



# Scalar conservation law in a bounded domain with strong source at boundary

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**Abstract.** We consider a scalar conservation law with source in a bounded open interval  $\Omega \subseteq \mathbb{R}$ . The equation arises from the macroscopic evolution of an interacting particle system. The source term models an external effort driving the solution to a given function  $\varrho$  with an intensity function  $V : \Omega \rightarrow \mathbb{R}_+$  that grows to infinity at  $\partial\Omega$ . We define the entropy solution  $u \in L^\infty$  and prove the uniqueness. When  $V$  is integrable,  $u$  satisfies the boundary conditions introduced by F. Otto (C. R. Acad. Sci. Paris, 322(1):729–734, 1996), which allows the solution to attain values at  $\partial\Omega$  different from the given boundary data. When the integral of  $V$  blows up,  $u$  satisfies an energy estimate and presents essential continuity at  $\partial\Omega$  in a weak sense.

**Keywords.** Scalar balance law, Initial–boundary value problem, Energy estimate, Doubling variable method.

## 1. Introduction

In this paper, we study the following initial-boundary value problem for a quasilinear scalar balance law in the bounded interval  $(0, 1) \subseteq \mathbb{R}$  given by

$$\begin{cases} \partial_t u + \partial_x [J(u)] + G = 0, & t > 0, x \in (0, 1), \\ u(0, \cdot) = u_0, \quad u(\cdot, 0) = \alpha, \quad u(\cdot, 1) = \beta. \end{cases} \quad (1.1)$$

where the source term  $G = G(t, x, u)$  reads

$$G(t, x, u) = V(x)(u - \varrho(t, x)), \quad (1.2)$$

and  $J, V, \varrho$  are nice functions defined respectively on  $\mathbb{R}, (0, 1)$  and  $\mathbb{R}_+ \times (0, 1)$ . Since the weak solution to (1.1) is not unique, we need to consider the *entropy solution* obtained through the *vanishing viscosity limit*. The entropy solution presents discontinuities both inside  $(0, 1)$  and at the boundaries. In particular, the values of  $u$  at  $\{0, 1\}$  can be different from the prescribed boundary data

$(\alpha, \beta)$ , so the boundary conditions are a priori formal. The first definition of the entropy solution is given in [2] for smooth  $u_0$  and homogeneous boundary  $(\alpha, \beta) \equiv (0, 0)$ . It is then generalized in [14, 15] to the case with  $u_0$ ,  $\alpha$  and  $\beta$  being  $L^\infty$  functions, see also [13, Section 2.6]. These definitions provide a set of possible boundary values, reflecting the formulation of *boundary layer* during the vanishing viscosity limit. We refer to [4, 5, 7, 16] and [6, Section 6.9] and references therein for more details and recent development.

Suppose that  $V(x) > 0$ , then  $G = G(t, x, u)$  satisfies that  $\partial_u G > 0$  and  $G(\cdot, \cdot, \varrho) \equiv 0$ , i.e.,  $G$  acts as a source (resp. sink) when  $u$  is less (resp. greater) than  $\varrho$ . When  $\varrho$  is a constant, (1.1)–(1.2) can be viewed as a conservation system with *relaxation* introduced in [11], with the first component degenerated to a stationary solution. In this paper, we aim at understanding the effect on the boundary discontinuities caused by *extremely strong perturbation*. Roughly speaking, suppose that  $V \rightarrow \infty$  as  $x \rightarrow 0, 1$  and choose  $\varrho$  that is compatible to the boundary data:  $\varrho|_{x=0} = \alpha$ ,  $\varrho|_{x=1} = \beta$ . We define the  $L^\infty$  entropy solution and prove the well-posedness. We then investigate its behavior near the boundaries and show that the appearance of discontinuity is dependent on the *integrability* of  $V$ . Generally speaking,

- If  $V$  is integrable, the boundary condition provides a set of possible values for  $u$  at  $x = 0$  (resp.  $x = 1$ ) which can be different from  $\alpha$  (resp.  $\beta$ ). The compatibility conditions are not necessary here.
- If the integral of  $V$  is divergent at  $x \in \{0, 1\}$ ,  $u$  satisfies an energy estimate which prescribes the boundary values in a weak sense, and one always observes continuous flux at the boundaries.

### 1.1. Physical motivation

The equation studied in this paper arises naturally from the *hydrodynamic limit* for asymmetric exclusion process with open boundaries [1, 18–20]. It is an open interacting particle system that describes the dynamics of stochastic lattice gas with hard core repulsion. Observed at properly chosen macroscopic space-time scale, the particle density evolves with a balance law with boundary conditions.

Consider the one-dimensional finite lattice  $\Lambda_N = \{1, \dots, N - 1\}$ . A variable  $\eta_i$  is assigned to each site  $i \in \Lambda_N$ , with  $\eta_i = 0$  if the site is empty and  $\eta_i = 1$  if it is occupied by a particle. The configuration is denoted by

$$\eta = (\eta_1, \dots, \eta_{N-1}) \in \{0, 1\}^{\Lambda_N}. \quad (1.3)$$

The dynamics is described as following. If there is a particle at site  $i$ , it waits for a random time  $\tau$  distributed as  $P(\tau > t) = e^{-t}$  and jumps to another *vacant* site  $i' > i$  on its right with probability  $p_\gamma(i' - i)$ , where

$$p_\gamma(k) := \frac{c_\gamma \mathbf{1}_{k>0}}{k^{1+\gamma}}, \quad c_\gamma^{-1} = \sum_{k=1}^{\infty} \frac{1}{k^{1+\gamma}}, \quad (1.4)$$

and  $\gamma > 1$  is a constant. We assume that the waiting times for all particles and all jumps are independent.

To model the boundary effects, we attach the system with two *infinitely extended reservoirs*. Suppose that one box containing infinitely many particles is placed at each site  $j \in \mathbb{Z}$ ,  $j \leq 0$ . The particles can enter and exit  $\Lambda_N$  obeying the following rules. Particles in the box  $j < 0$  can jump to any empty site  $i \in \Lambda_N$  with rate  $\alpha p_\gamma(|i - j|)$ , and particle at site  $i \in \Lambda_N$  can jump back to the box  $j < 0$  with rate  $(1 - \alpha)p_\gamma(|i - j|)$ . Here,  $\alpha \in (0, 1)$  is a given deterministic number that stands for the density of the reservoirs. Similar reservoirs with density  $\beta \in (0, 1)$  are placed at sites  $j \in \mathbb{Z}$ ,  $j \geq N$ .

Let  $L_{\text{exc},N}$ ,  $L_{-,N}$  and  $L_{+,N}$  be the infinitesimal generators of the exclusion dynamics, left and right reservoirs, respectively. For  $f : \{0, 1\}^{\Lambda_N} \rightarrow \mathbb{R}$ , they are precisely given by

$$\begin{aligned} L_{\text{exc},N}f(\eta) &= \sum_{i,i' \in \Lambda_N} c(i,i',\eta)[f(\eta^{i,i'}) - f(\eta)], \\ L_{-,N}f(\eta) &= \sum_{j \leq 0} \sum_{i \in \Lambda_N} c_-(i,j,\eta)[f(\eta^i) - f(\eta)], \\ L_{+,N}f(\eta) &= \sum_{j \geq N} \sum_{i \in \Lambda_N} c_+(i,j,\eta)[f(\eta^i) - f(\eta)], \end{aligned} \tag{1.5}$$

where  $\eta^{i,i'}$  is the configuration obtained by exchanging  $\eta_i$  and  $\eta_{i'}$  in  $\eta$ ,  $\eta^i$  is the one obtained by flipping  $\eta_i$  to  $1 - \eta_i$  in  $\eta$ , and

$$\begin{aligned} c(i,i',\eta) &= p_\gamma(i' - i)\eta_i(1 - \eta_{i'}), \\ c_-(i,j,\eta) &= \alpha p_\gamma(|i - j|)(1 - \eta_i) + (1 - \alpha)p_\gamma(|i - j|)\eta_i, \\ c_+(i,j,\eta) &= \beta p_\gamma(|i - j|)(1 - \eta_i) + (1 - \beta)p_\gamma(|i - j|)\eta_i. \end{aligned} \tag{1.6}$$

Consider the Markov process  $\{\eta(t) = \eta^N(t); t \geq 0\}$  generated by

$$L_N = N L_{\text{exc},N} + N^\gamma (L_{-,N} + L_{+,N}). \tag{1.7}$$

The factor  $N$  means that the dynamics of exclusion on  $\Lambda_N$  is accelerated to the hyperbolic scale  $Nt$ . Meanwhile,  $N^\gamma$  corresponds to a different scale for the reservoirs, for which the reason will be clarified later.

Assume some  $u_0 \in L^\infty((0, 1))$ , such that

$$u_0^N(x) := \sum_{i=1}^{N-1} \eta_i^N(0) \chi_{[\frac{i}{N} - \frac{1}{2N}, \frac{i}{N} + \frac{1}{2N})}(x) \xrightarrow{N \rightarrow \infty} u_0(x) \tag{1.8}$$

in probability, which precisely means that

$$\lim_{N \rightarrow \infty} P \left\{ \left| \int_0^1 u_0^N(x)g(x)dx - \int_0^1 u_0(x)g(x)dx \right| > \delta \right\} = 0 \tag{1.9}$$

for any  $\delta > 0$  and continuous function  $g$ . The *hydrodynamic limit* corresponds to the convergence that for almost every  $t > 0$ ,

$$u^N(t, x) := \sum_{i=1}^{N-1} \eta_i^N(t) \chi_{[\frac{i}{N} - \frac{1}{2N}, \frac{i}{N} + \frac{1}{2N})}(x) \xrightarrow{N \rightarrow \infty} u(t, x) \tag{1.10}$$

in probability. Since  $\gamma > 1$ ,  $p_\gamma$  possesses finite first moment:  $\mathbf{p}_\gamma := \sum_{k>0} kp_\gamma(k) < \infty$ . Hence, without considering the effects of reservoirs,  $u$  is the entropy solution to (see [17]):

$$\partial_t u + \mathbf{p}_\gamma \partial_x [u(1 - u)] = 0, \quad t > 0, \quad x \in (0, 1). \tag{1.11}$$

To investigate the effect of the left reservoirs, observe that

$$\begin{aligned} L_{-,N}[\eta_i] &= \sum_{j \leq 0} c_\alpha(i, j, \eta)(1 - 2\eta_i) \\ &= (\alpha - \eta_i) \sum_{k \geq i} p_\gamma(k) \approx (\alpha - \eta_i) \frac{c_\gamma}{i^\gamma \gamma}. \end{aligned} \tag{1.12}$$

The factor  $N^\gamma$  is chosen to get the non-trivial limit

$$\begin{aligned} N^\gamma L_{-,N} &\left[ \frac{1}{N} \sum_{i=1}^{N-1} \eta_i(t) g\left(\frac{i}{N}\right) \right] \\ &\approx \frac{1}{N} \sum_{i=1}^{N-1} (\alpha - \eta_i) \frac{c_\gamma N^\gamma}{i^\gamma \gamma} g\left(\frac{i}{N}\right) \rightarrow \frac{c_\gamma}{\gamma} \int_0^1 \frac{(\alpha - u)g}{x^\gamma} dx. \end{aligned} \tag{1.13}$$

Similar argument works for the right reservoir. Putting them together, we obtain formally the following hydrodynamic equation

$$\partial_t u + \mathbf{p}_\gamma \partial_x [u(1 - u)] + \frac{c_\gamma}{\gamma} \left[ \frac{u - \alpha}{x^\gamma} + \frac{u - \beta}{(1 - x)^\gamma} \right] = 0, \tag{1.14}$$

for  $x \in (0, 1)$ , with the natural initial and boundary conditions

$$u(0, x) = u_0(x), \quad u(t, 0) = \alpha, \quad u(t, 1) = \beta. \tag{1.15}$$

The source term can be written as  $V(x)(u - \varrho(x))$ , where

$$V = \frac{c_\gamma}{\gamma} \left[ \frac{1}{x^\gamma} + \frac{1}{(1 - x)^\gamma} \right], \quad \varrho = \frac{\alpha(1 - x)^\gamma + \beta x^\gamma}{x^\gamma + (1 - x)^\gamma}. \tag{1.16}$$

Conservation law with general  $V$  and  $\varrho$  can be modelled by exclusion process with *Glauber dynamics*, see [20] for details.

Note that in (1.8), the total variation of the initial empirical density  $u_0^N$  can grow in order  $\mathcal{O}(N)$ . For this reason, we focus on constructing the entropy solution in  $L^\infty$  space, rather than in the space of bounded-variation functions.

**Remark 1.1.** Assume some  $t_0 > 0$  such that (1.14) has a classical solution for  $t < t_0$ . Using the method of characteristics, one obtains the characteristic equation associated to (1.14):

$$x(0) = x_0 \in (0, 1), \quad x'(t) = \mathbf{p}_\gamma [1 - 2u(t, x(t))]. \tag{1.17}$$

Let  $v(t) := u(t, x(t))$  for  $t \in [0, t_0]$ , then

$$v(0) = u_0(x_0), \quad v'(t) = V(x(t)) [\varrho(x(t)) - v(t)], \tag{1.18}$$

where  $V(x)$  and  $\varrho(x)$  are functions given by (1.16). Hence, we formally obtain the second-order ordinary differential equation for the characteristic:

$$\begin{cases} x''(t) + V(x(t))x'(t) = \mathfrak{p}_\gamma V(x(t))[1 - 2\varrho(x(t))], \\ x(0) = x_0, \quad x'(0) = \mathfrak{p}_\gamma(1 - u_0(x_0)). \end{cases} \quad (1.19)$$

The classical solution is then determined by (1.18) along these lines.

## 2. Model and main results

Denote  $\Sigma = \mathbb{R}_+ \times (0, 1)$ . Through this paper, we consider the equation (1.1)–(1.2) on  $\Sigma$ . The following conditions are always assumed.

- (h1)  $J \in C^1(\mathbb{R}; \mathbb{R})$ .  
 (h2)  $V \in C((0, 1); \mathbb{R}_+)$  satisfies that

$$\lim_{x \rightarrow 0^+} V(x) = \lim_{x \rightarrow 1^-} V(x) = \infty. \quad (2.1)$$

- (h3) The initial data  $u_0$ , the boundary data  $\alpha, \beta$  and  $\varrho$  are measurable, essentially bounded functions on  $(0, 1)$ ,  $\mathbb{R}_+$  and  $\Sigma$ , respectively.

Our first aim is to define the unique entropy solution to (1.1)–(1.2) in  $L^\infty(\Sigma)$ . The concept of *Lax entropy–flux pair* plays a central role.

**Definition 2.1.** A function  $f \in C^2(\mathbb{R})$  is called a Lax entropy associated to (1.1) and  $q \in C^2(\mathbb{R})$  is called the corresponding flux, if

$$f''(u) \geq 0, \quad q'(u) = f'(u)J'(u), \quad \forall u \in \mathbb{R}. \quad (2.2)$$

As mentioned before, the properties of the entropy solution rely heavily on the integrability of  $V$ . Hereafter, we distinguish two cases.

### 2.1. Integrable case

The source  $G$  is called integrable when  $V$  belongs to  $L^1((0, 1))$ . In this case, we begin with Otto's definition of boundary entropy and the corresponding flux [15].

**Definition 2.2.**  $(F, Q) \in C^2(\mathbb{R}^2; \mathbb{R}^2)$  is called a *boundary entropy–flux pair*, if the next two conditions are satisfied.

1.  $(f, q) := (F, Q)(\cdot, k)$  is a Lax entropy–flux pair for all  $k \in \mathbb{R}$ ,
2.  $F(k, k) = \partial_u F(u, k)|_{u=k} = Q(k, k) = 0$  for all  $k \in \mathbb{R}$ .

The definition of entropy solution to (1.1) for the integrable case is similar to the case without  $V$  (see, e.g., [13, Definition 2.7.2, Theorem 2.7.31]) or with bounded  $V$  (see, e.g., [5, Definition 2.1]).

**Definition 2.3.** Assume  $V \in L^1((0, 1))$ . The entropy solution to (1.1) is a function  $u \in L^\infty(\Sigma)$  that satisfies the generalized entropy inequality

$$\begin{aligned} & \int_0^1 F(u_0, k)\varphi(0, \cdot)dx + \iint_\Sigma [F(u, k)\partial_t\varphi + Q(u, k)\partial_x\varphi]dxdt \\ & \geq \iint_\Sigma \partial_u F(u, k)V(x)(u - \varrho)\varphi dxdt \\ & \quad - M \int_0^T [F(\alpha, k)\varphi(\cdot, 0) + F(\beta, k)\varphi(\cdot, 1)]dt, \end{aligned} \tag{2.3}$$

for all boundary entropy–flux pairs  $(F, Q)$ ,  $k \in \mathbb{R}$ , and  $\varphi \in \mathcal{C}_c^2(\mathbb{R}^2)$  such that  $\varphi \geq 0$ . In (2.3), the constant  $M$  is given by

$$M := \sup \left\{ |J'(u)|; |u| \leq \text{esssup} \{ |\varrho|, |\alpha|, |\beta|, |u_0| \} \right\}. \tag{2.4}$$

As a standard result, the smooth entropy–flux pairs in Definition 2.3 can be replaced by non-smooth ones, and the initial condition holds in  $L^1$ .

**Definition 2.4.** For  $(u, k) \in \mathbb{R}^2$ , define

$$\eta(u, k) := |u - k|, \quad \xi(u, k) := \text{sgn}(u - k)[J(u) - J(k)]. \tag{2.5}$$

The pair  $(\eta, \xi)$  is called the *Kruzhkov entropy–flux pair*.

**Proposition 2.5.** Assume  $V \in L^1((0, 1))$ . The entropy solution is equivalently defined as  $u \in L^\infty(\Sigma)$  such that

$$\begin{aligned} & \int_0^1 |u_0 - k|\varphi(0, \cdot)dx + \iint_\Sigma [|u - k|\partial_t\varphi + \xi(u, k)\partial_x\varphi]dxdt \\ & \geq \iint_\Sigma \text{sgn}(u - k)V(x)(u - \varrho)\varphi dxdt \\ & \quad - M \int_0^T [|\alpha - k|\varphi(\cdot, 0) + |\beta - k|\varphi(\cdot, 1)]dt, \end{aligned} \tag{2.6}$$

for all  $k \in \mathbb{R}$  and  $\varphi \in \mathcal{C}_c^2(\mathbb{R}^2)$  such that  $\varphi \geq 0$ . Moreover,

$$\text{esslim}_{t \rightarrow 0+} \int_0^1 |u(t, x) - u_0(x)|dx = 0. \tag{2.7}$$

Using the methods in [13, Section 2.7 & 2.8], we obtain the well-posedness of  $u$  and an explicit expression for the boundary conditions.

**Proposition 2.6.** Assume that  $V \in L^1((0, 1))$ , then (1.1) admits a unique entropy solution  $u \in L^\infty(\Sigma)$ .

**Proposition 2.7.** Let  $u$  be as in Definition 2.3. For all  $0 < s < t$  and boundary entropy–flux pairs  $(F, Q)$ ,

$$\begin{aligned} & \text{esslim}_{x \rightarrow 0+} \int_s^t Q(u(r, x), \alpha(r))dr \leq 0, \\ & \text{esslim}_{x \rightarrow 1-} \int_s^t Q(u(r, x), \beta(r))dr \geq 0. \end{aligned} \tag{2.8}$$

### 2.2. Non-integrable case

The source  $G$  is called non-integrable when the integral of  $V$  is infinite. In this case, the singular points of the integral of  $V$  can only be  $\{0, 1\}$ . We will see later in Remark 2.10 that, when the integral of  $V$  is divergent at only one of them, the equation can be treated as a mixed boundary problem with one side integrable and the other side non-integrable. Hence, we assume without loss of generality that

$$\int_0^y V(x)dx = \int_{1-y}^1 V(x)dx = \infty, \quad \forall y \in (0, 1). \tag{2.9}$$

Also assume the compatibility conditions: for all  $T > 0$

$$\begin{aligned} \lim_{y \rightarrow 0^+} \int_0^T \int_0^y V(x) [\varrho(t, x) - \alpha(t)]^2 dx dt &= 0, \\ \lim_{y \rightarrow 0^+} \int_0^T \int_{1-y}^1 V(x) [\varrho(t, x) - \beta(t)]^2 dx dt &= 0. \end{aligned} \tag{2.10}$$

Notice that (2.10) is generally true in the integrable case, since  $\varrho$ ,  $\alpha$  and  $\beta$  are essentially bounded.

**Definition 2.8.** Assume (2.9) and (2.10). The entropy solution to (1.1) is a function  $u \in L^\infty(\Sigma)$  that satisfies the following conditions.

(EB) The energy bound: for all  $T > 0$ ,

$$\int_0^T \int_0^1 V(x) [u(t, x) - \varrho(t, x)]^2 dx dt < \infty. \tag{2.11}$$

(EI) The generalized entropy inequality

$$\begin{aligned} \int_0^1 f(u_0) \varphi(0, \cdot) dx + \iint_{\Sigma} [f(u) \partial_t \varphi + q(u) \partial_x \varphi] dx dt \\ \geq \iint_{\Sigma} f'(u) V(x) (u - \varrho) \varphi dx dt, \end{aligned} \tag{2.12}$$

for all Lax entropy–flux pairs  $(f, q)$  and all  $\varphi \in C_c^2(\mathbb{R} \times (0, 1))$  such that  $\varphi \geq 0$ ,

**Remark 2.9.** Despite that (2.12) contains no boundary condition, it turns out that the entropy solution is unique, see Theorem 2.12 and 2.13 below. Indeed, from (2.10) and (2.11),

$$\begin{aligned} \lim_{y \rightarrow 0^+} \int_0^T \int_0^y V(x) [u(t, x) - \alpha(t)]^2 dx dt &= 0, \\ \lim_{y \rightarrow 0^+} \int_0^T \int_{1-y}^1 V(x) [u(t, x) - \beta(t)]^2 dx dt &= 0. \end{aligned} \tag{2.13}$$

Given (2.9), the necessary boundary information is contained here. More details can be found in (2.17) and Lemma 3.2.

**Remark 2.10.** Indeed, the boundary conditions at  $x = 0$  and  $x = 1$  are treated separately. Hence, if  $V$  is integrable at  $x = 0$  (resp.  $x = 1$ ) but not at  $x = 1$  (resp.  $x = 0$ ), the entropy solution is defined by (2.11) and (2.3) for all  $\varphi \in C_c^2(\mathbb{R} \times (-\infty, 1))$  (resp.  $C_c^2(\mathbb{R} \times (0, \infty))$ ) such that  $\varphi \geq 0$ .

Similarly to the integrable case, we can define the entropy solution using the Kruzhkov entropy instead.

**Proposition 2.11.** *Assume (2.9) and (2.10). The entropy solution is equivalently defined as  $u \in L^\infty(\Sigma)$  satisfying (EB) and*

$$\begin{aligned} \int_0^1 |u_0 - k|\varphi(0, \cdot)dx + \iint_{\Sigma} [|u - k|\partial_t\varphi + \xi(u, k)\partial_x\varphi] dxdt \\ \geq \iint_{\Sigma} \text{sgn}(u - k)V(x)(u - \varrho)\varphi dxdt, \end{aligned} \tag{2.14}$$

for all  $k \in \mathbb{R}$  and  $\varphi \in C_c^2(\mathbb{R} \times (0, 1))$  such that  $\varphi \geq 0$ . Furthermore, the initial data is attained in the sense of (2.7).

We are now ready to state our main results.

**Theorem 2.12.** *(Uniqueness) Assume (2.10). Instead of (2.9), assume that  $V$  satisfies a stronger condition at the boundaries:*

$$\limsup_{y \rightarrow 0^+} \frac{1}{y^2} \int_0^y \left[ \frac{1}{V(x)} + \frac{1}{V(1-x)} \right] dx < \infty. \tag{2.15}$$

Then, there is at most one  $u \in L^\infty(\Sigma)$  that satisfies Definition 2.8.

Observe that for any  $\delta > 0$  and  $y \in (0, 1)$ ,

$$\begin{aligned} \int_0^T \frac{1}{y} \int_0^y |u(t, x) - \alpha(t)| dxdt \\ \leq \frac{1}{4\delta} \int_0^T \int_0^y V(x)(u - \alpha)^2 dxdt + \frac{T\delta}{y^2} \int_0^y \frac{1}{V(x)} dx. \end{aligned} \tag{2.16}$$

Taking  $y \rightarrow 0$  and choosing  $\delta$  arbitrarily small, (2.15) suggests that the boundary conditions in (1.1) hold in the sense of space-time average:

$$\lim_{y \rightarrow 0} \int_0^T \frac{1}{y} \int_0^y |u(t, x) - \alpha(t)| dxdt = 0, \tag{2.17}$$

and similarly for  $\beta(t)$ . The convergence in (2.17) can be significantly improved under extra conditions.

**Theorem 2.13.** *Let  $\alpha(t) \equiv \alpha$ ,  $\beta(t) \equiv \beta$  be almost everywhere constants and  $u$  satisfy (EB) and (EI). Assume (2.15) and for all  $T > 0$  that*

$$\begin{aligned} \lim_{y \rightarrow 0^+} \int_0^T \int_0^y V(x)|\varrho(t, x) - \alpha| dxdt = 0, \\ \lim_{y \rightarrow 0^+} \int_0^T \int_{1-y}^1 V(x)|\varrho(t, x) - \beta| dxdt = 0. \end{aligned} \tag{2.18}$$



Then, for all  $0 \leq s < t$  and Lax entropy–flux pairs  $(f, q)$ ,

$$\begin{aligned} \operatorname{esslim}_{x \rightarrow 0^+} \int_s^t q(u(r, x)) dr &= (t - s)q(\alpha), \\ \operatorname{esslim}_{x \rightarrow 1^-} \int_s^t q(u(r, x)) dr &= (t - s)q(\beta). \end{aligned} \tag{2.19}$$

**Corollary 2.14.** *Assume the same conditions as in Theorem 2.13 and that  $J$  is convex (or concave), then for all  $0 \leq s < t$ ,*

$$\begin{aligned} \operatorname{esslim}_{x \rightarrow 0^+} \int_s^t u(r, x) dr &= (t - s)\alpha, \\ \operatorname{esslim}_{x \rightarrow 1^-} \int_s^t u(r, x) dr &= (t - s)\beta. \end{aligned} \tag{2.20}$$

Finally, the existence of the entropy solution for non-integrable source with smooth coefficients and boundary data is established below.

**Theorem 2.15.** *(Existence) Assume that  $J \in C^2(\mathbb{R})$ ,  $V \in C^2((0, 1))$ ,  $\alpha, \beta \in C_b^2(\mathbb{R}_+)$  satisfy (2.9) and (2.10). Moreover, suppose that for each  $T > 0$ , there is a family of functions  $\{\varrho^\varepsilon; \varepsilon > 0\}$  such that the following conditions are fulfilled.*

- (i) For each  $\varepsilon > 0$ ,  $\varrho^\varepsilon \in C^2(\Sigma_T)$  and  $\varrho^\varepsilon \rightarrow \varrho$  in  $L^2(\Sigma_T)$ .
- (ii)  $\|\varrho^\varepsilon\|_{L^\infty(\Sigma_T)} \leq \|\varrho\|_{L^\infty(\Sigma_T)}$ ,  $\sup_{\varepsilon > 0} \|\varrho^\varepsilon\|_{H^1(\Sigma_T)} < \infty$  and

$$\sup_{\varepsilon > 0} \iint_{\Sigma_T} V(x) [\varrho^\varepsilon(t, x) - \varrho(t, x)]^2 dx dt < \infty. \tag{2.21}$$

Then, (1.1) admits an entropy solution in Definition 2.8.

*Example 2.16.* Recall (1.14) with boundary conditions (1.15). When  $\gamma \in (0, 1)$ , the source is integrable. When  $\gamma \geq 1$ , the source is non-integrable and the conditions in Theorem 2.12 and 2.15 are satisfied. Hence, the particle density evolves macroscopically with the unique entropy solution.

**Remark 2.17.** The method presented for integrable  $V$  can be extended to scalar balance laws in spatial dimensions  $d \geq 2$ , see, e.g., [14, 15]. For the multi-dimensional non-integrable case, one can construct an entropy solution satisfying an energy estimate similar to (2.11) via the standard vanishing viscosity limit. However, the corresponding uniqueness remains open.

### 2.3. Organization of the paper

The arguments for the integrable case are largely the same as those used in [13, Section 2.7, 2.8], see also [5]. Hence, we only summarize the ideas briefly. The focus is the non-integrable case. In Sect. 4, we prove Theorem 2.12 exploiting Kruzhkov’s doubling of variables technique. In Sect. 5, we prove Theorem 2.13 via an  $L^1$ -refinement of the energy bound. In Sect. 6, we prove Theorem 2.15 with vanishing viscosity method. Proposition 2.11 and some preliminary results are proved in Sect. 3.

### 2.4. Notations

For a measure space  $(X; \mu)$  and  $p \geq 1$ , let

$$L^p(X; \mu) = \{f; \|f\|_{L^p(X; \mu)} < \infty\}, \quad \|f\|_{L^p(X; \mu)}^p = \int_X |f|^p d\mu. \tag{2.22}$$

For  $p = \infty$ ,  $L^\infty(X; \mu)$  stands for the space of essentially bounded measurable functions and  $\|\cdot\|_{L^\infty(X; \mu)}$  is the essential supremum norm. When  $X \subseteq \mathbb{R}^d$  and  $\mu$  is the Lebesgue measure, we use the abbreviations  $L^p(X)$  and  $\|\cdot\|_{L^p(X)}$ .

Recall that  $\Sigma = \mathbb{R}_+ \times (0, 1)$  and denote by  $\nu$  the  $\sigma$ -finite measure on  $\Sigma$  given by  $\nu(dxdt) = V(x)dxdt$ . For  $T > 0$ , let  $\Sigma_T = (0, T) \times (0, 1)$ . With some abuse of notations, the restriction of  $\nu$  on  $\Sigma_T$  is still denoted by  $\nu$ .

Let  $(f, q)$  be either a Lax entropy–flux pair or the Kruzhkov entropy–flux pair  $(\eta, \xi)(\cdot, k)$ . For  $\varphi \in C_c^2(\mathbb{R}^2)$ , the entropy product of  $(f, q)$  is defined as

$$\begin{aligned} E_\varphi^{(f, q)}(u) &:= \iint_\Sigma [f(u)\partial_t\varphi + q(u)\partial_x\varphi] dxdt \\ &\quad - \iint_\Sigma f'(u)V(x)(u - \varrho)\varphi dxdt. \end{aligned} \tag{2.23}$$

We identify  $\partial_u\eta(u, k) = \text{sgn}(u - k)$  for  $\eta = |u - k|$ . Notice that the last integral in (2.23) is well-defined if and only if  $f'(u)(u - \varrho)\varphi \in L^1(\Sigma; \nu)$ .

### 3. Preliminary results

First, we verify the alternative definitions of the entropy solution with (2.6) and (2.14). The integrability of  $V$  is irrelevant here.

**Lemma 3.1.** *For  $u \in L^\infty(\Sigma)$ , (2.3) holds for all Lax entropy–flux pairs if and only it holds for the Kruzhkov entropy–flux pair and all  $k \in \mathbb{R}$ .*

*Proof.* Choose  $g \in C^2(\mathbb{R})$  such that  $g(0) = g'(0) = 0$ ,  $g(2) = g'(2) = 1$ ,  $g''(u) \geq 0$  and  $g(u) = g(-u)$ . For  $\varepsilon > 0$ , define

$$\begin{aligned} F_\varepsilon(u, k) &:= \begin{cases} |u - k| - \varepsilon, & |u - k| > 2\varepsilon, \\ \varepsilon g(\varepsilon^{-1}(u - k)), & |u - k| \leq 2\varepsilon, \end{cases} \\ Q_\varepsilon(u, k) &:= \int_k^u \partial_u F_\varepsilon(w, k) J'(w) dw. \end{aligned} \tag{3.1}$$

Observe that  $(F_\varepsilon, Q_\varepsilon)(\cdot, k)$  is a Lax entropy–flux pair for each  $\varepsilon$ ,  $(F_\varepsilon, Q_\varepsilon)(\cdot, k) \rightrightarrows (\eta, \xi)(\cdot, k)$  as  $\varepsilon \rightarrow 0$ , and

$$\partial_u F_\varepsilon(u, k) = \begin{cases} \text{sgn}(u - k), & |u - k| > 2\varepsilon, \\ g'(\varepsilon^{-1}(u - k)), & |u - k| \leq 2\varepsilon. \end{cases} \tag{3.2}$$

Suppose that (2.3) holds for all Lax entropy–flux pairs  $(f, q)$ . To get (2.6), it suffices to take  $(F_\varepsilon, Q_\varepsilon)(\cdot, k)$  in (2.3) and let  $\varepsilon \rightarrow 0$ . On the other hand, assume (2.6) for all  $k \in \mathbb{R}$ . Since  $k$  can be chosen smaller than  $-||u||_{L^\infty(\Sigma)}$ , (2.3) is true for the linear entropy  $f = u$  and the corresponding flux  $q = J$ . Then, one only needs to use the fact that any Lax entropy–flux pair  $(f, q)$  is contained in the convex hull of  $(\eta, \xi)(\cdot, k)$  and  $(u, J)$ .  $\square$

The proofs of Proposition 2.5 and 2.11 are standard. Below we assume the non-integrable case and prove Proposition 2.11 as an example.

*Proof.* The first argument follows directly from the previous lemma. To verify the  $L^1$ -continuity at  $t = 0$ , we use the idea in [13, Lemma 2.7.34, 2.7.41]. For any  $\phi \in \mathcal{C}_c^2(\mathbb{R})$  and  $\psi \in \mathcal{C}_c^2((0, 1))$  such that  $\phi, \psi \geq 0$ , let  $\varphi = \phi(t)\psi(x)$ . The entropy product in (2.23) reads

$$E_\varphi^{(\eta, \xi)(\cdot, k)}(u) = \iint_\Sigma |u - k| \phi' \psi \, dx dt + \iint_\Sigma [\xi(u, k) \psi' - \operatorname{sgn}(u - k) V(x)(u - \varrho) \psi] \phi \, dx dt. \tag{3.3}$$

Recall that  $u_0 \in L^\infty((0, 1))$  and  $u \in L^\infty(\Sigma)$ . Let  $M = \|u_0\|_{L^\infty} + \|u\|_{L^\infty}$  and for  $k \in [-M, M]$ ,  $|\xi(u, k)| \leq 2 \sup_{[-M, M]} |J|$ . Hence, the second line above is bounded by

$$\left[ C_M \sup |\psi'| + (\|u\|_{L^\infty} + \|\varrho\|_{L^\infty}) \int_0^1 V(x) \psi(x) dx \right] \int_0^\infty \phi(t) dt. \tag{3.4}$$

Since  $\psi$  is compactly supported within  $(0, 1)$ ,

$$E_\varphi^{(\eta, \xi)(\cdot, k)}(u) \leq \iint_\Sigma |u - k| \phi' \psi \, dx dt + C \int_0^\infty \phi(t) dt. \tag{3.5}$$

The generalized entropy inequality (2.14) then yields that

$$\phi(0) \int_0^1 |u_0(x) - k| \psi(x) dx + \int_0^\infty F_{k, \psi}(t) \phi'(t) dt \geq 0, \tag{3.6}$$

where the function  $F_{k, \psi} : (0, \infty) \rightarrow \mathbb{R}$  is defined as

$$F_{k, \psi}(t) := \int_0^1 |u(t, x) - k| \psi(x) dx - Ct. \tag{3.7}$$

From (3.6), after a possible modification on a set of zero measure,  $F_{k, \psi}$  is non-increasing on  $(0, \infty)$ , and

$$\operatorname{esslim}_{t \rightarrow 0^+} F_{k, \psi}(t) \leq \int_0^1 |u_0(x) - k| \psi(x) dx. \tag{3.8}$$

In other words, for all  $\psi \in \mathcal{C}_c^2((0, 1))$  such that  $\psi \geq 0$ ,

$$\operatorname{esslim}_{t \rightarrow 0^+} \int_0^1 |u(t, \cdot) - k| \psi \, dx \leq \int_0^1 |u_0 - k| \psi \, dx. \tag{3.9}$$

By a standard density argument, (3.9) holds for  $\psi \in L^1((0, 1))$  such that  $\psi \geq 0$ . One can approximate  $v \in L^\infty((0, 1))$  by simple functions taking only rational values to get

$$\operatorname{esslim}_{t \rightarrow 0^+} \int_0^1 |u(t, \cdot) - v| \psi \, dx \leq \int_0^1 |u_0 - v| \psi \, dx. \tag{3.10}$$

The result then follows by simply taking  $v = u_0$  and  $\psi \equiv 1$ . □

Next, we focus on the boundaries in the non-integrable case. Pick a function  $\psi \in C^\infty(\mathbb{R})$  such that

$$\text{supp } \psi \in (0, \infty), \quad \psi|_{x \geq 1} \equiv 1. \tag{3.11}$$

For  $\varepsilon > 0$  and  $x \in [0, 1]$ , define

$$\psi_\varepsilon(x) := \psi\left(\frac{x}{\varepsilon}\right)\mathbf{1}_{\{x < \varepsilon\}} + \mathbf{1}_{\{\varepsilon \leq x \leq 1-\varepsilon\}} + \psi\left(\frac{1-x}{\varepsilon}\right)\mathbf{1}_{\{x > 1-\varepsilon\}}. \tag{3.12}$$

Then,  $\psi_\varepsilon \in C_c^\infty((0, 1))$  and  $\psi_\varepsilon \rightarrow \mathbf{1}_{(0,1)}$  in  $L^1((0, 1))$  as  $\varepsilon \rightarrow 0$ .

**Lemma 3.2.** *Suppose that (2.15) holds and  $u \in L^\infty(\Sigma)$  satisfies (2.13). Fix some  $T > 0$  and recall that  $\Sigma_T = (0, T) \times (0, 1)$ . Let  $g$  be a measurable function on  $\mathbb{R}^3$  such that*

$$|g(t, x, w) - g(t, x', w')| \leq C(|x - x'| + |w - w'|) \tag{3.13}$$

for all  $(t, x), (t, x') \in \Sigma_T$  and  $|w|, |w'| \leq \|u\|_{L^\infty(\Sigma_T)}$ . Then,

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Sigma_T} g(t, x, u)\psi'_\varepsilon dxdt = \int_0^T [g(\cdot, 0, \alpha(\cdot)) - g(\cdot, 1, \beta(\cdot))] dt. \tag{3.14}$$

*Proof.* From the definition of  $\psi_\varepsilon$ ,

$$\iint_{\Sigma_T} g(\cdot, \cdot, u)\psi'_\varepsilon dxdt = \int_0^T \left( \int_0^\varepsilon + \int_{1-\varepsilon}^1 \right) g(\cdot, \cdot, u)\psi'_\varepsilon dxdt. \tag{3.15}$$

Noting that the integral of  $\psi'_\varepsilon(x)$  from 0 to  $\varepsilon$  is 1,

$$\begin{aligned} & \left| \int_0^T \int_0^\varepsilon g(t, x, u(t, x))\psi'_\varepsilon(x) dxdt - \int_0^T g(t, 0, \alpha(t)) dt \right| \\ & \leq \int_0^T \int_0^\varepsilon |g(t, x, u(t, x)) - g(t, 0, \alpha(t))| \psi'_\varepsilon(x) dxdt \end{aligned} \tag{3.16}$$

The condition of  $g$  together with the fact that  $|\psi'_\varepsilon| \leq C\varepsilon^{-1}$  yields that the last line is bounded from above by

$$\frac{C}{\varepsilon} \int_0^T \int_0^\varepsilon (x + |u(t, x) - \alpha(t)|) dxdt. \tag{3.17}$$

Applying Cauchy–Schwarz inequality, we obtain the upper bound

$$\frac{CT\varepsilon}{2} + \frac{1}{\delta} \int_0^T \int_0^\varepsilon V(x)(u - \alpha)^2 dxdt + \frac{C^2T\delta}{4\varepsilon^2} \int_0^\varepsilon \frac{1}{V(x)} dx, \tag{3.18}$$

for any  $\delta > 0$ . Taking first  $\varepsilon \rightarrow 0$  and then  $\delta$  sufficiently small, we have

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Sigma_T} g(t, x, u(t, x))\psi'_\varepsilon dxdt = \int_0^T g(t, 0, \alpha(t)) dt. \tag{3.19}$$

The integral over  $(1 - \varepsilon, 1)$  can be treated similarly. □

### 4. Uniqueness of the entropy solution

In this section, we first prove the uniqueness of the entropy solution in the non-integrable case, then briefly summarize the difference when  $V$  is integrable. In the non-integrable case, the uniqueness is a direct consequence of the stability below.

**Theorem 4.1.** *Assume (2.10), (2.15) and let  $u, v$  satisfy Definition 2.8 with the same  $(\alpha, \beta)$  and different  $(u_0, \varrho), (v_0, \varrho_*)$ , respectively. Then, for almost all  $t > 0$ ,*

$$\int_0^1 |u(t, \cdot) - v(t, \cdot)| dx \leq \int_0^1 |u_0 - v_0| dx + \iint_{\Sigma_t} V(x) |\varrho - \varrho_*| dx ds. \tag{4.1}$$

The next Kruzhkov-type lemma plays a key role in the proof. Since we choose the test function  $\varphi$  to be compactly supported in  $\Sigma$ , the integrability of  $V$  is indeed irrelevant to either the statement or the proof.

**Lemma 4.2.** *For all  $\varphi \in C_c^2(\Sigma)$  such that  $\varphi \geq 0$ ,*

$$\iint_{\Sigma} [ |u - v| \partial_t \varphi + \xi(u, v) \partial_x \varphi + |\varrho - \varrho_*| V \varphi ] dx dt \geq 0. \tag{4.2}$$

*Proof.* Let  $T > 0$  be fixed and we verify (4.2) for  $\varphi \in C_c^2(\Sigma_T)$ . Let  $\phi \in C_c^\infty(\mathbb{R})$  be a mollifier such that

$$\text{supp } \phi \subseteq (-1, 1), \quad \phi(-\tau) = \phi(\tau), \quad \int_{\mathbb{R}} \phi(\tau) d\tau = 1. \tag{4.3}$$

For  $\varepsilon > 0$ , let  $\phi_\varepsilon(\tau, \zeta) = \varepsilon^{-2} \phi(\varepsilon^{-1} \tau) \phi(\varepsilon^{-1} \zeta)$  and define

$$\Phi_\varepsilon(t, x, s, y) := \varphi \left( \frac{t+s}{2}, \frac{x+y}{2} \right) \phi_\varepsilon \left( \frac{t-s}{2}, \frac{x-y}{2} \right). \tag{4.4}$$

Without loss of generality, fix  $T = 1$ . Since  $\varphi \in C_c^2(\Sigma_1)$ , choose  $\delta > 0$  such that  $\text{supp } \varphi \subseteq [\delta, 1 - \delta]^2$ . The support of  $\Phi_\varepsilon$  is then contained in

$$\begin{aligned} t + s &\in [2\delta, 2 - 2\delta], & t - s &\in (-2\varepsilon, 2\varepsilon), \\ x + y &\in [2\delta, 2 - 2\delta], & x - y &\in (-2\varepsilon, 2\varepsilon). \end{aligned} \tag{4.5}$$

Direct computation shows that  $\Phi_\varepsilon \in C_c^2(\Sigma_1^2)$  for all  $\varepsilon \in (0, \delta)$ .

Hereafter, we assume  $\varepsilon \in (0, \delta)$ . Fixing  $(s, y) \in \Sigma_1$  and applying (2.14) with  $k = v(s, y)$  and  $\varphi = \Phi_\varepsilon(\cdot, \cdot, s, y) \in C_c^2(\Sigma_1)$ ,

$$\begin{aligned} E_{\Phi_\varepsilon(\cdot, \cdot, s, y)}^{(\eta, \xi)(\cdot, v(s, y))}(u) &= \iint_{\Sigma_1} |u - v(s, y)| \partial_t \Phi_\varepsilon(\cdot, \cdot, s, y) dx dt \\ &\quad + \iint_{\Sigma_1} \xi(u, v(s, y)) \partial_x \Phi_\varepsilon(\cdot, \cdot, s, y) dx dt \\ &\quad - \iint_{\Sigma_1} \text{sgn}(u - v(s, y)) V(x) (u - \varrho) \Phi_\varepsilon(\cdot, \cdot, s, y) dx dt \geq 0. \end{aligned} \tag{4.6}$$

Similar inequality holds for  $v = v(s, y)$ ,  $k = u(t, x)$  and  $\varphi = \Phi_\varepsilon(t, x, \cdot, \cdot)$ . Denote  $u_1 = u(t, x)$ ,  $v_1 = v(s, y)$ , then

$$\iint_{\Sigma_1} E_{\Phi_\varepsilon(\cdot, \cdot, s, y)}^{(\eta, \xi)(\cdot, v(s, y))}(u) ds dy + \iint_{\Sigma_1} E_{\Phi_\varepsilon(t, x, \cdot, \cdot)}^{(\eta, \xi)(\cdot, u(t, x))}(v) dx dt$$

$$\begin{aligned}
 &= \iiint_{\Sigma_1^2} \left\{ |u_1 - v_1|(\partial_t + \partial_s)\Phi_\varepsilon + \xi(u_1, v_1)(\partial_x + \partial_y)\Phi_\varepsilon \right. \\
 &\quad \left. - \operatorname{sgn}(u_1 - v_1)[G(t, x, u_1) - G_*(s, y, v_1)]\Phi_\varepsilon \right\} dy ds dx dt \\
 &\geq 0,
 \end{aligned} \tag{4.7}$$

where  $G(t, x, u) = V(x)(u - \varrho(t, x))$  and  $G_*(s, y, v) = V(y)(v - \varrho_*(s, y))$ .

Introduce the coordinates  $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ ,  $\boldsymbol{\theta} = (\theta_1, \theta_2)$  given by

$$\boldsymbol{\lambda} = \left( \frac{t+s}{2}, \frac{x+y}{2} \right), \quad \boldsymbol{\theta} = \left( \frac{t-s}{2}, \frac{x-y}{2} \right). \tag{4.8}$$

Recall that  $\Phi_\varepsilon = \varphi(\boldsymbol{\lambda})\psi_\varepsilon(\boldsymbol{\theta})$ . Direct computation shows that

$$\begin{aligned}
 (\partial_t + \partial_s)\Phi_\varepsilon &= \phi_\varepsilon(\boldsymbol{\theta})\partial_{\lambda_1}\varphi(\boldsymbol{\lambda}), \\
 (\partial_x + \partial_y)\Phi_\varepsilon &= \phi_\varepsilon(\boldsymbol{\theta})\partial_{\lambda_2}\varphi(\boldsymbol{\lambda}).
 \end{aligned} \tag{4.9}$$

Define  $\Omega := \{(\boldsymbol{\lambda}, \boldsymbol{\theta}); \boldsymbol{\lambda} + \boldsymbol{\theta} \in [0, 1]^2, \boldsymbol{\lambda} - \boldsymbol{\theta} \in [0, 1]^2\}$  and

$$\begin{aligned}
 \mathcal{I} &= |u_1 - v_1|\partial_{\lambda_1}\varphi(\boldsymbol{\lambda}) + \xi(u_1, v_1)\partial_{\lambda_2}\varphi(\boldsymbol{\lambda}), \\
 \mathcal{G} &= \operatorname{sgn}(u_1 - v_1)[G(\boldsymbol{\lambda} + \boldsymbol{\theta}, u_1) - G_*(\boldsymbol{\lambda} - \boldsymbol{\theta}, v_1)]\varphi(\boldsymbol{\lambda}).
 \end{aligned} \tag{4.10}$$

Then, (4.7) is rewritten as  $\mathcal{T}_\varepsilon - \mathcal{R}_\varepsilon \geq 0$  for  $\varepsilon \in (0, \delta)$ , where

$$\mathcal{T}_\varepsilon = \int_\Omega \mathcal{I}(\boldsymbol{\lambda}, \boldsymbol{\theta})\phi_\varepsilon(\boldsymbol{\theta})d(\boldsymbol{\lambda}, \boldsymbol{\theta}), \quad \mathcal{R}_\varepsilon = \int_\Omega \mathcal{G}(\boldsymbol{\lambda}, \boldsymbol{\theta})\phi_\varepsilon(\boldsymbol{\theta})d(\boldsymbol{\lambda}, \boldsymbol{\theta}). \tag{4.11}$$

Using the argument in [10, Theorem 1], one can show that

$$\lim_{\varepsilon \rightarrow 0} \mathcal{T}_\varepsilon = \iint_{\Sigma_1} \mathcal{I}(\boldsymbol{\lambda}, \mathbf{0})d\boldsymbol{\lambda}, \tag{4.12}$$

see also [13, Lemma 2.5.21]. Decompose  $\mathcal{G}$  as  $\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3$ , where

$$\begin{aligned}
 \mathcal{G}_1 &= \operatorname{sgn}(u_1 - v_1)[G(\boldsymbol{\lambda}, u_1) - G_*(\boldsymbol{\lambda}, v_1)]\varphi(\boldsymbol{\lambda}), \\
 \mathcal{G}_2 &= \operatorname{sgn}(u_1 - v_1)[G(\boldsymbol{\lambda} + \boldsymbol{\theta}, u_1) - G(\boldsymbol{\lambda}, u_1)]\varphi(\boldsymbol{\lambda}), \\
 \mathcal{G}_3 &= \operatorname{sgn}(u_1 - v_1)[G_*(\boldsymbol{\lambda}, v_1) - G_*(\boldsymbol{\lambda} - \boldsymbol{\theta}, v_1)]\varphi(\boldsymbol{\lambda}).
 \end{aligned} \tag{4.13}$$

Recall that  $G(\boldsymbol{\lambda}, u_1) = V(\lambda_2)(u_1 - \varrho(\boldsymbol{\lambda}))$ . Then,

$$\begin{aligned}
 |\mathcal{G}_2| &\leq V(\lambda_2)|\varrho(\boldsymbol{\lambda} + \boldsymbol{\theta}) - \varrho(\boldsymbol{\lambda})|\varphi(\boldsymbol{\lambda}) \\
 &\quad + |V(\lambda_2 + \theta_2) - V(\lambda_2)||u_1 - \varrho(\boldsymbol{\lambda} + \boldsymbol{\theta})|\varphi(\boldsymbol{\lambda}).
 \end{aligned} \tag{4.14}$$

Since  $\operatorname{supp} \varphi \subseteq [\delta, 1 - \delta]^2$  and  $u, \varrho \in L^\infty(\Sigma_1)$ ,

$$\begin{aligned}
 \left| \int_\Omega \mathcal{G}_2(\boldsymbol{\lambda}, \boldsymbol{\theta})\phi_\varepsilon(\boldsymbol{\theta})d(\boldsymbol{\lambda}, \boldsymbol{\theta}) \right| &\leq \int_{\Sigma_1} \varphi(\boldsymbol{\lambda})d\boldsymbol{\lambda} \int_{\mathbb{R}^2} \phi_\varepsilon(\boldsymbol{\theta})d\boldsymbol{\theta} \\
 &\quad \left\{ C_\delta |\varrho(\boldsymbol{\lambda} + \boldsymbol{\theta}) - \varrho(\boldsymbol{\lambda})| + C|V(\lambda_2 + \theta_2) - V(\lambda_2)| \right\},
 \end{aligned} \tag{4.15}$$

with  $C_\delta := \sup\{V(x); \delta \leq x \leq 1 - \delta\}$ . For sufficiently small  $\varepsilon$ , both  $\varrho$  and  $V$  are bounded on  $[\delta - \varepsilon, 1 - \delta + \varepsilon]^2$ . Then, the definition of  $\phi_\varepsilon$  and the Lebesgue

differentiation theorem show that this term vanishes as  $\varepsilon \rightarrow 0$ . The integral of  $\mathcal{G}_3$  is treated similarly. Finally,

$$\mathcal{G}_1 = |u_1 - v_1|V(\lambda_2)\varphi(\boldsymbol{\lambda}) - \operatorname{sgn}(u_1 - v_1)V(\lambda_2)[\varrho(\boldsymbol{\lambda}) - \varrho_*(\boldsymbol{\lambda})]\varphi(\boldsymbol{\lambda}), \quad (4.16)$$

so that  $\mathcal{G}_1 \geq -V(\lambda_2)|\varrho(\boldsymbol{\lambda}) - \varrho_*(\boldsymbol{\theta})|\varphi(\boldsymbol{\lambda})$ . Therefore,

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{R}_\varepsilon \geq - \iint_{\Sigma_1} V(\lambda_2)|\varrho(\boldsymbol{\lambda}) - \varrho_*(\boldsymbol{\lambda})|\varphi(\boldsymbol{\lambda})d\boldsymbol{\lambda}. \quad (4.17)$$

Recall that from (4.7), we have  $\mathcal{T}_\varepsilon - \mathcal{R}_\varepsilon \geq 0$ . By (4.12) and (4.17),

$$\iint_{\Sigma_1} \mathcal{I}(\boldsymbol{\lambda}, \mathbf{0})d\boldsymbol{\lambda} \geq - \iint_{\Sigma_1} V(\lambda_2)|\varrho(\boldsymbol{\lambda}) - \varrho_*(\boldsymbol{\lambda})|\varphi(\boldsymbol{\lambda})d\boldsymbol{\lambda}. \quad (4.18)$$

The desired inequality follows directly. □

*Proof of Theorem 4.1.* First, observe that the estimate is trivial if the integral of  $V|\varrho - \varrho_*|$  is infinite. Hereafter, we assume that  $\varrho - \varrho_* \in L^1(\Sigma_T; \nu)$ , where  $\nu(dxdt) = V(x)dxdt$ .

Fix an arbitrary  $\phi \in C_c^2((0, T))$  such that  $\phi \geq 0$  and recall the function  $\psi_\varepsilon$  given by (3.12). Using Lemma 4.2 with  $\varphi = \phi(t)\psi_\varepsilon(x)$ ,

$$\iint_{\Sigma_T} [|u - v|\phi'\psi_\varepsilon + \xi(u, v)\psi'_\varepsilon\phi + |\varrho - \varrho_*|V\phi\psi_\varepsilon]dxdt \geq 0. \quad (4.19)$$

Taking  $\varepsilon \rightarrow 0$ , we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \iint_{\Sigma_T} (|u - v|\phi' + |\varrho - \varrho_*|V\phi)\psi_\varepsilon dxdt \\ = \iint_{\Sigma_T} (|u - v|\phi' + |\varrho - \varrho_*|V\phi)dxdt. \end{aligned} \quad (4.20)$$

Notice that the convergence of the second term follows from  $\varrho - \varrho_* \in L^1(\Sigma_T; \nu)$ . We are left with the integral of  $\xi(u, v)\psi'_\varepsilon\phi$ . From the construction of  $\psi'_\varepsilon$ , this term is identically 0 for  $x \in [\varepsilon, 1 - \varepsilon]$ . Using the same argument as in Lemma 3.2, for any  $\delta > 0$ ,

$$\int_0^T \int_0^\varepsilon \xi(u, v)\psi'_\varepsilon\phi dxdt \leq \frac{1}{\delta} \int_0^T \int_0^\varepsilon \xi^2(u, v)V dxdt + \frac{CT\delta^2}{4\varepsilon^2} \int_0^\varepsilon \frac{dx}{V}, \quad (4.21)$$

with a constant  $C$  depending on  $\phi$ . From (2.15), the second term vanishes as  $\delta \rightarrow 0$ , uniformly in  $\varepsilon$ . Also observe that

$$|\xi(u, v)| = |J(u) - J(v)| \leq C|u - v|. \quad (4.22)$$

Since  $u$  and  $v$  satisfy (2.13) with common boundary data  $\alpha$ , the first term vanishes as  $\varepsilon \rightarrow 0$  for any fixed  $\delta > 0$ . By repeating the argument for the integral on  $(1 - \varepsilon, 1)$ ,

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Sigma_T} \xi(u, v)\psi_\varepsilon\phi dxdt = 0. \quad (4.23)$$

Putting these estimates together,

$$\int_0^T \left[ \phi'(t) \int_0^1 |u - v|dx + \phi(t) \int_0^1 V(x)|\varrho - \varrho_*|dx \right] dt \geq 0, \quad (4.24)$$

for all  $\phi \in \mathcal{C}_c^2((0, T))$  such that  $\phi \geq 0$ . From this,

$$t \mapsto \int_0^1 |u(t, \cdot) - v(t, \cdot)| dx - \iint_{\Sigma_t} V(x) |\varrho - \varrho_*| dx ds \tag{4.25}$$

is an essentially decreasing function of  $t$ . It suffices to apply the  $L^1$ -continuity of the entropy solution at  $t = 0$ .  $\square$

When  $V \in L^1((0, 1))$ , let  $u, v$  be as in Definition 2.3 with  $(\alpha, \beta, \varrho, u_0)$  and  $(\alpha_*, \beta_*, \varrho_*, v_0)$ , respectively. Instead of Lemma 4.2, the uniqueness follows from the next lemma.

**Lemma 4.3.** *For all  $\varphi \in \mathcal{C}_c^2(\mathbb{R}_+ \times \mathbb{R})$  such that  $\varphi \geq 0$ ,*

$$\begin{aligned} & \iint_{\Sigma} [|u - v| \partial_t \varphi + \xi(u, v) \partial_x \varphi + |\varrho - \varrho_*| V \varphi] dx dt \\ & \geq M \int_0^\infty [|\alpha - \alpha'| \varphi(\cdot, 0) + |\beta - \beta'| \varphi(\cdot, 1)] dt, \end{aligned} \tag{4.26}$$

where the constant  $M$  is the supreme of  $|J'|$  between the essential infimum and supremum of  $(\alpha, \alpha', \beta, \beta')$ .

The proof goes in the same line as that of Lemma 4.2, with the boundary terms treated with the argument used in [13, Theorem 2.7.28]. The only difference is that, when estimating  $\mathcal{G}_2$ , the support of  $\varphi$  contains boundary points. Observe that  $|\mathcal{G}_2|$  is bounded from above by

$$C_\varphi [V(\lambda_2) |\varrho(\boldsymbol{\lambda} + \boldsymbol{\theta}) - \varrho(\boldsymbol{\lambda})| + |V(\lambda_2 + \theta_2) - V(\lambda_2)|]. \tag{4.27}$$

As  $V$  is integrable, almost every point in  $(0, 1)$  is a Lebesgue point of  $V$ . This assures that the integral in (4.15) vanishes when  $\varepsilon \rightarrow 0$ .

### 5. Flux at boundary

This section is devoted to the identification of the behavior of the entropy solution at the boundaries. For the integrable case, (2.8) follows from (2.6) and exactly the same argument as used in [13, Theorem 2.7.31], so we focus on the non-integrable case and prove Theorem 2.13. Hereafter, always assume (2.15) and that  $\alpha$  and  $\beta$  are almost everywhere constant functions.

**Lemma 5.1.** *Assume (2.10). Let  $u$  be as in Definition 2.8 and  $(F, Q)$  be any boundary entropy-flux pair. Then, for  $\varphi \in \mathcal{C}_c^2(\mathbb{R} \times (-\infty, 1))$  such that  $\varphi \geq 0$ , we have  $\partial_u F(u, \alpha)(u - \varrho)\varphi \in L^1(\Sigma; \nu)$  and*

$$E_\varphi^{(F, Q)(\cdot, \alpha)}(u) + \int_0^1 F(u_0, \alpha) \varphi(0, \cdot) dx \geq 0. \tag{5.1}$$

*Similar result holds at the right boundary: for  $\varphi \in \mathcal{C}_c^2(\mathbb{R} \times (0, \infty))$  such that  $\varphi \geq 0$ , we have  $\partial_u F(u, \beta)(u - \varrho)\varphi \in L^1(\Sigma; \nu)$  and*

$$E_\varphi^{(F, Q)(\cdot, \beta)}(u) + \int_0^1 F(u_0, \beta) \varphi(0, \cdot) dx \geq 0. \tag{5.2}$$



*Proof.* Let  $K = (-\infty, T] \times (-\infty, y]$  for some  $T > 0$  and  $y < 1$ . Denote  $(f, q) = (F, Q)(\cdot, \alpha)$ . By its definition,  $|f'(u)| \leq C_F|u - \alpha|$ . Then,

$$\begin{aligned} |f'(u)(u - \varrho)\mathbf{1}_K| &\leq C_F|(u - \alpha)(u - \varrho)\mathbf{1}_K| \\ &\leq C_F(|(u - \varrho)^2\mathbf{1}_K| + |(\varrho - \alpha)(u - \varrho)\mathbf{1}_K|). \end{aligned} \tag{5.3}$$

By (2.11) and (2.10), both terms belong to  $L^1(\Sigma; \nu)$ . Hence, for all  $\varphi \in C_c^2(\mathbb{R} \times (-\infty, 1))$ ,  $f'(u)(u - \varrho)\varphi \in L^1(\Sigma; \nu)$ .

Fix  $\varphi \in C_c^2(\mathbb{R} \times (-\infty, 1))$  such that  $\varphi \geq 0$ . Let  $\varphi_\varepsilon = \varphi\psi_\varepsilon$ , where  $\psi_\varepsilon = \psi_\varepsilon(x)$  is given by (3.12). Since  $\alpha$  is constant, from (2.12),

$$E_{\varphi_\varepsilon}^{(f,q)}(u) + \int_0^1 f(u_0)\varphi_\varepsilon(0, \cdot)dx \geq 0. \tag{5.4}$$

Taking  $\varepsilon \rightarrow 0$ , it is straightforward to see that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^1 f(u_0)\varphi_\varepsilon(0, \cdot)dx + \iint_{\Sigma} f(u)\partial_t\varphi_\varepsilon dxdt \\ = \int_0^1 f(u_0)\varphi(0, \cdot)dx + \iint_{\Sigma} f(u)\partial_t\varphi dxdt. \end{aligned} \tag{5.5}$$

Using Lemma 3.2, since  $q(\alpha) = Q(\alpha, \alpha) = 0$  and  $\varphi(t, 1) = 0$ ,

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Sigma} q(u)\partial_x\varphi_\varepsilon dxdt = \iint_{\Sigma} q(u)\partial_x\varphi dxdt. \tag{5.6}$$

Recall that  $G(\cdot, \cdot, u) = V(x)(u - \varrho)$  and  $f'(u)(u - \varrho)\varphi \in L^1(\Sigma; \nu)$ , the dominated convergence theorem yields that

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Sigma} f'(u)G(\cdot, \cdot, u)\varphi_\varepsilon dxdt = \iint_{\Sigma} f'(u)G(\cdot, \cdot, u)\varphi dxdt. \tag{5.7}$$

Putting them together, we obtain the first assertion in the lemma. The second one follows similarly.  $\square$

To continue, we make use of the condition (2.18) to refine the energy bound (2.11) to the following  $L^1$ -integrability.

**Proposition 5.2.** *Assume (2.18), then  $u - \varrho \in L^1(\Sigma_T; \nu)$ , i.e.,*

$$\iint_{\Sigma_T} V(x)|u(t, x) - \varrho(t, x)| dxdt < \infty, \quad \forall T > 0. \tag{5.8}$$

*Proof.* Thanks to (2.18), it suffices to prove for  $y \in (0, 1)$  that

$$\int_0^T \int_0^y V(x)|u(t, x) - \alpha| dxdt < \infty, \tag{5.9}$$

and the similar bound for  $\beta$ . Recall the functions  $(F_\varepsilon, Q_\varepsilon)(\cdot, k)$  defined in (3.1) and observe that  $(F_\varepsilon, Q_\varepsilon)$  forms a boundary entropy–flux pair for fixed  $\varepsilon$ . Pick some  $T_* > T$  and  $y_* \in (y, 1)$ , the previous lemma yields that

$$E_{\varphi}^{(F_\varepsilon, Q_\varepsilon)(\cdot, \alpha)}(u) + \int_0^1 F_\varepsilon(u_0, \alpha)\varphi(0, \cdot)dx \geq 0, \tag{5.10}$$

for all  $\varphi \in C_c^2((-\infty, T_*) \times (-\infty, y_*))$  such that  $\varphi \geq 0$ . Hence,

$$\sup_{\varepsilon > 0} \iint_{\Sigma} G(\cdot, \cdot, u) \partial_u F_\varepsilon(u, \alpha) \varphi \, dxdt < \infty. \tag{5.11}$$

Fix such a  $\varphi$  and decompose  $G(\cdot, \cdot, u) \partial_u F_\varepsilon(u, \alpha) \varphi$  to

$$V \partial_u F_\varepsilon(u, \alpha) (u - \alpha) \varphi - V \partial_u F_\varepsilon(u, \alpha) (\varrho - \alpha) \varphi. \tag{5.12}$$

Since  $|\partial_u F_\varepsilon|$  is bounded by 1 uniformly in  $\varepsilon$ , by (2.18),

$$\begin{aligned} & \left| \iint_{\Sigma} V \partial_u F_\varepsilon(u, \alpha) (\varrho - \alpha) \varphi \, dxdt \right| \\ & \leq \|\varphi\|_{L^\infty(\Sigma)} \int_0^{T_*} \int_0^{y_*} V(x) |\varrho(t, x) - \alpha| \, dxdt \end{aligned} \tag{5.13}$$

is bounded from above uniformly in  $\varepsilon$ . Therefore,

$$\sup_{\varepsilon > 0} \iint_{\Sigma} V \partial_u F_\varepsilon(u, \alpha) (u - \alpha) \varphi \, dxdt < \infty. \tag{5.14}$$

From the construction of  $F_\varepsilon$ ,  $\partial_u F(u, \alpha) (u - \alpha) \geq 0$  for  $u \in \mathbb{R}$  and  $\partial_u F_\varepsilon(u, \alpha) = \text{sgn}(u - \alpha)$  if  $|u - \alpha| > 2\varepsilon$ . Then, for each fixed  $\varepsilon$ ,

$$\mathbf{1}_{\{|u-\alpha|>2\varepsilon\}} V |u - \alpha| \varphi \leq V \partial_u F_\varepsilon(u, \alpha) (u - \alpha) \varphi, \tag{5.15}$$

and in consequence,

$$\sup_{\varepsilon > 0} \iint_{\Sigma} \mathbf{1}_{\{|u-\alpha|>2\varepsilon\}} V |u - \alpha| \varphi \, dxdt < \infty. \tag{5.16}$$

Monotonic convergence theorem then yields that

$$\iint_{\Sigma} V(x) |u(t, x) - \alpha| \varphi(t, x) \, dxdt < \infty. \tag{5.17}$$

The proof is concluded by choosing  $\varphi$  such that  $\varphi|_{(0,T) \times (0,y)} \equiv 1$ . □

**Remark 5.3.** Assume the conditions in Theorem 2.13. Due to Proposition 5.2, (2.12) in Definition 2.3 can be generalized to

$$\begin{aligned} E_\varphi^{(f,q)}(u) &+ \int_0^1 f(u_0) \varphi(0, \cdot) \, dx \\ &\geq q(\beta) \int_0^\infty \varphi(t, 1) \, dt - q(\alpha) \int_0^\infty \varphi(t, 0) \, dt, \end{aligned} \tag{5.18}$$

for all Lax entropy–flux pairs  $(f, q)$  and  $\varphi \in C_c^2(\mathbb{R}^2)$  such that  $\varphi \geq 0$ . The same generalization works for (2.14).

Now we can state the proof of Theorem 2.13.

*Proof.* Pick nonnegative functions  $\phi \in C_c^2((0, T))$ ,  $\psi \in C_c^2(\mathbb{R})$  and define  $\varphi = \phi(t)\psi(x)$ . From the previous remark,

$$E_\varphi^{(f,q)}(u) + [\psi(0)q(\alpha) - \psi(1)q(\beta)] \int_0^T \phi(t) \, dt \geq 0. \tag{5.19}$$

There is a constant  $C = C(f, \phi)$ , such that

$$E_\phi^{(f,q)}(u) \leq \iint_{\Sigma_T} [q(u)\psi'\phi + C(V|u - \varrho| + 1)\psi] dxdt. \tag{5.20}$$

Due to the integrability proved in Proposition 5.2,

$$F_{q,\phi}(x) := \int_0^\infty q(u(t, x))\phi(t)dt - C \int_0^T \int_0^x (V(y)|u(t, y) - \varrho(t, y)| + 1)dy, \tag{5.21}$$

is well-defined as a measurable function on  $(0, 1)$ . If  $\psi(1) = 0$ ,

$$\psi(0)q(\alpha) \int_0^T \phi(t)dt + \int_0^1 F_{q,\phi}(x)\psi'(x)dx \geq 0. \tag{5.22}$$

This holds for all nonnegative  $\psi \in \mathcal{C}_c^2((-\infty, 1))$ , so  $F_{q,\phi}$  is non-increasing after possible modification on a Lebesgue null subset of  $(0, 1)$ . Hence,

$$\text{esslim}_{x \rightarrow 0^+} F_{q,\phi}(x) = \text{esslim}_{x \rightarrow 0^+} \int_0^\infty q(u(t, x))\phi(t)dt \tag{5.23}$$

exists for all  $\phi \in \mathcal{C}_c^\infty((0, T))$  such that  $\phi \geq 0$ . For all  $(s, t) \subseteq (0, T)$ , the result extends to  $\phi = \mathbf{1}_{(s,t)}$  with standard argument. The first equation in (2.19) then follows from (5.9) and the fact that  $V$  is not integrable on any neighbor of 0. The second one is proved similarly.  $\square$

When  $J$  is convex or concave, more information can be extracted from (2.19) by exploiting the idea in [8, 12].

*Proof of Corollary 2.14.* Let  $\mathcal{Q}$  be a countable set of functions such that (2.19) holds. The choice of  $\mathcal{Q}$  will be specified later. Fix an interval  $(s, t)$ , there exists a subset  $\mathcal{E} \subseteq (0, 1)$  with Lebesgue measure 0, such that

- (i)  $\|u(\cdot, x)\|_{L^\infty((s,t))} \leq \|u\|_{L^\infty((s,t) \times (0,1))}$  for all  $x \in (0, 1) \setminus \mathcal{E}$ ;
- (ii)  $(t - s)q(\alpha) = \lim_{x \in (0,1) \setminus \mathcal{E}, x \rightarrow 0^+} \int_{(s,t)} q(u(r, x))dr$  for all  $q \in \mathcal{Q}$ .

Denote  $m = \|u\|_{L^\infty((s,t) \times (0,1))}$ . For any sequence  $x_n \in (0, 1) \setminus \mathcal{E}$  such that  $x_n \rightarrow 0$ , we can find a subsequence  $x'_n$  and a family  $\{\mu_r\}_{r \in (s,t)}$  of probability measures, such that  $\mu_r([-m, m]) = 1$  and for each  $q \in \mathcal{Q}$ ,

$$(t - s)q(\alpha) = \int_s^t \int_{\mathbb{R}} q(z)\mu_t(dz)dr. \tag{5.24}$$

In other words,  $u(\cdot, x'_n)$  converges to  $\{\mu_r\}$  as  $n \rightarrow \infty$  in the weak- $\star$  topology of  $L^\infty((s, t))$ . To show (2.20), we need to show that

$$\mu_r(\{\alpha\}) = 1 \quad \text{for almost all } r \in (s, t). \tag{5.25}$$

For each rational number  $\delta$ , define

$$\begin{aligned} f_{-, \delta}(u) &:= \mathbf{1}_{u \leq \delta} |u - \delta|, & q_{-, \delta}(u) &:= \mathbf{1}_{u \leq \delta} (J(\delta) - J(u)), \\ f_{+, \delta}(u) &:= \mathbf{1}_{u \geq \delta} |u - \delta|, & q_{+, \delta}(u) &:= \mathbf{1}_{u \geq \delta} (J(u) - J(\delta)). \end{aligned} \tag{5.26}$$

It is easy to show that we can choose  $\mathcal{Q}$  to contain all  $q_{\pm, \delta}$ , so (5.24) holds for them. Observe that (5.25) is straightforward if  $J$  is monotonically increasing

(or decreasing) on  $[-m, m]$ . Indeed, suppose that  $J' \geq 0$  on  $[-m, m]$ . For  $\delta < \alpha$ ,  $q_{-, \delta}(u) > q_{-, \delta}(\alpha)$  for  $u < \delta$  and  $q_{-, \delta}(u) = q_{-, \delta}(\alpha)$  for  $u \geq \delta$ . Therefore,  $\mu_r([-m, \delta]) = 0$ . Similarly,  $\mu_r((\delta, m]) = 0$  for  $\delta > \alpha$ . As  $\delta$  can be any rational number, (5.25) holds. The case  $J$  is decreasing is similar.

Hereafter, we assume that  $J$  is concave and attaches its maximum at  $m_* \in [-m, m]$ . Suppose that  $\alpha \leq m_*$ , by the argument above

$$\mu_r([\alpha, \alpha_*]) = 1 \quad \text{for almost all } r \in (s, t), \tag{5.27}$$

where  $\alpha_* > \alpha$  is the only point that  $J(\alpha) = J(\alpha')$ . For  $\delta > \alpha_*$ ,  $q_{-, \delta}(u) \leq q_{-, \delta}(\alpha)$  on  $[\alpha, \alpha_*]$  with equality holds only for  $u = \alpha, \alpha_*$ . Therefore, (5.27) holds with  $[\alpha, \alpha_*]$  is replaced by  $\{\alpha, \alpha_*\}$ . Finally, let  $\mathcal{Q}$  also contain some Lax flux  $q$  such that  $q(\alpha_*) > q(\alpha)$  strictly, so (5.25) holds.  $\square$

### 6. Existence of the entropy solution

In this section, we fix some  $T > 0$  and construct an entropy solution on  $\Sigma_T$  via the vanishing viscosity limit. With the uniqueness proved in Theorem 2.12, we obtain an entropy solution on  $\Sigma$ . As before, we focus on the non-integrable case and then summarize the argument for the integrable case.

For the non-integrable case, assume that the conditions in Theorem 2.15 hold. For each  $\varepsilon > 0$ , the viscosity problem is constructed as

$$\begin{cases} \partial_t u^\varepsilon + \partial_x [J(u^\varepsilon)] + G^\varepsilon(t, x, u^\varepsilon) = \varepsilon \partial_x^2 u^\varepsilon, & (t, x) \in \Sigma_T, \\ u^\varepsilon(0, x) = u_0^\varepsilon(x), \quad u^\varepsilon(t, 0) = \alpha(t), \quad u^\varepsilon(t, 1) = \beta(t), \end{cases} \tag{6.1}$$

where  $G^\varepsilon(t, x, u) := V(x)(u - \varrho^\varepsilon(t, x))$  with  $\varrho^\varepsilon$  in Theorem 2.15,  $u_0^\varepsilon \in C^2([0, 1])$  approximates  $u_0$  in  $L^2((0, 1))$  and

$$u_0^\varepsilon(0) = \alpha(0), \quad u_0^\varepsilon(1) = \beta(0). \tag{6.2}$$

It admits a classical solution  $u^\varepsilon = u^\varepsilon(t, x)$  that satisfies

- (v1)  $u^\varepsilon - \varrho^\varepsilon \in L^2(\Sigma_T; \nu)$ , and
- (v2) for all  $\varphi \in C_c^2((-\infty, T) \times (0, 1))$ ,

$$\begin{aligned} & \int_0^1 u_0^\varepsilon \varphi(0, \cdot) dx + \iint_{\Sigma_T} [u^\varepsilon \partial_t \varphi + \varepsilon u^\varepsilon \partial_x^2 \varphi + J(u^\varepsilon) \partial_x \varphi] dx dt \\ & = \iint_{\Sigma_T} G(\cdot, \cdot, u^\varepsilon) \varphi dx dt. \end{aligned} \tag{6.3}$$

Some useful properties of  $u^\varepsilon$  are collected in Appendix A.

**Theorem 6.1.** *Along proper subsequence of  $\varepsilon \rightarrow 0$ ,  $u^\varepsilon$  converges to some  $u \in L^\infty(\Sigma_T)$  with respect to the weak- $\star$  topology of  $L^\infty(\Sigma_T)$ . Furthermore, the limit point satisfies (EB) for the given  $T$  and (EI) for all Lax entropy-flux pairs  $(f, q)$  and  $\varphi \in C_c^2((-\infty, T) \times \mathbb{R})$  such that  $\varphi \geq 0$ .*

Recall that a Young measure  $\mu = \{\mu_{t,x}; (t, x) \in \Sigma_T\}$  is a family of probability measures on  $\mathbb{R}$  such that  $(t, x) \mapsto \mu_{t,x}(A)$  is a measurable map from  $\Sigma_T$  to  $[0, 1]$  for any Borel subset  $A$  of  $\mathbb{R}$ . For continuous function  $h$ , define

$$\bar{h} : \Sigma_T \ni (t, x) \mapsto \int_0^1 h(z)\mu_{t,x}(dz). \tag{6.4}$$

In view of Lemma A.1,  $\|u^\varepsilon\|_{L^\infty(\Sigma_T)}$  is uniformly bounded. According to the fundamental theorem of Young measure, we obtain a  $\mu = \{\mu_{t,x}; (t, x) \in \Sigma_T\}$  as a subsequential limit point of  $u^\varepsilon$  in the following sense: for all  $h \in \mathcal{C}(\mathbb{R})$  and  $\varphi \in L^1(\Sigma_T)$ ,

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Sigma_T} h(u^\varepsilon)\varphi(t, x)dxdt = \iint_{\Sigma_T} \bar{h}(t, x)\varphi(t, x)dxdt. \tag{6.5}$$

We also have  $\mu_{t,x}([-m, m]) = 1$ , where  $m = \sup_{\varepsilon > 0} \|u^\varepsilon\|_{L^\infty(\Sigma_T)}$ .

*Proof of Theorem 6.1.* First, from Lemma A.2 and [3, Proposition 4.1],

$$\iint_{\Sigma_T} V(x) \left[ \int_0^1 [z - \varrho(t, x)]^2 \mu_{t,x}(dz) \right] dxdt < \infty. \tag{6.6}$$

For all Lax entropy–flux pairs  $(f, q)$ , from (6.1) we have

$$\begin{aligned} \partial_t[f(u^\varepsilon)] + \partial_x[q(u^\varepsilon)] &= f'(u^\varepsilon)\{\partial_t u^\varepsilon + \partial_x[J(u^\varepsilon)]\} \\ &= \varepsilon f'(u^\varepsilon)\partial_x^2 u^\varepsilon - f'(u^\varepsilon)G^\varepsilon(\cdot, \cdot, u^\varepsilon). \end{aligned} \tag{6.7}$$

Since  $f'' \geq 0$ ,  $f'(u^\varepsilon)\partial_x^2 u^\varepsilon \leq \varepsilon \partial_x^2[f(u^\varepsilon)]$ . Therefore,

$$\partial_t[f(u^\varepsilon)] + \partial_x[q(u^\varepsilon)] + f'(u^\varepsilon)G^\varepsilon(\cdot, \cdot, u^\varepsilon) \leq \varepsilon \partial_x^2[f(u^\varepsilon)]. \tag{6.8}$$

Recall the entropy product defined in (2.23). For  $\varphi \in \mathcal{C}_c^2((-\infty, T) \times (0, 1))$  such that  $\varphi \geq 0$ , we have

$$\begin{aligned} E_\varphi^{(f,q)}(u^\varepsilon) + \int_0^1 f(u_0^\varepsilon)\varphi(0, \cdot)dx \\ \geq \varepsilon \iint_{\Sigma_T} \partial_x[f(u^\varepsilon)]\partial_x\varphi dxdt - \iint_{\Sigma_T} f'(u^\varepsilon)V(\varrho^\varepsilon - \varrho)\varphi dxdt. \end{aligned} \tag{6.9}$$

In view of the condition (i) in Theorem 2.15 and Lemma A.2, the two terms in the right-hand side vanish as  $\varepsilon \rightarrow 0$ . We then obtain from (6.5) that

$$\begin{aligned} \iint_{\Sigma_T} [\bar{f}\partial_t\varphi + \bar{q}\partial_x\varphi - (\bar{g} - \bar{f}'\varrho)V\varphi] dxdt \\ \geq - \int_0^1 f(u_0)\varphi(0, \cdot)dx, \quad \text{where } g(u) := uf'(u). \end{aligned} \tag{6.10}$$

Observe that (6.6) and (6.10) can be viewed as the measure-valued version of (2.11) and (2.12), respectively. Hence, the main task is to show that, the Young measure  $\mu$  is concentrated on some  $u \in L^\infty(\Sigma_T)$ :

$$\mu_{t,x}(dz) = \delta_{u(t,x)}(dz) \quad \text{for almost all } (t, x) \in \Sigma_T. \tag{6.11}$$

To do this, we exploit the *compensated compactness* argument, see, e.g., [9, Section 5.D]. Define two sequences  $\Phi_\varepsilon, \Psi_\varepsilon : \Sigma_T \rightarrow \mathbb{R}^2$  by

$$\Phi_\varepsilon := (f(u^\varepsilon), q(u^\varepsilon)), \quad \Psi_\varepsilon := (-J(u^\varepsilon), u^\varepsilon). \tag{6.12}$$

Since  $\{\Phi_\varepsilon; \varepsilon > 0\}$  and  $\{\Psi_\varepsilon; \varepsilon > 0\}$  are bounded,  $\{\operatorname{div}\Phi_\varepsilon; \varepsilon > 0\}$  and  $\{\operatorname{curl}\Psi_\varepsilon; \varepsilon > 0\}$  are bounded in  $W^{-1,p}(\Sigma_T)$  for any  $p > 2$ . Notice that

$$\begin{aligned} \operatorname{div}\Phi_\varepsilon &= \partial_t[f(u^\varepsilon)] + \partial_x[q(u^\varepsilon)] \\ &= \varepsilon\partial_x^2[f(u^\varepsilon)] - \varepsilon f''(u^\varepsilon)(\partial_x u^\varepsilon)^2 - V f'(u^\varepsilon)(u^\varepsilon - \varrho^\varepsilon); \\ \operatorname{curl}\Psi_\varepsilon &= \partial_t u^\varepsilon + \partial_x[J(u^\varepsilon)] = \varepsilon\partial_x^2 u^\varepsilon - V(u^\varepsilon - \varrho^\varepsilon). \end{aligned} \tag{6.13}$$

Fix any  $\delta > 0$  and define  $\Sigma_T^\delta = (\delta, T - \delta) \times (\delta, 1 - \delta)$ . We claim that both  $\{\operatorname{div}\Phi_\varepsilon; \varepsilon > 0\}$  and  $\{\operatorname{curl}\Psi_\varepsilon; \varepsilon > 0\}$  are precompact in  $H^{-1}(\Sigma_T^\delta)$ . Indeed, we have seen from Lemma A.2 that  $\varepsilon\partial_x^2[f(u^\varepsilon)]$  vanishes as  $\varepsilon \rightarrow 0$  in  $H^{-1}(\Sigma_T^\delta)$  and  $\{\varepsilon f''(u^\varepsilon)(\partial_x u^\varepsilon)^2; \varepsilon > 0\}$  is a bounded sequence in  $L^1(\Sigma_T^\delta)$ . On the other hand, as  $V \leq C_\delta$  on  $[\delta, 1 - \delta]$ ,  $\{V f'(u^\varepsilon)(u^\varepsilon - \varrho^\varepsilon); \varepsilon > 0\}$  is also a bounded sequence in  $L^1(\Sigma_T^\delta)$ . Thanks to [9, Corollary 1.C.1], the claim holds for  $\{\operatorname{div}\Phi_\varepsilon; \varepsilon > 0\}$ . For  $\{\operatorname{curl}\Psi_\varepsilon; \varepsilon > 0\}$ , the argument is similar.

Now, the Div-Curl lemma [9, Theorem 5.B.4] yields that

$$\lim_{\varepsilon \rightarrow 0} (\Phi_\varepsilon \cdot \Psi_\varepsilon) = (\bar{f}, \bar{q}) \cdot (-\bar{J}, \bar{u}) = \bar{u}\bar{q} - \bar{J}\bar{f}, \tag{6.14}$$

weakly as distributions on  $\Sigma_T^\delta$ . Meanwhile, (6.5) with  $h = zq(z) - J(z)f(z)$  gives us that for all  $\varphi \in L^1(\Sigma_T^\delta)$ ,

$$\lim_{\varepsilon \rightarrow 0} \iint_{\Sigma_T^\delta} (\Phi_\varepsilon \cdot \Psi_\varepsilon) \varphi \, dxdt = \iint_{\Sigma_T^\delta} \bar{h} \varphi \, dxdt. \tag{6.15}$$

Hence, the *Tartar’s factorization* holds almost everywhere in  $\Sigma_T^\delta$ :

$$\int_0^1 (J - \bar{J})(f - \bar{f}) \, d\mu_{t,x} = \int_0^1 (z - \bar{u})(q - \bar{q}) \, d\mu_{t,x}. \tag{6.16}$$

As  $\delta > 0$  is arbitrary, we obtain (6.16) for all Lax entropy–flux pairs  $(f, q)$  and almost all  $(t, x) \in \Sigma_T$ . Standard argument then proves (6.11).  $\square$

For the integrable case, the approach is slightly different. Assume that  $V \in L^1((0, 1))$  and  $(\varrho, \alpha, \beta, u_0)$  are essentially bounded functions. For  $\varepsilon > 0$ , pick  $\varrho^\varepsilon \in \mathcal{C}^2(\Sigma_T)$ ,  $\alpha^\varepsilon, \beta^\varepsilon \in \mathcal{C}^2([0, T])$  and  $u_0^\varepsilon \in \mathcal{C}^2([0, 1])$  as a mollification of  $\varrho, \alpha, \beta$  and  $u_0$ :

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left\{ \int_0^1 (u_0^\varepsilon - u_0)^2 \, dx + \iint_{\Sigma_T} V(\varrho^\varepsilon - \varrho)^2 \, dxdt \right. \\ \left. + \int_0^T [(\alpha^\varepsilon - \alpha)^2 + (\beta^\varepsilon - \beta)^2] \, dt \right\} = 0, \end{aligned} \tag{6.17}$$

and for each  $\varepsilon > 0$ ,

$$\begin{aligned} \varrho^\varepsilon(t, 0) &= \alpha^\varepsilon(t), & \varrho^\varepsilon(t, 1) &= \beta^\varepsilon(t), & \forall t \in [0, T], \\ u_0^\varepsilon(0) &= \alpha^\varepsilon(0), & u_0^\varepsilon(1) &= \beta^\varepsilon(0). \end{aligned} \tag{6.18}$$

The viscosity problem for integrable case reads

$$\begin{cases} \partial_t u^\varepsilon + \partial_x [J(u^\varepsilon)] + G^\varepsilon(t, x, u^\varepsilon) = \varepsilon \partial_x^2 u^\varepsilon, \\ u^\varepsilon(0, x) = u_0^\varepsilon(x), & u^\varepsilon(t, 0) = \alpha^\varepsilon(t), & u^\varepsilon(t, 1) = \beta^\varepsilon(t). \end{cases} \tag{6.19}$$

where  $G^\varepsilon(t, x, u) := V(x)(u - \varrho^\varepsilon(t, x))$ .

Let  $u^\varepsilon$  be the classical solution and consider the limit  $\varepsilon \rightarrow 0$  as in the non-integrable case. To deal with the discontinuities formulated at the boundaries in this limit procedure, define for each  $\varepsilon > 0$  that

$$g_\varepsilon(x) := \begin{cases} 1 - e^{-\frac{x}{\varepsilon}}, & x \in [0, \frac{1}{2}), \\ 1 - e^{-\frac{1-x}{\varepsilon}}, & x \in (\frac{1}{2}, 1]. \end{cases} \tag{6.20}$$

For boundary entropy–flux  $(F, Q)$  and  $k \in \mathbb{R}$ , denote  $(f, q) = (F, Q)(\cdot, k)$ . For  $\varphi \in C_c^2((-\infty, T) \times \mathbb{R})$ , let  $\varphi_\varepsilon = \varphi g_\varepsilon$  and observe that

$$\begin{aligned} \mathcal{E}(\varepsilon) &:= E_{\varphi_\varepsilon}^{(f, q)}(u^\varepsilon) - \iint_{\Sigma_T} q(u^\varepsilon) \varphi g_\varepsilon' dx dt \\ &= \iint_{\Sigma_T} [f(u^\varepsilon) \partial_t \varphi + q(u^\varepsilon) \partial_x \varphi - f'(u^\varepsilon) G(\cdot, \cdot, u^\varepsilon)] g_\varepsilon dx dt. \end{aligned} \tag{6.21}$$

Following the manipulation in [13, Theorem 2.8.4], we show that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \left\{ \int_0^1 f(u_0) \varphi(0, \cdot) dx + \mathcal{E}(\varepsilon) \right\} \\ \geq -M \int_0^T [f(\alpha) \varphi(\cdot, 0) + f(\beta) \varphi(\cdot, 1)] dt, \end{aligned} \tag{6.22}$$

when  $\varphi \geq 0$ . From this, we obtain the measure-valued version of (2.3) for the subsequential weak- $\star$  limit of  $u^\varepsilon$ . The application of compensated compactness argument is exactly the same as in the non-integrable case.

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## Appendix A: The parabolic problem with non-integrable $V$

Assume the conditions in Theorem 2.15. Without loss of generality, assume that  $0 \leq \varrho^\varepsilon, u_0^\varepsilon \leq 1$  on  $[0, T] \times [0, 1]$ . Let  $u^\varepsilon$  be the solution to the parabolic equation (6.1). We collect and prove some useful estimates for  $u^\varepsilon$ .

**Lemma A.1.** *For all  $\varepsilon > 0$  and  $(t, x) \in \Sigma_T$ ,  $0 \leq u^\varepsilon \leq 1$ .*

*Proof.* Use the short notation  $\langle \cdot, \cdot \rangle$  to denote the inner product in  $L^2(\Sigma_T)$ . First assume that  $J$  is globally Lipschitz continuous:

$$|J(u) - J(u')| \leq M|u - u'|.$$

Let  $v = (u^\varepsilon - 1)^+$  and note that  $v|_{t=0} = v|_{x=0,1} = 0$ . Hence,

$$\begin{aligned} \langle \partial_t u^\varepsilon, v \rangle &= \langle \partial_t v, v \rangle = \frac{1}{2} \int_0^1 v^2(T, x) dx, \\ \langle \partial_x [J(u^\varepsilon)], v \rangle &= -\langle J(u) - J(1), \partial_x v \rangle \geq -M \langle |u - 1|, |\partial_x v| \rangle \\ &= -M \langle v, |\partial_x v| \rangle \geq -\varepsilon \|\partial_x v\|_{L^2(\Sigma_T)}^2 - \frac{M}{4\varepsilon} \|v\|_{L^2(\Sigma_T)}^2, \\ \varepsilon \langle \partial_x^2 u^\varepsilon, v \rangle &= \varepsilon \langle \partial_x^2 v, v \rangle = -\varepsilon \|\partial_x v\|_{L^2}^2. \end{aligned}$$



Testing (6.1) with  $v$ , as  $\langle G(\cdot, \cdot, u), v \rangle \geq 0$ ,

$$\int_0^1 v^2(T, x) dx \leq \frac{M}{2\varepsilon} \|v\|_{L^2(\Sigma_T)}^2 = \frac{M}{2\varepsilon} \iint_{\Sigma_T} v^2(t, x) dx dt.$$

Grönwall’s inequality shows that  $v(T, \cdot) \equiv 0$ , i.e.,  $u^\varepsilon(T, \cdot) \leq 1$  almost everywhere. Similar argument with  $v = (u^\varepsilon)^-$  shows that  $u^\varepsilon(T, \cdot) \geq 0$ . Since  $T$  appeared above can be replaced with any  $t \in (0, T)$ ,  $0 \leq u^\varepsilon(t, x) \leq 1$  for all  $(t, x) \in \Sigma_T$ .

If  $J$  is not globally Lipschitz continuous, construct  $J_*$ , such that  $J_* = J$  on  $[0, 1]$  and  $J_*$  is globally Lipschitz continuous. The above proof shows that we can replace  $J$  with  $J_*$  and the solution would not be affected.  $\square$

**Lemma A.2.** *Recall that  $\|\cdot\|_{L^2(\Sigma_T)}$  is the  $L^2$  norm with respect to the Lebesgue measure, while  $\|\cdot\|_{L^2(\Sigma_T; \nu)}$  is the  $L^2$  norm with respect to  $d\nu = V(x) dx dt$ .*

$$\sup_{\varepsilon > 0} \left\{ \varepsilon \|\partial_x u^\varepsilon\|_{L^2(\Sigma_T)}^2 + \|u^\varepsilon - \varrho\|_{L^2(\Sigma_T; \nu)}^2 \right\} < \infty. \tag{A.1}$$

*Proof.* First, from condition (ii) in Theorem 2.15, the continuous function  $\varrho^\varepsilon$  satisfies (2.10), so that  $\varrho^\varepsilon(t, 0) = \alpha(t)$ ,  $\varrho^\varepsilon(t, 1) = \beta(t)$  for all  $t \in [0, T]$ . Also, thanks to (ii), it suffices to verify (A.1) with  $\varrho$  replaced by  $\varrho^\varepsilon$ .

Let  $w^\varepsilon = u^\varepsilon - \varrho^\varepsilon$ . As both  $u^\varepsilon$  and  $\varrho^\varepsilon$  are uniformly bounded,

$$\begin{aligned} \langle \partial_t u^\varepsilon, w^\varepsilon \rangle &= \frac{1}{2} \int_0^1 (u^\varepsilon - \varrho^\varepsilon)^2 \Big|_{t=0}^{t=T} dx + \langle \partial_t \varrho^\varepsilon, u^\varepsilon - \varrho^\varepsilon \rangle \\ &\geq -C(1 + \|\partial_t \varrho^\varepsilon\|_{L^2(\Sigma_T)}), \\ \langle \partial_x [J(u^\varepsilon)], w^\varepsilon \rangle &= \int_0^T [\mathcal{J}(\cdot, \cdot, u^\varepsilon)] \Big|_{x=0}^{x=1} dt + \langle \partial_x \varrho^\varepsilon, J(u^\varepsilon) \rangle \\ &\geq -C(1 + \|\partial_x \varrho^\varepsilon\|_{L^2(\Sigma_T)}), \end{aligned}$$

where  $\mathcal{J} = \mathcal{J}(t, x, u)$  is given by

$$\mathcal{J}(t, x, u) := \int_0^u w J'(w) dw - \varrho^\varepsilon(t, x) J(u).$$

Noting that  $w^\varepsilon|_{x=0,1} = 0$ ,

$$\begin{aligned} \langle \partial_x^2 u^\varepsilon, w^\varepsilon \rangle &= \langle \partial_x u^\varepsilon, -\partial_x w^\varepsilon \rangle = \langle \partial_x u^\varepsilon, \partial_x \varrho^\varepsilon \rangle - \|\partial_x u^\varepsilon\|_{L^2(\Sigma_T)}^2 \\ &\leq \frac{1}{2} \|\partial_x \varrho^\varepsilon\|_{L^2(\Sigma_T)}^2 - \frac{1}{2} \|\partial_x u^\varepsilon\|_{L^2(\Sigma_T)}^2. \end{aligned}$$

Testing the equation with  $w^\varepsilon$ , we get

$$\frac{\varepsilon}{2} \|\partial_x u^\varepsilon\|_{L^2(\Sigma_T)}^2 + \iint_{\Sigma_T} V(x) (w^\varepsilon)^2 dx dt \leq C(1 + \|\varrho^\varepsilon\|_{H^1(\Sigma_T)}).$$

The estimate holds since  $\varrho^\varepsilon$  is uniformly bounded in  $H^1(\Sigma_T)$ .  $\square$

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