivalent $\mathtt{x}$-saturated normal forms are also $\sim_{\mathtt{RBB}}$ equivalent.

Definition 10 *(Saturation). Consider a normal form N. We say that:*

- *N is* η-saturated *iff:*
  - *whenever $N \xrightarrow{\mu} N_1 \xrightarrow{\varepsilon} N_2$, then N has a summand $\mu.N_2$ and N_2 is η-saturated; and*
  - *whenever $N \xrightarrow{x} c_1 \xrightarrow{\varepsilon} c_2$, then $c_2 = x_{\mathtt{d}} \parallel N_2$, N has a summand $x \mathbin{⫫} N_2$ and N_2 is η-saturated.*
- *N is* $\mathtt{d}$-saturated *iff:*
  - *whenever $N \xrightarrow{\varepsilon} N_1 \xrightarrow{\mu} N_2$, then N has a summand $\mu.N_2$ and N_2 is $\mathtt{d}$-saturated; and*
  - *whenever $N \xrightarrow{\varepsilon} N_1 \xrightarrow{x} c$, then $c = x_{\mathtt{d}} \parallel N_2$, N has a summand $x \mathbin{⫫} N_2$ and N_2 is $\mathtt{d}$-saturated.*
- *N is* $\mathtt{w}$-saturated *iff it is both η-saturated and $\mathtt{d}$-saturated, i.e., iff:*
  - *whenever $N \overset{\mu}{\Longrightarrow} N'$, then N has a summand $\mu.N'$ and N' is $\mathtt{w}$-saturated; and*
  - *whenever $N \overset{x}{\Longrightarrow} c$, then $c = x_{\mathtt{d}} \parallel N'$, N has a summand $x \mathbin{⫫} N'$ and N' is $\mathtt{w}$-saturated.*

Lemma 14. *Let $\mathtt{x} \in \{\eta, \mathtt{d}, \mathtt{w}\}$. If N is a $\mathtt{x}$-saturated normal form, then all the normal forms occurring in its derivatives are also $\mathtt{x}$-saturated.*

The following remark highlights the role of axiom TL2 in the technical development of the completeness results.

Remark 3. *Consider the normal form $x \mathbin{⫫} (\tau.N + M)$, which is a substitution instance of the left hand side of axiom TL2. We remark that $x \mathbin{⫫} (\tau.N + M)$ is not η-saturated. In fact, according to the decomposition of the semantics of CCS$_L$ configurations given in Section 5.1, we have that*

$$x \mathbin{⫫} (\tau.N + M) \xrightarrow{x} x_{\mathtt{d}} \parallel (\tau.N + M) \xrightarrow{\tau} x_{\mathtt{d}} \parallel N.$$

However, it is not possible to derive a transition $x \mathbin{⫫} (\tau.N + M) \xrightarrow{x} x_{\mathtt{d}} \parallel N$.

Consider now the corresponding substitution instance of the right hand side of axiom TL2, namely the normal form $x \mathbin{⫫} (\tau.N + M) + x \mathbin{⫫} N$. This time, thanks to presence of the summand $x \mathbin{⫫} N$, the η-saturation requirement is satisfied, since

$$x \mathbin{⫫} (\tau.N + M) + x \mathbin{⫫} N \xrightarrow{x} x_{\mathtt{d}} \parallel N.$$

A first important property is that, using $\mathcal{E}^{\mathtt{L}}_{\mathtt{RXB}}$, every normal form can be proved equal to an $\mathtt{x}$-saturated normal form.

Proposition 6. *Let $\mathtt{x} \in \{\eta, \mathtt{d}, \mathtt{w}\}$. For every CCS$_L$ normal form N there exists an $\mathtt{x}$-saturated normal form M such that $\mathcal{E}^{\mathtt{L}}_{\mathtt{RXB}} \vdash N \approx M$.*

Proof. We prove the statement only for $\mathtt{x} = \mathtt{w}$, as the proofs for η and $\mathtt{d}$ are simpler and can be seen as subcases of the proof for $\mathtt{w}$ (with, roughly, cases (1a) and (2a) corresponding to the proof for η, and cases (1b) and (2b) to that for $\mathtt{d}$). To this end, it is enough to show that for every simple normal form S it holds that

1. Whenever $S \overset{\mu}{\Longrightarrow} N$, then $\mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}} \vdash S \approx S + \mu.N$.
2. Whenever $S \overset{x}{\Longrightarrow} c$, then $c = x_{\mathtt{d}} \parallel N$ and $\mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}} \vdash S \approx S + x \mathbin{⫫} N$.

We will prove the two properties by induction over the total number n of τ-moves (possibly excluding μ) in the weak transitions $S \overset{\mu}{\Longrightarrow} N$ and $S \overset{x}{\Longrightarrow} c$.

1. Assume that $S \overset{\mu}{\Longrightarrow} N$.
   Base case $n = 0$. In this case, we have that $S \xrightarrow{\mu} N$ and, thus, $S = \mu.N$. It is then immediate to check that
   $$\mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}} \vdash S \approx S + S \approx S + \mu.N \ \ .$$
   Inductive step $n > 0$. We distinguish two cases:
   (a) $S \xrightarrow{\mu} N' \xrightarrow{\varepsilon} N$. Since $n > 0$, there is a simple normal form S' that is a summand of N' and such that $S' \overset{\tau}{\Longrightarrow} N$. By induction, we get that $\mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}} \vdash S' \approx S' + \tau.N$ and, consequently, $N' \approx N' + \tau.N$. Then we have
   $$\begin{aligned} \mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}} \vdash \ S &\approx \mu.N' \\ &\approx \mu.(N' + \tau.N) \\ &\overset{\text{(T3)}}{\approx} \mu.(N' + \tau.N) + \mu.N \\ &\approx \mu.N' + \mu.N \\ &\approx S + \mu.N \ \ . \end{aligned}$$

(b) $S \xrightarrow{\tau} N' \overset{\mu}{\Longrightarrow} N$. As $N' \overset{\mu}{\Longrightarrow} N$, then there is a simple normal form S' that is a summand of N' and such that $S' \overset{\mu}{\Longrightarrow} N$. Since the number of the τ-moves in $N' \overset{\mu}{\Longrightarrow} N$ is strictly smaller than n, by induction we get that $S' \approx S' + \mu.N$ and, thus, that $N' \approx N' + \mu.N$. Then we have

$$\begin{aligned}\mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}} \vdash S &\approx \tau.N' \\ &\overset{\text{(T2)}}{\approx} \tau.N' + N' \\ &\approx \tau.N' + N' + \mu.N \\ &\overset{\text{(T2)}}{\approx} \tau.N' + \mu.N \\ &\approx S + \mu.N \enspace .\end{aligned}$$

This completes the proof of the inductive step in this case, and we are done.

2. Assume that $S \overset{x}{\Longrightarrow} c$.
Base case $n = 0$. In this case, we have that $S \xrightarrow{x} c$ for some configuration c. By Definition 6, this is only possible if $S = x \mathbin{\lfloor\!\lfloor} N$ for some normal form N. Hence, from the rules in Table 7, we infer that $c = x_{\mathrm{d}} \parallel N$. It is then immediate to check that

$$\mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}} \vdash S \approx S + S \approx S + x \mathbin{\lfloor\!\lfloor} N \enspace .$$

Inductive step $n > 0$. We distinguish two cases:

(a) $S \xrightarrow{x} c' \xrightarrow{\varepsilon} c$. From $S \xrightarrow{x} c'$ we can reason as in the base case, and infer that $S = x \mathbin{\lfloor\!\lfloor} N'$ and $c' = x_{\mathrm{d}} \parallel N'$. As $x_{\mathrm{d}} \overset{\tau}{\not\longrightarrow}$ and $n > 0$, $c' \overset{\tau}{\Longrightarrow} c$ is due to a simple normal form S' that is a summand of N' and such that $S' \overset{\tau}{\Longrightarrow} N$, for some normal form N, and $c = x_{\mathrm{d}} \parallel N$. By induction, we get that $\mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}} \vdash S' \approx S' + \tau.N$ and, consequently, $N' \approx N' + \tau.N$. Then we have

$$\begin{aligned}\mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}} \vdash S &\approx x \mathbin{\lfloor\!\lfloor} N' \\ &\approx x \mathbin{\lfloor\!\lfloor} (N' + \tau.N) \\ &\overset{\text{(TL2)}}{\approx} x \mathbin{\lfloor\!\lfloor} (N' + \tau.N) + x \mathbin{\lfloor\!\lfloor} N \\ &\approx x \mathbin{\lfloor\!\lfloor} N' + x \mathbin{\lfloor\!\lfloor} N \\ &\approx S + x \mathbin{\lfloor\!\lfloor} N \enspace .\end{aligned}$$

(b) $S \xrightarrow{\tau} N' \overset{x}{\Longrightarrow} c$. As $N' \overset{x}{\Longrightarrow} c$ and $x \overset{\tau}{\not\longrightarrow}, x_{\mathrm{d}} \overset{\tau}{\not\longrightarrow}$, then there is a simple normal form S' that is a summand of N' and such that $S' \overset{x}{\Longrightarrow} c$. Since the number of the τ-moves in $S' \overset{x}{\Longrightarrow} c$ is strictly smaller than n, by induction we get that $c = x_{\mathrm{d}} \parallel N$, $S' \approx S' + x \mathbin{\lfloor\!\lfloor} N$ and, thus, that $N' \approx N' + x \mathbin{\lfloor\!\lfloor} N$. Then we have

$$\begin{aligned}\mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}} \vdash S &\approx \tau.N' \\ &\overset{\text{(T2)}}{\approx} \tau.N' + N' \\ &\approx \tau.N' + N' + x \mathbin{\lfloor\!\lfloor} N \\ &\overset{\text{(T2)}}{\approx} \tau.N' + x \mathbin{\lfloor\!\lfloor} N \\ &\approx S + x \mathbin{\lfloor\!\lfloor} N \enspace . \quad \square\end{aligned}$$

Remark 4. *As briefly outlined at the beginning of the proof of Proposition 6, cases* (1a) *and* (2a) *in the proof allow us to show that S is η-saturated, since they allow us to "eliminate" the τ-moves that follow the transition from S. Specifically, case* (1a) *uses axiom T3, which is included in $\mathcal{E}^{\mathtt{L}}_{\mathtt{R\eta B}}$ (and $\mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}}$) but not in $\mathcal{E}^{\mathtt{L}}_{\mathtt{RDB}}$. Conversely, cases* (1b) *and* (2b) *subsume the case that S is* d*-saturated, since they allow us to "eliminate" the τ-moves that precede, respectively, the μ-transition and the x-transition from S. Both cases rely on the use of axiom T2, which is included in $\mathcal{E}^{\mathtt{L}}_{\mathtt{RDB}}$ (and $\mathcal{E}^{\mathtt{L}}_{\mathtt{RWB}}$) but not in $\mathcal{E}^{\mathtt{L}}_{\mathtt{R\eta B}}$.*

The following proposition is the crux in the derivation of the completeness result: given two $\mathtt{x}$-saturated normal forms N, M and the distinguishing substitution $*$ associated to them, we have that whenever $*(N)$ and $*(M)$ are $\mathtt{x}$ bisimilar, then they are also branching bisimilar.

Proposition 7. *Let $\mathtt{x} \in \{\eta, \mathtt{d}, \mathtt{w}\}$, N, M be two $\mathtt{x}$-saturated normal forms, and let $*$ be a distinguishing substitution defined over them. Then $*(N) \sim_{\mathtt{XB}} *(M)$ implies $*(N) \sim_{\mathtt{BB}} *(M)$.*

Proof. We expand only the case of $\mathtt{x} = \mathtt{w}$. The cases $\mathtt{x} = \eta, \mathtt{d}$ can be treated in a similar fashion.

The proof proceeds by induction on $\mathrm{size}(N)+\mathrm{size}(M)$. Assume $*(N) \sim_{\texttt{WB}} *(M)$ and $*(N) \xrightarrow{\mu} p$ for some μ and p. We shall prove that $*(M)$ can match that transition modulo $\sim_{\texttt{BB}}$. Since $*(N) \sim_{\texttt{WB}} *(M)$, we can distinguish two cases:

1. $\mu = \tau$ and $p \sim_{\texttt{WB}} *(M)$. Since $*(x) \overset{\tau}{\not\longrightarrow}$ for all variables x, Lemma 13 yields that $N \xrightarrow{\tau} N'$ and $p = *(N')$ for some normal form N'. By Lemma 14, N' is $\texttt{w}$-saturated. Thus, since $\mathrm{size}(N') < \mathrm{size}(N)$, the inductive hypothesis yields that $*(N') \sim_{\texttt{BB}} *(M)$, and we are done in this case.
2. There are processes q_1, q_2, q such that $*(M) \xrightarrow{\varepsilon} q_1 \xrightarrow{\mu} q_2 \xrightarrow{\varepsilon} q$ and $p \sim_{\texttt{WB}} q$. We first consider the case that $\mu = a$, and then deal with the case $\mu \neq a$.
   (a) Assume $\mu = a$. Since $*(N) \xrightarrow{a} p$, we can distinguish two cases according to Lemma 13 and the form of $*$:
      i. $N \xrightarrow{a} N'$ and $p = *(N')$ for a normal form N'. By using Lemma 13 repeatedly, we have that $M \xrightarrow{\varepsilon} M_1$ for some normal form M_1 with $q_1 = *(M_1)$. Then, either $M_1 \xrightarrow{a} M_2$ for some normal form M_2 such that $q_2 = *(M_2)$, or $M_1 \xrightarrow{y} y_{\mathrm{d}} \parallel M_2$ for some variable y and normal form M_2 such that $q_2 = \varphi_{\lceil y \rceil} \parallel *(M_2)$.

      In the former case, by Lemma 13 we obtain that $M_2 \xrightarrow{\varepsilon} M'$ for some normal form M' such that $q = *(M')$. Since M' is $\texttt{w}$-saturated, we obtain that $a.M'$ is a summand of M, and $*(M) \xrightarrow{a} *(M') = q$. Moreover, by Lemma 14, N' and M' are $\texttt{w}$-saturated. Thus, the inductive hypothesis applied to $p = *(N') \sim_{\texttt{WB}} *(M') = q$ yields that $*(N') \sim_{\texttt{BB}} *(M')$, and we are done in this case.

      We now argue that the latter case $*(M) \xrightarrow{\varepsilon} *(M_1) \xrightarrow{a} \varphi_{\lceil y \rceil} \parallel *(M_2) = q_2$ leads to a contradiction. Since $q_2 \xrightarrow{\varepsilon} q$ and $\varphi_{\lceil y \rceil} \overset{\tau}{\not\longrightarrow}$, we have that $*(M_2) \xrightarrow{\varepsilon} q'$ for some q' such that $q = \varphi_{\lceil y \rceil} \parallel q'$. By Lemma 13, $M_2 \xrightarrow{\varepsilon} M'$ for some normal form M' such that $q' = *(M')$. So $p = *(N') \sim_{\texttt{WB}} \varphi_{\lceil y \rceil} \parallel *(M') = q$. By the unique decomposition result for $\sim_{\texttt{WB}}$ and the fact that $\varphi_{\lceil y \rceil}$ is indecomposable modulo $\sim_{\texttt{WB}}$ (Remark 2), we have that $\varphi_{\lceil y \rceil}$ is a parallel component of the unique decomposition modulo $\sim_{\texttt{WB}}$ of $*(N')$. As $\varphi_{\lceil y \rceil}$ is also indecomposable modulo $\sim_{\texttt{BB}}$ (Lemma 7.1), we can conclude that it is also a parallel component of the unique decomposition modulo $\sim_{\texttt{BB}}$ of $*(N')$. Hence, by Lemma 6 we obtain that

      $$\mathrm{bd}_{\texttt{BB}}(*(N')) \geq \lceil y \rceil . \tag{10}$$

      Now,

      $$\begin{aligned} \lceil y \rceil > W \geq w(N) & \quad \text{by the construction of } * \\ \geq w(a.N') & \quad \text{by definition of width} \\ \geq w(N') & \quad \text{by definition of width} \\ \geq \mathrm{bd}_{\texttt{BB}}(*(N')) & \quad \text{by Lemma 8} \\ \geq \lceil y \rceil & \quad \text{by Equation (10)} \end{aligned}$$

      which is a contradiction.
      This completes the analysis of the case in which $N \xrightarrow{a} N'$.
      ii. $N \xrightarrow{x} x_{\mathrm{d}} \parallel N'$ for some x, N' such that $p = \varphi_{\lceil x \rceil} \parallel *(N')$. Then $*(M) \xrightarrow{\varepsilon} q_1 \xrightarrow{a} q_2 \xrightarrow{\varepsilon} q$ for some q_1, q_2, q with $p = \varphi_{\lceil x \rceil} \parallel *(N') \sim_{\texttt{WB}} q$. By applying a symmetric reasoning to the one used in the previous case, we obtain that it cannot be the case that $M \overset{a}{\Longrightarrow} M'$ for some normal form M' with $*(M') = q$. Hence, we focus on the case in which $M \overset{y}{\Longrightarrow} y_{\mathrm{d}} \parallel M'$ for some y, M' such that $q = \varphi_{\lceil y \rceil} \parallel *(M')$. Since M is $\texttt{w}$-saturated, we have that $y \mathbin{\lfloor\!\lfloor} M'$ is a summand of M and M' is also $\texttt{w}$-saturated. Similarly, $x \mathbin{\lfloor\!\lfloor} N'$ is a summand of N and N' is $\texttt{w}$-saturated. Since $p = \varphi_{\lceil x \rceil} \parallel *(N') \sim_{\texttt{WB}} \varphi_{\lceil y \rceil} \parallel *(M') = q$, and both $\varphi_{\lceil x \rceil}$ and $\varphi_{\lceil y \rceil}$ are indecomposable modulo $\sim_{\texttt{WB}}$ (Remark 2), we can distinguish two cases:
         A. $\varphi_{\lceil x \rceil} \sim_{\texttt{WB}} \varphi_{\lceil y \rceil}$. By compositionality of $\sim_{\texttt{BB}}, \sim_{\texttt{WB}}$ with respect to parallel composition, and Corollary 2 for $\sim_{\texttt{WB}}$, we obtain $*(N') \sim_{\texttt{WB}} *(M')$. This means that, by induction, $*(N') \sim_{\texttt{BB}} *(M')$. Moreover, by construction of $\lceil \cdot \rceil$, we also obtain $x = y$, so that $\varphi_{\lceil x \rceil} \sim_{\texttt{BB}} \varphi_{\lceil y \rceil}$. Hence, $p \sim_{\texttt{BB}} q$. Since $x \mathbin{\lfloor\!\lfloor} M'$ is a summand of M, we have that $*(M) \xrightarrow{a} q$ and we are done in this case.
         B. $\varphi_{\lceil y \rceil}$ is a component of the unique parallel decomposition of $*(N')$ with respect to $\sim_{\texttt{WB}}$. Since $\varphi_{\lceil y \rceil}$ is also indecomposable modulo $\sim_{\texttt{BB}}$ (Lemma 7.1), we can conclude that $\varphi_{\lceil y \rceil}$ is also a component of the unique parallel decomposition of $*(N')$ with respect to $\sim_{\texttt{BB}}$. Hence, Lemma 6 yields:

         $$\mathrm{bd}_{\texttt{BB}}(*(N')) \geq \lceil y \rceil . \tag{11}$$

         But then,

         $$\begin{aligned} \lceil y \rceil > W \geq w(N) & \quad \text{by the construction of } * \\ \geq w(x \mathbin{\lfloor\!\lfloor} N') & \quad \text{by definition of width} \\ \geq w(N') & \quad \text{by definition of width} \end{aligned}$$

Table 8
Equational basis modulo rooted branching bisimilarity over $\mathrm{CCS}_{\mathrm{LC}}$.

Axiom system $\mathcal{E}_{\mathrm{RBB}}$: $\mathcal{E}_{\mathrm{B}} \cup \{\mathrm{TB}, \mathrm{TL1}\}$	
Additional equation derivable from $\mathcal{E}_{\mathrm{RBB}}$	
DT3	$\tau x \mid y \approx \mathbf{0}$

$$\geq \mathrm{bd}_{\mathrm{BB}}(*(N')) \qquad \text{by Lemma 8}$$

$$\geq \lceil y \rceil \qquad \text{by Equation (11)}$$

which is a contraction.

This completes the analysis of the case in which $N \xrightarrow{x} \mathrm{x}_{\mathrm{d}} \parallel N'$.

(b) The case of $\mu \neq a$ is easier, and follows from the same reasoning used in the case of $N \xrightarrow{a} N'$.

Summarising, we have shown that whenever $*(N) \xrightarrow{\mu} p$, then either

- $\mu = \tau$ and $p \sim_{\mathrm{BB}} *(M)$, or
- there is a process q such that $*(M) \xrightarrow{\mu} q$ and $p \sim_{\mathrm{BB}} q$.

By symmetry of $\sim_{\mathrm{WB}}$, this completes the proof. ☐

In light of Proposition 7, we can extend the property to the rooted versions of the considered equivalences.

Proposition 8. *Let $\mathrm{x} \in \{\eta, \mathrm{d}, \mathrm{w}\}$, N, M be two* x*-saturated normal forms, and let $*$ be a distinguishing substitution defined over them. Then $*(N) \sim_{\mathrm{RXB}} *(M)$ implies $*(N) \sim_{\mathrm{RBB}} *(M)$.*

Proof. We prove the statement only for $\mathrm{x} = \mathrm{w}$, as the proofs for the other two cases are similar.

Assume $*(N) \sim_{\mathrm{RWB}} *(M)$. Then, whenever $*(N) \xrightarrow{\mu} p$, we have that $*(M) \overset{\mu}{\Longrightarrow} q$ for some $q \sim_{\mathrm{WB}} p$. We can then proceed as in the proof of Proposition 7 to show that since M is w-saturated we get that $*(M) \xrightarrow{\mu} q$. Moreover, according to the form of the summand of N that generated the μ-transition, we get that:

- If $p = *(N')$ for a w-saturated normal form N', then $q = *(M')$ for a w-saturated normal form M' such that $*(N') \sim_{\mathrm{WB}} *(M')$. Hence, by Proposition 7, $p \sim_{\mathrm{BB}} q$ in this case.
- If $p = \varphi_{\lceil x \rceil} \parallel *(N')$ for some variable x and w-saturated normal form N', then $q = \varphi_{\lceil x \rceil} \parallel *(M')$ for a w-saturated normal form M' such that $*(M') \sim_{\mathrm{WB}} *(N')$. Hence, Proposition 7 gives $*(N') \sim_{\mathrm{BB}} *(M')$, from which we can infer that $p \sim_{\mathrm{BB}} q$ holds also in this case.

We have therefore obtained that whenever $*(N) \xrightarrow{\mu} p$, then there is a process q such that $*(M) \xrightarrow{\mu} q$ and $p \sim_{\mathrm{BB}} q$. By symmetry, the same property can be inferred for all transition from $*(M)$, thus giving that $*(N) \sim_{\mathrm{RBB}} *(M)$. ☐

5.3. The completeness results

The completeness of $\mathcal{E}^{\mathrm{L}}_{\mathrm{RXB}}$ for $\sim_{\mathrm{RXB}}$ over $\mathrm{CCS}_{\mathrm{L}}$ is then a direct consequence of the completeness of $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}}$ for $\sim_{\mathrm{RBB}}$ over $\mathrm{CCS}_{\mathrm{L}}$, and the other results presented in this section. Notice that if $\mathcal{A}_\tau$ is finite, then the axiom schemata T1 and T2 generate finitely many axioms, and thus $\mathcal{E}^{\mathrm{L}}_{\mathrm{RXB}}$ is finite in this case.

Theorem 4. *Let $\mathrm{x} \in \{\eta, \mathrm{d}, \mathrm{w}\}$. Let t, u be $\mathrm{CCS}_{\mathrm{L}}$ terms. If $t \sim_{\mathrm{RXB}} u$, then $\mathcal{E}^{\mathrm{L}}_{\mathrm{RXB}} \vdash t \approx u$.*

Proof. Assume that $t \sim_{\mathrm{RXB}} u$. By Corollary 3 and Proposition 6, we have that $\mathcal{E}^{\mathrm{L}}_{\mathrm{RXB}} \vdash t \approx N$ and $\mathcal{E}^{\mathrm{L}}_{\mathrm{RXB}} \vdash u \approx M$ for some x-saturated normal forms N, M. By the soundness of $\mathcal{E}_{\mathrm{RXB}}$, these imply that $N \sim_{\mathrm{RXB}} M$, which, in particular, gives $*(N) \sim_{\mathrm{RXB}} *(M)$, for the distinguishing substitution $*$ defined over them. Hence, by Proposition 8, it holds that $*(N) \sim_{\mathrm{RBB}} *(M)$. From the proof of the completeness of $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}}$ (Theorem 2), we can then infer that $\mathcal{E}^{\mathrm{L}}_{\mathrm{RBB}} \vdash N \approx M$. By Lemma 10, we get that $\mathcal{E}^{\mathrm{L}}_{\mathrm{RXB}} \vdash N \approx M$, and we can therefore conclude that $\mathcal{E}^{\mathrm{L}}_{\mathrm{RXB}} \vdash t \approx u$. ☐

6. Completeness for rooted branching bisimilarity over $\mathrm{CCS}_{\mathrm{LC}}$

In this section, we apply the strategy discussed in Section 4.1 to show that the axiom system $\mathcal{E}_{\mathrm{RBB}} = \mathcal{E}_{\mathrm{B}} \cup \{\mathrm{TB}, \mathrm{TL1}\}$ presented in Table 8 (with axioms TB and TL1 from Table 5) is a *complete axiomatisation of rooted branching equivalence* over $\mathrm{CCS}_{\mathrm{LC}}$. However, we have not been able to obtain the result in its full generality, but only under the assumption that the set $\mathcal{A}$ of action names is infinite. As we will see, this limitation is due to the complexity induced by communication between parallel components, and it will require a rather involved technical development of our proofs, mainly based on the novel notion of *potential of a variable in a term* (Definition 12 below).

Further considerations on this limitation, and a discussion on the reasons why we were not able to obtain the completeness result in the case with finitely many actions can be found in Section 7.

6.1. Soundness and normal forms over CCS_{LC}

The soundness of $\mathcal{E}_{\text{RBB}}$ modulo $\sim_{\text{RBB}}$ is immediate.

Theorem 5. *The axiom system $\mathcal{E}_{\text{RBB}}$ is sound modulo $\sim_{\text{RBB}}$ over CCS_{LC}.*

We then proceed to study normal forms over CCS_{LC}.

Definition 11 *(Normal form over CCS_{LC}). The set of normal forms over CCS_{LC} is generated by the following grammar:*

$$S ::= \mu.N \quad | \quad x \mathbin{\|\!\lfloor} N \quad | \quad (x \mid \alpha) \mathbin{\|\!\lfloor} N \quad | \quad (x \mid y) \mathbin{\|\!\lfloor} N$$

$$N ::= \mathbf{0} \quad | \quad S \quad | \quad N + N,$$

where $x, y \in \mathcal{V}$, $\mu \in \mathcal{A}_\tau$ and $\alpha \in \mathcal{A} \cup \overline{\mathcal{A}}$.

Proposition 9. *For every term t there is a normal form N such that $\mathcal{E}_{\text{RBB}} \vdash t \approx N$.*

Proof. The proof can be carried out by induction on the size of terms exactly like in the proof of Lemma 4.4 in [9], and therefore we omit it. ☐

6.2. Distinguishing substitutions and potential

According to the strategy in Section 4.1, we now need to define a distinguishing substitution $\triangle$ for normal forms over CCS_{LC}. To take into account the additional complexity in the structure of normal forms given by the presence of communication, the definition of $\triangle$ will depend on a finite subset $\mathcal{A}' \subset \mathcal{A} \cup \overline{\mathcal{A}}$ that is closed under $\bar{\cdot}$. Specifically, following [9], given a set of actions $\mathcal{A}' = \{\alpha_1, \dots, \alpha_n\}$, we define the distinguishing substitution $\triangle$ as follows:

$$\triangle(x) = \sum_{\ell=1}^{n} \alpha_\ell . a_{x,\ell} + \sum_{\ell=1}^{n} \alpha_\ell . b_{x,\ell}$$

where $a_{x,\ell}, b_{x,\ell} \notin \mathcal{A}' \cup \{\tau\}$ are two distinct actions in $\mathcal{A}$ associated uniquely with every pair x, ℓ. Note that these actions exist because $\mathcal{A}$ is infinite. Moreover, we use these special actions to introduce the notion of *potential* of a variable x in a process p. Roughly, for each pair x, ℓ, the potential gives the maximum number of weak transitions labelled with $a_{x,\ell}$ (respectively, $b_{x,\ell}$) that process p can perform in a row.

Definition 12 *(Potential). Let $x \in \mathcal{V}$, $\ell \in \{1, \dots, n\}$ and p be a CCS_{LC} process. The* a, ℓ-potential of x in p, *denoted $\pi_p^{a,\ell}(x)$, is the maximum $k \geq 0$ such that $p(\xrightarrow{\varepsilon}\xrightarrow{a_{x,\ell}})^k$. Similarly, the* b, ℓ-potential of x in p, *denoted $\pi_p^{b,\ell}(x)$, is the maximum $k \geq 0$ such that $p(\xrightarrow{\varepsilon}\xrightarrow{b_{x,\ell}})^k$.*

Before diving into the technical motivation behind this definition, we present two general properties of potential.

Lemma 15. *Let $c \in \{a, b\}$, $x \in \mathcal{V}$, and $\ell \in \{1, \dots, n\}$. If $p \xrightarrow{\varepsilon}\xrightarrow{c_{x,\ell}} p'$ then $\pi_{p'}^{c,\ell}(x) < \pi_p^{c,\ell}(x)$. Moreover, there is a process p' such that $p \xrightarrow{\varepsilon}\xrightarrow{c_{x,\ell}} p'$ and $\pi_{p'}^{c,\ell}(x) = \pi_p^{c,\ell}(x) - 1$.*

Proposition 10. *Let p and q be two CCS_{LC} processes such that $p \sim_{\text{BB}} q$. Then $\pi_p^{a,\ell}(x) = \pi_q^{a,\ell}(x)$ and $\pi_p^{b,\ell}(x) = \pi_q^{b,\ell}(x)$ for all x, ℓ.*

Proof. Follows from the fact that branching bisimilar processes have the same weak traces [23]. ☐

We are interested in the potential of variables in closed instances of normal forms. Specifically, given a normal form N and the distinguishing substitution $\triangle$ associated with it, the potential of x in $\triangle(N)$ will allow us to determine whether a τ-transition from $\triangle(N)$ is due to a summand $\tau. \triangle(N')$, or to summands of the forms $\triangle((x \mid \alpha) \mathbin{\|\!\lfloor} N')$ and $\triangle((x \mid y) \mathbin{\|\!\lfloor} N')$, as in the latter case the transition is certainly state changing modulo $\sim_{\text{BB}}$. To see why, we first notice that, for all pairs x, ℓ, the a, ℓ-potential of x in $\triangle(N)$ coincides with its b, ℓ-potential.

Lemma 16. *Let N be a CCS$_{LC}$ normal form such that the actions occurring in it are included in $\mathcal{A}' \cup \{\tau\}$, with $\mathcal{A}' = \{\alpha_1, \dots, \alpha_n\}$. Then $\pi^{a,\ell}_{\triangle(N)}(x) = \pi^{b,\ell}_{\triangle(N)}(x)$ for all $x \in \mathcal{V}, \ell \in \{1, \dots, n\}$.*

Proof. The lemma follows directly from the definition of the distinguishing substitution $\triangle$, the fact that $a_{x,\ell}, b_{x,\ell}$ are distinct and unique for each pair x, ℓ, and the fact that they cannot synchronise with any action occurring in $\triangle(N)$, nor between themselves. $\square$

Given the definition of $\triangle$, whenever $\triangle(x) \xrightarrow{\alpha_\ell} q$ for some $\ell \in \{1, \dots, n\}$, we have that either $q = a_{x,\ell}$ and $\pi^{b,\ell}_q(x) = 0$, or $q = b_{x,\ell}$ and $\pi^{a,\ell}_q(x) = 0$. This implies that if the α_ℓ-transition is performed in the context of a communication step, then the target process can potentially perform fewer $b_{x,\ell}$- (respectively $a_{x,\ell}$-) labelled weak transitions in a row. Since potential is preserved under branching bisimilarity, the τ-transition induced by the communication cannot be state-preserving modulo $\sim_{\mathrm{BB}}$. We can therefore conclude that if $\triangle(N) \xrightarrow{\tau} p$ for some p such that $p \sim_{\mathrm{BB}} \triangle(N)$ then the τ-transition is not derived from a communication involving a variable.

Corollary 4. *Let N be a CCS$_{LC}$ normal form such that the actions occurring in it are included in $\mathcal{A}' \cup \{\tau\}$, with $\mathcal{A}' = \{\alpha_1, \dots, \alpha_n\}$. If $\triangle(N) \xrightarrow{\tau} p$ and $p \sim_{\mathrm{BB}} \triangle(N)$, then $p = \triangle(N')$ for a summand $\tau.N'$ of N.*

Proof. Assume, towards a contradiction, that there is summand $S = (x \mid \alpha) \mathbin{\|\!\!\lfloor} N'$ of N such that $\triangle(S) \xrightarrow{\tau} p$ and $p \sim_{\mathrm{BB}} \triangle(N)$. The proof for a summand $S = (x \mid y) \mathbin{\|\!\!\lfloor} N''$ is similar, and therefore omitted. Moreover, we can assume, without loss of generality, that $p = a_{x,\ell} \parallel \triangle(N')$, for the unique ℓ such that $\alpha_\ell = \overline{\alpha}$, as the proof for the case $p = b_{x,\ell} \parallel \triangle(N')$ follows by applying a symmetrical argument.

As $p = a_{x,\ell} \parallel \triangle(N')$ we have that $\pi^{b,\ell}_p(x) = \pi^{b,\ell}_{\triangle(N')}(x)$. However, since due to summand S we get $\triangle(N) \xrightarrow{\tau} b_{x,\ell} \parallel \triangle(N')$, we can immediately conclude that $\pi^{b,\ell}_{\triangle(N)}(x) \geq \pi^{b,\ell}_{\triangle(N')}(x) + 1 > \pi^{b,\ell}_p(x)$. By Proposition 10, this contradicts $\triangle(N) \sim_{\mathrm{BB}} p$. $\square$

The properties and results discussed above allow us to identify the syntactic structure of a normal form N from the behaviour of $\triangle(N)$, i.e., we can prove that $\triangle$ is indeed distinguishing over CCS$_{\mathrm{LC}}$.

Proposition 11. *Let N and M be CCS$_{LC}$ normal forms such that the actions occurring in them are included in $\mathcal{A}' \cup \{\tau\}$. If $\triangle(N) \sim_{\mathrm{BB}} \triangle(M)$ then $\mathcal{E}_{\mathrm{RBB}} \vdash \mu.N \approx \mu.M$, for every $\mu \in \mathcal{A}_\tau$.*

Proof. Suppose that $\triangle(N) \sim_{\mathrm{BB}} \triangle(M)$. To prove that $\mathcal{E}_{\mathrm{RBB}} \vdash \mu.N \approx \mu.M$ for every $\mu \in \mathcal{A}_\tau$, we proceed by induction on $\mathrm{size}(N) + \mathrm{size}(M)$.

Let I, J, H, K, $\bar{I}$, $\bar{J}$, $\bar{H}$, $\bar{K}$ be disjoint sets, $\mu_i, \nu_{\bar{i}} \in \mathcal{A}' \cup \{\tau\}$, $x_j, x_h, x_k, x'_k, y_{\bar{j}}, y_{\bar{h}}, y_{\bar{k}}, y'_{\bar{k}} \in \mathcal{V}$ and N_i, N_j, N_h, N_k, $M_{\bar{i}}$, $M_{\bar{j}}$, $M_{\bar{h}}$, $M_{\bar{k}}$ be normal forms for all $i \in I$, $j \in J$, $h \in H$ $k \in K$, $\bar{i} \in \bar{I}$, $\bar{j} \in \bar{J}$, $\bar{h} \in \bar{H}$ $\bar{k} \in \bar{K}$, such that

$$N \approx \sum_{i \in I} \mu_i.N_i + \sum_{j \in J} x_j \mathbin{\|\!\!\lfloor} N_j + \sum_{h \in H} (x_h \mid \alpha_h) \mathbin{\|\!\!\lfloor} N_h + \sum_{k \in K} (x_k \mid x'_k) \mathbin{\|\!\!\lfloor} N_k$$

$$M \approx \sum_{\bar{i} \in \bar{I}} \nu_{\bar{i}}.M_{\bar{i}} + \sum_{\bar{j} \in \bar{J}} y_{\bar{j}} \mathbin{\|\!\!\lfloor} M_{\bar{j}} + \sum_{\bar{h} \in \bar{H}} (y_{\bar{h}} \mid \beta_{\bar{h}}) \mathbin{\|\!\!\lfloor} M_{\bar{h}} + \sum_{\bar{k} \in \bar{K}} (y_{\bar{k}} \mid y'_{\bar{k}}) \mathbin{\|\!\!\lfloor} M_{\bar{k}}.$$

From $\triangle(N) \sim_{\mathrm{BB}} \triangle(M)$, we have that whenever $\triangle(N) \xrightarrow{\mu} p$, then either $\mu = \tau$ and $p \sim_{\mathrm{BB}} \triangle(M)$, or $\triangle(M) \xrightarrow{\varepsilon} q' \xrightarrow{\mu} q$ with $q' \sim_{\mathrm{BB}} \triangle(N)$ and $q \sim_{\mathrm{BB}} p$. For the latter case we remark that, by Corollary 4 and the definition of $\triangle$, none of the (possible) τ-moves in the sequence $\triangle(M) \xrightarrow{\varepsilon} q'$ can be due to variables. This means that all (possible) τ-moves in that sequence stem from τ-prefixes. Hence, we can distinguish the following three cases:

1. There is no $i \in I$ such that $\mu_i = \tau$ and $\triangle(N_i) \sim_{\mathrm{BB}} \triangle(M)$, and there is no $\bar{i} \in \bar{I}$ such that $\nu_{\bar{i}} = \tau$ and $\triangle(N) \sim_{\mathrm{BB}} \triangle(M_{\bar{i}})$.
   Given the considerations above, in this case, as $\triangle(N) \sim_{\mathrm{BB}} \triangle(M)$, each transition $\triangle(N) \xrightarrow{\mu} p$ must be matched by a transition $\triangle(M) \xrightarrow{\mu} q$ for some q such that $p \sim_{\mathrm{BB}} q$. We can now distinguish four cases, according to the form of the syntactic summand S of N inducing the transition. In each case, we will prove that each summand of N is provably equal to some summand of M, and vice versa.
   (a) $S = \mu_i.N_i$ for some $i \in I$. Then $\triangle(N) \xrightarrow{\mu_i} \triangle(N_i)$, giving that there is a syntactic summand S' of M such that $\triangle(S') \xrightarrow{\mu_i} q$ for some q such that $q \sim_{\mathrm{BB}} \triangle(N_i)$. By Proposition 10, $\pi^{c,\ell}_{\triangle(N_i)}(x) = \pi^{c,\ell}_q(x)$ for $c = a, b$, for all $\ell \in \{1, \dots, n\}, x \in \mathcal{V}$, and since $\triangle(S) \xrightarrow{\mu_i} \triangle(N_i)$ is not due to a variable, we can apply the same reasoning used in the proof of Corollary 4 to deduce that also $\triangle(S') \xrightarrow{\mu_i} q$ cannot be due to a variable. Hence, $S' = \nu_{\bar{i}}.M_{\bar{i}}$ for some $\bar{i} \in \bar{I}$ such that $\nu_{\bar{i}} = \mu_i$ and $\triangle(N_i) \sim_{\mathrm{BB}} \triangle(M_{\bar{i}})$. Since the above reasoning does not depend on the choice of the particular index $i \in I$, we can deduce that for each $i \in I$ there is a $\bar{i} \in \bar{I}$ such that $\mu_i = \nu_{\bar{i}}$ and $\triangle(N_i) \sim_{\mathrm{BB}} \triangle(M_{\bar{i}})$. Symmetrically, for each summand $\nu_{\bar{i}}.M_{\bar{i}}$ there is a summand $\mu_i.N_i$ such that $\mu_i = \nu_{\bar{i}}$ and $\triangle(N_i) \sim_{\mathrm{BB}} \triangle(M_{\bar{i}})$. As, for each such pair of indexes, the sum of the sizes of N_i and $M_{\bar{i}}$ is strictly smaller than $\mathrm{size}(N) + \mathrm{size}(M)$, by induction we get that

$$\mathcal{E}_{\mathrm{RBB}} \vdash \mu_i.N_i \approx \mu_i.M_{\bar{i}} \overset{(\mu_i = \nu_{\bar{i}})}{\approx} \nu_{\bar{i}}.M_{\bar{i}}. \tag{12}$$

(b) $S = x_j \mathbin{\|\!\lfloor} N_j$ for some $j \in J$. Then $\triangle(S) \xrightarrow{\alpha_\ell} a_{x_j,\ell} \parallel \triangle(N_j)$ for any $\ell \in \{1,\dots,n\}$. This transition is then matched by a syntactic summand S' of M such that $\triangle(S') \xrightarrow{\alpha_\ell} q$ for some q such that $q \sim_{\mathrm{BB}} a_{x_j,\ell} \parallel \triangle(N_j)$. Firstly, we notice that, since $\alpha_\ell \neq \tau$, we have that S' cannot be of the form $S' = (y_{\bar{h}} \mid \beta_{\bar{h}}) \mathbin{\|\!\lfloor} M_{\bar{h}}$ for some $\bar{h} \in \bar{H}$, or of the form $S' = (y_{\bar{k}} \mid y'_{\bar{k}}) \mathbin{\|\!\lfloor} M_{\bar{k}}$ for some $\bar{k} \in \bar{K}$. Since, moreover, $\pi_q^{b,\ell}(x_j) = \pi_{\triangle(N_j)}^{b,\ell}(x_j)$ and $\pi_q^{a,\ell}(x_j) = \pi_{\triangle(N_j)}^{a,\ell}(x_j) + 1 = \pi_{\triangle(N_j)}^{b,\ell}(x_j) + 1 > \pi_q^{b,\ell}(x_j)$, from Lemma 16 we infer that $S' = \alpha_\ell.M'$, for some normal form M', gives a contradiction with $\triangle(M') = q \sim_{\mathrm{BB}} a_{x_j,\ell} \parallel \triangle(N_j)$. Hence, $S' = y_{\bar{j}} \mathbin{\|\!\lfloor} M_{\bar{j}}$ for some $\bar{j} \in \bar{J}$, and $\triangle(S') \xrightarrow{\alpha_\ell} q$ with either $q = a_{y_{\bar{j}},\ell} \parallel \triangle(M_{\bar{j}})$ or $q = b_{y_{\bar{j}},\ell} \parallel \triangle(M_{\bar{j}})$. We proceed to argue that the latter possibility can be immediately discarded. In fact, we recall that $a_{x_j,\ell}, a_{y_{\bar{j}},\ell}, b_{y_{\bar{j}},\ell}$ are all distinct. Then, by Lemma 16,

$$\pi_{\triangle(M_{\bar{j}})}^{b,\ell}(y_{\bar{j}}) = \pi_{\triangle(M_{\bar{j}})}^{a,\ell}(y_{\bar{j}})$$

and

$$\pi_{\triangle(N_j)}^{b,\ell}(y_{\bar{j}}) = \pi_{\triangle(N_j)}^{a,\ell}(y_{\bar{j}}),$$

giving thus

$$\pi_q^{b,\ell}(y_{\bar{j}}) = 1 + \pi_{\triangle(M_{\bar{j}})}^{b,\ell}(y_{\bar{j}}).$$

By Proposition 10, $q = b_{y_{\bar{j}},\ell} \parallel \triangle(M_{\bar{j}}) \sim_{\mathrm{BB}} a_{x_j,\ell} \parallel \triangle(N_j)$ implies

$$\pi_{b_{y_{\bar{j}},\ell} \parallel \triangle(M_{\bar{j}})}^{a,\ell}(y_{\bar{j}}) = \pi_{a_{x_j,\ell} \parallel \triangle(N_j)}^{a,\ell}(y_{\bar{j}}).$$

Since

$$\pi_{b_{y_{\bar{j}},\ell} \parallel \triangle(M_{\bar{j}})}^{a,\ell}(y_{\bar{j}}) = \pi_{\triangle(M_{\bar{j}})}^{a,\ell}(y_{\bar{j}})$$

and

$$\pi_{a_{x_j,\ell} \parallel \triangle(N_j)}^{a,\ell}(y_{\bar{j}}) = \pi_{\triangle(N_j)}^{a,\ell}(y_{\bar{j}}),$$

we get that

$$\pi_q^{b,\ell}(y_{\bar{j}}) > \pi_{\triangle(M_{\bar{j}})}^{a,\ell}(y_{\bar{j}}) = \pi_{\triangle(N_j)}^{b,\ell}(y_{\bar{j}}) = \pi_{a_{x_j,\ell} \parallel \triangle(N_j)}^{b,\ell}(y_{\bar{j}})$$

contradicting $q \sim_{\mathrm{BB}} a_{x_j,\ell} \parallel \triangle(N_j)$. Hence, $q = a_{y_{\bar{j}},\ell} \parallel \triangle(M_{\bar{j}})$ and we can use a similar argument as above to prove that if $y_{\bar{j}} \neq x_j$, then we get a contradiction with $q \sim_{\mathrm{BB}} a_{x_j,\ell} \parallel \triangle(N_j)$. Therefore, $q = a_{x_j,\ell} \parallel \triangle(M_{\bar{j}})$. Since $a_{x_j,\ell}$ is indecomposable, by Corollary 2 we have that $q \sim_{\mathrm{BB}} a_{x_j,\ell} \parallel \triangle(N_j)$ implies $\triangle(N_j) \sim_{\mathrm{BB}} \triangle(M_{\bar{j}})$. As the sum of the sizes of N_j and $M_{\bar{j}}$ is strictly smaller than $\mathrm{size}(N) + \mathrm{size}(M)$, by induction we obtain that

$$\mathcal{E}_{\mathrm{RBB}} \vdash \tau.N_j \approx \tau.M_{\bar{j}}. \tag{13}$$

Then, we have

$$\begin{aligned} x_j \mathbin{\|\!\lfloor} N_j &\overset{(\mathrm{TL1})}{\approx} x_j \mathbin{\|\!\lfloor} \tau.N_j \\ &\overset{(13)}{\approx} x_j \mathbin{\|\!\lfloor} \tau.M_{\bar{j}} \\ &\overset{(\mathrm{TL1})}{\approx} x_j \mathbin{\|\!\lfloor} M_h \\ &\overset{(x_j = y_{\bar{j}})}{\approx} y_{\bar{j}} \mathbin{\|\!\lfloor} M_{\bar{j}}. \end{aligned}$$

Summarising, by symmetry, we have obtained that for each $j \in J$ (respectively, $\bar{j} \in \bar{J}$) there is a $\bar{j} \in \bar{J}$ (respectively, $j \in J$) such that:

$$\mathcal{E}_{\mathrm{RBB}} \vdash x_j \mathbin{\|\!\lfloor} N_j \approx y_{\bar{j}} \mathbin{\|\!\lfloor} M_{\bar{j}}. \tag{14}$$

(c) $S = (x_h \mid \alpha_h) \mathbin{\|\!\lfloor} N_h$ for some $h \in H$. Then $\triangle(S) \xrightarrow{\tau} a_{x_h,\ell} \parallel \triangle(N_h)$ for the unique $\ell \in \{1,\dots,n\}$ such that $\alpha_\ell = \overline{\alpha_h}$, and there is a syntactic summand S' of M such that $\triangle(S') \xrightarrow{\tau} q$ with $q \sim_{\mathrm{BB}} a_{x_h,\ell} \parallel \triangle(N_h)$.c

Since $\alpha_\ell \neq \tau$ for all $\ell \in \{1,\dots,n\}$, we can immediately exclude that S' is of the form $y_{\bar{j}} \lfloor\!\lfloor M_{\bar{j}}$ for some $\bar{j} \in \bar{J}$. Since, moreover, $\pi_q^{b,\ell}(x_h) = \pi^{b,\ell}_{\triangle(N_h)}(x_h)$ and $\pi_q^{a,\ell}(x_h) = \pi^{a,\ell}_{\triangle(N_h)}(x_h) + 1 = \pi^{b,\ell}_{\triangle(N_h)}(x_h) + 1 > \pi_q^{b,\ell}(x_h)$, from Lemma 16 we infer that $S' = \tau.M'$, for some normal form M', would contradict $\triangle(M') = q \sim_{\mathrm{BB}} a_{x_h,\ell} \parallel \triangle(N_h)$. We now proceed to argue that $S' \neq (y_{\bar{k}} \mid y'_{\bar{k}}) \lfloor\!\lfloor M_{\bar{k}}$ for all $\bar{k} \in \bar{K}$. Assume, towards a contradiction, that S' is of that form for some $\bar{k} \in \bar{K}$, so that $\triangle(S') \xrightarrow{\tau} a_{y_{\bar{k}},\ell_1} \parallel a_{y'_{\bar{k}},\ell_2} \parallel \triangle(M_{\bar{k}})$ for some $\ell_1, \ell_2 \in \{1,\dots,n\}$ such that $\alpha_{\ell_1} = \overline{\alpha_{\ell_2}}$. Then $a_{y_{\bar{k}},\ell_1} \parallel a_{y'_{\bar{k}},\ell_2} \parallel \triangle(M_{\bar{k}}) \sim_{\mathrm{BB}} a_{x_h,\ell} \parallel \triangle(N_h)$. We can now proceed as in item 1b above and use the uniqueness of actions $a_{x_h,\ell}, a_{y_{\bar{k}},\ell_1}, a_{y'_{\bar{k}},\ell_2}$ and the potentials of the related variables in the two processes to obtain a contradiction.

Hence, $S' = (y_{\bar{h}} \mid \beta_{\bar{h}}) \lfloor\!\lfloor M_{\bar{h}}$ for some $\bar{h} \in \bar{H}$ such that $a_{x_h,\ell} \parallel \triangle(N_h) \sim_{\mathrm{BB}} c_{y_{\bar{h}},\ell'} \parallel \triangle(M_{\bar{h}})$, where $c \in \{a,b\}$ and ℓ' is the unique $\ell' \in \{1,\dots,n\}$ such that $\alpha_{\ell'} = \overline{\beta_{\bar{h}}}$. Also in this case, the same reasoning used in item 1b allows us to infer that $x_h = y_{\bar{h}}$ and $\ell = \ell'$. Consequently, $\alpha_h = \beta_{\bar{h}}$. Moreover, from $a_{x_h,\ell} \parallel \triangle(N_h) \sim_{\mathrm{BB}} a_{x_h,\ell} \parallel \triangle(M_{\bar{h}})$ and Corollary 2 we infer that $\triangle(N_h) \sim_{\mathrm{BB}} \triangle(M_{\bar{h}})$. Since the sum of the sizes of N_h and $M_{\bar{h}}$ is strictly smaller than $\mathrm{size}(N) + \mathrm{size}(M)$, by induction we get that

$$\mathcal{E}_{\mathrm{RBB}} \vdash \tau.N_h \approx \tau.M_{\bar{h}}. \tag{15}$$

Then, we have

$$\begin{aligned}(x_h \mid \alpha_h) \lfloor\!\lfloor N_h &\overset{\text{(TL1)}}{\approx} (x_h \mid \alpha_h) \lfloor\!\lfloor \tau.N_h \\ &\overset{(15)}{\approx} (x_h \mid \alpha_h) \lfloor\!\lfloor \tau.M_{\bar{h}} \\ &\overset{\text{(TL1)}}{\approx} (x_h \mid \alpha_h) \lfloor\!\lfloor M_{\bar{h}} \\ &\overset{(x_h = y_{\bar{h}},\, \alpha_h = \beta_{\bar{h}})}{\approx} (y_{\bar{h}} \mid \beta_{\bar{h}}) \lfloor\!\lfloor M_{\bar{h}}.\end{aligned}$$

Summarising, by symmetry, we have obtained that for each $h \in H$ (respectively, $\bar{h} \in \bar{H}$) there is a $\bar{h} \in \bar{H}$ (respectively, $h \in H$) such that:

$$\mathcal{E}_{\mathrm{RBB}} \vdash (x_h \mid \alpha_h) \lfloor\!\lfloor N_h \approx (y_{\bar{h}} \mid \beta_{\bar{h}}) \lfloor\!\lfloor M_{\bar{h}}. \tag{16}$$

(d) $S = (x_k \mid x'_k) \lfloor\!\lfloor N_k$ for some $k \in K$. We can proceed as in item 1c to prove that $S' = (y_{\bar{k}} \mid y'_{\bar{k}}) \lfloor\!\lfloor M_{\bar{k}}$ for some $\bar{k} \in \bar{K}$ such that $x_k = y_{\bar{k}}$, $x'_k = y'_{\bar{k}}$ and $\triangle(N_k) \sim_{\mathrm{BB}} \triangle(M_{\bar{k}})$. Moreover, since the sum of the sizes of N_k and $M_{\bar{k}}$ is strictly smaller than $\mathrm{size}(N) + \mathrm{size}(M)$, by induction we get that

$$\mathcal{E}_{\mathrm{RBB}} \vdash \tau.N_k \approx \tau.M_{\bar{k}}. \tag{17}$$

Then, we have

$$\begin{aligned}(x_k \mid x'_k) \lfloor\!\lfloor N_k &\overset{\text{(TL1)}}{\approx} (x_k \mid x'_k) \lfloor\!\lfloor \tau.N_k \\ &\overset{(17)}{\approx} (x_k \mid x'_k) \lfloor\!\lfloor \tau.M_{\bar{k}} \\ &\overset{\text{(TL1)}}{\approx} (x_k \mid x'_k) \lfloor\!\lfloor M_{\bar{k}} \\ &\overset{(x_k = y_{\bar{k}},\, x'_k = y'_{\bar{k}})}{\approx} (y_{\bar{k}} \mid y'_{\bar{k}}) \lfloor\!\lfloor M_{\bar{k}}.\end{aligned}$$

Summarising, for each $k \in K$ (respectively, $\bar{k} \in \bar{K}$) there is a $\bar{k} \in \bar{K}$ (respectively, $k \in K$) such that:

$$\mathcal{E}_{\mathrm{RBB}} \vdash (x_k \mid x'_k) \lfloor\!\lfloor N_k \approx (y_{\bar{k}} \mid y'_{\bar{k}}) \lfloor\!\lfloor M_{\bar{k}}. \tag{18}$$

Equations (12)–(18) taken together give $\mathcal{E}_{\mathrm{RBB}} \vdash N \approx M$, from which it is immediate to infer $\mathcal{E}_{\mathrm{RBB}} \vdash \mu.N \approx \mu.M$, for every $\mu \in \mathcal{A}_\tau$, and the proof is complete in this case.

2. Assume now that $\mu_i = \tau$ and $\triangle(N_i) \sim_{\mathrm{BB}} \triangle(M)$ for some $i \in I$, and that $\nu_{\bar{i}} = \tau$ and $\triangle(N) \sim_{\mathrm{BB}} \triangle(M_{\bar{i}})$ for some $\bar{i} \in \bar{I}$. Clearly, we have that $\triangle(N_i) \sim_{\mathrm{BB}} \triangle(M) \sim_{\mathrm{BB}} \triangle(N) \sim_{\mathrm{BB}} \triangle(M_{\bar{i}})$, and $\mathrm{size}(N_i) + \mathrm{size}(M_{\bar{i}}), \mathrm{size}(N_i) + \mathrm{size}(M), \mathrm{size}(N) + \mathrm{size}(M_{\bar{i}}) < \mathrm{size}(N) + \mathrm{size}(M)$, so that by induction we obtain

$$\mathcal{E}_{\mathrm{RBB}} \vdash \mu.N_i \approx \mu.M_{\bar{i}} \qquad \mathcal{E}_{\mathrm{RBB}} \vdash \mu.N \approx \mu.M_{\bar{i}} \qquad \mathcal{E}_{\mathrm{RBB}} \vdash \mu.N_i \approx \mu.M$$

from which $\mathcal{E}_{\mathrm{RBB}} \vdash \mu.N \approx \mu.M$ can be inferred, and the proof is complete in this case.

3. Assume that there is an index $i \in I$ such that $\mu_i = \tau$ and $\triangle(N_i) \sim_{\mathrm{BB}} \triangle(M)$, but there is no $\bar{i} \in \bar{I}$ such that $\nu_{\bar{i}} = \tau$ and $\triangle(N) \sim_{\mathrm{BB}} \triangle(M_{\bar{i}})$. (The symmetric case can be treated similarly and it is therefore omitted.) For every summand $\tau.N_i$ of N with $\triangle(N_i) \sim_{\mathrm{BB}} \triangle(M)$ we have that the sum of the sizes of N_i and M is strictly smaller than $\mathrm{size}(N) + \mathrm{size}(M)$. Hence, by induction we obtain that $\mathcal{E}_{\mathrm{RBB}} \vdash \tau.N_i \approx \tau.M$ for all such summands. Thus, possibly applying axioms A0–A3, we can infer that

$$\mathcal{E}_{\mathrm{RBB}} \vdash N \approx \tau.M + N' \tag{19}$$

where

$$N' = \sum_{i \in I_N} \mu_i.N_i + \sum_{j \in J} x_j \mathbin{\lfloor\!\lfloor} N_j + \sum_{h \in H} (x_h \mid \alpha_h) \mathbin{\lfloor\!\lfloor} N_h + \sum_{k \in K} (x_k \mid x'_k) \mathbin{\lfloor\!\lfloor} N_k$$

with $I_N = \{i \in I \mid \mu_i \neq \tau \vee \triangle(N_i) \nsim_{\mathrm{BB}} \triangle(M)\}$. Given the condition on the indexes in I_N, and considering that there is no $\bar{i} \in \bar{I}$ such that $\nu_{\bar{i}} = \tau$ and $\triangle(N) \sim_{\mathrm{BB}} \triangle(M_{\bar{i}})$, it is immediate to verify that, for each μ, whenever $\triangle(N') \xrightarrow{\mu} p$ then $\triangle(M) \xrightarrow{\mu} q$ for some q such that $p \sim_{\mathrm{BB}} q$. In particular, by applying the same reasoning used in the analysis of case 1 above, we have:

- for each $i \in I_N$ there is a $\bar{i}_i \in \bar{I}$ such that
$$\mathcal{E}_{\mathrm{RBB}} \vdash \mu_i.N_i \approx \nu_{\bar{i}_i}.M_{\bar{i}_i};$$
- for each $j \in J$ there is a $\bar{j}_j \in \bar{J}$ such that
$$\mathcal{E}_{\mathrm{RBB}} \vdash x_j \mathbin{\lfloor\!\lfloor} N_j \approx y_{\bar{j}_j} \mathbin{\lfloor\!\lfloor} M_{\bar{j}_j};$$
- for each $h \in H$ there is a $\bar{h}_h \in \bar{H}$ such that
$$\mathcal{E}_{\mathrm{RBB}} \vdash (x_h \mid \alpha_h) \mathbin{\lfloor\!\lfloor} N_h \approx (y_{\bar{h}_h} \mid \beta_{\bar{h}_h}) \mathbin{\lfloor\!\lfloor} M_{\bar{h}_h};$$
- for each $k \in K$ there is a $\bar{k}_k \in \bar{K}$ such that
$$\mathcal{E}_{\mathrm{RBB}} \vdash (x_k \mid x'_k) \mathbin{\lfloor\!\lfloor} N_k \approx (y_{\bar{k}_k} \mid y'_{\bar{k}_k}) \mathbin{\lfloor\!\lfloor} M_{\bar{k}_k}.$$

Summarising, we have obtained that

$$\mathcal{E}_{\mathrm{RBB}} \vdash M \approx N' + M' \tag{20}$$

where

$$\begin{aligned} M' = {} & \sum\{\nu_{\bar{i}}.M_{\bar{i}} \mid \bar{i} \neq \bar{i}_i \text{ for all } i\} + \sum\{y_{\bar{j}} \mathbin{\lfloor\!\lfloor} M_{\bar{j}} \mid \bar{j} \neq \bar{j}_j \text{ for all } j\} + \\ & \sum\{(y_{\bar{h}} \mid \beta_{\bar{h}}) \mathbin{\lfloor\!\lfloor} M_{\bar{h}} \mid \bar{h} \neq \bar{h}_h \text{ for all } h\} + \sum\{(y_{\bar{k}} \mid y'_{\bar{k}}) \mathbin{\lfloor\!\lfloor} M_{\bar{k}} \mid \bar{k} \neq \bar{k}_k \text{ for all } k\}. \end{aligned}$$

Then:

$$\begin{aligned} \mathcal{E}_{\mathrm{RBB}} \vdash \mu.N & \overset{(19)}{\approx} \mu.(\tau.M + N') \\ & \overset{(20)}{\approx} \mu.(\tau.(N' + M') + N') \\ & \overset{(\mathrm{TB})}{\approx} \mu.(N' + M') \\ & \overset{(20)}{\approx} \mu.M \end{aligned}$$

and the claim follows also in this case. □

6.3. The completeness result over CCS$_{LC}$

The completeness of the axiom system $\mathcal{E}_{\mathrm{RBB}}$ then follows from Proposition 9 and Proposition 11.

Theorem 6. *Let t, u be CCS$_{LC}$ terms. If $t \sim_{\mathrm{RBB}} u$, then $\mathcal{E}_{\mathrm{RBB}} \vdash t \approx u$.*

Proof. Assume that $t \sim_{\mathrm{RBB}} u$. By Proposition 9, there exist normal forms N and M such that $\mathcal{E}_{\mathrm{RBB}} \vdash t \approx N$ and $\mathcal{E}_{\mathrm{RBB}} \vdash u \approx M$, with

$$\begin{aligned} N & \approx \sum_{i \in I} \mu_i.N_i + \sum_{j \in J} x_j \mathbin{\lfloor\!\lfloor} N_j + \sum_{h \in H} (x_h \mid \alpha_h) \mathbin{\lfloor\!\lfloor} N_h + \sum_{k \in K} (x_k \mid x'_k) \mathbin{\lfloor\!\lfloor} N_k \\ M & \approx \sum_{\bar{i} \in \bar{I}} \nu_{\bar{i}}.M_{\bar{i}} + \sum_{\bar{j} \in \bar{J}} y_{\bar{j}} \mathbin{\lfloor\!\lfloor} M_{\bar{j}} + \sum_{\bar{h} \in \bar{H}} (y_{\bar{h}} \mid \beta_{\bar{h}}) \mathbin{\lfloor\!\lfloor} M_{\bar{h}} + \sum_{\bar{k} \in \bar{K}} (y_{\bar{k}} \mid y'_{\bar{k}}) \mathbin{\lfloor\!\lfloor} M_{\bar{k}}, \end{aligned}$$

where all the $N_i, N_j, N_h, N_k, M_{\bar{i}}, M_{\bar{j}}, M_{\bar{h}}, M_{\bar{k}}$ are in normal form. To prove the claim, it is then enough to prove that $\mathcal{E}_{\mathrm{RBB}} \vdash N \approx M$, since, be transitivity, this would imply $\mathcal{E}_{\mathrm{RBB}} \vdash t \approx u$. To this end, we show that $\mathcal{E}_{\mathrm{RBB}} \vdash N \approx N + M$, since, by symmetry of $\sim_{\mathrm{RBB}}$, this also gives $\mathcal{E}_{\mathrm{RBB}} \vdash M \approx N + M$ and thus that $\mathcal{E}_{\mathrm{RBB}} \vdash N \approx M$.

By the soundness of $\mathcal{E}_{\mathrm{RBB}}$ (Theorem 5), $t \approx N$ and $u \approx M$ imply that $t \sim_{\mathrm{RBB}} N$ and $u \sim_{\mathrm{RBB}} M$, thus giving $N \sim_{\mathrm{RBB}} M$. From $N \sim_{\mathrm{RBB}} M$, it follows that $\triangle(N) \sim_{\mathrm{RBB}} \triangle(M)$. We can then apply the same reasoning used in case 1 of the proof of Proposition 11 and obtain that:

- For each index $\bar{i} \in \bar{I}$ there is an index $i_{\bar{i}} \in I$ such that $\mu_{i_{\bar{i}}} = \nu_{\bar{i}}$ and $\triangle(N_{i_{\bar{i}}}) \sim_{\mathrm{BB}} \triangle(M_{\bar{i}})$. Then, for each $\bar{i} \in \bar{I}$, by Proposition 11 we obtain that

$$\mathcal{E}_{\mathrm{RBB}} \vdash \mu_{i_{\bar{i}}}.N_{i_{\bar{i}}} \approx \mu_{i_{\bar{i}}}.M_{\bar{i}} = \nu_{\bar{i}}.M_{\bar{i}}. \tag{21}$$

- For each $\bar{j} \in \bar{J}$ there is a $j_{\bar{j}} \in J$ such that $y_{\bar{j}} = x_{j_{\bar{j}}}$ and $\triangle(M_{\bar{j}}) \sim_{\mathrm{BB}} \triangle(N_{j_{\bar{j}}})$. Then, for each $\bar{j} \in \bar{J}$, by Proposition 11 and axiom TL1 we obtain that

$$\mathcal{E}_{\mathrm{RBB}} \vdash x_{j_{\bar{j}}} \mathbin{\|\!\!\lfloor} N_{j_{\bar{j}}} \approx x_{j_{\bar{j}}} \mathbin{\|\!\!\lfloor} \tau.N_{j_{\bar{j}}} \approx y_{\bar{j}} \mathbin{\|\!\!\lfloor} \tau.M_{\bar{j}} \approx y_{\bar{j}} \mathbin{\|\!\!\lfloor} M_{\bar{j}}. \tag{22}$$

- For each $\bar{h} \in \bar{H}$ there is a $h_{\bar{h}} \in H$ such that $y_{\bar{h}} = x_{h_{\bar{h}}}$, $\beta_{\bar{h}} = \alpha_{h_{\bar{h}}}$ and $\triangle(M_{\bar{h}}) \sim_{\mathrm{BB}} \triangle(N_{h_{\bar{h}}})$. Then, for each $\bar{h} \in \bar{H}$, by Proposition 11 and axiom TL1 we obtain that

$$\mathcal{E}_{\mathrm{RBB}} \vdash (x_{h_{\bar{h}}} \mid \alpha_{h_{\bar{h}}}) \mathbin{\|\!\!\lfloor} N_{h_{\bar{h}}} \approx (x_{h_{\bar{h}}} \mid \alpha_{h_{\bar{h}}}) \mathbin{\|\!\!\lfloor} \tau.N_{h_{\bar{h}}} \approx (y_{\bar{h}} \mid \beta_{\bar{h}}) \mathbin{\|\!\!\lfloor} \tau.M_{\bar{h}} \approx (y_{\bar{h}} \mid \beta_{\bar{h}}) \mathbin{\|\!\!\lfloor} M_{\bar{h}}. \tag{23}$$

- For each $\bar{k} \in \bar{K}$ there is a $k_{\bar{k}} \in K$ such that $y_{\bar{k}} \mid y'_{\bar{k}} \approx x_{k_{\bar{k}}} \mid x'_{k_{\bar{k}}}$, modulo C1, and $\triangle(M_{\bar{k}}) \sim_{\mathrm{BB}} \triangle(N_{k_{\bar{k}}})$. Then, for each $\bar{k} \in \bar{K}$, by Proposition 11 and axiom TL1 we obtain that

$$\mathcal{E}_{\mathrm{RBB}} \vdash (x_{k_{\bar{k}}} \mid x'_{k_{\bar{k}}}) \mathbin{\|\!\!\lfloor} N_{k_{\bar{k}}} \approx (x_{k_{\bar{k}}} \mid x'_{k_{\bar{k}}}) \mathbin{\|\!\!\lfloor} \tau.N_{k_{\bar{k}}} \approx (y_{\bar{k}} \mid y'_{\bar{k}}) \mathbin{\|\!\!\lfloor} \tau.M_{\bar{k}} \approx (y_{\bar{k}} \mid y'_{\bar{k}}) \mathbin{\|\!\!\lfloor} M_{\bar{k}}. \tag{24}$$

The fact that $\mathcal{E}_{\mathrm{RBB}} \vdash N \approx N + M$ then immediately follows from Equations (21)–(24). ☐

7. Complications for the cases of branching and η-bisimilarity

In this section, we discuss, at a high level, some of the technical subtleties that prevented us from obtaining the completeness result over $\mathrm{CCS_{LC}}$ in full generality. As we will see, the additional complexity is entailed by the τ-transitions induced from communication steps.

7.1. Branching bisimilarity

The first consequence of introducing communication in the language is that we can no longer rely on the notion of branching degree of a process to infer the syntactic structure of normal forms from the behaviour of their closed instances.

Consider the substitution $*$ introduced in Section 4.3. The feature that made $*$ *distinguishing* was that we could guarantee that a derivative of $*(M)$, for a $\mathrm{CCS_L}$ normal form M, had a large branching degree if and only if it was the target of a transition induced by a closed instance of a variable occurring in M. However, a $\mathrm{CCS_{LC}}$ normal form N can be such that the branching degree of $*(N)$ is large even if it still has to perform the actions induced by the closed instances of variables. For example, $*(\tau.(x \mid y))$ and $*(x \mid y)$ have the same branching degree. In other words, due to state preserving τ-transitions, we cannot guarantee that $*(N)$ has a large branching degree because it has actually reached a process that has that degree syntactically. This causes various issues in the proof of a variant of Proposition 5 over $\mathrm{CCS_{LC}}$, especially in the initial argument stating that, given $*(N) \sim_{\mathrm{BB}} *(M)$, whenever $*(N) \xrightarrow{\mu} p$ then M simulates the transition via $*(M) \xrightarrow{\varepsilon} q' \xrightarrow{\mu} q$, where none of the τ-transitions from $*(M)$ to q' is due to a communication involving a variable.

We remark that this issue is not strictly related to the precise definition of the substitution $*$ (indeed, $\sigma(\tau.(x \mid y))$ and $\sigma(x \mid y)$ are branching bisimilar for all closed substitutions σ), but rather to the fact that as long as we consider finitely many actions, we cannot define a closed substitution that prevents (derivatives of) closed instances of variables to communicate in any possible parallel composition context. (Of course, to avoid communications one could map variables into processes that can perform only τ-transitions, but this would not be useful to identify the structure of normal forms under weak semantics.) For this reason, the notion of potential, which we used in Section 6 to deal with the case of infinitely many actions, would not provide insightful results under the assumption that $\mathcal{A}$ is finite. All our attempts to find alternative measures leading to the definition of a distinguishing substitution for $\mathrm{CCS_{LC}}$ normal forms when only finitely many actions are considered have thus far been unsuccessful.

In [12] a semantics of open terms is defined along the lines of the decomposition discussed in Section 5. The authors define (rooted) branching bisimilarity over configurations and prove that two configurations are equivalent in this semantics if and only if for all closed substitutions their closed instances are pairwise equivalent in the classic (rooted) branching bisimulation semantics over closed terms. The completeness proof then uses the fact that transitions over configurations disclose their syntactic structure. A similar technique would not work in our setting. One could indeed define a transition relation over $\mathrm{CCS_{LC}}$ configurations as done in [5]. However, each induced (rooted) weak bisimulation-based equivalence over configurations results in a relation that is finer than the corresponding (rooted) equivalence over closed terms. Informally, this is due to the fact that the special variables x_μ (the $\mathrm{CCS_{LC}}$ counterpart of the variables x_{d} we defined for $\mathrm{CCS_L}$ configurations) occurring in configurations can be mapped by substitutions into arbitrary terms that are not related to the substitution of the triggering variable x. Moreover, transitions over configurations always distinguish τ-prefixes from τ-transitions inferred from communications, making thus the relations too fine (especially in the cases of η, delay and weak bisimulation).

7.2. η-bisimilarity

Our conjecture is that the axiom system $\mathcal{E}_{\mathrm{R\eta B}} = \mathcal{E}_{\mathrm{RBB}} \cup \{\mathrm{T3}, \mathrm{TL2}\}$ is a complete axiomatisation for rooted η-bisimulation over $\mathrm{CCS}_{\mathrm{LC}}$ with infinitely many actions.

However, the difficulties in the definition of a semantics for $\mathrm{CCS}_{\mathrm{LC}}$ configurations described above have so far prevented us from applying the strategy based on saturation, from Section 5, to derive the completeness result for rooted η-bisimilarity over $\mathrm{CCS}_{\mathrm{LC}}$ with infinitely many actions from the one obtained for rooted branching bisimilarity over the same language.

Unfortunately, also our attempts to provide a direct proof of the completeness result for rooted η-bisimilarity were not successful. The main issue is that our argument based on the *potential of a variable in a process* is not robust against τ-moves generated by a communication involving a variable in this semantics. For instance, consider the simple normal form $S = x \mathbin{\|\!\lfloor} N$. According to the definition of $\triangle$, we have that $\triangle(S)$ has $2n$ possible outgoing transitions: n labelled with actions $\alpha_1, \dots, \alpha_n$ each leading to $a_{x,\ell} \parallel \triangle(N)$, and n, with the same labels, leading to $b_{x,\ell} \parallel \triangle(N)$. We recall that $a_{x,\ell}, b_{x,\ell}$ are unique for each pair x, ℓ and cannot synchronise with any other action occurring in the term. Let us fix an ℓ, and consider the α_ℓ-transition to $a_{x,\ell} \parallel \triangle(N)$. This transition can be matched, in the η-bisimulation sense, by a summand of the form $\alpha_\ell.((x \mid \overline{\alpha_\ell}) \mathbin{\|\!\lfloor} N)$. In fact, the communication between $\triangle(x)$ and α_ℓ is hidden in the sequence of trailing τ-moves that are allowed by η-bisimilarity. Indeed, this summand cannot match all transitions from S, but we can easily imagine to a have a similar summand for each $\ell \in \{1, \dots, n\}$. This means that we cannot construct a summand-by-summand η-bisimulation between two normal forms, which is fundamental to prove a version of Proposition 11 in this semantics. We leave the quest for an alternative definition of a distinguishing substitution allowing us to overcome this technical issue as avenue for future work.

8. Concluding remarks

In this paper we have studied the equational characterisation of CCS parallel composition modulo four classic bisimulation-based weak congruences. We focused on two variants of the parallel operator: pure interleaving and full merge.

In the former case, we have proved that the use of Bergstra and Klop's left merge auxiliary operator, is sufficient to obtain (finite) complete axiomatisations of CCS modulo the considered congruences. Indeed, in [5] it is proved that this operator is also necessary in the case of rooted branching bisimilarity, and in [6] the same conclusion is drawn for the other three weak congruences considered in this paper.

The case of full merge turned out to be more complex than expected. We showed that by adding Bergstra and Klop's communication merge operator, together with left merge, to CCS we can obtain a completeness result for rooted branching bisimilarity, but only under the assumption that the set of actions is infinite. The completeness of $\sim_{\mathrm{RBB}}$ over CCS with left merge, communication merge, and finitely many actions remains an open problem, as well as the completeness of $\sim_{\mathrm{R\eta B}}$ over the same language with either finitely or infinitely many actions. Specifically, we plan to investigate the following conjectures:

Conjecture 1. The axiom system $\mathcal{E}_{\mathrm{RBB}}$, presented in Table 8, is a finite complete axiomatisation of rooted branching bisimilarity over $\mathrm{CCS}_{\mathrm{LC}}$ over a finite set of actions.

Conjecture 2. The axiom system $\mathcal{E}_{\mathrm{R\eta B}} = \mathcal{E}_{\mathrm{RBB}} \cup \{\mathrm{T3}, \mathrm{TL2}\}$, with T3 and TL2 from Table 6, is a complete axiomatisation of rooted η-bisimilarity over $\mathrm{CCS}_{\mathrm{LC}}$ with infinitely many actions.

Conjecture 3. The axiom system $\mathcal{E}_{\mathrm{R\eta B}} = \mathcal{E}_{\mathrm{RBB}} \cup \{\mathrm{T3}, \mathrm{TL2}, \mathrm{E1}\}$, with T3 and TL2 from Table 6, and E1 defined as follows

$$\sum_{a \in \mathcal{A}} (x \mid a) \mathbin{\|\!\lfloor} ((y \mid \overline{a}) \mathbin{\|\!\lfloor} z) \approx \sum_{a \in \mathcal{A}} (x \mid a) \mathbin{\|\!\lfloor} ((y \mid \overline{a}) \mathbin{\|\!\lfloor} z) + (x \mid y) \mathbin{\|\!\lfloor} z$$

is a finite complete axiomatisation of rooted η-bisimilarity over $\mathrm{CCS}_{\mathrm{LC}}$ with finitely many actions.

In Section 5 we considered the language $\mathrm{CCS}_{\mathrm{L}}$ and we derived the completeness of the axiomatisations proposed for $\sim_{\mathrm{RWB}}$, $\sim_{\mathrm{RDB}}$ and $\sim_{\mathrm{R\eta B}}$ from the completeness of the axiomatisation for $\sim_{\mathrm{RBB}}$. In Section 7, we discussed why a similar strategy cannot be applied to obtain a complete axiomatisation for $\sim_{\mathrm{R\eta B}}$ over $\mathrm{CCS}_{\mathrm{LC}}$. Hence, it is natural to wonder what happens to $\sim_{\mathrm{RWB}}$ and $\sim_{\mathrm{RDB}}$.

The reason why we could not derive completeness results for those equivalences over $\mathrm{CCS}_{\mathrm{LC}}$ is that, when we consider that language, then the root condition alone is not enough to ensure the congruence property for weak and delay equivalence. In fact, due to the presence of the communication merge, rooted weak bisimilarity is *not* a congruence over $\mathrm{CCS}_{\mathrm{LC}}$ when its operational semantics is defined as in Table 2 (see, e.g., [1,18,26] for more details). For instance, we have that $\tau.a \sim_{\mathrm{RWB}} \tau.a + a$, but $\mathbf{0} \sim_{\mathrm{RWB}} \tau.a \mid \overline{a}.b \not\sim_{\mathrm{RWB}} (\tau.a + a) \mid \overline{a}.b \sim_{\mathrm{RWB}} \tau.b$. For $\sim_{\mathrm{RDB}}$, we have a similar outcome (see, e.g., [26]).

Informally, the issue is that the use of sequences of transitions $\xrightarrow{\varepsilon}\xrightarrow{\mu}\xrightarrow{\varepsilon}$ (respectively, $\xrightarrow{\varepsilon}\xrightarrow{\mu}$) in the definition of $\sim_{\mathrm{RWB}}$ (respectively, $\sim_{\mathrm{RDB}}$) clashes with the definition of the operational semantics of $\mathrm{CCS}_{\mathrm{LC}}$ in terms of *single-step* transitions. In fact, in the former case, the presence of τ*-moves* can be disregarded since, as unobservable actions, they cannot allow an external observer to distinguish two processes. Conversely, in the latter case, the presence of a τ*-prefix* may prevent the derivation of an observable transition.

Interestingly, the *single-step* semantics of $\mathrm{CCS}_{\mathrm{LC}}$ stems from [15], but the original semantics of communication merge given in [14, 18] was defined in (rooted) weak bisimulation semantics, i.e., by taking weak transitions into account. Later on, in [1,29] a SOS formulation of this original semantics was given. Specifically, in [1] an operational semantics CCS was given directly in terms of *weak transitions*: SOS rules are built using literals of the form $t \overset{\mu}{\Longrightarrow} t'$ that, essentially, encode the sequence of (classic) literals $t \xrightarrow{\varepsilon}\xrightarrow{\mu}\xrightarrow{\varepsilon} t'$

in a single expression. Weak bisimilarity is then defined as the largest symmetric relation over processes such that $t \sim_{\mathtt{WB}} u$ if, and only if, whenever $t \overset{\mu}{\Longrightarrow} t'$ then, either $\mu = \tau$ and $t' \sim_{\mathtt{WB}} u$, or $u \overset{\mu}{\Longrightarrow} u'$ for some u' such that $t' \sim_{\mathtt{WB}} u'$. Processes t, u are then *rooted weak bisimilar*, notation $t \sim_{\mathtt{RWB}} u$, if, and only if, whenever $t \overset{\mu}{\Longrightarrow} t'$ (respectively, $u \overset{\mu}{\Longrightarrow} u'$) then $u \overset{\mu}{\Longrightarrow} u'$ (respectively, $t \overset{\mu}{\Longrightarrow} t'$) for some u' (respectively, t') such that $t' \sim_{\mathtt{WB}} u'$.

Given the differences in the proposed semantics, we have that some axioms that are sound modulo *strong* bisimilarity (and thus also modulo $\sim_{\mathtt{RBB}}$) over $\mathrm{CCS}_{\mathrm{LC}}$, become unsound modulo rooted weak bisimilarity in the new setting: this is the case of axioms C5, H, and D5 in Table 4. As a consequence, we cannot exploit the completeness of the axiomatisation for rooted branching bisimilarity (in the case the set of actions in infinite) to derive a complete axiomatisation for rooted weak bisimilarity.

Since (rooted) delay bisimilarity abstracts from the τ-moves that are performed *before* the observable action, it is natural to conjecture that we can build a new semantics for it based on sequences of transitions of the form $\xrightarrow{\varepsilon}\xrightarrow{\mu}$.

We leave as an avenue for future research the investigation of axiomatisations for rooted delay and weak bisimilarity built over the modified semantics.

We remark that although the results presented in this paper have been obtained by combining and adapting existing proof techniques, to our mind this does not make them trivial, as they still require a carefully crafted technical development to deal with the subtleties of weak semantics (as exemplified by the decomposition results in Section 5.1 and the use of the novel notion of potential in Section 6). This is also the reason why we decided to restrict our attention to the recursion, restriction, and relabelling free fragment of CCS. In particular, obtaining axiomatisations of congruences that are complete over open terms in languages that include parallel composition operators is hard even without recursion. Thus, we found it sensible to first study the problem in a simplified, recursion free, setting. The equations in the complete axiomatisations we presented in this paper remain sound over arbitrary processes and can therefore be used to reason about processes with recursion, together with classic inference rules and equations to deal with that construct. Achieving completeness, however, is highly non-trivial when one combines parallel composition, recursion, prefixing and non-deterministic choice arbitrarily and soon one goes beyond the realm of purely equational logic (see, e.g., the discussion in Section 5.2 of [3]). We leave the extension of our results to the recursion-free version CCS as an avenue for future research.

CRediT authorship contribution statement

Luca Aceto: Writing – review & editing, Writing – original draft, Supervision, Funding acquisition. **Valentina Castiglioni:** Writing – review & editing, Writing – original draft, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis. **Anna Ingólfsdóttir:** Supervision. **Bas Luttik:** Writing – review & editing, Writing – original draft, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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