Gran Sasso Science Institute

MATHEMATICS OF NATURAL, SOCIAL AND LIFE SCIENCES
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Cycle XXXI - AY 2015/2018

Nonlinear Schrödinger equations and quantum fluids non-vanishing at infinity: Incompressible limit and quantum vortices

PHD CANDIDATE

Lars Eric Hientzsch

PhD Thesis submitted
6 September 2019

ADVISOR

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Meinen Eltern

Preamble

This thesis concerns the mathematical analysis of some hydrodynamic models describing quantum fluids, namely fluids whose macroscopic behavior still exhibits quantum effects. The prototype model for such fluids is the quantum hydrodynamic (QHD) system arising as model in the description of phenomena like superfluidity, Bose-Einstein condensation (BEC), superconductivity, quantum plasmas and semi conductor devices. From a mathematical point of view, the QHD system is given by a compressible Euler system augmented by a stress tensor accounting for the quantum features in the fluid and which depends on the density and its derivatives. Stress tensors of this kind also appear in the theory of capillarity developed by Korteweg, one refers to these systems as Euler–Korteweg and Navier–Stokes-Korteweg systems when inviscid or viscous respectively. Motivated by the analysis of some physically relevant solutions like quantized vortices, we consider the system on the whole space complemented with non-zero boundary conditions at infinity. The Cauchy problem for finite and infinite energy weak solutions (including vortex solutions) is investigated. Our method relies on the equivalence between QHD systems and NLS type equations through the Madelung transforms and the polar factorization method that renders the equivalence rigorous for rough solutions. We are thus led to study the well-posedness theory in the energy space for a class of nonlinear Schrödinger equations with non-zero boundary conditions at infinity that we develop ad-hoc. Moreover, we consider the asymptotic behavior of weak solutions to the QHD system in a suitable scaling regime that is linked to the study of quantized vortices and can be interpreted as quantum counterpart of the low Mach number limit of classical fluid dynamics. The dispersion relation of acoustic waves turns out to be characterized by the Bogoliubov dispersion relation. In the scaling regime, the dynamics of vortex solutions can be asymptotically described by the Kirchhoff-Onsager ODE system.
Secondly, we study the quantum Navier-Stokes (QNS) equations that can be understood as a viscous regularization of the QHD system with density dependent viscosity tensor. Physically, it is motivated by applications in the modeling of dissipative quantum fluids and as a showcase model for capillary fluids. We introduce global existence of finite energy weak solutions of the quantum Navier-Stokes system with non-trivial far-field behavior. In contrast to the results for the QHD system, the analysis of the QNS system is entirely based on techniques from fluid dynamics. Finally, we investigate the low Mach number limit and prove strong convergence to weak solutions of the incompressible Navier-Stokes equations for general ill-prepared data.

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Contents

Introduction

This introduction aims to provide a (very) brief review of the physical theory of quantum fluids and its hallmarks accounting for the choice of the models considered in this thesis. Subsequently, we highlight the key features of the mathematical analysis and the author's contributions.

0.1 Review of the physical theory

The first experimental realization of liquefied helium at very low temperatures (around $T \sim 4K$) by Onnes 1908 [151] was rewarded with a Nobel prize and triggered an extensive research, both experimental and theoretical in low temperature physics. In 1938, Allen and Misener [5] and Kapitza [114] discovered independently that at a temperature $T = 2,17K$ liquid helium undergoes a second-order phase transition, superfluidity - a new state of matter - had been discovered. The critical temperature is called λ-point because of the shape of the specific heat capacity experiencing a peak at that temperature. Superfluid flow exhibits several striking properties among which inviscid flow through narrow capillaries and irrotationality below a critical speed. It was suspected that superfluidity is due to the quantum-mechanical properties of the fluid emerging macroscopically. In early attempts, London [137] proposed to explain this phenomena by a recent theory developed by Einstein to describe ideal Bose gases. In 1925, Einstein inspired by a work of Bose, had argued that for gases of non-interacting particles a phase transition occurs at very low temperatures. A macroscopically relevant fraction of the gas is condensed, namely the atoms occupy the state of lowest energy. By the time, the predicted critical temperature was way beyond reach of experimental methods. However, London [137] argued that superfluidity and Bose-Einstein condensation are linked even though only a small fraction ($\sim 10\%$) of superfluid helium is condensed and the system is strongly interacting while Einstein's theory deals with non-interacting gases. These observations inspired Tisza and Landau [117] to design the two-fluid model describing the hydrodynamic behavior of superfluid flow. The description consists in two interpenetrating fluid components, the normal one with density and velocity field (ρ_n, v_n) and the superfluid one characterized by (ρ_s, v_s). The total density of the fluid is given by $\rho = \rho_n + \rho_v$. The densities ρ_n, ρ_s depend on the temperature as follows. For temperatures above the λ-point, there is no superfluid component, the entire fluid is described as normal fluid, namely $\rho = \rho_n$. The superfluid density increases

while lowering the temperature and at the absolute zero $T = 0$ reaches the total density of the fluid $\rho = \rho_s$, the normal fluid density vanishes. A milestone in the theory was reached by Bogoliubov [34] in 1947; he introduced a microscopic theory for weakly interacting Bose gases based on the concepts of Bose-Einstein condensation. In particular, his theory elucidating the excitation spectrum of a weakly interacting Bose gas proved to be compatible with the criteria of superfluidity of Landau. Intense theoretical research has been carried out to both explore and interlink the phenomena of superfluidity and Bose-Einstein condensation. We stress that, even though superfluidity and Bose-Einstein condensation are closely connected, these are distinct phenomena; to illustrate their conceptual difference is way beyond the scope of this introduction and we refer the reader for instance to [160]. Continuing our short walk through the history of the physical discoveries, the next cornerstone of superfluid flow was predicted by Onsager [152] in 1949 and Feynman [80] in 1955: the appearance of quantized vortices. Their experimental discovery is due to Hall and Vinen [99] in 1956 and they have directly been observed by Packard and Sanders [156] in 1972.
The experimental validation of the Bose-Einstein condensation theory has only been realized in recent years. Indeed, in order to reach the temperature threshold of around $200nK$ for the occurrence of Bose-Einstein condensation of hydrogen atoms highly advanced laser-cooling techniques are required. Their development has been been rewarded with the Nobel prize in 1997. The realization of BEC has been achieved in 1995 by Cornell and Wieman (JILA) [64] and shortly after Ketterle (MIT) [116] earning them the Nobel prize of physics in 2001. This breakthrough has triggered a tremendous interest in both theoretical and experimental investigation of Bose-Einstein condensation. We refer the reader again to [160] for more details.

In the following, we sketch some of the cornerstones of the theory that will appear in mathematical context in the body of this thesis. In their seminal papers, Gross [90] and Pitaevskii [159], have independently shown that a weakly interacting Bose gas at zero temperature is characterized by an macroscopic order parameter ψ that satisfies the so called Gross-Pitavskii equation. The equation also well describes superfluid helium II close to the λ-point [87], however we emphasize that the coefficients have a different physical meaning. In the early stages of quantum mechanics, Madelung [138] argued that the linear Schrödinger equation admits an alternative hydrodynamic formulation by decomposing the wave-function in its magnitude and its phase. The velocity field is then associated to the phase gradient of the wave-function. This analogy has been exploited by Landau [117, 126] to investigate nonlinear phenomena in superfluidity. Starting from the Gross-Pitaevskii equation, one is led to a compressible Euler equation augmented by a dispersive stress tensor that sometimes is called quantum pressure term and describes forces due to the spatial variations of the magnitude of the wave-function on small scales. The velocity field is formally associated to the phase gradient and therefore describes an irrotational flow away from vacuum regions. The fact that the wave-function is single-valued leads to the quantization of circulation computed on a small contour around a vortex that can be seen as a topological defect, namely a jump in the phase. The classical pressure and the quantum pressure act on different length-scales, the former dominates if the density of the fluid varies at large length scales while the latter is relevant when the density varies on length scales proportional to the *healing length* of the condensate, typically of order 10^{-6}m for Bose-Einstein condensates and 10^{-10}m for superfluid helium [74, 160]. If one linearizes the hydrodynamic equations around a constant density equilibrium state, then it becomes clear that the quantum pressure alters the dispersion of sound waves. Indeed, for high frequencies or short wave-length, that is at small length scales up to order of the healing length, the dispersion is not linear but rather quadratic. This discovery had been a milestone in the description of the theory, it is a key feature of the Bogoliubov theory of 1947. The dispersion relation characterizes elementary excitations in a weakly interacting Bose gas and creates a strong link with the Landau criterion of superfluidity based on elementary excitations in the superfluid. The core size of a quantized vortex is proportional to the healing length, a feature that together with the quantized circulation distinguishes the quantum vortex essentially from a classical vortex. Both properties are due to the restrictions imposed by the properties of the order parameter. Quantized vortices are believed to be the key in the understanding of quantum turbulence. In their conclusions of [113], the authors argue that since quantum turbulence is reduced to quantized vortices and their entanglements it might be more accessible in some respects compared to classical turbulence. At the same time, a better understanding of quantum turbulence is expected to yield some insight also to classical turbulence strongly motivating the study of hydrodynamic systems of quantum fluids. The dynamics of quantized vortices is completely different in two and three dimensions. While complicated entanglement of vortex filaments may happen in three dimensions, it is meaningful to think about vortices in two dimensions as point vortices given the order of the healing length [18]. If one assumes the healing length to be infinitesimally small compared to the distance between vortices, then it turns out that their dynamics is asymptotically described by the dynamics of classical point vortices through the Kirchhoff-Onsager law. To conclude, we mention that the QHD system has also been derived in various context beyond the description of superfluidity: it has e.g. been used for the description of quantum plasmas [98] and semi conductor devices [83] where the derivation of the QHD system is obtained by means of a momentum expansion of the Wigner-Boltzmann equation. We also point out that similar models were derived by taking into account further correction terms like for instance viscous tensors [91]. The derivation is based on a Wigner-Fokker-Planck model. The approach based on the Wigner equation provides an alternative mathematical derivation of various (dissipative) quantum fluid models including the quantum Navier-Stokes equations. We refer the reader to the review paper [111]. These considerations underline the contextualization of the considered models, namely QHD and QNS in the class of Euler–Korteweg and Navier–Stokes–Korteweg fluids that model capillary effects in fluids. Their name is borrowed from Korteweg who first introduced fluid dynamics equations augmented by stress tensor describing capillary effects [121].

0.2 Mathematical analysis

We introduce the models subject to our study and elucidate the main techniques used for their mathematical analysis. The results obtained in this thesis are highlighted. As already pointed out, the prototype model for quantum fluids, i.e. compressible, inviscid fluids subject to quantum (dispersive) effects is the quantum hydrodynamic system (QHD) that reads

$$\begin{cases} \partial_t \rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}\left(\frac{J \otimes J}{\rho}\right) + \nabla p(\rho) = \frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases} \tag{0.2.1}$$

The dynamics is considered on the whole space $\mathbf{R}^d$ for $d = 2, 3$ and equipped with non-vanishing boundary conditions at infinity

$$\rho(x) \to 1, \qquad |x| \to \infty. \tag{0.2.2}$$

The unknowns are the mass density ρ and the current density J, the pressure is denoted by $p(\rho)$. The boundary condition is motivated by applications such as Bose-Einstein condensation [90, 159, 160] and superfluidity close to the λ-point [87] described by the Gross-Pitaevskii theory. In addition, the non-zero boundary conditions give rise to rich behavior for the equations, namely a variety of special solutions in form of coherent structures. We mention vortex solutions [90, 159, 87], travelling waves [18] and dark solitons in nonlinear optics [122]. The nonlinear third-order dispersive tensor on the right-hand side takes in account for the quantum effects of fluid. Beyond superfluidity and BEC, the QHD system appears in various applications such as superconductivity [79] quantum plasmas [98] and in the modelling of semi conductor devices [83]. In a more general framework, the QHD system belongs to the class of Euler-Korteweg fluids describing inviscid capillary fluids. Capillarity effects in fluid flow are mathematically described by a stress tensor depending on the density and its derivatives. If additionally the phenomena is of viscous nature, we call these Navier-Stokes-Korteweg systems, namely

$$\begin{aligned} &\partial_t \rho + \operatorname{div}(\rho u) = 0, \\ &\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = 2\nu \operatorname{div}(\mathbb{S}) + \kappa^2 \operatorname{div}(\mathbb{K}), \end{aligned} \tag{0.2.3}$$

where the viscous stress tensor $\mathbb{S} = \mathbb{S}(\nabla u)$ is given by

$$\mathbb{S} = h(\rho)\mathbf{D}u + g(\rho) \operatorname{div} u\mathbf{I},$$

while the capillary term $\mathbb{K} = \mathbb{K}(\rho, \nabla\rho)$ reads

$$\mathbb{K} = \left(\rho \operatorname{div}(k(\rho)\nabla\rho) - \frac{1}{2}(\rho k'(\rho) - k(\rho)|\nabla\rho|^2)\right)\mathbb{I} - k(\rho)\nabla\rho \otimes \nabla\rho.$$

The capillary tensor is referred to as Korteweg tensor [121], see also [165, 166]. The family of Navier-Stokes-Korteweg equations has rigorously been derived in [75] and more recently in [101]. The quantum Navier-Stokes equation that we study below is obtained from (0.2.3) by setting $h(\rho) = \rho$, $g(\rho) = 0$ and $k(\rho) = \frac{1}{\rho}$. If we additionally set $\nu = 0$, we recover the QHD system. The results of this thesis can hence be interpreted in a framework that is not restricted to the modelling of quantum fluids but rather constituted by the family capillary fluids. Turning our attention to system (0.2.1), we observe that quantum correction term, sometimes also referred to as quantum pressure or Bohm potential, may - under suitable regularity assumptions - also be written in different ways

$$\frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \frac{1}{4}\nabla\Delta\rho - \operatorname{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}) = \frac{1}{4}\operatorname{div}(\rho\nabla^2\log\rho). \tag{0.2.4}$$

Further, we notice that system (0.2.1) is Hamiltonian, whose energy

$$\mathcal{E}(\rho, J) = \int \frac{1}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}\frac{|J|^2}{\rho} + F(\rho)\mathrm{d}x, \tag{0.2.5}$$

is formally conserved along the flow of solutions. The internal energy is characterized by the identity

$$F(\rho) = \rho\int_1^\rho \frac{p(s)}{s^2}\mathrm{d}s - (\rho - 1), \tag{0.2.6}$$

and incorporates the boundary conditions at infinity (0.2.2), such boundary conditions at infinity are usually called far-field behavior in literature. Our analysis is concerned with the physically relevant barotropic γ-pressure laws, i.e. $p(\rho) = \frac{1}{\gamma}\rho^\gamma$ with $\gamma > 1$ for $d = 2$ and $1 < \gamma < 3$ for $d = 3$ so that the internal energy equals

$$F(\rho) = \frac{1}{\gamma(\gamma-1)}(\rho^\gamma - 1 - \gamma(\rho - 1)).$$

From a mathematical point of view, system (2.0.1) is a compressible Euler system augmented by the nonlinear stress tensor (0.2.4) encoding the quantum effects. Heuristically, working with weak solutions of finite energy only provides a priori estimates on $\sqrt{\rho}u$ and $\nabla\sqrt{\rho}$ in $L^2(\mathbf{R}^d)$ and $\rho - 1$ in $L^\gamma_2(\mathbf{R}^d)$ that are not sufficient to define the velocity field u a.e. in $\mathbf{R}^d$. This is due to appearance of vacuum that on its turn can not be controlled. In general, the density ρ only has Sobolev regularity and the nodal set $\{\rho = 0\}$ might be quite irregular. Given the well-known difficulties in the analysis of weak solutions to compressible Euler equations, this suggests that an analysis by means of classical fluid dynamics equations faces various obstacles. To overcome these difficulties, one commonly exploits the analogy between system (0.2.1) and a family of nonlinear Schrödinger equations. This analogy was already established by Madelung [138] since the early days of quantum mechanics and has also been exploited by Landau [117, 126] to investigate nonlinear phenomena in superfluidity. Let us consider the following nonlinear Schrödinger equation of Gross-Pitaevskii type

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + F'(|\psi|^2)\psi, \tag{0.2.7}$$

where F is the internal energy (0.2.6) and complemented with non-trivial boundary conditions at infinity, i.e.

$$|\psi(x)| \to 1 \quad \text{as} \quad |x| \to \infty.$$

The Hamiltonian associated to the evolution equation (0.2.7) reads

$$\mathcal{E}(\psi) = \int_{\mathbf{R}^d} \frac{1}{2}|\nabla\psi|^2 + F(|\psi|^2)\mathrm{d}x, \tag{0.2.8}$$

For $\gamma = 2$, (1.1.1) reduces to the Gross-Pitaevskii equation [90, 159] and the Hamiltonian is given by the celebrated Ginzburg-Landau energy functional [88]. Coming back to the analogy with the hydrodynamic formulation, the approach based on the Madelung transform consists in expressing the wave function in terms of its amplitude and phase $\psi = \sqrt{\rho}e^{iS}$. This ansatz plugged in (0.2.7) and separating real and imaginary parts leads to the system

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\nabla S) = 0, \\ \partial_t S + \frac{1}{2}|\nabla S|^2 + f(\rho) = \frac{1}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}. \end{cases}$$

Defining the velocity field to be the phase gradient $u = \nabla S$, writing the equation for velocity u and multiplying by the mass density ρ yields

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = \rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right).$$

If we further define the current as $J = \rho u$, then (ρ, J) satisfies system (0.2.1). The Madelung transform is valid for smooth solutions away from vacuum, a feature that also constitutes its main drawback. Indeed the phase cannot be uniquely defined where the wave function vanishes. Ruling out the presence of vacuum excludes in particular quantized vortices being coherent structures appearing in the vacuum region of the fluid and one of the most striking phenomena in superfluidity and Bose-Einstein condensation, see e.g. [87, 159]. On the level of the wave-function dynamics, close to the vacuum the phase can only experience jumps by integer multiples of 2π in order for ψ to be single-valued. Consequently, if one considers a contour around a vacuum point, the circulation of the velocity field must be quantized, i.e.

$$\oint_C v \cdot \mathrm{d}\vec{l} = 2\pi\frac{\hbar}{m}n.$$

If $n \neq 0$, then the fluid has a quantized vortex in that vacuum point of degree (winding number) n and a quantum of vorticity is given by $\frac{\hbar}{m}$. Unfortunately, albeit giving a very accurate description of the specific phenomena, this approach is not well suited for rough solutions having only the regularity given by the finite energy framework. For this reason, in this thesis we follow the approach developed in [11], which provides a rigorous framework for finite energy weak solutions by using a polar factorisation method. The hydrodynamic variables $(\rho, \Lambda = \frac{J}{\sqrt{\rho}})$ are defined in a self-consistent way not limited by the appearance of vacuum regions. Global in time existence of finite energy weak solutions to (0.2.1) is proved in [11, 12] for vanishing

boundary conditions at infinity but for fairly general pressure laws and without any further smallness or regularity assumptions. For that purpose, the method in [11, 12] relies on the well-posedness of the Cauchy problem for the underlying effective wave-function dynamics. When (0.2.1) is studied with vanishing boundary conditions at infinity, the H^1-theory for the well-posedness of the corresponding nonlinear Schrödinger equation is satisfactorily well developed, see e.g. [52, 167]. When complemented with non-vanishing boundary conditions, the well-posedness in the respective energy space of (0.2.7) with nonlinearity $F'(|\psi|^2)\psi$ where F is defined by (0.2.6) is only known [84] for $\gamma = 2$. This motivates us to investigate the Cauchy Problem for the nonlinear Schrödinger equation with nonlinearity (0.2.6) and non-zero boundary conditions at infinity for the suitable range of γ. In Theorem 1.1.1, we prove the global well-posedness of the wave-function dynamics in the energy space, which in this context is not a Banach space - for instance in the three-dimensional setting it may rather be seen as an affine space. Consequently, the standard Kato method should be adapted to this situation. On the other hand, the considered equations exhibit a very rich dynamics described by special solutions. In this regards, our well-posedness results complements the known existence results of travelling waves [141, 58, 148, 59], vortex solutions [90, 159], vortex rings [29], dark solitons [122] to mention only some of them. Turning to the Cauchy Problem for (0.2.1), we also need to introduce a suitable polar decomposition for wave-functions in the energy space. As the stability properties of the polar decomposition are crucial, we need to rely on the metric of the energy space combined with a local argument compensating for the lack of integrability. Our first result states global existence of finite energy weak solutions to (0.2.1) with far-field behavior (0.2.2), see Theorem 2.1.3. The constructed solutions allow for vacuum regions and in particular satisfy a generalized irrotational condition that is compatible with the presence of quantized vortices. In a second moment, we aim to study solutions including quantized vortices. Such solutions are in general only locally of finite energy and hence not covered by Theorem 2.1.3. Based on the existence result for vortex solutions to the Gross-Pitaevskii equations in [32] and a further generalization of the polar factorization, our second result for the QHD system states global in time existence for a suitable class of weak solutions of infinite energy including vortex solutions, see Theorem 2.1.4.
The second model we study is given by the quantum Navier-Stokes describing a viscous regularization of the QHD system (0.2.1),

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = 2\nu \operatorname{div}(\rho \mathbf{D} u) + 2\kappa^2 \rho \nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right). \end{cases} \tag{0.2.9}$$

where $(t,x) \in [0,T) \times \mathbf{R}^d$ and complemented with the far-field behavior

$$\rho \to 1 \qquad \text{as} \quad |x| \to \infty. \tag{0.2.10}$$

The unknowns are the mass density ρ and the velocity field of the fluid u. We consider a barotropic flow with pressure law described by $p(\rho) = \frac{1}{\gamma}\rho^\gamma$. The energy associated to (0.2.9) reads

$$E(t) = \int_{\mathbf{R}^d} \frac{1}{2}\rho|u|^2 + 2\kappa^2|\nabla\sqrt{\rho}|^2 + F(\rho)\mathrm{d}x, \tag{0.2.11}$$

where the internal energy is given by (0.2.6). System (0.2.9) arises as model for a dissipative quantum fluid. Mathematically, it has been derived from a Chapman-Enskog expansion for the Wigner equation with a BGK term [46, 112], see also [111] where several dissipative quantum fluid models are derived by means of a moment closure of (quantum) kinetic equations with appropriate choices of the collision terms. We emphasize that the QNS system can feature as model for Navier–Stokes–Korteweg fluids. The far-field behavior is motivated by the study of singular limits and the analogy with its inviscid counterpart, the QHD system with non-trivial far-field. Indeed, in the inviscid limit, namely $\nu \to 0$, one formally recovers system (0.2.1), the vanishing viscosity limit has recently been investigated [40] to construct dissipative weak solutions to (0.2.1) as limit of dissipative weak solutions of (0.2.9). In the absence of capillary effects, namely $\kappa = 0$, the QNS system reduces to the compressible Navier-Stokes system with density dependent viscosity [38], see also [129, 170] for global existence of weak solutions. From a mathematical point of view, the analysis of viscous compressible fluids with density dependent viscosity is significantly different from the constant viscosity case. Indeed, by a formal computation one is led to the energy inequality,

$$E(t) + 2\nu\int_0^T\int_{\mathbf{R}^d} \rho|\mathbf{D}u|^2\mathrm{d}x\mathrm{d}t \leq E(0).$$

We notice that the dissipation is degenerate and hence the natural uniform estimates seem not to be sufficient to define and control the velocity u and its gradient ∇u a.e. on $\mathbf{R}^d$ due to the appearance of vacuum. Contrarily to that, when dealing with constant densities, the energy inequality yields a L^2 bound for ∇u that allows one to define the velocity field a.e. on $\mathbf{R}^d$. By consequence, the Lions-Feireisl theory, see [135, 77] does not apply in the context of compressible fluids with density dependent viscosity. The loss of control for the velocity field and the positivity of the density is somehow reminiscent to what we observed for the QHD system. However, the analogy to nonlinear Schrödinger equations is lost when introducing viscosity and we need to implement fluid dynamics techniques. In the context of Navier-Stokes-Korteweg fluids, this lack of uniform bounds has been compensated for by the introduction of a new entropy [38, 39], the Bresch-Desjardins entropy valid provided that the viscosity and capillary tensors enjoy a particular structure. The constraint on the viscosity tensor usually reads $g(\rho) = \rho h'(\rho) - h(\rho)$. In recent years, considerable effort has been spent in the mathematical analysis of Korteweg fluids and in generalizing the entropy methods, see [37, 40, 42, 14] and references therein. Global in time existence of finite energy weak solutions to the QNS system on the torus $\mathbf{T}^d$ for $d = 2, 3$ has been obtained in [13] and [125] following different approaches. The former rather relies on the structure of the equation and requires a constraint on the viscosity coefficient w.r.t. to the capillary coefficient while the latter uses a truncated formulation of the equations. The solutions to the truncated system enjoy suitable stability properties by means of which global weak solutions of the QNS system are

constructed. By implementing an invading domains approach based on the result of [125], we show that there exists global finite energy weak solutions of (0.2.9) on $\mathbf{R}^d$ for $d = 2, 3$ and with far-field behavior (0.2.10), see Theorem 3.1.2. As byproduct, we can show the same result for the compressible Navier-Stokes equations with density dependent viscosity, see Corollary 3.1.3. Further, we recover the analogue of [13, 125] on the whole space, namely global in time existence of finite energy weak solutions to QNS with vanishing boundary conditions at infinity, see Theorem 3.1.4. For our method, the construction of suitable periodic initial data on the sequence of invading tori is crucial, in particular in view of the far-field behavior (3.0.2). In this context, we remark that the density dependent viscosity and capillary effects allow for an control of the density in Sobolev norms that proves useful in the analysis of the Cauchy Problem as it provides better information on its far-field behavior compared to the constant viscosity case with far-field described in [135].
Next, we investigate the low Mach number limit for the QNS system. We denote by ε the scaled Mach number, so that in the suitable scaling regime system (0.2.9) reads,

$$\begin{cases} \partial_t\rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \frac{1}{\varepsilon^2}\nabla P(\rho_\varepsilon) = 2\nu \operatorname{div}(\rho_\varepsilon \mathbf{D} u_\varepsilon) + 2\kappa^2\rho_\varepsilon\nabla\left(\frac{\Delta\sqrt{\rho_\varepsilon}}{\sqrt{\rho_\varepsilon}}\right), \end{cases} \tag{0.2.12}$$

The rescaled internal energy is given by

$$F_\varepsilon = \frac{\rho^\gamma - 1 - \gamma(\rho - 1)}{\varepsilon^2\gamma(\gamma - 1)}.$$

In the low Mach number regime, i.e. in the limit as $\varepsilon \to 0$, the dynamics of (4.0.3) is formally governed by the incompressible Navier-Stokes equations,

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu\Delta u, \qquad \operatorname{div} u = 0. \tag{0.2.13}$$

For general ill-prepared initial data of finite energy, we prove strong convergence of finite energy weak solutions towards weak solutions of the incompressible Navier Stokes equations, see Theorem 4.1.2. We deal with a general class of weak solutions which are not smooth enough to allow the use of relative entropy techniques [76]. It is crucial to exploit the bounds yielded by the energy estimates and the Bresch-Desjardins entropy methods [38]. The presence of the Korteweg tensor in system (0.2.12) also affects the low Mach number limit through the analysis of the acoustic waves. Indeed, denoting the density fluctuations $\sigma_\varepsilon = \frac{\rho_\varepsilon - 1}{\varepsilon}$, momentum $m_\varepsilon = \rho_\varepsilon u_\varepsilon$ and linearizing around the constant density state 1 we obtain the linear system

$$\begin{cases} \partial_t\sigma_\varepsilon + \frac{1}{\varepsilon}\operatorname{div}(m_\varepsilon) = 0, \\ \partial_t(m_\varepsilon) + \frac{1}{\varepsilon}\nabla\left(1 - \kappa^2\varepsilon^2\Delta\right)\sigma_\varepsilon = G_\varepsilon, \end{cases}$$

that can be symmetrized by considering

$$\tilde{\sigma}_\varepsilon = (1 - \varepsilon^2\kappa^2\Delta)^{\frac{1}{2}}\sigma_\varepsilon, \qquad \tilde{m}_\varepsilon = (-\Delta)^{-\frac{1}{2}} \operatorname{div} m_\varepsilon,$$

so that

$$\begin{cases} \partial_t \tilde{\sigma_\varepsilon} + \frac{1}{\varepsilon}(-\Delta)^{\frac{1}{2}}(1-\kappa^2\varepsilon^2\Delta)^{\frac{1}{2}}\tilde{m_\varepsilon} = 0, \\ \partial_t \tilde{m_\varepsilon} - \frac{1}{\varepsilon}(-\Delta)^{\frac{1}{2}}(1-\kappa^2\varepsilon^2\Delta)^{\frac{1}{2}}\tilde{\sigma_\varepsilon} = \tilde{G_\varepsilon} \end{cases} \tag{0.2.14}$$

It turns out that $(\tilde{\sigma_\varepsilon}, \tilde{m_\varepsilon})$ controls $(\sigma_\varepsilon, \mathbf{Q}(m_\varepsilon))$ where $\mathbf{Q}$ denotes the Leray-Helmholtz projection on the irrotational part and is therefore sufficient for the control of the decay of acoustic waves. Further, the evolution of system (0.2.14) is characterized by a semi-group operator e^{-itH_ε} where $H_\varepsilon = \frac{1}{\varepsilon}\sqrt{(-\Delta)(1-\varepsilon^2\kappa^2)\Delta}$. Hence, the dispersion of acoustic waves is enhanced by the capillary tensor. As the viscosity does not alter the linearized system, it agrees with the one for the QHD system. This explains why the dispersion relation associated to H_ε is given by the scaled Bogoliubov dispersion relation. Compared to a wave-like dispersion in a fluid without capillary effects, the acoustic dispersion is enhanced and captured by suitable Strichartz estimates. The analysis related to the scaled Bogoliubov dispersion relation can be regarded as the $\varepsilon-$version of the results in [95]. In particular, we stress that since the dispersion relation is not homogeneous, we cannot obtain our estimates just by a rescaling argument and we need to adapt the proof in [95]. Once we have a satisfactory control of the acoustic dispersion, the uniform estimates deriving from energy and entropy methods provide the compactness leading to the strong convergence towards a weak solution to the incompressible Navier–Stokes equation. If the initial data are prepared in such a way that the formation of an initial layer is ruled out, then we obtain convergence towards a Leray solution, see Proposition 4.1.4.

Finally, we study the counter-part of the low Mach number limit also for the QHD system (0.2.1). The equations (0.2.1) and the analogue (0.2.7) as stated are given in the dimensionless scaling where the characteristic spatial length scale over which the density varies is referred to as *healing length* ξ and is typically very small. As already introduced in the previous paragraph, the healing length describes the length scale of density variations for which the quantum pressure is relevant and also determines the core size of a quantized vortex - constituting one of the essential differences to a vortex in a classical fluid. To illustrate this, we notice that the prototype of a vortex solution is given by

$$\sqrt{\rho} = f(|x|), \qquad \Lambda = 2\pi\kappa f(|x|)\frac{x^\perp}{|x|^2},$$

where f is a smooth radial profile such that $f(0) = 0$ and $f(|x|) \to 1$ as $|x| \to \infty$. The properties of the radial profile are well-known [103], the profile f approaches 1 at a distance proportional to ξ away from the core $f(0) = 0$. We consider the scaling regime in which the vortex core size being proportional to the healing length is small compared to the distance between vortices. This motivates to set $\varepsilon = \xi$ and to scale (0.2.7) by $x' = \frac{x}{\varepsilon}$ and $t' = \frac{t}{\varepsilon^2}$. On the level of the wave-function dynamics, this limit is fairly well-understood [28]. The method relies on refined estimates on the Jacobian of the wave-function that turns out to equal the vorticity $\nabla \wedge J$ modulo a factor 2. It is well-known that for well-prepared vortex configurations the ε-limit is characterized by the Kirchhoff-Onsager ODE system [140] for point vortices for $d = 2$. Ginzburg-Landau vortices in the ε-limit have extensively been studied in various different contexts see e.g. [25, 2, 162, 63, 133, 145, 105] and references therein. On the hydrodynamic level, the scaled QHD system (0.2.1) reads,

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div} J_\varepsilon = 0 \\ \partial_t J_\varepsilon + \operatorname{div} (\Lambda_\varepsilon \otimes \Lambda_\varepsilon) + \dfrac{1}{\varepsilon^2} \nabla p(\rho_\varepsilon) = \dfrac{1}{2} \rho_\varepsilon \nabla \left(\dfrac{\Delta \sqrt{\rho_\varepsilon}}{\sqrt{\rho_\varepsilon}} \right), \end{cases} \tag{0.2.15}$$

and is reminiscent of the low Mach number scaling for classical fluid. The scaled energy functional is given by

$$\int_{\mathbf{R}^d} \frac{1}{2} |\nabla \sqrt{\rho_\varepsilon}|^2 + \frac{1}{2} |\Lambda_\varepsilon|^2 + \frac{1}{\varepsilon^2} F(\rho_\varepsilon) \mathrm{d}x. \tag{0.2.16}$$

If the system (0.2.16) is linearized around the constant density state $\rho = 1$, we recover the analogue of (0.2.14). Here $\kappa = 1$ due to the chosen scaling and the dispersion relation for acoustic waves is precisely given by the Bogoliubov dispersion relation corresponding to H_ε. The augmented dispersion relation has its origin in the presence of the quantum pressure term modifying the dispersive behavior for high wave-numbers or respectively at small length-scales. The threshold wave-number at which the behavior undergoes a change from linear to quadratic dispersion is given by $\frac{1}{\varepsilon} = \frac{1}{\xi}$ that corresponds to the observation that the quantum pressure is relevant on small length scales of order ξ. Heuristically, the density is expected to be almost constant in the ε-limit and the dynamics can be asymptotically described by the incompressible Euler equation. We rigorously show that as $\varepsilon \to 0$, finite energy weak solutions to (0.2.15) converge to the trivial solution $(\rho = 1, u = 0)$, see Theorem 5.1.1. This is due to the fact that the solutions for the QHD system, consistent with the physical theory, satisfy a generalized irrotational condition. This rules out any rotational flow in ε-limit. Further for $d = 2$, we study the ε-limit for the dynamics of vortices. Here, we rephrase the result of [28] in the hydrodynamic framework yielding that suitable well-prepared vortex configurations are asymptotically described by the Kirchhoff-Onsager ODE system for point vortices.

0.3 Outline of the thesis

The thesis is divided in two parts; the first one is dedicated to the analysis of the Cauchy Problems while in the second we investigate the low Mach number limit for the QNS system and its quantum counter-part for the QHD system.

Chapter 1 is devoted to the well-posedness theory of a class of nonlinear Schrödinger equations with non-vanishing boundary conditions at infinity. The result is based on a joint work in progress with P. Antonelli and P. Marcati. Beyond the well-posedness theory, we review known results of special solutions and large time asymptotics that complement our analysis of the Cauchy problem. Finally, we review the existence result of vortex-like solutions to the Gross-Pitaevskii equation proved in [32].

In Chapter 2, we investigate the Cauchy Problem of the quantum hydrodynamic (QHD) system posed on $\mathbf{R}^d$ for $d = 2, 3$ with non-trivial far-field behavior. We introduce global in time

existence of finite and infinite energy weak solutions. Our analysis exploits the well-posedness theory for the nonlinear Schrödinger equations introduced in the preceding chapter. The last part of the chapter is dedicated to special solutions to the QHD system such as travelling waves and vortex solutions. The latter can be considered in the framework of the existence result for weak solutions of infinite energy. These results have been announced in [10] and are obtained in joined work with P. Antonelli and P. Marcati.
Chapter 3 introduces the global in time existence of finite energy weak solutions of the Quantum Navier-Stokes (QNS) equations and is fruit of a work in progress with P. Antonelli and S. Spirito.
Chapter 4 mainly follows [9]. We discuss the low Mach number limit for finite energy weak solutions for general ill-prepared data and prove strong convergence to weak solutions of the Navier-Stokes equations. For well-prepared data, convergence to Leray solutions is achieved. The result is also to appear in the proceeding [8].
The final Chapter 5 concerns the scaling limit for the QHD system that is suitable for the study of the vortex dynamics and somehow reminiscent of the low Mach number limit in classical fluids. We show that any finite energy weak solution, namely a solution without vortices, converges to the constant state ($\rho = 1, u = 0$). In a second moment, we rephrase the result of [28] in the hydrodynamic framework provided by the QHD system.

0.4 Notations

We list the notations of function spaces and operators used in the following. We denote

- by $\mathcal{D}(\mathbf{R}_+ \times \mathbf{R}^3)$ the space of test functions $C_0^\infty(\mathbf{R}_+ \times \mathbf{R}^3)$ and by $\mathcal{D}'(\mathbf{R}_+ \times \mathbf{R}^3)$ the space of distributions. The duality bracket between $\mathcal{D}$ and $\mathcal{D}'$ is denoted by $\langle \cdot, \cdot \rangle$,

- for $0 < T \leq \infty$ by $L^p(\mathbf{R}^d)$ for $1 \leq p \leq \infty$ the Lebesgue space with norm $\|\cdot\|_{L^p}$ and by $L^p(0,T;L^q(\mathbf{R}^d))$ the space of functions $u : (0,T) \times \mathbf{R}^d \to \mathbf{R}^n$ with norm
$$\|u\|_{L^pL^q} = \left(\int_0^T \left| \int_{\mathbf{R}^d} |u(t,x)|^q \right|^{\frac{p}{q}} \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{p}}.$$
If $T = \infty$, we write $L^p(\mathbf{R}_+; L^q(\mathbf{R}^d))$. Further, we denote by $L^{p-}(0,T;L^q(\mathbf{R}^d))$, the functions f such that $f \in L^{p'}(0,T;L^q(\mathbf{R}^d))$ for any $1 \leq p' < p$.

- the non-homogeneous Sobolev space by $W^{k,p} = (I - \Delta)^{-\frac{k}{2}} L^p(\mathbf{R}^d)$ and $H^k(\mathbf{R}^d) = W^{k,2}(\mathbf{R}^d)$. Its dual will be denoted by $W^{-k,p'}$ with p' being the Hölder conjugate of p. The homogeneous spaces are denoted by $\dot{W}^{k,p}(\mathbf{R}^d) = (-\Delta)^{-\frac{k}{2}} L^p(\mathbf{R}^d)$ and $\dot{W}^{k,p}(\mathbf{R}^d) = \dot{H}^k(\mathbf{R}^d)$, and the dual space $\dot{W}^{-k,p'}$,

- by $X^k(\mathbf{R}^d)$ the Zhidkov space [175] defined for $k \in \mathbf{N}$ as
$$X^k(\mathbf{R}^d) = \left\{ f \in L^\infty \, : \, \nabla^\alpha f \in L^2(\mathbf{R}^d) \; \forall |\alpha| \leq k \right\},$$

- by $L^p_2(\mathbf{R}^d)$ the Orlicz space defined as
$$L^p_2(\mathbf{R}^d) = \left\{ f \in L^1_{loc} \,:\, |f|\chi_{\{|f|\leq\frac{1}{2}\}} \in L^2(\mathbf{R}^d),\, |f|\chi_{\{|f|\geq\frac{1}{2}\}} \in L^p(\mathbf{R}^d) \right\},$$
we refer to [1, 135] for details.

- by $B^s_{q,r}(\mathbf{R}^d)$ the non-homogeneous Besov space and by $\dot{B}^s_{q,r}$ the homogeneous Besov space, see [22]. We denote by $B^{-s}_{q',r'}$ the dual space of $B^s_{q,r}$,

- by $\mathbf{P}$ and $\mathbf{Q}$ the Helmholtz–Leray projectors on divergence-free and gradient vector fields, respectively:
$$\mathbf{Q} = \nabla\Delta^{-1}\,\mathrm{div}, \qquad\qquad \mathbf{P} = \mathbf{I} - \mathbf{Q},$$
For $f \in W^{k,p}(\mathbf{R}^d)$ with $1 < p < \infty$ and k the operators $\mathbf{P}, \mathbf{Q}$ can be expressed as composition of Riesz multipliers and are bounded linear operators on $W^{k,p}(\mathbf{R}^d)$.

- the Fourier transform of f by $\hat{f} := \mathcal{F}(f)$ and the inverse Fourier transform by $f^\vee$.

- the frequency cut-off $P_N(f) = (\phi_N(\xi)\hat{f})^\vee$, where ϕ is a smooth frequency cut-off compactly supported in $supp(\phi) \subset \{\frac{1}{2}N \leq |\xi| \leq 2N\}$. Similarly, by $P_{\leq N}(f)$ we denote the projection on frequencies of order $|\xi| \leq N$.

- finally the symmetric part of the gradient is denoted by $\mathbf{D}u = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and the asymmetric part by $\mathbf{A}u = \frac{1}{2}(\nabla u - (\nabla u)^T)$

In what follows C will be any uniform constant independent from δ, ε or n.

Part I

Cauchy Theory

CHAPTER 1

Nonlinear Schrödinger equations with non-vanishing boundary conditions at infinity

Abstract

We introduce a well-posedness theory in the energy space for a class of nonlinear Schrödinger equations with non-vanishing boundary conditions at infinity. Compared to previous results [84, 82], we relax the regularity assumptions on the nonlinearity significantly. Section 1.1 concerns the well-posedness and is composed by several parts. In a first moment, we review the properties of the energy space and the action of the Schrödinger group on the energy space. Secondly, we investigate the problem for $d = 2$ and subcubic nonlinearities for $d = 3$. Thirdly, we study supercubic nonlinearities for $d = 3$ that require a different approach exploiting the particular structure of the energy space for $d = 3$. In Section 1.2, we briefly review known results regarding travelling waves and scattering. Section 1.3 recalls the existence result of vortex-like solutions of infinite energy introduced in [32].

This chapter presents a joint work with P. Antonelli and P. Marcati and is devoted to study Cauchy problem for a class of nonlinear Schrödinger equations, namely we consider

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + F'(|\psi|^2)\psi, \tag{1.0.1}$$

with non-trivial boundary conditions at infinity, i.e.

$$|\psi(x)| \to 1 \quad \text{as} \quad |x| \to \infty.$$

The boundary condition is motivated by a variety of applications. System (1.0.1) arises as model for superfluidity in Helium II [126] close to the λ-point [87, 159] and for quantum vortices [23, 159] as well as in the description of Bose-Einstein condensates [90, 159, 89, 160]. Moreover, it has served to investigate dark solitons in nonlinear optics [120, 122, 158]. For a more detailed introduction to the physical phenomena described by (1.0.1) we refer the reader to the Introduction of this thesis. Here, our aim is to provide a well-posedness theory in the energy space for (1.0.1) for a general class of nonlinearities developed ad hoc for its application to the existence theory of the hydrodynamic system (2.0.1) exposed in the subsequent chapter. More precisely, we consider (1.0.1) as Hamiltonian equations associated to the Hamiltonian

$$\mathcal{E}(\psi) = \int_{\mathbf{R}^d} \frac{1}{2}|\nabla\psi|^2 + F(|\psi|^2)\mathrm{d}x, \tag{1.0.2}$$

where is given by (0.2.6). In this respect, the internal energy is chosen such that it is non-negative, convex, has minimum at $\rho = 1$ and satisfies

$$p(\rho) = F'(\rho)\rho - F(\rho) = \frac{1}{\gamma}\rho^{\gamma}.$$

Here, $\gamma > 1$ for $d = 2$ and $1 < \gamma < 3$ for $d = 3$. In short, the choice for (1.0.2) corresponds to the hydrodynamic system considered with a γ-pressure law. The well-posedness result for (1.0.1) will yield the existence of finite energy weak solutions to (2.0.1) discussed in Chapter 2. We retain that the well-posedness theory of (1.0.1) with the given nonlinearity F' is of interest beyond its applications to the Cauchy Theory of the QHD system. Regarding the mathematical analysis of the problem, the loss of integrability of ψ requires a Cauchy theory for initial data in the energy space. For the Gross-Pitaevskii equation, namely for $\gamma = 2$, a local in time result in Zhidkov spaces has been introduced in [174] for $d = 1$, see also [175], and in [81] in the multi-dimensional case. In [30] it had been proven that the Gross-Pitaevskii equation is well-posed in $1+H^1(\mathbf{R}^d)$ for $d = 2, 3$. Global well-posedness in the energy space has been introduced in [84]. Subsequently, in [82, 148] a more general class of C^3-nonlinearities, respectively C^2-nonlinearities, has been studied. In [82], the author decomposes the initial datum ψ_0 in the energy space as sum of a fixed smooth bounded function ϕ element of the energy space and a H^1-datum and develops an H^1-theory for the affine problem $\phi + w$ that ultimately yields the well-posedness result in the energy space. The class of nonlinearities associated to (1.0.2) is such that its gradient is not locally Lipschitz for γ close to 1 and therefore requires a different approach for both local and global theory. The bound $\gamma < 3$ is due to the fact that the equation becomes energy critical for $\gamma = 3$ and $d = 3$. For $d = 3$, equation (1.0.1) equipped with a cubic-quintic nonlinearity, thus energy critical, is shown to be well-posed in [118]. The cubic-quintic model is of great physical interest in nonlinear optics [120]. In the limit case $\gamma = 1$, the nonlinearity in (1.0.1) becomes logarithmic. To the best of our knowledge, well-posedness in the energy space for (1.0.1) complemented with non-vanishing boundary conditions at infinity has not been solved. For vanishing boundary conditions at infinity, the Cauchy Problem has been addressed in [53, 52] and more recently in [51].

Our approach is inspired by the techniques developed in [84] and [82] combined with some new ingredients dealing with the low regularity of the nonlinearity. To that end, we proceed in two steps. We firstly deal with the case $d = 2$ and subcubic nonlinearities in $d = 3$. In this regime, we are able to perform the local well-posedness theory directly in the energy space. Our approach does not require a decomposition as in [82], we exploit the properties of the energy space and the action of the Schrödinger semigroup on the energy space as discussed in [84] combined with classical tools for nonlinear Schrödinger equations. This allows us to perform the proof directly in the energy space. In a second moment, we deal with the case $d = 3$ and $2 \leq \gamma < 3$. In this setting, we crucially rely on the special structure of the energy space in the three-dimensional setting that has been pointed out in [84]. For $d = 3$ the energy space can be up to a phase shift identified with $1 + \dot{H}^1(\mathbf{R}^3)$ so that one may work in the affine space $1 + \dot{H}^1(\mathbf{R}^3)$ and develop a suitable $\dot{H}^1(\mathbf{R}^3)$-theory by means of Strichartz estimates. The existence of finite energy solutions that do not belong to $1 + L^2(\mathbf{R}^d)$, see for example the travelling waves constructed in [57], further motivates the well-posedness theory in the energy space.

After setting up the well-posedness theory, we review some known results concerning special solutions and large time behavior of (1.0.1). In sharp contrast to its counterpart with vanishing conditions at infinity, the defocusing nonlinear Schrödinger equation for which scattering in the energy space is known [86], system (1.1.1) admits a variety of special solutions. In [59, 142] the authors show that for any speed $0 < c < v_s$, where v_s denotes the sound speed, there exist travelling wave solutions to (1.0.1). For $d = 2$, these are in particular minimizers of the energy functional for solutions under a prescribed constraint on the total momentum [59] and therefore rule out a scattering theory. The situation for $d = 3$ is different. There are no travelling waves below a certain energy threshold for (1.0.1), see [141] and at least for $\gamma = 2$, it has been shown that under suitable regularity and smallness assumptions solutions exhibit scattering, see [95, 96, 97] and the more recent paper [92]. Moreover, for $\gamma = 2$ equation (1.1.1) admits vortex solutions. In this regard, for the convenience of the reader in view of its application to the QHD system we briefly recall the main results of [32] in Section 1.3.

This chapter is organized as follows. Section 1.1 tackles the well-posedness issue for equation (1.1.1). We start by introducing some preliminary results regarding nonlinear Schrödinger equations and recall the key properties of the energy space in Section 1.1.1. The local in time Cauchy theory follows a different approach for $d = 2$ and $d = 3$ for subcubic nonlinearities presented in Section 1.1.2 and for supercubic nonlinearities in $d = 3$ provided in Section 1.1.4. We discuss special solutions and large time behavior in Section 1.2.1. Finally, we consider vortex solutions in Section 1.3 reviewing the main result of [32].

Preliminary results

This paragraph introduces some notation and recalls basic facts regarding the Cauchy theory for (non)linear Schrödinger equations. We refer the reader to the monographs [52] and [167] for more details. We consider the Cauchy Problem

$$\begin{cases} i\partial_t\psi = -\frac{1}{2}\Delta\psi + \mathcal{N}(\psi), \\ \psi(0,x) = \psi_0, \end{cases} \tag{1.0.3}$$

for a given nonlinearity $\mathcal{N}$ and initial data ψ_0 in the energy space. We consider (1.0.3) for a class of nonlinearities such that it describes the evolution equation of the Hamiltonian (1.0.2). The problem can be reformulated in an equivalent way [52] as integral equation, that we refer to as *Duhamel formula*,

$$\psi(t) = e^{\frac{i}{2}t\Delta}\psi_0 - i\int_0^t e^{\frac{i}{2}(t-s)\Delta}\mathcal{N}(\psi)(s)\mathrm{d}s,$$

where $e^{\frac{i}{2}t\Delta}$ denotes the semigroup operator characterizing the linear Schrödinger evolution. Given a metric space X and initial data $\psi_0 \in X$ we say that ψ is a local strong solution if there exists $T > 0$ such that $\psi \in C([-T,T];X)$ and the integral equation (1) is well-defined and satisfies for all $t \in [-T,T]$. We aim to provide local and global well-posedness for (1.0.3) in the energy space. The Cauchy Problem is locally well-posed in a space X if for any $T > 0$ there exists a unique local in time strong solutions such that the blow-up alternative is satisfied and the solutions depends continuously on the initial data w.rt. to a suitable topology. We say that the problem is globally well-posed if T can be arbitrarily large. The precise notion of well-posedness in the energy space we use in this chapter is quite technical and introduced by Theorem 1.1.1.

A fundamental tool for the Cauchy theory of nonlinear Schrödinger and dispersive equations in general are the well-known Strichartz estimates. The local existence result relies on Strichartz estimates. We say that a pair (q,r) is (Schrödinger) admissible if $q, r \geq 2$ such that

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \qquad (q,r,d) \neq (2,\infty,2),$$

and we recall the well-known Strichartz estimates, see [115] and references therein.

Lemma 1.0.1. *Let $d = 2,3$ and (q,r) be an admissible pair. Then the linear propagator satisfies,*

$$\|e^{\frac{i}{2}t\Delta}u\|_{L^q([-T,T];L^r(\mathbf{R}^d))} \leq C\|u\|_{L^2(\mathbf{R}^d)},$$

and for any (q_1,r_1) admissible pair one has

$$\left\|\int_0^t e^{\frac{i}{2}(t-s)\Delta}f(s)\mathrm{d}s\right\|_{L^q([-T,T];L^r(\mathbf{R}^d)} \leq C\|f\|_{L^{q_1'}([-T,T];L^{r_1'}(\mathbf{R}^d))}. \tag{1.0.4}$$

Given a time interval $I = [-T,T]$, it is convenient to introduce the Strichartz space $S^0(I \times \mathbf{R}^d)$ characterised by the norm

$$\|u\|_{S^0} := \sup_{(q,r) admissible} \|u\|_{L^q(I;L^r(\mathbf{R}^d))}.$$

We notice that since $(q,r) = (\infty,2)$ is admissible one has

$$\|u\|_{C(I;L^2(\mathbf{R}^d))} \lesssim \|u\|_{S^0}. \tag{1.0.5}$$

Moreover, we introduce the dual space $N^0 = (S^0(I \times \mathbf{R}^d))^*$ satisfying the estimate

$$\|f\|_{N^0} \lesssim \|f\|_{L^{q_1'}(I;L^{r_1'}(\mathbf{R}^d))}, \tag{1.0.6}$$

for any admissibile pair (q_1,r_1). Further, in order to discuss the well-posedness theory for (1.1.59) in the energy space, we also work with the function space $S^1(I \times \mathbf{R}^d)$ and $N^1(I \times \mathbf{R}^d)$ defined by the norms

$$\|u\|_{S^1} = \|u\|_{S^0} + \|\nabla u\|_{S^0}, \qquad \|F\|_{N^1} = \|F\|_{N^0} + \|\nabla F\|_{N^0}. \tag{1.0.7}$$

With the Strichartz estimates at hand, the typical scheme for a fixed-point argument for H^1-theory reads as follows. Firstly, one observes that for u defined by the Duhamel formula and $u_0 \in H^1$ one has,

$$\|u\|_{S^1} \lesssim \|u_0\|_{H^1} + \|\mathcal{N}\|_{N^1}.$$

The problem is thus reduced to a suitable control of the nonlinearity $\mathcal{N}$. Secondly, we show that the solution map defines a contraction on a ball of S^1.
The present setting requires several modifications due to the fact that the energy space is not included in any L^p space due to the non-vanishing boundary conditions at infinity. More precisely, we consider wave-functions such that ψ lies in a subspace of $X^1(\mathbf{R}^d) + H^1(\mathbf{R}^d)$, where $X^k(\mathbf{R}^d)$ for $k \in \mathbf{N}$ denotes the Zhidkov space, see Section 0.4 and [175]. For k positive integer, the space X^k is defined as

$$X^k(\mathbf{R}^d) := \left\{\psi \in L^\infty(\mathbf{R}^d) \quad : \quad \nabla^\alpha \psi \in L^2(\mathbf{R}^d),\ \forall\, 1 \leq |\alpha| \leq k\right\}.$$

For a sum of Banach spaces, the norm is defined by

$$\|\psi\|_{X^1+H^1} = \inf\left\{\|\psi_1\|_{X_1} + \|\psi_2\|_{H^1} \quad : \quad \psi = \psi_1 + \psi_2\right\}.$$

The space $X^1(\mathbf{R}^d) + H^1(\mathbf{R}^d)$ enjoys a continuous embedding in $L^\infty(\mathbf{R}^d) + L^p(\mathbf{R}^d)$ with $2 \leq p \leq 2^*$. We state the following Lemma that will be used repeatedly in the present chapter. Let $\chi \in C_c^\infty(\mathbf{C}, \mathbf{R})$ be a smooth cut-off function such that

$$\chi(z) = 1 \quad |z| \leq 2, \qquad \chi(z) \leq 1 \quad z \in \mathbf{C}, \qquad supp(\chi) \subset B_3(0). \tag{1.0.8}$$

In particular, given a wave-function $\psi : \mathbf{R}^d \to \mathbf{C}$ we introduce

$$\psi_{low} := \chi(\psi)\psi, \qquad \psi_{high} := (1 - \chi(\psi))\psi, \tag{1.0.9}$$

that enjoy the following properties.

Lemma 1.0.2 ([84])**.** *Let χ be as in* (1.0.8)*. If $\psi \in X^1 + H^1$ such that $|\psi|^2 - 1 \in L^2(\mathbf{R}^d)$ then $\psi_{low} \in X^1(\mathbf{R}^d)$ and $\psi_{high} \in H^1(\mathbf{R}^d)$.*

The statement is provided Lemma 2.1 in [84].

1.1 Well-posedness in the energy space for nonlinear Schrödinger equations with non-vanishing initial data at infinity

In this section, we investigate the Cauchy Problem for a class of nonlinear Schrödinger equations with non-vanishing density at infinity given by,

$$\begin{cases} i\partial_t \psi = -\frac{1}{2}\Delta\psi + F'(|\psi|^2)\psi, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{1.1.1}$$

where the initial data satisfies the boundary condition

$$|\psi_0(x)| \to 1, \quad \text{as} \quad |x| \to \infty. \tag{1.1.2}$$

Assuming that the boundary condition ρ_0 at infinity equals 1 is not restrictive since one may always renormalize it by a scaling argument. The associated Hamiltonian is given by

$$\mathcal{E}(\psi) = \int_{\mathbf{R}^d} \frac{1}{2}|\nabla\psi|^2 + F(|\psi|^2)\mathrm{d}x,$$

where $F(\cdot)$ is the renormalized internal energy defined in (0.2.6). We are led to consider nonlinearities of the type

$$F'(|\psi|^2) = f(|\psi|^2) = \frac{1}{\gamma-1}\left(|\psi|^{2(\gamma-1)} - 1\right), \qquad \text{where} \quad \begin{cases} \gamma > 1 & d = 2, \\ 1 < \gamma < 3 & d = 3. \end{cases} \tag{1.1.3}$$

We observe that F is non-negative, convex and such that $F(1) = 0$ is the global minimum. The nonlinearity f satisfies $f(1) = 0$ and $f'(1) > 0$ and is thus defocusing. The class of nonlinearities (1.1.3) we are interested in is not covered by the mentioned results [84, 82, 148] when $\gamma \neq 2$ and $\gamma < 3$. The present theory is robust enough to treat more general nonlinearities, for the sake of simplicity and conciseness in the exposition we restrict ourselves to the class of (1.1.3). We provide a self-contained presentation of the well-posedness theory for the Cauchy Problem (1.1.1) in the space

$$\mathbb{E}_2 = \left\{\psi \in H^1_{loc}(\mathbf{R}^d) \quad : \quad \mathcal{E}_2(\psi) < \infty\right\}, \tag{1.1.4}$$

where

$$\mathcal{E}_2(\psi) = \int_{\mathbf{R}^d} \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}\left(|\psi|^2 - 1\right)^2 \mathrm{d}x. \tag{1.1.5}$$

We refer to (1.1.4) as energy space and observe that (1.1.4) is a complete metric space [84] equipped with the distance function

$$d_{\mathbb{E}_2}(\psi, \tilde{\psi}) := \|\psi - \tilde{\psi}\|_{X^1 + H^1(\mathbf{R}^d)} + \||\psi|^2 - |\tilde{\psi}|^2\|_{L^2(\mathbf{R}^d)}. \tag{1.1.6}$$

The main result of this section states well-posedness of (1.1.1) in the energy space.

Theorem 1.1.1. *Let $d = 2, 3$, let us assume $\gamma > 1$ for $d = 2$ and $1 < \gamma < 3$ for $d = 3$. Then for any $\psi_0 \in \mathbb{E}_2$ there exists a unique strong solution $\psi \in C(\mathbf{R}; \mathbb{E}_2)$ to* (1.1.1)*. Moreover, it enjoys the following properties;*

1. *for every time $t \in \mathbf{R}$, $\mathcal{E}(t) = \mathcal{E}(0)$ with $\mathcal{E}$ defined in* (1.0.2)*,*
2. *if additionally $\Delta\psi_0 \in L^2(\mathbf{R}^d)$, then for any $0 < T < \infty$, one has $\Delta\psi \in L^\infty([-T, T], L^2(\mathbf{R}^d))$,*
3. *one has $\psi - e^{\frac{i}{2}t\Delta}\psi_0 \in C(\mathbf{R}, H^1(\mathbf{R}^d))$,*

4. *for every $R > 0$, let $\psi_0 \in \mathbb{E}_2$ such that $\mathcal{E}_2(\psi_0), \mathcal{E}_2(\psi_0^n) \leq R$ and $d_{\mathbb{E}_2}(\psi_0, \psi_0^n) \to 0$ as $n \to \infty$. Then, for any $0 < T < \infty$, we have*

$$\sup_{|t| \leq T} d_{\mathbb{E}_2}(\psi(t), \psi^n(t)) \to 0 \qquad \text{as} \quad n \to \infty,$$

where ψ, ψ^n are solutions to (1.1.1) with initial data ψ_0, ψ_0^n respectively.

Remark 1.1.2. Several remarks are in order. In the case $\gamma = 2$, Theorem 1.1.1 has already been proven in [84]. In particular, the author obtains Lipschitz dependence on the initial data that can not be expected for small γ due to the low regularity of the nonlinearity. We recover the Lipschitz stability estimate for $\gamma \geq \frac{3}{2}$, see Proposition 1.1.18. For $d = 3$ and $\gamma = 3$ equation (1.1.1) is energy critical. Local well-posedness in the energy space for small data has been shown in [82]. Exploiting the particular structure of the energy space for $d = 3$ and the perturbation Lemma introduced in [168], one can infer well-posedness for the energy critical problem in the energy space. This has been pointed out in [118] where the authors show global well-posedness for the problem with cubic-quintic nonlinearity.

Let us briefly comment on the method of the proof. We distinguish two cases,

(i) For $d = 2$, and for $d = 3$ with $1 < \gamma \leq 2$, we infer local well-posedness in the energy space by a direct argument. We show that there exists a solution $\psi \in \mathbb{E}_2$ to (1.1.1) with initial data ψ_0 such that $\psi - \psi_0 \in C(\mathbf{R}; H^1(\mathbf{R}^d))$. With the given restriction on γ, the nonlinearity grows sufficiently slowly so that one may work in L^2-based spaces only. Subsequently, we infer uniqueness and continuous dependence on the initial data. The latter requires a careful analysis due to the low regularity properties of F' for small γ.

(ii) For $d = 3$ and $2 \leq \gamma < 3$, two observation are crucial. The first is that the energy space can be identified with $\mathbb{E}_2(\mathbf{R}^3) = 1 + \dot{H}^1(\mathbf{R}^3)$ so that one may develop an $\dot{H}^1$-theory for the affine space. Further, for $\gamma \geq \frac{3}{2}$, the nonlinearity is locally Lipschitz on bounded sets of $\mathbb{E}_2$.

The second case can not be treated by the same method as the first case. For γ close to the threshold $\gamma = 3$ being energy critical we need to exploit the membership of $\nabla\psi$ in the whole range of Strichartz spaces. The bounds in L^2-based spaces used in the former case turn out to be insufficient. Conversely, the former case can not be tackled by the method of the latter since the bounds on the gradient of ψ in Strichartz spaces crucially rely on the special structure of the energy space for $d = 3$ and the fact that $\gamma \geq \frac{3}{2}$. In neither of the cases, our method requires a decomposition of the wave-function in a smooth part and an H^1-function as in [82, 148].

The rest of this section is as follows. Firstly, we review the key properties of the energy space. Secondly, we provide a well-posedness result for the case $\gamma > 1$ if $d = 2$ and $1 < \gamma \leq 2$ if $d = 3$. Local well-posedness is shown in Section 1.1.2 and global well-posedness in Section 1.1.3. The case $2 \leq \gamma \leq 3$ for $d = 3$ is tackled in Section 1.1.4. Here, we exploit the special structure of $\mathbb{E}_2$ in $d = 3$ and the fact that $\gamma \geq 2$ implies better regularity properties of the nonlinearities.

1.1.1 Properties of the energy space $\mathbb{E}_2$

We recall some of the properties of the space $\mathbb{E}_2$ presented in [84] to which we refer the interested reader. The space (1.1.4) is a complete metric space equipped with the distance function (1.1.6), however it is not a vector space. It turns out that the space $\mathbb{E}_2$ coincides with the energy space for (1.1.1), namely the set of wave-functions ψ for which (1.0.2) is finite, see Lemma 1.1.13.

Lemma 1.1.3 ([84]). *The following hold true*

1. $\mathbb{E}_2 \subset X^1(\mathbf{R}^d) + H^1(\mathbf{R}^d)$,
2. $\mathbb{E}_2 + H^1(\mathbf{R}^d) \subset \mathbb{E}_2$, *in particular for* $\psi \in \mathbb{E}_2$, $v \in H^1(\mathbf{R}^d)$
$$\||\psi+v|^2-1\|_{L^2} \leq \||\psi|^2-1\|_{L^2} + C\left(1+\sqrt{\mathcal{E}_2(\psi)}\right)\left(\|v\|_{L^2}+\|v\|_{L^4}+\|v\|_{L^4}^2\right). \quad (1.1.7)$$
3. *for* $\psi \in \mathbb{E}_2$, *let* ψ_{high}, ψ_{low} *be defined by* (1.0.9). *Then* $\psi_{low} \in X^1(\mathbf{R}^d)$ *and* $\psi_{high} \in H^1(\mathbf{R}^d)$. *In particular*
$$\|\psi_{low}\|_{X^1} \leq C\sqrt{\mathcal{E}_2(\psi)}, \qquad \|\psi_{high}\|_{H^1} \leq C\sqrt{\mathcal{E}_2(\psi)}. \quad (1.1.8)$$
4. *For* $\psi, \tilde{\psi} \in \mathbb{E}_2$ *and* $v, \tilde{v} \in H^1$ *one has*
$$\begin{aligned} d_{\mathbb{E}_2}(\psi+v, \tilde{\psi}+\tilde{v}) \leq & C(1+\|v\|_{H^1}+\|\tilde{v}\|_{H^1})d_{\mathbb{E}}(\psi,\tilde{\psi}) \\ & + (1+\sqrt{\mathcal{E}(\psi)}+\sqrt{\mathcal{E}(\tilde{\psi})}+\|v\|_{H^1}+\|\tilde{v}\|_{H^1})\|v-\tilde{v}\|_{H^1} \end{aligned}$$

Due to Sobolev embedding, for $d=3$, the structure of the energy space is characterized as follows. We notice that whenever $f \in \mathcal{D}'(\mathbf{R}^d)$ with $\nabla f \in L^p(\mathbf{R}^d)$ with $p<d$ then there exists $c \in \mathbf{R}$ such that $f+c \in L^{p^*}$, see Theorem 4.5.9 in [104]. For $d=3$, this yields $f-c \in L^6(\mathbf{R}^3)$. To see that this is false for $d=2$, consider $e^{i\log(2+|x|)^\alpha}$. For $\alpha<\frac{1}{2}$, one has $e^{i\log(2+|x|)^\alpha} \in \mathbb{E}_2$. Denote
$$\dot{H}^1(\mathbf{R}^3) = \left\{v \in L^6(\mathbf{R}^3): \quad \nabla v \in L^2(\mathbf{R}^3)\right\},$$
the completion of $C_0^\infty(\mathbf{R}^3)$ with the norm $\|\nabla v\|_{L^2}$. We introduce
$$\mathcal{F}_c = \left\{v \in \dot{H}^1(\mathbf{R}^3) \quad : \quad |v|^2 + 2Re(c^{-1}v) \in L^2\right\}. \quad (1.1.9)$$

One readily checks that
$$\tilde{\delta}(u,v) = \|\nabla u - \nabla v\|_{L^2} + \||u|^2 + 2Re(c^{-1}u) - |u|^2 - 2Re(c^{-1}v) - |v|^2\|_{L^2}$$

defines a distance function on $\mathcal{F}_c$.

Proposition 1.1.4 ([84])**.** *For $d = 3$, the energy space can be identified with the set of functions*

$$\mathbb{E} = \{\psi = c + v, \quad c \in \mathbb{C}, |c| = 1, \quad v \in F_c\}. \tag{1.1.10}$$

Moreover the metric function $d_{\mathbb{E}_2}$ is equivalent to

$$\delta(c+v, \tilde{c}+\tilde{v}) = |c-\tilde{c}| + \|\nabla v - \nabla \tilde{v}\|_{L^2} + \left\||v|^2 + 2Re(c^{-1}v) - |\tilde{v}|^2 - 2Re(\tilde{c}^{-1}\tilde{v})\right\|_{L^2}. \tag{1.1.11}$$

Remark 1.1.5. The particular structure of the energy space allows for an alternative approach to solve the Cauchy Problem in dimension $d = 3$. This has been pointed out in [84] for the Gross-Pitaevskii equation and has also been implemented in [118] for cubic-quintic nonlinearities and far-field behavior (1.1.2). Here, we will follow this approach to tackle the case $d = 3$ and $2 < \gamma < 3$.

Any function in the energy space $\mathbb{E}_2$ may be approximated by a sequence of smooth functions.

Lemma 1.1.6. *Let $\psi \in \mathbb{E}_2$, then there exists $\{\psi_n\}_{n\in N} \subset C^\infty$ such that*

$$d_{\mathbb{E}_2}(\psi, \psi_n) \to 0,$$

as $n \to 0$. Moreover, for any $\psi \in \mathbb{E}$, there exists $\varphi \in C_b^\infty(\mathbf{R}^d) \cap \mathbb{E}_2$ such that $\nabla\varphi \in H^\infty(\mathbf{R}^d)$ and such that

$$\psi - \varphi \in H^1(\mathbf{R}^d). \tag{1.1.12}$$

The first statement is proven in [84] by considering the convolution with a standard mollification kernel and the second statement follows from Proposition 1.1. in [82].

1.1.2 Local well-posedness in $2d$ and subcubic nonlinearities in $3d$

By Definition, a function $\psi \in C(\mathbf{R}, \mathbb{E}_2(\mathbf{R}^d))$ is a local strong solution to (1.1.1) if there exists $T > 0$ such that it satifies the integral equation (1.1.1) described by the the Duhamel formula

$$\psi(t) = e^{\frac{i}{2}t\Delta}\psi_0 - i\int_0^t e^{\frac{i}{2}(t-s)\Delta}\mathcal{N}(\psi)(s)\mathrm{d}s,$$

for all $t \in [-T, T]$, where $\mathcal{N}(\psi) = F'(|\psi|^2)\psi$. Our aim is to show local well-posedness for (1.1.1) by performing a fixed point argument. We start by recalling the action of the linear Schrödinger group on the space $X^k(\mathbf{R}^d) + H^k(\mathbf{R}^d)$.

Lemma 1.1.7 ([84])**.** *Let d be a positive integer. For every k, for every $t \in \mathbf{R}$, the operator $e^{\frac{i}{2}t\Delta}$ maps $X^k(\mathbf{R}^d) + H^k(\mathbf{R}^d)$ into itself respecting the estimates*

$$\|e^{\frac{i}{2}t\Delta}f\|_{X^k+H^k} \leq C\,(1+t)^{\frac{1}{2}}\,\|f\|_{X^k+H^k}, \tag{1.1.13}$$

and

$$\|e^{\frac{i}{2}t\Delta}f - f\|_{L^2} \leq C|t|^{\frac{1}{2}}\|\nabla f\|_{L^2}. \tag{1.1.14}$$

Moreover, if $f \in X^k(\mathbf{R}^d) + H^k(\mathbf{R}^d)$, the map $t \in \mathbf{R} \mapsto e^{\frac{i}{2}t\Delta}f \in X^k(\mathbf{R}^d) + H^k(\mathbf{R}^d)$ is continuous.

Next, we review the action of the linear Schrödinger group on the space $\mathbb{E}_2$.

Proposition 1.1.8 ([84]). *Let $d = 2, 3$. For every $t \in \mathbf{R}$, the linear propagator $e^{\frac{i}{2}t\Delta}$ maps $\mathbb{E}_2$ to itself and for every $\psi \in \mathbb{E}_2$ the map $t \in \mathbf{R} \mapsto e^{\frac{i}{2}t\Delta}\psi_0 \in \mathbb{E}_2$ is continuous. Moreover, given $R > 0$, $T > 0$ there exists $C > 0$ such that for every $\psi_0, \tilde{\psi}_0 \in \mathbb{E}_2$ with $\mathcal{E}_2(\psi_0) \leq R$, $\mathcal{E}_2(\tilde{\psi}_0) \leq R$ one has*

$$\sup_{|t|\leq T} d_{\mathbb{E}_2}(e^{\frac{i}{2}t\Delta}\psi_0, e^{\frac{i}{2}t\Delta}\tilde{\psi}_0) \leq C d_{\mathbb{E}_2}(\psi_0, \tilde{\psi}_0). \tag{1.1.15}$$

Further, given $R > 0$, there exists $T(R) > 0$ such that, for every $\psi_0 \in \mathbb{E}_2$ with $\mathcal{E}_2(\psi_0) \leq R$, we have

$$\sup_{|t|\leq T(R)} \mathcal{E}_2(e^{\frac{i}{2}t\Delta}\psi_0) \leq 2R.$$

Next, we wish to show that the non-homogeneous term appearing in the Duhamel formula is bounded in $L^\infty([-T,T]; H^1(\mathbf{R}^d))$. We collect some properties of the nonlinearity $\mathcal{N}(\psi) = F'(|\psi|^2)\psi$ with F' defined in (1.1.3). We compute that

$$\nabla\mathcal{N}(\psi) = \left(|\psi|^{2(\gamma-1)} - 1\right)\nabla\psi + (\gamma-1)|\psi|^{2(\gamma-1)}\left(1 + \frac{\psi^2}{|\psi|^2}\right)\nabla\overline{\psi}. \tag{1.1.16}$$

The quantity $\frac{\psi}{|\psi|}$ can be seen as the polar-factor of the wave-function ψ and will be rigorously introduced in Section 2.2.
Let $\chi \in C_c^\infty(\mathbf{C}, \mathbf{R})$ be defined by (1.0.8). We denote

$$\mathcal{N}_1(\psi) := \mathcal{N}(\psi)\chi(\psi), \qquad \mathcal{N}_2(\psi) = \mathcal{N}(\psi)(1 - \chi(\psi)). \tag{1.1.17}$$

From the inequality

$$\left||\psi|^{2(\gamma-1)}\psi - |\tilde{\psi}|^{2(\gamma-1)}\tilde{\psi}\right| \leq C\left(|\psi|^{2(\gamma-1)} + |\tilde{\psi}|^{2(\gamma-1)}\right)|\psi - \tilde{\psi}|,$$

we conclude

$$\begin{aligned} \left|\mathcal{N}_1(\psi) - \mathcal{N}_1(\tilde{\psi})\right| &\leq C\left|\psi - \tilde{\psi}\right|, \\ \left|\mathcal{N}_2(\psi) - \mathcal{N}_2(\tilde{\psi})\right| &\leq \left(1 + |\psi|^{2(\gamma-1)} + |\tilde{\psi}|^{2(\gamma-1)}\right)\left|\psi - \tilde{\psi}\right|, \end{aligned} \tag{1.1.18}$$

The next Lemma summarizes the needed properties of the nonlinearity $\mathcal{N}(\psi)$.

Lemma 1.1.9. *Let $d = 2, 3$ and $\gamma > 1$ then $\mathcal{N} \in C^1(\mathbf{R}^2, \mathbf{R}^2)$. Further if $d = 2, 3$ and $1 < \gamma \leq 2$, one has for every $\psi \in \mathbb{E}_2$ that*

$$\begin{aligned} \|\mathcal{N}_1(\psi)\|_{L^2(\mathbf{R}^d)} &\leq C\||\psi|^2 - 1\|_{L^2(\mathbf{R}^d)}, \\ \|\mathcal{N}_2(\psi)\|_{L^{\frac{6}{5}}(\mathbf{R}^d)} &\leq C\mathcal{E}_2(\psi)\||\psi|^2 - 1\|_{L^2(\mathbf{R}^d)}, \end{aligned} \tag{1.1.19}$$

and

$$\begin{aligned} \|\nabla\mathcal{N}_1(\psi)\|_{L^2(\mathbf{R}^d)} &\leq C\|\nabla\psi\|_{L^2(\mathbf{R}^d)}, \\ \|\nabla\mathcal{N}_2(\psi)\|_{L^{\frac{6}{5}}(\mathbf{R}^d)} &\leq C\mathcal{E}_2(\psi)\|\nabla\psi\|_{L^2(\mathbf{R}^d)}. \end{aligned} \tag{1.1.20}$$

If $\gamma > 2$ and $d = 2$, then

$$\begin{aligned} \|\mathcal{N}_1(\psi)\|_{L^2(\mathbf{R}^2)} &\leq \||\psi|^2 - 1\|_{L^2(\mathbf{R}^2)}, \\ \|\mathcal{N}_2(\psi)\|_{L^{\frac{6}{5}}(\mathbf{R}^d)} &\leq \mathcal{E}_2(\psi)^{(\gamma-1)}\||\psi|^2 - 1\|_{L^2}, \end{aligned} \tag{1.1.21}$$

and

$$\begin{aligned} \|\nabla\mathcal{N}_1(\psi)\|_{L^2(\mathbf{R}^2)} &\leq C\|\nabla\psi\|_{L^2(\mathbf{R}^2)}, \\ \|\nabla\mathcal{N}_2(\psi)\|_{L^{\frac{6}{5}}(\mathbf{R}^d)} &\leq \mathcal{E}_2(\psi)^{(\gamma-1)}\|\nabla\psi\|_{L^2}. \end{aligned} \tag{1.1.22}$$

Proof. Let us consider $\mathcal{N}$ as $\mathcal{N} : \mathbf{R}^2 \to \mathbf{R}^2$. Then its Jacobian reads

$$D\mathcal{N}(u) = \left(\gamma|u|^{2(\gamma-1)} - 1\right)\left(Id - \frac{u \otimes u}{|u|^2}\right),$$

hence $\mathcal{N} \in C^1(\mathbf{R}^2; \mathbf{R}^2)$. We deal separately with the cases $1 < \gamma \leq 2$ for $d = 2, 3$ and $\gamma > 2$ for $d = 2$ only.

Case 1: Let $d = 2, 3$ and $1 < \gamma \leq 2$. We observe that,

$$\left||\psi|^{2(\gamma-1)} - 1\right| \leq \left||\psi|^2 - 1\right|, \tag{1.1.23}$$

and thus $|\psi|^{2(\gamma-1)} - 1 \in L^2(\mathbf{R}^d)$. From (1.1.17), we conclude that

$$|\mathcal{N}_1(\psi)| \leq \left||\psi|^2 - 1\right| \in L^2(\mathbf{R}^d).$$

Let ψ_{low} and ψ_{high} be as defined in (1.0.9). From the inequality

$$|\mathcal{N}_2(\psi)| \leq \left||\psi|^2 - 1\right| |\psi_{high}|,$$

and the observation, that if $\psi \in \mathbb{E}_2$ then $(1 - \chi(\psi))\psi \in L^6(\mathbf{R}^d)$ thanks to (1.1.8), we conclude that

$$\begin{aligned} \|\mathcal{N}_2\|_{L^{\frac{6}{5}}} &\leq \|\psi_{high}\|_{L^6}^2 \||\psi|^2 - 1\|_{L^2} \\ &\leq C\mathcal{E}_2(\psi)\||\psi|^2 - 1\|_{L^2}. \end{aligned}$$

In view of (1.1.16), we have that

$$\|\nabla\mathcal{N}_1\|_{L^2(\mathbf{R}^d)} \leq C\|\nabla\psi\|_{L^2(\mathbf{R}^d)}.$$

For the respective bound of $\nabla\mathcal{N}_2$, we use again that $\psi_{high} \in L^6(\mathbf{R}^d)$ and thus

$$\begin{aligned} \|\nabla\mathcal{N}_2(\psi)\|_{L^{\frac{6}{5}}(\mathbf{R}^d)} &\leq C\|\psi_{high}\|_{L^6(\mathbf{R}^d)}^2 \|\nabla\psi\|_{L^2(\mathbf{R}^d)} \\ &\leq C\mathcal{E}_2(\psi)\|\nabla\psi\|_{L^2(\mathbf{R}^d)}. \end{aligned}$$

Case: $\gamma > 2$ and $d = 2$.
Since $\gamma > 2$, the function $s \mapsto s^{\gamma-1}$ is convex and therefore We notice that,

$$\|\mathcal{N}_1(\psi)\|_{L^2(\mathbf{R}^2)} \leq C\||\psi|^2 - 1\|_{L^2(\mathbf{R}^2)},$$

and

$$\begin{aligned}\|\mathcal{N}_2(\psi)\|_{L^{\frac{3}{2}}(\mathbf{R}^d)} &\le \|\psi_{high}\|^{2(\gamma-1)}_{L^{12(\gamma-1)}(\mathbf{R}^2)}\||\psi|^2-1\|_{L^2(\mathbf{R}^2)},\\ &\le \mathcal{E}_2(\psi)^{(\gamma-1)}\||\psi|^2-1\|_{L^2(\mathbf{R}^2)}.\end{aligned}$$

Arguing similarly, we conclude

$$\|\nabla\mathcal{N}_1(\psi)\|_{L^2} \le C\|\nabla\psi\|_{L^2},$$

and

$$\begin{aligned}\|\nabla\mathcal{N}_2(\psi)\|_{L^{\frac{6}{5}}(\mathbf{R}^2)} &\le C\|\psi_{high}\|^{2(\gamma-1)}_{L^{6(\gamma-1)}(\mathbf{R}^2)}\|\nabla\psi\|_{L^2}\\ &\le \mathcal{E}_2(\psi)^{(\gamma-1)}\|\nabla\psi|\|_{L^2(\mathbf{R}^2)}.\end{aligned}$$

□

We are now ready to tackle the issue of local well-posedness for (1.1.1). We anticipate that the continuous dependence on the initial data differs significantly from the classical approach as consequence of technicalities arising from the low regularity of the nonlinearity combined with the lack of integrability of ψ.

Proposition 1.1.10. *Let $d=2,3$, $\gamma>1$ if $d=2$ and $1<\gamma<2$ if $d=3$. For every $R>0$, there exists $T>0$ such that for every $\psi_0\in\mathbb{E}_2$ with $\mathcal{E}_2(\psi_0)\le R$ there exists a unique solution $\psi\in C([-T,T];\mathbb{E}_2)$ of* (1.1.1) *with $\psi(0)=\psi_0$ and the follwing blow-up alternative is satisfied: either $T=+\infty$ or*

$$\lim_{t\nearrow T}\mathcal{E}(t)=+\infty.$$

Finally, if ψ_0^n is such that $\mathcal{E}(\psi_0^n)\le R$ and $d_{\mathbb{E}_2}(\psi_0^n,\psi_0)$ converges to 0 as n goes to infinity, then $\sup_{t\in[-T,T]}d_{\mathbb{E}_2}(\psi_n,\psi)$ converges to 0 as n goes to infinity.

As it will become clear from the Definition of the solution map (1.1.24), to implement the existence argument directly in the energy space comes with the price of the restriction of $1<\gamma\le 2$ for $d=3$. Indeed, otherwise one would need to require that $\nabla\psi_0\in L^p$ for some $p>2$. For $d=2$, this difficulty can be circumvented by means of Sobolev embeddings. Another way to avoid this issue, is to work with a smooth decomposition as in [82]. In that way, one is forced to work in an affine spaces instead of working directly in the energy space as done in [84]. Moreover, the approach in [82] requires nonlinearities to be of C^3-regularity.

Proof. **Local existence.** Fix $M>0$ and $T>0$ to be chosen later. To show local existence, given $\psi_0\in\mathbb{E}_2$ such that $\mathcal{E}_2(\psi_0)\le R$, we implement a fixed point argument for the map

$$S(u)(t)=e^{\frac{i}{2}t\Delta}\psi_0-\psi_0-i\int_0^t e^{\frac{i}{2}(t-s)\Delta}\mathcal{N}(\psi_0+u)(s)\mathrm{d}s=:e^{\frac{i}{2}t\Delta}\psi_0-\psi_0+\Phi(\psi_0+u),\quad(1.1.24)$$

in

$$X_T=\left\{u\in L^\infty([-T,T];H^1(\mathbf{R}^d))\quad:\quad\|u\|_{X_T}\le M\right\},$$

We define the distance function d for $u, v \in X_T$ as

$$d(u,v) = \|u - v\|_{L^\infty L^2}.$$

The space (X_T, d) is a complete metric space, indeed it suffices to check that X_T is closed. Let $\{u_n\}_{n\in\mathbf{N}} \subset X_T$ be a Cauchy sequence for the metric d. Then there exists $u \in L^\infty([-T,T]; L^2(\mathbf{R}^d))$ such that $u_n \to u$ in $L^\infty([-T,T]; L^2(\mathbf{R}^d))$. Since u_n is uniformly bounded in X_T, by lower-semicontinuity of norms we infer that

$$\|\nabla u\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))} \leq \liminf_{n\to\infty} \|\nabla u_n\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))},$$

and thus $u \in X_T$. We notice that, if $\mathcal{E}_2(\psi_0) \leq R$ and $u \in X_T$, then thanks to (1.1.7) and Minkowski inequality we obtain

$$\begin{aligned} \sqrt{\mathcal{E}_2(\psi_0 + u)} &\leq \sqrt{\mathcal{E}_2(\psi_0)} + C\left(1 + \sqrt{\mathcal{E}_2(\psi_0)}\right)\|u\|_{H^1} + \|u\|_{H^1}^2 \\ &\leq \sqrt{R} + C(1+\sqrt{R})M + M^2 =: C(M,R). \end{aligned} \tag{1.1.25}$$

Next we show that S defined in (1.1.24) maps X_T onto X_T. Let $u \in X_T$, then

$$\begin{aligned} &\|S(u)\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))} \\ &\leq \|e^{\frac{i}{2}t\Delta}\psi_0 - \psi_0\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))} + \|\Phi(\psi_0+u)\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))}. \end{aligned}$$

From (1.1.14), we conclude that

$$\|e^{\frac{i}{2}t\Delta}\psi_0 - \psi_0\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))} \leq CT^{\frac{1}{2}}\|\nabla\psi_0\|_{L^2}. \tag{1.1.26}$$

To estimate the nonlinear term, we distinguish the cases $1 < \gamma \leq 2$ and $\gamma > 2$.
Case $d = 2,3$ *and* $1 < \gamma \leq 2$.
In the former case, thanks to the bounds (1.1.19), we conclude that for q' such that $(q, 6)$ is an admissible pair,

$$\begin{aligned} &\|\Phi(\psi_0+u)\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))} \\ &\leq \|\mathcal{N}_1(\psi_0+u)\|_{L^1([-T,T];L^2(\mathbf{R}^d))} + \|\mathcal{N}_2(\psi_0+u)\|_{L^{q'}([-T,T];L^{\frac{6}{5}}(\mathbf{R}^d))} \\ &\leq C\left(T + T^{\frac{1}{q'}}\sup_t \mathcal{E}_2(\psi_0+u)\right)\||\psi_0+u|^2 - 1\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))}. \end{aligned}$$

The inequality (1.1.25) yields that

$$\|\Phi(\psi_0+u)\|_{L^\infty([-T,T];L^2(\mathbf{R}^d)} \leq C\left(T + T^{\frac{1}{q'}}C(M,R)^2\right)C(M,R). \tag{1.1.27}$$

It remains to infer the desired bound on $\nabla S(u)$. Let q_1 be such that $(q_1, \frac{3}{2})$ is an admissible pair. Then, it follows from (1.1.20) that

$$\begin{aligned} \|\nabla S(u)\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))} &\leq C\|\nabla\psi_0\|_{L^2_x} + \|\nabla\mathcal{N}_1(\psi_0+u)\|_{L^1_tL^2_x} + \|\nabla\mathcal{N}_2(\psi_0+u)\|_{L^{q'}L_x^{\frac{6}{5}}} \\ &\leq C\|\nabla\psi_0\|_{L^2_x} + C\left(T + T^{\frac{1}{q'}}\sup_t \mathcal{E}_2(\psi)\right)\left(\|\nabla\psi_0\|_{L^2} + \|\nabla u\|_{L^\infty([-,T,T];L^2(\mathbf{R}^d))}\right). \end{aligned}$$

Hence,

$$\|\nabla S(u)\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))} \le C\sqrt{R} + C\left(T + T^{\frac{1}{q'}} C(R,M)^2\right)(\sqrt{R}+M). \tag{1.1.28}$$

We conclude that

$$S(u) \in C([-T,T]; H^1(\mathbf{R}^d)),$$

and summing up (1.1.26), (1.1.27), (1.1.28) we obtain that

$$\|S(u)\|_{X_T} \le C(1+T^{\frac{1}{2}})\sqrt{R} + C\left(T + T^{\frac{1}{q'}} C(R,M)^2\right) C(R,M).$$

Next, we check that the map S introduced in (1.1.24) defines a contraction on (X_T, d). Let $u_1, u_2 \in X_T$ and denote

$$\psi_1 = \psi_0 + u_1, \qquad \psi_2 = \psi_0 + u_2,$$

we observe that since

$$|\mathcal{N}_1(\psi_1) - \mathcal{N}_1(\psi_2)| \le C\,|u_1 - u_2|\,,$$

and

$$\begin{aligned}&|\mathcal{N}_2(\psi_1) - \mathcal{N}_2(\psi_2)| \\ &\qquad \le C\left(1 + |\psi_{1,high}|^2) + |\psi_{2,high}|^2)\right)|u_1 - u_2|\,,\end{aligned}$$

we have that

$$\|\mathcal{N}_1(\psi_1) - \mathcal{N}_1(\psi_2)\|_{L^1L^2} \le CT\|u_1 - u_2\|_{L^\infty L^2}, \tag{1.1.29}$$

and

$$\begin{aligned}&\|\mathcal{N}_2(\psi_1) - \mathcal{N}_2(\psi_2)\|_{L^{q'}L^{\frac{6}{5}}} \\ &\le CT^{\frac{1}{q'}}\left(\|\psi_1\|^2_{L^\infty(X_1+H^1)} + \|\psi_2\|^2_{L^\infty(X_1+H^1)}\right)\|u_1 - u_2\|_{L^\infty L^2}.\end{aligned} \tag{1.1.30}$$

We compute

$$\begin{aligned} d(S(u_1), S(u_2)) &= \left\| -i\int_0^t \left(\mathcal{N}(\psi_1) - \mathcal{N}(\psi_2)\right)(s)\mathrm{d}x \right\|_{L^\infty([-T,T],L^2(\mathbf{R}^d))} \\ &\le \|\mathcal{N}_1(\psi_1) - \mathcal{N}_1(\psi_2)\|_{L^1_tL^2_x} + \|\mathcal{N}_2(\psi_1) - \mathcal{N}_2(\psi_2)\|_{L^{q'}_tL^{\frac{6}{5}}_x} \\ &\le C\left(T + T^{\frac{1}{q'}}\left(\sup_t \mathcal{E}_2(\psi_1)^2 + \sup_t \mathcal{E}_2(\psi_2)^2\right)\right)\|u_1 - u_2\|_{L^\infty L^2} \\ &\le C\left(T + T^{\frac{1}{q}} C(R,M)^2\right) d(u_1, u_2). \end{aligned}$$

We now choose M and T. Given ψ_0 such that $\mathcal{E}_2(\psi_0) \le R$, we fix $M > 0$ such that

$$M = 4C\sqrt{R},$$

and we notice that there exists $0 < T \le 1$ sufficiently small such that

$$C\left(T + T^{\frac{1}{q}} C(R,M)^2\right) \le \frac{1}{2},$$

and

$$C(T + T^{\frac{1}{q'}} C(R, M)^2) C(M, R) \leq \frac{1}{2} M,$$

where $C(M, R)$ as defined in (1.1.25). Hence, S maps X_T onto X_T and defines a contraction on X_T. The Banach fixed point Theorem yields a unique $u \in X_T$ such that $\psi_0 + u$ is solution to (1.1.1). In particular, the embedding $\mathbb{E}_2 + H^1 \subset \mathbb{E}_2$ yields that $\psi_0 + u \in C([-T, T]; \mathbb{E}_2)$.
Case: $d = 2$ and $\gamma > 2$. The proof follows the same lines. Combining the non-homogeneous Strichartz estimates (1.0.4) and (1.1.21) we obtain that

$$\|\Phi(\psi_0 + u)\|_{L^\infty[-T,T];L^2(\mathbf{R}^2))} \leq C \left(T + T^{\frac{2}{3}} \sup_t \mathcal{E}_2(\psi_0 + u)^{\gamma - 1} \right) \||\psi_0 + u|^2 - 1\|_{L^2}. \quad (1.1.31)$$

Again exploiting the Strichartz estimates (1.0.4) and the nonlinear estimate (1.1.22), we conclude that

$$\begin{aligned} \|\nabla S(u)\|_{L^\infty[-T,T];L^2(\mathbf{R}^2))} &\leq C(1 + T^{\frac{1}{2}}) \|\nabla \psi_0\|_{L^2} \\ &+ C \left(T + T^{\frac{2}{3}} \sup_t \mathcal{E}_2(\psi_0 + u)^{\gamma - 1} \right) \left(\|\nabla \psi_0\|_{L^\infty[-T,T];L^2(\mathbf{R}^2))} + \|\nabla u\|_{L^\infty[-T,T];L^2(\mathbf{R}^2))} \right). \end{aligned} \quad (1.1.32)$$

Therefore, combining (1.1.26), (1.1.31) and (1.1.32), we infer that

$$\|S(u)\|_{X_T} \leq C(1 + T^{\frac{1}{2}}) \sqrt{R} + C \left(T + T^{\frac{2}{3}} C(M, R)^{2(\gamma - 1)} \right) C(M, R).$$

Next, we check that S defines a contraction on X_T. To that end, we observe that (1.1.29) still holds. The inequality (1.1.30) is replaced by

$$\|\mathcal{N}_2(\psi_1) - \mathcal{N}_2(\psi_2)\|_{L^{\frac{3}{2}} L^{\frac{6}{5}}} \leq C T^{\frac{2}{3}} \left(\|\psi_1\|^{2(\gamma-1)}_{L^\infty X^1 + H^1} + \|\psi_2\|^{2(\gamma-1)}_{L^\infty X^1 + H^1} \right) \|\psi_1 - \psi_2\|_{L^\infty L^2} \quad (1.1.33)$$

Combining (1.1.29) and (1.1.30), we compute

$$d(S(u_1), S(u_2)) \leq C \left(T + T^{\frac{2}{3}} C(M, R)^{2(\gamma - 1)} \right) d(u, v).$$

Thus, proceeding as in the previous case, we define $M = 4C\sqrt{R}$ and we conclude that there exits $T > 0$ only depending on R such that S maps X_T onto itself and defines a contraction on X_T. Thus, $\psi_0 + u \in C([-T, T]; \mathbb{E}_2)$ solves (1.1.1).
Uniqueness. Let $\psi_1, \psi_2 \in C([-T, T], \mathbb{E}_2)$ be two solutions to (1.1.1) with initial data $\psi_1(0) = \psi_2(0) = \psi_0 \in \mathbb{E}_2$. One has that

$$\psi_1(t) - \psi_2(t) = -i \int_0^t e^{\frac{i}{2}(t-s)\Delta} \left(\mathcal{N}(\psi_1) - \mathcal{N}(\psi_2) \right)(s) \mathrm{d}s. \quad (1.1.34)$$

We start by discussing the case $1 < \gamma \leq 2$. We recall that (1.1.29) and (1.1.30) yield

$$\|\mathcal{N}_1(\psi_1) - \mathcal{N}_1(\psi_2)\|_{L^1 L^2} \leq C \|\psi_1 - \psi_2\|_{L^1_x L^2_t}, \quad (1.1.35)$$

and

$$\begin{aligned}
&\|\mathcal{N}_2(\psi_1)-\mathcal{N}_2(\psi_2)\|_{L_t^{q'}L_x^{\frac{6}{5}}}\\
&\leq C\left\|\left(|\psi_{high,1}|^{2(\gamma-1)}+|\psi_{high,2}|^{2(\gamma-1)}(1-\chi(\psi_2))\right)|\psi_1-\psi_2|\right\|_{L_t^{q'}L_x^{\frac{6}{5}}} \qquad (1.1.36)\\
&\leq C\left(\|\psi_1\|^{\gamma-1}_{L_t^\infty(X^1+H^1)}+\|\psi_2\|^{\gamma-1}_{L_t^\infty(X^1+H^1)}\right)\|\psi_1-\psi_2\|_{L_t^\infty L_x^2}.
\end{aligned}$$

Therefore, upon applying the Strichartz estimate (1.0.4) and the bounds on $\mathcal{N}$ we conclude that

$$\|\psi_1-\psi_2\|_{L^\infty L^2}\leq C\left(T^{\frac{1}{q'}}+\|\psi_1\|^{\gamma-1}_{L^\infty(X^1+H^1)}+\|\psi_2\|^{\gamma-1}_{L^\infty(X^1+H^1)}\right)\|\psi_1-\psi_2\|_{L^{q'}L^2}. \qquad (1.1.37)$$

Hence, we deduce that $\|\psi_1(t)-\psi_2(t)\|_{L^2}=0$ for a.e. $t\in[-T,T]$ and hence $\psi_1=\psi_2$ a.e. on $[-T,T]\times\mathbf{R}^d$, see for example Lemma 4.2.2 in [52]. For the case $d=2$ and $\gamma>2$ it suffices to replace (1.1.30) by (1.1.33) and to notice that one may proceed accordingly to infer the respective version of (1.1.37) yielding uniqueness of solutions on $[-T,T]\times\mathbf{R}^d$.

Blow-up alternative.

Let $\psi_0\in\mathbb{E}_2$ and define

$$\begin{aligned}
T_{max}(\psi_0)&=\sup\{T>0 \quad : \quad \text{there exists solution to (1.1.1) on} \quad [0,T]\},\\
T_{min}(\psi_0)&=\sup\{T>0 \quad : \quad \text{there exists solution to (1.1.1) on} \quad [-T,0]\}.
\end{aligned}$$

From the previous step, we infer $\psi\in C((-T_{min},T_{max});\mathbb{E}_2)$ solution of (1.1.1). Proceeding by contradiction, we assume that $T_{max}<\infty$ and that there exists $0<M<\infty$ and a sequence t_n converging to T_{max} such that $\mathcal{E}_2(\psi(t_n))\leq M$ for all $n\in\mathbf{N}$. Let k be such that $T(M)+t_k>T_{\max}(\psi_0)$. Then starting from t_k, Step 1 provides a solution up to $t_k+T(M)$. This violates the maximality assumption and we conclude that

$$\mathcal{E}_2(\psi(t))\to\infty, \qquad \text{as} \quad t\to T_{\max}.$$

The statement for T_{min} is shown analogously.

The proof of the continuous dependence on the initial data of the solution requires some auxiliary Lemmas and is postponed after Lemma 1.1.12. □

Next, we introduce estimates on the nonlinear flow in Strichartz norms that are required for the proof of the continuous dependence on the initial data. The estimates used for the contraction argument in the proof of Proposition 1.1.10 are not sufficient since they only allow to control the difference of solutions ψ_1,ψ_2 provided that $\psi_1-\psi_2\in L_t^\infty L_x^2(\mathbf{R}^d)$. We split the nonlinearity to analyse separately its behavior when the unkowns are small or large. Let $\eta\in C_c^\infty(\mathbf{R})$ such that $\eta(z)=1$ for $|z|\leq\frac{1}{4}$ and $supp(\eta)\subset B_{\frac{1}{2}}(0)$ and define

$$\mathcal{N}_{sing}(\psi):=\mathcal{N}(\psi)\eta(|\psi|), \qquad \mathcal{N}_{reg}(\psi):=\mathcal{N}(\psi)(1-\eta(|\psi|)), \qquad (1.1.38)$$

The cut-off function η is used to isolate the singular behavior of $|\psi|^{2(\gamma-1)}$, namely the set on which the Lipschitz constant blows up. The choice of the support of η ensures that this set is of finite Lebesgue measure. We underline that η and χ as in (1.0.8) are not to be confused. We recall that the indices *high/low* indicate functions to which we applied the cut-off χ while *reg / sing* are used for functions to which η has been applied. Given $\psi \in X^1 + H^1$, the cut-off function χ allows to split $\psi = \psi_{low} + \psi_{high}$ so that $\psi_{low} \in X^1$ and $\psi_{high} \in H^1$. Similarly to the contraction argument, the properties of the regularity $\mathcal{N}$ do not allow to directly control $\|\nabla\Phi(\psi) - \nabla\Phi(\psi_2)\|_{L_t^\infty L_x^2}$ for $\psi_1, \psi_2 \in \mathbb{E}_2$, hence motivating the following estimates.

Lemma 1.1.11. *Let $d = 2, 3$ and $1 < \gamma < 2$ for $d = 3$ and $\gamma > 1$ with $\gamma \neq 2$ for $d = 2$, further let $R > 0$ be fixed and $r = 4$ if $\gamma < 2$ and $r = 2\gamma$ if $\gamma > 2$. Given $\psi, \tilde{\psi} \in \mathbb{E}_2$ such that $\mathcal{E}_2(\psi) \leq R$ and $\mathcal{E}_2(\tilde{\psi}) \leq R$, there exists $\alpha > 0$ such that*

$$\begin{aligned} &\left\|\mathcal{N}(\psi) - \mathcal{N}(\tilde{\psi})\right\|_{N^0([-T,T]\times\mathbf{R}^d)} \\ &\leq CT^\alpha R^{\max(\frac{1}{2},\gamma-1)} \left(\|\psi - \tilde{\psi}\|_{L^\infty([-T,T];L^\infty+L^2(\mathbf{R}^d))} + \left\||\psi|^2 - |\tilde{\psi}|^2\right\|_{L_t^{\frac{4}{d}} L_x^2} + \|\psi - \tilde{\psi}\|_{L_t^q(L_x^\infty + L_x^r)}\right), \end{aligned} \tag{1.1.39}$$

Proof. We consider separately the cases $1 < \gamma < 2$ and $\gamma > 2$. In the former, for $\psi_{low}, \tilde{\psi}_{low}$ one has

$$\left||\psi_{low}|^{2(\gamma-1)}\psi_{low} - |\tilde{\psi}_{low}|^{2(\gamma-1)}\tilde{\psi}_{low}\right| \leq C|\psi_{low} - \tilde{\psi}_{low}|;$$

from which we infer that for $\psi, \tilde{\psi}$, it holds

$$\left|\mathcal{N}_{sing}(\psi) - \mathcal{N}_{sing}(\tilde{\psi})\right| \leq C\left|\psi - \tilde{\psi}\right|. \tag{1.1.40}$$

Further, for $\psi \in \mathbb{E}_2$ we denote $\mathcal{A} = \{x \in \mathbf{R}^d : |\psi| < \frac{1}{4}\}$, then the Chebycheff inequality yields

$$\mu(\mathcal{A}) \leq \mu\left(\{||\psi|^2 - 1||^2 > \frac{1}{2}\}\right) \leq 2\mathcal{E}_2(\psi) \leq 2R.$$

Thus, for $\psi, \tilde{\psi} \in \mathbb{E}_2$ such that $\mathcal{E}_2(\psi) \leq R$ and $\mathcal{E}_2(\tilde{\psi}) \leq R$ we obtain

$$\left\|\left|\mathcal{N}_{sing}(\psi) - \mathcal{N}_{sing}(\tilde{\psi})\right|\right\|_{L^1([-T,T];L^2(\mathbf{R}^d))} \leq CT\sqrt{R}\|\psi - \tilde{\psi}\|_{L^\infty([-T,T];L^\infty+L^2(\mathbf{R}^d))}. \tag{1.1.41}$$

While to deal with $\mathcal{N}_{reg}$, we write

$$\begin{aligned} &\left|\mathcal{N}_{reg}(\psi) - \mathcal{N}_{reg}(\tilde{\psi})\right| \\ &\leq \left||\psi|^{2(\gamma-1)}(1-\eta(\psi)) - |\tilde{\psi}|^{2(\gamma-1)}(1-\eta(\tilde{\psi}))\right||\psi| + \left|\left(|\tilde{\psi}|^{2(\gamma-1)} - 1\right)(\psi - \tilde{\psi})\right|, \end{aligned} \tag{1.1.42}$$

The first contribution is controlled by Hölder continuity and local Lipschitz continuity as

$$\begin{aligned} &\left||\psi|^{2(\gamma-1)}(1-\eta(\psi)) - |\tilde{\psi}|^{2(\gamma-1)}(1-\eta(\tilde{\psi}))\right||\psi| \\ &\leq C\left(|\psi|^{2(\gamma-2)}(1-\eta(\psi)) + |\tilde{\psi}|^{2(\gamma-2)}(1-\eta(\tilde{\psi}))\right)\left||\psi|^2 - |\tilde{\psi}|^2\right||\psi_{low}| \\ &+ C\left||\psi|^2 - |\tilde{\psi}|^2\right|^{\gamma-1}|\psi_{high}| \\ &\leq C\left||\psi|^2 - |\tilde{\psi}|^2\right||\psi_{low}| + \left||\psi|^2 - |\tilde{\psi}|^2\right|^{\gamma-1}|\psi_{high}|, \end{aligned}$$

where we used that $1 < \gamma < 2$. Next, we define

$$\frac{1}{r_1'} = \frac{\gamma - 1}{2} + \frac{1}{6} < \frac{2}{3}.$$

Let q_1' such that (q_1, r_1) is admissible, then $q_1' < \frac{12}{12-d}$. Hence, there exits $\alpha = \alpha(q_1) > \frac{3-d}{3}$ such that,

$$\begin{aligned} &\left\| \left| |\psi|^{2(\gamma-1)}(1-\eta(\psi)) - |\tilde{\psi}|^{2(\gamma-1)}(1-\eta(\tilde{\psi})) \right| |\psi| \right\|_{L_t^1 L_x^2 + L_t^{q_1'} L_x^{r_1'}} \\ &\leq C \left(T^{\frac{4}{4-d}} + T^{\alpha} \| |\psi(1-\chi(\psi)) \|_{L_t^\infty L_x^6} \right) \left\| |\psi|^2 - |\tilde{\psi}|^2 \right\|_{L_t^{\frac{4}{d}} L_x^2}. \end{aligned} \tag{1.1.43}$$

Concerning the second contribution, we notice that for the specified range of γ estimate (1.1.23) is valid and thus

$$\left| \left(|\tilde{\psi}|^{2(\gamma-1)} - 1 \right) (\psi - \tilde{\psi}) \right| \leq C \left| \left(|\tilde{\psi}|^2 - 1 \right) (\psi - \tilde{\psi}) \right|.$$

Therefore,

$$\left\| \left(|\tilde{\psi}|^{2(\gamma-1)} - 1 \right) (\psi - \tilde{\psi}) \right\|_{L_t^1 L_x^2 + L_t^{\frac{12}{12-d}} L_x^{\frac{3}{2}}} \leq C \left(T^{\frac{1}{q}} + T^{\frac{12-d}{12} - \frac{1}{q}} \right) \| |\tilde{\psi}|^2 - 1 \|_{L_t^\infty L_x^2} \| \psi - \tilde{\psi} \|_{L_t^q (L_x^\infty + L_x^r)}.$$

We observe that $\frac{12-d}{12} - \frac{1}{q} = \frac{24-5d}{24} > 0$. Finally, we conclude that there exists $\alpha > 0$ such that

$$\begin{aligned} &\left\| \mathcal{N}_{reg}(\psi) - \mathcal{N}_{reg}(\tilde{\psi}) \right\|_{N^0} \leq C\sqrt{R} T^{\alpha} \Big(\left\| |\psi|^2 - |\tilde{\psi}|^2 \right\|_{L_t^{\frac{4}{d}} L_x^2} \\ &+ \| \psi - \tilde{\psi} \|_{L_t^q (L_x^\infty + L_x^r)} \Big). \end{aligned} \tag{1.1.44}$$

For $\gamma > 2$ and $d = 2$, the inequality (1.1.41) is still valid. Instead we replace (1.1.42) by

$$\begin{aligned} \left| \mathcal{N}_{reg}(\psi) - \mathcal{N}_{reg}(\tilde{\psi}) \right| &\leq \left| |\psi|^{2(\gamma-1)}(1-\eta(\psi)) - |\tilde{\psi}|^{2(\gamma-1)}(1-\eta(\tilde{\psi})) \right| |\psi| \\ &+ \left| \left(|\tilde{\psi}|^{2(\gamma-1)} - 1 \right) (\psi - \tilde{\psi}) \right|, \end{aligned}$$

followed by

$$\begin{aligned} &\left| |\psi|^{2(\gamma-1)}(1-\eta(\psi)) - |\tilde{\psi}|^{2(\gamma-1)}(1-\eta(\tilde{\psi})) \right| |\psi| \\ &\leq C_\gamma |\psi| \left(|\psi|^{2(\gamma-2)} + |\tilde{\psi}|^{2(\gamma-2)} \right) \left| |\psi|^2 - |\tilde{\psi}|^2 \right|, \end{aligned}$$

where

$$C_\gamma |\psi| \left(|\psi|^{2(\gamma-2)} + |\tilde{\psi}|^{2(\gamma-2)} \right) \in L^\infty(\mathbf{R}^2) + L^p(\mathbf{R}^2),$$

for any $1 \leq p < \infty$ due to the Sobolev embedding $H^1(\mathbf{R}^2) \hookrightarrow L^p(\mathbf{R}^2)$. Thus, we may choose p such that

$$\begin{aligned} &\left\| \left| |\psi|^{2(\gamma-1)}(1-\eta(\psi)) - |\tilde{\psi}|^{2(\gamma-1)}(1-\eta(\tilde{\psi})) \right| |\psi| \right\|_{L_t^1 L_x^2 + L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \leq T^{\frac{1}{2}} \| |\psi|^2 - |\tilde{\psi}|^2 \|_{L_t^2 L_x^2} \\ &+ C T^{\frac{1}{4}} \left(\sup_t \mathcal{E}_2(\psi) \left(\sup_t \mathcal{E}_2(\psi)^{\gamma-2} + \sup_t \mathcal{E}_2(\tilde{\psi})^{\gamma-2} \right) \right) \| |\psi|^2 - |\tilde{\psi}|^2 \|_{L_t^2 L_x^2}. \end{aligned}$$

Similarly, we observe that

$$|\tilde{\psi}|^{2(\gamma-1)} - 1 \in L^2(\mathbf{R}^2) + L^p(\mathbf{R}^2),$$

for any $1 \leq p < 2$. This allows to conclude that there exists $\theta > 0$

$$\begin{aligned}&\left\|(|\tilde{\psi}|^{2(\gamma-1)} - 1)(\psi - \tilde{\psi})\right\|_{L_t^1 L^2 + L_t^{\frac{4}{3}} L_x^{\frac{4}{3}} + L_t^{\frac{3}{2}} L_x^{\frac{6}{5}}}\\ &\leq T^\theta \left(\sup_t \mathcal{E}_2(\tilde{\psi}) + \sup_t \mathcal{E}_2(\tilde{\psi})^{(\gamma-1)}\right) \|\psi - \tilde{\psi}\|_{L_t^q(L_x^\infty + L_x^r)}.\end{aligned}$$

Finally, we obtain that there exists $\alpha > 0$ such that

$$\|\mathcal{N}_{reg}(\psi) - \mathcal{N}_{reg}(\tilde{\psi})\|_{N^0} \leq CT^\theta R^{\gamma-1}(\left(\||\psi|^2 - |\tilde{\psi}|^2\|_{L_t^2 L_x^2} + \|\psi - \tilde{\psi}\|_{L_t^q(L_x^\infty + L_x^r}\right).$$

□

Concatenating the Strichartz estimates (1.0.4) and Lemma 1.1.11 gives the following.

Lemma 1.1.12. *Let $\gamma > 1$ with $\gamma \neq 2$ for $d = 2$ and $1 < \gamma < 2$ for $d = 3$, further Φ as defined in (1.1.24) and $R > 0$ be fixed. Given $\psi, \tilde{\psi} \in \mathbb{E}_2$ such that $\mathcal{E}_2(\psi) \leq R$ and $\mathcal{E}_2(\tilde{\psi}) \leq R$, there exists $C = C(R) > 0$ and $\alpha > 0$ such that*

$$\begin{aligned}&\|\Phi(\psi) - \Phi(\psi_n)\|_{S^0([-T,T]\times\mathbf{R}^d)}\\ &\leq C(R)T^\alpha \left(\|\psi - \tilde{\psi}\|_{L_t^\infty(L_x^\infty + L_x^2)} + \|\psi - \tilde{\psi}\|_{L_t^q(L_x^\infty + L_x^r)} + \||\psi|^2 - |\tilde{\psi}|^2\|_{L_t^2 L_x^2}\right).\end{aligned} \tag{1.1.45}$$

We are now in position to complete the proof of Proposition 1.1.10.

Proof of Proposition 1.1.10 continued. We prove **continuous dependence on the initial data**.

We assume $\gamma \neq 2$. For $\gamma = 2$ the nonlinearity is algebraic and Lipschitz dependence on the initial data is proved by Lemma 4 in [84]. Given $\psi_0 \in \mathbb{E}_2$, let $\{\psi_0^n\}_{n\in N} \subset \mathbb{E}_2$ be such that

$$d(\psi_0^n, \psi_0) \to 0, \quad \text{as} \quad n \to \infty.$$

Let $\psi_n \in C([-T_n, T_n]; \mathbb{E}_2)$ be the unique maximal solution such that $\psi_n(0) = \psi_0^n$. There exists N_0 such that for all $n \geq N_0$, one has that $\mathcal{E}_2(\psi_0^n) \leq 2\mathcal{E}_2(\psi_0)$. It follows that there exists $T = T(2\mathcal{E}_2(\psi_0))$ such that for sufficiently large n, the solutions ψ and ψ_n are defined on $[-T, T]$ and moreover,

$$\sup_{|t|\leq T} (\mathcal{E}_2(\psi_n(t)) + \mathcal{E}_2(\psi(t))) \leq C_T \mathcal{E}_2(\psi_0).$$

By using (1.1.15), we know that for n sufficiently large we have

$$\sup_{|t|\leq T} d_{\mathbb{E}_2}(e^{\frac{i}{2}t\Delta}\psi_0, e^{\frac{i}{2}t\Delta}\psi_0^n) \leq C d_{\mathbb{E}_2}(\psi_0, \psi_0^n), \tag{1.1.46}$$

for n sufficiently large. To compensate for the lack of a Lipschitz regularity of $\nabla\mathcal{N}$, we proceed in several steps.

1. Let $r=4$ if $\gamma<2$ and $r=2\gamma$ if $\gamma>2$, further let q such that (q,r) is an admissible pair. Then, there exists $T'>0$ such that

$$\begin{aligned}&\|\psi_n-\psi\|_{L^\infty([-T',T'];L^\infty+L^2)}+\|\psi_n-\psi\|_{L^q([-T',T'];L^\infty+L^r)}\\&+\||\psi_n|^2-|\psi|^2\|_{L^{\frac{4}{d}}([-T',T'];L^2)}\\&\leq Cd_{\mathbb{E}_2}(\psi_0^n,\psi_0).\end{aligned}\tag{1.1.47}$$

2. Provided (1.1.47) holds, we infer that

$$\|\nabla\psi_n-\nabla\psi\|_{L^\infty([-T',T'];L^2)}\to 0,\tag{1.1.48}$$

as n goes to ∞.

3. We combine the previous steps to conclude that

$$\sup_{|t|\leq T'} d_{\mathbb{E}_2}(\psi_n(t),\psi(t))\to 0.\tag{1.1.49}$$

Step 1 We show (1.1.47). We dispose of the first contribution, there exists $\theta>0$ such that,

$$\begin{aligned}&\|\psi-\psi_n\|_{L^\infty([-T,T];L^\infty+L^2(\mathbf{R}^d))}\\&\leq\|e^{\frac{i}{2}t\Delta}\psi_0-e^{\frac{i}{2}t\Delta}\psi_0^n\|_{L^\infty_t([-T,T];L^\infty_x+L^2_x(\mathbf{R}^d))}+\|\Phi(\psi)-\Phi(\psi_n)\|_{L^\infty([-T,T],L^2(\mathbf{R}^d))}\\&\leq Cd_{\mathbb{E}_2}(\psi_0,\psi_0^n)+C\|\Phi(\psi)-\Phi(\psi_n)\|_{L^\infty([-T,T],L^2(\mathbf{R}^d))}\end{aligned}\tag{1.1.50}$$

where we used (1.1.15) to control the linear part of the solution and (1.1.45) to bound the non-homogeneous contribution $\Phi(\psi)-\Phi(\psi_n)$.
Similarly,

$$\begin{aligned}&\|\psi-\psi_n\|_{L^q([-T,T];L^\infty+L^r(\mathbf{R}^d))}\\&\leq\|e^{\frac{i}{2}t\Delta}\psi_0-e^{\frac{i}{2}t\Delta}\psi_0^n\|_{L^q([-T,T];L^\infty+L^r(\mathbf{R}^d))}+\|\Phi(\psi)-\Phi(\psi_n)\|_{L^q([-T,T],L^r(\mathbf{R}^d)}\end{aligned}$$

Applying (1.1.46) yields

$$\begin{aligned}\|e^{\frac{i}{2}t\Delta}\psi_0-e^{\frac{i}{2}t\Delta}\psi_0^n\|_{L^q_t(L^\infty_x+L^r_x)}&\leq T^{\frac{1}{q}}d_{\mathbb{E}_2}\left(e^{\frac{i}{2}t\Delta}\psi_0,e^{\frac{i}{2}t\Delta}\psi_0^n\right)\\&\leq CT^{\frac{1}{q}}d_{\mathbb{E}_2}(\psi_0,\psi_0^n).\end{aligned}$$

Proceeding as before, we recover

$$\|\psi-\psi_n\|_{L^q([-T,T];L^\infty+L^r(\mathbf{R}^d))}\leq CT^{\frac{1}{q}}d(\psi_0,\psi_0^n)+\|\Phi(\psi)-\Phi(\psi_n)\|_{L^q([-T,T];L^r(\mathbf{R}^d))}.\tag{1.1.51}$$

We notice that

$$\begin{aligned}\left||\psi_n|^2-|\psi|^2\right|\leq&\left||e^{\frac{i}{2}t\Delta}\psi_0|^2-|e^{\frac{i}{2}t\Delta}\psi_0^n|^2\right|+\left|2Re\left(e^{-it\Delta}\overline{\psi_0^n}\left(\Phi(\psi_n)-\Phi(\psi)\right)\right)\right|\\&+\left|2Re\left(e^{-it\Delta}(\overline{\psi_0^n}-\overline{\psi_0})\Phi(\psi)\right)\right|\\&+\left(|\Phi(\psi_n)|+|\Phi(\psi)|\right)|\Phi(\psi_n)-\Phi(\psi)|\,.\end{aligned}$$

We control the terms separately. From (1.1.46), one has that

$$\left\| |e^{\frac{i}{2}t\Delta}\psi_0|^2 - |e^{\frac{i}{2}t\Delta}\psi_0^n|^2 \right\|_{L_t^{\frac{4}{d}} L_x^2} \le CT^{\frac{d}{4}} d(\psi_0, \psi_0^n).$$

Next,

$$\begin{aligned}
&\left\| 2Re\left(e^{-it\Delta}\overline{\psi_0^n}\left(\Phi(\psi_n) - \Phi(\psi)\right)\right)\right\|_{L_t^{\frac{4}{d}} L_x^2} \\
&\le T^{\frac{4}{d}} \sup_t \mathcal{E}_2(e^{\frac{i}{2}t\Delta}\psi_0^n)\|\Phi(\psi_n) - \Phi(\psi)\|_{L_t^\infty L_x^2} + T^{\frac{d}{8}} \sup_t \mathcal{E}_2(e^{\frac{i}{2}t\Delta}\psi_0^n)^{\frac{1}{2}} \|\Phi(\psi_n) - \Phi(\psi)\|_{L_t^{\frac{8}{d}} L_x^4}
\end{aligned}$$

Using the previous inequalities, we infer that there exists $\theta_1 > 0$ such that

$$\begin{aligned}
&\left\| 2Re\left(e^{-it\Delta}\overline{\psi_0^n}\left(\Phi(\psi_n) - \Phi(\psi)\right)\right)\right\|_{L_t^{\frac{4}{d}} L_x^2} \\
&\le C(R)T^{\theta_1}\left(\|\psi - \psi_n\|_{L_t^\infty(L_x^\infty + L_x^2)} + \|\psi - \psi_n\|_{L_t^\infty(L_x^\infty + L_x^r)} + \||\psi|^2 - |\psi_n|^2\|_{L_t^q L_x^2}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left\| 2Re\left(e^{-it\Delta}(\overline{\psi_0^n} - \overline{\psi_0})\Phi(\psi)\right)\right\|_{L_t^{\frac{q}{2}} L_x^2} \\
&\le C\left(T^{\frac{d}{4}}\|\Phi(\psi)\|_{L_t^\infty L_x^2} + T^{\frac{d}{8}}\|\Phi(\psi)\|_{L_t^{\frac{8}{d}} L_x^4}\right) d_{\mathbb{E}_2}(\psi_0, \psi_0^n).
\end{aligned}$$

Finally, there exists $\theta_2 > 0$ such that

$$\begin{aligned}
&\|\left(|\Phi(\psi_n)| + |\Phi(\psi)|\right)|\Phi(\psi_n) - \Phi(\psi)|\|_{L_t^{\frac{q}{2}} L_x^2} \\
&\le \left(\|\Phi(\psi_n)\|_{L_t^q L_x^4} + \|\Phi(\psi)\|_{L_t^q L_x^4}\right)\|\Phi(\psi_n) - \Phi(\psi)\|_{L_t^q L_x^4} \\
&\le CT^{\theta_2}\left(\sup_t \mathcal{E}_2(\psi(t)) + (\sup_t \mathcal{E}_2(\psi_n(t))\right) \\
&\times \left(\|\psi - \psi_n\|_{L_t^\infty(L_x^\infty + L_x^2)} + \|\psi - \psi_n\|_{L_t^\infty(L_x^\infty + L_x^r)} + \||\psi|^2 - |\psi_n|^2\|_{L_t^q L_x^2}\right).
\end{aligned}$$

Combining the previous four inequalities, we infer that there exists $\theta > 0$

$$\begin{aligned}
&\left\||\psi_n|^2 - |\psi|^2\right\|_{L_t^q L_x^2} \le C(R)(1 + T^\theta) d_{\mathbb{E}_2}(\psi_0, \psi_0^n) \\
&+ C(R)T^\theta\left(\|\psi - \psi_n\|_{L_t^\infty(L_x^\infty + L_x^2)} + \|\psi - \psi_n\|_{L_t^\infty(L_x^\infty + L_x^r)} + \||\psi|^2 - |\psi_n|^2\|_{L_t^q L_x^2}\right).
\end{aligned} \tag{1.1.52}$$

Summing up (1.1.50), (1.1.51) and (1.1.52) and applying (1.1.45) yields that there exists $\theta > 0$ such that

$$\begin{aligned}
&\|\psi_n - \psi\|_{L^\infty([-T,T];L^\infty + L^2(\mathbf{R}^d))} + \|\psi_n - \psi\|_{L^q([-T,T];L^\infty + L^r(\mathbf{R}^d))} + \||\psi_n|^2 - |\psi|^2\|_{L^{\frac{4}{d}}([-T,T];L^2(\mathbf{R}^d))} \\
&\le C(1 + T^{\frac{1}{q}}) d_{\mathbb{E}_2}(\psi_0^n, \psi_0) + C(R)T^\theta\Big(\|\psi_n - \psi\|_{L^\infty([-T,T];L^\infty + L^2(\mathbf{R}^d))} + \|\psi_n - \psi\|_{L^q([-T,T];L^\infty + L^r(\mathbf{R}^d))} \\
&+ \||\psi_n|^2 - |\psi|^2\|_{L^{\frac{4}{d}}([-T,T];L^2(\mathbf{R}^d))}\Big).
\end{aligned}$$

For $T > 0$ sufficiently small depending on R, the desired inequality (1.1.47) follows and Step 1 is complete.

Step 2.
We recall that the linear propagator $e^{\frac{i}{2}t\Delta}$ and ∇ commute, so that

$$\nabla\psi - \nabla\psi_n = e^{\frac{i}{2}t\Delta}\left(\nabla\psi_0 - \nabla\psi_0^n\right) - i\int_0^t e^{\frac{i}{2}(t-s)\Delta}\left(\nabla\mathcal{N}(\psi) - \nabla\mathcal{N}(\psi_n)\right)(s)\mathrm{d}s.$$

We denote $\mathcal{N}'(\psi) = D\mathcal{N}(\psi)$, Lemma 1.1.9 states that $\mathcal{N} \in C^1(\mathbf{R}^2, \mathbf{R}^2)$ and thus we have that $\nabla\mathcal{N}(\psi) = \mathcal{N}'(\psi)\nabla\psi$ for a.e. $\psi \in \mathbf{C}$, see for instance Remark 1.3.(v) in [52]. We estimate the difference of the linear terms as

$$\left\|e^{\frac{i}{2}t\Delta}\left(\nabla\psi_0 - \nabla\psi_0^n\right)\right\|_{L^\infty([-T,T],L^2(\mathbf{R}^d))} \leq Cd_{\mathbb{E}_2}(\psi_0, \psi_0^n), \tag{1.1.53}$$

since $e^{\frac{i}{2}t\Delta}$ is an isometry on $L^2(\mathbf{R}^d)$. We split the non-homogeneous term in

$$\begin{aligned}
&\left\|i\int_0^t e^{\frac{i}{2}(t-s)\Delta}\left(\nabla\mathcal{N}(\psi) - \nabla\mathcal{N}(\psi_n)\right)(s)\mathrm{d}s\right\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))}\\
&\leq \|\mathcal{N}_1'(\psi_n)(\nabla\psi - \nabla\psi_n)\|_{L^1([-T,T];L^2(\mathbf{R}^d))} + \|\mathcal{N}_2'(\psi_n)(\nabla\psi - \nabla\psi_n)\|_{L^{q_1'}([-T,T];L^{\frac{6}{5}}(\mathbf{R}^d))}\\
&+ \left\|\left(\mathcal{N}_1'(\psi) - \mathcal{N}_1'(\psi_n)\right)\nabla\psi\right\|_{L^1([-T,T];L^2(\mathbf{R}^d))} + \left\|\left(\mathcal{N}_2'(\psi) - \mathcal{N}_2'(\psi)\right)\nabla\psi_n\right\|_{L^{q_1'}([-T,T];L^{\frac{6}{5}}(\mathbf{R}^d))}\\
&\leq C\left(T + T^{\frac{1}{q'}}\mathcal{E}_2(\psi)^\alpha\right)\|\nabla\psi - \nabla\psi_n\|_{L^\infty([-T,T],L^2(\mathbf{R}^d))}\\
&+ \left\|\left(\mathcal{N}_1'(\psi) - \mathcal{N}_1'(\psi_n)\right)\nabla\psi\right\|_{L^1([-T,T];L^2(\mathbf{R}^d))} + \left\|\left(\mathcal{N}_2'(\psi) - \mathcal{N}_2'(\psi)\right)\nabla\psi_n\right\|_{L^{q_1'}([-T,T];L^{\frac{6}{5}}(\mathbf{R}^d))},
\end{aligned} \tag{1.1.54}$$

where q_1 such that $(q_1, 6)$ is admissibile and $\alpha = \max\{1, \gamma - 1\}$. Thus for $T > 0$ sufficiently small so that

$$C\left(T + T^{\frac{1}{q'}}\mathcal{E}_2(\psi)^\alpha\right) \leq \frac{1}{2}.$$

We conclude that combining (1.1.53) and (1.1.54) that

$$\begin{aligned}
&\|\nabla\psi - \nabla\psi_n\|_{L^\infty([-T,T],L^2(\mathbf{R}^d))} \leq Cd_{\mathbb{E}_2}(\psi_0, \psi_0^n)\\
&+ \left\|\left(\mathcal{N}_1'(\psi) - \mathcal{N}_1'(\psi_n)\right)\nabla\psi\right\|_{L^1([-T,T];L^2(\mathbf{R}^d))} + \left\|\left(\mathcal{N}_2'(\psi) - \mathcal{N}_2'(\psi)\right)\nabla\psi_n\right\|_{L^{q_1'}([-T,T];L^{\frac{6}{5}}(\mathbf{R}^d))}.
\end{aligned}$$

To prove (1.1.48) we are left to show that the second line of the right-hand side converges to 0 as n goes to infinity. We proceed by contradiction assuming that there exists a subsequence still denoted ψ_n such that for all n,

$$\left\|\left(\mathcal{N}_1'(\psi) - \mathcal{N}_1'(\psi_n)\right)\nabla\psi\right\|_{L^1_tL^2_x} + \left\|\left(\mathcal{N}_2'(\psi) - \mathcal{N}_2'(\psi)\right)\nabla\psi_n\right\|_{L^{q_1'}_tL^{\frac{6}{5}}_x} \geq \varepsilon. \tag{1.1.55}$$

Inequality (1.1.47) implies that up to passing to a subsequence, still denoted ψ_n, we have ψ_n converges to ψ a.e. on $(-T,T) \times \mathbf{R}^d$. Further, there exists $\phi \in L^\infty((-T,T); L^{2^*}(\mathbf{R}^d))$ with $2^* = \frac{2d}{d-2}$ such that $|\psi_n|(1 - \chi(\psi_n)) \leq \phi$ a.e. on $(-T,T) \times \mathbf{R}^d$. Since $\mathcal{N} \in C^1(\mathbf{R}^2, \mathbf{R}^2)$, one has that

$$\begin{aligned}
\left|\left(\mathcal{N}_1'(\psi) - \mathcal{N}_1'(\psi_n)\right)\nabla\psi\right| &\to 0 \quad \text{a.e. in} \quad (-T,T) \times \mathbf{R}^d,\\
\left|\left(\mathcal{N}_2'(\psi) - \mathcal{N}_2'(\psi_n)\right)\nabla\psi\right| &\to 0 \quad \text{a.e. in} \quad (-T,T) \times \mathbf{R}^d.
\end{aligned}$$

Moreover, we control

$$\left|\left(\mathcal{N}_1'(\psi)-\mathcal{N}_1'(\psi_n)\right)\nabla\psi\right| \leq C|\nabla\psi| \in L^1((-T,T);L^2(\mathbf{R}^d)),$$
$$\left|\left(\mathcal{N}_2'(\psi)-\mathcal{N}_2'(\psi_n)\right)\nabla\psi\right| \leq C\left(|\psi|^{2(\gamma-1)}+|\phi|^{2(\gamma-1)}\right)|\nabla\psi| \in L^{q_1'}((-T,T);L^{\frac{6}{5}}(\mathbf{R}^d)),$$

so that the dominated convergence Theorem then implies that (1.1.55) is violated and Step 2 is complete.

Step 3. It remains to show that

$$\left\||\psi|^2-|\psi_n|^2\right\|_{L^\infty([-T,T];L^2(\mathbf{R}^d))} \to 0.$$

Inequality (1.1.15) implies that

$$\begin{aligned}&\left\||\psi|^2-|\psi_n|^2\right\|_{L_t^\infty L_x^2} \leq C\left(1+\|\psi-\psi_0\|_{L_t^\infty H_x^1}+\|\psi_n-\psi_n^0\|_{L_t^\infty H_x^1}\right)d_{\mathbb{E}_2}(\psi_0,\psi_0^n)\\&+\left(1+\sup_t\mathcal{E}_2(\psi)+\sup_t\mathcal{E}_2(\psi_n)+\|\psi-\psi_0\|_{L_t^\infty H_x^1}+\|\psi_n-\psi_n^0\|_{L_t^\infty H_x^1}\right)\\&\times\|\psi-\psi_0-(\psi_n-\psi_n^0)\|_{L_t^\infty H_x^1}.\end{aligned}$$

Hence, it suffices to show that,

$$\|\psi-\psi_0-(\psi_n-\psi_n^0)\|_{L_t^\infty H_x^1} \to 0.$$

We notice that (1.1.48) yields

$$\|\nabla\left(\psi-\psi_0-(\psi_n-\psi_n^0)\right)\|_{L_t^\infty L_x^2} \leq C\|\nabla\psi_n-\nabla\psi\|_{L_t^\infty L_x^2}+\|\nabla\psi_0-\nabla\psi_0^n\|_{L_x^2} \to 0,$$

thanks to (1.1.48) from Step 2. We are left to show that

$$\|\psi-\psi_0-(\psi_n-\psi_n^0)\|_{L_t^\infty L_x^2} \to 0.$$

To that end we observe that

$$\psi-\psi_0-(\psi_n-\psi_n^0)=e^{\frac{i}{2}t\Delta}(\psi_0-\psi_0^n)-(\psi_0-\psi_0^n)-i\int_0^t e^{\frac{i}{2}(t-s)\Delta}\left(\mathcal{N}(\psi)-\mathcal{N}(\psi)\right)(s)\mathrm{d}s.$$

Lemma 1.1.8 yields that

$$\left\|e^{\frac{i}{2}t\Delta}(\psi_0-\psi_0^n)-(\psi_0-\psi_0^n)\right\|_{L_t^\infty L_x^2} \leq CT^{\frac{1}{2}}\|\nabla(\psi_0-\psi_0^n)\|_{L_x^2} \to 0.$$

Applying (1.1.45) or (1.1.45) allows to conclude,

$$\left\|-i\int_0^t e^{\frac{i}{2}(t-s)\Delta}\left(\mathcal{N}(\psi)-\mathcal{N}(\psi)\right)(s)\mathrm{d}s\right\|_{L_t^\infty L_x^2} \to 0.$$

Step 3 is complete and hence we have shown continuous dependence on the initial data w.r.t. to the topology of $\mathbb{E}_2$ induced by the metric $d_{\mathbb{E}_2}$. □

1.1.3 Global well-posedness in $2d$ and for subcubic nonlinearities in $3d$

In this paragraph, we extend the local theory to a global well-posedness result. For that purpose, we firstly show that the energy as defined in (1.0.2) controls $\mathcal{E}_2(\psi)$ and vice versa. Secondly, we prove that the energy is conserved by the flow for regular solutions. Finally, persistence of the regularity together with an approximation argument yields global existence for solutions in the energy space.

We start by observing that if ψ is of finite energy, namely $\mathcal{E}(\psi) < \infty$ with $\mathcal{E}$ defined in (1.0.2), then $\psi \in \mathbb{E}_\gamma$ given by

$$\mathbb{E}_\gamma = \left\{ \psi \in H^1_{loc}(\mathbf{R}^d) \quad : \quad \nabla\psi \in L^2(\mathbf{R}^d), \quad |\psi|^2 - 1 \in L^\gamma_2(\mathbf{R}^d) \right\}. \tag{1.1.56}$$

Indeed, the property $F(|\psi|^2) \in L^1(\mathbf{R}^d)$, for F defined in (1.0.2), is equivalent to $|\psi|^2 - 1 \in L^\gamma_2(\mathbf{R}^d)$, where L^γ_2 denotes the Orlicz space, as will follow from Lemma 1.1.13, see Chapter 5 in [135] and Appendix A in [9]. In particular, we recall that $F(\rho)$ is equivalent to $|\rho - 1|^2$ if $|\rho - 1| < \frac{1}{2}$ and equivalent to $|\rho - 1|^\gamma$ if $|\rho - 1| > \frac{1}{2}$. The choice of the exponent γ therefore influences the behavior for $\rho \to 0$ and $\rho \to \infty$ while the behavior close to the minimum $\rho = 1$ remains quadratic.

Lemma 1.1.13. *Let*

$$V(\psi) = \int_{\mathbf{R}^d} F(|\psi|^2)\mathrm{d}x,$$

then $V \in C^1(\mathbb{E}_2; \mathbf{R})$ and the first variation satisfies $\frac{\delta V}{\delta \overline{\psi}} = \mathcal{N}(\psi)$. Further, if $\gamma > 1$ for $d = 2$ and $1 < \gamma < 3$ for $d = 3$, then $\mathbb{E}_\gamma = \mathbb{E}_2$.

This can be seen as the analogous statement compared to the subcritical defocusing nonlinear Schrödinger equation where $H^1(\mathbf{R}^d)$ coincides with the space of functions for which the Hamiltonian is finite.

Proof. Since $\gamma > 1$, the function $F : \mathbf{R}_+ \to R_+$ is convex, non-negative and achieves its global minimum in $F(1) = 0$. In particular, $F \in C^1(\mathbf{R}, \mathbf{R}_+)$. Next, we check that $V : \mathbb{E}_2 \to \mathbf{R}_+$ is well-defined. Let $\psi \in \mathbb{E}_2$. In order to conclude that $V(\psi) \leq \mathcal{E}_2(\psi)$, we use that

$$F(|\psi|^2) \in L^1(\mathbf{R}^d) \quad \Leftrightarrow \quad |\psi|^2 - 1 \in L^\gamma_2(\mathbf{R}^d).$$

Namely, we check that there exists $C > 0$ such that

$$\int_{\mathbf{R}^d} \left||\psi|^2 - 1\right|^2 \mathbf{1}_{\{||\psi|^2-1| \leq \frac{1}{2}\}} + \left||\psi|^2 - 1\right|^\gamma \mathbf{1}_{\{||\psi|^2-1| \geq \frac{1}{2}\}}\mathrm{d}x \leq C. \tag{1.1.57}$$

If $\gamma \leq 2$ the conclusion is immediate. If $\gamma \geq 2$, we observe that $\{||\psi|^2 - 1| \geq \frac{1}{2}\}$ is of finite Lebesgue measure. Moreover, since

$$\int_{\mathbf{R}^d} |\nabla|\psi||^2 \,\mathrm{d}x \leq C \int_{\mathbf{R}^d} |\nabla\psi|^2 \,\mathrm{d}x < +\infty,$$

and

$$||\psi|-1| \leq \left||\psi|^2-1\right|,$$

for $x \in \{||\psi|^2-1| \leq \frac{1}{2}\}$ we have that $|\psi|-1 \in H^1(\mathbf{R}^d)$. Thus, by Sobolev embedding we infer that

$$\int_{\mathbf{R}^d} \left||\psi|^2-1\right|^{\gamma} \mathbf{1}_{\{||\psi|^2-1| \geq \frac{1}{2}\}} \leq C.$$

Conversely, arguing analogously it is clear that if $|\psi|^2-1 \in L^2_\gamma$ with $\gamma \geq 2$ then (1.1.57) implies that $\psi \in \mathbb{E}_2$ provided that $2\gamma \leq 2^*$. If $\gamma < 2$, then we exploit that by Markov-inequality the set $\mathcal{A} = \left\{x \in \mathbf{R}^d \quad : \quad \left||\psi|^2-1\right| > \frac{1}{2}\right\}$ is of finite Lebesgue measure and we proceed as before using Sobolev embedding. Thus,

$$\int_{\mathbf{R}^d} \left||\psi|^2-1\right|^2 \mathbf{1}_{\{||\psi|^2-1| \geq \frac{1}{2}\}} < +\infty.$$

It follows, $|\psi|^2-1 \in L^2(\mathbf{R}^d)$. □

We are now ready to show persistence of the regularity and conservation of the energy (1.0.2) for regular solutions to (1.1.1).

Lemma 1.1.14. *Let $d = 2, 3$ and $\gamma > 1$ if $d = 2$ and $1 < \gamma \leq 2$ if $d = 3$. Let $\psi_0 \in \mathbb{E}_2$ be such that $\Delta\psi_0 \in L^2(\mathbf{R}^d)$. Then there exists a unique solution $\psi \in C(\mathbf{R}; \mathbb{E}_2)$ to (1.1.1) such that for any $0 < T < \infty$ one has $\Delta\psi \in L^\infty([-T, T]; L^2(\mathbf{R}^d))$ and $\partial_t\psi \in L^\infty([-T, T]; L^2(\mathbf{R}^d))$ Moreover,*

$$\mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0),$$

for all $t \in \mathbf{R}$.

Proof. Let $\psi_0 \in \mathbb{E}_2$ and $\Delta\psi_0 \in L^2(\mathbf{R}^d)$. Proposition 1.1.10 provides a $T > 0$ such that there exists a unique $\psi \in C([-T, T]; \mathbb{E}_2)$ strong solution to (1.1.1) with initial data $\psi(0) = \psi_0$. The blow-up alternative yields that there exists $T_1 > 0$ and $C = C(\mathcal{E}_2(\psi_0)) > 0$ such that for all $t \in [-T_1, T_1]$,

$$\mathcal{E}_2(t) \leq C.$$

In particular, exploiting continuity in time we conclude

$$i\partial_t\psi(0) = -\frac{1}{2}\Delta\psi_0 + \mathcal{N}(\psi_0).$$

We claim that $\partial_t\psi(0) \in L^2(\mathbf{R}^d)$. Indeed, for $\mathcal{N}_1, \mathcal{N}_2$ defined in (1.1.17), one has

$$\|\mathcal{N}_1(\psi_0)\|_{L^2(\mathbf{R}^d)} \leq \||\psi_0|^2-1\|_{L^2},$$

and

$$\|\mathcal{N}_2(\psi_0)\|_{L^2} \leq C\||\psi_0|^{2\gamma-1}\|_{L^2(supp(\chi(\psi))^c)} \leq \|\psi_0\|^{2(\gamma-1)}_{L^{2(2\gamma-1)}(supp(\chi(\psi))^c)} \leq \|\psi_0\|^{2\gamma-1}_{X^2+H^2},$$

provided $1 < \gamma < \infty$. In particular, we observe that $\|\mathcal{N}_2(\psi_0)\|_{L^2} \leq \sup_t \mathcal{E}_2(\psi)^6$ provided that $\gamma \leq 2$. It follows $\partial_t\psi(0) \in L^2(\mathbf{R}^d)$. Denote $\phi_0 := \partial_t\psi(0)$ and $\phi := \partial_t\psi$. Differentiating (1.1.1) in time yields

$$i\partial_t\phi = -\frac{1}{2}\Delta\phi + \left(\gamma|\psi|^{2(\gamma-1)} - 1\right)\phi + (\gamma-1)|\psi|^{2(\gamma-2)}\psi^2\overline{\phi}. \tag{1.1.58}$$

Here, we have used that $\partial_t\mathcal{N}(\psi) = \mathcal{N}'(\psi)\partial_t\psi$ a.e. on $(-T,T)\times\mathbf{R}^d$ from Lemma 1.1.9. For $\chi \in C_c^\infty(\mathbf{C},\mathbf{R})$ as in (1.1.17), we have that

$$\left\|\left(\left(\gamma|\psi|^{2(\gamma-1)} - 1\right)\phi + (\gamma-1)|\psi|^{2(\gamma-2)}\psi^2\overline{\phi}\right)\chi(\psi)\right\|_{L_t^1L_x^2} \leq \|\phi\|_{L_t^1L_x^2},$$

and for $r=6$ and q such that (q,r) is admissible,

$$\begin{aligned}
&\left\|\left(\left(\gamma|\psi|^{2(\gamma-1)} - 1\right)\phi + (\gamma-1)|\psi|^{2(\gamma-2)}\psi^2\overline{\phi}\right)(1-\chi(\psi))\right\|_{L_t^{q'}L_x^{\frac{6}{5}}} \\
&\leq T^{\frac{1}{q'}}\|\psi(1-\chi(\psi)\|^3_{L_t^\infty L^{6(\gamma-1)}}\|\phi\|_{L_t^\infty L_x^2} \\
&\leq \sup_t \mathcal{E}_2(\psi)^{2(\gamma-1)}\|\phi\|_{L_t^\infty L_x^2},
\end{aligned}$$

Thus, there exists $0 < T' < T_1$ such that

$$\|\phi\|_{L^\infty([-T',T'];L^2(\mathbf{R}^d))} \leq \|\phi_0\|_{L^2} + \left(T' + T'^{\frac{1}{q'}}\sup_{|t|\leq T'}\mathcal{E}_2(\psi)(t)^3\right)\|\phi\|_{L^\infty([-T',T'];L^2(\mathbf{R}^d))},$$

and

$$\left(T' + T'^{\frac{1}{q'}}\sup_{|t|\leq T'}\mathcal{E}_2(\psi)(t)^3\right) < \frac{1}{2}.$$

Thus,

$$\|\phi\|_{L^\infty([-T',T'];L^2(\mathbf{R}^d))} \leq 2\|\phi_0\|_{L^2}.$$

We stress that $T' < T_1$ only depends on $\mathcal{E}_2(\psi_0)$. Further, if $\gamma \leq 2$, by using the equation we recover

$$\begin{aligned}
\|\Delta\psi\|_{L^\infty[-T',T'];L^2(\mathbf{R}^d))} &\leq \|\phi\|_{L^\infty[-T,T];L^2(\mathbf{R}^d))} + \|\mathcal{N}(\psi)\|_{L^\infty([-T',T'];L^2(\mathbf{R}^d))} \\
&\leq \|\phi\|_{L^\infty([-T',T'];L^2(\mathbf{R}^d))} + \sup_t \mathcal{E}_2(\psi)(t) + \sup_t \mathcal{E}_2(\psi)(t)^{2\gamma-1}.
\end{aligned}$$

If $\gamma > 2$, one has that

$$\|\mathcal{N}_2(\psi)\|_{L^\infty([-T',T'];L^2(\mathbf{R}^d))} \leq \|\psi_{high}\|^{2\gamma-1}_{L^\infty([-T',T'];L^{2(2\gamma-1)}(\mathbf{R}^d))}.$$

If $d=2$ the right-hand side is bounded due to the Sobolev embedding. If $d=3$, we apply the Gagliardo-Nirenberg inequality to infer that

$$\|\psi_{high}\|_{L^\infty([-T',T'];L^{2(2\gamma-1)}(\mathbf{R}^3))} \leq \|\Delta\psi\|^\theta_{L^\infty([-T',T'];L^2(\mathbf{R}^3))}\|\psi_{high}\|^{1-\theta}_{L^\infty([-T',T'];L^6(\mathbf{R}^3))},$$

with $\theta = \frac{\gamma-2}{2\gamma-1}$. Therefore,

$$\|\mathcal{N}_2(\psi)\|_{L^\infty[-T',T'];L^2(\mathbf{R}^3))} \leq \|\Delta\psi\|^{\gamma-2}_{L^\infty[-T',T'];L^2(\mathbf{R}^3))}\|\psi_{high}\|^{\gamma+1}_{L^\infty[-T',T'];L^6(\mathbf{R}^3))},$$

and by consequence,

$$\begin{aligned}\|\Delta\psi\|_{L^\infty[-T',T'];L^2(\mathbf{R}^3))} \leq \|\phi\|_{L^\infty[-T',T'];L^2(\mathbf{R}^3))} + \sup_t \mathcal{E}_2(\psi)^{\frac{1}{2}} \\ + \sup_t \mathcal{E}_2(\psi)^{\gamma+1}\|\Delta\psi\|^{\gamma-2}_{L^\infty[-T',T'];L^2(\mathbf{R}^3))}.\end{aligned}$$

Hence $\Delta\psi \in L^\infty[-T',T'];L^2(\mathbf{R}^d))$ for all $\gamma > 1$ if $d = 2$ ad for $1 < \gamma \leq 3$ if $d = 3$. Next, we show that $\mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0)$ for all $t \in [-T',T']$. To that end, we compute the L^2 scalar product of (1.1.1) and $\partial_t\psi$ and take the real part to infer

$$0 = Re\,\langle i\partial_t\psi, \partial_t\psi\rangle = Re\,\langle -\Delta\psi + \mathcal{N}(\psi), \partial_t\psi\rangle\,,$$

for any $t \in [-T',T']$. We notice that all products are well-defined and a.e. on $\mathbf{R}^d$ one has that

$$\int_{\mathbf{R}^d} Re(\overline{\mathcal{N}(\psi)}\partial_t\psi)\mathrm{d}x = \int_{\mathbf{R}^d}(|\psi|^{2(\gamma-1)} - 1)Re(\overline{\psi}\partial_t\psi)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^d} F(|\psi|^2)\mathrm{d}x.$$

where we used Lemma 1.1.13. We conclude that for all $t \in [-T',T']$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^d}\frac{1}{2}|\nabla\psi|^2 + F(|\psi|^2)\mathrm{d}x.$$

By consequence, we have $\mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0)$ for all $t \in [-T',T']$. Lemma 1.1.13 states that $\mathcal{E}(\psi)$ and $\mathcal{E}_2(\psi)$ are equivalent and hence since $T' > 0$ only depends on $\sup_{t\in[-T',T']}\mathcal{E}_2(t)$ and

$$\mathcal{E}_2(\psi(t)) \sim \mathcal{E}(\psi(t) = \mathcal{E}(\psi_0),$$

we may therefore iterate the above procedure and Proposition 1.1.10 to conclude that for any $T > 0$, there exists a solution $\psi \in C([-T,T];\mathbb{E}_2)$ to (1.1.1) with $\partial_t\psi \in L^\infty([-T,T];L^2(\mathbf{R}^d))$ and $\Delta\psi \in L^\infty([-T,T];L^2(\mathbf{R}^d))$ and moreover for all $t \in [-T,T]$ one has $\mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0)$. It remains to show that ψ is a global solution, namely $\psi \in C(\mathbf{R},\mathbb{E}_2)$. By contradiction, assume that there exists a maximal time $T < \infty$ such that $\psi \in C([-T,T];\mathbb{E}_2)$ and $T < \infty$. In view of the blow-up alternative provided by Proposition 1.1.10, we conclude that

$$\lim_{t\nearrow T}\mathcal{E}_2(\psi(t)) = \infty.$$

Since again $\mathcal{E}_2(\psi(t)) \sim \mathcal{E}(\psi)(t)$, we obtain a contradiction to the conservation of energy. By consequence, $T = \infty$ and the proof is complete. □

Combining Lemma 1.1.14 and Proposition 1.1.10 yields the global well-posedness result in the space $\mathbb{E}_2$.

Corollary 1.1.15. *Let $d = 2, 3$ and $\gamma > 1$ if $d = 2$ and $1 < \gamma \leq 2$ if $d = 3$. Then* (1.1.1) *is globally wellposed in $\mathbb{E}_2$. In particular,*

$$\mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0),$$

for all $t \in \mathbf{R}$.

The statement follows from a standard approximation argument.

Proof. Let $\psi_0 \in \mathbb{E}_2(\mathbf{R}^d)$, then thanks to Proposition 1.1.10 there exits $T > 0$ and a unique solution $\psi \in C([-T, T]; \mathbb{E}_2)$ to (1.1.1). Moreover, Lemma 1.1.6 states that there exists $\{\psi_0^n\} \subset C^\infty$ such that $\nabla\psi_0^n \in L^2(\mathbf{R}^d)$ uniformly and $d(\psi_0, \psi_0^n)$ converges to 0 as n goes to infinity. Lemma 1.1.14 provides a sequence of global solutions $\psi_n \in C(\mathbf{R}, \mathbb{E}_2)$ such that $\mathcal{E}(\psi_n(t)) = \mathcal{E}(\psi_0^n)$ for all n. Relying on the continuous dependence on the initial data, we conclude that

$$\sup_{t\in[-T,T]} d(\psi(t), \psi_n(t)) \to 0 \qquad \text{as} \quad n \to \infty.$$

Hence, $\mathcal{E}(\psi_n)(t) \to \mathcal{E}(\psi(t))$ for all $t \in (-T, T)$. Iterating the local existence result of Proposition 1.1.10 together with the blow-up alternative yield the global existence, i.e. $\psi \in C(\mathbf{R}, \mathbb{E}_2)$. $\square$

1.1.4 Well-posedness theory in $3d$ for supercubic nonlinearities

This paragraph covers the $3d$ well-posedness theory of (1.1.1) for $2 \leq \gamma < 3$. The theory exploits the particular structure of the energy space (1.1.4) for $d = 3$ and relies on the regularity properties of the nonlinearity $\mathcal{N}$, namely that in this regime $\nabla\mathcal{N}$ is locally Lipschitz. Indeed, recalling (1.1.16), one may write $\nabla\mathcal{N}(c + u) = F_1(c + u)\nabla u + F_2(c + u)\overline{\nabla u}$, where $F_j(z) = O(|z|^{2(\gamma-1)})$. Further, we have $\nabla F_j(z) = O(|z|^{2\gamma-3})$, so that for $\gamma \geq \frac{3}{2}$ we conclude that $\nabla\mathcal{N}$ is locally Lipschitz. This motivates us to present an existence result for $\gamma \in [\frac{3}{2}, 3)$ including an alternative proof of Proposition 1.1.10 for the range $\frac{3}{2} \leq \gamma \leq 2$. For the energy-critical case $\gamma = 3$ we refer to [82] and [118]. In [82], local well-posedness in the energy space for small data is proven, in [118] the authors obtain global well-posedness for cubic-quintic nonlinearities. With the analoguous approach one may show global well-posedness for quintic nonlinearities. This Section firstly gives a more precise characterization of the energy space for $d = 3$ and then tackles the well-posedness problem.

Structure of the energy space for $d = 3$

We provide the following result being a special case of Lemma 1.1.3 for $d = 3$.

Corollary 1.1.16. *The following hold true*

1. $\mathcal{F}_c \subset X^1 + H^1(\mathbf{R}^3)$,

2. *$\mathcal{F}_c + H^1(\mathbf{R}^3) \subset \mathcal{F}_c$, in particular for $u \in \mathcal{F}_c$, $v \in H^1$*

$$\||c+v+u|^2-1\|_{L^2} \leq \||c+v|^2-1\|_{L^2} + C\left(1+\sqrt{\mathcal{E}_2(c+v)}\right)\left(\|u\|_{L^2}+\|u\|_{L^4}+\|u\|_{L^4}^2\right).$$

3. *For $c+v, \tilde{c}+\tilde{v} \in \mathbb{E}_2$ and $u, \tilde{u} \in H^1$ one has*

$$\begin{aligned}\delta_{c,\tilde{c}}(c+v+u, \tilde{c}+\tilde{v}+\tilde{u}) \leq & C(1+\|u\|_{H^1}+\|\tilde{u}\|_{H^1})\delta_{c,\tilde{c}}(c+v+, \tilde{c}+\tilde{v}) \\ & + (1+\sqrt{\mathcal{E}(c+v)}+\sqrt{\mathcal{E}(\tilde{c}+\tilde{v})}+\|u\|_{H^1}+\|\tilde{u}\|_{H^1})\|u-\tilde{u}\|_{H^1}.\end{aligned}$$

Well-posedness theory

Given $\psi_0 \in \mathbb{E}_2(\mathbf{R}^3)$, there exists $c \in \mathbf{C}$ with $|c| = 1$ and $u_0 \in \mathcal{F}_c$ such that $\psi_0 = c + u_0$. We show that the dynamics of equation (1.1.1) is charaterised by $\psi(t) = c+u(t)$ with $u \in C(\mathbf{R}, \mathcal{F}_c)$ unique solution to

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u + (|c+u|^{2(\gamma-1)}-1)(c+u), \\ u(0,x) = u. \end{cases} \tag{1.1.59}$$

For that purpose, we proceed in several steps:

1. We introduce a global well-posedness theory for (1.1.59) in $\mathcal{F}_c$.

2. Given $u \in C(\mathbf{R}, \mathcal{F}_c)$ solution to (1.1.59), the function $\psi := c+u \in C(\mathbf{R}, \mathbb{E}_2(\mathbf{R}^3))$ defines a solution to (1.1.1).

3. We prove uniqueness for solutions to (1.1.1) in $C(\mathbf{R}, \mathbb{E}_2(\mathbf{R}^3))$.

Our approach distinguishes itself from the approach chosen in [82]. In [82], the author decomposes a given in initial datum in $\psi_0 = \phi_0 + u_0$ such that $\phi_0 \in C_b^\infty$ and $u_0 \in H^1$. Subsequently, an H^1-theory for (1.1.59) with c replaced by u_0 is developed. This approach, requiring higher regularity assumptions on F', might be carried out in any dimension, while the decomposition $\psi_0 = c + u_0$ used here is only available in $d = 3, 4$.

Well-posedness for (1.1.59)

This section is inspired by the H^1-theory for the sub-critical NLS in $3d$ as presented in Paragraph 3.3. in [167]. To meet notations in literature, we fix the exponents,

$$p := 2\gamma - 1, \qquad q = 10, \quad r = \frac{30}{13},$$

so that $2 \leq p < 5$ and (q, r) is a Strichartz admissible pair. The equation (1.1.59) becomes energy critical for $\gamma = 3$ or $p = 5$ respectively. We recall that the spaces N^0 and N^1 are defined in (1.0.6) and (1.0.7) respectively.

Lemma 1.1.17. *Let $\frac{3}{2} \leq \gamma < 3$ and $\mathcal{N}_1, \mathcal{N}_2$ as in* (1.1.17)*. Denote*

$$\alpha_1 = 1 - \frac{p}{10}, \qquad \alpha_2 = \frac{1}{2} - \frac{p}{10}.$$

Then for $c \in \mathbf{C}$ with $|c| = 1$ and $u \in \mathcal{F}_c(\mathbf{R}^3) \cap \dot{W}^{1,r}$, one has

$$\|\mathcal{N}(c+u)\|_{L_t^1 L_x^2} \leq CT\||c+u|^2 - 1\|_{L_t^\infty L_x^2} + CT^{1-\frac{p}{10}}\|u\|^{p-1}_{L_t^{10} L_x^{\frac{5}{2}(p-1)}}\|u\|_{L_{t,x}^{10}}.$$

For the gradient one has that,

$$\begin{aligned} &\|\nabla \mathcal{N}_1(c+u)\|_{L_t^1 L_x^2(\mathbf{R}^3)} \leq CT\|\nabla u\|_{L_t^\infty L_x^2(\mathbf{R}^3)}, \\ &\|\nabla \mathcal{N}_2(c+u)\|_{L_t^2 L_x^{\frac{6}{5}}(\mathbf{R}^3)} \leq CT^{\alpha_2}\|u\|^{p-1}_{L_t^{10} L_x^{\frac{5}{2}(p-1)}(\mathbf{R}^3)}\|\nabla u\|_{L_t^q L_x^r(\mathbf{R}^3)}. \end{aligned} \tag{1.1.60}$$

Moreover, there exists $\alpha > 0$ such that for $u, v \in \mathcal{F}_c(\mathbf{R}^3) \cap \dot{W}^{1,r}$ with

$$\sup_{t \in [-T,T]} \mathcal{E}_2(c+u) + \|u\|_{L^q \dot{W}^{1,r}} \leq M,$$
$$\sup_{t \in [-T,T]} \mathcal{E}_2(c+v) + \|v\|_{L^q \dot{W}^{1,r}} \leq M,$$

one has

$$\begin{aligned} &\|\mathcal{N}(c+u) - \mathcal{N}(c+v)\|_{N^0([-T,T]\times\mathbf{R}^d)} \leq C(T + T^{\frac{3}{4}})\delta_{c,c}(u,v) \\ &+ T^{\alpha_2}M^{p-1}\||c+u|^2 - |c+v|^2\|_{L_t^\infty L_x^2} + +T^{\alpha_2}M\|u-v\|_{L_t^\infty L_x^2}, \end{aligned} \tag{1.1.61}$$

and

$$\begin{aligned} \|\nabla \mathcal{N}(c+u) - \nabla \mathcal{N}(c+u)\|_{N^1([-T,T]\times\mathbf{R}^3)} &\leq C_\gamma T\|\nabla u - \nabla v\|_{L_t^\infty L_x^2} \\ &+ C_\gamma T^{\alpha_2}M^{p-1}\|\nabla u - \nabla v\|_{L_t^{10} L_x^r} + C_\gamma T^{\frac{3}{4}}M\|u-v\|_{L_t^\infty L_x^6} + C_\gamma T^{\alpha_2}M\|u-v\|_{L_t^{10} L_x^{10}}. \end{aligned}$$

Proof. Let $\mathcal{N}_1, \mathcal{N}_2$ be as in (1.1.17). We observe that Lemma 1.1.3 states that $(c+u)(1-\chi(c+u)) \in H^1(\mathbf{R}^3)$ and thus in $L^r(\mathbf{R}^3)$ for all $2 \leq r \leq 6$. Further, $u \in \mathcal{F}_c(\mathbf{R}^3) \cap \dot{W}^{1,r}$ implies that $u \in L^{10}(\mathbf{R}^3)$ and thus by interpolation $(c+u)(1-\chi(c+u)) \in L^r(\mathbf{R}^3)$ for any $2 \leq r \leq 10$ provided $u \in \mathcal{F}_c(\mathbf{R}^3) \cap \dot{W}^{1,r}$. This fact will be used repeatedly during the proof. It is immediate to see that

$$\|\mathcal{N}_1(c+u)\|_{L_t^1 L_x^2} \leq CT\||c+u|^2 - 1\|_{L_t^\infty L_x^2}.$$

We observe that the support of $1 - \chi(c+u)$ is of finite Lebesgue measure and moreover $|c+u| \leq 2|u|$ on that set. Taking in account the fact that $u \in L^r_{loc}(\mathbf{R}^3)$ for $2 \leq r \leq 10$, we conclude that

$$\|\mathcal{N}_2(c+u)\|_{L_t^1 L_x^2} \leq CT^{1-\frac{p}{10}}\|u\|^{p-1}_{L_t^{10} L_x^{\frac{5}{2}(p-1)}(supp((1-\chi(c+u)))}\|u\|_{L_t^{10} L_x^{10}}.$$

We proceed to bound

$$\|\nabla \mathcal{N}_1(c+u)\|_{L_t^1 L^2} \leq CT\|\nabla u\|_{L_t^\infty L^2}.$$

In order to bound $\nabla\mathcal{N}_2$, we compute that

$$\|\nabla\mathcal{N}_2(c+u)\|_{L_t^2 L_x^{\frac{6}{5}}} \leq T^{\alpha_2}\|\nabla\mathcal{N}_2(c+u)\|_{L_t^{\frac{10}{p}} L_x^{\frac{6}{5}}} \leq C_\gamma T^{\alpha_2}\|(1-\chi(c+u))u\|^{p-1}_{L_t^{10} L_x^{\frac{5}{2}(p-1)}}\|\nabla u\|_{L_t^{10} L_x^r}.$$

Let $u, v \in \mathcal{F}_c$, then

$$\begin{aligned}
&\|\mathcal{N}_1(c+u)-\mathcal{N}_1(c+v)\|_{L_t^1 L_x^2 + L_t^{\frac{4}{3}} L_x^{\frac{3}{2}}} \\
&\leq C\left(T\left\||c+u|^2-|c+v|^2\right\|_{L_t^\infty L_x^2} + T^{\frac{3}{4}}\||c+v|^2-1\|_{L_t^\infty L_x^2}\|u-v\|_{L_t^\infty L_x^6}\right) \\
&\leq CT\left\||c+u|^2-|c+v|^2\right\|_{L_t^\infty L_x^2} + CT^{\frac{3}{4}}\sqrt{\mathcal{E}_2(c+v)}\|\nabla u-\nabla v\|_{L_t^\infty L_x^2}.
\end{aligned}$$

We derive the pointwise estimate,

$$\begin{aligned}
|\mathcal{N}_2(c+u)-\mathcal{N}_2(c+v)| \leq &\left(|(c+u)_{high}|^{p-3}+|(c+v)_{high}|^{p-3}\right)|(c+u)_{high}|\left||(c+u)_{high}|^2-|(c+v)_{high}|^2\right| \\
&+\left|(|(c+v)_{high}|^{p-1}-1)(u-v)\right|.
\end{aligned}$$

Thus, if $\frac{3}{2} \leq \gamma \leq 2$, we conclude that

$$\begin{aligned}
&\|\mathcal{N}_2(c+u)-\mathcal{N}_2(c+v)\|_{L_t^{\frac{4}{3}} L_x^{\frac{3}{2}}} \\
&\leq CT^{\frac{3}{4}}\sup_t\sqrt{\mathcal{E}_2(c+u)}\left(\||c+u|^2-|c+v|^2\|_{L_t^\infty L_x^2}+\|u-v\|_{L_t^\infty L_x^6}\right).
\end{aligned}$$

For, $\gamma \geq 2$ implying $p \geq 3$ we obtain that there exits $\alpha, \beta > 0$ and $\theta_1, \theta_2 > 0$ such that

$$\begin{aligned}
&\|\mathcal{N}_2(c+u)-\mathcal{N}_2(c+v)\|_{L_t^{\frac{10}{9}} L_x^{\frac{30}{17}} + L_t^2 L_x^{\frac{6}{5}}} \\
&\leq CT^\beta\left(\sup_t \mathcal{E}_2(c+u)^{\theta_1}\|u\|^{1-\theta_1}_{L_t^q \dot{W}_x^{1,r}}\right) \\
&\times\left(\left(\sup_t \mathcal{E}_2(c+u)^{\theta_1}\|u\|^{1-\theta_1}_{L_t^q \dot{W}_x^{1,r}}\right)^{p-2}+\left(\sup_t \mathcal{E}_2(c+v)^{\theta_1}\|v\|^{1-\theta_1}_{L_t^q \dot{W}_x^{1,r}}\right)^{p-2}\right) \\
&\times\||c+u|^2-|c+v|^2\|_{L_t^\infty L_x^2}+T^\alpha\left(\sup_t \mathcal{E}_2(c+u)^{\theta_2}\|u\|^{1-\theta_2}_{L_t^q \dot{W}_x^{1,r}}\right)\|u-v\|_{L_t^\infty L_x^2}.
\end{aligned}$$

To show the local Lipschitz bound on the gradient of $\mathcal{N}(\cdot)$, we have the pointwise estimates

$$\begin{aligned}
&\Big|\left(\gamma|c+u|^{p-1}-1\right)\nabla u-\left(\gamma|c+v|^{p-1}-1\right)\nabla v \\
&+(\gamma-1)\left(|c+u|^{p-3}(c+u)^2\overline{\nabla u}-|c+v|^{p-3}(c+v)^2\overline{\nabla v}\right)\Big| \\
&\leq\left(\gamma|c+u|^{p-1}-1+(\gamma-1)|c+u|^{p-1}\right)|\nabla u-\nabla v| \\
&+(2\gamma-1)\left(1+|u|^{p-2}+|v|^{p-2}\right)|u-v||\nabla v|.
\end{aligned}$$

We notice that

$$\gamma|c+u|^{p-1}-1+(\gamma-1)|c+u|^{p-1} \in L^\infty(\mathbf{R}^3)+L^{\frac{5}{2}(p-1)}(\mathbf{R}^3),$$

and

$$(2\gamma-1)\left(1+|u|^{p-2}+|v|^{p-2}\right)\in L^\infty+L^{\frac{5(p-1)}{2(p-2)}}(\mathbf{R}^3).$$

Therefore, for $\alpha=\frac{1}{2}-\frac{p}{10}$,

$$\begin{aligned}
&\|\nabla\mathcal{N}(c+u)-\nabla\mathcal{N}(c+u)\|_{L^1_tL^2_x+L_t^{\frac{4}{3}}L_x^{\frac{3}{2}}+L^2_tL_x^{\frac{6}{5}}}\\
&\le C_\gamma T\|\nabla u-\nabla v\|_{L^\infty_tL^2_x}+C_\gamma T^\alpha\left(\|u\|^{p-1}_{L^{10}_tL_x^{\frac{5}{2}(p-1)}}+\|v\|^{p-1}_{L^{10}_tL_x^{\frac{5}{2}(p-1)}}\right)\|\nabla u-\nabla v\|_{L^{10}_tL^r_x}\\
&+C_\gamma T^{\frac{3}{4}}\|u-v\|_{L^\infty_tL^6_x}\|\nabla v\|_{L^\infty_tL^2_x}+C_\gamma T^\alpha\left(\|u\|^{p-2}_{L^{10}_tL_x^{\frac{10}{3}(p-2)}}+\|v\|^{p-2}_{L^{10}_tL_x^{\frac{10}{3}(p-2)}}\right)\\
&\times\|u-v\|_{L^{10}_tL^{10}_x}\|\nabla v\|_{L^{10}_tL^r_x}.
\end{aligned}$$

□

We are now in position to show local well-posedness of (1.1.59)

Proposition 1.1.18. *Let $\frac{3}{2}\le\gamma<3$. Then the equation* (1.1.59) *is locally well-posed in $\mathcal{F}_c(\mathbf{R}^3)$. Moreover, for every $R>0$ there exists $T>0$ and $C>0$ such that for all $u_0,v_0\in\mathcal{F}_c$ with $\mathcal{E}_2(c+u_0)\le R$ and $\mathcal{E}_2(c+v_0)\le R$, one has*

$$\sup_{|t|\le T}\delta_{c,c}(u(t),v(t))\le C\delta_{c,c}(u_0,v_0).$$

Proof. **Local existence**
Given $c\in\mathbf{C}$ such that $|c|=1$ and $u_0\in\mathcal{F}_c$ with $\sqrt{\mathcal{E}_2(c+u_0)}\le R$, we perform a fixed point argument for the solution map

$$S(u)(t)=e^{\frac{i}{2}t\Delta}u_0-i\int_0^t e^{\frac{i}{2}(t-s)\Delta}\mathcal{N}(c+u)(s)\mathrm{d}s, \tag{1.1.62}$$

in

$$X_T=\{u\in L^\infty\mathcal{F}_c\cap L^q\dot{W}^{1,r}\quad:\quad\sup_{|t|\le T}\sqrt{\mathcal{E}_2(c+u)}+\|\nabla u\|_{L^q([-T,T];L^r(\mathbf{R}^3))}\le M\}.$$

To that end, we first check that $S:X_T\to X_T$. From Lemma 1.1.17, it follows

$$\begin{aligned}
\|\nabla S(u)\|_{L^\infty_tL^2_x}&\le C\|\nabla u_0\|_{L^2}\\
&+C_\gamma\left(T\|\nabla u\|_{L^\infty_tL^2_x}+T^\alpha\|u\|^{p-1}_{L^{10}_tL_x^{\frac{5}{2}(p-1)}}\|\nabla u\|_{L^q_tL^r_x}\right),
\end{aligned}$$

where $\alpha=\frac{1}{2}-\frac{p}{10}>0$. The same argument yields

$$\begin{aligned}
\|\nabla S(u)\|_{L^q_tL^r_x}&\le C\|\nabla u_0\|_{L^2}\\
&+C_\gamma\left(T\|\nabla u\|_{L^\infty_tL^2_x}+T^\alpha\|u\|^{p-1}_{L^{10}_tL_x^{\frac{5}{2}(p-1)}}\|\nabla u\|_{L^q_tL^r_x}\right).
\end{aligned}$$

Next, we observe that

$$c+u=c+u_0+\left(e^{\frac{i}{2}t\Delta}u_0-u_0\right)-i\int_0^t e^{\frac{i}{2}(t-s)\Delta}\mathcal{N}(c+u)(s)\mathrm{d}s,$$

with $c+u_0\in\mathbb{E}_2$ and $u-u_0\in L^\infty([-T,T];H^1)$. Indeed, thanks to (1.1.14) and $u_0\in X^1+H^1$, we infer that

$$\|e^{\frac{i}{2}t\Delta}u_0-u_0\|_{H^1}\leq C(1+|T|^{\frac{1}{2}})\|\nabla u_0\|_{L^2}.$$

Moreover, agaim from Lemma 1.1.17 we obtain

$$\begin{aligned}\|i\int_0^t e^{\frac{i}{2}(t-s)\Delta}\mathcal{N}(c+u)(s)\mathrm{d}s\|_{L_t^\infty L_x^2}\leq C_\gamma\Big(&T\||c+u|^2-1\|_{L_t^\infty L_x^2}\\&+T^\alpha\|u(1-\chi(c+u))\|^{p-1}_{L_t^{10}L_x^{\frac{5}{2}(p-1)}}\|u\|_{L_t^{10}L_x^{10}}\Big),\end{aligned}$$

where $\alpha=1-\frac{p}{10}$. Thus

$$\begin{aligned}\|u-u_0\|_{L_t^\infty H_x^1}&\leq C(1+T^{\frac{1}{2}})\|\nabla u_0\|_{L^2}\\&+C_\gamma\left(T\||c+u|^2-1\|_{L_t^\infty L_x^2}+T^\alpha\|u(1-\chi(c+u))\|^{p-1}_{L_t^{10}L_x^{\frac{5}{2}(p-1)}}\|u\|_{L_t^{10}L_x^{10}}\right)\\&+C_\gamma\left(T\|\nabla u\|_{L_t^\infty L_x^2}+T^\alpha\|u\|^{p-1}_{L_t^{10}L_x^{\frac{5}{2}(p-1)}}\|\nabla u\|_{L_t^qL_x^r}\right)\\&\leq C(1+T^{\frac{1}{2}})\sqrt{\mathcal{E}_2(c+u_0)}+C_\gamma\left(T\sqrt{\mathcal{E}_2(c+u)}+T^\alpha\|\nabla u\|^p_{L_t^qL_x^r}\right).\end{aligned}$$

Therefore, from the second statement of Lemma 1.1.3 we conclude

$$\begin{aligned}\big\||c+u|^2-1\big\|_{L_t^\infty L_x^2}&\leq\||c+u_0|^2-1\|_{L^2}+C_\gamma\left(1+\sqrt{\mathcal{E}_2(c+u_0)}\right)\\&\times\left((1+T^{\frac{1}{2}})\sqrt{\mathcal{E}_2(c+u_0)}+T\sqrt{\mathcal{E}_2(c+u)}+T^\alpha\|\nabla u\|^p_{L_t^qL_x^r}\right)\\&+C_\gamma^2\left((1+T^{\frac{1}{2}})\sqrt{\mathcal{E}_2(c+u_0)}+\left(T\sqrt{\mathcal{E}_2(c+u)}+T^\alpha\|\nabla u\|^p_{L_t^qL_x^r}\right)\right)^2.\end{aligned}$$

Therefore, there exits $C>0$ and $C_\gamma>0$ such that

$$\begin{aligned}\|S(u)\|_{X_T}\leq CR+&C_\gamma(1+R)\left((1+T^{\frac{1}{2}})R+TM+T^\alpha M^p\right)\\&+C_\gamma^2(1+R)^2\left((1+T^{\frac{1}{2}})R+TM+T^\alpha M^p\right)^2.\end{aligned}$$

Next, we check that S defines a contraction on X_T. Let $u,v\in X_T$. Firstly, the non-homogeneous Strichartz estimates (1.0.4) and Lemma 1.1.17 yield

$$\begin{aligned}&\|\nabla u-\nabla v\|_{L_t^\infty L_x^2}+\|\nabla u-\nabla v\|_{L_t^qL_x^r}\leq\|\nabla\mathcal{N}(c+u)-\nabla\mathcal{N}(c+v)\|_{L_t^1L_x^2+L_t^2L^{\frac{6}{5}}(\mathbf{R}^3)}\\&\leq C_\gamma\Big(T\|\nabla(u-v)\|_{L_t^\infty L_x^2(\mathbf{R}^3)}+T^\alpha\|u\|^{p-1}_{L_t^{10}L_x^{\frac{5}{2}(p-1)}}\|\nabla(u-v)\|_{L_t^qL_x^r(\mathbf{R}^3)}\\&+T^\alpha\left(\|u\|^{p-2}_{L^{\frac{10}{3}(p-2)}}+\|v\|^{p-2}_{L_t^{10}L_x^{\frac{10}{3}(p-2)}}\right)\|u-v\|_{L_{t,x}^{10}}\|\nabla v\|_{L_t^qL_x^r(\mathbf{R}^3)}\Big)\\&\leq C_\gamma\left(T+T^\alpha M^{p-1}\right)d(u,v).\end{aligned}$$

Secondly, using again Lemma 1.1.3, we control

$$\begin{aligned}
&\left\| |c+u|^2 - |c+v|^2 \right\|_{L_t^\infty L_x^2} \\
&\leq \left(1 + 2\sqrt{\mathcal{E}_2(c+u_0)} + \|u-u_0\|_{L_t^\infty H_x^1} + \|v-u_0\|_{L_t^\infty H_x^1}\right) \|u-v\|_{L_t^\infty H_x^1} \\
&\leq C_\gamma (1 + 2R + 2(1+\sqrt{T})M + 2TM + 2T^\alpha M^p)(T + T^\alpha M^{p-1}) M d(u,v)
\end{aligned}$$

We summarize that

$$d(S(u), S(v)) \leq C_\gamma (1 + 2R + 2(1+\sqrt{T})M + 2TM + 2T^\alpha M^p)(T + T^\alpha M^{p-1}) M d(u,v).$$

Given $R > 0$, we may now choose M and T sufficiently small so that $S : X_T \to X_T$ and so that S defines a contraction on X_T.

Uniqueness Let $u, v \in C(0,T]; \mathcal{F}_c)$ be two solutions with initial data $u_0 \in \mathcal{F}_c$. We consider an interval $[0,\delta]$ with $0 < \delta < T$. Then, applying Lemma 1.1.17 and arguing in the same spirit as for the contraction argument,

$$\begin{aligned}
&\|\nabla u - \nabla v\|_{L_t^\infty([0,\delta];L^2)} + \|\nabla u - \nabla v\|_{L_t^q([0,\delta];L^r)} \\
&\leq C_\gamma (\delta + \delta^\alpha) \left(1 + \|u\|_{L_t^{10} L_x^{\frac{5}{2}(p-1)}} + \|v\|_{L_t^{10} L_x^{\frac{5}{2}(p-1)}} + \|\nabla u\|_{L_t^\infty L_x^2} + \|\nabla v\|_{L_t^q L_x^r}\right) \\
&\times \left(\|\nabla u - \nabla v\|_{L_t^\infty([0,\delta];L^2)} + \|\nabla u - \nabla v\|_{L_t^q([0,\delta];L^r)}\right).
\end{aligned}$$

Hence, for $\delta > 0$ sufficiently small we conclude that $\nabla u = \nabla v$ a.e. on $[0,\delta] \times \mathbf{R}^d$ and therefore $u = v$ a.e. on $[0,\delta] \times \mathbf{R}^d$. We may then iterate the argument to cover the whole interval $[0,T]$. Uniqueness backward in time is analogue.

Blow-up alternative is proven analogously to Proposition 1.1.10.

Continuous dependence on the initial data.

Let $u_0, v_0 \in \mathcal{F}_c$ such that $\mathcal{E}_2(c+u_0) \leq R$ and $\mathcal{E}_2(c+v_0) \leq R$. Let $T' > 0$ such that $E(c+u) \leq 3R$, $E(c+v) \leq 3R$ for all $|t| \leq T'$. Then,

$$\begin{aligned}
&\sup_t d_{\mathcal{F}_c}(u,v) \leq C(1 + \|u-u_0\|_{L_t^\infty H_x^1} + \|v-v_0\|_{L_t^\infty H_x^1}) d_{\mathcal{F}_c}(u_0,v_0) \\
&+ C\left(1 + E(c+u) + E(c+v) + \|u-u_0\|_{L_t^\infty H_x^1} + \|v-v_0\|_{L_t^\infty H_x^1}\right) \\
&\times \|u - u_0 - v + v_0\|_{L_t^\infty H_x^1}.
\end{aligned}$$

Combining the above arguments, upon choosing $0 < T < T'$ sufficiently small we obtain the desired Lipschitz estimate. □

Next, we extend the local result to a global existence result. We show the respective version of Lemma 1.1.14.

Lemma 1.1.19. *Let $\gamma \geq 2$ and $u_0 \in \mathcal{F}_c$ such that additionally $\Delta u_0 \in L^2(\mathbf{R}^3)$. Then the exists a unique solution $u \in C(\mathbf{R}, \mathcal{F}_c)$ such that $\Delta u \in L^\infty([-T,T]; L^2(\mathbf{R}^d))$ for all $T > 0$. Moreover,*

$$\mathcal{E}(u(t)) = \mathcal{E}(u_0),$$

for all $t \in \mathbf{R}$.

Proof. Given $u_0 \in \mathcal{F}_c \cap \dot{H}^2$, there exists $T_* > 0$ and a unique solution $u \in C([-T_*, T_*], \mathcal{F}_c)$ with initial data u_0. Moreover, there exists $T > 0$ and $C = C(\mathcal{E}_2(c+u_0))$ such that $\mathcal{E}_2(c+u(t)) \leq C$ for all $|t| \leq T$. We obtain that

$$\mathcal{N}(c+u_0) \in L^2(\mathbf{R}^3),$$

exploiting the fact that $u_0 \in L^\infty$. Proceeding as in the proof of Lemma 1.1.14 we infer that there exists $T_1 > 0$ such that $\partial_t u \in L^\infty([-T_1, T_1]; L^2(\mathbf{R}^3))$ and $\Delta u \in L^\infty([-T_1, T_1]; L^2(\mathbf{R}^3))$. Relying on the regularity properties of $\mathcal{N}$ introduced in Lemma 1.1.13, one concludes that the energy ins conserved for all $t \in [-T_1, T_1]$. Finally, since $\mathcal{E}_2(c+u) \sim \mathcal{E}(c+u)$ thanks to Lemma 1.1.13, we may iterate the local result to obtain global existence. □

The energy conservation for regular solutions allows to extend the local result of Proposition 1.1.18 to global well-posedness.

Corollary 1.1.20. *Let $d = 3$ and $\frac{3}{2} \leq \gamma < 3$ Then* (1.1.59) *is globally-wellposed in $\mathcal{F}_c$ and in particular,*

$$\mathcal{E}(c+u(t)) = \mathcal{E}(c+u_0),$$

for all $t \in \mathbf{R}$.

The proof is analogous to the proof of Corollary 1.1.15 and therefore omitted.

Well-posedness for (1.1.1)

The well-posedness result for the system (1.1.59) implies well-posedness for (1.1.1) in the energy space.

Proposition 1.1.21. *Let $\frac{3}{2} \leq \gamma < 3$. The problem* (1.1.1) *is globally well-posed in $\mathbb{E}_2(\mathbf{R}^3)$, namely given $\psi_0 \in \mathbb{E}_2$ there exits a unique solution $\psi \in C(\mathbf{R}, \mathbb{E}_2)$ with initial condition $\psi(0) = \psi_0$. Moreover, the following hold*

1. *for all $t \in \mathbf{R}$, one has $\mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0)$,*
2. *if moreover $\Delta\psi_0 \in L^2(\mathbf{R}^3)$, then for any $T > 0$ $\Delta\psi \in L^\infty([-T, T]; L^2(\mathbf{R}^3))$,*
3. *$\psi - e^{\frac{i}{2}t\Delta}\psi_0 \in C(\mathbf{R}, H^1(\mathbf{R}^3))$,*
4. *for every $R > 0$, for every $T > 0$, there exists $C > 0$ such that for every $\psi_0, \tilde{\psi_0} \in \mathbb{E}_2$ such that $\mathcal{E}_2(\psi_0) \leq R$ and $\mathcal{E}_2(\psi_0) \leq R$, the respective solutions $\psi, \tilde{\psi}$ satisfy,*

$$\sup_{|t| \leq T} d_{\mathbb{E}_2}(\psi(t), \tilde{\psi(t)}) \leq C d_{\mathbb{E}_2}(\psi_0, \tilde{\psi_0}).$$

We remark that Proposition 1.1.21 is a generalization of Theorem 1.1 in [84] providing the statement for $\gamma = 2$. Compared to the statement of Proposition 1.1.10, the continuous dependence on the initial data is upgraded to Lipschitz continuity of the flow. In the regime $\gamma \geq \frac{3}{2}$ the nonlinearity is locally Lipschitz on $\mathbb{E}_2$ allowing for an Lipschitz estimate to be proven.

Proof. **Existence**. Given $\psi_0 \in \mathbb{E}_2$ there exits $c \in \mathbf{C}$ with $|c| = 1$ and $u_0 \in \mathcal{F}_c$ such that $\psi_0 = c + u_0$. Proposition 1.1.10 together with Corollary 1.1.20 provide a unique global solution $u \in C(\mathbf{R}, \mathcal{F}_c)$ to (1.1.59). Thus, $\psi := c + u \in C(\mathbf{R}, \mathbb{E}_2)$ is a global solution to (1.1.1) with initial data $\psi(0) = \psi_0$. Moreover, $\mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0)$ for all times $t \in \mathbf{R}$.

Uniqueness. Given $\psi_0 \in \mathbb{E}_2$, we denote by ψ the solution constructed previously. Let $\tilde{\psi} \in C(\mathbf{R}, \mathbb{E}_2)$ be another solution to (1.1.1) such that $\tilde{\psi}(0) = \psi_0$. It suffices to show that $\psi - \tilde{\psi} \in C(\mathbf{R}, H^1(\mathbf{R}^3))$. Indeed, this implies that

$$\tilde{\psi} = \psi + (\tilde{\psi} - \psi) \in c + v + H^1 \subset c + \mathcal{F}_c.$$

Hence, $\tilde{\psi} - c \in \mathcal{F}_c$ is a solution to (1.1.59) and therefore in virtue of we infer that $\tilde{\psi} - c$ must coincide with the unique solution $v \in C(\mathbf{R}, \mathcal{F}_c)$.

Next, we prove that $\psi - \tilde{\psi} \in C(\mathbf{R}, H^1(\mathbf{R}^3))$. Since $\nabla\psi - \nabla\psi \in C(\mathbf{R}; L^2(\mathbf{R}^3))$ by hypothesis, it only remains to show that $\psi - \tilde{\psi} \in C(\mathbf{R}; L^2(\mathbf{R}^3))$. For that purpose, we recall that

$$\|\mathcal{N}_1(\psi)\|_{L_t^\infty L^2} \leq C \sup_t \mathcal{E}_2(\psi),$$

and for $\psi \in \mathbb{E}_2$ and $2 \leq p \leq 3$, i.e. $\frac{3}{2} \leq \gamma \leq 2$, one has

$$\|\mathcal{N}_2(\psi)\|_{L_t^\infty L^2} \leq C \sup_t \mathcal{E}_2(\psi),$$

while for $3 \leq p < 5$,

$$\|\mathcal{N}_2(\psi)\|_{L_t^\infty L^{\frac{6}{p}}} \leq C \sup_t \mathcal{E}_2(\psi)^p,$$

and $\frac{6}{5} \leq \frac{6}{p} < 2$. The case $p \leq 3$ is immediate, we only treat the case $3 \leq p < 5$, see also Lemma 1.1.17. We have

$$\begin{aligned}
\|\psi - \tilde{\psi}\|_{L_t^\infty L_x^2} &\leq \left\| \int_0^t e^{\frac{i}{2}(t-s)\Delta} \mathcal{N}(\psi) - \mathcal{N}(\tilde{\psi}) \mathrm{d}s \right\|_{L_t^\infty L_x^2} \\
&\leq C\|\mathcal{N}_1(\psi) - \mathcal{N}_1(\tilde{\psi})\|_{L_t^1 L_x^2} + C\|\mathcal{N}_2(\psi) - \mathcal{N}_2(\tilde{\psi})\|_{L_t^2 L_x^{\frac{6}{5}}} \\
&\leq CT(\sup_t \mathcal{E}_2(\psi) + \sup_t \mathcal{E}_2(\tilde{\psi}) + CT^{\frac{1}{2}} \left(\sup_t \mathcal{E}_2(\psi)^p + \sup_t \mathcal{E}_2(\tilde{\psi})^p \right).
\end{aligned}$$

Thus, $\psi - \tilde{\psi} \in C(\mathbf{R}; H^1(\mathbf{R}^3))$ and the uniqueness claim follows.

Lipschitz continuity of the flow. Fix $R > 0$, let $c + v_0, \tilde{c} + \tilde{v_0} \in \mathbb{E}_2$ be such that $\mathcal{E}_2(c + v_0) \leq R$ and $\mathcal{E}_2(\tilde{c} + \tilde{v_0}) \leq R$. Let $T > 0$ to be chosen later. Denote by $c + v, \tilde{c} + \tilde{v} \in C(\mathbf{R}, \mathbb{E}_2)$ the corresponding solutions. Then

$$\begin{aligned}
&\delta_{c,\tilde{c}}(c + v, \tilde{c} + \tilde{v}) \\
&\leq C \left(1 + \|v - e^{\frac{i}{2}t\Delta} v_0\|_{L_t^\infty H_x^1} + \|\tilde{v} - e^{\frac{i}{2}t\Delta} \tilde{v_0}\|_{L_t^\infty H_x^1} \right) \delta_{c,\tilde{c}}(c + e^{\frac{i}{2}t\Delta} v_0, \tilde{c} + e^{\frac{i}{2}t\Delta} \tilde{v}) \\
&+ \left(1 + \mathcal{E}(c + v) + \mathcal{E}(\tilde{c} + \tilde{v}) + \|v - e^{\frac{i}{2}t\Delta} v_0\|_{L_t^\infty H_x^1} + \|\tilde{v} - e^{\frac{i}{2}t\Delta} \tilde{v_0}\|_{L_t^\infty H_x^1} \right) \\
&\times \left\| v - \tilde{v} - e^{\frac{i}{2}t\Delta}(v_0 - \tilde{v_0}) \right\|_{L_t^\infty H_x^1}
\end{aligned}$$

Applying (1.1.15) to dispose of the linear terms and exploiting Lemma 1.1.17 for the nonlinear term, we conclude that for $T > 0$ depending only on $\mathcal{E}_2(\psi_0)$ small enough the desired Lipschitz estimate is valid. Relying on the conservation of energy, we may iterate the procedure to obtain the Lipschitz estimate for arbitrary times $T > 0$. □

1.2 Special solutions and large time asymptotics for NLS equations with non-vanishing boundary conditions at infinity

This paragraph contains a review of known results for the existence of special solutions, large time asymptotics and scaling regimes for the class of nonlinear Schrödinger equations (1.1.1) with nonlinearity given by (1.1.3) and far-field behavior (1.1.2). The equation (1.1.1) equipped with non zero conditions at infinity is known to possess a variety of special solutions such as e.g. travelling waves [142, 59], solitons [108, 109] and vortices [159, 154, 32, 29]. This is in sharp contrast to its defocusing counterpart with trivial conditions at infinity; scattering is known for the subcritical equation for $d = 1, 2, 3$, see [86], and thus no special solutions exist. The behavior of (1.1.1) with boundary conditions (1.1.2) highly depends on the dimension d. While scattering is ruled out by the existence of small energy travelling waves for $d = 2$, non-existence of travelling waves for small energy is known [141] for $d = 3$ and scattering has been proven for $\gamma = 2$ for small regular data and $d = 3$ and for $d = 4$. Extensive literature has been dedicated to the analysis of these questions for the Gross-Pitaevskii equation, i.e. when $\gamma = 2$ and for cubic-quintic nonlinearities.
To conclude, we remark that stability properties of special solutions have been extensively studied in literature. Their study is beyond the scope of the present work, we limit ourselves to provide an (incomplete) list of key references [108, 109, 122].

1.2.1 Travelling waves

Fixed $y \in S^{d-1}$, we say that a solution to (1.1.1) is a travelling wave if of the form

$$\Phi(t, x) = \psi(x - tcy),$$

where we refer to c as its velocity. Since (1.1.1) is invariant under rotations, we may assume without loss of generality that $y = (1, 0, ..., 0)$ and $c \geq 0$. The travelling wave is a solution to

$$-ic\partial_{x_1}\psi = -\Delta\Phi + \mathcal{N}(\Phi). \tag{1.2.1}$$

The problem of existence of travelling waves for nonlinear Schrödinger equations with non-vanishing conditions at infinity has received remarkable attention in literature. The study of existence of traveling waves and their properties such as stability for Gross-Pitaevskii type equations is often referred to as *Roberts program* due to a series papers by C.A. Jones, C.J. Putterman and P.H. Roberts, see [108, 109] and references therein, initiating the analysis of travelling waves. For an investigation of the existence problem of travelling waves for a class of general nonlinearities, including the one described by (1.1.3), we refer the interested reader to the series of papers [141, 142, 59] and references therein. Although the results in the series of papers allow for more nonlinearities, we restrict our review to the nonlinearity given by (1.1.3). This implies in particular that the respective potential, see (1.0.2) is non-negative. Extensive literature has been dedicated to the analysis of travelling waves for the Gross-Pitaevskii equation, namely $\gamma = 2$, see e.g. [30, 31, 27].

An important threshold for the existence of travelling waves for (1.1.1) is given by the sound speed at infinity $v_S = \sqrt{F''(1)}$. The sound speed is best intrepeted in the framework of QHD system (2.0.1) where the relation reads $v_s(\rho) = \sqrt{\frac{\partial p(\rho)}{\partial \rho}}$. It has been proven in [141] that no travelling waves of finite energy for (1.1.1) exists for supersonic speeds, namely for $c > v_s$ for $d = 2, 3$. Further, there does not exist any stationary finite energy solution to (1.2.1). More precisely,

$$\Delta\psi = \mathcal{N}(\psi),$$

does not admit any finite energy solution, i.e. $\psi \in \mathcal{E}$. This can be verified by means of Pohozaev identities; given ψ such that $\mathcal{E}(\psi) < +\infty$, one may show that

$$(d-2)\int_{\mathbf{R}^d} |\nabla\psi|^2 \mathrm{d}x + d\int_{\mathbf{R}^d} F(|\psi|^2)\mathrm{d}x = 0,$$

see Lemma 2.4 in [44]. Therefore, we obtain $\psi = 0$ provided the potential is non-negative as for (1.0.2). The picture for velocities $0 < c < v_S$ is completely different for $d = 2$ and $d = 3$ that we discuss separately. Our considerations are based on the series of papers [141, 142, 59], where the authors, see p. 5 in [59], require that

1. the function F' is continuous on $[0, \infty)$ and of class C^1 in a neighborhood of 1, $F'(1) = 0$ and $F'(1) > 0$;

2. the nonlinearity is subcritical, i.e. there exists $C > 0$ and $p_0 < \frac{2}{d-2}$ (with $p_0 < \infty$ if $d = 2$) such that

$$|F'(r)| \leq C(1 + r^{p_0}),$$

   for any $r \geq 0$.

3. the nonlinearity is non-degenerate, namely there exists $C, \alpha_0 > 0$ and $r_* > 1$ such that for any $r \geq r_*$ one has $F'(r) \geq C r^{\alpha^0}$;

4. F is of class C^2 near 1 and

$$F'(r) = (r-1) + \frac{1}{2}F'''(1)(r-1)^2 + \mathcal{O}((r-1)^3),$$

   for r close to 1.

One easily checks that the nonlinearity F' as defined in (1.1.3) verifies all assumptions. For $d = 2$, Theorem 0.1. in [59] implies that for any speed $0 < c < v_S$, there exists a travelling wave ψ of finite energy solving (1.2.1) and moreover has the following minimizing property.

Theorem 1.2.1 ([59]). *Let $d = 2$ and let F be given by* (1.1.3)*. Then, for any $q \in (0, \infty)$ there exists a travelling wave ψ of* (1.2.1) *with speed $c = c(\psi) \in (0, v_s)$ such that the momentum $Q(\psi) = q$. Moreover, ψ minimizes the energy $\mathcal{E}$ among all functions of momentum equal to q.*

For $d = 3$, the situation is fundamentally different, there exists an energy threshold so that there are no travelling waves with energy below the threshold. On the other hand, for all speeds $0 < c < v_s$ there exists travelling waves but with higher energy, see [142].

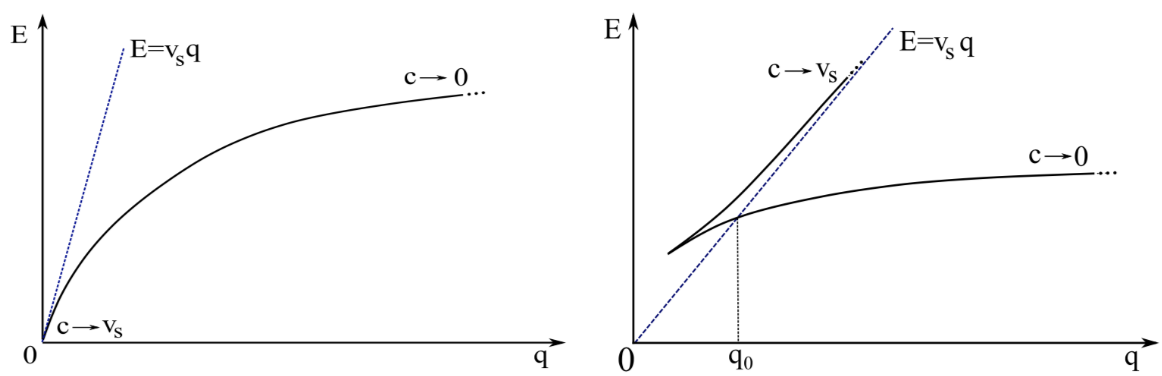

Figure 1.1: Energy (E) momentum (q) diagrams for (1.3.1): a) in dimension 2 and b) in dimension 3. The diagrams are a courtesy of D. Chiron, M. Maris [59]

.

This is compatible with the energy-momentum diagram, see Figure 1.1. The existing travelling waves at speeds close to the sound speed belong to the upper branch of large energy.

Theorem 1.2.2. *[142] Let $d = 3$ and F as defined in* (1.1.3)*, then for any $0 < c < v_S$, there exists a travelling wave solution ψ to* (1.2.1) *such that $\mathcal{E} < \infty$.*

Regarding travelling waves of small energy we recall the following.

Theorem 1.2.3. *[59] Let $d = 3$ and F as defined in* (1.1.3)*.*

1. *There exists $k_* > 0$ depending only on F such that if $c \in [0, v_s]$ and if $\psi \in \mathbb{E}_\gamma$ solution to* (1.2.1) *satisfying $\int_{\mathbf{R}^d} |\nabla\psi|^2 \mathrm{d}x < k_*$, then ψ is constant.*

2. *Moreover, there exists $l_* > 0$ depending only on F such that any solution $\psi \in \mathbb{E}_\gamma$ to* (1.2.1) *with $c \in [0, v_s]$ and $\int_{\mathbf{R}^d} \left(|\psi|^2 - 1\right)^2 \mathrm{d}x < l_*$ is constant.*

In particular, there exists $0 < c_0 < v_s$ such that there are no travelling waves of speed $c \in (c_0, v_s)$ which minimize the energy at fixed momentum.

We conclude this paragraph with a remark on the transsonic limit $c \to v_s$ and the limit as $c \to 0$. In the transonic limit, it is known that for $d = 2$, these travelling waves do exists and are rarefaction pulses asymptotically characterised by the ground states of the Kadomtsev-Petsviashvili I equation, see [58]. For $\gamma = 2$, this is due to [27]. For $d = 3$, Theorem 1.1. and Corollary 1.2. in [142] provide existence of travelling waves with speed close to v_s. The traveling waves constructed in [142] converge to ground state of the KP-I equation as $c \to v_S$, see Theorem 6 in [58] and are hence rarefaction pulses. Contrary to what happens for $d = 2$, the travelling waves for $d = 3$ have high energy of order $\frac{1}{\sqrt{v_s^2 - c^2}}$ as $c \to v_S$.
In the limit as $c \to 0$, one heuristically expects the energy to be diverging due to the absence of stationary solutions of finite energy. To the best of our knowledge, a complete description of this limit for general nonlinearities is open, see also p. 80 in [59]. However, for $\gamma = 2$ the limit regime has been investigated in [30] for $d = 2$.

Theorem 1.2.4 ([30])**.** *There exists $c_0 > 0$ such that for any $0 < c < c_0$, there exists a (non-constant) function solution to* (1.2.1) *and $\mathcal{E}(\psi) < +\infty$. Moreover,*

1. *there exists C_0, C_1 such that*

$$2\pi|\log c| + C_0 \leq \mathcal{E}(\psi) \leq 2\pi|\log c| + C_1,$$

2. *ψ has exactly two vortices, with degree ± 1, located at a distance $D \sim \frac{1}{c}$ as $c \to 0$,*

3. *ψ is smooth, i.e. $\psi \in C^\infty$.*

We refer the interested reader also to the review paper [31] and the more recent work [60] where the authors investigate numerically (new) multiple branches of travelling waves for the Gross-Pitaevskii equation (1.3.1) corresponding to excited states. For $d = 3$, [29, 56] provide travelling vortex rings solutions. Numerical evidence for their existence had been provided by [108]. Vortex solutions of infinite energy are discussed in Section 1.3.

1.2.2 Scattering

We start by observing that Theorem 1.2.1 rules out any scattering result for $d = 2$, since there exists travelling waves for arbitrarily small energy. For $d = 3$, it had been conjectured in [27] that below the threshold for which no travelling waves exists there are global dispersive solutions to the Gross-Pitaevskii equation. Indeed, Theorem 1.2.3 leaves room for a scattering theory, even for general nonlinearities. To the author's knowledge, no scattering theory is available for the general class of nonlinearities. In this direction, we mention the seminal paper [95] followed by [96, 97] where scattering was introduced for the Gross-Pitaevskii equation for $d \geq 3$ and subsequently in [96, 97] for small data under suitable regularity assumptions. More recently, in [92] the authors obtain scattering for small data in the energy space assuming additional angular regularity. Provided the solution is radial then scattering for solutions of small energy is proven. Concerning general nonlinearities, in [143] the authors investigate the global behavior for some generalized Gross-Pitaevskii equations. The class of nonlinearities considered is interesting as it modifies the behavior of the internal energy F close to the constant state 1 so that its behavior is no longer quadratic as it is the case for (1.1.2). This is also to be compared to the remark proceeding Lemma 1.1.13 and assumption 4 in Section 1.2.1.

1.3 Nonlinear Schrödinger equations with non-vanishing boundary conditions and infinite energy

In this section, we review extensions of the existence theory in the energy space to solutions of infinite energy as established in [32]. Here, we restrict ourselves to the Gross-Pitaevskii equation, namely $\gamma = 2$, reading

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + (|\psi|^2 - 1)\psi. \tag{1.3.1}$$

Moreover, we are interested in solutions with non-zero degree at infinity such as vortices. In literature, solutions that do not belong to the energy space are often referred to as solutions of infinite energy. e.g. in [157] the author shows the unconditional well-posedness of the equation (1.3.1) for a class of solutions of infinite energy but zero degree. Here, we only address the existence of the former type of infinite energy solutions.
For $d = 2$, there exists stationary vortex solution for (1.3.1) given by $U_0(x) = f(r)e^{im\theta}$ with $r = |x|$ and θ is defined by $e^{i\theta} = \frac{x}{|x|}$ identifying $\mathbf{C}$ and $\mathbf{R}^2$ and the radial profile f satisfies

$$\begin{cases} f''(r) + \frac{1}{r}f'(r) - \frac{d^2}{r^2}f(r) + 2\left(1 - f(r)^2\right)f(r) = 0 \\ f(0) = 0, \qquad f(r) \to 1 \quad \text{as} \quad r \to \infty. \end{cases} \tag{1.3.2}$$

The integer n is referred to as degree of the quantum vortex. Explicit computations show that the radial profile is such that $\nabla_x f \in L^2$ and $|\nabla U_0(x)| \sim \frac{d}{|x|}$ asymptotically as $|x| \to \infty$, see [103]. Therefore, $\mathcal{E}_2(U_0) = \infty$, compatible with the observation that there do not exist stationary solutions to (1.1.1) of finite energy, see Section 1.2.1. More in general, it can be proven that continuous wave functions that do not vanish at infinity and with non-trivial degree at infinity have infinite energy. By degree at infinity, we intend the winding number at infinity, i.e. for $r > 0$ large enough such that $\psi > 0$ on ∂B_r, we define the map $\psi_r : \partial B_r \sim S^1 \to S^1$ as $x \mapsto \frac{\psi(x)}{|\psi(x)|}$. The map $U_{0,r}$ then has topological degree m.

Lemma 1.3.1. *Let $\varphi : \mathbf{R}^2 \to \mathbf{C}$ be of class C^1 such that $|\varphi(x)| \to 1$ as $|x| \to \infty$ and such that the degree of φ at infinity is different from 0, then*

$$\int_{\mathbf{R}^2} \frac{1}{2}|\nabla\varphi|^2 \mathrm{d}x = +\infty.$$

For a proof, see for instance [153]. The question regarding the dynamics of several vortices arises naturally. Multi-vortex configurations may be investigated by considering initial data given by the product of several of these vortices, namely fix $a_1, ..., a_n$ and $m_1, ...m_n$ and consider

$$U_d = \prod_{j=1}^{n} f_j(x - a_j)e^{im_j\theta_j}, \tag{1.3.3}$$

with $\theta_j = \frac{x-a_j}{|x-a_j|}$ and for profiles f_j satisfying (1.3.2). The first issue to tackle is the Cauchy Problem for (1.3.1) in a suitable class of initial data allowing for multi-vortex configurations to be studied. In [32], Bethuel and Smets study the problem by considering initial data of the type $\psi = U_0 + v$, where U_0 can be chosen to be a fixed vortex-configuration $U_0 = f(r)e^{im\theta}$ or a multi-vortex configuration and v a H^1-perturbation. We postpone the qualitative analysis of their dynamics to Chapter 5. For this type of data the suitable framework considered in [32] is $\psi = U_0 + u \in \mathcal{V} + H^1$, where

$$\mathcal{V} := \left\{U \in L^\infty : \quad \nabla|U| \in L^2, \, \nabla^k U \in L^2, \text{for any } k \geq 2, \, (|U|^2 - 1) \in L^2\right\}. \tag{1.3.4}$$

It is easy to check that $U_0 \in \mathcal{V}$. We notice that

$$(\mathcal{V} + H^1) \cap \dot{H}^1(\mathbf{R}^d) = \mathbb{E}_2.$$

Indeed, this is an immediate consequence of Lemma 1.1.6. We report the following Gagliardo-Nirenberg type inequality for $U \in \mathcal{V}$.

Lemma 1.3.2 ([32]). *Let $U \in L^\infty(\mathbf{R}^2)$ such that $\Delta U \in L^2(\mathbf{R}^2)$, then $\nabla U \in L^4(\mathbf{R}^2)$. In particular,*

$$\|\nabla U\|_{L^4(\mathbf{R}^2)} \leq 18^{\frac{1}{4}} \|U\|^{\frac{1}{2}}_{L^\infty(\mathbf{R}^2)} \|\Delta U\|^{\frac{1}{2}}_{L^2(\mathbf{R}^2)}. \tag{1.3.5}$$

The existence result in [32] is proven by decomposing the initial data in $\psi_0 = U + w_0$. Subsequently, it is shown that the problem

$$i\partial_t w = -\frac{1}{2}\Delta w + F_U(w), w(0, x) = w_0 \in H^1(\mathbf{R}^2); \tag{1.3.6}$$

where $F_U(w) = -\frac{1}{2}\Delta U + (|U_0 + w|^2 - 1)(U + w)$, is well-posed in $H^1(\mathbf{R}^2)$. This approach is somehow reminiscent of the approach in [82] where the author solves (1.3.6) for a fixed $U \in C_b^\infty \cap \mathbb{E}_2$. A major difference compared to the well-posedness theory in the energy space consists in the uniqueness statement. While Theorem 1.1.1 provides uniqueness in $C([0, T], \mathbb{E}_2)$, in the case of solutions of infinite energy Theorem 1.3.3 only states uniqueness in the affine space $U + H^1(\mathbf{R}^2)$. The main novelty in the approach of [32] is the introduction of a renormalized energy functional needed to show the global existence result. The authors consider for a given decomposition $\psi = U + w$,

$$\mathcal{E}_U(\psi) = \int_{\mathbf{R}^2} \frac{1}{2}|\nabla w|^2 - \int_{\mathbf{R}^2} Re(\Delta U \cdot w) + \int_{R^2} \frac{1}{2}\left(|U + w|^2 - 1\right)^2. \tag{1.3.7}$$

We recall that for $\psi = U + v$ with $U \in \mathcal{V}$ and $v \in H^1(\mathbf{R}^2)$, the renormalized energy functional introduced in (1.3.7) is finite since $w \in H^1(\mathbf{R}^2)$ and $\Delta U \in L^2(\mathbf{R}^2)$. Similarly to Lemma 1.1.3 we see that $|\psi|^2 - 1 \in L^2(\mathbf{R}^2)$. We summarize the results of [32] in the following Theorem.

Theorem 1.3.3. *[32] Let $\psi_0 \in \mathcal{V} + H^1(\mathbf{R}^2)$. Then there exists a unique solution $t \mapsto \psi(t)$ of the Gross-Pitaevskii equation* (1.3.1) *such that $\psi(0) = \psi_0$ and $\psi(t) - \psi(0) \in C^0(\mathbf{R}, H^1(\mathbf{R}^2))$. Further,*

$$\frac{d}{dt}\mathcal{E}_U(\psi(t)) = 0 \quad on \quad \mathbf{R}.$$

In addition, if $w_0 \in H^2(\mathbf{R}^2)$, then $w \in C(\mathbf{R}; H^2(\mathbf{R}^2))$.
Moreover, if $\tilde{\psi}_0 = U + \tilde{w}_0 \in \mathcal{V} + H^1(\mathbf{R}^2)$ and $t \mapsto \psi(t)$ the respective solution, then

$$\|\psi(t) - \tilde{\psi}(t)\|_{H^1(\mathbf{R}^2)} \leq C(t, \phi_0, \mathcal{E}_U(\psi_0))\|\psi_0 - \tilde{\psi}_0\|_{H^1(\mathbf{R}^2)}.$$

As pointed out in [32], the renormalized energy functional that may defined equivalently to (1.3.7) under suitable decay assumptions on ∇U as follows. For a fixed decomposition $\psi = U + v$ and fixed $R > 0$ we define

$$\mathcal{E}_{R,U} = \int_{B(R)} \frac{1}{2}|\nabla\psi|^2 - |\nabla U|^2 + \frac{1}{2}\left(|\psi|^2 - 1\right)^2 \mathrm{d}x. \tag{1.3.8}$$

The functional is well-defined, integrating by parts we recover that

$$\mathcal{E}_{R,U} = \int_{B(R)} \frac{1}{2}|v|^2 - 2Re(\Delta U \cdot v)| + \frac{1}{2}\left(|\psi|^2 - 1\right)^2 \mathrm{d}x + \int_{\partial B_R} Re(\partial_r U \cdot v).$$

As observed in [32, 28], one has

$$\int_{\partial B_R} Re(\partial_r U \cdot v) \to 0,$$

as $R \to \infty$ provided that $|\nabla U(z)| \leq \frac{C}{\sqrt{|z|}}$ for $z \geq 1$. For a proof, see Lemma 3.3 in [28]. This motivates to define

$$\mathcal{V}_* = \left\{ U \in \mathcal{V} \quad : \quad |\nabla U(z)| \leq \frac{C}{\sqrt{|z|}}, \quad |z| \geq 1 \right\}. \tag{1.3.9}$$

We notice that the Definition depends on the choice of the decomposition for ψ. This alternative Definition is motivated by its application to the study of vortices. The multi-vortex configurations we have in mind lie in $\mathcal{V}_*$. For $\psi \in \mathcal{V}_* + H^1(\mathbf{R}^2)$, we have that

$$\begin{aligned}
&\int_{\mathbf{R}^2} \frac{1}{2}|\nabla w|^2 - \int_{\mathbf{R}^2} Re(\Delta U \cdot w) + \int_{R^2} \frac{1}{2}\left(|\phi_0 + w|^2 - 1\right)^2 \\
&= \lim_{R\to\infty} \int_{B(R)} \frac{1}{2}|\nabla\psi|^2 - |\nabla U|^2 + \frac{1}{2}\left(|\psi|^2 - 1\right)^2,
\end{aligned}$$

thus the renormalized energy may be defined equivalently by (1.3.8) if $\psi \in \mathcal{V}_* + H^1(\mathbf{R}^2)$. This second Definition is suitable for the analysis of the corresponding QHD system and will be used in Section 2. A similar approach has been followed in [155] to investigate the energy of vortex solutions to (1.3.1). There the authors renormalize the energy by subtraction of $\frac{d^2}{|x|^2}(1-\chi_{B_R})$ that corresponds for $d=2$ to the logarithmic divergence of the energy functional at infinity; we recall that $|\nabla U|^2 \sim \frac{d^2}{|x|^2}$ at infinity. Finally, we mention that the renormalized energy functional is unbounded from below for vortex configurations with total degree $d = deg(\psi, \infty) \geq 2$, see Lemma 4.5 in [28].

Solutions with non-zero degree at infinity for $d = 3$

We briefly mention known results for solution to (1.3.1) on $\mathbf{R}^3$ with infinity energy and non-zero degree at infinity. We recall that vortex rings of finite energy are known to exists [29], see also Section 1.2.1. Regarding solutions of infinite energy, in [57] the author shows that there exits vortex helices of infinite energy solving (1.3.1). Finally, straight vortex filaments can be seen as solutions to the two dimensional problem. It turns out that these three geometries - straight filaments, vortex rings and helices - are the relevant ones arising as concentration set for the vorticity in the scaling limit of Chapter 5, see also page 1560 in [57]. The phenomena of leapfrogging vortex rings has been studied [105].

CHAPTER 2

Existence of weak solutions to the QHD system with non-trivial far-field

Abstract

This chapter is concerned with the global in time existence of weak solutions to the QHD system on $\mathbf{R}^d$ for $d = 2, 3$ complemented with non-trivial far-field behavior. Our method exploits the underlying effective wave-function dynamics, for that purpose we introduce a suitable polar decomposition in Section 2.2. In Section 2.3 we prove global existence of finite energy weak solutions to the QHD system, while Section 2.4 is devoted to the analysis of weak solutions of infinite energy. Finally, we discuss special solutions including quantized vortices in Section 2.5.

This chapter is based on a work in collaboration with P. Antonelli and P. Marcati, some of the results have been announced in paragraph 7 of [10]. The goal of this chapter is to introduce global existence of weak solutions for the quantum hydrodynamic (QHD) system

$$\begin{cases} \partial_t \rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}\left(\dfrac{J \otimes J}{\rho}\right) + \nabla p(\rho) = \dfrac{1}{2}\rho \nabla \left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases} \tag{2.0.1}$$

posed on $\mathbf{R}^d$ with $d = 2, 3$ and equipped with non-vanishing boundary conditions at infinity

$$\rho(x) \to 1, \qquad |x| \to \infty. \tag{2.0.2}$$

The unknowns are the mass (or charge) density ρ and the current density J, the pressure term is denoted by $p(\rho)$. System (2.0.1) describes quantum fluids and arises e.g. as a model in superfluidity, Bose-Einstein condensates and nonlinear optics. The boundary condition is motivated by applications such as quantum vortices [159], superfluidity in Helium II close to the λ-point [87, 159], Bose-Einstein condensates [90, 159, 160] and dark solitons in nonlinear optics [122]. For a more detailed introduction to the physical applications of (2.0.1) we refer the reader to the Introduction of this thesis. We provide an existence theory for finite energy weak solutions and for weak solutions of infinite energy including quantized vortices. Our analysis of the Cauchy problem figures as starting point for a subsequent investigation of quantized vortices being in general of infinite energy. The term on the right hand side takes into account the quantum effects of the fluid and is a nonlinear third order (dispersive) term. Under suitable regularity assumptions, it may also be written in different ways

$$\frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)=\frac{1}{4}\nabla\Delta\rho-\operatorname{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho})=\frac{1}{4}\operatorname{div}(\rho\nabla^2\log\rho). \tag{2.0.3}$$

System (2.0.1) is Hamiltonian, whose energy

$$\mathcal{E}(\rho,J)=\int\frac{1}{2}|\nabla\sqrt{\rho}|^2+\frac{1}{2}\frac{|J|^2}{\rho}+F(\rho)\mathrm{d}x, \tag{2.0.4}$$

is formally conserved along the flow of solutions. The internal energy is characterized by the equation $p(\rho)=F'(\rho)\rho-F(\rho)$. Here, we consider barotropic fluids with pressure laws given by the γ-law. The internal energy then reads

$$F(\rho)=\frac{1}{\gamma(\gamma-1)}(\rho^\gamma-1-\gamma(\rho-1)).$$

Our analysis is based on the the properties of the nonlinear Schrödinger equation (1.1.1) for which we have proven global well-posedness in the energy space in Chapter 1. Indeed, as mentioned in the Introduction system (2.0.1) is closely related to (1.1.1) studied in the previous Chapter 1. Our method consists in developing a polar decomposition approach that allows us to introduce the hydrodynamic variables $\rho=|\psi|^2$ and $\Lambda=\frac{J}{\sqrt{\rho}}$ in a self-consistent way avoiding to define the velocity field in the vacuum region. The polar factorization method represents an adaption of the method introduced in [11, 12] where the authors show the existence of global finite energy weak solutions to (2.0.1) without far-field. Consequently, they may rely on H^1-wave functions for which the polar factorization is provided. Here, we deal wave-functions $\psi\in\mathbb{E}_2$ and $\psi\in\mathcal{V}+H^1$ respectively. The loss of integrability and the fact that these spaces are only metric but not Banach spaces require some new ingredients in analyzing the stability for the polar decomposition.

The quantum hydrodynamic system has widely been investigated in literature. Without far-field, the existence of finite energy weak solutions has been proven in [11, 12], see also references therein for previous results for more regular solutions. For $\gamma=1$, we refer to the model as isothermal quantum fluid and global existence of finite energy weak solutions has been shown in [50] in the case without far-field, see also [48] for the analysis of large time behavior. System (2.0.1) is closely related to the class of Euler-Korteweg fluids [19] where capillary effects need to be taken into account. Indeed, system (2.0.1) may be considered as Euler-Korteweg model with a particular choice of the capillary tensor. In [21] local existence and uniqueness of strong solutions for small and regular data for a class of E-K models including the QHD system with far-field (2.0.2) has been considered. The approach in [21] excludes the presence of vacuum. In [16], global existence and uniqueness for strong solutions that are irrotational and a small perturbation of the constant state is proven. The result requires irrotationality, smallness and high Sobolev regularity of the initial data yielding in particular that $0<c<\rho<C$ for some

$C, c > 0$. For system (2.0.1) with $\gamma = 2$, existence of global strong irrotational solutions has been proven in [17]. The approach of [17] requires the initial data to be a small perturbation of the constant state $\rho = 1$ in the sense that vacuum is excluded and the initial data is small in high Sobolev norms. Under further regularity assumptions, the solutions constructed in [17] are unique. In the framework of weak solutions to the E-K system satisfying the energy inequality, it has been pointed out that infinitely many solutions can be constructed for prescribed initial data by means of convex integration, see [70]. The result applies to the system posed on the torus $\mathbf{T}^d$. Weak-strong uniqueness has been pointed out in [85]. In [40], the authors introduce a relative entropy suitable for a general framework of Navier-Stokes and Euler-Korteweg fluids, further the weak-strong uniqueness result is recovered. In particular, in [40] it is shown how to construct dissipative weak solutions to (2.0.1) as limit of weak solutions to the quantum Navier-Stokes system (3.0.1). Finally, we mention that local well-posedness for smooth solutions to an incompressible inhomogeneous Euler-Korteweg model has been investigated in [47].

In the present chapter, we deal with finite energy weak solutions for initial data being merely of finite energy without further regularity or smallness assumptions. As it will become clear from the Definition 2.1.1, the weak solutions constructed here are weakly irrotational in the sense that a weaker condition on the vorticity $\nabla \wedge J$, called the generalized irrotationality condition is assumed. In a second moment, we provide a framework for weak solutions including quantized vortices that are in general of finite energy. For that purpose, we introduce a suitable notion of degree for vortex solutions connected to the idea of quantized circulation and prove global existence of solutions exhibiting vortices. In particular, there exists stationary vortex solutions to (2.0.1).

The chapter is organized as follows, the main results are stated in Section 2.1. Section 2.2 sets up the polar decomposition approach and introduces in a self consistent way the hydrodynamic variables. We give a suitable notion of vorticity. Subsequently Section 2.3 and 2.4 introduces global existence of finite energy weak solutions and weak solutions of infinite energy respectively. In Section 2.5.1 we discuss special solutions and large time asymptotics.

2.1 Definitions and main results

Weak solutions

We specify the class of weak solutions subject to our analysis. The pressure functions considered are given by the physically relevant γ-law, i.e. $p(\rho) = \frac{1}{\gamma}\rho^\gamma$. The internal energy density is characterized by the differential equation $p(\rho) = \rho F'(\rho) - F(\rho)$ and is non-negative, convex and the minimum is achieved for $\rho = 1$, i.e.

$$F(\rho) := \frac{\rho^\gamma - 1 - \gamma(\rho - 1)}{\gamma(\gamma - 1)}, \tag{2.1.1}$$

The energy functional (2.0.4) reads

$$\mathcal{E}(\rho, J) = \int \frac{1}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}\frac{|J|^2}{\rho} + F(\rho)\mathrm{d}x. \tag{2.1.2}$$

We recall that under suitable regularity assumptions the quantum pressure term can be re-written as

$$2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \operatorname{div}\left(\rho\nabla^2\log\rho\right) = \nabla\Delta\rho - 4\operatorname{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}).$$

Definition 2.1.1 (Finite energy weak solutions)**.** *Let $\rho_0, J_0 \in L^1_{loc}(\mathbf{R}^d)$, we say the pair (ρ, J) is a finite energy weak solution to the Cauchy problem for* (2.0.1) *with initial data*

$$\rho(0) = \rho_0, \qquad J(0) = J_0,$$

in the space-time slab $[0,T)\times\mathbf{R}^d$ if there exist two locally integrable functions

$$\sqrt{\rho} \in L^2_{loc}(0,T; H^1_{loc}(\mathbf{R}^d)), \quad \Lambda \in L^2_{loc}(0,T; L^2_{loc}(\mathbf{R}^d))$$

such that

(i) $\rho := (\sqrt{\rho})^2, J := \sqrt{\rho}\Lambda$*;*

(ii) $\forall\, \zeta \in \mathcal{C}_0^\infty([0,T)\times\mathbf{R}^d)$,

$$\int_0^T\int_{\mathbf{R}^d}\rho\partial_t\zeta + J\cdot\nabla\zeta\,\mathrm{d}x\mathrm{d}t + \int_{\mathbf{R}^d}\rho_0(x)\zeta(0,x)\,\mathrm{d}x = 0;$$

(iii) $\forall\zeta \in \mathcal{C}_0^\infty([0,T)\times\mathbf{R}^d;\mathbf{R}^d)$,

$$\begin{aligned}\int_0^T\int_{\mathbf{R}^d} J\cdot\partial_t\zeta + (\Lambda\otimes\Lambda):\nabla\zeta + p(\rho)\operatorname{div}\zeta + (\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}):\nabla\zeta \\ + \frac{1}{4}\rho\Delta\operatorname{div}\zeta\,\mathrm{d}x\mathrm{d}t + \int_{\mathbf{R}^d} J_0(x)\cdot\zeta(0,x)\,\mathrm{d}x = 0;\end{aligned}$$

(iv) (finite energy) the total energy defined as in (2.1.2) *is finite for almost every $t \in [0,T)$;*

(v) (generalized irrotationality condition) for almost every $t \in (0,T)$

$$\nabla\wedge J = 2\nabla\sqrt{\rho}\wedge\Lambda,$$

holds in the sense of distributions.

We say (ρ, J) is a global in time finite energy weak solution if we can take $T = \infty$ in the above definition.

Remark 2.1.2. For smooth solutions (ρ, J) the generalized irrotationality condition implies that the solution describes an irrotational flow. Indeed, in the smooth framework, one has $J = \rho v$ for a smooth velocity field v. It follows that $\nabla \wedge J = 2\nabla\sqrt{\rho} \wedge \sqrt{\rho}v$ implies $\rho\nabla \wedge v = 0$. This suggests that the previous Definition is the suitable weak formulation of the classical irrotationality condition $\rho\nabla \wedge v = 0$ valid away from vacuum in the Madelung transform approach. The generalized irrotationality condition is compatible with the physical phenomena we aim to describe such as superfluidity and Bose-Einstein condensation; the angular momentum is conserved and the vorticity is only carried by quantized vortices that are coherent structure appearing in the nodal set where the density of the fluid vanishes.

Main results

We give a precise formulation of our existence results that require the initial data to be consistent with a wave function $\psi : \mathbf{R}^d \to \mathbf{C}$. The respective function spaces have been introduced in Section 1.1 and Section 1.3.

Theorem 2.1.3. *Let $d = 2, 3$, $\gamma > 1$ for $d = 2$ and $1 < \gamma < 3$ for $d = 3$. Given $\psi_0 \in \mathbb{E}_\gamma$ defined in* (1.1.56)*, define $\rho_0 := |\psi_0|^2$ and $J_0 = Im(\overline{\psi_0}\nabla\psi_0)$. Then there exists a global in time finite energy weak solution of* (2.0.1) *with initial data*

$$\rho(0) = \rho_0, \qquad J(0) = J_0,$$

and such that

$$\sqrt{\rho} - 1 \in L^\infty(\mathbf{R}; H^1(\mathbf{R}^d)), \quad \Lambda \in L^\infty(\mathbf{R}; L^2(\mathbf{R}^d))$$

Moreover the energy (2.1.2) *is conserved for all times $t \in \mathbf{R}$, i.e. $\mathcal{E}(t) = \mathcal{E}(0)$.*

The statement of Theorem 2.1.3 can be generalized in the sense that the assumptions on the pressure law might be relaxed to more general barotropic pressure laws. Provided that the well-posedness theory in the energy space $\mathbb{E}_2$ for the corresponding nonlinear Schrödinger equation (1.1.1) is available, the method leading to Theorem 2.1.3 may be adapted. In particular, monotonicity and non-negativity of F are not needed. For instance, the well-posedness result for cubic-quintic nonlinearities [118] allows one to derive the existence of finite energy weak solutions for the respective QHD system.

Next, we provide an existence result regarding the Cauchy Problem for a class of initial data including vortices. As already mentioned before, when considering non-trivial boundary conditions at infinity (2.0.2) vortex solutions in general have infinite energy, see also Lemma 1.3.1 for the wave-function analogue. For this reason, in the next Theorem we consider weak solutions of infinite energy. More precisely, the solutions constructed in the Theorem below satisfy all the requirements of Definition 2.1.1 but statement (iv). We recall that the space $\mathcal{V}$ is defined in (1.3.4)

Theorem 2.1.4. *Let $d = 2$ and $\gamma = 2$ and let $\psi_0 \in \mathcal{V}(\mathbf{R}^2) + H^1(\mathbf{R}^2)$ and let the decomposition $\psi = U + w$ be fixed . Define $\rho_0 := |\psi_0|^2$ and $J_0 := Im(\overline{\psi_0}\nabla\psi_0)$. Then there exists a global weak solution of* (2.0.1)*. such that $\rho(0) = \rho_0$ and $J(0) = J_0$ and*

$$\sqrt{\rho} - 1 \in L^\infty(0, \infty; H^1(\mathbf{R}^2)) \qquad \Lambda \in L^\infty(0, \infty; L^2_{loc}(\mathbf{R}^2)).$$

If in addition, $U \in \mathcal{V}_*(\mathbf{R}^2)$, namely $|\Lambda_U(x)| \leq \frac{C}{\sqrt{|x|}}$ for $|x| >> 1$, then

$$E_{\Lambda_U}(t) = \lim_{R\to\infty} \int_{B_R(0)} \frac{1}{2}\left(|\nabla\sqrt{\rho}|^2 + |\Lambda|^2 - |\Lambda_U|^2\right) + \frac{1}{4}(\rho - 1)^2 \mathrm{d}x < \infty$$

and $E_{\Lambda_U}(t)$ is conserved for a. e. $t \in [0, T]$.

We refer the reader to Section 2.5.2 for an application of Theorem 2.1.4 to the study of quantized vortices.

Remark 2.1.5. The proof of Theorem 2.1.4 relies on the description of the wave-function dynamics provided by [32] and discussed in Section 1.3 . The local in time theory is inspired by the method of [82], namely to study the affine problem $\psi = U + w$ with $U \in \mathcal{V}(\mathbf{R}^2)$ and $w \in H^1(\mathbf{R}^2)$. Even though ∇U is not square integrable, it is regular, i.e. $\nabla^k U \in L^2(\mathbf{R}^2)$ for all $k \geq 2$, and therefore the strategy is comparable to the one of [82] where the wave-function is decomposed as $\psi = \phi + w$ with $\phi \in C_b^\infty(\mathbf{R}^d) \cap \mathbb{E}_2(\mathbf{R}^d)$ and $w \in H^1(\mathbf{R}^d)$. This suggests that the (local) existence result of [32], see also Theorem 1.3.3 might be extended to the case $\gamma \geq 2$ (ensuring $F'(r) \in C^1$). On its turn, this allows one to extend Theorem 2.1.4.

2.2 Polar decomposition approach

The correspondence between the wave function dynamics described by the nonlinear Schrödinger equation (1.1.1) and the hydrodynamic system (2.0.1) is established by means of the polar decomposition approach that associates to a wave function ψ the hydrodynamic variables $(\sqrt{\rho}, \Lambda)$. This method is not limited by vacuum regions including quantized vortices to be considered. While the Madelung transform approach, i.e. decomposing $\psi = |\psi| e^{i\frac{S}{\hbar}}$, is only valid for smooth, non-vanishing wave functions, the polar factor decomposition remains valid for wave-functions that are merely of finite energy. Here, the nodal set is of finite Lebesgue measure due to the assumption $|\psi|^2 - 1 \in L^2(\mathbf{R}^d)$ but is not a regular set [66]. Our method is based on the polar decomposition for H^1-functions as in [11, 12]. Dealing with wave-functions that are neither integrable due to (2.0.2) nor of finite energy requires substantial modifications of the method. In particular, neither the energy space $\mathbb{E}_2$ nor the space $\mathcal{V}$ are Banach spaces and thus there is no natural topology induced by a norm available. Still, the space $\mathbb{E}_2$ is a complete metric space, see Section 1.1.1, while the space $\mathcal{V}$ does not seem to be endowed with a nice topology, see also [32].
Our approach consists in factorizing the wave function ψ in its amplitude $\sqrt{\rho} = |\psi|$ and its

polar factor ϕ, namely a function with values in the unitary disk of the complex plane. Given any function $\psi \in L^2_{loc}(\mathbf{R}^d)$ we define the set

$$P(\psi) := \left\{ \phi \in L^\infty(\mathbf{R}^d) \ : \ \|\phi\|_{L^\infty} \leq 1, \, \psi = \sqrt{\rho}\phi \, \text{a. e. in} \, \mathbf{R}^d \right\},$$

where $\sqrt{\rho} := |\psi|$. It is clear that the polar factor is not uniquely defined on vacuum regions, but one has that $|\phi| = 1$ $\sqrt{\rho}\mathrm{d}x$ a. e. in $\mathbf{R}^d$ and the polar factor ϕ is uniquely defined $\sqrt{\rho}\mathrm{d}x$ a. e. in $\mathbf{R}^d$. This is - in a oversimplified context - somehow reminiscent of Y. Brenier's idea in [36]. We characterize the set $P(\psi)$ by a variational problem. To that end, given $\psi \in L^2_{loc}(\mathbf{R}^d)$ and $R > 0$ we define

$$P(\psi, B_R(0)) := \left\{ \phi \in L^\infty(\mathbf{R}^d) \ : \ \|\phi\|_{L^\infty} \leq 1, \, \psi = \sqrt{\rho}\phi \, \text{a. e. in} \, B_R \right\},$$

and consider the maximizer of the functional

$$\Phi_R[\phi] = Re \int_{B_R} \overline{\phi}\psi \mathrm{d}x$$

over the set

$$S_R := \{\phi \in L^2(B_R) \quad : \quad \|\phi\|_{L^\infty} \leq 1\}.$$

It is straightforward to check the following equivalence.

Lemma 2.2.1. *Fix a radius $R > 0$ and let $\psi \in L^2(B_R)$. Then the following are equivalent.*

1. *$\phi \in P(\psi, B_R(0))$,*

2. *ϕ is a maximizer of the functional of $\Phi_R[\cdot]$ over S_R.*

Proof. "$\Rightarrow$" We notice that $\Phi_R[\phi] \leq \|\phi\|_{L^\infty}\|\psi\|_{L^1}$. Let $\phi \in P_R(\psi)$, then $\psi = \phi|\psi|$ $\sqrt{\rho}$ a. e. in B_R and it follows that

$$\int_{B_R} Re(\overline{\phi}\psi)\mathrm{d}x = \int_{B_R} |\psi|\mathrm{d}x,$$

thus ϕ maximizes the functional Φ_R over S_R.
"$\Leftarrow$" Let ϕ be a maximizer of Φ_R over S_R. A maximizer does indeed exists for a linear functional to be maximized over a convex domain. Since ϕ is a maximizer one has that

$$Re \int_{B_R} \overline{\phi}\psi \mathrm{d}x = \int_{B_R} |\psi|\mathrm{d}x.$$

It is easy to check that $\phi \in P(\psi, B_R(0))$. □

The property of being a polar factor for a function $\psi \in L^2_{loc}(\mathbf{R}^d)$ is a local property as shows the following criterion.

Lemma 2.2.2. *Let $\psi \in L^2_{loc}(\mathbf{R}^d, \mathbf{C})$. The following are equivalent:*

1. *The function $\phi \in L^\infty(\mathbf{R}^d, \mathbf{C})$ is a polar factor for ψ, i. e. $\phi \in P(\psi)$.*

2. *There exists a covering of* $\mathbf{R}^d$ *by a countable union of balls* B_j *(of finite radius* r_j*) and* ϕ *such that* $\phi\big|_{B_j} \in P(\psi, B_j)$ *for every* $j \in \mathbf{N}$.

Proof. The implication (1) $\to$ (2) is obvious. The polar factor is uniquely defined $\sqrt{\rho}\mathrm{d}x$ a. e. in $\mathbf{R}^d$. Given a countable covering of $\mathbf{R}^d$, for every j, the restriction $\phi\big|_{B_j}$ is a polar factor of ψ on B_j.

Conversely, we claim that

$$\phi(x) := \phi_i(x), \qquad \text{where} \quad i = \min_j \{j \,|\, x \in B_j\},$$

is a polar factor of ψ, i. e. $\phi \in P(\psi)$. Indeed, ϕ is well-defined since on every B_j the polar factor is uniquely defined $\sqrt{\rho}\mathrm{d}x$ almost everywhere and does not depend on the particular choice of the covering. Given $i \neq j$ such that $A := B_i \cap B_j \neq \emptyset$ and $\mu(A) > 0$, let $\phi_i \in P(\psi, B_i)$ and $\phi_j \in P(\psi, B_j)$, then necessarily $\sqrt{\rho}\phi_i = \sqrt{\rho}\phi_j$ a. e. on A. Assume not, then it is easy to construct a contradiction to the maximizing property of polar factors introduced in Lemma 2.2.1. The set

$$B = \{x \in A \;:\quad \phi_i(x) \neq \phi_j(x)\}$$

is such that

$$\int_B \sqrt{\rho}\mathrm{d}x > 0,$$

by assumption. Further, since $\phi_i \in P(\psi, B_i)$ and $\phi_j \in P(\psi, B_j)$ and therefore maximize $\Phi[\cdot]$ on the respective domains one has that

$$\int_B \sqrt{\rho}\mathrm{d}x = \Phi_B[\phi_i] \neq \Phi_B[\phi_j] = \int_B \sqrt{\rho}\mathrm{d}x,$$

yielding the desired contradiction.

$\square$

With this criterion at hand, it is immediate to check that for any $\psi \in L^2_{loc}(\mathbf{R}^d)$ there exists a polar factor ϕ being uniquely determined $\sqrt{\rho}\mathrm{d}x$ everywhere.

2.2.1 Stability of the polar factorization for wave-functions in the energy space

Next, we show how to recast the hydrodynamic variables $(\sqrt{\rho}, \Lambda)$ given the wave function ψ. Although $\mathbb{E}_2 \subset \mathcal{V} + H^1$, see Lemma 1.1.6, we consider separately the cases $\psi \in \mathbb{E}_2$ and $\psi \in \mathcal{V} + H^1$. In the former case, we exploit the topology on $\mathbb{E}_2$ induced by the metric $d_{\mathbb{E}_2}$. In the latter, we prove H^1-stability for a fixed decomposition.

Lemma 2.2.3 (Stability in $\mathbb{E}$)**.** *Let* $\psi \in \mathbb{E}_2$, *let* $\sqrt{\rho} := |\psi|$ *be its amplitude and let* $\phi \in P(\psi)$ *be a polar factor associated to* ψ. *Then* $\sqrt{\rho} - 1 \in L^2$ *and* $\nabla\sqrt{\rho} = \mathrm{Re}\left(\bar{\phi}\nabla\psi\right) \in L^2(\mathbf{R}^d)$. *Moreover, if we define* $\Lambda := \mathrm{Im}\left(\bar{\phi}\nabla\psi\right)$, *then* $\Lambda \in L^2(\mathbf{R}^d)$ *and the following identity holds*

$$\mathrm{Re}\left(\nabla\bar{\psi} \otimes \nabla\psi\right) = \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} + \Lambda \otimes \Lambda \quad a.\ e.\ in\ \mathbf{R}^d. \tag{2.2.1}$$

Furthermore, if $\{\psi_n\} \subset \mathbb{E}$ *such that* $d_{\mathbb{E}}(\psi_n, \psi) \to 0$*, then the stability property holds*

$$\begin{aligned} &\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \quad \Lambda_n \to \Lambda, \quad in \quad L^2(\mathbf{R}^d), \\ &\rho_n - \rho \to 0 \quad in \quad L^2(\mathbf{R}^d), \qquad \sqrt{\rho_n} - \sqrt{\rho} \to 0 \quad in \quad H^1(\mathbf{R}^d). \end{aligned}$$

Finally, for $\psi \in \mathbb{E}_2$ *the current density* $J = \sqrt{\rho}\Lambda$ *satisfies,*

$$\nabla \wedge J = 2\nabla\sqrt{\rho} \wedge \Lambda \in L^1(\mathbf{R}^d),$$

Proof. Let $\psi \in \mathbb{E}_2$, then from Lemma 1.1.6 there exists a sequence $\{\psi_n\}_{n\in\mathbf{N}} \subset C^\infty(\mathbf{R}^d)$ of smooth functions such that

$$d_{\mathbb{E}_2}(\psi_n, \psi) \to 0,$$

as $n \to \infty$. Let

$$\phi_n = \begin{cases} \frac{\psi_n(x)}{|\psi_n(x)|} & \psi_n(x) \neq 0, \\ 0 & \psi_n(x) = 0, \end{cases}$$

it is immediate to check that for every $n \in \mathbf{N}$ one has $\phi_n \in P(\psi_n)$. Since $\|\phi_n\|_{L^\infty(\mathbf{R}^d)} \leq 1$ uniformly there exists $\phi \in L^\infty(\mathbf{R}^d)$ such that

$$\phi_n \overset{*}{\rightharpoonup} \phi.$$

It is straightforward to check that ϕ is a polar-factor for ψ, i. e. $\phi \in P(\psi)$, as it is sufficient to check this property locally, see Lemma 2.2.2. Moreover, we have that

$$\nabla\sqrt{\rho_n} \rightharpoonup \nabla\sqrt{\rho} \quad \text{in} \quad L^2(\mathbf{R}^3),$$

and

$$Re(\overline{\phi_n}\nabla\psi_n) \rightharpoonup \quad Re(\overline{\phi}\nabla\psi) \quad \text{in} \quad L^2(\mathbf{R}^3),$$

where we have used that $d_{\mathbb{E}_2}(\psi_n, \psi) \to 0$. Since $\nabla\sqrt{\rho_n} = Re(\overline{\phi_n}\nabla\psi_n)$ for smooth functions, we conclude that $\nabla\sqrt{\rho} = Re(\overline{\phi}\nabla\psi)$ a. e. in $\mathbf{R}^d$. This identity holds independently from the particular choice of the polar factor. Indeed, we have $\nabla\psi = 0$ a.e. on $\{\psi = 0\}$, see for instance Theorem 6.19 in [132], while ϕ is uniquely determined on $\{\psi \neq 0\}$ a.e. in $\mathbf{R}^d$. By using this property, we may then define $\Lambda := Im(\overline{\phi}\nabla\psi)$ without ambiguity given by the particular choice of the polar factor. Exploiting again that $\nabla\psi = 0$ a.e. on $\{\psi = 0\}$ and the fact that $|\phi| = 1$ $\sqrt{\rho}dx$ a.e. in $\mathbf{R}^d$, one obtains that

$$\begin{aligned} Re\left(\nabla\overline{\psi} \otimes \nabla\psi\right) &= Re\left(\phi\nabla\overline{\psi} \otimes \overline{\phi}\nabla\phi\right) = Re\left(\overline{\phi}\nabla\psi\right) \otimes Re\left(\overline{\phi}\nabla\psi\right) + Im\left(\overline{\phi}\nabla\psi\right) \otimes Im\left(\overline{\phi}\nabla\psi\right) \\ &= \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} + \Lambda \otimes \Lambda. \end{aligned}$$

Taking the trace yields,

$$|\nabla\psi|^2 = |\nabla\sqrt{\rho}|^2 + |\Lambda|^2 .$$

Similarly, since

$$\nabla\overline{\psi} \wedge \nabla\psi = \left(\phi\nabla\overline{\psi}\right) \wedge \left(\overline{\phi}\nabla\psi\right) = 2i\nabla\sqrt{\rho} \wedge \Lambda,$$

we recover the identity,

$$\nabla \wedge J = \nabla \wedge Im(\overline{\psi}\nabla\psi) = Im(\nabla\overline{\psi} \wedge \nabla\psi) = 2\nabla\sqrt{\rho} \wedge \Lambda.$$

The stability property is shown as follows, let $\{\psi_n\} \subset \mathbb{E}_2$ such that $d_{\mathbb{E}_2}(\psi_n, \psi) \to 0$. Then it is easy to check that

$$\begin{aligned} Re(\overline{\phi_n}\nabla\psi_n) &\rightharpoonup Re(\overline{\phi}\nabla\psi), \\ Im(\overline{\phi_n}\nabla\psi_n) &\rightharpoonup Im(\overline{\phi}\nabla\psi), \end{aligned}$$

Moreover,

$$\begin{aligned} \|\nabla\psi\|_{L^2}^2 = \|\nabla\sqrt{\rho}\|_{L^2}^2 + \|\Lambda\|_{L^2}^2 &\le \liminf_{n\to\infty} \left(\|\nabla\sqrt{\rho_n}\|_{L^2}^2 + \|\Lambda_n\|_{L^2}^2\right) \\ &= \lim_{n\to\infty} \|\nabla\psi_n\|_{L^2}^2 = \|\nabla\psi\|_{L^2}^2 \end{aligned}$$

Weak convergence together with convergence in norm implies strong convergence

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho} \qquad \Lambda_n \to \Lambda \quad \text{in} \quad L^2(\mathbf{R}^d).$$

The convergence $\rho_n - \rho \to 0$ in $L^2(\mathbf{R}^d)$ is an immediate consequence of the assumption. We denote

$$\mathcal{A} = \left\{x \in \mathbf{R}^d \ : \ |\psi|^2 \le \frac{1}{2}\right\}, \qquad \mathcal{A}_n = \left\{x \in \mathbf{R}^d \ : \ |\psi_n|^2 \le \frac{1}{2}\right\}.$$

From $\psi, \psi_n \in \mathbb{E}_2$ and the Chebyshev inequality we conclude that $\mathcal{A}$ and $\mathcal{A}_n$ are of finite Lebesgue measure for all $n \in \mathbf{N}$. Moreover, for $x \in (\mathcal{A} \cup \mathcal{A}_n)^c$ one has

$$||\psi| - |\psi_n|| \le C\left||\psi|^2 - |\psi_n|^2\right|.$$

Thus,

$$\|\sqrt{\rho_n} - \sqrt{\rho}\|_{L^2((\mathcal{A}\cup\mathcal{A}_n)^c)}^2 \le \|\rho_n - \rho\|_{L^2((\mathcal{A}\cup\mathcal{A}_n)^c)}^2$$

The convergence on the set $\mathcal{A} \cup \mathcal{A}_n$ of finite Lebesgue measure can be derived from the local strong convergence of $\nabla\sqrt{\rho_n}$ to $\nabla\sqrt{\rho}$. The proof is complete. □

2.2.2 Stability of the polar factorization for functions of infinite energy

Next, we introduce the polar decomposition and its stability for $\psi \in \mathcal{V} + H^1(\mathbf{R}^2)$ defined in (1.3.4). The next Lemma is the analogue of Lemma 2.2.3 and shows some local properties of the hydrodynamic quantities $(\sqrt{\rho}, \Lambda)$. The subsequent Lemma 2.2.5 then proves some global properties needed to define the renormalized energy functional.

Lemma 2.2.4 (Stability in $\mathcal{V} + H^1$)**.** *Let $\psi \in \mathcal{V}(\mathbf{R}^2) + H^1(\mathbf{R}^2)$, let $\sqrt{\rho} := |\psi|$ be its amplitude and let $\phi \in P(\psi)$ be a polar factor associated to ψ. Then $\sqrt{\rho} - 1 \in L^2(\mathbf{R}^2)$ and $\nabla\sqrt{\rho} = \mathrm{Re}\left(\overline{\phi}\nabla\psi\right)$. Moreover, if $\Lambda := \mathrm{Im}\left(\overline{\phi}\nabla\psi\right)$, then $\Lambda \in L^2_{loc}(\mathbf{R}^2)$ and the following identity holds*

$$\mathrm{Re}\left(\nabla\bar{\psi} \otimes \nabla\psi\right) = \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} + \Lambda \otimes \Lambda \quad a.\ e.\ in\, \mathbf{R}^2.$$

Furthermore, if $\{\psi_n\} \subset \mathcal{V}(\mathbf{R}^2) + H^1(\mathbf{R}^2)$ *such that* $\psi_n - \psi \to 0$ *in* $H^1(\mathbf{R}^2)$*, then the stability property holds*

$$\begin{aligned}
&\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \quad \Lambda_n \to \Lambda, \quad \mathit{in} \quad L^2_{loc}(\mathbf{R}^2),\\
&\rho - \rho_n \to 0 \quad \mathit{in} \quad L^2(\mathbf{R}^2), \qquad \sqrt{\rho} - \sqrt{\rho_n} \to 0 \quad \mathit{in} \quad L^2(\mathbf{R}^2).
\end{aligned}$$

Finally, for $\psi \in \mathcal{V}(\mathbf{R}^2) + H^1(\mathbf{R}^2)$ *the current density* $J = \sqrt{\rho}\Lambda$ *satisfies,*

$$\nabla \wedge J = 2\nabla\sqrt{\rho} \wedge \Lambda.$$

Proof. Given $\psi \in \mathcal{V} + H^1(\mathbf{R}^2)$, let $U \in \mathcal{V}$ and $w \in H^1(\mathbf{R}^2)$ such that $\psi = U + w$. We notice that inequality (1.1.7) yields that $\sqrt{\rho} - 1 \in L^2$. Further, there exists $\{\psi_n\} \subset C^\infty(\mathbf{R}^2)$ such that ψ_n converges to ψ in $H^1_{loc}(\mathbf{R}^d)$, the approximating sequence might for instance be constructed by convolution. Let

$$\phi_n = \begin{cases} \frac{\psi_n(x)}{|\psi_n(x)|} & \psi_n(x) \neq 0,\\ 0 & \psi_n(x) = 0, \end{cases}$$

it is immediate to check that for every $n \in \mathbf{N}$ one has $\phi_n \in P(\psi_n)$. Since $\|\phi_n\|_{L^\infty(\mathbf{R}^2)} \leq 1$ uniformly there exists $\phi \in L^\infty(\mathbf{R}^2)$ such that $\phi_n \overset{*}{\rightharpoonup} \phi$. Relying on Lemma 2.2.2 and a local argument, we obtain that $\phi \in P(\psi)$. Further, we infer that

$$\nabla\sqrt{\rho_n} \rightharpoonup \nabla\sqrt{\rho_n}, \quad Re(\overline{\phi_n}\nabla\psi_n) \rightharpoonup Re(\overline{\phi}\nabla\psi)$$

in $L^2_{loc}(\mathbf{R}^2)$. It follows, $\nabla\sqrt{\rho} = Re(\overline{\phi}\nabla\psi)$ a. e. in $\mathbf{R}^2$. Arguing as in the proof of Lemma 2.2.3, we observe that this identity is independent from the particular choice of the polar factor. Again without ambiguity, we define $\Lambda := Im(\overline{\phi}\nabla\psi)$. We proceed as in Proof of Lemma 2.2.3 to check that

$$Re\left(\nabla\overline{\psi} \otimes \nabla\psi\right) = Re\left(\overline{\phi}\nabla\psi\right) \otimes Re\left(\overline{\phi}\nabla\psi\right) + Im\left(\overline{\phi}\nabla\psi\right) \otimes Im\left(\overline{\phi}\nabla\psi\right),$$

a. e. in $\mathbf{R}^2$. Taking the trace yields,

$$|\nabla\psi|^2 = |\nabla\sqrt{\rho}|^2 + |\Lambda|^2.$$

As in the proof of Lemma 2.2.3 we recover the identity.

$$\nabla \wedge J = 2\nabla\sqrt{\rho} \wedge \Lambda.$$

The stability property is shown as follows, let $\{\psi_n\} \subset \mathcal{V} + H^1(\mathbf{R}^2)$ such that $\|\psi - \psi_n\|_{H^1(\mathbf{R}^2)} \to 0$. The fourth statement of Lemma 1.1.3 might be adapted so that

$$\||\psi_n|^2 - |\psi|^2\|_{L^2} \leq C\left(\||\psi|^2 - 1\|_{L^2} + \|\psi_n - \psi\|_{H^1}\right)\|\psi_n - \psi\|_{H^1}.$$

Thus, $\rho - \rho_n \to 0$ in $L^2(\mathbf{R}^d)$. Further, on any compact set $K \subset \mathbf{R}^2$,

$$\begin{aligned} Re(\overline{\phi_n}\nabla\psi_n) &\rightharpoonup Re(\overline{\phi}\nabla\psi), \\ Im(\overline{\phi_n}\nabla\psi_n) &\rightharpoonup Im(\overline{\phi}\nabla\psi), \end{aligned}$$

in $L^2(K)$. Moreover,

$$\begin{aligned} \|\nabla\psi\|_{L^2(K)}^2 = \|\nabla\sqrt{\rho}\|_{L^2(K)}^2 + \|\Lambda\|_{L^2(K)}^2 &\leq \liminf_{n\to\infty}\left(\|\nabla\sqrt{\rho_n}\|_{L^2(K)}^2 + \|\Lambda_n\|_{L^2(K)}^2\right) \\ &= \lim_{n\to\infty}\|\nabla\psi_n\|_{L^2(K)}^2 = \|\nabla\psi\|_{L^2(K)}^2 \end{aligned}$$

Weak convergence together with strong convergence in norm implies strong convergence

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \qquad \Lambda_n \to \Lambda, \quad \text{in} \quad L^2(K).$$

Proceeding as in Lemma 2.2.3, i.e. combining $\rho - \rho_n \to 0$ in $L^2(\mathbf{R}^d)$ and the local strong L^2-convergence of $\nabla\sqrt{\rho_n}$, we infer that $\sqrt{\rho_n} - \sqrt{\rho} \to 0$ strongly in $L^2(\mathbf{R}^2)$. The proof is complete. □

We introduce the renormalized energy functional for the QHD system being the analogue of (1.3.8), see also [32]. Further, we recall that $\mathcal{V}_*$ defined in (1.3.9) is the space such that $U \in \mathcal{V}$ and $|\nabla U(z)| \leq \frac{C}{\sqrt{|z|}}$ for $|z|$ large.

Lemma 2.2.5. *Let $\psi \in \mathcal{V} + H^1(\mathbf{R}^2)$, and define $\sqrt{\rho} = |\psi|$, then $\nabla\sqrt{\rho} = Re(\overline{\phi}\nabla\psi)$ a.e. in $\mathbf{R}^2$ and $\sqrt{\rho} - 1 \in H^1(\mathbf{R}^2)$. Moreover, if $\psi \in \mathcal{V}_* + H^1(\mathbf{R}^2)$ is decomposed as $\psi = U + w$, and let $\Lambda_U = \mathrm{Im}(\overline{\phi_U}\nabla U)$ for a polar factor ϕ_U of U. Then the renormalized energy functional defined as*

$$E_{\Lambda_U} = \lim_{R\to\infty}\int_{B_R(0)} \frac{1}{2}|\nabla\sqrt{\rho}|^2 + |\Lambda|^2 - |\Lambda_U|^2 + \frac{1}{2}(\rho - 1)^2 \mathrm{d}x, \tag{2.2.2}$$

is finite.

We notice that hence for $\psi \in \mathcal{V}(\mathbf{R}^2) + H^1(\mathbf{R}^2)$ one has $\sqrt{\rho} - 1 \in H^1(\mathbf{R}^2)$. In particular, together with Lemma 2.2.4, this implies that $\nabla\sqrt{\rho_n}$ converges weakly to $\nabla\sqrt{\rho}$ in $L^2(\mathbf{R}^2)$. However, it is in general not clear whether this weak convergence can be upgraded to strong convergence.

Proof. From Lemma 2.2.4, we have that $\nabla|\psi| = Re(\overline{\phi}\nabla\psi)$ holds a.e. in $\mathbf{R}^2$ for a polar factor ϕ of ψ. We recall that the identity does not depend on the particular choice of ϕ being uniquely defined $\sqrt{\rho}$ a.e. in $\mathbf{R}^d$ and that $\nabla\sqrt{\rho}$ vanishes a.e. on $\{\sqrt{\rho} = 0\}$. Given any decomposition $\psi = U + w$ with $U \in \mathcal{V}$ and $w \in H^1(\mathbf{R}^2)$ we denote by ϕ_U the polar factor of U being uniquely defined outside the nodal set of U. Moreover $\nabla U = 0$ a. e. on the nodal set of U due to the regularity properties of U. We denote $\sqrt{\rho_U} := |U| = \mathrm{Re}(\overline{\phi_U}U)$, thus $\nabla\sqrt{\rho_U} = \mathrm{Re}(\overline{\phi_U}\nabla U)$ and further we define $\Lambda_U = \mathrm{Im}(\overline{\phi_U}\nabla U)$. The quantities are well-defined in view of Lemma 2.2.4. Next, we show that $\nabla\sqrt{\rho} \in L^2(\mathbf{R}^2)$.

$$\int_{\mathbf{R}^2} |\nabla\sqrt{\rho}|^2 \mathrm{d}x = \int_{\mathbf{R}^2} |Re(\overline{\phi}\nabla\psi)|^2 \mathrm{d}x = \int_{\mathbf{R}^2} \left|Re((\overline{\phi}\nabla U - \overline{\phi_U}\nabla U + \overline{\phi_U}\nabla U + \overline{\phi}\nabla w)\right|^2 \mathrm{d}x$$
$$\leq C\left(\|\nabla\sqrt{\rho_U}\|_{L^2}^2 + \|\phi\|_{L^\infty}\|\nabla w\|_{L^2(\mathbf{R}^d)} + \int_{\mathbf{R}^d} \left|\mathrm{Re}\left((\overline{\phi} - \overline{\phi_u})\nabla U\right)\right|^2 \mathrm{d}x\right)$$

Hence, it suffices to bound the last integral. Let $c \in (0, \frac{1}{2}]$ and define

$$\mathcal{A} := \{x \in \mathbf{R}^2 : \quad ||U+w|^2 - 1| \geq c\}, \quad \mathcal{B} := \{x \in \mathbf{R}^2 : \quad ||U|^2 - 1| \geq c\},$$

and notice that $\mathcal{A}, \mathcal{B}$ are of of finite Lebesgue measure in virtue of the Chebyshev inequality and $U \in \mathcal{V}$ as well as $\psi \in \mathcal{V} + H^1$. We have that

$$\int_{\mathcal{A}\cup\mathcal{B}} \left|\mathrm{Re}\left((\overline{\phi} - \overline{\phi_U})\nabla U\right)\right|^2 \mathrm{d}x \leq C_{c,\mathcal{A},\mathcal{B}}\|\phi - \phi_U\|_{L^\infty(\mathcal{A}\cup\mathcal{B})}^2 \|\nabla U\|_{L^\infty(\mathcal{A}\cup\mathcal{B})}^2$$

Further, neither $|U+w|$ nor $|U|$ vanish on $(\mathcal{A}\cup\mathcal{B})^C$, thus the polar factors are uniquely defined on $(\mathcal{A}\cup\mathcal{B})^c$. We conclude that

$$\int_{\mathbf{R}^2\setminus(\mathcal{A}\cup\mathcal{B})} \left|\mathrm{Re}\left((\overline{\phi} - \overline{\phi_u})\nabla U\right)\right|^2 \mathrm{d}x \leq C\|\phi - \phi_U\|_{L^4(\mathbf{R}^2\setminus(\mathcal{A}\cup\mathcal{B}))}^2 \|\nabla U\|_{L^4(\mathbf{R}^2\setminus(\mathcal{A}\cup\mathcal{B}))}^2,$$

and recall that $\nabla U \in L^4(\mathbf{R}^2)$ from Lemma 3.3.17. To infer the L^4 bound for $\phi - \phi_U$, we notice that

$$\phi = \frac{U+w}{|U+w|}, \quad \phi_U = \frac{U}{|U|} \quad \text{a.e. on} \quad \mathbf{R}^2\setminus(\mathcal{A}\cup\mathcal{B}).$$

Since on $\mathbf{R}^2\setminus(\mathcal{A}\cup\mathcal{B})$, by Definition $||U+w| - 1| \leq c$ and $||U| - 1| \leq c$ one has

$$\left|\frac{U+w}{|U+w|} - \frac{U}{|U|}\right|^2 \leq C\left(|U+w|(|U| - |U+w|) + |w|^2\right) \leq C\left(|U| - |U+w| + |w|^2\right).$$

Moreover, on $(\mathcal{A}\cup\mathcal{B})^c$, one has

$$||U+w| - |U|| \leq C||U+w|^2 - |U|^2,$$

and therefore we conclude that

$$\|\phi - \phi_U\|_{L^4(\mathbf{R}^2\setminus(\mathcal{A}\cup\mathcal{B})}^2 \leq C_c\left(\||U+w|^2 - |U|^2\|_{L^2} + \|v\|_{L^4}^2\right)\|\nabla U\|_{L^4}^2 < +\infty.$$

In the last step we used that $|U|^2 - 1 \in L^2(\mathbf{R}^2)$ and $|U+w|^2 - 1 \in L^2(\mathbf{R}^2)$.
It remains to show that if $\psi \in \mathcal{V}_* + H^1(\mathbf{R}^2)$, then (2.2.2) is finite. For a fixed decomposition $\psi = U + w$, the renormalized energy functional is defined by (1.3.8) and finite. We invoke the identity

$$|\nabla\psi|^2 - |\nabla U|^2 = |\nabla\sqrt{\rho}|^2 - |\nabla\sqrt{\rho_U}|^2 + |\Lambda|^2 - |\Lambda_U|^2,$$

thus by recalling that $\nabla\sqrt{\rho} \in L^2(\mathbf{R}^2)$ and $\nabla|U| \in L^2(\mathbf{R}^2)$,

$$E_{\Lambda_U} = \lim_{R\to\infty}\int_{\mathbf{R}^2} \frac{1}{2}|\nabla\sqrt{\rho}|^2 - |\nabla|U||^2 + |\Lambda|^2 - |\Lambda_U|^2 + \frac{1}{2}(\rho-1)^2\mathrm{d}x,$$

is finite. □

Remark 2.2.6. Instead of assuming that $\psi \in \mathcal{V}_*$ in order to define the renormalized energy in Lemma 2.2.5, one may equivalently suppose that given the decomposition $\psi = U + w$, one has $|\Lambda_U| \leq \frac{C}{|x|}$ for $|x| \to \infty$. We recall that the Definitions given in (1.3.7) and (1.3.8) are equivalent provided that

$$\int_{\partial B_R} \mathrm{Re}(\partial_r U w) \to 0,$$

as $R \to \infty$. We write

$$\begin{aligned}\int_{\partial B_R} \mathrm{Re}(\partial_r U w) &= \int_{\partial B_R} \mathrm{Re}(\overline{\phi_U}\partial_r U \phi_U w),\\ &= \int_{\partial B_R} \mathrm{Re}(\overline{\phi_U}\partial_r U)\,\mathrm{Re}(\phi_U v) + \int_{\partial B_R} \mathrm{Im}(\overline{\phi_U}\partial_r U)\,\mathrm{Im}(\phi_U v)\\ &= \int_{\partial B_R} \mathrm{Re}(\phi_U w)\nabla\sqrt{\rho_U}\cdot n + \int_{\partial B_R} \mathrm{Im}(\phi_U w)\Lambda_U \cdot n.\end{aligned}$$

Since $\nabla\sqrt{\rho_U} = \nabla|U| \in L^2(\mathbf{R}^2)$ the first term in the sum goes to 0 as $R \to \infty$. Further, for any $w \in H^1(\mathbf{R}^2)$,

$$\lim_{R\to\infty}\int_{\partial B_R} |w|^2 = 0,$$

see Lemma 3.4 in [28]. The idea consists in showing that $f(R) = \int_{\partial B_R}|w|^2 \in W^{1,1}(\mathbf{R})$ and thus in $C_0(\mathbf{R})$. Hence, the boundary term goes to zero if there exists $C > 0$ such that

$$\int_{\partial B_R} |\Lambda_U|^2 \leq C,$$

uniformly in R. This is achieved by assuming $|\Lambda_U(z)| \leq \frac{C}{\sqrt{|z|}}$, compatible to the hypothesis $U \in \mathcal{V}_*$.

2.2.3 Continuity of the vorticity with respect to weak continuity

The following analysis on the continuity of the vorticity $\nabla \wedge J$ is needed in view of the study of the singular limit in Sections 5.1 and 5.2. We recall that from Lemma 2.2.4 and 2.2.3 respectively, we have that $\nabla \wedge J = 2\nabla\sqrt{\rho}\wedge\Lambda$. If $\psi \in \mathbb{E}_2$ then $\nabla\wedge J \in L^1(\mathbf{R}^d)$, while if $\psi \in \mathcal{V}(\mathbf{R}^2) + H^1(\mathbf{R}^2)$ we only have $\nabla \wedge J \in L^1_{loc}(\mathbf{R}^d)$.

Lemma 2.2.7. *Let $d = 2, 3$ and $\{\psi_n\} \subset H^1_{loc}(\mathbf{R}^d)$ be a sequence converging weakly to some $\psi \in H^1_{loc}(\mathbf{R}^d)$. Define $J_n := \mathrm{Im}(\overline{\psi_n}\nabla\psi_n)$ and $J := \mathrm{Im}(\overline{\psi}\nabla\psi)$.*
If there exists $C > 0$ uniform in n such that $\mathcal{E}(\psi_n) \leq C$, then

$$J_n \rightharpoonup J \quad in \quad L^p_{loc}(\mathbf{R}^d),$$

with $1 \leq p < 2$ *for* $d = 2$ *and for* $1 \leq p < \frac{3}{2}$ *for* $d = 3$*. Moreover,*

$$\nabla \wedge J_n \to \nabla \wedge J,$$
$$\nabla\sqrt{\rho_n} \wedge \Lambda_n \to \nabla\sqrt{\rho} \wedge \Lambda,$$

in $\mathcal{D}'$*.*

We notice that if $J = Im(\overline{\psi}\nabla\psi)$, then up to a factor 2 the vorticity of the current density $\nabla \wedge J$ equals the Jacobian $\mathrm{Jac}(\psi) = \det(\nabla\psi)$, namely

$$\frac{1}{2}\nabla \wedge J = \mathrm{Jac}(\psi).$$

For $d = 2$, the statement of the Lemma is an immediate consequence of the continuity of distributional Jacobians with respect to weak convergence. The distributional Jacobian is defined by

$$\int_{\mathbf{R}^d} [\mathrm{Jac}(\psi)]\phi \mathrm{d}x := \frac{1}{2}\int_{\mathbf{R}^d} J \cdot \nabla \wedge \phi, \qquad \phi \in C_c^1,$$

provided that $J \in L^1_{loc}(\mathbf{R}^d)$. We notice that for $\psi_n, \psi \in H^1_{loc}(\mathbf{R}^2)$ the distributional Jacobian is well-defined and coincides with $\mathrm{Jac}(\psi)$. Thus,

$$\nabla \wedge J_n = [\mathrm{Jac}(\psi_n)] \to [\mathrm{Jac}(\psi)] = \nabla \wedge J \qquad \text{in} \quad \mathcal{D}'.$$

For $d = 3$, assuming $\psi \in H^1(\mathbf{R}^3)$ is in general not sufficient neither to define nor to identify the distributional Jacobian and the Jacobian, while assuming for instance $\psi \in W^{1,d}_{loc}$ would be sufficient. More in general, the Jacobian $det(\nabla\psi)$ and the distributional $[\mathrm{Jac}(\psi)]$ coincide whenever the latter satisfies $[\mathrm{Jac}(\psi)] \in L^1_{loc}(\mathbf{R}^d)$, see [149]. This will become relevant in the context of vortex solutions addressed in Section 1.3 and in Section 2.5.2. Indeed, the prototype of an idealized vortex solutions is given by $\frac{x}{|x|}$ where we identify $\mathbf{C}$ with $\mathbf{R}^2$. It's Jacobian is 0 a.e. in $\mathbf{R}^d$ while its distributional Jacobian equals $[\mathrm{Jac}(\frac{x}{|x|})] = \delta_0$. We refer the reader to the review paper [146] and to [4] for Jacobians of functions with values in the sphere.
Here, we give a proof of the Lemma that does not rely on the continuity of Jacobians but exploits that J is consistent with a wave-function ψ.

Proof. The weak H^1_{loc}-convergence implies that ψ_n converges strongly to ψ in $L^r_{loc}(\mathbf{R}^d)$ for $2 \leq r \leq \frac{2d}{d-2}$ and $\nabla\psi_n$ converges weakly to $\nabla\psi$ in $L^2(\mathbf{R}^d)$. Hence

$$J_n \rightharpoonup J = \mathrm{Im}(\overline{\psi}\nabla\psi) \quad \text{in} \quad L^p_{loc}(\mathbf{R}^d),$$

for any $1 \leq p < 2$ if $d = 2$ and for any $1 \leq p < \frac{3}{2}$ if $d = 3$. Moreover, we have that

$$\nabla \wedge J_n \to \nabla \wedge J \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^d).$$

Since $J = \mathrm{Im}(\overline{\psi}\nabla\psi)$ with $\psi \in H^1_{loc}(\mathbf{R}^d)$, it follows from Lemma 2.2.4 that

$$\nabla \wedge J = \sqrt{\rho} \wedge \Lambda \in L^1(\mathbf{R}^d),$$

where $\sqrt{\rho}, \Lambda$ are the hydrodynamic variables associated to ψ. In particular,

$$\nabla\sqrt{\rho_n} \wedge \Lambda_n \to \nabla\sqrt{\rho} \wedge \Lambda \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^d).$$

□

2.3 Existence of finite energy weak solutions

This section is dedicated to the proof of Theorem 2.1.3. We construct a global in time finite energy weak solution to (2.0.1) with initial data (ρ_0, J_0) by combining the well-posedness theory for (1.1.1) introduced in Chapter 1 and the polar decomposition of Section 2.2. In order to derive global existence of finite energy weak solutions to (2.0.1), we do not require any further regularity or smallness assumptions beyond the ones imposed by the well-posedness theory for the corresponding NLS equation (1.1.1) and the polar decomposition of Lemma 2.2.3, namely, we work in the natural framework of initial data of finite energy. In particular, by identity (2.2.1), one has that the energy functional (2.1.2) equals the Hamiltonian (1.0.2). However, the initial data must be consistent with an initial wave function. While Lemma 2.2.3 provides a definition of the initial data for the hydrodynamic system on the basis of a given wave-function ψ_0, the possible appearance of vacuum generally rules out the converse. In addition, the generalized irrotationality condition is a necessary condition for the hydrodynamic quantities to be consistent with a wave function. At this stage, several questions arise. What are the sufficient assumptions on the initial data (ρ_0, J_0) in terms of regularity so that the wave-function can be reconstructed? In physics literature, this problem is referred to as *Pauli problem*, see [172]. From a mathematical point of view, this task is related to the lifting problem that has been investigated in [35, 43, 147], see also references therein. Lifting properties have also been exploited in the study of travelling wave solutions to (1.1.1) of transonic speeds, see [58]. Can the level of generality of initial data generated by a wave function among the set of initial data (ρ_0, J_0) of finite energy by determined? For a more detailed discussion of these issues, we refer the reader to [11], p. 659.
Finally, we remark that in order to give a rigorous derivation of system (2.0.1) we exploit the persistence of regularity properties of NLS equations, namely (1.1.1), which in the present setting means that if $\psi_0 \in X^2 \cap \mathbb{E}_2$ then $\psi \in X^2 \cap \mathbb{E}_2$ for all finite times. For NLS equations that do not enjoy this property the method may adapted by implementing a regularization procedure, see for instance [10] where a perturbation Lemma is used, or by relying on the integral formulation of the corresponding nonlinear Schrödinger equation [7].

Proof of Theorem 2.1.3. Given $\psi_0 \in \mathbb{E}_2$, Lemma 1.1.13 implies that $\psi_0 \in \mathbb{E}_2$. In virtue of Theorem 1.1.1 there exists a unique solution $\psi \in C(\mathbf{R}, \mathbb{E}_2)$ to (1.1.1) with initial data $\psi(0) = \psi_0$. Moreover, ψ is such that the energy $\mathcal{E}$ is conserved for all time, i.e. $\mathcal{E}(\psi(t)) = \mathcal{E}(\psi_0)$ for all $t \in \mathbf{R}$. Further, thanks to Lemma 1.1.6 there exists a sequence ψ_0^n such that $\Delta\psi_0^n \in L^2(\mathbf{R}^d)$ and $d_{\mathbb{E}_2}(\psi_0^n, \psi_0)$ converges to 0 as n goes to infinity. Theorem 1.1.1 provides a unique solution $\psi^n \in C(\mathbf{R}, \mathbb{E}_2)$ with initial data ψ_0^n that conserves the energy and enjoys the bound $\Delta\psi^n \in L^\infty([-T, T], L^2(\mathbf{R}^d))$ for any $T > 0$ finite. We define the initial data

$$\sqrt{\rho_0^n} := |\psi_0^n|, \qquad J_0^n := Im(\overline{\psi_0^n}\nabla\psi_0^n)$$

and we introduce

$$\sqrt{\rho_n} := |\psi_n|, \qquad J_n := Im(\overline{\psi_n}\nabla\psi_n).$$

We shall show that $(\rho_n := \sqrt{\rho_n}^2, J_n)$ is a global finite energy weak solution to (2.0.1) according to Definition 2.1.1. Firstly, we verify the required *a priori* bounds. Since $\nabla|\psi^n| \leq \nabla\psi^n$ a.e. on $\mathbf{R}^d$ and $\psi^n \in C(\mathbf{R}, \mathbb{E}_2)$ we conclude $\sqrt{\rho_n} - 1 \in L^\infty(\mathbf{R}, H^1(\mathbf{R}^d))$ and for any $T > 0$ finite that $\sqrt{\rho_n} - 1 \in L^\infty([0,T]; H^2(\mathbf{R}^d))$, see also Lemma 2.2.3. Similarly, by Definition $\Lambda_n \in L^\infty(\mathbf{R}, L^2(\mathbf{R}^d))$ and we infer that $J_n \in L^\infty([0,T]; H^1(\mathbf{R}^d))$. We prove that (ρ_n, J_n) is a finite energy weak solution of (2.0.1). Let $\zeta \in C_0^\infty([0,T) \times \mathbf{R}^d)$ and consider

$$
\begin{aligned}
\int_0^T \int_{\mathbf{R}^d} \rho_n \partial_t \zeta \mathrm{d}x\mathrm{d}t + \int_{\mathbf{R}^d} \rho_n^0(x)\eta(0,x)\mathrm{d}x &= -\int_0^T \int_{\mathbf{R}^d} \partial_t \rho_n \zeta \mathrm{d}x\mathrm{d}t = -\int_0^T \int_{\mathbf{R}^d} 2Im(\overline{\psi_n} i \partial_t \psi_n)\zeta \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\mathbf{R}^d} 2Im\left(\overline{\psi_n}(-\frac{1}{2}\Delta\psi_n + F'(|\psi_n|^2)\psi_n)\right)\zeta \mathrm{d}x\mathrm{d}t \\
&= -\int_0^T \int_{\mathbf{R}^d} (J_n) \cdot \nabla\zeta \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

We observe that since $\psi_n \in C([0,T], H^2_{loc}(\mathbf{R}^d))$ all integrals are well-defined and (ρ_n, J_n) satisfies the continuity equation. Similarly, we derive the momentum equation. Let $\zeta \in C_0^\infty([0,T) \times \mathbf{R}^d; \mathbf{R}^d)$, then

$$
\begin{aligned}
\int_0^T \int_{\mathbf{R}^d} \partial_t J_n \zeta \mathrm{d}x\mathrm{d}t &= \int_0^T \int_{\mathbf{R}^d} Im\left(\nabla\psi_n\left(-\frac{i}{2}\Delta\overline{\psi}_n + iF'(\rho_n)\overline{\psi}_n\right)\right)\zeta \mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \int_{\mathbf{R}^d} Im\left(\overline{\psi_n}\left(\frac{i}{2}\Delta\nabla\psi_n - i(\nabla F'(\rho_n))\psi_n - iF'(\rho_n)\nabla\psi_n\right)\right)\zeta \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\mathbf{R}^d} Im\left(-\frac{i}{2}\nabla\psi_n\Delta\overline{\psi_n} + \frac{i}{2}\overline{\psi}_n\Delta\nabla\psi_n\right)\zeta \mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathbf{R}^d} \rho_n \nabla F'(\rho_n)\zeta \mathrm{d}x\mathrm{d}t \\
&= -\int_0^T \int_{\mathbf{R}^d} Re\left(\nabla\overline{\psi_n} \otimes \nabla\psi_n\right)\zeta \mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \int_{\mathbf{R}^d} \frac{1}{4}\nabla\Delta\rho_n\zeta - \int_0^T \int_{\mathbf{R}^d} \rho_n \nabla F'(\rho_n)\zeta \mathrm{d}x\mathrm{d}t
\end{aligned}
$$

We stress that the quantity

$$
Im \int_{\mathbf{R}^d} \overline{\psi_n} \nabla\Delta\psi_n \zeta \mathrm{d}x,
$$

is meaningful as pairing between $\nabla\Delta\psi_n \in H^{-1}_{loc}(\mathbf{R}^d)$ and $\psi_n \in H^1_{loc}(\mathbf{R}^d)$. Using identity (2.0.3), (2.2.1) and $p(\rho) = F'(\rho)\rho - F(\rho)\rho$, we infer that

$$
\begin{aligned}
\int_{\mathbf{R}^d} J_{0,n}\zeta(0)\mathrm{d}x &+ \int_0^T \int_{\mathbf{R}^d} J_n \partial_t \zeta \mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \int_{\mathbf{R}^d} ((\Lambda_n \otimes \Lambda) + (\nabla\sqrt{\rho_n} \otimes \nabla\sqrt{\rho_n})) : \nabla\zeta + p(\rho)\operatorname{div}\zeta \\
&- \int_0^T \int_{\mathbf{R}^d} \frac{1}{4}\rho_n \Delta \operatorname{div}\zeta \mathrm{d}x\mathrm{d}t = 0
\end{aligned}
$$

Hence, (ρ_n, J_n) is a weak solution of (2.0.1). We introduce $\sqrt{\rho} = |\psi|$ and $J = Im(\overline{\psi}\nabla\psi)$ and it remains to show that (ρ, J) is a finite energy weak solution for the initial data (ρ_0, J_0). To that end, we observe that

$$\sqrt{\rho} - 1 \in L^\infty(\mathbf{R}; H^1(\mathbf{R}^d)), \qquad J \in L^\infty(\mathbf{R}, L^{\frac{3}{2}}(\mathbf{R}^d) + L^2(\mathbf{R}^d)).$$

In particular, for Λ as in Lemma 2.2.3, we have $\Lambda \in L^2(\mathbf{R}^d)$ and $J = \sqrt{\rho}\Lambda$. Since $d_{\mathbb{E}_2}(\psi_n^0, \psi^0) \to 0$, Theorem 1.1.1 ensures that for any $T > 0$ finite,

$$\sup_{|t|\leq T} d_{\mathbb{E}_2}(\psi_n(t), \psi(t)) \to 0, \qquad \text{as} \quad n \to \infty.$$

The stability property of the nonlinear Schrödinger equation combined with the stability of the polar decomposition provided by Lemma 2.2.3 w.r.t. to the topology of $\mathbb{E}_2$ yields that

$$\sqrt{\rho_n} - 1 \to \sqrt{\rho} - 1 \quad \text{in} \quad L^\infty([0,T]; H^1(\mathbf{R}^d)), \qquad \Lambda_n \to \Lambda \quad \text{in} \quad L^\infty([0,T]; L^2(\mathbf{R}^d)).$$

In particular, J_n converges strongly to J in $L^\infty([0,T]; L^{\frac{3}{2}}_{loc}(\mathbf{R}^d))$. Further, we observe that

$$\sup_{t\in[0,T]} \|\rho_n - \rho\|_{L^2(\mathbf{R}^d)} \leq \sup_{|t|\leq T} d_{\mathbb{E}_2}(\psi_n, \psi) \to 0.$$

One derives the respective convergences for the initial data analogously. The strong convergence results allow us to pass to the limit in the weak formulation of the equations and thus (ρ, J) is a weak solution to (2.0.1) complemented with initial data (ρ_0, J_0). Finally, we observe that the conservation of energy of ψ together with identity (2.2.1) yields conservation of energy (2.1.2) for (ρ, J) and for all times $t \in [0, T]$. It remains to check the generalized irrotationality condition. By definition, one has

$$\nabla \wedge J = Im(\nabla\overline{\psi} \wedge \nabla\psi), \quad a.e. \text{ on } \mathbf{R}^d,$$

and again from Lemma 2.2.3,

$$\nabla\overline{\psi} \wedge \nabla\psi = 2i\nabla\sqrt{\rho} \wedge \Lambda.$$

The proof is complete. □

2.4 Existence of weak solutions of infinite energy

Relying on this existence result for the underlying wave-function, we show that there exists a global weak solution to (2.0.1) equipped with nontrivial conditions at infinity (2.0.2) and with initial data of infinite energy. The class of initial data $\mathcal{V} + H^1(\mathbf{R}^2)$, with $\mathcal{V}$ defined in (1.3.4), includes in particular H^1-perturbations of vortex solutions.

The proof follows a similar scheme as the Proof of Theorem 2.1.3. Here $\gamma = 2$ and thus we rely on the existence results for (1.3.1) provided by Theorem 1.3.3, see also [32]. Regarding the stability of the polar factorization, we can not exploit the metric of the energy space but need to decompose the wave-functions in order to be able to exploit the H^1-topology.

Proof of Theorem 2.1.4. Given $\psi_0 \in \mathcal{V} + H^1(\mathbf{R}^2)$, we decompose the initial data in in $\psi_0 = U + w_0$ with $U \in \mathcal{V}$ and $w_0 \in H^1(\mathbf{R}^2)$. From Theorem 1.3.3, see also [32] we conclude that there exists a unique solution ψ to (1.3.1) such that $\psi - \psi_0 \in C(\mathbf{R}; H^1(\mathbf{R}^d))$. Moreover, given $\{w_{0,n}\}_{n\in\mathbf{N}} \subset H^2(\mathbf{R}^2)$ such that $w_{0,n} \to w_0$ in $H^1(\mathbf{R}^2)$ there exists a sequence $\psi_n = U + w_n$ of solutions to (1.3.1) such that $\psi_n - \psi_{0,n} \in C(\mathbf{R}, H^2(\mathbf{R}^d))$ where $\psi_{0,n} = U + w_{0,n}$ and

$$\|\psi_n - \psi\|_{L^\infty(0,T;H^1(\mathbf{R}^2))} \leq \|\psi_{n,0} - \psi_0\|_{H^1(\mathbf{R}^2)}.$$

From Poincaré inequality $\psi_n \in L^\infty(0,T; H^2_{loc}(\mathbf{R}^2))$ and thus we proceed as in Proof of Theorem 2.1.3. Defining $\rho_n := |\psi_n|^2$ and $J_n = Im(\overline{\psi_n}\nabla\psi_n)$, we infer that (ρ_n, J_n) is a global weak solution to (2.0.1). Let $\rho := |\psi|^2$ and $J_n = Im(\overline{\psi}\nabla\psi)$, we need to ensure that we are in position to pass to the limit in the weak formulation of the equations and to conclude that (ρ, J) is a weak solution of (2.0.1). To that end, we exploit the stability properties provided by Lemma 2.2.4; we obtain that

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho} \quad L^2_{loc}(\mathbf{R}^2), \qquad \Lambda_n \to \Lambda \quad L^2_{loc}(\mathbf{R}^2),$$

and

$$\sqrt{\rho_n} - \sqrt{\rho} \to \sqrt{\rho} \quad L^2(\mathbf{R}^2), \quad J_n \to J \quad L^q_{loc}(\mathbf{R}^2),$$

where $2 \leq p < \infty$ and $1 \leq q < 2$. Hence, the passage to the limit as n goes to ∞ in the weak formulation of the equations is justified. From Lemma 2.2.5, we infer that $\nabla\sqrt{\rho} \in L^2(\mathbf{R}^2)$. In virtue of Proposition 1.3.3, the renormalized energy (1.3.7) is conserved during the evolution of (1.3.1). If $\psi \in \mathcal{V}_* + H^1(\mathbf{R}^2)$ then the definitions (1.3.7) and (1.3.8) are equivalent, see Remark 2.2.6. Thus,

$$\lim_{R\to\infty}\int_{\mathbf{R}^2} \frac{1}{2}(|\nabla\sqrt{\rho}|^2 - |\nabla|U||^2 + |\Lambda|^2 - |\Lambda_U|^2 + \frac{1}{2}(\rho-1)\mathrm{d}x < +\infty, \tag{2.4.1}$$

and the renormalized energy is conserved for a.e. $t \in [0,T]$; indeed the renormalized energy functional (2.2.2) differs from (2.4.1) by $-\int_{\mathbf{R}^2}\frac{1}{2}|\nabla|U||^2\mathrm{d}x$ being finite and constant in time. □

2.5 Special solutions and large time asymptotics

This section aims to give a short (not exhaustive) overview about some special solutions to the QHD system (2.0.1). In particular, we notice that the properties of solutions (1.1.1) discussed in Section 1.2.1 can be translated to the class of weak solutions of (2.0.1) by means of Theorem 2.1.3. Thus, setting up the Cauchy theory for (2.0.1) is suitable for the study the hydrodynamic counterpart of a variety of special solutions to (1.1.1) such as travelling waves [30, 31, 27, 142, 59], travelling vortex ring [29, 56] and a travelling vortex-antivortex pair [108].

2.5.1 Solutions of finite energy

We show how to translate the results for (1.1.1) obtained in [142, 59] to travelling waves for (2.0.1). The method is illustrated at the example of Theorem 1.2.1.

Corollary 2.5.1. *Let $d = 2, 3$ for any $0 < c < v_s$, there exists a travelling wave solution (ρ, J) to* (2.0.1)*. Namely, there exists $(\sqrt{\rho}, \Lambda)$ such that $\sqrt{\rho} - 1 \in H^1(\mathbf{R}^d)$ and $\Lambda \in L^2(\mathbf{R}^d)$ and*

$$\rho = (\sqrt{\rho(x-ct)})^2, \qquad J = \sqrt{\rho(x-ct)}\Lambda(x-ct),$$

being solution to (2.0.1)*. In particular, for $d = 2$, there exists a travelling wave for initial data with arbitrarily small prescribed energy.*

Proof. In virtue of Theorem 1.2.1 and 1.2.2, see also [142, 59], there exists a travelling wave solution $\psi \in C(\mathbf{R}, \mathbb{E}_2)$ to (1.1.1). Thanks to Theorem 2.1.3 the travelling wave ψ induces a finite energy weak solution (ρ, J) to (2.0.1) of the form $\rho(t, x) = \rho(x - ct)$ and $J(t, x) = J(x - ct)$. □

For $d = 2$, this results rules out the possibility of a scattering theory for the QHD system. Recalling figure 1.1, we notice that the situation is different for $d = 3$. At least in the class of finite energy weak solutions provided by Theorem 2.1.3, Theorem 1.2.3 suggests that no travelling waves are to be expected at small energies. A major draw-back is clearly given by the fact that this approach does not enable us to construct travelling waves directly in the class of finite energy weak solutions to (2.0.1) but only in the subset of solutions that are consistent with a wave-function dynamics.

We remark that the stability of planar travelling waves in capillary fluids has been subject of the study in [20]. Small energy travelling waves for the Euler-Korteweg system for $d = 2$ have been investigated in [15]. The approach requires high regularity and smallness of the initial data. As mentioned in Section 1.2.1, in [29] the authors show existence of vortex rings of finite energy for (1.1.1) for $d = 3$. It would be interesting to investigate the corresponding weak solutions to (2.0.1). In the present work, we focus on vortex solution for $d = 2$ of infinite energy.

By means of the mentioned results in Section 1.2.2, in particular [95, 97], it is immediate to construct global dispersive solutions to (2.0.1). For the sake of conciseness, we illustrate the procedure for the result. This scattering theory is only valid for solutions to (2.0.1) that are consistent with an underlying wave-function dynamics. It would be of great interest for future investigations to prove scattering for solutions that are not necessarily induced by a wave function. It was shown in [16] that global strong irrotional solutions to the QHD system for $d \geq 3$ scatter under suitable smallness and regularity assumptions.

2.5.2 Vortex solutions

We discuss vortical solutions in the hydrodynamic framework provided by (2.0.1), we start by giving a heuristic approach that will be made rigorous in Proposition 2.5.2 by means of Theorem 2.1.4 and the stability of the polar decomposition introduced in Section 2.2. Quantized vortices for $d = 2$ and quantized vortex lines for $d = 3$ have been predicted by Onsager [152] in 1949 and Feynman [80] in 1955. Even though their approaches and the respective conclusions are somehow different, both authors suggest that the circulation in a superfluid is quantized. We refer the reader to Chapter 2 in [74] for an overview of these early results. The fluid is supposed to be irrotational in the absence of vortices and it was predicted that if v denotes the velocity field of the fluid, then for an arbitrary contour C around the vortex, the circulation equals

$$\Gamma = \oint_C v \cdot \mathrm{d}\vec{l} = 2\pi n \frac{\hbar}{m}, \tag{2.5.1}$$

for an integer n with $\hbar$ denoting the reduced Planck constant and m the exact mass of the bare helium atom. A quantum of circulation is therefore given by $\kappa = \frac{\hbar}{m}$. By passing to polar coordinates, we conclude from (2.5.1) that the azimuthal speed around the vortex reads

$$v_\theta = \kappa \frac{n}{r},$$

where $r = |x|$. This highlights a further difference compared to vortices in a classical fluid, e.g. a Rankine vortex, for which one usually supposes that flow close to the vortex core is governed by a rigid body motion, namely $v = \Omega \wedge r$ for some angular speed Ω. Hence, away from vacuum $\nabla \wedge v = 2\Omega \neq 0$ for classical fluids, in contradiction to the (generalized) irrotationality condition assumed for quantum fluids. These ideas can be made rigorous in the framework of the Gross-Pitaevskii theory developed by [90] and [159]. The latter states that the fluid is characterized by an order parameter ψ satisfying the Gross-Pitaevskii equation (1.3.1). We look for stationary vortex solutions to (1.3.1). To that end, if we assume the velocity field to be of the form $v = 2\pi\kappa \frac{x^\perp}{|x|^2}$ as suggested by (2.5.1) and think of v as phase gradient then we are led to plug into the Gross-Pitaevskii equation (1.3.1) the ansatz $\psi = f(|x|)e^{in\theta}$. In [87], see also [160], it was observed that $\psi = f(|x|)e^{in\theta}$ is a solution to (1.3.1) provided that the radial profile $f(|x|)$ satisfies (1.3.2). By means of the elliptic regularity, the velocity profile f can be seen to be regular and to enjoy the following asymptotic.

If n denotes the circulation integer, then f_n behaves like $f_n \sim r^{|n|}$ for $r \to 0$ and for $r > 1$ one has

$$1 - \frac{d^2}{2r^2} \leq f_n(r) \leq 1,$$

see e.g. [103]. The density profile is plotted in Figure 2.1. We recall that (1.3.1) has been scaled such that it is in dimensionless form, the characteristic length scale is given by the healing length ξ as discussed in the Introduction of this thesis. By consequence, the density profile reaches the unperturbed density $f = 1$ over a distance proportional to the healing length ξ, explaining its name.

We observe that for a vortex solution of the type $\psi = f_n(|x|)e^{in\theta}$, the hydrodynamic variables

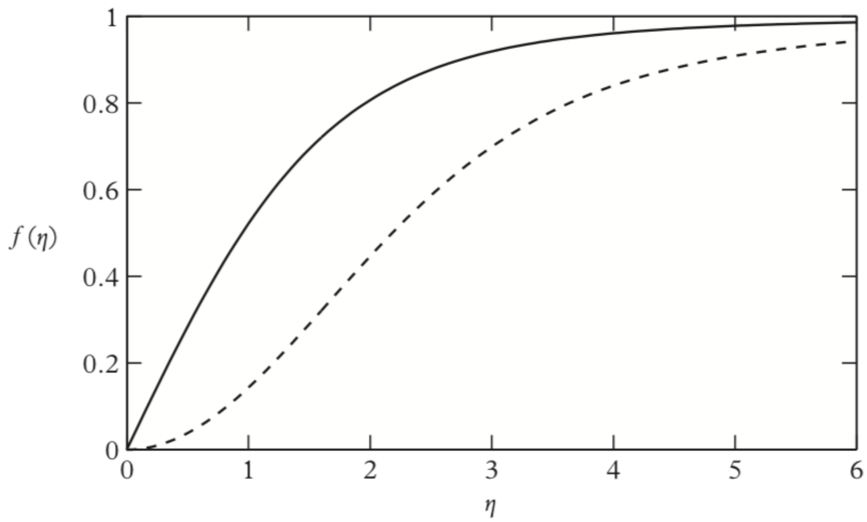

Figure 2.1: Density profile as function of $\eta = r$, here $n = 1$ solid line, $n = 2$ dashed line. The diagram is a courtesy of L. Pitaevskii, S. Stringari [160]

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defined by the polar factorization trivially read

$$\sqrt{\rho} = f_n(|x|), \qquad \Lambda(x) = n f_n(x)\frac{x^\perp}{|x|^2}. \tag{2.5.2}$$

Since for all $|x| > 0$ it holds $f(|x|) > 0$, for such a special solution one may actually define $v = \frac{\Lambda}{\sqrt{\rho}} = n\frac{x^\perp}{|x|^2}$ for $|x| > 0$. This yields for all $R > 0$,

$$\oint_{\partial B_R} v \cdot \mathrm{d}\vec{l} = 2\pi n,$$

namely the circulation is quantized. Since we consider the equations in dimensionless scaling, here a quantum of circulation equals $\kappa = 1$. It is not clear how to compute the (quantized) circulation for general hydrodynamic solutions $(\sqrt{\rho}, \Lambda)$, as one should need some regularity as well as control on the vacuum set. On the other hand, heuristically we expect that

$$\lim_{R\to\infty}\oint_{\partial B_R} v \cdot \mathrm{d}\vec{l} = \lim_{R\to\infty}\oint_{\partial B_R} J \cdot \mathrm{d}\vec{l},$$

since formally we have $\rho \to 1$ as $|x| \to \infty$. Moreover, by using Stokes' Theorem, we also have

$$\lim_{R\to\infty}\oint_{\partial B_R} J \cdot \mathrm{d}\vec{l} = \int_{\mathbf{R}^2} \nabla \wedge J \mathrm{d}x.$$

For this reason, it could be useful to define the degree at infinity in the following way

$$deg(J,\infty) = \frac{1}{2\pi}\lim_{R\to\infty}\oint_{\partial B_R} J \cdot \mathrm{d}\vec{l} = \frac{1}{2\pi}\lim_{R\to\infty}\int_{B_R} \nabla \wedge J \mathrm{d}x. \tag{2.5.3}$$

In the next Proposition we provide a straightforward example that for the vortex solution (2.5.2) this is indeed a satisfactory definition.

Proposition 2.5.2. *Let* $n \in \mathbf{Z}$ *and* f_n *solution to* (1.3.2). *Then*

$$\sqrt{\rho} = f_n(|x|), \qquad \Lambda = f_n(|x|)\frac{x^\perp}{|x|^2}, \qquad J = \sqrt{\rho}\Lambda$$

defines a stationary vortex solution to (2.0.1). *Moreover,*

$$deg(J, \infty) = n$$

and its vorticity is quantized in the sense that for $|x| > 0$ *one may define* $v = \frac{\Lambda}{\sqrt{\rho}}$ *and*

$$\oint_{\partial B_R} v \cdot \mathrm{d}\vec{l} = 2\pi n.$$

For fixed $R > 0$*, the energy of a vortex solution on* $B_R(0)$ *is given by*

$$\int_{B_R} \frac{1}{2}|\nabla\sqrt{\rho}|^2 + |\Lambda|^2 + \frac{1}{2}(\rho - 1)^2 \mathrm{d}x = 2\pi n^2 |\log R| + C, \tag{2.5.4}$$

for some universal constant $C > 0$.

The proof is immediate but we decide to provide it nevertheless to illustrate the above heuristic in the spcial case of a vortex solution.

Proof of Proposition. Let $n \in \mathbf{Z}$ and consider $\psi = f_n(|x|)e^{in\theta}$ such that f_n satisfies (1.3.2). Then ψ defines a stationary solution to (1.3.1), see [87]. By means of Theorem 2.1.4, if we define

$$\sqrt{\rho} = f_n(|x|), \qquad J = Im\left(f_n(|x|)e^{-in\theta}\nabla(f_n(|x|)e^{in\theta})\right) = df_n(|x|)^2\frac{x^\perp}{|x|^2},$$

then (ρ, J) is a stationary weak solution to (2.0.1) in virtue of Theorem 2.1.4. The renormalized energy functional (2.2.2) is finite and trivially conserved for all times. For fixed $R > 0$, the energy can be computed using the explicit representation of a vortex and is given by

$$\int_{B_R} \frac{1}{2}|\nabla\sqrt{\rho}|^2 + |\Lambda|^2 + \frac{1}{2}(\rho - 1)^2 \mathrm{d}x = 2\pi n^2 |\log R| + C,$$

for some universal constant $C > 0$. Finally, by exploiting the identity $\nabla \wedge J = 2\nabla\sqrt{\rho} \wedge \Lambda$ it is immediate to see that

$$\int_{B_R} \nabla \wedge J \mathrm{d}x = \int_0^R \int_0^{2\pi} 2f_n(|x|)f_n'(|x|)\mathrm{d}\theta\mathrm{d}r = 2\pi n f_n(R).$$

In virtue of Stokes' Theorem for $R > 0$,

$$\int_{B_R} \nabla \wedge J \mathrm{d}x = \oint_{\partial B_R} J \cdot \mathrm{d}\vec{l},$$

leading to

$$\lim_{R\to\infty} \oint_{\partial B_R} J \cdot \mathrm{d}\vec{l} = 2\pi n.$$

□

In particular, we notice that if $\psi \in \mathbb{E}_2 \cap C^1$, then one verifies that the associated current J satisfies

$$deg(J, \infty) = 0,$$

since otherwise $\mathcal{E}_2(\psi) = +\infty$, see also Lemma 1.3.1. At Sobolev regularity, the definition of degree is substantially more involved, since the nodal set is in general not regular. We refer the interested reader to [24, 6] for an mathematically rigorous investigation of the topological degree in this context. The dynamics of vortices for the Gross-Pitaevskii equation in the scaling limit considered in Chapter 5 has been proven in [28]. For a general introduction to topological degree theory see [45]. We refer to the seminal book [25] for a complete analysis of Ginzburg-Landau vortices.

To conclude, we summarize the key differences between classical and quantum vortices:

(i) the circulation of quantum vortices is quantized, namely equal to $2\pi n$ with $n \in \mathbf{Z}$, we refer to n as the degree of the vortex;

(ii) quantum vortices describe an irrotational flow away from the vortex core in contrast to classical vortices that generically are modelled by a rigid body motion around the vortex core. In classical turbulence, e.g. the Rankine vortex for a viscous fluid combines the behavior of a rigid body motion up to a threshold distance from the vortex core and an irrotational motion beyond that threshold, see also Figure 2.2.

(iii) the size of a quantum vortex core is determined and of the order of the healing length ξ.

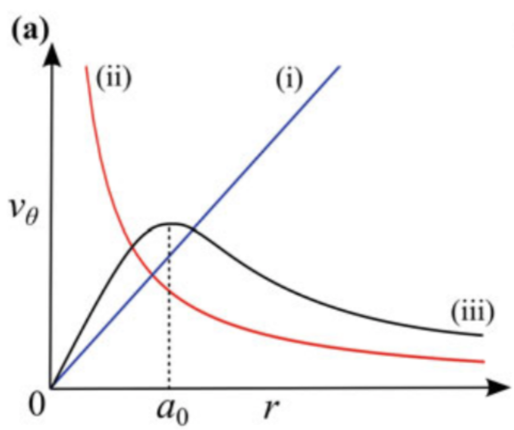

Figure 2.2: Angular velocity for (i) solid body rotation, (ii) vortex line in a condensate (irrotational flow), and (iii) flow around a hurricane or a bathtub vortex, which combines solid body rotation in the inner region $r << a_0$ and irrotational flow in the outer region $r >> a_0$. The figure is a courtesy of C. Barenghi and N. Parker [18]

We remark that Theorem 2.1.4 provides a suitable framework for the mathematical analysis of the dynamics for initial data given by multi-vortex configuration characterized by the vortex core centers $(a_1, ...a_n)$ and degrees $(m_1, ..., m_n)$ of the form

$$\sqrt{\rho} = \prod_{j=1}^{n} f_{m_j}(|x - a_j|), \qquad \Lambda = \left(\prod_{j=1}^{n} f_{m_j}(|x - a_j|) \right) \sum_{i}^{n} m_j \frac{|x - a_j|^{\perp}}{|x - a_j|^2}.$$

We postpone the discussion to Section 5.2 where we address the dynamics of small perturbations of such vortex initial data in a suitable scaling limit, namely in the regime when the healing length is small, $\xi \to 0$.

CHAPTER 3

Existence of finite energy weak solutions to the quantum Navier-Stokes equations with non-trivial far-field

Abstract

We introduce existence of global finite energy weak solutions to the quantum Navier-Stokes equations posed on the whole space with non trivial far-field behavior in dimensions $d = 2, 3$. The main results of this chapter are presented in Section 3.1. In Section 3.2, we construct a sequence of suitable periodic initial data on a sequence of growing tori and discuss existence of respective periodic solutions to a truncated formulation based on [125]. The periodic solutions are extended to approximate truncated weak solutions on the whole space in Section 3.3. Further, we show that the sequence converges to a truncated weak solution to QNS. The proof of the main results is given in Section 3.4, namely we show that truncated weak solutions are weak solutions. In Section 3.5, we show how to construct weak solutions satisfying a slightly stronger version of the energy inequality based on the result in [13]. This is motivated by its applications to chapter 4.

This chapter, based on a joint work in progress with P. Antonelli and S. Spirito, is devoted to the study the Cauchy problem for the Quantum-Navier-Stokes equations (QNS) posed on $(0, T) \times \mathbf{R}^d$,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = 2\nu \operatorname{div}(\rho \mathbf{D} u) + 2\kappa^2 \rho \nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right), \end{cases} \tag{3.0.1}$$

complemented with the far-field behavior

$$\rho \to 1 \qquad \text{as} \qquad |x| \to \infty. \tag{3.0.2}$$

The unknowns are given by the mass density ρ and the velocity field of the fluid u. We consider a pressure given by the γ-law with $\gamma > 1$. We refer to the coefficients ν and κ as viscosity and capillarity coefficients respectively. The energy we consider for system (3.0.1) is given by

$$E(t) = \int_{\mathbf{R}^d} \frac{1}{2}\rho|u|^2 + 2\kappa^2|\nabla\sqrt{\rho}|^2 + F(\rho)\mathrm{d}x, \tag{3.0.3}$$

where the internal energy is as defined in (0.2.5), namely

$$F(\rho) = \frac{\rho^\gamma - 1 - \gamma(\rho - 1)}{\gamma(\gamma - 1)}. \tag{3.0.4}$$

The integrability of (3.0.4) encodes the far-field condition for finite energy solutions. Our main result in this Chapter states the global existence of finite energy weak solutions to (3.0.1) with $\kappa \geq 0$ and far-field (3.0.2). We do not require any smallness or further regularity of the initial data, in particular vacuum regions are considered.
The quantum Navier-Stokes equations arise as model for a dissipative quantum fluid [111]. In this regard, it represents an approach that aims to include diffusive phenomena in the standard model for quantum fluids given by the QHD system (2.0.1) discussed in Chapter 2. It has been derived from a Chapman-Enskog expansion for the Wigner equation with a BGK term [46, 112], see also [111] where several dissipative quantum fluid models are derived by means of a moment closure of (quantum) kinetic equations with appropriate choices of the collision term. System (2.0.1) can also be recovered as inviscid limit of the QNS system (3.0.1) in the framework of solutions studied in [40]. From this perspective, the global existence of finite energy weak solutions to (3.0.1) with non-trivial far-field behavior can be seen as analogue result to the one for the inviscid counter-part (2.0.1). While the mathematical analysis of the inviscid system (2.0.1) exploits the analogy with a nonlinear Schrödinger equation, the analysis of the Cauchy Problem for (3.0.1) shares similarities with techniques developed in the context of capillary fluids. As already mentioned in the Introduction, system (3.0.1) belongs to the class of Navier-Stokes-Korteweg systems. In the general framework, the capillary tensor on the right-hand side of (3.0.1) is referred to as Korteweg tensor [121]. The family of Navier-Stokes-Korteweg equations has rigorously been derived in [75] and more recently in [101]. In the absence of capillary effects, namely $\kappa = 0$, system (3.0.1) reduces to the compressible Navier-Stokes equations with degenerate viscosity tensor [38]. Global existence of weak solutions on $\mathbf{R}^d$ for $d = 2, 3$ has been obtained in [170] and [129] with vanishing boundary conditions at infinity, i.e. for integrable densities. Concerning the Quantum-Navier Stokes system, the existence of finite energy weak solutions on the torus $\mathbf{T}^d$, for $d = 2, 3$ has been introduced in [13] and [125], see also references therein for previous results. The former approach relies on a suitable construction of approximate solutions which retains the same *a priori* bounds, while the latter exploits a truncated formulation of the problem and suitable compactness properties. Global existence of weak solutions to the isothermal quantum Navier-Stokes equations, i.e. $\gamma = 1$, has been introduced in [50]. The result of [50] is proven on $\mathbf{R}^d$ for $d = 2, 3$ with vanishing boundary conditions at infinity, the proof makes use on an invading domain approach. In general, the mathematical analysis of fluids with density dependent viscosity, even without capillary tensor, requires new tools compared to the case of constant viscosity (Lions [134, 135]-Feireisl theory [77]). This is due to a loss of control on uniform bounds on the velocity field in possible vacuum regions where the density vanishes. A crucial contribution in this regard has been made in [38] where the authors observe that

provided a particular structure of the equations additional information and uniform bounds can be obtained by means of the so called Bresch-Desjardins entropy [38]. We refer the reader also to the review paper [161] and the recent paper [42] considering the problem in a general framework regarding compressible Navier-Stokes equations with degenerate viscosity. When considered with capillary effects, the nonlinear structure of the dispersive tensor depending on the density and its derivatives entails additional mathematical difficulties, see for instance [38, 39, 41].

Turning to system, (3.0.1), the choice of the far-field behavior (3.0.2) is motivated by its applications to the study of singular limits such as the low Mach number limit exposed in Chapter 4. We remark that our result of global existence of finite energy weak solutions also applies to the compressible Navier-Stokes equations with degenerate viscosity tensor, namely for $\kappa = 0$. To the best of our knowledge, the Cauchy Problem for (3.0.1) with $\kappa \geq 0$ and far-field behavior (3.0.2) has not been previously investigated in literature in the class of weak solutions for $d = 2, 3$. Local strong solutions have been constructed in [94, 130, 131] for (3.0.1) with $\nu > 0$ and $\kappa = 0$. For $d = 1$, $\nu > 0$ and $\kappa = 0$, existence and uniqueness of global strong solutions with (3.0.2) has been shown in [144, 100].

We comment on the suitable mathematical framework for the analysis of weak solutions to (3.0.1). For finite energy weak solutions to (3.0.1) without further regularity assumptions neither u, ∇u nor $\frac{1}{\sqrt{\rho}}$ are defined almost everywhere due to the possible presence of vacuum - as it occurs for the analysis of the QHD system (2.0.1) studied in Chapter 2. For the inviscid system, we introduce the hydrodynamic variables by means of a polar factorization that is not limited by the appearance of vacuum, see Section 2.2. Due the these reasons, it turns out that also the Cauchy problem for (3.0.1) is best studied in terms of the more suitable variables $(\sqrt{\rho}, \Lambda)$. The same difficulty arises in the context of barotropic Navier-Stokes equations with density dependent viscosity, see for instance [129]. We define the tensor $\mathbf{T}_\nu \in L^2(\mathbf{R} \times \mathbf{R}^d)$ satisfying

$$\sqrt{\nu}\sqrt{\rho}\mathbf{T}_\nu = \nu\nabla(\rho u) - 2\nu\nabla\sqrt{\rho} \otimes \Lambda \qquad \text{in} \quad \mathcal{D}'((0,T) \times \mathbf{R}^d)). \tag{3.0.5}$$

By denoting $\mathbf{S}_\nu = \mathbf{T}_\nu^{sym}$, we see that for smooth solutions we have $\sqrt{\nu}\sqrt{\rho}\mathbf{S}_\nu = \nu\rho\mathbf{D}u$. Analogously, for the capillary tensor we use the identity (2.0.3) as well as the relation

$$\kappa\sqrt{\rho}\mathbf{S}_\kappa = \kappa^2\sqrt{\rho}\left(\nabla^2\sqrt{\rho} - 4(\nabla\rho^{\frac{1}{4}} \otimes \nabla\rho^{\frac{1}{4}})\right), \tag{3.0.6}$$

that is well-defined in view of the regularity inferred by the *a priori*, i.e. energy and Bresch-Desjardins entropy estimates. It is in general not possible to infer the usual energy inequality which reads

$$E(t) + 2\nu\int_0^t\int_{\mathbf{R}^d} \rho\,|\mathbf{D}u|^2\,\mathrm{d}x\mathrm{d}t \leq E(0). \tag{3.0.7}$$

The energy inequality (3.0.7) is replaced by its weaker formulation below (3.1.1). For the sake of consistency with the literature regarding (quantum) Navier-Stokes equations, we do not use the hydrodynamic variable Λ in this chapter. However, we stress that whenever the symbol

$\sqrt{\rho}u$ appears it should be read as Λ. Analogously, whenever the symbol $\rho\mathbf{D}u$ appears it has to be intended as in (3.0.5).

Our method consists in an invading-domains approach relying on the existence of weak solutions [125] that in addition satisfy a truncated formulation of the momentum equation. More precisely, given initial data of finite energy we construct periodic initial data on a sequence of invading domains. On each of them, [125] provides a periodic truncated weak solution. Next, we show that these periodic solutions provide a sequence of approximate solutions on the whole space. In the limit, we recover a truncated weak solution to (3.0.1) on the whole space that can be shown to be a finite energy weak solution.

This chapter is structured as follows. In Section 3.1, we give the precise definition of finite energy weak solution and state our main results. Section 3.2 is dedicated to the construction of suitable periodic initial data on growing tori $\mathbf{T}^d_n$ for which we postulate the existence of finite energy weak solutions based on [125]. In particular, the periodic weak solutions satisfy a truncated formulation of the momentum equation that will be crucial for the sequel. In section 3.3, we extend the sequence of truncated periodic solutions on invading domains to approximate truncated weak solutions on the whole space with far-field behavior (3.0.2). Further, we obtain an truncated finite energy weak solutions to (3.0.1) in the limit. Finally, we present the proof of the main Theorem in Section 3.4, namely we show that the obtained truncated weak solutions are finite energy weak solutions to (3.0.1). Section 3.5 discusses (3.1.1) and (3.1.2). More precisely, we elucidate the reason for the constant C appearing on the right-hand side of both inequalities. Further, we show that there exists - at least for $d = 3$ - finite energy weak solutions that satisfy (3.1.1) and (3.1.2) with $C = 1$. This information is essential in the context of the low Mach number limit, especially for Proposition 4.1.4.

3.1 Definition and main results

As already mentioned, the strategy for proving the existence Theorem 3.1.2 goes through constructing suitable solutions on domains $\mathbf{T}^d_n = \mathbf{R}^d/n\mathbf{Z}^d$ with $n \in \mathbf{N}$. By performing the limit as $n \to \infty$ and by using suitable truncations, we will then define weak solutions on the whole space $\mathbf{R}^d$. Hence, for the sake of generality we give the Definition of finite energy weak solutions for an arbitrary domain Ω, which will be $\Omega = \mathbf{R}^d, \mathbf{T}^d$ or $\Omega = \mathbf{T}^d_n$ respectively according to our purposes.

Definition 3.1.1. *A pair (ρ, u) with $\rho \geq 0$ is said to be a finite energy weak solution of the Cauchy Problem* (3.0.1) *posed on $[0,T) \times \Omega$ complemented with initial data (ρ_0, u_0) if*

(i) integrability conditions

$$\sqrt{\rho} \in L^2_{loc}((0,T)\times\Omega); \quad \sqrt{\rho}u \in L^2_{loc}((0,T)\times\Omega); \quad \nabla\sqrt{\rho} \in L^2_{loc}((0,T)\times\Omega);$$
$$\nabla\rho^{\frac{\gamma}{2}} \in L^2_{loc}((0,T)\times\Omega); \quad \mathbf{T}_\nu \in L^2_{loc}((0,T)\times\Omega); \quad \kappa\nabla^2\sqrt{\rho} \in L^2_{loc}((0,T)\times\Omega);$$
$$\sqrt{\kappa}\nabla\rho^{\frac{1}{4}} \in L^4_{loc}((0,T)\times\Omega)$$

(ii) continuity equation

$$\int_\Omega \rho_0\phi(0) + \int_0^T\int_\Omega \rho\phi_t + \sqrt{\rho}\sqrt{\rho}u\nabla\phi = 0,$$

for any $\phi \in C_c^\infty([0,T)\times\Omega)$.

(iii) momentum equation

$$\begin{aligned}
&\int_\Omega \rho_0u_0\psi(0) + \int_0^T\int_\Omega \sqrt{\rho}\sqrt{\rho}u\psi_t + (\sqrt{\rho}u\otimes\sqrt{\rho}u)\nabla\psi + \rho^\gamma \operatorname{div}\psi \\
&- 2\nu\int_0^T\int_\Omega (\sqrt{\rho}u\otimes\nabla\sqrt{\rho})\,\nabla\psi - 2\nu\int_0^T\int_\Omega (\nabla\sqrt{\rho}\otimes\sqrt{\rho}u)\,\nabla\psi \\
&+ \nu\int_0^T\int_\Omega \sqrt{\rho}\sqrt{\rho}u\Delta\psi + \nu\int_0^T\int_\Omega \sqrt{\rho}\sqrt{\rho}u\nabla\operatorname{div}\psi \\
&- 4\kappa^2\int_0^T\int_\Omega (\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho})\,\nabla\psi + 2\kappa^2\int_0^T\int_\Omega \sqrt{\rho}\nabla\sqrt{\rho}\nabla\operatorname{div}\psi = 0,
\end{aligned}$$

for any $\psi \in C_c^\infty([0,T)\times\Omega;\mathbf{R}^d)$.

(iv) there exists a tensor $\mathbf{T}_\nu \in L^2((0,T)\times\Omega)$ *satisfying identity* (3.0.5) *in* $\mathcal{D}'((0,T)\times\Omega)$ *such that the following energy inequality holds for a.e.* $t\in[0,T]$,

$$E(t) + 2\nu\int_0^t\int_\Omega |\mathbf{S}_\nu|^2\mathrm{d}x\mathrm{d}t \le CE(0), \tag{3.1.1}$$

where $\mathbf{S}_\nu$ *is the symmetric part of* $\mathbf{T}_\nu$ *and* E *as defined in* (3.0.3).

(v) Let

$$B(t) = \int_\Omega \frac{1}{2}\left|\sqrt{\rho}u\right|^2 + (2\kappa^2+4\nu^2)\left|\nabla\sqrt{\rho}\right|^2 + F(\rho)\mathrm{d}x.$$

Then for a.e. $t\in[0,T]$,

$$\begin{aligned}
&B(t) + \int_0^t\int_\Omega \frac{1}{2}\left|\mathbf{A}_\nu\right|^2\mathrm{d}x\mathrm{d}s + \nu\kappa^2\int_0^t\int_\Omega \left|\nabla^2\sqrt{\rho}\right|^2 + |\nabla\rho^{\frac{1}{4}}|^4\mathrm{d}x\mathrm{d}s + \nu\int_0^t\int_\Omega \left|\nabla\rho^{\frac{\gamma}{2}}\right|^2\mathrm{d}x\mathrm{d}s \\
&\le C\int_\Omega \frac{1}{2}\left|\sqrt{\rho_0}u_0\right|^2 + (2\kappa^2+4\nu)\left|\nabla\sqrt{\rho_0}\right|^2 + F(\rho_0)\mathrm{d}x,
\end{aligned} \tag{3.1.2}$$

where $\mathbf{A}_\nu = \mathbf{T}_\nu^{asym}$, *with* $\mathbf{T}_\nu$ *defined as in the previous point.*

Notice that the far-field behavior is encoded in the definition of the energy functional (3.0.3). Our main results states the existence of global finite energy weak solutions to (3.0.1) with non-vanishing density at infinity.

Theorem 3.1.2. *Let* $d = 2,3$ *and* $\gamma > 1$. *Given initial data* (ρ_0,u_0) *of finite energy, there exists a global finite energy weak solution to* (3.0.1) *on* $\mathbf{R}^d$ *satisfying the far-field condition* (3.0.2).

When $\kappa = 0$, system (3.0.1) reduces to the compressible Navier-Stokes equations with degenerate viscosity for which we have the following existence result

Corollary 3.1.3. *Let $d = 2, 3$ and $\kappa = 0$, $\gamma > 1$. Given initial data (ρ_0, u_0) of finite energy and BD entropy, there exists a global finite energy weak solution to* (3.0.1) *on $\mathbf{R}^d$ satisfying the far field condition* (3.0.2).

The statement of Theorem 3.1.2 remains true if we consider system (3.0.1) on $\mathbf{R}^d$ with trivial far-field behavior, i.e.

$$\rho \to 0, \quad \text{as} \quad |x| \to \infty.$$

In this case, the internal energy does not need to be renormalized and is given by

$$F(\rho) = \frac{1}{\gamma - 1}\rho^\gamma. \tag{3.1.3}$$

The following is the analogue result on the whole space to [13, 125] for $\kappa > 0$ and [170] for $\kappa = 0$ respectively.

Theorem 3.1.4. *Let $d = 2, 3$ and $\nu > 0, \kappa \geq 0$ and $\gamma > 1$. Given initial data (ρ_0, u_0) of finite energy and BD entropy, there exists a global finite energy weak solution to* (3.0.1) *on $\mathbf{R}^d$ with trivial far-field behavior.*

The proof can adapted with minor modifications, we therefore only provide a sketch of the proof. Gaining integrability of the density removes technical difficulties and simplifies both the construction of initial data and the extensions of periodic solutions to functions on the whole space.

3.1.1 Truncation functions

Next, we introduce the cut-off functions by means of which we shall construct the aforementioned solutions to the truncated formulation of (3.0.1). Here, we work with a specific choice for the truncation function β that is suitable for our purpose.

Definition 3.1.5. *Let $\overline{\beta} : \mathbf{R} \to \mathbf{R}$ be an even positive compactly supported smooth function such that $\overline{\beta}(z) = 1$ for $z \in [-1, 1]$ and supp$(\overline{\beta}) \subset (-2, 2)$ and $0 \leq \overline{\beta} \leq 1$. Further, we define $\tilde{\beta} : \mathbf{R} \to \mathbf{R}$ as*

$$\tilde{\beta}(z) = \int_0^z \overline{\beta}(s)\mathrm{d}s.$$

For any $\delta > 0$, we define $\overline{\beta}_\delta(z) := \overline{\beta}(\delta z)$ and $\tilde{\beta}_\delta(z) := \frac{1}{\delta}\tilde{\beta}(\delta z)$. Given $y \in \mathbf{R}^3$, we denote

$$\hat{\beta}_\delta(z) := \prod_{l=1}^{3} \overline{\beta}_\delta(y_l),$$

further

$$\beta_\delta^1(y) = \int_0^{y_1} \hat{\beta}_\delta(y_1', y_2, y_3)\mathrm{d}y_1' = \tilde{\beta}_\delta(y_1)\overline{\beta}_\delta(y_2)\overline{\beta}_\delta(y_3).$$

The functions $\beta_\delta^2(y), \beta_\delta^3(y)$ are defined analogously.

We drop the indices of β_δ^l whenever this does not cause ambiguity in order to simplify notations. We summarize some properties of the truncation functions introduced in Definition 3.1.5.

Lemma 3.1.6. *Let $\delta > 0$, β_δ^l as in Definition 3.1.5 and $M := \|\beta\|_{W^{2,\infty}}$. Then, there exists $C = C(M) > 0$ such that the following bounds hold.*

1. *for $1 \leq l \leq d$,*

$$\|\beta_\delta^l\|_{L^\infty} \leq \frac{C}{\delta}, \qquad \|\nabla \beta_\delta^l\|_{L^\infty} \leq C, \qquad \|\nabla^2 \beta_\delta^l\|_{L^\infty} \leq C\delta,$$

2. *As δ goes to 0, for every $y \in \mathbf{R}^d$,*

$$\beta_\delta^l(y) \to y_l, \qquad \nabla \beta_\delta^l(y) \to e_l,$$

where $e_l \in \mathbf{R}^d$ such that $e_l^i = 1$ for $i = l$ and $e_l^i = 0$ otherwise.

3.2 Solutions on periodic domains

3.2.1 Construction of periodic initial data

Given initial data of finite energy $(\sqrt{\rho_0}, \sqrt{\rho_0} u_0)$ for the problem (3.0.1) posed on $\mathbf{R}^d$ with far-field boundary conditions (3.0.2), we construct a sequence of initial data to the periodic problem on $\mathbf{T}_n^d$, where we define the scaled torus as

$$\mathbf{T}_n^d = \mathbf{R}^d / n\mathbf{Z}^d.$$

To that end, we consider a smooth cut-off function $\chi \in C_c^\infty(\mathbf{R}^d)$ with *supp*$(\chi) \subset [-1,1]^d$ such that

$$\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]^d} \leq \chi \leq \mathbf{1}_{[-1,1]^d},$$

and the rescaling $\chi_n(x) = \chi(\frac{x}{n})$. We observe that $\|\nabla \chi_n\|_{L^\infty(\mathbf{R}^d)} = O(\frac{1}{n})$. We define the sequence of initial data $(\sqrt{\rho_n^0}, \sqrt{\rho_n^0} u_n^0)$ on $\mathbf{T}_n^d$ as follows

$$\sqrt{\rho_n^0} = \sqrt{\rho^0}\chi_n + (1 - \chi_n), \qquad \sqrt{\rho_n^0} u_n^0 = \sqrt{\rho^0} u^0 \mathbf{1}_{(-n,n)^d}. \tag{3.2.1}$$

We observe that due to the choice of the cut-off functions, the sequence of initial data $(\sqrt{\rho_{0,n}}, \Lambda_{0,n})$ can be extended to periodic functions and may therefore be considered as functions defined on $\mathbf{T}_n^d$. A similar construction has recently been used in [49] for the existence of weak solutions to isothermal fluids. In the present setting, we additionally need to take into account the non-vanishing conditions at infinity leading to a lack of integrability. We collect some of its properties. For that purpose, we recall the uniform estimates satisfied by the initial data $(\sqrt{\rho_0}, \Lambda_0)$ on $\mathbf{R}^d$. More details are provided in Section 4.2. For a generic pair $(\sqrt{\rho}, \sqrt{\rho} u)$ of finite energy we have the following bounds.

Lemma 3.2.1. *Let $(\sqrt{\rho}, \sqrt{\rho}u)$ be such that $E(\sqrt{\rho}, \sqrt{\rho}u) < +\infty$. Then*

$$\sqrt{\rho} - 1 \in H^1(\mathbf{R}^d), \quad \rho - 1 \in L_2^\gamma(\mathbf{R}^d) \cap L^2(\mathbf{R}^d), \quad \sqrt{\rho}u \in L^2(\mathbf{R}^d).$$

Proof. The bound $E(\sqrt{\rho}, \sqrt{\rho}u) < +\infty$ implies $\nabla\sqrt{\rho} \in L^2(\mathbf{R}^d)$. By a convexity argument, see Section 4.2 or also [136], the bound $F(\rho) \in L^1(\mathbf{R}^d)$ yields the estimate in the Orlicz space $\rho - 1 \in L_2^\gamma(\mathbf{R}^d)$, i.e. there exists $C > 0$ such that

$$\int_{\mathbf{R}^d} |\rho - 1|^2 \mathbf{1}_{\{|\rho-1| \leq \frac{1}{2}\}} + |\rho - 1|^\gamma \mathbf{1}_{\{|\rho-1| > \frac{1}{2}\}} \mathrm{d}x \leq C.$$

If $\gamma \geq 2$, this implies $\rho - 1 \in L^2(\mathbf{R}^d)$. The inequality $|\sqrt{\rho} - 1| \leq |\rho - 1|$ yields $\sqrt{\rho} - 1 \in L_2^\gamma(\mathbf{R}^d)$. If $\gamma < 2$, we notice that the set $\{|\rho - 1| > \frac{1}{2}\}$ is of finite Lebesgue measure and $\nabla\sqrt{\rho} \in L^2(\mathbf{R}^d)$ implies $\sqrt{\rho} - 1 \in L^2_{loc}(\mathbf{R}^d)$. Hence, there exists $C_1 > 0$ such that

$$\int_{\mathbf{R}^d} |\sqrt{\rho} - 1|^2 \mathbf{1}_{\{|\rho-1| > \frac{1}{2}\}} \mathrm{d}x \leq C_1,$$

therefore $\sqrt{\rho} - 1 \in H^1(\mathbf{R}^d)$. For real $s \geq 0$ such that $|s^2 - 1| > \frac{1}{2}$, there exists $c > 0$ such that $(s^2 - 1)^2 \leq c(s - 1)^4$ implying that there exists $C_2 > 0$, such that

$$\int_{\mathbf{R}^d} |\rho - 1|^2 \mathbf{1}_{\{|\rho-1| > \frac{1}{2}\}} \mathrm{d}x \leq \int_{\mathbf{R}^d} |\sqrt{\rho} - 1|^4 \mathbf{1}_{\{|\rho-1| > \frac{1}{2}\}} \mathrm{d}x \leq C_2.$$

Hence, we conclude $\rho - 1 \in L^2(\mathbf{R}^d)$. The remaining bounds are immediate consequences of the finite energy assumption. □

Next, we infer uniform bounds (in n) for the sequence of periodic initial data defined in (3.2.1).

Lemma 3.2.2. *Given $(\sqrt{\rho_0}, \sqrt{\rho_0}u_0)$ such that $E_{\mathbf{R}^d}(\sqrt{\rho_0}, \sqrt{\rho_0}u_0) < \infty$, the sequence of initial data $(\sqrt{\rho_0^n}, \sqrt{\rho_0^n}u_0^n)$ defined by (3.2.1) satisfies the following. For every n there exists $C_n > 0$ such that denoting $\rho_{0,n} = (\sqrt{\rho_0^n})^2$, one has*

$$\int_{\mathbf{T}_n^d} \frac{1}{2} \left|\nabla\sqrt{\rho_0^n}\right|^2 + \frac{1}{2} \left|\sqrt{\rho_0^n}u_0^n\right|^2 + \frac{1}{\gamma - 1} \rho_0^{n,\gamma} \mathrm{d}x \leq Cn^d, \tag{3.2.2}$$

$$\lim_{n\to\infty} \int_{\mathbf{T}_n^d} \frac{1}{2} \left|\sqrt{\rho_0^n} - 1\right|^2 = \int_{\mathbf{R}^d} \frac{1}{2} |\sqrt{\rho_0} - 1|^2 \mathrm{d}x, \tag{3.2.3}$$

$$\lim_{n\to\infty} \int_{\mathbf{T}_n^d} \frac{1}{2} \left|\nabla\sqrt{\rho_0^n}\right|^2 = \int_{\mathbf{R}^d} \frac{1}{2} |\nabla\sqrt{\rho_0}|^2 \mathrm{d}x, \tag{3.2.4}$$

$$\lim_{n\to\infty} \int_{\mathbf{T}_n^d} F(\rho_0^n) = \int_{\mathbf{R}^d} F(\rho_0) \mathrm{d}x, \tag{3.2.5}$$

$$\lim_{n\to\infty} \int_{\mathbf{T}_n^d} \frac{1}{2} \rho_n^0 \left|u_n^0\right|^2 = \int_{\mathbf{R}^d} \frac{1}{2} \rho^0 |u^0|^2 \mathrm{d}x, \tag{3.2.6}$$

In particular, there exists $C > 0$ such that $E_{T_n^d}(\sqrt{\rho_0^n}, \sqrt{\rho_0^n}u_0^n) \leq C$ uniformly in n, with E as defined in (3.0.3) *and*

$$\lim_{n\to\infty} E_{T_n^d}\left(\sqrt{\rho_0^n}, \sqrt{\rho_0^n}u_0^n\right) = E(\sqrt{\rho_0}, \sqrt{\rho_0}u_0).$$

We emphasize that the constant C_n on the right-hand side of (3.2.2) is not uniform in n in view of the far-field condition (3.0.2).

Proof. Given $(\sqrt{\rho^0}, \sqrt{\rho^0}u^0)$ of finite energy let the sequence $(\sqrt{\rho^0_n}, \sqrt{\rho^0_n}u^n_0)$ be defined by (3.2.1). Inequality (3.2.2) will then follow from the convergences (3.2.4), (3.2.5) and (3.2.6). Since (3.2.3) and (3.2.4) imply that $\|\sqrt{\rho^n_0}-1\|_{H^1(\mathbf{T}^d_n)} \leq C$ uniformly in n, we conclude that $\sqrt{\rho_{0,n}} \in H^1(\mathbf{T}^d_n)$ for all $n \in \mathbf{N}$. From (3.2.5), one has that $\rho^n_0 - 1 \in L^\gamma_2(\mathbf{T}^d_n)$, thus also $\rho^\gamma_{0,n} \in L^1(\mathbf{T}^d_n)$ by checking that

$$\int_{\mathbf{T}^d_n} (\rho^n_0)^\gamma \mathbf{1}_{\{|\rho^n_0-1|\leq\frac{1}{2}\}}\mathrm{d}x \leq \int_{\mathbf{T}^d_n} 1 + L_\gamma|\rho-1|\mathbf{1}_{\{|\rho^n_0-1|\leq\frac{1}{2}\}}\mathrm{d}x \leq C_{n,d},$$

where $C_{n,d}$ is proportional to the volume of $\mathbf{T}^d_n$. Thus, it only remains to prove (3.2.3) - (3.2.6). Let us show (3.2.3). We notice that

$$\sqrt{\rho^n_0} - 1 = (\sqrt{\rho_0}-1)\chi_n,$$

and thus

$$|\sqrt{\rho^n_0}-1| \leq |\sqrt{\rho_0}-1|.$$

We conclude by means of the dominated convergence Theorem that

$$\lim_{n\to\infty}\int_{\mathbf{T}^d_n}\left|\sqrt{\rho^n_0}-1\right|^2\mathrm{d}x = \int_{\mathbf{R}^d}|\sqrt{\rho_0}-1|^2\mathrm{d}x$$

Similarly in order to derive (3.2.4), we denote $\omega_n = supp(\nabla\chi_n)$ and observe that ω_n has measure of order $O(n^d)$. We have that for all $n \in \mathbf{N}$,

$$\begin{aligned}
&\int_{\mathbf{T}^d_n}\frac{1}{2}|\nabla\sqrt{\rho^n_0}|^2\mathrm{d}x = \int_{\mathbf{R}^d}\frac{1}{2}\left|(\nabla\sqrt{\rho_0})\chi_n + (\sqrt{\rho_0}-1)\nabla\chi_n\right|^2\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbf{R}^d}\chi_n^2\left|\nabla\sqrt{\rho_0}\right|^2 + 2\chi_n(\sqrt{\rho_0}-1)\nabla\sqrt{\rho_0}\cdot\nabla\chi_n + (\sqrt{\rho_0}-1)^2|\nabla\chi_n|^2\mathrm{d}x\\
&\leq \int_{\mathbf{R}^d}\frac{1}{2}|\nabla\sqrt{\rho_0}|^2\mathrm{d}x + \left(\|\nabla\sqrt{\rho_0}\|_{L^2(\omega_n)}\|\sqrt{\rho}_0-1\|_{L^6(\omega_n)}\|\chi_n\nabla\chi_n\|_{L^3(\omega_n)}\right)\\
&+ \frac{1}{2}\|\nabla\chi_n\|^2_{L^\infty(\omega_n)}\|\sqrt{\rho_0}-1\|^2_{L^2(\omega_n)}\\
&\leq \int_{\mathbf{R}^d}\frac{1}{2}|\nabla\sqrt{\rho_0}|^2\mathrm{d}x + n^{\frac{d-3}{3}}\|\nabla\sqrt{\rho_0}\|_{L^2(\omega_n)}\|\sqrt{\rho}_0-1\|_{L^6(\omega_n)} + \frac{1}{2}n^{-2}\|\sqrt{\rho_0}-1\|^2_{L^2(\omega_n)}.
\end{aligned}$$

We observe that

$$\limsup_{n\to\infty}\left(n^{\frac{d-3}{3}}\|\nabla\sqrt{\rho_0}\|_{L^2(\omega_n)}\|\sqrt{\rho}_0-1\|_{L^6(\omega_n)}\right) = 0$$

Indeed, one has that $\sqrt{\rho_0}-1 \in H^1(\mathbf{R}^d)$ and therefore

$$\limsup_{n\to\infty}\|\sqrt{\rho_0}-1\|_{L^p(\omega_n)} = 0,$$

for all $2 \leq p \leq p^*$ as consequence of the decay of the tails of L^p-functions. It is immediate to see,

$$\limsup_{n\to\infty}\left(\frac{1}{2}n^{-2}\|\sqrt{\rho_0}-1\|^2_{L^2(\omega_n)}\right)=0$$

The dominated convergence Theorem yields that

$$\lim_{n\to\infty}\int_{\mathbf{T}^d_n}\frac{1}{2}|\nabla\sqrt{\rho_0^n}|^2\mathrm{d}x=\int_{\mathbf{R}^d}\frac{1}{2}|\nabla\sqrt{\rho_0}|^2\mathrm{d}x$$

For (3.2.5), we observe that $\sqrt{\rho_{0,n}}$ converges pointwise to $\sqrt{\rho_0}$ a.e. on $\mathbf{R}^d$. Since, $F(\cdot)$ is convex on $\mathbf{R}_{\geq 0}$, it follows that $F(\rho_0^n)$ converges pointwise to $F(\rho_0)$ a. e. on $\mathbf{R}^d$. We aim to show the desired inequality and the convergence by introducing G defined as

$$G(x)=\begin{cases}C & \text{if} \quad x\in\{\sqrt{\rho_0}\leq\frac{1}{\sqrt{2}}\},\\ C|\sqrt{\rho_0}-1|^2 & \text{if} \quad x\in\{\frac{1}{\sqrt{2}}\leq\sqrt{\rho_0}\leq 1\},\\ F(\rho_0) & \text{if} \quad x\in\{\sqrt{\rho_0}\geq 1\},\end{cases}$$

We notice that $G:\mathbf{R}_{\geq 0}\to\mathbf{R}_{\geq 0}$ and $G\in L^1(\mathbf{R}^3)$. Splitting the integral, we have that the contribution coming from the domain $\{\sqrt{\rho_0}\leq\frac{1}{\sqrt{2}}\}$ is finite since the volume of the set is bounded. The second contribution is bounded as $\sqrt{\rho_0}-1\in H^1(\mathbf{R}^d)$ and the last is bounded as $F(\rho_0)\in L^1(\mathbf{R}^d)$.
We claim that $F(\sqrt{\rho_0^n}^2)(x)\leq G(x)$ for all $x\in\mathbf{T}^d_n$ and $n\in\mathbf{N}$. Indeed, if $x\in\{\sqrt{\rho_0}\leq\frac{1}{\sqrt{2}}\}$, then $x\in\{0\leq\sqrt{\rho_0^n}\leq 1\}$ for all $n\in\mathbf{N}$. Thus, on the given set $F(\sqrt{\rho_0^n}^2)\leq F(0)=\gamma-1$. Next, if $x\in\{\frac{1}{\sqrt{2}}\leq\sqrt{\rho_0}\leq 1\}$ then $x\in\{\frac{1}{\sqrt{2}}\leq\sqrt{\rho_0^n}\leq 1\}$ for all $n\in\mathbf{N}$. Thus, on the prescribed domain

$$F(\sqrt{\rho_n}^2)\leq C\left|\sqrt{\rho_0^n}^2-1\right|^2=\left|(\sqrt{\rho_0}-1)^2\chi_n^2+2(\sqrt{\rho_0}-1)\chi_n\right|^2\leq C|\sqrt{\rho}_0-1|^2,$$

as $|\sqrt{\rho_0}-1|\leq 1$. If $x\in\{\sqrt{\rho_0}\geq 1\}$, then in particular $x\in\{\sqrt{\rho_0^n}\geq 1\}$ for all $n\in\mathbf{N}$. Thus, the concatenation $F((\sqrt{\rho_0^n})^2)$ is a convex function on the set $\{\sqrt{\rho_0}\geq 1\}$ and

$$F(\sqrt{\rho_0^n}^2)\leq\chi_nF(\sqrt{\rho_0}^2)+(1-\chi_n)F(1)\leq F(\rho_0).$$

We may therefore apply the dominated convergence Theorem to obtain

$$\lim_{n\to\infty}\int_{\mathbf{T}^d_n}F(\rho_0^n)\mathrm{d}x=\int_{\mathbf{R}^d}F(\rho_0)\mathrm{d}x.$$

The convergence (3.2.6) is immediate. □

3.2.2 Existence of solutions on periodic domains

In this section, we discuss the existence of a sequence of weak solutions to the system (3.0.1) on $\mathbf{T}^d_n$ with initial data $(\sqrt{\rho_0^n},\sqrt{\rho_0^n}u_0^n)$ provided by Lemma 3.2.2. For that purpose, we rely on the result in [125], where the authors show global existence of weak solutions to (3.0.1) posed on $[0,T)\times\mathbf{T}_n^d$ for $\gamma>1$, $\nu>0$ and $\kappa\geq 0$ complemented with initial data of finite energy. In a first step, the authors construct weak solutions to a truncated formulation of the momentum equation; these are referred to as renormalized solutions and are obtained as limit of a sequence of weak solutions to the truncated formulation of the system augmented by drag forces. The use of a truncated solutions of the momentum equation is crucial for the construction considering that as a sequence of truncated solutions enjoys suitable compactness properties. These compactness properties will be pivotal to our approach for the proof of Theorem 3.1.2. The second step consists in showing that solutions to the truncated formulation also are also weak solutions to (3.0.1). Following a different approach, namely constructing smooth approximate weak solutions to (3.0.1), the existence of weak solutions to (3.0.1) posed on $\mathbf{T}_n^d$ satisfying properties (i)-(iii) has also been obtained in [13] provided that $\kappa<\nu$ and $\gamma>1$ for $d=2$ and $\kappa^2<\nu<\frac{9}{8}\kappa^2$ as well as $1<\gamma<3$ for $d=3$. It can rigorously be shown that the solutions constructed in [13] actually are finite energy weak solutions to (3.0.1) on $\mathbf{T}_n^d$ in the sense of Definition 3.1.1. In particular, the solutions provided in [13] satisfy (3.1.1) and (3.1.2) with $C=1$, see also Section 3.5. Following the arguments in [125] and Section 3.5, it can be checked that the solutions provided in [125] are finite energy weak solution in the sense of Definition 3.1.1, namely also satisfy the inequalities (3.1.1) and (3.1.2). We define

$$\kappa_n=\begin{cases}\kappa_n=\kappa & \text{if} \quad \kappa>0,\\ \kappa_n=\frac{1}{n} & \text{if} \quad \kappa=0.\end{cases} \tag{3.2.7}$$

Next, we give the Definition of finite energy weak solution to the truncated formulation based on the notion of renormalized solution in [125]. The truncation functions are introduced in 3.1.5.

Definition 3.2.3. *Let $d=2,3$. Given a domain Ω, a sequence $(\sqrt{\rho_n},\sqrt{\rho_n}u_n)$ is called approximate truncated weak solution to* (3.0.1) *on $(0,T)\times\Omega$ with initial data (ρ_0^n,u_0^n) if the following are satisfied.*

1. *integrability conditions*

$$\sqrt{\rho_n}\in L^2_{loc}((0,T)\times\Omega);\quad \sqrt{\rho_n}u_n\in L^2_{loc}((0,T)\times\Omega);\quad \nabla\sqrt{\rho_n}\in L^2_{loc}((0,T)\times\Omega);$$
$$\nabla\rho_n^{\frac{\gamma}{2}}\in L^2_{loc}((0,T)\times\Omega);\quad \mathbf{T}_{\nu,n}\in L^2_{loc}((0,T)\times\Omega);\quad \kappa_n\nabla^2\sqrt{\rho_n}\in L^2_{loc}((0,T)\times\Omega);$$
$$\sqrt{\kappa_n}\nabla\rho_n^{\frac{1}{4}}\in L^4_{loc}((0,T)\times\Omega)$$

2. *(approximate continuity equation) there exists a sequence of distributions $D_n\in\mathcal{D}'([0,T)\times\Omega)$ such that $D_n\to 0$ in $\mathcal{D}'$ as $n\to\infty$ and*

$$\int_\Omega\rho_0^n\psi(0,x)\mathrm{d}x+\int_\Omega\rho_n\psi_t+\sqrt{\rho_n}\sqrt{\rho}_nu_n\cdot\nabla\psi\mathrm{d}x\mathrm{d}t=\langle D_n,\psi\rangle\,, \tag{3.2.8}$$

for any $\psi\in C_c^\infty([0,T)\times\Omega)$.

3. *(approximate momentum equation and compatibility conditions) for any truncation function β as in Definition 3.1.5, there exist sequences of measures $\mu^n_\beta, \overline{\mu}^n_\beta$ obeying the bound*

$$\|\mu^n_\beta\|_{\mathcal{M}} + \|\overline{\mu}^n_\beta\|_{\mathcal{M}} \leq C\|\beta''\|_{L^\infty},$$

uniformly in n and distributions $G_n, K_n, V_n \in \mathcal{D}'([0,T)\times\Omega)$ such that $G_n, K_n, V_n \to 0$ in $\mathcal{D}'$ as $n \to \infty$ and such that

$$\begin{aligned}
&\int_\Omega \rho^n_0 \beta(u^n_0)\psi(0,x)\mathrm{d}x + \int_0^T\int_\Omega \rho_n\beta(u_n)\partial_t\psi + \rho_n\beta(u_n)u_n\cdot\nabla\psi \mathrm{d}x\mathrm{d}t \\
&- 2\int_0^T\int_\Omega \left(\sqrt{\nu_n}\mathbf{S}_{\nu,n}\beta'(u_n) + \kappa_n\sqrt{\rho_n}\mathbf{S}_{\kappa,n}\beta'(u_n))\cdot\nabla\psi + 2\rho_n^{\frac{\gamma}{2}}\nabla\rho_n^{\frac{\gamma}{2}}\right)\beta'(u_n)\cdot\nabla\psi\mathrm{d}x\mathrm{d}t \\
&= \left\langle \mu^n_\beta + G_n, \psi\right\rangle,
\end{aligned} \tag{3.2.9}$$

with $\mathbf{S}_{\nu,n}$ being the symmetric part of $\mathbf{T}_{\nu,n}$ satisfying the compatibility condition

$$\sqrt{\nu_n}\sqrt{\rho_n}[\mathbf{T}_{\nu,n}]_{ik} = \nu\partial_j(\rho\beta_i'(u_n)u_{k,n}) - 2\nu\sqrt{\rho_n}u_{k,n}\beta_i'(u_n)\partial_j\sqrt{\rho_n} + \overline{\mu}^n_\beta + V_n$$

in $\mathcal{D}'([0,T)\times\Omega)$ and the capillary tensor $\mathbf{S}_{\kappa,n}$ satisfying

$$\kappa_n\mathbf{S}_{\kappa,n} = \kappa_n^2\left(\sqrt{\rho_n}\left(\nabla^2\sqrt{\rho_n} - 4(\nabla\rho_n^{\frac{1}{4}}\otimes\nabla\rho_n^{\frac{1}{4}}\right)\right) + K_n. \tag{3.2.10}$$

in $\mathcal{D}'([0,T)\times\Omega)$.

Finally, we say that $(\sqrt{\rho_n}, \sqrt{\rho_n}u_n)$ is a sequence of truncated weak solutions if $D_n = G_n = K_n = V_n = 0$. Further, a truncated weak solution is called finite energy truncated weak solution if in addition (3.1.1) *and* (3.1.2) *are satisfied.*

Based on the aforementioned result of [125] and above considerations, we postulate the following existence result.

Theorem 3.2.4. *Let $\gamma > 1, \nu > 0$, κ_n as defined in* (3.2.7) *and $(\sqrt{\rho^n_0}, \sqrt{\rho^n_0}u^n_0)$ be provided by Lemma 3.2.2. Then there exists a sequence $(\sqrt{\rho_n}, \sqrt{\rho_n}u_n)$ of finite energy truncated weak solutions to* (3.0.1) *on $(0,T)\times\mathbf{T}^d_n$ with defect measures R^n_β, $\overline{R}^n_\beta$.*

Several remarks are in order.

1. In (3.1.1) and (3.1.2) the pressure term $\frac{1}{\gamma-1}\rho^\gamma$, has been replaced by the internal energy $F(\rho)$. On a bounded domain Ω, one easily checks that $F(\rho_n) \in L^1(\Omega)$ is equivalent to $\rho_n \in L^\gamma(\Omega)$.

2. The bounds on the measures $R^n_\beta, \overline{R}^n_\beta$ are uniform in n since only depending on the second derivatives of β being bounded in virtue of Lemma 3.1.6.

3. We comment on the energy and entropy inequalities. The solutions provided by Theorem 3.2.4 satisfy the following bounds uniformly in n. Firstly, since the solutions provided by Theorem 3.2.4 satisfy the energy (3.1.1) and BD entropy inequality (3.1.2) and we infer from Lemma 3.2.2 that

$$\limsup_{n\to\infty}\left(E_{\mathbf{T}_n^d}(t)(\sqrt{\rho_n},\sqrt{\rho_n}u_n)+\int_0^T\int_{\mathbf{T}_n^d}|\mathbf{S}_{\nu,n}|^2\mathrm{d}x\mathrm{d}t\right)\leq\limsup_{n\to\infty}E_{\mathbf{T}_n^d}(\sqrt{\rho_0^n},\sqrt{\rho_0^n}u_0^n)$$
$$\leq C\int_{\mathbf{R}^d}\frac{1}{2}\rho_0|u_0|^2+2\kappa^2|\nabla\sqrt{\rho_0}|^2+F(\rho_0)\mathrm{d}x. \tag{3.2.11}$$

Secondly, from Theorem 3.2.4 and again from Lemma 3.2.2 we conclude that there exists $C>0$ such that

$$\limsup_{n\to\infty}\left(B_{\mathbf{T}_n^d}(t)+\int_0^T\left(\nu|\nabla\rho_n^{\frac{\gamma}{2}}|^2+\nu\kappa_n^2(|\nabla\rho_n^{\frac{1}{4}}|^4+|\nabla^2\sqrt{\rho_n}|^2)+|\mathbf{T}_{\nu,n}|^2\right)\mathrm{d}x\mathrm{d}t\right)$$
$$\leq C\limsup_{n\to\infty}B_{\mathbf{T}_n^d}(\sqrt{\rho_0^n},\sqrt{\rho_0^n}u_0^n)$$
$$\leq C\int_{\mathbf{R}^d}\frac{1}{2}\rho_0|u_0|^2+(2\kappa^2+4v^2)|\nabla\sqrt{\rho_0}|^2+F(\rho_0)\mathrm{d}x. \tag{3.2.12}$$

3.3 Extension to approximate solutions and convergence

In this section, we show that there exists a finite energy truncated weak solution to (3.0.1) on the whole space with far-field condition (3.0.2). The strategy of our method consists in several steps.

(i) We extend the sequence of periodic solutions provided by Theorem 3.2.4 to a sequence of functions defined on the whole space.

(ii) We prove that the extension are approximate truncated solutions to (3.0.1) on $\mathbf{R}^d$ according to Definition 3.2.3. As n goes to ∞ we obtain a finite energy truncated weak solution to (3.0.1).

Given a sequence of approximate truncated solutions on the invading domains $\mathbf{T}_n^d$ provided by Theorem 3.2.4, we define the density and momenta $(\rho_n:=(\sqrt{\rho_n})^2, m_n:=\sqrt{\rho_n}\sqrt{\rho_n}u_n)$. We extend (ρ_n,m_n) by the stationary solution $(\rho=1,m=0)$ on $\mathbf{R}^d\backslash[-n,n]^d$. Let $\eta_n\in C_c^\infty(\mathbf{R}^d)$ be a smooth cut-off function such that

$$\mathbf{1}_{[-n+\frac{1}{2},n-\frac{1}{2}]^d}\leq\eta_n\leq\mathbf{1}_{[-n,n]^d},$$

and denote $\mathcal{Q}_n=[-n,n]^d\backslash(-n+\frac{1}{2},n-\frac{1}{2})^d$ so that $supp(\nabla\eta_n)\subset\mathcal{Q}_n$. We introduce,

$$\begin{aligned}&\widetilde{\rho_n}:=\rho_n\eta_n+(1-\eta_n),\quad \tilde{m}_n=m_n\eta_n\\&\widetilde{\mathbf{S}_{\nu,n}}=\mathbf{S}_{\nu,n}\eta_n,\quad\widetilde{\mathbf{T}_{\nu,n}}=\mathbf{T}_{\nu,n}\eta_n.\end{aligned} \tag{3.3.1}$$

Further, we denote

$$\tilde{u}_n = \begin{cases} \frac{\tilde{m}_n(t,x)}{\tilde{\rho}_n(t,x)} & \text{if} \quad (t,x) \in \{\tilde{\rho}_n > 0\}, \\ 0 & \text{if} \quad (t,x) \in \{\tilde{\rho}_n = 0\}. \end{cases}$$

Finally, we define

$$\tilde{\rho}_n^0 = (\sqrt{\rho_n^0})^2 \eta_n + (1 - \eta_n), \qquad \tilde{m}_n^0 = m_n^0 \eta_n, \tag{3.3.2}$$

and

$$\tilde{u}_n^0 = \begin{cases} \frac{\tilde{m}_n^0(x)}{\tilde{\rho}_n^0(x)} & \text{if} \quad x \in \{\tilde{\rho}_n^0 > 0\}, \\ 0 & \text{if} \quad x \in \{\tilde{\rho}_n^0 = 0\}. \end{cases}$$

The main result of this Section is the following.

Theorem 3.3.1. *Let $d = 2, 3$, $\gamma > 1$, $\nu > 0$ and κ_n as defined in* (3.2.7). *Then $(\tilde{\rho}_n, \tilde{u}_n)$ defined in* (3.3.1) *is an approximate truncated weak solution to* (3.0.1) *with initial data $(\tilde{\rho}_n^0, \tilde{u}_n^0)$ given by* (3.3.2) *and viscosity and capillary tensor $\widetilde{\mathbf{T}_{\nu,n}}$ and $\widetilde{\mathbf{S}_{\kappa,n}}$ respectively. Further, the measures $\mu_\beta^n, \overline{\mu_\beta^n}$ satisfy*

$$\mu_\beta^n = R_\beta^n \eta_n, \quad \overline{\mu_\beta^n} = \overline{R}_\beta^n \eta_n,$$

with $R_\beta^n, \overline{R}_\beta^n$ provided by Theorem 3.2.4.
Moreover, as n goes to infinity $(\tilde{\rho}_n, \tilde{u}_n, \widetilde{\mathbf{T}_{\nu,n}}, \widetilde{\mathbf{S}_{\kappa,n}})$ converges to a finite energy truncated weak solution (ρ, u) with initial data (ρ^0, u^0) and with viscosity and capillary tensors $(\mathbf{T}_\nu, \mathbf{S}_\kappa)$ weak L^2-limits of $\widetilde{\mathbf{T}_{\nu,n}}, \widetilde{\mathbf{S}_{\nu,n}}$ respectively. More precisely, $\tilde{\rho}_n$ converges strongly to ρ in $C(\mathbf{R}_+; L^p_{loc}(\mathbf{R}^d))$ for $1 < p < \sup\{3, \gamma\}$, the momenta $\tilde{m}_n$ converge strongly to m in $L^2_{loc}(\mathbf{R}_+; L^p_{loc}(\mathbf{R}^d))$ in $1 \le p < \frac{3}{2}$ and $\tilde{\rho}_n u_n \beta(\tilde{u}_n)$ converges strongly to $\rho u \beta(u)$ in $L^2_{loc}(\mathbf{R}_+; L^2_{loc}(\mathbf{R}^d)$.

We notice that the measures are well-defined on $(0, T) \times \mathbf{R}^d$ taking into account the support properties of η_n. We start by collecting the needed uniform estimates that will follow from (3.2.11) as well as (3.2.12). These will be used to show the first part of Theorem 3.3.1. Subsequently, we provide convergence results for the sequence being needed for the passage to the limit.

Lemma 3.3.2. *The extensions introduced in* (3.3.1) *obey the following uniformly in n,*

$$\begin{aligned} &\tilde{\rho}_n - 1 \in L^\infty(\mathbf{R}_+; L^2(\mathbf{R}^d), \quad F(\tilde{\rho}_n) \in L^\infty_{loc}(\mathbf{R}_+; L^1(\mathbf{R}^d)), \\ &\sqrt{\tilde{\rho}_n} - 1 \in L^\infty_{loc}(\mathbf{R}_+; H^1(\mathbf{R}^d)), \quad \tilde{m}_n \in L^\infty(\mathbf{R}_+, L^{\frac{3}{2}}(\mathbf{R}^d) + L^2(\mathbf{R}^d)), \\ &\sqrt{\tilde{\rho}_n} \tilde{u}_n \in L^\infty_{loc}(\mathbf{R}_+; L^2(\mathbf{R}^d)), \quad \widetilde{\mathbf{T}_{\nu,n}} \in L^2_{loc}(\mathbf{R}_+; L^2(\mathbf{R}^d)). \end{aligned} \tag{3.3.3}$$

Moreover, there exists $\sqrt{\rho}, m, \mathbf{T}_\nu, \mathbf{S}_\nu, \mathbf{S}_\kappa$ such that

$$\begin{aligned} &\sqrt{\tilde{\rho}_n} - 1 \rightharpoonup \sqrt{\rho} - 1 \quad \text{in} \quad L^\infty(\mathbf{R}_+; H^1(\mathbf{R}^d)), \quad \tilde{m}_n \rightharpoonup m \quad \text{in} \quad L^\infty(\mathbf{R}_+; L^{\frac{3}{2}} + L^2(\mathbf{R}^d)), \\ &\widetilde{\mathbf{T}_{\nu,n}} \rightharpoonup \mathbf{T}_\nu \quad \text{in} \quad L^2_{loc}(\mathbf{R}_+; L^2(\mathbf{R}^d)), \quad \widetilde{\mathbf{S}_{\nu,n}} \rightharpoonup \mathbf{S}_\nu \quad \text{in} \quad L^2_{loc}(\mathbf{R}_+; L^2(\mathbf{R}^d)). \end{aligned} \tag{3.3.4}$$

If $\kappa > 0$, one has additionally that

$$\kappa\nabla^2\sqrt{\tilde{\rho}_n} \in L^2_{loc}(\mathbf{R}_+; L^2(\mathbf{R}^d)), \quad \kappa^{\frac{1}{4}}\nabla\tilde{\rho}_n^{\frac{1}{4}} \in L^4_{loc}(\mathbf{R}_+; L^4(\mathbf{R}^d)), \tag{3.3.5}$$

and $\widetilde{\mathbf{S}_{\kappa,n}}, \mathbf{S}_\kappa \in L^2_{loc}(\mathbf{R}_+; L^2(\mathbf{R}^d))$ such that $\widetilde{\mathbf{S}_{\kappa,n}} \rightharpoonup \mathbf{S}_\kappa$ in $L^2_{loc}(\mathbf{R}_+; L^2(\mathbf{R}^d))$. Finally, for a.e. $t \in [0,T)$, one has that

$$\begin{aligned}&\limsup_{n\to\infty}\Big(\int_{R^d}\frac{1}{2}\tilde{\rho}_n|\tilde{u}_n|^2 + 2\kappa_n^2|\nabla\sqrt{\tilde{\rho}_n}|^2 + F(\tilde{\rho}_n)\mathrm{d}x + 2\nu\int_0^T\int_{\mathbf{R}^d}|\widetilde{\mathbf{S}_{\nu,n}}|^2\mathrm{d}x\mathrm{d}t\Big)\\ &\le C\int_{\mathbf{R}^d}\frac{1}{2}\rho_0|u_0|^2 + 2\kappa^2|\nabla\sqrt{\rho_0}|^2\mathrm{d}x,\end{aligned} \tag{3.3.6}$$

and

$$\begin{aligned}&\limsup_{n\to\infty}\Big(\int_{\mathbf{R}^d}\frac{1}{2}\left|\sqrt{\tilde{\rho}_n}\tilde{u}_n(t)\right|^2 + (\kappa^2+2\nu^2)|\nabla\sqrt{\tilde{\rho}_n}(t)|^2 + F((\tilde{\rho}_n)^2)(t)\mathrm{d}x\\ &+2\nu\int_0^T\int_{\mathbf{R}^d}|\widetilde{\mathbf{T}_{\nu,n}}|^2\mathrm{d}x\mathrm{d}t + \nu\int_0^t\int_{\mathbf{R}^d}|\tilde{\rho}_n^{\frac{\gamma}{2}}|^2\mathrm{d}x\mathrm{d}t + \nu\kappa\int_0^T\int_{\mathbf{R}^d}|\nabla\tilde{\rho}_n^{\frac{1}{4}}|^4 + |\nabla^2\sqrt{\tilde{\rho}_n}|^2\Big)\\ &\le C\left(1+\int_{\mathbf{R}^d}\frac{1}{2}\rho_0|u_0|^2 + (2\kappa^2+4v^2)|\nabla\sqrt{\rho_0}|^2 + F(\rho_0)\mathrm{d}x\right).\end{aligned} \tag{3.3.7}$$

Proof. We start by showing the bounds (3.3.3). One has

$$\limsup_{n\to\infty}\|\tilde{\rho}_n - 1\|_{L^\infty_t L^2_x} = \limsup_{n\to\infty}\|(\rho_n-1)\eta_n\|_{L^\infty_t L^2_x} \le C\limsup_{n\to\infty}\|(\rho_n-1)\|_{L^\infty_t L^2_x(\mathbf{T}^d_n)},$$

that is bounded in view of (3.2.11) and (3.2.12). Indeed, proceeding as in the proof of Lemma 3.2.1, one obtains that the right hand side is bounded by the sum of $F(\rho_n) \in L^\infty(\mathbf{R}_+; L^1(\mathbf{T}^d_n))$ and $\nabla\sqrt{\rho_n} \in L^\infty(\mathbf{R}_+; L^1(\mathbf{T}^d_n)))$. Those, in their turn, are uniformly bounded in virtue of (3.2.11) and (3.2.12). Next, due to convexity of the renormalized internal energy and $F(1) = 0$, one has

$$F(\tilde{\rho}_n) \le \eta_n F(\rho_n),$$

which yields

$$\limsup_{n\to\infty}\|F(\tilde{\rho}_n)\|_{L^\infty(\mathbf{R}_+;L^1(\mathbf{R}^d))} \le \limsup_{n\to\infty}\|F(\rho_n)\|_{L^\infty(\mathbf{R}_+;L^1(\mathbf{T}^d_n))},$$

being bounded again by (3.2.11). The pointwise inequality

$$|\sqrt{\tilde{\rho}_n} - 1| \le |\tilde{\rho}_n - 1|,$$

yields the bound $\sqrt{\tilde{\rho}_n} - 1 \in L^\infty(\mathbf{R}_+; L^2(\mathbf{R}^d))$ uniformly. Next,

$$\begin{aligned}&\limsup_{n\to\infty}\|\nabla\sqrt{\tilde{\rho}_n}\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{R}^d))}\\ &=\limsup_{n\to\infty}\left\|\frac{1}{2\sqrt{\rho_n\eta_n + (1-\eta_n)}}\left(\eta_n\nabla\rho_n + (\rho_n-1)\nabla\eta_n\right)\right\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{R}^d))}\\ &\le C\limsup_{n\to\infty}\left(\|\frac{\sqrt{\eta_n}}{2\sqrt{\rho_n}}\nabla\rho_n\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{R}^d)} + \|\frac{(\rho_n-1)}{2\sqrt{(\rho_n-1)\eta_n+1}}\nabla\eta_n\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{R}^d))}\right)\\ &\le C\limsup_{n\to\infty}\|\nabla\sqrt{\rho_n}\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{T}^d_n))},\end{aligned}$$

where we used that

$$
\begin{aligned}
&\limsup_{n\to\infty} \| \frac{\rho_n - 1}{\sqrt{(\rho_n-1)\eta_n+1}} \nabla \eta_n \|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{T}_n^d))} \\
&\leq C \limsup_{n\to\infty} \left(\left\| (\rho_n - 1) \mathbf{1}_{\{\rho_n > \frac{1}{4}\}} \nabla \eta_n \right\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{T}_n^d))} + \left\| \frac{(\rho_n-1)}{\sqrt{1-\eta_n}} \mathbf{1}_{\{\rho_n < \frac{1}{4}\}} \nabla \eta_n \right\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{T}_n^d))} \right) \\
&\leq C' \limsup_{n\to\infty} \left(\left\| (\rho_n - 1) \mathbf{1}_{\{\rho_n > \frac{1}{4}\}} \nabla \eta_n \right\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{T}_n^d))} + \left\| \frac{(\sqrt{\rho_n}-1)}{\sqrt{1-\eta_n}} \mathbf{1}_{\{\rho_n < \frac{1}{4}\}} \nabla \eta_n \right\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{T}_n^d))} \right) \\
&\leq C' \limsup_{n\to\infty} \Big(\left\| (\rho_n - 1) \mathbf{1}_{\{\rho_n > \frac{1}{4}\}} \right\|_{L^\infty(\mathbf{R}_+;L^2(\mathcal{Q}_n))} \\
&+ \frac{1}{2} \| \sqrt{\rho_n} - 1 \|_{L^\infty(\mathbf{R}_+;L^4(\mathcal{Q}_n))} \left\| \nabla \sqrt{1-\eta_n} \right\|_{L^\infty(\mathbf{R}_+;L^4(\mathcal{Q}_n))} \Big) \\
&= 0,
\end{aligned}
$$

following from the integrability properties of $\rho_n - 1$, the L^∞-bound and the support properties for the cut-off η_n and its gradient. The bound on the momenta is an immediate consequence of (3.2.11) by observing that $|\tilde{m_n}| \leq |m_n|$ on $[-n,n]^d$ and $\tilde{m_n} = 0$ on $\mathbf{R}^d \setminus [-n,n]^d$. The bound on $\widetilde{\mathbf{T}_{\nu,n}}$ is analogous. Next, we show the bound on $\nabla \tilde{\rho}_n^{\frac{\gamma}{2}}$. If $\gamma \geq 3$, then $f(t) = t^{\frac{\gamma-1}{2}}$ is convex and therefore

$$
\begin{aligned}
\limsup_{n\to\infty} \| \nabla \tilde{\rho}_n^{\frac{\gamma}{2}} \|_{L^2_{t,x}} &= \limsup_{n\to\infty} \| \gamma (\sqrt{\tilde{\rho}_n})^{\gamma-1} \nabla \sqrt{\tilde{\rho}_n} \|_{L^2_{t,x}} \\
&\leq \limsup_{n\to\infty} \| \gamma \left((\eta_n \rho_n^{\frac{\gamma-1}{2}} + (1-\eta_n)) \nabla \sqrt{\tilde{\rho}_n} \right) \|_{L^2_{t,x}},
\end{aligned}
$$

and proceeding analogously as in the bound for $\nabla \sqrt{\tilde{\rho}_n}$ we conclude by invoking (3.2.12). If $1 < \gamma < 3$, we use that $f(t) = t^{\frac{\gamma-1}{2}}$ is a concave function s.t. $f(0) = 0$ and therefore sub-additive and proceed as in the previous case. We conclude that

$$
\limsup_{n\to\infty} \| \nabla \tilde{\rho}_n^{\frac{\gamma}{2}} \|_{L^2(0,T;L^2(\mathbf{R}^d))} \leq C \limsup_{n\to\infty} \| \nabla \rho_n{}^{\frac{\gamma}{2}} \|_{L^2(0,T;L^2_x(\mathbf{T}_n^d))}.
$$

Finally, we show the bounds (3.3.5),

$$
\limsup_{n\to\infty} \| \nabla \tilde{\rho}_n^{\frac{1}{4}} \|_{L^4_{t,x}} \leq C \limsup_{n\to\infty} \left(\left\| \frac{\eta_n}{4(\rho_n\eta_n + 1 - \eta_n)^{\frac{3}{4}}} \nabla \rho_n \right\|_{L^4_{t,x}} + \left\| \frac{(\rho_n - 1)}{4(\rho_n\eta_n + 1 - \eta_n)^{\frac{3}{4}}} \nabla \eta_n \right\|_{L^4_{t,x}} \right)
$$

The first term is controlled by

$$
\limsup_{n\to\infty} \left\| \frac{\eta_n}{4(\rho_n\eta_n + 1 - \eta_n)^{\frac{3}{4}}} \nabla \rho_n \right\|_{L^4_{t,x}} \leq \left\| \eta_n^{\frac{1}{4}} \nabla \rho_n^{\frac{1}{4}} \right\|_{L^4_t L^4(\mathbf{T}_n^d)},
$$

that is uniformly bounded from (3.2.12). The second term is estimated as

$$\begin{aligned}
&\limsup_{n\to\infty}\left\|\frac{(\rho_n-1)}{4(\rho_n\eta_n+1-\eta_n)^{\frac{3}{4}}}\nabla\eta_n\right\|_{L^4_{t,x}}\\
&\leq \limsup_{n\to\infty}\left(\|\mathbf{1}_{\{\rho_n-1<0\}}\frac{1}{4(1-\eta_n)^{\frac{3}{4}}}\nabla\eta_n\|_{L^4L^4(\mathcal{Q}_n)}+\|\mathbf{1}_{\{\rho_n-1>0\}}\frac{(\rho_n-1)^{\frac{1}{4}}}{4\eta_n^{\frac{3}{4}}}\nabla\eta_n\|_{L^4L^4(\mathcal{Q}_n)}\right)\\
&\leq C\left(1+\limsup_{n\to\infty}(\|\rho_n-1\|_{L^4L^2(\mathcal{Q}_n)}\right)=C.
\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\|\nabla\tilde{\rho}_n^{\frac{1}{4}}\|_{L^4(0,T;L^4(\mathbf{R}^d))}\leq C\limsup_{n\to\infty}\|\nabla\tilde{\rho}_n^{\frac{1}{4}}\|_{L^4(0,T;L^4(\mathbf{T}^d_n))}.$$

It remains to bound $\nabla^2\sqrt{\tilde{\rho}_n}$ in $L^2(0,T;L^2(\mathbf{R}^d))$. To that end we compute that

$$\begin{aligned}
\nabla^2\sqrt{\tilde{\rho}_n}&=\frac{1}{2\sqrt{\tilde{\rho}_n}}\eta_n\nabla^2\rho_n-\frac{1}{4}\frac{\eta_n^2}{\tilde{\rho}_n^{\frac{3}{2}}}(\nabla\rho_n)^2\\
&+\frac{1}{2}\frac{\rho_n}{\sqrt{\tilde{\rho}_n}}\nabla^2\eta_n-\frac{1}{4}\frac{(\rho_n-1)^2}{\tilde{\rho}_n^{\frac{3}{2}}}(\nabla\eta)^2\\
&+\frac{1}{\sqrt{\tilde{\rho}_n}}\nabla\eta_n\nabla\rho_n-\frac{1}{2}\frac{\eta_n(\rho_n-1)}{\tilde{\rho}_n^{\frac{3}{2}}}\nabla\eta_n\nabla\rho_n.
\end{aligned}$$

Proceeding analogously as for the previous bound, the L^2L^2-norm of the RHS of the first line is bounded by $\|\nabla^2\sqrt{\rho_n}\|_{L^2(0,T;L^2(\mathbf{T}^d_n))}$ that again is uniformly bounded in view of (3.2.11). The other terms can be controlled by exploiting the properties of η_n and (3.2.11) and (3.2.12) holding uniformly in n. Finally, we observe that since

$$\left|\sqrt{\tilde{\rho}_n}\tilde{u}_n\right|=\left|\frac{m_n\eta_n}{\sqrt{\rho_n\eta_n+(1-\eta_n)}}\right|\leq\sqrt{\eta_n}\left|\frac{m_n}{\sqrt{\rho_n}}\right|,$$

one has

$$\int_{\mathbf{R}^d}\frac{1}{2}\tilde{\rho}_n|\tilde{u}_n|^2\mathrm{d}x\leq\int_{\mathbf{T}^d_n}\frac{1}{2}\sqrt{\rho_n}\,|u_n|^2\,\mathrm{d}x$$

Therefore, combing the previous inequalities with (3.2.11) and (3.2.12) we conclude that inequalities (3.3.6) and (3.3.7) are satisfied for a.e. $t\in[0,T)$. Hence, the uniform bounds (3.3.3) follow. □

The uniform bounds lead to the following convergence results.

Lemma 3.3.3. *The following convergences hold up to subsequences.*

1. $\tilde{\rho}_n\to\rho$ *strongly in* $C(\mathbf{R}_+;L^p_{loc}(\mathbf{R}^d))$ *for* $1<p<\sup\{3,\gamma\}$.

2. $\tilde{m_n}\to m$ *strongly in* $L^2_{loc}(\mathbf{R}_+;L^p_{loc}(\mathbf{R}^d))$ *in* $1\leq p<\frac{3}{2}$.

3. $\nabla\rho_n^{\frac{\gamma}{2}} \rightharpoonup \nabla\rho^{\frac{\gamma}{2}}$ *weakly in* $L^2(0,T;L^2_{loc}(\mathbf{R}^d))$.

If $\kappa > 0$, *then*

1. $\sqrt{\tilde{\rho}_n} \to \sqrt{\rho}$ *strongly in* $L^2_{loc}(0,T;H^1_{loc}(\mathbf{R}^d))$,
2. $\nabla^2\sqrt{\tilde{\rho}_n} \rightharpoonup \nabla^2\sqrt{\rho}$ *weakly in* $L^2_{loc}(0,T;L^2_{loc}\mathbf{R}^d))$.

Proof. 1. One has that uniformly in n,

$$\begin{aligned}\|\partial_t\tilde{\rho}_n\|_{L^\infty(\mathbf{R}_+;W^{-1,\frac{3}{2}}_{loc}(\mathbf{R}^d))} &= \|\eta_n\partial_t\rho_n\|_{L^\infty(\mathbf{R}_+;W^{-1,\frac{3}{2}}_{loc}(\mathbf{R}^d))}\\ &\leq \|m_n\|_{L^\infty(\mathbf{R}_+;L^{\frac{3}{2}}_{loc}(\mathbf{T}^d_n))},\end{aligned}$$

being bounded in virtue of (3.2.11). On the other hand, $\tilde{\rho}_n \in L^\infty(\mathbf{R}_+;L^\gamma_{loc}\cap L^3_{loc}(\mathbf{R}^d))$ and $\nabla\tilde{\rho}_n \in L^2(\mathbf{R}^d) + L^{\frac{3}{2}}(\mathbf{R}^d)$ and hence $\tilde{\rho}_n \in L^\infty(\mathbf{R}_+;W^{1,\frac{3}{2}}_{loc}(\mathbf{R}^d))$. Thus, we conclude from the Aubin-Lions Lemma that $\tilde{\rho}_n - 1$ is compact in $C(\mathbf{R}_+;L^p_{loc}(\mathbf{R}^d))$ for any $1 \leq p < \sup\{3,\gamma\}$.

2. Since $\partial_t\tilde{m_n} = \eta_n\partial_t m_n$, from the second equation of (3.0.1) we infer that $\partial_t(\rho_n u_n)$ is uniformly bounded in $L^2(\mathbf{R}_+;H^{-s}_{loc}(\mathbf{R}^d)$ for s large enough by applying the uniform bounds of (3.2.11) and (3.2.12). From (3.3.3), we have that $\tilde{m_n} \in L^\infty(\mathbf{R}_+;L^{\frac{3}{2}} + L^2(\mathbf{R}^d)$ and $\nabla\tilde{m_n} = \eta_n\nabla m_n + m_n\nabla\eta_n$ is bounded in $L^2(0,T;L^1_{loc}(\mathbf{R}^d))$, thus $\tilde{m_n} \in L^2(0,T;W^{1,1}_{loc}(\mathbf{R}^d))$. The Aubin-Lions lemma implies that $\tilde{m_n}$ is compact in $L^2(0,T;L^p_{loc}(\mathbf{R}^d))$ for $1 \leq p < \frac{3}{2}$.

3. The uniform bound $\nabla\tilde{\rho}_n^{\frac{\gamma}{2}} \in L^2(0,T;L^2(\mathbf{R}^d))$ implies that up to passing to subsequences the sequence converges weakly with the weak limit being identified by $\nabla\rho^{\frac{\gamma}{2}}$ by means of point (1).

If $\kappa > 0$, we have $\nabla\sqrt{\tilde{\rho}_n} \in L^2(0,T;L^2(\mathbf{R}^d)$ additionally to $\sqrt{\tilde{\rho}_n} - 1 \in L^\infty(\mathbf{R}_+;H^1(\mathbf{R}^d))$. Hence, $\nabla^2\sqrt{\tilde{\rho}_n}$ converges weakly to $\nabla^2\sqrt{\rho}$ in $L^2_{loc}(0,T;L^2_{loc}(\mathbf{R}^d))$ up to subsequences. Further, by combining the strong convergence of $\tilde{\rho}_n$ and the bounds on the second order derivatives of $\sqrt{\tilde{\rho}_n}$ we obtain that

$$\sqrt{\tilde{\rho}_n} \to \sqrt{\rho} \qquad \text{in} \quad L^2_{loc}(0,T;H^1_{loc}(\mathbf{R}^d)).$$

□

Lemma 3.3.4. *Let* $f \in C \cap L^\infty(\mathbf{R}^d;\mathbf{R})$ *and let* $(\tilde{\rho}_n,\tilde{u}_n)$ *be as defined in* (3.3.1) *and let* u *be defined as*

$$u = \begin{cases} \frac{m(t,x}{\rho(t,x)} & (t,x) \in \{\rho > 0\},\\ 0 & (t,x) \in \{\rho = 0\}.\end{cases} \tag{3.3.8}$$

Then,

1. *for any* $0 < \alpha < \frac{5\gamma}{3}$, *one has* $\tilde{\rho}_n^\alpha f(\tilde{u}_n) \to \rho^\alpha f(u)$ *in* $L^p((0,T)\times\mathbf{R}^d)$ *with* $1 \leq p < \frac{5\gamma}{3\alpha}$,

2. $\tilde{\rho}_n u_n f(\tilde{u}_n) \to \rho u f(u)$ *strongly in* $L^2_{loc}(\mathbf{R}_+; L^2_{loc}(\mathbf{R}^d)$,

Proof. We recall that (1) and (2) of Lemma 3.3.3 imply that $\tilde{\rho}_n$ converges to ρ a.e. in $(0,T)\times\mathbf{R}^d$ and $\tilde{m_n}$ converges to m a.e. in $(0,T)\times\mathbf{R}^d$. The Fatou Lemma implies that

$$\int_{\mathbf{R}_+}\int_{\mathbf{R}^d}\liminf_{n\to\infty}\frac{\tilde{m_n}^2}{\tilde{\rho}_n}\mathrm{d}x\mathrm{d}t \le \liminf_{n\to\infty}\int_{\mathbf{R}_+}\int_{\mathbf{R}^d}\frac{\tilde{m_n}^2}{\tilde{\rho}_n}\mathrm{d}x\mathrm{d}t$$
$$\le \liminf_{n\to\infty}\int_{\mathbf{R}_+}\int_{\mathbf{R}^d}\eta_n|\sqrt{\rho_n}u_n|^2\mathrm{d}x\mathrm{d}t \le \liminf_{n\to\infty}\int_{\mathbf{R}_+}\int_{\mathbf{T}^d_n}|\sqrt{\rho_n}u_n|^2\mathrm{d}x\mathrm{d}t < +\infty,$$

due to inequality (3.2.11). Hence $m=0$ on the null set of ρ and by consequence $\sqrt{\rho}u \in L^\infty(0,T;L^2(\mathbf{R}^d))$. Moreover, since $\sqrt{\tilde{\rho}_n}\tilde{u}_n$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbf{R}^d))$ it converges weakly-$*$ to some limit function $\Lambda \in L^\infty(0,T;L^2(\mathbf{R}^d))$. By uniqueness of weak-limits and the a.e. convergence of $\tilde{\rho}_n, \tilde{m_n}$ we recover, $m=\rho u=\sqrt{\rho}\Lambda$.
We show (1). On $\{\rho>0\}$, we have that

$$\tilde{\rho}_n^\alpha f(\tilde{u}_n) \to \rho^\alpha f(u), \quad \text{a.e. in} \quad \{\rho>0\}.$$

As by hypothesis $f \in L^\infty(\mathbf{R}^d;\mathbf{R})$ and $\alpha>0$ we have that

$$|\tilde{\rho}_n^\alpha f(u_n)| \le |\tilde{\rho}_n|^\alpha \|f\|_{L^\infty} \to 0 \quad \text{a.e. in} \quad \{\rho=0\}.$$

It follows, that $\tilde{\rho}_n^\alpha f(\tilde{u}_n)$ converges to $\rho^\alpha f(u)$ a.e. in $(0,T)\times\mathbf{R}^d$. From (3.3.3), we have that $\tilde{\rho}_n^{\frac{\gamma}{2}} \in L^\infty(\mathbf{R}_+;L^2_{loc}(\mathbf{R}^d))\cap L^2(0,T;L^6_{loc}(\mathbf{R}^d))$ uniformly. By interpolation $\tilde{\rho}_n^{\frac{\gamma}{2}} \in L^{\frac{10}{3}}(0,T;L^{\frac{10}{3}}_{loc}(\mathbf{R}^d))$. Together with Vitali's convergence theorm, this yields strong convergence of $\tilde{\rho}_n^\alpha f(\tilde{u}_n)$ in $L^p_{loc}((0,T)\times\mathbf{R}^d))$ for $0<\alpha<\frac{5\gamma}{3}$ and $1\le p<\frac{5\gamma}{3\alpha}$.
Next, we show (2). We use again that $\tilde{\rho}_n$ and $\tilde{m_n}$ converge a.e. in $(0,T)\times\mathbf{R}^d$ and conclude that

$$\tilde{\rho}_n\tilde{u}_n f(\tilde{u}_n) \to m f(u) \qquad \text{a.e. in} \quad \{\rho>0\},$$
$$|\tilde{\rho}_n\tilde{u}_n f(\tilde{u}_n)| \le |\tilde{m_n}|\,\|f\|_{L^\infty} \to 0 \qquad \text{a.e. in} \quad \{\rho=0\}.$$

Vitali's convergence theorem together with the uniform bounds from Lemma 3.3.2 yield the strong convergence in $L^2_{loc}(\mathbf{R}_+;L^2_{loc}(\mathbf{R}^d)$. ☐

We are now in position to proof Theorem 3.3.1.

Proof. Let $(\tilde{\rho^0_n},\tilde{m^0_n})$ and $\tilde{u^0_n}$ be defined by (3.3.2) and $(\tilde{\rho}_n,\tilde{m}_n)$ be defined by (3.3.1). The required uniform bounds are consequence of (3.3.3) and (3.3.5). We compute

$$\int_{\mathbf{R}^d}\tilde{\rho_{0,n}}\psi(0,x)\mathrm{d}x + \int_0^T\int_{\mathbf{R}^d}\tilde{\rho}_n\psi_t + \tilde{m_n}\cdot\nabla\psi\mathrm{d}x\mathrm{d}t$$
$$= \int_{\mathbf{R}^d}\rho_0^n\eta_n\psi\mathrm{d}x + \int_0^T\int_{R^d}\rho_n\partial_t(\eta_n\psi) + m_n\cdot\nabla(\eta_n\psi)\mathrm{d}x\mathrm{d}t$$
$$+ \int_{\mathbf{R}^d}(1-\eta_n)\psi(0,x)\mathrm{d}x + \int_0^T\int_{\mathbf{R}^d}\partial_t\left((1-\eta_n)\psi\right) - m_n\psi\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T\int_{\mathbf{R}^d}m_n\psi\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}t,$$

where we used that $\eta_n\psi \in C_c^\infty([0,T)\times \mathbf{T}_n^d)$ and thus the second line is the weak formulation of the continuity equation for (ρ_n, m_n) on $\mathbf{T}_n^d$ for an admissible test-function. We denote

$$\langle D_n, \psi\rangle = \int_0^T \int_{\mathbf{R}^d} m_n \psi \cdot \nabla \eta_n \mathrm{d}x\mathrm{d}t, \tag{3.3.9}$$

and observe that D_n is well-defined and uniformly bounded in $\mathcal{D}'([0,T)\times \mathbf{R}^d)$ since $m_n \in L^\infty(0,T;L^{\frac{3}{2}}(\mathbf{T}_n^d))$ and $\eta_n \in C_c^\infty(\mathbf{R}^d)$. Further, as $supp(\nabla\eta_n) \subset \mathcal{Q}_n$, we conclude $supp(D_n) \subset \mathcal{Q}_n$. This allows to infer that D_n converges to 0 in $\mathcal{D}'$ as it is uniformly bounded and for n sufficiently large $supp(\psi)\cap supp(D_n) = \emptyset$. We proceed to verify that $(\tilde{\rho}_n, \tilde{u}_n)$ is an approximate solution to the truncated formulation of the momentum equation. One has that,

$$\begin{aligned} &\int_{\mathbf{R}^d} \tilde{\rho}_n^0 \beta(\tilde{u}_n^0)\psi(0,x)\mathrm{d}x = \int_{\mathbf{R}^d} \rho_n^0 \beta(u_n^0)(\eta_n\psi_0)\mathrm{d}x \\ &+ \int_{\mathbf{R}^d} \rho_n^0 \left(\beta(\tilde{u}_n^0) - \beta(u_n^0)\right)\eta_n\psi(0,x) + (1-\eta_n)\psi(0,x)\mathrm{d}x. \end{aligned} \tag{3.3.10}$$

Next, one has

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}^d} \tilde{\rho}_n \beta(\tilde{u}_n)\partial_t\psi \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbf{R}^d} \rho_n \beta(u_n)\partial_t(\eta_n\psi)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbf{R}^d} \rho_n \left(\beta(\tilde{u}_n) - \beta(u_n)\right)\partial_t(\eta_n\psi) + (1-\eta_n)\beta(\tilde{u}_n)\partial_t\psi(0,x)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.3.11}$$

Similarly,

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}^d} \tilde{\rho}_n \tilde{u}_n \beta(\tilde{u}_n)\cdot\nabla\psi \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbf{R}^d} \rho_n u_n \beta(u_n)\cdot\nabla(\psi\eta_n)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbf{R}^d} \rho_n u_n \left(\left(\beta(\tilde{u}_n) - \beta(u_n)\right)\nabla(\psi\eta_n) - \beta(\tilde{u}_n)\psi\nabla\eta_n\right)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.3.12}$$

The pressure time is dealt with as follows,

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}^d} \tilde{\rho}_n^{\frac{\gamma}{2}} \nabla \tilde{\rho}_n^{\frac{\gamma}{2}} \beta'(\tilde{u}_n)\psi \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbf{R}^d} \rho_n^{\frac{\gamma}{2}} \nabla \rho_n^{\frac{\gamma}{2}} \beta'(u_n)\eta_n\psi \mathrm{d}x\mathrm{d}t \\ &- \int_0^T \int_{\mathbf{R}^d} \left(\tilde{\rho}_n^\gamma - \eta_n\rho_n^\gamma\right)\nabla(\beta'(\tilde{u}_n)\psi)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbf{R}^d} \rho_n^\gamma \beta'(\tilde{u}_n)\psi\nabla\eta_n + \rho_n^{\frac{\gamma}{2}} \nabla \rho_n^{\frac{\gamma}{2}} (\beta(\tilde{u}_n) - \beta(u_n))\eta_n\psi \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.3.13}$$

The second line of (3.3.13) can be controlled bu using the convexity of $s \mapsto s^\gamma$ as follows

$$\left|\int_0^T \int_{\mathbf{R}^d} \left(\tilde{\rho}_n^\gamma - \eta_n\rho_n^\gamma\right)\nabla(\beta'(\tilde{u}_n)\psi)\mathrm{d}x\mathrm{d}t\right| \leq \int_0^T \int_{\mathbf{R}^d} \left|(1-\eta_n)\nabla(\beta'(\tilde{u}_n)\psi)\right| \mathrm{d}x\mathrm{d}t.$$

The third line is easily seen to be uniformly bounded and to have support contained in $\mathcal{Q}_n$ due to the properties of $\nabla\eta_n$ and $\eta_n(\beta(\tilde{u}_n) - \beta(u_n)) \neq 0$ only on $\mathcal{Q}_n$. For the viscosity tensor we recover,

$$\int_0^T 2\nu\sqrt{\tilde\rho_n}\widetilde{\mathbf{S}_{\nu,n}}\beta'(\tilde u_n)\nabla\psi \mathrm{d}x\mathrm{d}t = \int_0^T 2\nu\sqrt{\rho_n}\mathbf{S}_{\nu,n}\beta'(u_n)\nabla(\psi\eta_n)\mathrm{d}x\mathrm{d}t$$
$$-\int_0^T\int_{\mathbf{R}}^d 2\nu\sqrt{\tilde\rho_n}\mathbf{S}_{\nu,n}\beta'(\tilde u_n)\psi\nabla\eta_n + 2v\left(\sqrt{\tilde\rho_n}\beta'(\tilde u_n)-\sqrt{\rho_n}\beta'(u_n)\right)\mathbf{S}_{\nu,n}\nabla(\psi\eta_n)\mathrm{d}x\mathrm{d}t. \tag{3.3.14}$$

Similarly, for the capillary tensor, one has

$$\int_0^T 2\kappa^2\sqrt{\tilde\rho_n}\widetilde{\mathbf{S}_{\kappa,n}}\beta'(\tilde u_n)\nabla\psi \mathrm{d}x\mathrm{d}t = \int_0^T 2\kappa^2\sqrt{\rho_n}\mathbf{S}_{\kappa,n}\beta'(u_n)\nabla(\psi\eta_n)\mathrm{d}x\mathrm{d}t$$
$$-\int_0^T\int_{\mathbf{R}}^d 2\kappa^2\sqrt{\tilde\rho_n}\mathbf{S}_{\kappa,n}\beta'(\tilde u_n)\psi\nabla\eta_n + 2\kappa^2\left(\sqrt{\tilde\rho_n}\beta'(\tilde u_n-\sqrt{\rho_n}\beta'(u_n)\right)\mathbf{S}_{\kappa,n}\nabla(\psi\eta_n)\mathrm{d}x\mathrm{d}t. \tag{3.3.15}$$

Summing up equations from (3.3.10) to (3.3.15) yields that

$$\begin{aligned}&\int_{\mathbf{R}^d}\tilde\rho_{0,n}\tilde{u_{0,n}} + \int_0^t\int_{\mathbf{R}^d}\tilde\rho_n\beta(\tilde u_n)\partial_t\psi + \tilde\rho_n\tilde u_n\beta(\tilde u_n)\cdot\nabla\psi - \tilde\rho_n^{\frac{\gamma}{2}}\nabla\tilde\rho_n^{\frac{\gamma}{2}}\beta'(\tilde u_n)\psi\mathrm{d}x\mathrm{d}t\\ &-\int_0^t\int_{\mathbf{R}^d}\left(2\nu\sqrt{\tilde\rho_n}\widetilde{\mathbf{S}_{\nu,n}} + 2\kappa^2\sqrt{\tilde\rho_n}\tilde{S_{\kappa,n}}\right)\beta'(\tilde u_n)\cdot\nabla\psi\mathrm{d}x\mathrm{d}t = \left\langle R_\beta^n,\eta_n\psi\right\rangle + \langle G_n,\psi\rangle,\end{aligned} \tag{3.3.16}$$

where R_β^n is the measure provided by Theorem 3.2.4 and G_n is a distribution such that

$$\begin{aligned}\langle G_n,\psi\rangle = &\int_{\mathbf{R}^d}\rho_n^0\left(\beta(\tilde u_n^0)-\beta(u_n^0)\right)\eta_n\psi(0,x)\mathrm{d}x\\ &+\int_0^T(\beta(\tilde u_n)-\beta(u_n))\left(\rho_n\partial_t(\eta_n\psi)+\rho_n u_n\cdot\nabla(\psi\eta_n)\right)-\rho_n u_n\beta(\tilde u_n)\psi\nabla\eta_n\mathrm{d}x\mathrm{d}t\\ &-\int_0^T\int_{\mathbf{R}^d}\left(2\nu\mathbf{S}_{\nu,n}+2\kappa^2\mathbf{S}_{\kappa,n}\right)\sqrt{\tilde\rho_n}\beta'(\tilde u_n)\psi\nabla\eta_n\mathrm{d}x\mathrm{d}t\\ &+2\int_0^T\int_{\mathbf{R}^d}\left(\sqrt{\tilde\rho_n}\beta'(\tilde u_n)-\sqrt{\rho_n}\beta'(u_n)\right)\left(\nu\mathbf{S}_{\nu,n}+\kappa\mathbf{S}_{\kappa,n}\right)\nabla(\psi\eta_n)\mathrm{d}x\mathrm{d}t\\ &-\int_0^T\int_{\mathbf{R}^d}\left(\tilde\rho_n^\gamma-\eta_n\rho_n^\gamma\right)\nabla(\beta'(\tilde u_n)\psi)\mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_{\mathbf{R}^d}\rho_n^\gamma\beta'(\tilde u_n)\psi\nabla\eta_n+\rho_n^{\frac{\gamma}{2}}\nabla\rho_n^{\frac{\gamma}{2}}(\beta(\tilde u_n)-\beta(u_n))\eta_n\psi\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.3.17}$$

From the uniform bounds provided by Lemma 3.3.2, the properties of β and η_n and the fact that $supp(\tilde u_n)\subset[-n,n]^d$, we conclude that $supp(G_n)\subset\mathcal{Q}_n$ and thus that there exists a uniform constant $C>0$ such that,

$$|\langle G_n,\psi\rangle|\le C\|\psi\|_{C_c^\infty}.$$

In particular, arguing as for D_n, we observe that G_n converges to 0 in $\mathcal{D}'$ since for n large enough $supp(G_n)\cap supp(\psi)=\emptyset$. It remains to check that the compatibility conditions for the tensors $\widetilde{\mathbf{S}_{\nu,n}}$ and $\widetilde{\mathbf{S}_{\kappa,n}}$ are satisfied. Let $\psi \in C_c^\infty([0,T)\times\mathbf{R}^d)$, then

$$
\begin{aligned}
&\int_0^T\int_{\mathbf{R}^d}\sqrt{\nu}\sqrt{\tilde\rho_n}\beta'(\tilde u_n)[\widetilde{\mathbf{T}_{\nu,n}}]_{jk}\psi\mathrm{d}x\mathrm{d}t = \int_0^T\int_{\mathbf{R}^d}\sqrt{\nu}\sqrt{\rho_n}\beta'(u_n)[\mathbf{T}_{\nu,n}]_{jk}(\eta_n\psi)\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbf{R}^d}\sqrt{\nu}\left(\sqrt{\tilde\rho_n}\beta'(\tilde u_n)-\sqrt{\rho_n}\beta'(u_n)\right)[\mathbf{T}_{\nu,n}]_{jk}(\eta_n\psi)\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_{\mathbf{R}^d}\nu\left(\partial_j(\rho_n\beta_i'(u_n)u_{n,k})-2\sqrt{\rho_n}u_n\beta_i'(u_n)\partial_j\sqrt{\rho_n}+\overline{R_\beta}\right)(\eta_n\psi)\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbf{R}^d}\sqrt{\nu}\left(\sqrt{\tilde\rho_n}\beta'(\tilde u_n)-\sqrt{\rho_n}\beta'(u_n)\right)[\mathbf{T}_{\nu,n}]_{jk}(\eta_n\psi)\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_{\mathbf{R}^d}\nu\left(\partial_j(\tilde\rho_n\beta_i'(\tilde u_n)\tilde u_{n,k})-2\sqrt{\tilde\rho_n}u_n\beta_i'(\tilde u_n)\partial_j\sqrt{\tilde\rho_n}\right)\psi+\overline{R_\beta}\eta_n\psi\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbf{R}^d}\nu\left(\eta_n\partial_j(\rho_n\beta_i'(u_n)u_{n,k})-\partial_j(\tilde\rho_n\beta_i'(\tilde u_n)\tilde u_{n,k})\right)\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\int_{\mathbf{R}^d}2\nu\left(\eta_n\left(\sqrt{\rho_n}u_n\beta_i'(u_n)\partial_j\sqrt{\rho_n}\right)+\sqrt{\tilde\rho_n}u_n\beta_i'(\tilde u_n)\partial_j\sqrt{\tilde\rho_n}\right)\psi\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbf{R}^d}\sqrt{\nu}\left(\sqrt{\tilde\rho_n}\beta'(\tilde u_n)-\sqrt{\rho_n}\beta'(u_n)\right)[\mathbf{T}_{\nu,n}]_{jk}(\eta_n\psi)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Thus, there exists a distribution V_n with $supp(V_n)\subset\mathcal{Q}_n$ and such that for any $\psi\in C_c^\infty([0,T)\times\mathbf{R}^d)$ one has

$$
\begin{aligned}
&\int_0^t\int_{\mathbf{R}^d}\sqrt{\nu}\sqrt{\tilde\rho_n}\beta'(\tilde u_n)[\widetilde{\mathbf{T}_{\nu,n}}]_{jk}\psi\mathrm{d}x\mathrm{d}t\\
&=\nu\int_0^T\int_{\mathbf{R}^d}\left(\partial_j(\tilde\rho_n\beta_i'(\tilde u_n)\tilde u_{n,k})-2\sqrt{\tilde\rho_n}u_n\beta_i'(\tilde u_n)\partial_j\sqrt{\tilde\rho_n}\right)\psi\mathrm{d}x\mathrm{d}t\\
&+\int_0^t\int_{\mathbf{R}^d}\overline{R_\beta}\eta_n\psi+V_n\psi\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Moreover, there exists a uniform $C>0$ such that

$$|\langle V_n,\psi\rangle|\le C\|\psi\|_{C_c^\infty},$$

for any $\psi\in C_c^\infty([0,T)\times\mathbf{R}^d)$. The uniform bound in $\mathcal{D}'$ together with the support properties imply that V_n converges to 0 as $n\to\infty$. Arguing similarly, we recover for the capillary tensor

$$
\begin{aligned}
&\int_0^T\int_{\mathbf{R}^d}2\kappa\sqrt{\tilde\rho_n\widetilde{\mathbf{S}_{\kappa,n}}}\psi\mathrm{d}x\mathrm{d}t=\kappa^2\int_0^t\int_{\mathbf{R}^d}\sqrt{\rho_n}\left(\nabla^2\sqrt{\rho_n}-4(\nabla\rho_n^{\frac14}\otimes\nabla\rho_n^{\frac14})\right)\eta_n\psi\mathrm{d}x\mathrm{d}t\\
&+2\kappa\int_0^T\int_{\mathbf{R}^d}(\sqrt{\tilde\rho_n}-\sqrt{\rho_n})\mathbf{S}_{\kappa,n}\eta_n\psi\mathrm{d}x\mathrm{d}t\\
&=\kappa^2\int_0^t\int_{\mathbf{R}^d}\sqrt{\tilde\rho_n}\left(\nabla^2\sqrt{\tilde\rho_n}-4(\nabla\tilde\rho_n^{\frac14}\otimes\nabla\tilde\rho_n^{\frac14})\right)\psi\mathrm{d}x\mathrm{d}t\\
&+\kappa^2\int_0^t\int_{\mathbf{R}^d}\left(\sqrt{\rho_n}\left(\nabla^2\sqrt{\rho_n}-4(\nabla\rho_n^{\frac14}\otimes\nabla\rho_n^{\frac14})\right)\eta_n-\sqrt{\tilde\rho_n}\left(\nabla^2\sqrt{\tilde\rho_n}-4(\nabla\tilde\rho_n^{\frac14}\otimes\nabla\tilde\rho_n^{\frac14})\right)\right)\psi\mathrm{d}x\mathrm{d}t\\
&+2\kappa\int_0^T\int_{\mathbf{R}^d}(\sqrt{\tilde\rho_n}-\sqrt{\rho_n})\mathbf{S}_{\kappa,n}\eta_n\psi\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

Hence, the error is given by a distribution K_n such that $supp(K_n) \subset \mathcal{Q}_n$ and K_n is uniformly bounded in $\mathcal{D}'$ and converges to 0 as n goes to infinity. We conclude that $(\tilde{\rho}_n, \tilde{u}_n)$ as defined in (3.3.1) is an approximate truncated weak solution to (3.0.1) with initial data (3.3.2).
It remains to perform the limit as n goes to infinity. We start by verifying that the inequalities (3.1.1) and (3.1.2) are satisfied. Since (3.3.6) and (3.3.7) are verified, the inequalities follow from the weak convergences provided by (3.3.4), (3.3.5), the second statement of Lemma 3.3.4, lower semi-continuity of norms and the following observation. Denote by Λ the weak-$*$ $L^\infty L^2$-limit of $\sqrt{\tilde{\rho}_n \tilde{u}_n}$. Then, since $\Lambda = \sqrt{\rho} u$ whenever $\rho \neq 0$, we infer for a.e. $t \in [0, T)$,

$$\int_{\mathbf{R}^d} \frac{1}{2} \rho |u|^2(t) \mathrm{d}x \leq \int_{\mathbf{R}^d} \frac{1}{2} |\Lambda|^2(t) \mathrm{d}x.$$

Next, we wish to pass to the limit in (3.2.8). We check that

$$\begin{aligned}\int_{\mathbf{R}^d} \tilde{\rho}_n^0 \psi(0,x) \mathrm{d}x &= \int_{\mathbf{R}^d} \rho_n^0 \psi(0,x) \mathrm{d}x + \int_{\mathbf{R}^d} (\rho_n^0 - 1)(1 - \eta_n) \psi(0,x) \mathrm{d}x \\ &\to \int_{\mathbf{R}^d} \rho^0 \psi(0,x) \mathrm{d}x,\end{aligned}$$

where the first term converges thank to Lemma 3.2.2 and the second term converges to 0 since for any ψ there exists $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$, one has $supp(\psi) \cap supp((1-\eta_n)) = \emptyset$. We may thus pass to the limit on the left-hand-side. Thus (ρ, u) satisfies the continuity equation of (3.0.1). We proceed to the limit of (3.2.9). Firstly, we infer that

$$\int_{\mathbf{R}^d} \tilde{\rho}_n^0 \beta(\tilde{u}_n^0) \psi(0,x) \mathrm{d}x \to \int_{\mathbf{R}^d} \rho^0 \beta(u_n^0) \psi(0,x) \mathrm{d}x,$$

by splitting the integral as in (3.3.10), applying Lemma 3.2.2 to the first term and arguing with the support properties to dispose of the remainder. Indeed, $(\beta(\tilde{u}_n) - \beta(u_n))\eta_n \neq 0$ only on $\mathcal{Q}_n$ and thus

$$\int_{\mathbf{R}^d} \rho_n^0 (\beta(\tilde{u}_n) - \beta(u_n)) \eta_n \psi(0,x) \mathrm{d}x \to 0.$$

The last term converges to 0 as $\beta \in L^\infty$ and for sufficiently large n $supp(\psi) \cap supp((1-\eta_n)) = \emptyset$. By applying the convergence results of Lemma 3.3.3 and Lemma 3.3.4, we may pass to the limit on the left-hand-side of (3.3.16). We refer the reader to [125] for more details. Since $R_\beta^n \eta_n$ is uniformly bounded, there exists a measure μ_β such that

$$\left\langle R_\beta^n \eta_n, \psi \right\rangle \to \langle \mu_\beta, \psi \rangle .$$

Thus, (ρ, u) satisfies (3.2.8), (3.2.9) with $F = G = 0$. Similarly, since $\overline{R_\beta}^n \eta_n$ is uniformly bounded, there exists a measure $\overline{\mu_\beta}$ such that

$$\left\langle \overline{R_\beta}^n \eta_n, \psi \right\rangle \to \langle \overline{\mu_\beta}, \psi \rangle .$$

Given that V_n, K_n converge to 0 in the limit as $n \to \infty$, we infer that the compatibility conditions are satisfied in the limit. $\square$

3.4 Proof of the main Theorems

It remains to show that the truncated weak solution (ρ, u) provided by Theorem 3.3.1 is a finite energy weak solution to (3.0.1) with (3.0.2). We proceed as in [125]. Being a local argument, the non trivial far-field entails only minor changes and therefore we omit details.

Proof of Theorem 3.1.2. Given initial data $(\sqrt{\rho_0}, \sqrt{\rho_0}u_0)$ of finite energy, i.e. such that (3.0.3) is bounded, concatenating Lemma 3.2.2, Theorem 3.2.4 and Theorem 3.3.1 provides a truncated weak solution (ρ, u) to (3.0.1) with far-field condition (3.0.2) such that (3.1.1) and (3.1.2) are satisfied. It remains to show that (ρ, u) is a finite energy weak solution to (3.0.1) according to Definition 3.1.1. For that purpose, we recall that β in (3.2.9) is as stated in Definition 3.1.5. From Lemma 3.1.6 and the uniform bounds , we conclude by applying the dominated convergence Theorem that for any $1 \leq l \leq 3$ and any compact set $K \subset \mathbf{R}^d$,

$$
\begin{aligned}
&\rho\beta_\delta^l(u) \to \rho u^l \quad \text{in} \quad L^1((0,T) \times K), \qquad \sqrt{\rho}\beta_\delta^l(u) \to \sqrt{\rho}u^l \quad \text{in} \quad L^2(0,T) \times K),\\
&\rho^{\frac{\gamma}{2}}\nabla\rho^{\frac{\gamma}{2}}\nabla\beta_\delta^l(u) \to \rho^{\frac{\gamma}{2}}\nabla\rho^{\frac{\gamma}{2}} \quad \text{in} \quad L^1((0,T) \times K),\\
&\sqrt{\rho}(\sqrt{\nu}\mathbf{S}_\nu + \kappa\mathbf{S}_\kappa)\nabla\beta_\delta^l(u) \to \sqrt{\rho}(\sqrt{\nu}\mathbf{S}_\nu + \kappa\mathbf{S}_\kappa) \quad \text{in} \quad L^1((0,T) \times K).
\end{aligned}
$$

Further we have that

$$\|\mu_\beta\|_{\mathcal{M}} + \|\overline{\mu}_\beta\|_{\mathcal{M}} \leq 2M.\delta.$$

Thus, performing the δ-limit in (3.2.9) yields a weak solution to the momentum equation of (3.0.1) which completes the proof. □

Next, we comment on the proof of Corollary 3.1.3.

Proof of Corollary 3.1.3. If $\kappa = 0$, the BD entropy of the initial data yields a $L^2(\mathbf{R}^d)$ bound for $\nabla\sqrt{\rho_0}$. We may then construct periodic initia data on $\mathbf{T}_n^d$ by means of Lemma 3.2.2. Concatenating Theorem 3.2.4 and Theorem 3.3.1 provides a finite energy weak solution to (3.0.1) that additionally satisfies the truncated formulation. The capillary tensor $\widetilde{\mathbf{S}_{\kappa,n}}$ is uniformly bounded in $L^2_{loc}(0,T; L^2(\mathbf{R}^d))$ and since $\kappa_n \to 0$ the corresponding contribution in (3.3.16) satisfies

$$\int_0^T \int_{\mathbf{R}^d} 2\kappa_n\widetilde{\mathbf{S}_{\kappa,n}}\beta'(\tilde{u}_n) \cdot \nabla\psi \mathrm{d}x\mathrm{d}t \to 0.$$

Thus, the pair $(\sqrt{\rho}, u)$ is a truncated weak solution to (3.0.1) with $\kappa = 0$. Proceeding as in the proof of Theorem 3.1.2 we carry out the δ-limit to obtain a finite energy weak solution. □

Finally, we sketch the proof of Theorem 3.1.4. It follows the strategy of proof of Theorem 3.1.2. Minor modifications are necessary to the different far-field behavior. However, gaining integrability for ρ in the present setting simplifies the proof.

Proof of Theorem 3.1.4. If the system is considered with trivial far-field behavior, then the internal energy is given by (3.1.3). Let $(\sqrt{\rho^0}, \sqrt{\rho^0}u^0)$ be initial data of finite energy and BD-entropy. One easily verifies that Lemma 3.2.2 is still valid and provides periodic initial data.

We then concatenated Theorem 3.2.4 and Theorem 3.3.1 to obtain a truncated weak solution on the whole space, where one only needs to substitute the renormalized internal energy by (3.1.3) and adapt the related uniform bounds. The δ-limit is then performed analogously to the proof of Theorem 3.1.2. □

3.5 Energy and BD entropy inequality

Definition 3.1.1 requires a finite energy weak solution to satisfy the energy inequality and entropy inequality as stated in (3.1.1) and (3.1.2) respectively. Both inequalities include a constant $C \geq 1$ on the right-hand side. This is motivated by the fact that the result in [125], which we exploit in order to get a truncated weak solution in each periodic domain $\mathbf{T}_n^d$, does not yield a solution with the natural energy inequality, but only a general estimate as in (3.1.1) and (3.1.2). In view of the analysis of the low Mach number limit, we address the question whether there exists weak solutions that satisfy (3.1.1) and (3.1.2) with $C = 1$. For suitable well-prepared data, we aim to show that finite energy weak solutions strongly converge to a Leray solutions of the incompressible Navier-Stokes equations in the low Mach number limit. In this section, we prove that the solutions to (3.0.1) in $\mathbf{T}^d$ constructed in [13] do satisfy (3.1.1) and (3.1.2) with $C = 1$. The approach in [13] requires a constraint on the coefficients ν and κ but on the other hand the weak solutions constructed there are obtained as limit of smooth solutions to an approximating system retaining the same *a priori* bounds as (3.0.1).
Based on this existence result on the torus, one may then construct finite energy weak solutions on $\mathbf{R}^3$ with far-field (3.0.2) that satisfy the desired energy inequality and entropy inequality. Indeed, the procedure of invading domains does not entail additional constant on the right-hand side of the energy inequality, see also Lemma 3.2.2.
We shall consider initial data (ρ^0, u^0) of finite energy, i.e. $E(\rho^0, u^0) < +\infty$ satisfying the following assumptions,

$$\begin{aligned} \rho^0 \geq 0 \quad \text{in} \quad \mathbf{T}^3, \\ \rho \in L^1(\mathbf{T}^3) \cap L^\gamma(\mathbf{T}^3), \\ \nabla\sqrt{\rho} \in L^2(\mathbf{T}^3), \end{aligned} \tag{3.5.1}$$

and

$$\begin{aligned} u_0 = 0 \quad \text{on} \quad \{\rho^0 = 0\}, \\ \sqrt{\rho^0} u^0 \in L^2(\mathbf{T}^3) \cap L^{2+}(\mathbf{T}^3). \end{aligned} \tag{3.5.2}$$

Theorem 3.5.1. *Let $d = 3$. Let ν, κ and γ positive such that $\kappa^2 < \nu^2 < \frac{9}{8}\kappa^2$ and $1 < \gamma < 3$. Then for any $0 < T < \infty$ there exists a finite energy weak solution (ρ, u) of (3.0.1) on $(0,T)\times\mathbf{T}^3$ with initial data (ρ^0, u^0) of finite energy satisfying (3.5.1) and (3.5.2). In particular, (ρ, u) satisfies (3.1.1) and (3.1.2). Moreover for a.e. $0 \leq s < t < T$ one has*

$$E(t) + \int_s^t |\mathbf{S}_\nu(t')|^2 \mathrm{d}x\mathrm{d}t' \leq E(s). \tag{3.5.3}$$

Firstly, we recall needed uniform estimates and compactness results obtained in [13] and secondly, we show Proposition 3.5.7 and Proposition 3.5.8 that imply Theorem 3.5.1. The weak solution provided in [13] is obtained as limit of a sequence of approximating solutions $\{(\rho_\delta, u_\delta)\}_\delta$ satisfying the following system.

$$\begin{cases} \partial_t \rho_\delta + \operatorname{div} \rho_\delta u_\delta = 0 \\ \partial_t(\rho_\delta u_\delta) + \operatorname{div}(\rho_\delta u_\delta \otimes u_\delta) + \nabla\left((\rho_\delta)^\gamma + P_\delta(\rho_\delta)\right) + \tilde{p}_\delta(\rho_\delta) u_\delta = \kappa^2 \operatorname{div} \mathcal{K}_\delta + 2\nu \operatorname{div}(\mathcal{S}_\delta), \end{cases} \tag{3.5.4}$$

with initial data

$$\begin{aligned} \rho_\delta(0,x) &= \rho_\delta^0(x), \\ \rho u(0,x) &= \rho_\delta^0(x) u_\delta^0(x). \end{aligned} \tag{3.5.5}$$

The approximating viscosity term is defined as

$$\mathcal{S}_\delta = h_\delta(\rho_\delta) \mathbf{D} u_\delta + g_\delta(\rho_\delta) \operatorname{div} u_\delta \mathbf{I},$$

with

$$h_\delta = \rho_\delta + \delta(\rho_\delta)^{\frac{7}{8}} + \delta(\rho_\delta)^\gamma, \qquad g_\delta = \rho_\delta h_\delta'(\rho_\delta) - h_\delta(\rho_\delta). \tag{3.5.6}$$

The approximating dispersive term reads

$$\operatorname{div} \mathcal{K}_\delta = 2\rho_\delta \nabla \left(\frac{h_\delta'(\rho_\delta) \operatorname{div}(h_\delta'(\rho_\delta) \nabla \sqrt{\rho_\delta})}{\sqrt{\rho_\delta}} \right).$$

Initial data

Next, we specify the initial data for which the system (3.5.4) is considered. Given initial data (ρ^0, u^0) of finite energy satisfying (3.5.1) and (3.5.2) one may construct a sequence of smooth initial data $(\rho^{0,\delta}, u^{0,\delta})$ such that

$$\begin{aligned} &\rho_\delta^0 \to \rho^0 \quad \text{strongly in} \quad L^1(\mathbf{T}^d), \\ &\{\rho_\delta^0\}_\delta \quad \text{uniformly bounded in} \quad L^1 \cap L^\gamma(\mathbf{T}^d), \\ &\{h_\delta(\rho_\delta^0) \nabla \sqrt{\rho_\delta^0}\}_\delta \quad \text{uniformly bounded in} \quad L^2 \cap L^{2+\eta}(\mathbf{T}^d), \\ &h_\delta(\rho_\delta^0) \nabla \sqrt{\rho^{\delta,0}} \to \nabla \sqrt{\rho^0} \quad \text{strongly in} \quad L^2(\mathbf{T}^d) \\ &\{\sqrt{\rho_\delta^0} u_\delta^0\}_\delta \quad \text{uniformly bounded in} \quad L^2 \cap L^{2+\eta}(\mathbf{T}^d), \\ &\rho_\delta^0 u_\delta^0 \to \rho^0 u^0 \quad \text{in} \quad L^1(\mathbf{T}^d), \\ &f_\delta(\rho_\delta^0) \to 0 \quad \text{strongly in} \quad L^1(\mathbf{T}^d) \end{aligned} \tag{3.5.7}$$

In virtue of Theorem 6 in [13], there exists a global smooth solution to the Cauchy problem (3.5.4) equipped with initial data as specified as in (3.5.7).

Proposition 3.5.2. *Let $\nu, \kappa > 0$ such that $\kappa^2 < \nu^2 < \frac{9}{8}\kappa^2$ and $\gamma \in (1,3)$. Then, for $\delta > 0$ sufficiently small, there exists a global smooth solution of* (3.5.4) *with initial data* (3.5.5).

***A priori* bounds**

We provide the uniform estimates that will be needed subsequently, for their proof we refer the reader to [13]. These bounds are obtained from the energy equality and the Bresch-Desjardins entropy inequality for the system (3.5.4). The energy functional for the approximating system (3.5.4) is defined for $t \in [0, T)$ as

$$E_\delta(t) = \int_{\mathbf{T}^d} \frac{1}{2}\rho_\delta |u_\delta|^2 + \frac{\kappa^2}{2}|h'_\delta(\rho_\delta)\nabla\sqrt{\rho_\delta}|^2 + \frac{1}{(\gamma-1)}(\rho_\delta)^\gamma + f_\delta(\rho_\delta)\mathrm{d}x. \tag{3.5.8}$$

Firstly, we recall the energy inequality for the system (3.5.4).

Lemma 3.5.3. *Let (ρ_δ, u_δ) be a global smooth solution of* (3.5.4)*. Then for any* $0 \le s < t \le T$ *and (ρ_δ, u_δ),*

$$E_\delta(t) + 2\nu \int_0^t \int_{\mathbf{T}^d} h_\delta(\rho_\delta)|\mathbf{D}u_\delta|^2 + g_\delta(\rho_\delta)|\operatorname{div} u_\delta|^2 + \tilde{p}_\delta(\rho_\delta)|u_\delta|^2 \mathrm{d}x\mathrm{d}t = E_\delta(s) \tag{3.5.9}$$

Secondly, given a smooth solution (ρ_δ, u_δ) of (3.5.4), we introduce the effective velocity $v_\delta = u_\delta + c\nabla\phi_\delta(\rho_\delta)$, for some suitable constant c. Then (ρ_δ, v_δ) is a smooth solution of the viscous Euler system

$$\begin{cases} \partial_t\rho_\delta + \operatorname{div}(\rho_\delta v_\delta) = c\Delta h_\delta(\rho_\delta), \\ \partial_t(\rho_\delta v_\delta) + \operatorname{div}(\rho_\delta v_\delta \otimes v_\delta) + \nabla(\rho_\delta)^\gamma + \tilde{\lambda}\nabla p_\delta(\rho_\delta) - c\Delta(h_\delta(\rho_\delta)v_\delta) + \tilde{p}(\rho_\delta)v_\delta \\ -2(\nu - c)\operatorname{div}(h_\delta(\rho_\delta)\mathbf{D}v_\delta) - 2(\nu-c)\nabla(g_\delta(\rho_\delta)\operatorname{div} v_\delta) - \tilde{\kappa}^2 \operatorname{div}\mathcal{K}_\delta = 0, \end{cases} \tag{3.5.10}$$

where the function ϕ is defined as in [13] and

$$\begin{cases} \mu = \nu - \sqrt{\nu^2 - \kappa^2}, \\ \tilde{\kappa}^2 = \kappa^2 - 2\nu c + c^2, \\ \tilde{\lambda} = (\mu - c)/\mu. \end{cases}$$

The Bresch-Desjardins entropy is defined as

$$B_\delta(t) = \int_{\mathbf{T}^d} \frac{1}{2}\rho_\delta|v_\delta|^2 + \frac{(\rho_\delta)^\gamma}{(\gamma-1)} + \tilde{\lambda} f_\delta(\rho_\delta) + 2\tilde{\kappa}^2 \left|h'_\delta(\rho)\nabla\sqrt{\rho_\delta}\right|^2. \tag{3.5.11}$$

By Proposition 2 in [13], any smooth solution (ρ_δ, v_δ) of (3.5.10) satisfies the related energy inequality.

Lemma 3.5.4. *Let (ρ_δ, u_δ) be a global smooth solution of* (3.5.4)*. Given $c \in (0, \mu)$, the pair*

(ρ, v_δ) *is a smooth solution of* (3.5.10) *and the BD entropy inequality is satisfied ,*

$$
\begin{aligned}
&B_\delta(t) + c\int_0^t\int_{\mathbf{T}^d} h_\delta(\rho_\delta)|\mathbf{A}v_\delta|^2 \mathrm{d}x dt + (2\nu - c)\int_0^t\int_{\mathbf{T}^d} h_\delta(\rho_\delta)|Dv_\delta|^2 + g_\delta(\rho_\delta)|\operatorname{div} v_\delta|^2 \mathrm{d}x \mathrm{d}t \\
&\quad + \int_0^t\int_{\mathbf{T}^d} \tilde{p_\delta}|v_\delta|^2 \mathrm{d}x dt + c\gamma\int_0^t\int_{\mathbf{T}^d} h'_\delta(\rho_\delta)|\nabla\rho_\delta|^2(\rho_\delta)^{\gamma-2}\mathrm{d}x\mathrm{d}t + c\tilde{\lambda}\int_0^t\int_{\mathbf{T}^d} h'_\delta(\rho_\delta)|\nabla\rho_\delta|^2 f''_\delta(\rho_\delta)\mathrm{d}x\mathrm{d}t \\
&\quad + c\tilde{\kappa}^2\int_0^t\int_{\mathbf{T}^d} h_\delta(\rho_\delta)|\nabla^2\phi_\delta(\rho_\delta)|^2\mathrm{d}x\mathrm{d}t + c\tilde{\kappa}^2\int_0^t\int_{\mathbf{T}^d} g_\delta(\rho_\delta)|\Delta\phi_\delta(\rho_\delta)|^2\mathrm{d}x\mathrm{d}t \\
&\quad \leq B_\delta(0)
\end{aligned}
\tag{3.5.12}
$$

We summarize the needed *a priori bounds* that are consequences of Lemma 3.5.3 and Lemma 3.5.4.

Lemma 3.5.5. *Let* (ρ_δ, u_δ) *be a smooth solution of* (3.5.4) *with initial data satisfying* $\rho_\delta > 0$ *and assumptions in 3.5 , then there exists* $C > 0$ *independent from* δ *such that*

$$\sup_t \int \rho_\delta|u_\delta|^2\mathrm{d}x \leq C, \qquad \sup_t \int \left|h'_\delta(\rho_\delta)\nabla\sqrt{\rho_\delta}\right|^2 \mathrm{d}x \leq C \tag{3.5.13}$$

$$\sup_t \int (\rho_\delta + (\rho_\delta)^\gamma)\,\mathrm{d}x \leq C \qquad \int\int h_\delta(\rho_\delta)|\mathbf{D}u_\delta|^2\mathrm{d}x\mathrm{d}t \leq C \tag{3.5.14}$$

$$\sup_t \int f_\delta(\rho_\delta)\mathrm{d}x \leq C, \qquad \int\int |\tilde{p}(\rho_\delta)|\,|u_\delta|^2\,\mathrm{d}x\mathrm{d}t. \tag{3.5.15}$$

In particular,

$$\sup_t \int |\nabla\sqrt{\rho_\delta}|^2\,\mathrm{d}x \leq C, \qquad \int\int \rho_\delta\,|\mathbf{D}u_\delta|^2\,\mathrm{d}x\mathrm{d}t. \tag{3.5.16}$$

Moreover, we recall the following convergence results from [13].

Lemma 3.5.6. *Let* (ρ_δ, u_δ) *be a smooth solution of* (3.5.4). *Then*

$$
\begin{aligned}
h_\delta(\rho_\delta) - \rho_\delta \to 0 &\quad \textit{strongly in} \quad L^1((0,T)\times\mathbf{T}^d), \\
h'_\delta(\rho_\delta)\sqrt{\rho_\delta} \to \sqrt{\rho} &\quad \textit{strongly in} \quad L^2((0,T)\times\mathbf{T}^d), \\
h'_\delta(\rho_\delta)\nabla\sqrt{\rho_\delta} \to \nabla\sqrt{\rho} &\quad \textit{strongly in} \quad L^2((0,T)\times\mathbf{T}^d), \\
(\rho_\delta)^\gamma \to \rho^\gamma &\quad \textit{strongly in} \quad L^1((0,T)\times\mathbf{T}^d), \\
p(\rho_\delta) \to 0 &\quad \textit{strongly in} \quad L^1((0,T)\times\mathbf{T}^d), \\
\tilde{p}(\rho_\delta) \to 0 &\quad \textit{strongly in} \quad L^1((0,T)\times\mathbf{T}^d), \\
\sqrt{\rho_\delta}u_\delta \to \sqrt{\rho}u &\quad \textit{strongly in} \quad L^2((0,T)\times\mathbf{T}^d),
\end{aligned}
\tag{3.5.17}
$$

Given the construction of the sequence of approximating solutions done in [13],in what follows we show that finite energy weak solutions (ρ, u) obtained as limit of (ρ_δ, u_δ) satisfy (3.1.1). The key-point consists in observing that, the energy dissipation for the approximating system (3.5.4) is bounded from below by

$$\int_0^t\int_{\mathbf{T}^d} \rho_\delta|\mathbf{D}u_\delta|^2\mathrm{d}x\mathrm{d}t \leq \int_0^t\int_{\mathbf{T}^d} h_\delta(\rho_\delta)|\mathbf{D}u_\delta|^2 + g_\delta(\rho_\delta)|\operatorname{div} u_\delta|^2 + \tilde{p}_\delta(\rho_\delta)|u_\delta|^2\mathrm{d}x\mathrm{d}t, \tag{3.5.18}$$

Indeed, by definition $\gamma > 1$ and

$$h_\delta(\rho_\delta) \geq \rho_\delta \geq 0, \qquad g_\delta(\rho_\delta) = -\frac{1}{8}\varepsilon(\rho_\delta)^{\frac{7}{8}} + \varepsilon(\gamma-1)(\rho_\delta)^\gamma.$$

Given a vector valued function u such that $\nabla u \in L^2(\mathbf{T}^d)$, it holds

$$\|\operatorname{div} u\|_{L^2(\mathbf{T}^d)} \leq C\sqrt{d}\|\mathbf{D}u\|_{L^2(\mathbf{T}^d)}.$$

Therefore,

$$\begin{aligned}
&\int_0^t \int_{\mathbf{T}^d} h_\delta(\rho_\delta)|\mathbf{D}u_\delta|^2 + g_\delta(\rho_\delta)|\operatorname{div} u_\delta|^2 \mathrm{d}x\mathrm{d}t \\
&\geq \int_0^t \int_{\mathbf{T}^d} \rho_\delta|\mathbf{D}u_\delta|^2 + \varepsilon\left((\rho_\delta)^{\frac{7}{8}} + (\rho_\delta)^\gamma\right)|\mathbf{D}u_\delta|^2 - \frac{3}{8}\varepsilon(\rho_\delta)^{\frac{7}{8}}|\mathbf{D}u_\delta|^2 \mathrm{d}x\mathrm{d}t \\
&\geq \int_0^t \int_{\mathbf{T}^d} \rho_\delta|\mathbf{D}u_\delta|^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Observing that $\tilde{p}(\rho_\delta)|u_\delta|^2 \geq 0$, we conclude (3.5.18).

Proposition 3.5.7. *Let (ρ, u) be a finite energy weak solution of* (3.0.1) *on $(0,T) \times \mathbf{T}^3$ obtained as limit of a sequence $\{(\rho_\delta, u_\delta)\}$ smooth solution of* (3.5.4). *Then (ρ, u) satisfies* (3.1.1).

Proof. First, we observe that from (3.5.7) we conclude that

$$\begin{aligned}
&\int_{\mathbf{T}^d} \frac{1}{2}\rho^{\delta,0}|u^{\delta,0}|^2 + \frac{\kappa^2}{2}|h'_\delta(\rho^{\delta,0})\nabla\sqrt{\rho^{\delta,0}}|^2 + \frac{1}{(\gamma-1)}(\rho^{\delta,0})^\gamma + f_\delta(\rho^{\delta,0})\mathrm{d}x \\
&\to \int_{\mathbf{T}^d} \frac{1}{2}\rho^0|u^0|^2 + \frac{\kappa^2}{2}|\nabla\sqrt{\rho^0}|^2 + \frac{1}{\gamma-1}(\rho^0)^\gamma \mathrm{d}x.
\end{aligned}$$

Next, we notice that since $\sqrt{\rho_\delta}\mathbf{D}u_\delta \in L^2(0,T;L^2(\mathbf{T}^d))$, there exists $\mathbf{S}_\nu \in L^2(0,T;L^2(\mathbf{T}^d)$ such that $\sqrt{\rho_\delta}\mathbf{D}u_\delta \rightharpoonup \mathbf{S}_\nu$ weakly in $L^2(0,T;L^2(\mathbf{T}^d)$ as $\delta \to 0$. By exploiting the lower semi-continuity of the energy functional, we conclude

$$\begin{aligned}
&\int_{\mathbf{T}^d} \frac{1}{2}\rho|u|^2 + \frac{\kappa^2}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{\gamma-1}\rho^\gamma \mathrm{d}x + 2\nu\int_0^t \int_{\mathbf{T}^d} |\mathbf{S}_\nu|^2 \,\mathrm{d}x\mathrm{d}t \\
&\leq \liminf_{\delta\to 0} E_\delta(t) + 2\nu\int_0^t \int_{\mathbf{T}^d} h_\delta(\rho_\delta)|\mathbf{D}u_\delta|^2 + g_\delta(\rho_\delta)|\operatorname{div} u_\delta|^2 + \tilde{p}_\delta(\rho_\delta)|u_\delta|^2 \mathrm{d}x\mathrm{d}t \\
&\leq \liminf_{\delta\to 0} E_\delta(0) = E(0)
\end{aligned}$$

It remains to check that $\mathbf{S}_\nu$ is such that (3.0.5) is satisfied. From Lemma 3.5.6 we conclude that $\sqrt{\rho_\delta}\mathbf{D}u_\delta \rightharpoonup \sqrt{\rho}\mathbf{S}_\nu$ in $L^1((0,T) \times \mathbf{T}^d)$. We are left to show that

$$\sqrt{\rho}\mathbf{S}_\nu = (\nabla(\rho u) - \nabla\sqrt{\rho} \otimes \sqrt{\rho}u)^{sym} \quad \text{in} \quad \mathcal{D}'((0,T) \times \mathbf{T}^d).$$

Let $\phi \in \mathcal{D}((0,T)\times \mathbf{T}^d)$ and consider

$$\langle \sqrt{\rho_\delta}\mathbf{D}u_\delta, \phi\rangle = \langle (\nabla(\rho_\delta u_\delta))^{sym}, \phi\rangle - 2\langle (\nabla\sqrt{\rho_\delta}\otimes\sqrt{\rho_\delta}u_\delta)^{sym}, \phi\rangle$$

From $\rho_\delta u_\delta \to \rho u$ in $L^1((0,T)\times\mathbf{T}^d)$ as well as $\nabla\sqrt{\rho_\delta}\to\nabla\sqrt{\rho}$ and $\sqrt{\rho_\delta}u_\delta \to \sqrt{\rho}u$ both strongly in $L^2((0,T)\times\mathbf{T}^d)$ we conclude that

$$(RHS) \to \nabla(\rho u)^{sym} - 2\left(\nabla\sqrt{\rho_\delta}\otimes\sqrt{\rho_\delta}u_\delta\right)^{sym}$$

in $\mathcal{D}'((0,T)\times\mathbf{T}^d)$. □

Proposition 3.5.8. *Let (ρ,u) be a finite energy weak solution of* (3.0.1) *on* $(0,T)\times\mathbf{T}^3$ *obtained as limit of a sequence* $\{(\rho_\delta,u_\delta)\}$ *smooth solution of* (3.5.4). *Then (ρ,u) satisfies* (3.1.2).

Proof. We denote by $\mathbf{A}_\nu$ the weak $L^2(0,T;L^2(\mathbf{T}^d))$-limit of $\sqrt{\rho_\delta}\mathbf{A}u_\delta$. Similarly to the considerations made for $\mathbf{S}_\nu$, we show that $\sqrt{\rho}\mathbf{A}_\varepsilon = \sqrt{\rho}\mathbf{T}_\nu^{asymm}$ with $\mathbf{T}_\nu$ as defined in (3.0.5). Thus,

$$\|\mathbf{A}_\nu\|^2_{L^2(0,T;L^2(\mathbf{T}^d))} \le \liminf_{\delta\to0}\|\sqrt{\rho_\delta}\mathbf{A}u_\delta\|^2_{L^2(0,T;L^2(\mathbf{T}^d))} \le \liminf_{\delta\to0}\|\sqrt{h_\delta(\rho_\delta)}\mathbf{A}u_\delta\|^2_{L^2(0,T;L^2(\mathbf{T}^d))}$$

Further, we have that $\nabla^2\sqrt{\rho_\delta}$ converges weakly to $\nabla^2\sqrt{\rho}$ in $L^2(0,T;L^2(\mathbf{T}^d))$ from Lemma 3.5.6, thus by Lemma 5.3 in [13] we conclude that

$$\|\nabla^2\sqrt{\rho}\|^2_{L^2(0,T;L^2(\mathbf{T}^d))} \le \liminf_{\delta\to0} C\int_0^t\int_{\mathbf{T}^d} h_\delta(\rho_\delta)\left|\nabla^2\phi_\delta(\rho_\delta)\right|^2 + g_\delta(\rho_\delta)|\Delta\phi_\delta(\rho_\delta)|^2 \mathrm{d}x\mathrm{d}t.$$

Moreover,

$$c\gamma\int_0^t\int_{\mathbf{T}^d}|\nabla\rho|^2\rho^{\gamma-2}\mathrm{d}x\mathrm{d}t \le \liminf_{\delta\to0} c\gamma\int_0^t\int_{\mathbf{T}^d} h'_\delta(\rho_\delta)|\nabla\rho_\delta|^2(\rho_\delta)^{\gamma-2}\mathrm{d}x\mathrm{d}t.$$

By observing that $B_\delta(0)\to B(0)$ and exploiting lower semi-continuity of norms, we infer (3.1.2). □

Part II

Low Mach number limit and its quantum counterpart

CHAPTER 4

Low Mach number limit for the Quantum Navier-Stokes equations

Abstract

In this chapter, we investigate the low Mach number limit for the quantum Navier-Stokes system considered in the three-dimensional space. For general ill-prepared initial data of finite energy, we prove strong convergence of finite energy weak solutions towards weak solutions of the incompressible Navier Stokes equations. Section 4.1 introduces the suitable scaling, the notion of solutions and the main results. In Section 4.2 we collect the required uniform bounds. Section 4.3 concerns the study of the linearized system of acoustic waves that exploits the enhanced dispersion given by the Bogoliubov dispersion relation. Once we have a control of the acoustic dispersion, the compactness provided by the uniform bounds leads to the strong convergence towards a weak solution to the incompressible Navier–Stokes equation in Section 4.4. Finally, we present the detailed dispersive analysis of the semigroup operator associated to the Bogoliubov dispersion relation in Section 4.5.

This chapter, based on [9] in collaboration with P. Antonelli and P. Marcati, presents the analysis the low Mach number limit the Quantum-Navier-Stokes equations (QNS) studied in the previous chapter. The system is considered on $(0,T)\times\mathbf{R}^3$,

$$\begin{cases} \partial_t\rho+\operatorname{div}(\rho u)=0, \\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\nabla P(\rho)=2\nu\operatorname{div}(\rho\mathbf{D}u)+2\kappa^2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases} \tag{4.0.1}$$

the unknowns are given by the mass density ρ and the fluid velocity field u. We consider a barotropic pressure given by the usual γ-law, i.e. $P(\rho)=\frac{1}{\gamma}\rho^\gamma$, with $\gamma>1$, ν and κ denote the viscosity and capillarity coefficients respectively. The energy we consider for this system (4.0.3) is given by

$$E(t)=\int_{\mathbf{R}^3}\frac{1}{2}\rho|u|^2+2\kappa^2|\nabla\sqrt{\rho}|^2+F(\rho)\mathrm{d}x, \tag{4.0.2}$$

where the internal energy takes the form (3.0.4). The finite energy assumption, more precisely the integrability of F encodes the far-field behavior,

$$\rho\to 1 \qquad \text{as} \quad |x|\to\infty.$$

As already said, system (4.0.1) enters the more general class of Navier-Stokes-Korteweg systems, see also the Introduction and Chapter 3.

After a suitable rescaling (see subsection 4.1.1), the system (4.0.1) reads,

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \frac{1}{\varepsilon^2}\nabla P(\rho_\varepsilon) = 2\nu \operatorname{div}(\rho_\varepsilon \mathbf{D} u_\varepsilon) + 2\kappa^2 \rho_\varepsilon \nabla\left(\frac{\Delta\sqrt{\rho_\varepsilon}}{\sqrt{\rho_\varepsilon}}\right), \end{cases} \tag{4.0.3}$$

with initial data

$$\rho_\varepsilon(0,x) = \rho_{\varepsilon,0},$$

$$(\rho_\varepsilon u_\varepsilon)(0,x) = \rho_{\varepsilon,0} u_{\varepsilon,0},$$

where $\varepsilon \ll 1$ is the scaled Mach number. Therefore the rescaled internal energy becomes

$$F_\varepsilon(\rho_\varepsilon) = \frac{\rho_\varepsilon^\gamma - 1 - \gamma(\rho_\varepsilon - 1)}{\varepsilon^2 \gamma(\gamma-1)}. \tag{4.0.4}$$

In the low Mach number regime, i.e. in the limit as $\varepsilon \to 0$, the dynamics of (4.0.3) is formally governed by the incompressible Navier-Stokes equations,

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u, \qquad \operatorname{div} u = 0. \tag{4.0.5}$$

The aim of this chapter is to rigorously study this limit in its full generality, i.e. by considering arbitrary finite energy initial data without imposing further regularity or smallness assumptions and in particular without being well-prepared. This class of initial data cannot provide in the limit smooth solutions to incompressible Navier-Stokes, for this reason we have to exclude the use of relative entropy methods [73], [76]. As it emerged during the analysis of the Cauchy Problem of (4.0.3) in Chapter 3, the system entails some mathematical difficulties due to the possible appearance of vacuum regions. Indeed, the degenerate viscosity prevents a suitable control of the velocity field in the vacuum. In particular this yields some problems in establishing the necessary compactness estimates on the convective term $\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon$. Global in time existence of finite energy weak solutions to (4.0.3) with far-field behavior is provided by Theorem 3.1.2. The proof of Theorem 3.1.2 is based on an invading domain approach based on the existence result in [125]. Besides the mathematical difficulties originated from the degenerate viscosity, in this framework we also have to cope with the lack of integrability of the mass density due to non-trivial boundary conditions at infinity. The precise notion of solution is given in Definition 4.1.1 below.
One of the main tools in this Chapter is provided by a class of suitable Strichartz estimates, that allow to capture more accurately the different dispersive scales involved in the propagation of the acoustic waves, as a consequence of the specific dispersion relation. Indeed, contrarily to the classical case where the fluctuations evolve accordingly to the classical wave equation [136, 68, 169], here in our problem the presence of the quantum term contributes in a non-trivial way to the dispersion relation, especially at high frequencies. The dispersion relation inferred here, see formula (4.3.1) below, is strictly related to the Bogoliubov spectrum describing excitations in a Bose-Einstein condensate, which predicts the superfluid behavior of the gas [34, 33, 163]. This is somehow reminiscent of the analysis of fluctuations done when studying

the quasi-neutral limit for a class of Navier-Stokes-Korteweg systems [71, 72]. The analysis related to the dispersion relation (4.3.1) can be regarded as the $\varepsilon-$version of the results in [95]. We will present this analysis in Section 4.5. Here we remark that since the dispersion relation (4.3.1) is not homogeneous, we cannot obtain our estimates by a rescaling argument and we need to adapt the proof in [95]. On the other hand, it should be remarked that if we perform a frequency splitting as in [26], then the estimates (4.3.9), (4.3.10) deteriorate at low frequencies. A more detailed explanation can be found in Section 4.3 and Section 4.5.

Furthermore, in the limit we recover a weak solution of the incompressible Navier-Stokes equation $u \in L^{\infty}(0,T;L^2(\mathbf{R}^3)) \cap L^2(0,T,\dot{H}^1(\mathbf{R}^3))$. We remark that we are able to obtain bounds on the gradient of the limiting solution to (4.0.5), even though at fixed $\varepsilon > 0$ only a weak version of the energy inequality (4.1.4) is available. The mathematical motivations for this have been addressed in Chapter 3. The validity of (4.1.4) is clarified in Section 4.1, see also the discussion in Section 3.5. However, this weak version of the energy inequality will anyway yield the aforementioned natural bounds on the gradient of the velocity field in the low Mach number limit. In fact, thanks to some uniform bounds satisfied by the momentum density, we can also infer further smoothing properties for the limiting solution to (4.0.5), see Theorem 4.1.2 and Proposition 4.4.5 for more details. Due to the presence of an initial layer which cannot be avoided for general ill-prepared data, the weak solution enters the Leray class only if further assumptions on the initial data are made. More precisely, only for well-prepared data it is possible to show that the solutions obtained in the limiting procedure satisfy the energy inequality.

The study of singular limits for fluid dynamical equations occupies a vast portion of mathematical literature, for a more comprehensive introduction to the topic we address the reader to the monograph [76] and the reviews [3, 138]. Our method shares some similarities with [68] which studies the compressible Navier-Stokes equations on the whole space. Indeed, there the authors exploit some Strichartz type estimates to analyse the acoustic waves. On the other hand, for the QNS system the dispersion relation is modified and reads as in formula (4.3.1); thus for high frequencies the fluctuations appearing in classical fluid dynamics and in system (4.0.1) differ considerably. Recently, the incompressible limit for a similar system has been investigated in [173] and later in [123]. In both papers the quantum Navier-Stokes system is augmented by adding a damping term in the momentum equation. This extra term allows to circumvent mathematical difficulties related to the lack of control of the velocity field in the vacuum. Moreover both papers deal with smooth local in time solutions for the limiting incompressible dynamics. By using this further regularity assumption, it is then possible for them to exploit a relative entropy method.

Here, we tackle the problem from a different perspective, namely we retrieve global weak solutions in the limit rather than convergence to the unique local strong solution to the limiting system. Moreover, while in [123, 173] the fluctuations are studied by using a wave-like dispersion as for classical fluid dynamical systems, here we consider the full dispersion relation determined by the Bogoliubov spectrum (4.3.1) and the enhanced dispersion leads to a better

control on the fluctuations. This is achieved by carrying out a refined analysis on the dispersive properties of the acoustic waves that together with new uniform estimates enables us to study the low Mach number limit for general ill-prepared initial data without regularity or smallness assumptions and without damping. For the inviscid system, i.e. the QHD system, the low Mach number limit with ill-prepared data has been studied in [73] on the torus. The authors consider - differently from the solutions considered in Chapter 2 - solutions that are not generalized irrotational. As we will elucidate in Chapter 5, the analogue of the low Mach number system for the QHD system (5.0.2) should rather be interpreted as scaling limit where the scaling parameter is given by the healing length being the characteristic length scale of the system.
This paper is organized as follows, we introduce notations and preliminary results in Section 4.1. Subsequently, the needed *a priori* estimates are provided in Section 4.2. This is particularly relevant since finite energy weak solutions of (4.0.3) only obey a weak form of the energy inequality, for a detailed discussion see Chapter 3 and in particular Section 3.5. Section 4.3 is dedicated to the analysis of the acoustic waves. The strong convergence of finite energy weak solutions of (4.0.3) to weak solutions of (4.0.4) is achieved in Section 4.4 by means of an Aubin-Lions compactness argument. Furthermore, we investigate the regularity properties of the limit u and show that u lies in the class of Leray solutions under suitable additional assumptions. Section 4.5 is devoted to the proof of the dispersive estimates.

4.1 Definitions and main results

In what follows C will be any constant independent from ε.

4.1.1 Scaling

Different scalings are reasonable see for instance the review papers [3, 138] and references therein. To recast the introduced scaling (4.0.3) of system (4.0.1), one starts writing the equations by re-scaling each length scale by its characteristic value (dimensionless scaling) and we assume the Mach number to be small. We expect the fluid to behave like an incompressible fluid on large time scales when the density is almost constant and the velocity is small. Thus, we introduce the change of variable and unknowns,

$$t \mapsto \varepsilon t, \qquad u \mapsto \varepsilon u_{\varepsilon}.$$

Moreover, the viscosity and capillarity coefficients scale as

$$\nu \mapsto \varepsilon \nu, \qquad \kappa \mapsto \varepsilon \kappa.$$

Weak solutions

The global in time existence of finite energy weak solutions to (4.0.3) is provided by Theorem 3.1.2. As pointed out in Chapter 3, the degenerate viscosity prevents the velocity field to be uniquely determined in the vacuum region; indeed system (4.0.3) lacks bounds for u_ε. Following up on the analysis of the Cauchy problem in Chapter 3, we notice that in this framework (ρ, Λ) are the suitable variables to tackle the problem. We encounter similar difficulties also in the analysis of the inviscid system (2.0.1). Mathematically speaking, this means that whenever the symbol ρ_ε appears, it should be read as $\rho_\varepsilon = (\sqrt{\rho_\varepsilon})^2$ and similarly for the momentum density $m_\varepsilon = \rho_\varepsilon u_\varepsilon = \sqrt{\rho_\varepsilon}\Lambda_\varepsilon$. At no moment neither the velocity field u_ε nor its gradient ∇u_ε are defined a.e. in $\mathbf{R}^3$. Analogue to (3.0.5), the tensor $\mathbf{T}_\varepsilon$ is defined through the following identity

$$\sqrt{\rho_\varepsilon}\mathbf{T}_\varepsilon = \nabla(\rho_\varepsilon u_\varepsilon) - 2\nabla\sqrt{\rho_\varepsilon} \otimes \Lambda_\varepsilon, \tag{4.1.1}$$

in $\mathcal{D}'((0,T)\times\mathbf{R}^3)$ and

$$\mathbf{S}_\varepsilon = \mathbf{T}_\varepsilon{}^{sym}, \tag{4.1.2}$$

in $\mathcal{D}'((0,T)\times\mathbf{R}^3)$. Under suitable assumptions on the mass density ρ the quantum pressure term can be alternatively rewritten as in (2.0.3) and (3.0.6). In this way the equation for the momentum density in (4.0.3) reads

$$\partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}\left(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon + 4\kappa^2\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\right) + \frac{1}{\varepsilon^2}\nabla P(\rho_\varepsilon) - 2\nu \operatorname{div}(\sqrt{\rho_\varepsilon}\mathbf{S}_\varepsilon) - \kappa^2\nabla\Delta\rho_\varepsilon = 0.$$

In particular, we observe that, at present it is not clear whether arbitrary finite energy weak solutions to (4.0.3) satisfy the following energy inequality,

$$E(t) + 2\nu\int_0^t\int_{\mathbf{R}^3}\rho_\varepsilon|\mathbf{D}u_\varepsilon|^2\mathrm{d}x\mathrm{d}t' \le E(0), \tag{4.1.3}$$

We therefore work with the weaker form of the energy inequality (4.1.4) whose validity has been discussed in Section 3.5.

For the sake of consistency with Chapter 3 and the literature on (quantum) Navier–Stokes equations, we avoid the use of Λ_ε in the present chapter. The above considerations motivate the notion of solution in terms of the mathematical unknowns $\sqrt{\rho_\varepsilon}$ and $\sqrt{\rho_\varepsilon}u_\varepsilon$ instead of the physical unknowns of density ρ_ε and momentum m_ε. For the convenience of the reader and in order to introduce the ε-dependent objects, we recall the definition of finite energy weak solutions.

Definition 4.1.1. *A pair $(\rho_\varepsilon, u_\varepsilon)$ with $\rho_\varepsilon \ge 0$ is said to be a finite energy weak solution of the Cauchy Problem* (4.0.3) *if*

(i) integrability conditions

$$\sqrt{\rho_\varepsilon} \in L^2_{loc}((0,T)\times\mathbf{R}^3); \qquad \sqrt{\rho_\varepsilon}u_\varepsilon \in L^2_{loc}((0,T)\times\mathbf{R}^3);$$
$$\nabla\sqrt{\rho_\varepsilon} \in L^2_{loc}((0,T)\times\mathbf{R}^3);$$

(ii) continuity equation

$$\int_{\mathbf{R}^3}\rho_\varepsilon^0\phi(0) + \int_0^T\int_{\mathbf{R}^3}\rho_\varepsilon\phi_t + \sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon}u_\varepsilon\nabla\phi = 0,$$

for any $\phi \in C_c^\infty([0,T)\times\mathbf{R}^3)$.

(iii) momentum equation

$$\int_{\mathbf{R}^d}\rho_\varepsilon^0 u_\varepsilon^0\psi(0)+\int_0^T\int_{\mathbf{R}^d}\sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon}u_\varepsilon\psi_t+(\sqrt{\rho_\varepsilon}u_\varepsilon\otimes\sqrt{\rho_\varepsilon}u_\varepsilon)\nabla\psi+\frac{1}{\varepsilon^2}\rho_\varepsilon^\gamma\operatorname{div}\psi$$
$$-2\nu\int_0^T\int_{\mathbf{R}^d}\left(\sqrt{\rho_\varepsilon}u_\varepsilon\otimes\nabla\sqrt{\rho_\varepsilon}\right)\nabla\psi-2\nu\int_0^T\int_{\mathbf{R}^d}\left(\nabla\sqrt{\rho_\varepsilon}\otimes\sqrt{\rho_\varepsilon}u_\varepsilon\right)\nabla\psi$$
$$+\nu_\varepsilon\int_0^T\int_{\mathbf{R}^d}\sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon}u_\varepsilon\Delta\psi+\nu_\varepsilon\int_0^T\int_{\mathbf{R}^d}\sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon}u_\varepsilon\nabla\operatorname{div}\psi$$
$$-4\kappa^2\int_0^T\int_{\mathbf{R}^d}\left(\nabla\sqrt{\rho_\varepsilon}\otimes\nabla\sqrt{\rho_\varepsilon}\right)\nabla\psi+2\kappa^2\int_0^T\int_{\mathbf{R}^d}\sqrt{\rho_\varepsilon}\nabla\sqrt{\rho_\varepsilon}\nabla\operatorname{div}\psi=0,$$

for any $\psi\in C_c^\infty([0,T)\times\mathbf{R}^3;\mathbf{R}^3)$.

(iv) there exists a tensor $\mathbf{T}_\varepsilon\in L^2((0,T)\times\mathbf{R}^3)$ *satisfying identity* (4.1.1) *in* $\mathcal{D}'((0,T)\times\mathbf{R}^3)$ *such that the following energy inequality holds for a.e.* $t\in[0,T]$,

$$E(t)+2\nu\int_0^t\int_{\mathbf{R}^3}|\mathbf{S}_\varepsilon|^2\mathrm{d}x\mathrm{d}t\le E(0), \tag{4.1.4}$$

where $\mathbf{S}_\varepsilon$ *is the symmetric part of* $\mathbf{T}_\varepsilon$, *i.e.* $\mathbf{S}_\varepsilon=\mathbf{T}_\varepsilon^{sym}$.

(v) Define

$$B_\varepsilon(t)=\int_{\mathbf{R}^3}\frac12\left|\sqrt{\rho_\varepsilon}u_\varepsilon\right|^2+2(\kappa^2+4\nu^2)\left|\nabla\sqrt{\rho_\varepsilon}\right|^2+F_\varepsilon(\rho_\varepsilon)\mathrm{d}x.$$

Then the Bresch-Desjardins entropy inequality holds for a.e. $t\in[0,T]$,

$$\begin{aligned}&B_\varepsilon(t)+\int_0^t\int_{\mathbf{R}^3}\frac12\left|\mathbf{A}_\varepsilon\right|^2\mathrm{d}x\mathrm{d}s+\nu\kappa^2\int_0^t\int_{\mathbf{R}^3}\left|\nabla^2\sqrt{\rho_\varepsilon}\right|^2+\left|\nabla\rho_\varepsilon^{\frac14}\right|^4\mathrm{d}x\mathrm{d}s+\frac{\nu\gamma}{\varepsilon^2}\int_0^t\int_{\mathbf{R}^3}\left|\nabla\rho_\varepsilon^{\frac{\gamma}{2}}\right|^2\mathrm{d}x\mathrm{d}s\\&\le\int_{\mathbf{R}^3}\frac12\left|\sqrt{\rho_\varepsilon^0}u_\varepsilon^0\right|^2+(2\kappa^2+4\nu^2)\left|\nabla\sqrt{\rho_\varepsilon^0}\right|^2+F_\varepsilon(\rho_\varepsilon^0)\mathrm{d}x,\end{aligned} \tag{4.1.5}$$

where $\mathbf{A}_\varepsilon=\mathbf{T}_\varepsilon^{asym}$, *with* $\mathbf{T}_\varepsilon$ *defined as in the previous point.*

We remark that Definition 4.1.1 differs from scaled version of Definition 3.1.1 by the constants $C>0$ appearing in (3.1.1) and (3.1.2). Here, we require $C=1$. In general, it is not clear whether the solutions provided by Theorem 3.1.2 of the previous chapter satisfy the energy and entropy inequality with $C=1$. This is due to the fact that Theorem 3.1.2 is proven by an invading domain approach that is based on the periodic solutions introduced in [125]. The result in [125] is on its turn obtained by means of a truncation procedure during which the appearance of constant C in the inequalities occurs. However, as pointed out in Section 3.5, there exists weak solutions to (4.0.3) compatible with Definition 4.1.1. Indeed, based on the result of [13] one may construct suitable finite energy weak solutions satisfying (4.1.4) and (4.1.5) at the expense of a restriction on the coefficients ν and κ.

Main result

Let us specify the assumptions on the initial data for the system (4.0.3). We consider initial data $(\rho_\varepsilon^0, u_\varepsilon^0)$ such that

$$\|\nabla\sqrt{\rho_\varepsilon^0}\|_{L^2(\mathbf{R}^3)} \leq C, \quad \|\sqrt{\rho_\varepsilon^0}u_\varepsilon^0\|_{L^2(\mathbf{R}^3)} \leq C, \quad \|F_\varepsilon(\rho_\varepsilon^0)\|_{L^1(\mathbf{R}^3)} \leq C, \tag{4.1.6}$$

where is C independent on $\varepsilon > 0$. Furthermore, we assume that

$$\sqrt{\rho_\varepsilon^0}u_\varepsilon^0 \rightharpoonup u_0 \quad \text{in } L^2(\mathbf{R}^3). \tag{4.1.7}$$

With this definition at hand, we now state the main Theorem characterising the low Mach number regime for (4.0.3).

Theorem 4.1.2. *Let $1 < \gamma < 3$, let $(\rho_\varepsilon, u_\varepsilon)$ be a finite energy weak solution of (4.0.3) with initial data satisfying (4.1.6) and (4.1.7) and let $0 < T < \infty$ be an arbitrary time. Then $\rho_\varepsilon - 1$ converges strongly to 0 in $L^\infty(0,T;L^2(\mathbf{R}^3)) \cap L^4(0,T;H^s(\mathbf{R}^3))$ for any $0 \leq s < 1$. For any subsequence (not relabeled) $\sqrt{\rho_\varepsilon}u_\varepsilon$ converging weakly to u in $L^\infty(0,T,L^2(\mathbf{R}^3))$, then $u \in L^\infty(0,T;L^2(\mathbf{R}^3)) \cap L^2(0,T;\dot{H}^1(\mathbf{R}^3))$ is a global weak solution to the incompressible Navier-Stokes equation (4.0.5) with initial data $u_{|t=0} = \mathbf{P}(u_0)$ and $\sqrt{\rho_\varepsilon}u_\varepsilon$ converges strongly to u in $L^2(0,T;L^2_{loc}(\mathbf{R}^3))$.*
Moreover, $\mathbf{Q}(\rho_\varepsilon u_\varepsilon)$ converges strongly to 0 in $L^2(0,T,L^q(\mathbf{R}^3))$ for any $2 < q < \frac{9}{4}$. Finally the limiting solution u also satisfies $u \in L^{\frac{4}{1+4s}-}(0,T;H^s(\mathbf{R}^3))$, for $0 \leq s \leq \frac{1}{2}$.

Remark 4.1.3. Let us remark that in order for the limiting function u to satisfy the energy inequality, i.e. to be a Leray weak solution [127], stronger assumptions on the initial data $(\rho_\varepsilon^0, u_\varepsilon^0)$ are needed. Indeed the initial total energy for the compressible system in general does not converge, as $\varepsilon \to 0$, to the initial energy for (4.0.5), which would be given by $\frac{1}{2}\int |\mathbf{P}u_0|^2$. The excess energy determines an initial layer which cannot be avoided for ill-prepared data. On the other hand, if we require

$$\begin{aligned} &\sqrt{\rho_\varepsilon^0}u_\varepsilon^0 \to u_0 = \mathbf{P}(u_0) \quad \text{strongly in} \quad L^2(\mathbf{R}^3), \\ &F_\varepsilon(\rho_\varepsilon^0) \to 0 \quad \text{strongly in} \quad L^1(\mathbf{R}^3), \\ &\nabla\sqrt{\rho_\varepsilon^0} \to 0 \quad \text{strongly in} \quad L^2(\mathbf{R}^3) \end{aligned} \tag{4.1.8}$$

then the following Proposition holds true.

Proposition 4.1.4. *Under the same assumptions of Theorem 4.1.2, let $\rho_\varepsilon^0, u_\varepsilon^0$ further satisfy (4.1.8). Then the limiting solution u to (4.0.5) satisfies the energy inequality*

$$\int_{\mathbf{R}^3} \frac{1}{2}|u(t)|^2 \mathrm{d}x + \nu \int_0^t \int_{\mathbf{R}^3} |\nabla u|^2 \mathrm{d}x\mathrm{d}t' \leq \int_{\mathbf{R}^3} \frac{1}{2}|u_0|^2 \mathrm{d}x, \tag{4.1.9}$$

for almost every $t \in [0,T]$.

4.2 Uniform estimates

In this Section, we start our analysis on the low Mach number limit by inferring some uniform estimates for finite energy weak solutions to (4.0.3). In our framework we need to take into account the non trivial boundary conditions for the mass density. For this reason, we provide some estimates on the quantities $\sqrt{\rho_\varepsilon}-1$ and $\rho_\varepsilon-1$. Furthermore, the lack of control for ∇u_ε in the vacuum will be compensated by the bounds inferred from (4.1.4) and (4.1.5).
The techniques to obtain integrability properties for $\sqrt{\rho_\varepsilon}-1$ are in sense reminiscent of the analysis of the energy space $\mathbb{E}_2$ in Section 1.1.1 and the study of the far-field for (2.0.1) in Chapter 2. Indeed, we shall use repeatedly the following observation, see for instance Theorem 4.5.9 in [104]: if $f\in\mathcal{D}'(\mathbf{R}^d)$ with $\nabla f\in L^p(\mathbf{R}^d)$ for $p<d$, then there exists a constant c such that $f-c\in L^{p^*}(\mathbf{R}^d)$, where p^* is the critical Sobolev exponent. The condition $p<d$ is sharp.
Firstly, we state some *a priori* bounds on the initial data.

4.2.1 Initial data of finite energy

If the initial data $(\rho_\varepsilon^0,u_\varepsilon^0)$ is assumed to be of finite energy, i.e. $E(\rho_\varepsilon^0,u_\varepsilon^0)<\infty$, then

$$\nabla\sqrt{\rho_\varepsilon^0}\in L^2(\mathbf{R}^3),\qquad \sqrt{\rho_\varepsilon^0}u_\varepsilon^0\in L^2(\mathbf{R}^3),\qquad \frac{\rho_\varepsilon^0-1-\gamma(\rho_\varepsilon^0-1)}{\varepsilon^2\gamma(\gamma-1)}\in L^1(\mathbf{R}^3).$$

This implies additional bounds which we summarize.

Lemma 4.2.1. *If the initial data $(\rho_\varepsilon^0,u_\varepsilon^0)$ is of finite energy, then there exists $C>0$ independent from $\varepsilon>0$ such that*

(i) $\|\rho_\varepsilon^0-1\|_{L^2}\leq C\varepsilon^\beta$, *where*

$$\beta=\beta(\gamma)=\begin{cases}\frac{2}{(6-\gamma)} & \gamma<2;\\ 1 & \gamma\geq 2;\end{cases}\tag{4.2.1}$$

(ii) $\sqrt{\rho_\varepsilon^0}-1\in H^1(\mathbf{R}^3)$ *and in particular for $2\leq p<6$ we have* $\|\sqrt{\rho_\varepsilon^0}-1\|_{L^p}\leq C\varepsilon^{\alpha(p)}$, *where*

$$\alpha(p)=\begin{cases}\frac{2(6-p)}{p(6-\gamma)} & \gamma<2;\\ \frac{(6-p)}{2p} & \gamma\geq 2;\end{cases}\tag{4.2.2}$$

(iii) $\rho_\varepsilon^0u_\varepsilon^0\in L^2(\mathbf{R}^3)+L^{\frac{3}{2}}(\mathbf{R}^3)$. *In particular $\rho_\varepsilon^0u_\varepsilon^0\in H^{-s}(\mathbf{R}^3)$ with $s>\frac{1}{2}$.*

Proof. We prove the first statement following the method in [136]. By convexity of the function $s\to s^\gamma-1-\gamma(s-1)$ for $\gamma>1$ and the fact that the internal energy, as defined in (4.0.4), satisfies $F_\varepsilon(\rho_\varepsilon^0)\in L^1(\mathbf{R}^3)$) one concludes that

$$\int_{\mathbf{R}^3}\left|\rho_\varepsilon^0-1\right|^2\mathbf{1}_{|\rho_\varepsilon^0-1|\leq\frac{1}{2}}+\left|\rho_\varepsilon^0-1\right|^\gamma\mathbf{1}_{|\rho_\varepsilon^0-1|>\frac{1}{2}}\mathrm{d}x\leq C\varepsilon^2\tag{4.2.3}$$

and when $\gamma \geq 2$ one has

$$\int_{\mathbf{R}^d} (\rho_\varepsilon - 1)^2 \mathrm{d}x \leq C\varepsilon^2.$$

Upon observing that

$$\sqrt{\rho_\varepsilon} - 1 = \frac{\rho_\varepsilon - 1}{1 + \sqrt{\rho_\varepsilon}} \leq (\rho_\varepsilon - 1),$$

we obtain,

$$\int_{\mathbf{R}^3} \left|\sqrt{\rho_\varepsilon}^0 - 1\right|^2 \mathbf{1}_{|\rho_\varepsilon^0 - 1| \leq \frac{1}{2}} + \left|\sqrt{\rho_\varepsilon^0} - 1\right|^\gamma \mathbf{1}_{|\rho_\varepsilon^0 - 1| > \frac{1}{2}} \mathrm{d}x \leq C\varepsilon^2, \tag{4.2.4}$$

in particular $\sqrt{\rho_\varepsilon^0} - 1 \in L_2^m$. From what we said before, the bound $\nabla\sqrt{\rho_\varepsilon} \in L^2(\mathbf{R}^3)$ implies there exists $c > 0$ such that $\sqrt{\rho_\varepsilon^0} - c \in L^6(\mathbf{R}^3)$. It is easy to conclude that necessarily $c = 1$ since $\sqrt{\rho_\varepsilon^0} - 1 \in L_2^m(\mathbf{R}^3)$ and by interpolation with (4.2.4) we have

$$\int_{\mathbf{R}^3} \left|\sqrt{\rho_\varepsilon}^0 - 1\right|^p \mathbf{1}_{|\rho_\varepsilon^0 - 1| \leq \frac{1}{2}} + \left|\sqrt{\rho_\varepsilon^0} - 1\right|^p \mathbf{1}_{|\rho_\varepsilon^0 - 1| > \frac{1}{2}} \mathrm{d}x \leq C\varepsilon^{p\alpha(p)},$$

for any $2 \leq p \leq 6$, where $\alpha(p) = \frac{2(6-p)}{p(6-\gamma)}$ for $\gamma < 2$ and $\alpha(p) = \frac{6-p}{2p}$ for $\gamma \geq 2$. In particular,

$$\|\sqrt{\rho_\varepsilon^0} - 1\|_{L^2(\mathbf{R}^3)} \leq C\varepsilon^\alpha,$$

where

$$\alpha(2) = \begin{cases} \frac{4}{6-\gamma} & \gamma < 2, \\ 1 & \gamma \geq 2. \end{cases}$$

Therefore, $\sqrt{\rho_\varepsilon^0} - 1 \in H^1(\mathbf{R}^3)$. Next we show that for $1 < \gamma < 2$, one has

$$\|\rho_\varepsilon^0 - 1\|_{L^2(\mathbf{R}^3)} \leq C\varepsilon^{2\alpha(4)},$$

In view of (4.2.3), it is sufficient to show that the second term in (4.2.3) is bounded by $C\varepsilon^\alpha$. To that end we observe that for $|z - 1| \geq \frac{1}{2}$, one has

$$(z^2 - 1)^2 \leq 25(z - 1)^4.$$

Hence,

$$\int_{\mathbf{R}^3} \left|\rho_\varepsilon^0 - 1\right|^2 \mathbf{1}_{|\rho_\varepsilon^0 - 1| > \frac{1}{2}} \mathrm{d}x \leq \int_{\mathbf{R}^3} \left|\sqrt{\rho_\varepsilon^0} - 1\right|^4 \mathbf{1}_{|\rho_\varepsilon^0 - 1| > \frac{1}{2}} \mathrm{d}x \leq C\varepsilon^{4\alpha(4)}.$$

Since for $1 < \gamma < 2$, we have $2\alpha(4) = \frac{2}{6-\gamma}$, the statement (i) follows. Finally, to prove (iii) we notice that if the initial data is of finite energy then we obtain

$$\sqrt{\rho_\varepsilon^0} - 1 \in H^1(\mathbf{R}^3), \qquad \sqrt{\rho_\varepsilon^0} u_\varepsilon^0 \in L^2(\mathbf{R}^3),$$

which allow us to conclude

$$\rho_\varepsilon^0 u_\varepsilon^0 = \sqrt{\rho_\varepsilon^0} u_\varepsilon^0 + (\sqrt{\rho_\varepsilon}^0 - 1)\sqrt{\rho_\varepsilon^0} u_\varepsilon^0 \in L^2(\mathbf{R}^3) + L^{\frac{3}{2}}(\mathbf{R}^3).$$

$\square$

4.2.2 Uniform estimates on the solution

By Definition 4.1.1, the finite energy weak solution $(\rho_\varepsilon, u_\varepsilon)$ we consider satisfies the energy inequality (4.1.4) and the BD entropy type inequality (4.1.5) that imply the following *a priori* estimates listed below.

Lemma 4.2.2. *If $(\rho_\varepsilon, u_\varepsilon)$ is a finite energy weak solution of (4.0.3), then there exists $C > 0$ independent from $\varepsilon > 0$ such that*

(i) $\|\rho_\varepsilon - 1\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{R}^3))} \leq C\varepsilon^\beta$, *for $\beta(\gamma)$ defined as in (4.2.1)*

(ii) $\|\frac{1}{\varepsilon}\nabla\rho_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\mathbf{R}_+;L^2(\mathbf{R}^3))} \leq C$,

(iii) $\sqrt{\rho_\varepsilon} - 1 \in L^\infty(\mathbf{R}_+; H^1(\mathbf{R}^3))$ *and in particular for $2 \leq p < 6$ and for $\alpha(p)$ defined as in (4.2.2) it holds* $\|\sqrt{\rho_\varepsilon} - 1\|_{L^\infty(\mathbf{R}_+;L^p(\mathbf{R}^3))} \leq C\varepsilon^{\alpha(p)}$,

(iv) $\|\nabla^2\sqrt{\rho_\varepsilon}\|_{L^2(\mathbf{R}_+;L^2(\mathbf{R}^3))} \leq C$,

(v) for any $0 \leq s < 2$ and $2 \leq p < \frac{4}{s}$, there exists $0 < \beta(p,s) < 2$ such that $\|\sqrt{\rho_\varepsilon} - 1\|_{L^p(\mathbf{R}_+;H^s(\mathbf{R}^3))} \leq C\varepsilon^\beta$. *Moreover, for $1 < s \leq 2$,*
$\|\sqrt{\rho_\varepsilon} - 1\|_{L^{\frac{2}{s-1}}(\mathbf{R}_+;H^s(\mathbf{R}^3))} \leq C$. *In particular,* $\sqrt{\rho_\varepsilon} - 1 \in L^2(\mathbf{R}_+; L^\infty(\mathbf{R}^3))$.

(vi) $\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{R}^3))} \leq C$,

Proof. The first and the third statement are proven similarly to Lemma 4.2.1 exploiting the fact that $F_\varepsilon \in L^\infty(\mathbf{R}_+; L^1(\mathbf{R}^3))$. The remaining statements except the fifth are direct consequences of inequalities (4.1.4) and (4.1.5). Statement (v) follows by interpolation of (ii) and (iv). For $0 < s < 2$ and $0 < \theta < 1$ there exists $\beta > 0$ such that

$$\|\sqrt{\rho_\varepsilon} - 1\|_{L^p(\mathbf{R}_+;H^s(\mathbf{R}^d))} \leq C\,\|\sqrt{\rho_\varepsilon} - 1\|^\theta_{L^2(\mathbf{R}_+;H^2(\mathbf{R}^3))}\,\|\sqrt{\rho_\varepsilon} - 1\|^{1-\theta}_{L^\infty(\mathbf{R}_+;L^2(\mathbf{R}^3))} \leq C\varepsilon^\beta,$$

where $s = 2\theta$ and p such that $p \leq \frac{2}{\theta}$. In particular if $s > \frac{3}{2}$ this yields for any $2 \leq p < \frac{8}{3}$ that $\sqrt{\rho_\varepsilon} - 1$ converges strongly to 0 in $L^p(0,T; L^\infty(\mathbf{R}^3))$. By interpolation between the bounds $\sqrt{\rho_\varepsilon} - 1 \in L^\infty(\mathbf{R}_+; H^1(\mathbf{R}^3))$ and $\nabla^2(\sqrt{\rho_\varepsilon} - 1) \in L^2(\mathbf{R}_+; L^2(\mathbf{R}^3))$, one may infer the slightly stronger bound $\sqrt{\rho_\varepsilon} - 1 \in L^{\frac{2}{s-1}}(\mathbf{R}_+; H^s(\mathbf{R}^3))$ for $1 < s \leq 2$. ☐

4.2.3 Bounds on density fluctuation σ_ε and momentum m_ε

Next, we provide bounds on the density fluctuation $\sigma_\varepsilon := \frac{\rho_\varepsilon - 1}{\varepsilon}$.

Lemma 4.2.3. *If $(\rho_\varepsilon, u_\varepsilon)$ is a finite energy weak solution of (4.0.3), then for any $0 < T < \infty$, σ_ε satisfies*

(i) $\sigma_\varepsilon^0 \in L^2_m(\mathbf{R}^3)$ *with* $m = \min\{2, \gamma\}$;

(ii) $\sigma_\varepsilon \in L^\infty(\mathbf{R}_+; L^2_m(\mathbf{R}^3))$;

(iii) $\varepsilon\nabla\sigma_\varepsilon \in L^\infty((\mathbf{R}_+; L^2(\mathbf{R}^3)) + L^4(0,T; L^2(\mathbf{R}^3))$ *and* $\varepsilon\sigma_\varepsilon \in L^4(0,T; H^1(\mathbf{R}^3))$*;*

(iv) $\varepsilon\nabla^2\sigma_\varepsilon \in L^2(0,T; L^2(\mathbf{R}^3)) + L^{\frac{4}{3}}(0,T; L^2(\mathbf{R}^3))$ *and* $\varepsilon\nabla^2\sigma_\varepsilon \in L^{\frac{4}{3}}(0,T; H^2(\mathbf{R}^3))$*;*

(v) In particular, if $\gamma = 2$*, then* $\sigma_\varepsilon \in L^2(0,T,H^1(\mathbf{R}^3))$*.*

All the previous bounds are uniform in $\varepsilon > 0$*.*

Proof. The first bound follows from

$$\int_{\mathbf{R}^d} \left|\rho_\varepsilon^0 - 1\right|^2 \mathbf{1}_{|\rho_\varepsilon^0 - 1| \le \frac{1}{2}} + \left|\rho_\varepsilon^0 - 1\right|^\gamma \mathbf{1}_{|\rho_\varepsilon^0 - 1| > \frac{1}{2}} \mathrm{d}x \le C\varepsilon^2; \tag{4.2.5}$$

and similarly the second.
The third bound follows by observing $\varepsilon\nabla\sigma_\varepsilon = 2\sqrt{\rho_\varepsilon}\nabla\sqrt{\rho_\varepsilon}$ and applying the bounds for $\sqrt{\rho_\varepsilon} - 1$ of Lemma 4.2.2. In particular,

$$\|\varepsilon\nabla\sigma_\varepsilon\|_{L^4(0,T;L^2)} \le \|\nabla\sqrt{\rho_\varepsilon}\|_{L^4(0,T;L^2)} + \|\sqrt{\rho_\varepsilon} - 1\|_{L^4(0,T;L^\infty)}\|\nabla\sqrt{\rho_\varepsilon}\|_{L^\infty(0,T;L^2)}.$$

Hence, we conclude that $\varepsilon\sigma_\varepsilon \in L^4(0,T; L^6(\mathbf{R}^3)$. By interpolation with $\sigma_\varepsilon \in L^\infty(L^2 + L^\gamma(\mathbf{R}^3))$, it follows $\varepsilon\sigma_\varepsilon \in L^4(\mathbf{R}_+; L^2(\mathbf{R}^3))$. We remark that if $\gamma \ge 2$, then $\sigma_\varepsilon \in L^\infty(0,\infty; L^2(\mathbf{R}^3))$ and the interpolation is not needed. Thus, $\varepsilon\sigma_\varepsilon \in L^4(0,T; H^1(\mathbf{R}^3))$. Similarly for,

$$\varepsilon\nabla^2\sigma_\varepsilon = 2\sqrt{\rho_\varepsilon}\nabla^2\sqrt{\rho_\varepsilon} + 2\left|\nabla\sqrt{\rho_\varepsilon}\right|^2,$$

we conclude exploiting again the bounds on $\sqrt{\rho_\varepsilon} - 1$. Moreover, for $\gamma = 2$, the estimate $\frac{1}{\varepsilon}\nabla\rho_\varepsilon^{\frac{\gamma}{2}}$ allows us to conclude that $\sigma_\varepsilon \in L^2(0,T,H^1(\mathbf{R}^3))$. □

Corollary 4.2.4. *If* $(\rho_\varepsilon, u_\varepsilon)$ *is a finite energy weak solution of* (4.0.3)*, then for any* $0 < T < \infty$*,*

$$\rho_\varepsilon u_\varepsilon \in L^{4-}(0,T; L^2(\mathbf{R}^3)).$$

Proof. It is sufficient to write $\rho_\varepsilon u_\varepsilon = \sqrt{\rho_\varepsilon}u_\varepsilon + (\sqrt{\rho_\varepsilon} - 1)\sqrt{\rho_\varepsilon}u_\varepsilon$ and to see that

$$\begin{aligned}\|\rho_\varepsilon u_\varepsilon\|_{L^{4-}(0,T;L^2(\mathbf{R}^3))} &\le C\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty(0,T;L^2)}\left(T^{\frac{1}{4}+} + \|\sqrt{\rho_\varepsilon} - 1\|_{L^{4-}L^\infty}\right)\\ &\le C'(1 + T^{\frac{1}{4}+}),\end{aligned}$$

since $\sqrt{\rho_\varepsilon} - 1 \in L^{4-}(0,T; L^\infty(\mathbf{R}^3))$ from Lemma 4.2.2. □

The Corollary 4.2.4 together with the a priori bounds on $\sqrt{\rho_\varepsilon} - 1$ and $\mathbf{S}_\varepsilon$ allow us to prove a stronger estimate on $\rho_\varepsilon u_\varepsilon$ by splitting high and low frequencies.

Proposition 4.2.5. *If* $(\rho_\varepsilon, u_\varepsilon)$ *is a finite energy weak solution of* (4.0.3)*, then for any* $0 < T < \infty$ *and for any* $0 \le s \le \frac{1}{2}$ *and* $1 \le p < \frac{4}{1+4s}$*,*

$$\rho_\varepsilon u_\varepsilon \in L^p(0,T; H^s(\mathbf{R}^3)), \tag{4.2.6}$$

where the bound is uniform in $\varepsilon > 0$*. In particular for any* $0 \le s_1 < \frac{1}{4}$*, one has* $\rho_\varepsilon u_\varepsilon \in L^2(0,T; H^{s_1}(\mathbf{R}^3))$*.*

The following Lemma is necessary for the proof of Proposition 4.2.5.

Lemma 4.2.6. *There exist* $f_1 \in L^2(\mathbf{R}_+; W^{1,\frac{3}{2}}(\mathbf{R}^3))$ *and* $f_2 \in L^{\frac{4}{3}-}(0,T; H^1(\mathbf{R}^3))$ *such that*

$$\rho_\varepsilon u_\varepsilon = f_1 + f_2, \qquad a.e.\ in\ \mathbf{R}^3.$$

Proof. From (4.1.1), the following identity holds in $\mathcal{D}'$,

$$\nabla(\rho_\varepsilon u_\varepsilon) = \sqrt{\rho_\varepsilon}\mathbf{T}_\varepsilon + \nabla\sqrt{\rho_\varepsilon} \otimes \sqrt{\rho_\varepsilon}u_\varepsilon = \mathbf{T}_\varepsilon + (\sqrt{\rho_\varepsilon} - 1)\mathbf{T}_\varepsilon + 2\nabla\sqrt{\rho_\varepsilon} \otimes \sqrt{\rho_\varepsilon}u_\varepsilon,$$

and from the bounds of Lemma 4.2.2 we deduce that $\nabla(\rho_\varepsilon u_\varepsilon) \in L^2(\mathbf{R}_+; L^{\frac{3}{2}}(\mathbf{R}^3) + L^{\frac{4}{3}-}(0,T; L^2(\mathbf{R}^3))$. Indeed,

$$\begin{aligned}
&\|\nabla\sqrt{\rho_\varepsilon} \otimes \sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^2_t L^{\frac{3}{2}}_x} \leq C\|\nabla\sqrt{\rho_\varepsilon}\|_{L^2_t L^6_x}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty_t L^2_x} \\
&\|(\sqrt{\rho_\varepsilon} - 1)\mathbf{T}_\varepsilon\|_{L^{\frac{4}{3}-}_t L^2_x} + \|\mathbf{T}_\varepsilon\|_{L^{\frac{4}{3}-}_t L^2_x} \leq C_T(1 + \|\sqrt{\rho_\varepsilon} - 1\|_{L^{4-}_t L^\infty_x})\|\mathbf{T}_\varepsilon\|_{L^2_t L^2_x},
\end{aligned}$$

Therefore there exist g_1, g_2 such that

$$\nabla(\rho_\varepsilon u_\varepsilon) = g_1 + g_2 \quad \text{a.e. in } \mathbf{R}^3 \text{ and } \quad g_1 \in L^2(\mathbf{R}_+; L^{\frac{3}{2}}(\mathbf{R}^3)), \quad g_2 \in L^{\frac{4}{3}-}(0,T; L^2(\mathbf{R}^3))$$

We show that g_1, g_2 can always be chosen such that they are gradient fields. If not, then given a decomposition g_1, g_2, we observe that

$$0 = \mathbf{P}(\nabla(\rho_\varepsilon u_\varepsilon)) = \mathbf{P}(g_1) + \mathbf{P}(g_2), \tag{4.2.7}$$

where $\mathbf{P}$ denotes the Leray projector onto divergence free vector fields. We define $\tilde{g_1} := \mathbf{Q}(g_1)$ and $\tilde{g_2} := g_2 + \mathbf{P}(g_1)$. From (4.2.7) this implies that

$$\tilde{g_2} = g_2 + \mathbf{P}(g_1) = g_2 - \mathbf{P}(g_2) = \mathbf{Q}(g_2).$$

Hence, there exist $\tilde{f}_1, \tilde{f}_2$ such that

$$\tilde{g_1} = \nabla\tilde{f}_1, \qquad \tilde{g_2} = \nabla\tilde{f}_2, \quad \text{a.e. in } \mathbf{R}^3,$$

and $\nabla(\rho_\varepsilon u_\varepsilon) = \tilde{g_1} + \tilde{g_2}$. We recall again that for any distribution f such that $\nabla f \in L^p(\mathbf{R}^d)$ with $p < d$, there exists a $c \in \mathbf{R}$ such that $f - c \in L^{p^*}(\mathbf{R}^d)$ with the Sobolev exponent $p^* = \frac{dp}{d-p}$. Thus, there exist real numbers c_1, c_2 such that

$$\tilde{f}_1 - c_1 \in L^2(\mathbf{R}_+; L^3(\mathbf{R}^3)), \quad \text{and} \qquad \tilde{f}_2 - c_2 \in L^{\frac{4}{3}-}(0,T; L^6(\mathbf{R}^3)).$$

Define,

$$f_1 = \tilde{f}_1 - c_1 - P_{\leq 1}\left(\tilde{f}_1 - c_1\right), \qquad f_2 = \tilde{f}_2 - c_2 + P_{\leq 1}\left(\tilde{f}_1 - c_1\right).$$

We claim that $f_1 \in L^{\frac{3}{2}}(\mathbf{R}^3)$. Indeed, since $\dot{B}^0_{\frac{3}{2},\frac{3}{2}} \hookrightarrow L^{\frac{3}{2}}$, we have

$$\begin{aligned}
\|f_1\|_{L^{\frac{3}{2}}} \leq C\|f_1\|_{\dot{B}^0_{\frac{3}{2},\frac{3}{2}}} &= \left(\sum_{j>0} 2^{-\frac{3}{2}j}2^{\frac{3}{2}j}\|P_j f_1\|^{\frac{3}{2}}_{L^{\frac{3}{2}}}\right)^{\frac{2}{3}} \\
&\leq C\|f_1\|_{\dot{B}^1_{\frac{3}{2},2}} \leq C\|\nabla f_1\|_{L^{\frac{3}{2}}},
\end{aligned}$$

where we used in the last step that $L^{\frac{3}{2}} \hookrightarrow \dot{B}^0_{\frac{3}{2},2}$. We conclude that $f_1 \in L^2(\mathbf{R}_+; W^{1,\frac{3}{2}}(\mathbf{R}^3))$. Next, we check that $f_2 \in L^{\frac{4}{3}-}(0,T; H^1(\mathbf{R}^3))$. An application of Bernstein's inequality gives

$$\|P_{\leq N}(\tilde{f}_1 - c_1)\|_{L^6} \leq CN^{\frac{1}{2}}\|P_{\leq N}(\tilde{f}_1 - c_1)\|_{L^3},$$

therefore $f_2 \in L^6$. Again by Bernstein's inequality, we control

$$\|\nabla f_2\|_{L^2} \leq C\left(\|\nabla(\tilde{f}_2 - c_2)\|_{L^2} + C\|\nabla(P_{\leq 1}(\tilde{f}_1 - c_1))\|_{L^{\frac{3}{2}}}\right)$$

Thus $f_2 \in L^6$ and $\nabla f_2 \in L^2$. Since

$$\nabla\left(\rho_\varepsilon u_\varepsilon - f_1 - f_2\right) = 0 \quad \text{a.e. in } \; \mathbf{R}^3,$$

we infer

$$\rho_\varepsilon u_\varepsilon - f_1 - f_2 = C \quad \text{a.e. in } \; \mathbf{R}^3.$$

The Sobolev embedding yields that since $f_1 \in L^2(\mathbf{R}_+; W^{1,\frac{3}{2}}(\mathbf{R}^3))$ also $f_1 \in L^2(\mathbf{R}_+; L^2(\mathbf{R}^3))$, thus by Corollary 4.2.4,

$$f_2 + C = \rho_\varepsilon u_\varepsilon - f_1 \in L^2(0,T; L^2(\mathbf{R}^3)).$$

This implies $f_2 + C \in L^{\frac{4}{3}-}(0,T; H^1(\mathbf{R}^3))$ and in particular $f_2 + C \in L^{\frac{4}{3}-}(0,T; L^6(\mathbf{R}^3))$, again by Sobolev embedding. We recover that necessarily $C = 0$. Finally,

$$\rho_\varepsilon u_\varepsilon = f_1 + f_2 \quad \text{a.e. in } \; \mathbf{R}^3,$$

where $f_1 \in L^2(0,T; W^{1,\frac{3}{2}}(\mathbf{R}^3))$ and $f_2 \in L^{\frac{4}{3}-}(0,T; H^1(\mathbf{R}^3))$. $\square$

The statement of Proposition 4.2.5 follows by interpolation.

Proof of Proposition 4.2.5. By Lemma 4.2.6, we have that $\rho_\varepsilon u_\varepsilon = f_1 + f_2$, where $f_1 \in L^{\frac{4}{3}}(0,T; W^{1,\frac{3}{2}})$ and $f_2 \in L^{\frac{4}{3}-}(0,T; H^1)$. By Sobolev embedding $f_1 \in L^{\frac{4}{3}}(0,T; H^{\frac{1}{2}})$, thus we conclude $\rho_\varepsilon u_\varepsilon \in L^{\frac{4}{3}-}(0,T; H^{\frac{1}{2}}(\mathbf{R}^3))$. By interpolation we conclude that for $0 \leq s \leq \frac{1}{2}$ it follows

$$\|\rho_\varepsilon u_\varepsilon\|_{L^p_t H^s_x} \leq C\|\rho_\varepsilon u_\varepsilon\|^{2s}_{L^{\frac{4}{3}-}H^{\frac{1}{2}}}\|\rho_\varepsilon u_\varepsilon\|^{1-2s}_{L^{4-}L^2}$$

for p such that $p < \frac{4}{1+4s}$, that completes the proof. $\square$

4.3 Acoustic waves

This section is devoted to the analysis of the acoustic waves in the system. For highly subsonic flows they undergo rapid oscillations in time, so that one expects the acoustic waves to converge weakly to 0. Furthermore, we will see that the dispersion relation satisfied by the fluctuations around the incompressible flow is not given by the classical waves but by the (scaled) Bogoliubov dispersion relation [34], which in our system reads

$$\omega(\xi) = \frac{1}{\varepsilon}\sqrt{|\xi|^2 + \varepsilon^2\kappa^2|\xi|^4}, \tag{4.3.1}$$

see (4.3.8) below.
To perform this analysis we use identity (2.0.3) and rewrite system (4.0.3) as

$$\begin{cases} \partial_t\rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \frac{1}{\varepsilon^2}\nabla P(\rho_\varepsilon) = 2\nu \operatorname{div}(\rho_\varepsilon \mathbf{D}u_\varepsilon) - 4\kappa^2 \operatorname{div}\left(\nabla\sqrt{\rho_\varepsilon}\otimes\nabla\sqrt{\rho_\varepsilon}\right) + \kappa^2\nabla\Delta\rho_\varepsilon, \end{cases} \tag{4.3.2}$$

where we recall that the term $\rho_\varepsilon \mathbf{D}u_\varepsilon$ should be interpreted as in (4.1.2). We notice that, by using (4.0.4) we can write

$$\frac{1}{\varepsilon^2}\nabla P(\rho_\varepsilon) = \frac{1}{\gamma\varepsilon^2}\nabla\rho_\varepsilon^\gamma = \frac{1}{\varepsilon}\nabla\sigma_\varepsilon + (\gamma-1)\nabla F_\varepsilon,$$

so that (4.3.2) reads

$$\begin{cases} \partial_t\sigma_\varepsilon + \frac{1}{\varepsilon}\operatorname{div}(m_\varepsilon) = 0, \\ \partial_t m_\varepsilon + \frac{1}{\varepsilon}\nabla\left(1-\kappa^2\varepsilon^2\Delta\right)\sigma_\varepsilon = G_\varepsilon, \end{cases} \tag{4.3.3}$$

and

$$G_\varepsilon = \operatorname{div}\left(-\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon - 4\kappa^2\nabla\sqrt{\rho_\varepsilon}\otimes\nabla\sqrt{\rho_\varepsilon} + 2\nu\rho_\varepsilon\mathbf{D}u_\varepsilon - (\gamma-1)F_\varepsilon\mathbb{I}\right). \tag{4.3.4}$$

Projecting onto irrotational vector fields we obtain the system describing acoustic waves

$$\begin{cases} \partial_t\sigma_\varepsilon + \frac{1}{\varepsilon}\operatorname{div}(\mathbf{Q}m_\varepsilon) = 0, \\ \partial_t\mathbf{Q}(m_\varepsilon) + \frac{1}{\varepsilon}\nabla\left(1-\kappa^2\varepsilon^2\Delta\right)\sigma_\varepsilon = \mathbf{Q}(G_\varepsilon). \end{cases} \tag{4.3.5}$$

The initial datum for (4.3.5) is given by

$$\sigma_\varepsilon^0 = \frac{\rho_\varepsilon^0 - 1}{\varepsilon}, \qquad m_\varepsilon^0 = \rho_\varepsilon^0 u_\varepsilon^0,$$

where we observe that by Lemma 4.2.1,

$$\sigma_\varepsilon^0 \in H^{-\frac{3}{2}}(\mathbf{R}^3), \qquad m_\varepsilon^0 \in H^{-\frac{1}{2}}(\mathbf{R}^3). \tag{4.3.6}$$

The main result of this section shows the strong convergence to 0 of the acoustic waves.

Theorem 4.3.1. *Let $(\rho_\varepsilon, u_\varepsilon)$ be a finite energy weak solution of (4.0.3). Then, for any $0 < T < \infty$*

(i) the density fluctuations $\rho_\varepsilon - 1$ converge strongly to 0 in $C(0,T;L^2(\mathbf{R}^3))$ and in $L^4(0,T;H^s(\mathbf{R}^3))$ for any $s \in (-\frac{3}{2}, 1)$,

(ii) If $\gamma = 2$, then σ_ε converges strongly to 0 in $L^2(0,T;L^q(\mathbf{R}^3))$ for any $2 < q < 6$,

(iii) for any $2 < q < \frac{9}{4}$ *there exists* $\delta > 0$ *such that* $\mathbf{Q}(m_\varepsilon)$ *converges strongly to* 0 *in* $L^2(0,T;B^{\delta}_{q,2}(\mathbf{R}^3))$.

In order to infer estimates on $(\sigma_\varepsilon, \mathbf{Q}m_\varepsilon)$ by studying (4.3.5), we derive Strichartz estimates for a symmetrization of the linearised system (4.3.5) that will ultimately imply the convergence of $(\sigma_\varepsilon, \mathbf{Q}m_\varepsilon)$. More precisely, we define

$$\tilde{\sigma_\varepsilon} := (1-\varepsilon^2\kappa^2\Delta)^{\frac{1}{2}}\sigma_\varepsilon \qquad\qquad \tilde{m_\varepsilon} := (-\Delta)^{-\frac{1}{2}}\operatorname{div} m_\varepsilon,$$

and check that if $(\sigma_\varepsilon, m_\varepsilon)$ is a solution of (4.3.5) then $(\tilde{\sigma_\varepsilon}, \tilde{m_\varepsilon})$ satisfies the symmetrized system

$$\begin{cases} \partial_t\tilde{\sigma_\varepsilon} + \frac{1}{\varepsilon}(-\Delta)^{\frac{1}{2}}(1-\kappa^2\varepsilon^2\Delta)^{\frac{1}{2}}\tilde{m_\varepsilon} = 0, \\ \partial_t\tilde{m_\varepsilon} - \frac{1}{\varepsilon}(-\Delta)^{\frac{1}{2}}(1-\kappa^2\varepsilon^2\Delta)^{\frac{1}{2}}\tilde{\sigma_\varepsilon} = \tilde{F}_\varepsilon, \end{cases} \tag{4.3.7}$$

where $\tilde{F}_\varepsilon = (-\Delta)^{-\frac{1}{2}}\operatorname{div} F_\varepsilon$. Hence, the linear evolution is characterised by the unitary semigroup e^{-itH_ε}, where

$$H_\varepsilon = \frac{1}{\varepsilon}\sqrt{(-\Delta)(1-(\varepsilon\kappa)^2\Delta)} \tag{4.3.8}$$

is a self-adjoint operator with Fourier multiplier given by (4.3.1). In what follows, we are going to provide a class of Strichartz estimates for the linear propagator e^{-itH_ε} which will yield a control of some mixed space-time norms of $(\tilde{\sigma_\varepsilon}, \tilde{m_\varepsilon})$ in terms of the (scaled) Mach number. An interpolation argument exploiting the *a priori* estimates introduced in Section 4.2 gives the final result. The Strichartz estimates are stated in the framework of Besov spaces, for the sake of conciseness we postpone their proof to the Section 4.5.
Before stating the next Proposition, we recall that a pair of Lebesgue exponents (p,q) is called Schrödinger admissible if $2 \le p, q \le \infty$ and $\frac{2}{p} + \frac{3}{q} = \frac{3}{2}$.

Proposition 4.3.2. *Let* $\varepsilon > 0$, *fix* $\alpha > 0$ *arbitrarily small and let* (p,q), (p_1,q_1) *be two admissible pairs. Then the following estimates hold true*

$$\|e^{itH_\varepsilon}f\|_{L^p_tB^0_{q,2}} \le C\varepsilon^\alpha\|f\|_{B^\alpha_{2,2}}, \tag{4.3.9}$$

$$\left\|\int_0^t e^{i(t-s)H_\varepsilon}F(s)\mathrm{d}s\right\|_{L^p_tB^0_{q,2}} \le C\varepsilon^\alpha\|F\|_{L^{p_1'}B^\alpha_{q_1',2}}. \tag{4.3.10}$$

Proposition 4.3.2 will be proved in Section 4.5, in fact it will be a consequence of the more general Proposition 4.5.10.
Let us remark that the case $\varepsilon = 1$ was already studied in [95], where the authors infer dispersive estimates for the propagator e^{itH_1} in order to study scattering properties for the Gross-Pitaevskii equation. In our case we need to keep track of the $\varepsilon-$dependence of the estimates, in order to show the convergence to zero of the acoustic part. However, since $H_\varepsilon = H_\varepsilon(\sqrt{-\Delta})$ is a non-homogeneous function of $\sqrt{-\Delta}$, it is not possible to obtain a decay in ε by simply rescaling the estimates in [95]. This is for example different from what happens

for classical fluids [68] where the wave-like acoustic dispersion yields the convergence to zero by scaling the estimates and by considering the fast dynamics for the fluctuations.
On the other hand here we can exploit that the Strichartz estimates associated to the operator (4.3.8) are sligthly better than the ones for the Schrödinger operator close to the Fourier origin. This fact is also noticed in [95] for H_1. By exploiting this regularizing effect, Proposition 4.3.2 somehow improves a class of similar estimates inferred in [26] in another context (the linear wave regime for the Gross-Pitaevskii equation). Indeed the authors of [26] consider H_ε in two different regimes: for low frequencies below the threshold $\frac{1}{\varepsilon}$ the operator behaves like the wave operator, while above the threshold it is Schrödinger-like. In this way the low frequency part experiences a derivative loss, due to the wave-type dispersive estimates inferred.
Here we do not split H_ε in low and high frequencies, nevertheless we prove the convergence to zero of the acoustic part by only losing a small amount of derivatives.
In order to apply the estimates (4.3.9), (4.3.10) to system (4.3.5), we first need to bound F_ε defined in (4.3.4) in suitable spaces.

Lemma 4.3.3. *If $(\rho_\varepsilon, u_\varepsilon)$ is a finite energy solution to* (4.0.3)*, then one has,*

(i) $F_{\varepsilon,1} = \operatorname{div}\left(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon + 4\kappa^2 \nabla\sqrt{\rho_\varepsilon} \otimes \nabla\sqrt{\rho_\varepsilon}\right) + (\gamma - 1)\nabla F_\varepsilon \in L^\infty(0, T; B^{-s}_{2,2}(\mathbf{R}^3))$*, for* $s > \frac{5}{2}$*,*

(ii) $F_{\varepsilon,2} = 2\nu \operatorname{div}(\rho_\varepsilon \mathbf{D} u_\varepsilon) \in L^{\frac{4}{3}}(0, T; B^{-1}_{2,2}(\mathbf{R}^3))$.

Proof. We shall use repeatedly the embeddings $L^1(\mathbf{R}^3) \hookrightarrow H^{-s}(\mathbf{R}^3) = B^{-s}_{2,2}(\mathbf{R}^3)$ valid for $s > \frac{3}{2}$. For the first statement, we observe that

$$(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + 4\kappa^2 (\nabla\sqrt{\rho_\varepsilon} \otimes \nabla\sqrt{\rho_\varepsilon}) + F_\varepsilon \in L^\infty(0, T; L^1(\mathbf{R}^3)),$$

and thus $G_{\varepsilon,1} \in L^\infty(0, T; H^{-s}(\mathbf{R}^3))$ for $s > \frac{5}{2}$. In particular this implies $G_{\varepsilon,1} \in L^\infty(0, T; B^{-s-\frac{3}{2}+\frac{3}{q}}_{q,2}(\mathbf{R}^3))$ for $q \geq 2$ and $s > \frac{5}{2}$.
Regarding the second statement, we observe that

$$\|\sqrt{\rho_\varepsilon}\mathbf{S}_\varepsilon\|_{L^{\frac{4}{3}-}L^2} \leq C_T \left(\|\mathbf{S}_\varepsilon\|_{L^2_{t,x}} + \|\sqrt{\rho_\varepsilon} - 1\|_{L^{4-}_t L^\infty_x} \|\mathbf{S}_\varepsilon\|_{L^2_{t,x}} \right),$$

and thus $F_{\varepsilon,2} \in L^\infty(0, T; H^{-1}(\mathbf{R}^3))$. □

Remark 4.3.4. Here, we need to use Strichartz estimates in non-homogeneous spaces. This is due to the fact that L^1 has no embedding in any homogeneous Besov space but $\dot{B}^0_{1,\infty}$.

By combining the dispersive estimates of Proposition 4.3.2 and the bounds in Lemma 4.3.3 we can then infer the convergence to zero of $(\sigma_\varepsilon, \mathbf{Q}m_\varepsilon)$.

Proposition 4.3.5. *Let $(\sigma_\varepsilon, m_\varepsilon)$ be solution of* (4.3.3) *with initial data $(\sigma_{\varepsilon,0}, m_{\varepsilon,0})$. Then for any $s \in \mathbf{R}$, $\alpha > 0$ arbitrarily small and for any admissible pairs $(p, q), (p_1, q_1)$, the following estimate holds true*

$$\|(\sigma_\varepsilon, \mathbf{Q}(m_\varepsilon))\|_{L^p_t B^{s-\alpha}_{q,2}} \leq C_T \left(\varepsilon^\alpha \|(\sigma_{\varepsilon,0}, m_{\varepsilon,0})\|_{H^s} + \varepsilon^\alpha \|F_\varepsilon\|_{L^{p_1'} B^s_{q_1',2}} \right). \tag{4.3.11}$$

Moreover, for any admissible pair (p,q) and any $s>5/2$,

$$\|(\sigma_\varepsilon, \mathbf{Q}(m_\varepsilon))\|_{L^p_t B^{-s-\alpha}_{q,2}} \le C_T \varepsilon^\alpha. \tag{4.3.12}$$

The condition $s>5/2$ is due to the low regularity of the nonlinearity in (4.3.5).

Proof. The inequality (4.3.11) follows from (4.3.9), (4.3.10) and the observation that for any $s\in\mathbf{R}$ and $1<q\le\infty$ one has that

$$\|\sigma_\varepsilon\|_{L^p_t B^s_{q,2}} \le C\|\tilde{\sigma_\varepsilon}\|_{L^p_t B^s_{q,2}}, \tag{4.3.13}$$

and

$$\|\mathbf{Q} m_\varepsilon\|_{L^p_t B^s_{q,2}} \le C\|\tilde{m_\varepsilon}\|_{L^p_t B^s_{q,2}}. \tag{4.3.14}$$

Indeed, to check (4.3.13) we define the operator $T(f) = (1-\varepsilon^2\kappa^2\Delta)^{-\frac12} f$. By means of Bernstein inequalities and straight forward estimates on the derivatives of the symbol of T, we conclude that $T : B^s_{q,2} \to B^s_{q,2}$ is bounded for any $s\in\mathbf{R}$.
The inequality (4.3.14), follows from observing that the projection on the gradient part $\mathbf{Q}$ is given by a matrix valued Fourier multiplier $m(\xi) = \frac{\xi_k\xi_j}{|\xi|^2}$ while the change of variables $(-\Delta)^{-\frac12}\,\mathrm{div}$ corresponds to the multiplier $\frac{\xi_j}{|\xi|}$. Hence, the operator $\tilde{m_\varepsilon} \mapsto \mathbf{Q} m_\varepsilon$ is a Fourier multiplier of degree 0. Therefore for any $1\le q,r\le\infty$ and $s\in\mathbf{R}$ one has

$$\|\mathbf{Q} m_\varepsilon\|_{B^s_{q,r}} \le C\|\tilde{m_\varepsilon}\|_{B^s_{q,r}}.$$

Similarly, it is easy to check that

$$\|\tilde{m}_{\varepsilon,0}\|_{B^s_{q,r}} \le C\|m_{\varepsilon,0}\|_{B^s_{q,r}}.$$

and that

$$\|\tilde{\sigma}_{\varepsilon,0}\|_{B^s_{2,2}} \le C\left(\|P_{\le\frac1\varepsilon}(\sigma_{\varepsilon,0})\|_{B^s_{2,2}} + \|P_{>\frac1\varepsilon}(\varepsilon\nabla\sigma_{\varepsilon,0})\|_{B^s_{2,2}}\right),$$

where we recall that $\sigma_{\varepsilon,0}\in B^{-s}_{2,2}$ for $s<\frac32$ and $\varepsilon\nabla\sigma_{\varepsilon,0}\in H^{-\frac12-}$ uniformly in ε in virtue of Lemma 4.2.1. It remains to prove (4.3.12). From (4.3.6), we have that $\sigma_{\varepsilon,0}, m_{\varepsilon,0}\in H^{\tilde{s}}$ if $\tilde{s}<-\frac32$. Lemma 4.3.3 yields $G_\varepsilon = G_{\varepsilon,1}+G_{\varepsilon,2}$ with $G_{\varepsilon,1}\in L^\infty(0,\infty;B^{-s}_{2,2}(\mathbf{R}^3))$ for $s>\frac52$ and $G_{\varepsilon,2}\in L^{\frac43}(0,T;B^{-1}_{2,2}(\mathbf{R}^3))$. Hence, for $(p_1,q_1)=(\infty,2)$ we obtain

$$\|G_\varepsilon\|_{L^{p_1'}(0,T;B^{-s}_{q_1',2})} \le C_T,$$

provided $s>\frac52$. Thus for any admissible pair (p,q) and $s>\frac52$, one has that

$$\|(\sigma_\varepsilon, \mathbf{Q}(m_\varepsilon))\|_{L^p_t B^{-s-\alpha}_{q,2}} \le C_T\varepsilon^\alpha\left(\|(\sigma^0_\varepsilon, m^0_\varepsilon)\|_{B^{-s}_{2,2}} + \|G_\varepsilon\|_{L^1 B^{-s}_{2,2}}\right) \le C_T'\varepsilon^\alpha$$

This completes the proof. □

Proof of Theorem 4.3.1.

Proof. The statements of point (i) follow from the uniform bounds established in Lemma 4.2.3. The first statement is immediate and the second statement follows from the bound $\varepsilon\sigma_\varepsilon \in L^4(\mathbf{R}_+; H^1(\mathbf{R}^3))$ and $\sigma_\varepsilon \in L^\infty(0,T; H^{-s}(\mathbf{R}^3))$ for $s > \frac{3}{2}$. By interpolation $\varepsilon\sigma_\varepsilon \to 0$ in $L^4(0,T; H^s(\mathbf{R}^3))$ for any $s \in (-\frac{3}{2}, 1)$. The statement (ii) is inferred by observing that from Lemma 4.2.3 one has $\sigma_\varepsilon \in L^2(0,T; H^1(\mathbf{R}^3))$ if $\gamma = 2$. Interpolation with (4.3.12) yields that $\sigma_\varepsilon \to 0$ in $L^2(0,T; L^q(\mathbf{R}^3))$.
Finally, to obtain a bound on $\mathbf{Q}m_\varepsilon$, we interpolate between the *a priori* bound (4.2.6) and the inequality (4.3.12) that holds for any $\tilde{s} > \frac{5}{2}$, any $\tilde{\alpha} > 0$ sufficiently small. Bound (4.2.6) implies that $m_\varepsilon \in L^{\frac{4}{1+4s}-}(0,T; B_{q,2}^{s-3(\frac{1}{2}-\frac{1}{q})}(\mathbf{R}^3))$ for any $q \geq 2$ and $0 \leq s \leq \frac{1}{2}$. By interpolation, we have that for $r = \theta(-\tilde{s} - \tilde{\alpha}) + (1-\theta)(s - 3(\frac{1}{2} - \frac{1}{q}))$ it holds

$$\begin{aligned}
\|\mathbf{Q}(m_\varepsilon)\|_{L_t^p B_{q,2}^r} &\leq \left\| \|\mathbf{Q}(m_\varepsilon)\|_{B_{q,2}^{r_1}}^{\theta} \|\mathbf{Q}(m_\varepsilon)\|_{B_{q,2}^{r_2}}^{1-\theta} \right\|_{L_t^p} \\
&\leq \|\mathbf{Q}(m_\varepsilon)\|_{L_t^{p_1} B_{q,2}^{r_1}}^{\theta} \|\mathbf{Q}(m_\varepsilon)\|_{L_t^{p_2} B_{q,2}^{r_2}}^{1-\theta},
\end{aligned}$$

Hence, choosing $\tilde{\alpha} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)$, we look for (θ, s, r, q) such that $0 < \theta < 1$, $0 < s \leq \frac{1}{2}$, $2 < q \leq 6$ and moreover

$$\theta > \frac{\frac{11}{4} - \frac{1}{2q}}{s + \frac{5}{2q} + \frac{5}{4}} \quad \Leftrightarrow \quad r > 0,$$

as well as

$$\frac{1}{p} = \theta\left(s + \frac{3}{2q} - \frac{1}{2}\right) + \frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right).$$

We compute that for $2 < q < \frac{9}{4}$ there exists $\alpha > 0$ and (θ, s, r) such that the above requirements are met and moreover $r > \alpha$ and $p \geq 2$. We find that

$$\|\mathbf{Q}m_\varepsilon\|_{L^2(0,T;B_{q,2}^\alpha)} \leq C_T \varepsilon^{(1-\theta)\alpha}.$$

□

4.4 Convergence to the limiting system

The uniform bound $\rho_\varepsilon u_\varepsilon \in L^{\frac{4}{1+4s}}(0,T; H^s(\mathbf{R}^3))$ for $0 \leq s \leq \frac{1}{2}$ shown in Proposition 4.2.5 implies that up to passing to a subsequence there exists $u \in L^{\frac{4}{1+4s}}(0,T; H^s(\mathbf{R}^3))$ such that $\rho_\varepsilon u_\varepsilon \rightharpoonup u$. We decompose $\rho_\varepsilon u_\varepsilon = m_\varepsilon = \mathbf{Q}(m_\varepsilon) + \mathbf{P}(m_\varepsilon)$ by means of the Helmholtz projection operator and analyse the convergence of the incompressible part $\mathbf{P}(m_\varepsilon)$.

Proposition 4.4.1. *Under the assumptions of Theorem 4.1.2, $\mathbf{P}(m_\varepsilon)$ converges strongly to u in $L^2(0,T; L^2_{loc}(\mathbf{R}^3))$ as ε goes to 0. Further, m_ε converges strongly to u in $L^2(0,T; L^2_{loc}(\mathbf{R}^3))$.*

Proof. From (4.2.6), we have $\mathbf{P}(m_\varepsilon) \in L^p(0,T;H^s(\mathbf{R}^3))$ for $0 \le s \le \frac{1}{2}$ and $1 \le p < \frac{4}{1+4s}$. Thus, there exists $\tilde{u} \in L^p(0,T;H^s(\mathbf{R}^3))$ such that $\mathbf{P}(m_\varepsilon) \rightharpoonup \tilde{u}$ weakly in $L^p(0,T;H^s(\mathbf{R}^3))$. Moreover, from

$$\partial_t \mathbf{P}(m_\varepsilon) + \mathbf{P}\left(\operatorname{div}\left(m_\varepsilon \otimes u_\varepsilon\right)\right) = 2\nu \mathbf{P}\left(\operatorname{div}(\sqrt{\rho_\varepsilon}\mathbf{S}_\varepsilon)\right) + \kappa^2 \mathbf{P}\left(\operatorname{div}\left(\nabla\sqrt{\rho_\varepsilon} \otimes \nabla\sqrt{\rho_\varepsilon}\right)\right),$$

with $\mathbf{S}_\varepsilon$ defined in (4.1.2), we conclude that $\partial_t \mathbf{P}(m_\varepsilon) \in L^2(0,T;H^{-s}(\mathbf{R}^3)$ for any $s > \frac{5}{2}$. Indeed, it suffices to observe that from the energy bounds of Lemma 4.2.2 we have $\nabla\sqrt{\rho_\varepsilon} \in L^\infty(0,T;L^2(\mathbf{R}^3))$, $\mathbf{S}_\varepsilon \in L^2(0,T;L^2(\mathbf{R}^3))$ and $\sqrt{\rho_\varepsilon}u_\varepsilon \in L^\infty(0,T;L^2(\mathbf{R}^3))$. In the virtue of the Aubin-Lions Lemma, we infer from $\mathbf{P}(m_\varepsilon) \in L^p(0,T;H^s(\mathbf{R}^3))$ with $0 \le s \le \frac{1}{2}$ and $1 \le p < \frac{4}{1+4s}$ and from $\partial_t \mathbf{P}(m_\varepsilon) \in L^2(0,T;H^{-s}(\mathbf{R}^3))$ for $s > \frac{5}{2}$ that if $\tilde{u}$ is the weak limit of $\mathbf{P}(m_\varepsilon)$ then

$$\mathbf{P}(m_\varepsilon) \to \tilde{u} \quad \text{strongly in} \quad L^2(0,T;L^2_{loc}(\mathbf{R}^3)).$$

We recall that $m_\varepsilon \rightharpoonup u$ weakly in $L^p(0,T;H^s(\mathbf{R}^3))$. In order to conclude that the sequence $\{m_\varepsilon\}$ converges strongly to u in $L^2(0,T;L^2_{loc}(\mathbf{R}^3))$, i. e. $u = \tilde{u}$, it remains to show that

$$\mathbf{Q}(m_\varepsilon) \to 0 \quad \text{strongly in} \quad L^2(0,T;L^2_{loc}(\mathbf{R}^3)).$$

From Theorem 4.3.1, one has that $\mathbf{Q}(m_\varepsilon)$ converges strongly to 0 in $L^2(0,T;B^\delta_{q,2}(\mathbf{R}^3))$ for some $q > 2$ and $\delta > 0$. We notice that $B^\delta_{q,2}(\mathbf{R}^3)$ is continuously embedded in L^q, thus $\mathbf{Q}(m_\varepsilon) \to 0$ in $L^2(0,T;L^q(\mathbf{R}^3))$ for some $q > 2$ and therefore in particular in $L^2(0,T;L^2_{loc}(\mathbf{R}^3))$. □

The strong convergence of $\sqrt{\rho_\varepsilon} - 1$ provided by Lemma 4.2.2 allows us to infer the strong convergence of $\sqrt{\rho_\varepsilon}u_\varepsilon$ to u.

Corollary 4.4.2. *Under the assumptions of Theorem 4.1.2, $\sqrt{\rho_\varepsilon}u_\varepsilon$ converges strongly to u in $L^2(0,T;L^2_{loc}(\mathbf{R}^3))$.*

Proof. It suffices to consider for any compact $K \subset \mathbf{R}^n$,

$$\begin{aligned}
\|\sqrt{\rho_\varepsilon}u_\varepsilon - u\|_{L^2(0,T;L^2(K))} &\le \|\rho_\varepsilon u_\varepsilon - u\|_{L^2(0,T;L^2(K))}\| + \|(1-\sqrt{\rho_\varepsilon})\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^2(0,T;L^2(K))} \\
&\le \|\rho_\varepsilon u_\varepsilon - u\|_{L^2(0,T;L^2(K))} \\
&\quad + C\|(1-\sqrt{\rho_\varepsilon})\|_{L^2(0,T;L^\infty(K))}\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty(0,T;L^2(K))} \\
&\le C\left(\|\rho_\varepsilon u_\varepsilon - u\|_{L^2(0,T;L^2(K))}\| + \varepsilon^\beta\right),
\end{aligned}$$

for some $\beta > 0$, where we used the convergence provided by Lemma 4.2.2 in the last step. □

The obtained compactness enables us to pass to limit in the weak formulation of the equations.

Lemma 4.4.3. *Under the assumptions of Theorem 4.1.2, the limit function u is a weak solution of* (4.0.5) *with initial data $u\big|_{t=0} = \mathbf{P}(u_0)$.*

Proof. Let $\phi \in C_c^\infty([0,T) \times \mathbf{R}^3; \mathbf{R}^3)$. We infer from Lemma 4.2.2 and Corollary 4.4.2, that passing to the limit $\varepsilon \to 0$ in the continuity equation

$$\int_{\mathbf{R}^3} \rho_{\varepsilon,0}\phi(0) + \int_0^T \int_{\mathbf{R}^3} \rho_\varepsilon \phi_t + \sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon} u_\varepsilon \nabla \phi = 0,$$

yields $\operatorname{div} u = 0$ in $\mathcal{D}'\left((0,T) \times \mathbf{R}^3\right)$. We consider the weak formulation of the momentum equation projected onto divergence free vector fields, let $\psi \in C_c^\infty([0,T) \times \mathbf{R}^3; \mathbf{R}^3)$ such that $\operatorname{div} \psi = 0$, and consider

$$\begin{aligned}
&\int_{\mathbf{R}^3} \rho_{\varepsilon,0} u_{\varepsilon,0} \psi(0) + \int_0^T \int_{\mathbf{R}^3} \sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon} u_\varepsilon \psi_t + (\sqrt{\rho_\varepsilon} u_\varepsilon \otimes \sqrt{\rho_\varepsilon} u_\varepsilon) \nabla \psi \\
&- 2\nu \int_0^T \int_{\mathbf{R}^3} (\sqrt{\rho_\varepsilon} u_\varepsilon \otimes \nabla \sqrt{\rho_\varepsilon}) \nabla \psi - 2\nu \int_0^T \int_{\mathbf{R}^3} (\nabla \sqrt{\rho_\varepsilon} \otimes \sqrt{\rho_\varepsilon} u_\varepsilon) \nabla \psi \\
&+ \nu \int_0^T \int_{\mathbf{R}^3} \sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon} u_\varepsilon \Delta \psi - 4\kappa^2 \int_0^T \int_{\mathbf{R}^3} (\nabla \sqrt{\rho_\varepsilon} \otimes \nabla \sqrt{\rho_\varepsilon}) \nabla \psi = 0,
\end{aligned} \tag{4.4.1}$$

Invoking Lemma 4.2.2 and Corollary 4.4.2, one concludes that the (4.4.1) converges to

$$\int_{\mathbf{R}^3} \mathbf{P}(u_0)\psi(0) + \int_0^T \int_{\mathbf{R}^3} u\psi_t + (u \otimes u)\nabla \psi + \nu \int_0^T \int_{\mathbf{R}^3} u \Delta \psi = 0.$$

Moreover we used that $\rho_{\varepsilon,0} u_{\varepsilon,0}$ converges weakly to u_0 in $L^{\frac{3}{2}-}_{loc}(\mathbf{R}^3)$ as consequence from (4.1.7) and Lemma 4.2.1. We conclude by recalling that ψ is divergence free. Therefore, there exists a function p defined on $(0,T) \times \mathbf{R}^3$ such that u is solution of

$$\partial_t u + u \cdot \nabla u + \nabla p = 2\nu \Delta u, \quad \operatorname{div} u = 0 \quad \text{in } \mathcal{D}'\left((0,T) \times \mathbf{R}^3\right),$$

with initial data $\mathbf{P}(u_0)$, where we recall that by (4.1.7) we assumed $\sqrt{\rho_\varepsilon^0} u_\varepsilon^0 \rightharpoonup u_0$ in $L^2(\mathbf{R}^3)$. $\square$

As we already said, at fixed $\varepsilon > 0$ the finite energy weak solutions $(\rho_\varepsilon, u_\varepsilon)$ to (4.0.3) satisfy a weak version of the energy inequality due to the degenerate viscosity, namely

$$E(t) + 2\nu \int_0^t |\mathbf{S}_\varepsilon|^2 \, \mathrm{d}s\mathrm{d}x \leq E(0),$$

where $\mathbf{S}_\varepsilon$ is given by (4.1.2). We remark that in fact in the limit as $\varepsilon \to 0$ it is possible to recover the usual energy dissipation. More precisely, the uniform boundedness of $\mathbf{S}_\varepsilon \in L^2(0,T,L^2(\mathbf{R}^3))$ only yields that $\mathbf{S}_\varepsilon \rightharpoonup S$ weakly in $L^2((0,T) \times \mathbf{R}^3)$. In the next Proposition we show that in fact we have $S = \frac{1}{2}\mathbf{D}u$. Moreover, by assuming the initial data to be well-prepared then by the convergence of the total energy at initial time we can also show that the limit function u obtained is indeed a Leray solution.

Proposition 4.4.4. *Under the assumptions of Theorem 4.1.2, let $\mathbf{S}_\varepsilon$ be as defined in* (4.1.2) *then*

$$\mathbf{S}_\varepsilon \rightharpoonup \mathbf{D}u \qquad \text{in} \quad L^2((0,T) \times \mathbf{R}^3).$$

Consequently, the limiting u *solution to* (4.0.5) *satisfies* $u \in L^\infty(0,T;L^2(\mathbf{R}^3)) \cap L^2(0,T;\dot{H}^1(\mathbf{R}^3))$. *If additionally,* $(\rho_\varepsilon^0, u_\varepsilon^0)$ *satisfies* (4.1.8)*, then* u *is a Leray solution of* (4.0.5)*, i.e. it satisfies* (4.1.9).

Proof. In virtue of Lemma 4.4.3 the limit function u is a weak solution of (4.0.5) with initial data $u\big|_{t=0} = \mathbf{P}(u_0)$. Next, we show that $\mathbf{S}_\varepsilon \rightharpoonup \mathbf{D}u$ in $L^2(0,T;L^2(\mathbf{R}^3))$. From Lemma 4.2.2, one has that $\mathbf{S}_\varepsilon \rightharpoonup S$ weakly in $L^2((0,T)\times\mathbf{R}^3)$. Moreover, $\sqrt{\rho_\varepsilon}\mathbf{S}_\varepsilon \to S$ in $\mathcal{D}'((0,T)\times\mathbf{R}^3)$. Indeed, let us write $\sqrt{\rho_\varepsilon}\mathbf{S}_\varepsilon = \mathbf{S}_\varepsilon + (\sqrt{\rho_\varepsilon}-1)\mathbf{S}_\varepsilon$. The second term converges to 0 in $\mathcal{D}'((0,T)\times\mathbf{R}^3)$ since $\sqrt{\rho_\varepsilon}-1 \to 0$ strongly in $L^\infty(0,T,L^q(\mathbf{R}^3))$ for $2 \le q < 6$ from Lemma 4.2.2. On the other hand, from (4.1.2) we infer that $\sqrt{\rho_\varepsilon}\mathbf{S}_\varepsilon \to \mathbf{D}u$ in $\mathcal{D}'((0,T)\times\mathbf{R}^3)$. Indeed, from Proposition 4.4.1, we have $\nabla(\rho_\varepsilon u_\varepsilon) \to \nabla u$ in $\mathcal{D}'((0,T)\times\mathbf{R}^3)$ and from $\nabla\sqrt{\rho_\varepsilon} \to 0$ in $L^2(0,T;L^2(\mathbf{R}^3))$ by Lemma 4.2.2, it follows $\nabla\sqrt{\rho_\varepsilon}\otimes\sqrt{\rho_\varepsilon}u_\varepsilon \to 0$ in $L^2(0,T,L^1(\mathbf{R}^3))$. Thus $S = \mathbf{D}u \in L^2(0,T;L^2(\mathbf{R}^3))$. We observe that for $u \in H^1(\mathbf{R}^3)$ such that $\operatorname{div} u = 0$, one has

$$\int_{\mathbf{R}^3} |\nabla u|^2 \,\mathrm{d}x = 2\int_{\mathbf{R}^3} |\mathbf{D}u|^2 \,\mathrm{d}x.$$

Finally, by lower semi-continuity we conclude that

$$\begin{aligned}
&\int_{\mathbf{R}^3} \frac{1}{2}|u|^2 \mathrm{d}x + \nu\int_0^t\int_{\mathbf{R}^3} |\nabla u|^2 \,\mathrm{d}x\mathrm{d}t \\
&\le \liminf_{\varepsilon\to 0} \int_{\mathbf{R}^3} \frac{1}{2}\rho_\varepsilon|u_\varepsilon|^2 + \kappa^2|\nabla\sqrt{\rho_\varepsilon}|^2 \mathrm{d}x + 2\nu\int_0^t\int_{\mathbf{R}^3} |\mathbf{S}_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t \\
&\le \int_{\mathbf{R}^3} \frac{1}{2}\rho_\varepsilon^0|u_\varepsilon^0|^2 + \kappa^2|\nabla\sqrt{\rho_\varepsilon^0}|^2 + F_\varepsilon^0 \mathrm{d}x.
\end{aligned}$$

Thus, $u \in L^\infty(0,T;L^2(\mathbf{R}^3))$ and $\nabla u \in L^2(0,T;L^2(\mathbf{R}^3)$. The additional uniform estimate is provided by Proposition 4.2.5. In order to conclude (4.1.9), it remains to show that,

$$\int_{\mathbf{R}^3} \frac{1}{2}\rho_\varepsilon^0|u_\varepsilon^0|^2 + \kappa^2|\nabla\sqrt{\rho_\varepsilon^0}|^2 + F_\varepsilon^0 dx \to \int_{\mathbf{R}^3} \frac{1}{2}|u^0|^2 \mathrm{d}x.$$

If the initial data satisfies (4.1.8), the proof is complete. □

Finally, we stress that, since the bounds obtained in Proposition 4.2.5 are uniform in $\varepsilon > 0$, they are also inherited by the solution to (4.0.5) obtained in the limit. Next Proposition proves the last statement of Theorem 4.1.2.

Proposition 4.4.5. *Let* u *be the solution to* (4.0.5) *obtained in the limit. Then for any* $0 < T < \infty$ *it satisfies* $u \in L^p(0,T;H^s(\mathbf{R}^3))$*, with* $0 \le s \le \frac{1}{2}$ *and* $1 \le p < \frac{1}{1+4s}$.

4.5 Strichartz estimates for the acoustic wave system

The main purpose of this Section is to give a proof of Proposition 4.3.2 (see Proposition 4.5.10 below), that is we want to study the dispersive properties satisfied by solutions to system (4.3.7). Even if this chapter only studies the three dimensional setting for the sake of completeness the whole analysis is carried out in the general d-dimensional setting ($d \geq 2$). As already mentioned, for $\varepsilon = 1$, the dispersive analysis associated to the operator H_1 has been carried out in [95, 96, 97]. In this study, we need to carefully track down the ε-dependence on the estimates as the (scaled) Mach number ε not only determines a time scale but also a frequency threshold such that the operator behaves differently. This is due to the non-homogenity of the dispersion relation and is opposite to the analysis of low Mach number limit in classical fluid dynamics where the Mach number ε only determines the time scale. The dispersive analysis for non-homogeneous symbols has been investigated in more general framework also in [93, 61, 92].

4.5.1 Dispersive estimate

In what follows we are going to prove the $L^\infty - L^1$ dispersive estimate associated for the semigroup e^{itH_ε}. For the convenience of the reader, we recall the stationary phase estimate in [95].

Proposition 4.5.1 ([95]). *Let $\phi(r) \in C^\infty(0,\infty)$ satisfy the following.*

(i) $\phi'(r), \phi''(r) > 0$ *for all* $r > 0$.

(ii) $\phi'(r) \sim \phi'(s)$ *and* $\phi''(r) \sim \phi''(s)$ *for all* $0 < s < r < 2s$.

(iii) $\left|\phi^{(k+1)}(r)\right| \lesssim \frac{\phi'(r)}{r^k}$ *for all* $r > 0$ *and* $k \in \mathbf{N}$.

Let $\chi(r)$ be a dyadic cut-off function with support around $r \sim R$ and that satisfies

$$|\chi^{(k+1)}(r)| \lesssim R^{-k}.$$

These estimates are supposed to hold uniformly for r and R, but may depend on k. Then if

$$I_\phi(t,x,R) := \int_{\mathbf{R}^d} e^{it\phi(|\xi|)+ix\cdot\xi}\, d\xi$$

we have

$$\sup_{x\in\mathbf{R}^d} |I_\phi(t,x,R)| \lesssim t^{-\frac{d}{2}} \left(\frac{\phi'(R)}{R}\right)^{-\frac{d-1}{2}} \left(\phi''(R)\right)^{-\frac{1}{2}} \tag{4.5.1}$$

Several observations are in order. We define

$$h(r) = \det\left(\mathrm{Hess}(\phi(r))\right), \tag{4.5.2}$$

and exploiting that ϕ is a radial function we compute

$$h(r) = \left(\frac{\phi'(r)}{r}\right)^{d-1} \phi''(r),$$

so that the right hand side of (4.5.1) involves $h(R)^{-1/2}$. This is consistent with the general theory for stationary phase estimates, see for instance Theorem 7.7.6 in [104]. Furthermore, from Proposition 2 in [61] it follows that the dispersive estimate (4.5.1) is sharp in the sense that there exists t_0 and R_0 such that for all $|t| > |t_0|$ and $R > R_0$ there exists $C > 0$ such that

$$\sup_{x\in\mathbf{R}^d} |I_\phi(t,x,R)| \geq Ct^{-\frac{d}{2}} \left(\frac{\phi'(R)}{R}\right)^{-\frac{d-1}{2}} \left(\phi''(R)\right)^{-\frac{1}{2}}. \tag{4.5.3}$$

In [95], the estimate (4.5.1) has been applied to the pseudo-differential operator H_1, i.e. $\phi(r) = r\sqrt{1+\kappa^2 r^2}$. We remark that the dispersive estimate for the symbol ω defined by the Bogliubov dispersion relation (4.3.1) can be obtained from estimate (4.5.1) by defining

$$\phi_\varepsilon(r) = \frac{1}{\varepsilon^2}\phi(\varepsilon r).$$

Indeed, we notice that $\omega(r) = \phi_\varepsilon(r)$ and after a computation that

$$\det\left(\mathrm{Hess}(\phi_\varepsilon(r))\right) = h(\varepsilon r),$$

as well as

$$I_{\phi_\varepsilon}(t,x,R) := \int_{\mathbf{R}^d} e^{ix\xi+it\phi_\varepsilon(r)}\chi(r)d\xi = \varepsilon^{-d} I_\phi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \varepsilon R\right), \tag{4.5.4}$$

by rescaling. We remark that the scaling affects the support of frequencies. Finally, to track down the ε-dependence in the dispersive estimate it is enough to study the properties of the Hessian matrix of ϕ_ε in terms of its determinant $h(\varepsilon r)$.

Lemma 4.5.2. *Let h be defined as in (4.5.2). There exists $C > 0$ such that for any $\lambda \in [0,\infty]$,*

$$0 \leq h(\lambda)^{-\frac{1}{2}} \leq C\frac{1}{\kappa^{\frac{d}{2}}}\left(\frac{\kappa\lambda}{\sqrt{1+(\kappa\lambda)^2}}\right)^{\frac{d-2}{2}} \tag{4.5.5}$$

For $d = 2$, there exists $C > 0$ such that for any $\lambda \in [0,\infty)$,

$$\frac{1}{\sqrt{3}} \leq h(\lambda)^{-\frac{1}{2}} \leq C.$$

Proof. This follows from immediate computations. □

The information on the (scaled) function h allows to derive the dispersive estimate for the symbol ϕ_ε.

Corollary 4.5.3. *Let $\phi_\varepsilon(r) = \frac{1}{\varepsilon}r\sqrt{1+(\varepsilon\kappa)^2 r^2}$, $R > 0$ be given and let $\chi(r) \in C_c(0,\infty)$ be as in Proposition 4.5.1. Then there exists a constant $C > 0$ such that*

$$\sup_{x\in\mathbf{R}^d} |I_{\phi_\varepsilon}(t,x,R)| \leq Ct^{-\frac{d}{2}}\left(\frac{\phi_\varepsilon'(R)}{R}\right)^{-\frac{(d-1)}{2}} \phi_\varepsilon''(R)^{-\frac{1}{2}} \tag{4.5.6}$$

In particular, this implies there exists $C > 0$ independent from ε such that,

$$\sup_{x\in\mathbf{R}^d} |I_{\phi_\varepsilon}(t,x,R)| \leq Ct^{-\frac{d}{2}}h(\varepsilon R)^{-\frac{1}{2}} \leq C't^{-\frac{d}{2}} \tag{4.5.7}$$

Proof. For fixed $\varepsilon > 0$, the assumptions of Proposition 4.5.1 on $\phi_\varepsilon(r)$ are met and (4.5.6) follows. By using (4.5.4) and (4.5.1) we obtain

$$\begin{aligned}\sup_{x\in\mathbf{R}^d} |I_{\phi_\varepsilon}(t,x,R)| &= \varepsilon^{-d} \sup_{x\in\mathbf{R}^d} \left| I_\phi(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \varepsilon R) \right| \\ &\leq Ct^{-\frac{d}{2}} \left(\frac{\phi'(\varepsilon R)}{\varepsilon R} \right)^{-\frac{d-1}{2}} \left(\phi''(\varepsilon R)\right)^{-\frac{1}{2}} = Ct^{-\frac{d}{2}} h(\varepsilon R)^{-\frac{1}{2}}.\end{aligned}$$

To conclude the second estimate, it is enough to observe that for $d \geq 2$ there exists $C > 0$ such that $h(r)^{-\frac{1}{2}} \leq C$ uniformly on $(0,\infty)$ as consequence of Lemma 4.5.2. □

The estimate in (4.5.7) implies that the operator H_ε has the same dispersive properties as the Schrödinger operator. As a consequence (4.5.7) would yield Schrödinger type dispersive estimates for frequency localized functions. However, from (4.5.5) we shall infer that in fact, for $d > 2$, we can derive better estimates, due to the regularizing effect of $\frac{(\varepsilon r)}{\sqrt{1+(\varepsilon r)^2}}$ when εr is small. This has already been pointed out in [95] for the operator e^{itH_1} and is explained by a different curvature of the geometric surface $|\xi|\sqrt{1+|\xi|^2}$ with respect to $|\xi|^2$. We reformulate this observation in the next Corollary.

Corollary 4.5.4. *Let $d \geq 2$, $\phi_\varepsilon(r) = \frac{1}{\varepsilon} r\sqrt{1+(\varepsilon\kappa)^2 r^2}$, $R > 0$ be given and let $\chi(r) \in C_c(0,\infty)$ be as in Proposition 4.5.1. Then there exists a constant $C > 0$ such that*

$$\sup_{x\in\mathbf{R}^d} |I_{\phi_\varepsilon}(t,x,R)| \leq \frac{C}{\kappa^{\frac{d}{2}}} t^{-\frac{d}{2}} \left(\frac{\varepsilon\kappa R}{\sqrt{1+(\varepsilon\kappa R)^2}} \right)^\delta, \tag{4.5.8}$$

for any $0 \leq \delta \leq \frac{d-2}{2}$.

Remark 4.5.5. We emphasize that the RHS in the estimate (4.5.8) blows up as κ goes to 0. This reflects the contribution of the quantum pressure term to the dispersion relation. In the absence of the quantum pressure, i.e. $\kappa = 0$, one recovers a linear dispersion relation for which wave-type dispersive estimates are to be expected and (4.5.8) being of Schrödinger type does not hold. However, here we consider the system (4.0.3) for fixed and bounded κ and thus we set $\kappa = 1$ in the following, see also the scaling chosen in Section 4.1.1.

This motivates to define the pseudo-differential operator U_ε corresponding to the Fourier multiplier

$$m(\varepsilon\xi) = \frac{(\varepsilon\xi)}{\sqrt{1+(\varepsilon|\xi|)^2}}, \quad \text{as} \quad U_\varepsilon := (-\varepsilon^2\Delta)^{\frac{1}{2}}(1-\varepsilon^2\Delta)^{-\frac{1}{2}}. \tag{4.5.9}$$

In particular, this allows to gain the factor ε^δ in the estimate at the expense of a factor R^δ corresponding to a loss of derivatives, see inequality (4.5.10).

4.5.2 Strichartz estimates

Next, we infer the needed Strichartz estimates from the dispersive estimate (4.5.8). First, we recall the definition of admissible exponents.

Definition 4.5.6. *We say the pair of exponents (p, q) is Schrödinger admissible if $2 \leq p, q \leq \infty$,*

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$$

and $(p, q, d) \neq (2, \infty, 2)$.

The first step consists in showing a pointwise in time estimate.

Lemma 4.5.7 (Pointwise estimate)**.** *For fixed $\varepsilon > 0$ and $R > 0$, let $f \in L^1(\mathbf{R}^d)$ such that supp$(\hat{f}) \subset \{\xi \in \mathbf{R}^d : \frac{1}{2}R \leq |\xi| \leq 2R\}$. The following estimate holds for any $2 \leq q \leq \infty$:*

$$\|e^{itH_\varepsilon} f\|_{L^q(\mathbf{R}^d)} \leq Ct^{-d\left(\frac{1}{2}-\frac{1}{q}\right)} \|U_\varepsilon^{\delta\left(1-\frac{2}{q}\right)} f\|_{L^{q'}(\mathbf{R}^d)},$$

and consequently

$$\|e^{itH_\varepsilon} f\|_{L^q(\mathbf{R}^d)} \leq Ct^{-d\left(\frac{1}{2}-\frac{1}{q}\right)} (\varepsilon R)^{\delta\left(1-\frac{2}{q}\right)} \|f\|_{L^{q'}(\mathbf{R}^d)}. \tag{4.5.10}$$

Proof. The operator e^{itH_ε} is unitary on L^2 therefore there exists $C_1 > 0$ not depending on ε, R so that

$$\|e^{itH_\varepsilon} f\|_{L^2(\mathbf{R}^d)} \leq C_1 \|f\|_{L^2},$$

Furthermore, Corollary 4.5.4 guarantees that there exists $C_2 > 0$ not depending on ε, R

$$\|e^{itH_\varepsilon} f\|_{L^\infty(\mathbf{R}^d)} \leq C_2 t^{-\frac{d}{2}} \|U_\varepsilon^\delta f\|_{L^1(\mathbf{R}^d)}.$$

By a standard interpolation argument we conclude the proof. Estimate (4.5.10) follows from

$$\left(\frac{\varepsilon R}{\sqrt{1+\varepsilon^2 R^2}}\right)^{\delta\left(1-\frac{2}{q}\right)} \leq (\varepsilon R)^{\delta\left(1-\frac{2}{q}\right)}.$$

□

Next, we show Strichartz estimates localized in frequencies on dyadic blocks.

Lemma 4.5.8. *For $d \geq 2$, $\varepsilon, R > 0$ and $0 < \delta \leq \frac{d-2}{2}$, let $f \in L^2(\mathbf{R}^d)$ and $F \in L^{p'}(0, T; L^{q'})$ such that supp$(\hat{f})$, supp$(\hat{F}(t)) \subset \{\xi \in \mathbf{R}^d : \frac{1}{2}R \leq |\xi| \leq 2R\}$ Then there exists a constant $C > 0$ independent from T, ε such that for any (p, q), (p_1, q_1) admissible pairs,*

$$\|e^{itH_\varepsilon} f\|_{L_t^p L_x^q} \leq C \|U_\varepsilon^{\delta\left(\frac{1}{2}-\frac{1}{q}\right)} f\|_{L^2}, \tag{4.5.11}$$

$$\left\|\int_{\mathbf{R}} e^{-itH_\varepsilon} F(t)\, dt\right\|_{L^2} \leq C \|U_\varepsilon^{\delta\left(\frac{1}{2}-\frac{1}{q}\right)} F\|_{L_t^{p'} L_x^{q'}}. \tag{4.5.12}$$

Moreover,

$$\begin{aligned} \left\| \int_{\mathbf{R}} e^{i(t-s)H_\varepsilon} F(s) \mathrm{d}s \right\|_{L^p_t L^q_x} &\leq C^2 \| U_\varepsilon^{\delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)} F\|_{L^{p_1'}_t L^{q_1'}_x}, \\ \left\| \int_{s<t} e^{i(t-s)H_\varepsilon} F(s) \mathrm{d}s \right\|_{L^p_t L^q_x} &\leq C^2 \| U_\varepsilon^{\delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)} F\|_{L^{p_1'}_t L^{q_1'}_x}. \end{aligned} \tag{4.5.13}$$

Proof. Given (4.5.8) and considering the fact that e^{itH_ε} is an isometry on $L^2(\mathbf{R}^d)$, we observe that Theorem 1 of [115] applies. We notice that the constants in the estimates (4.5.11) are identical as coming from an abstract duality argument. □

We remark that for $\varepsilon = 1$, we recover the Strichartz estimates provided by Theorem 2.1 in [95].

Proposition 4.5.9. *Let $d \geq 2$, $\varepsilon > 0$, $0 < \delta \leq \frac{d-2}{2}$. Then there exists a constant $C > 0$ independent from T, ε such that for any (p, q), (p_1, q_1) admissible pairs,*

$$\| e^{itH_\varepsilon} f\|_{L^p_t \dot{B}^0_{q,2}} \leq C \| U_\varepsilon^{\delta\left(\frac{1}{2}-\frac{1}{q}\right)} f\|_{L^2}, \tag{4.5.14}$$

and

$$\left\| \int_{s<t} e^{i(t-s)H_\varepsilon} F(s) \mathrm{d}s \right\|_{L^p_t \dot{B}^0_{q,2}} \leq C \| U_\varepsilon^{\delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)} f\|_{L^{p_1'} \dot{B}^0_{q_1',2}}. \tag{4.5.15}$$

Proof. By the scaling $t' = \frac{t}{R^2}$ and $x' = \frac{x}{R}$, for $R \in \mathbf{R}$, we achieve that $P_R(e^{it'H_\varepsilon} f)(t', x')$ is spectrally supported in the annulus $\{\xi \in \mathbf{R}^d : \frac{1}{2} \leq |\xi| \leq 2\}$. Therefore, we infer from (4.5.11) that

$$\begin{aligned} \| P_R \left(e^{itH_\varepsilon} f \right) \|_{L^p_t L^q_x} &= R^{\frac{d}{q}+\frac{2}{p}} \left\| P_1 \left(e^{it'H_\varepsilon} f \right) \right\|_{L^p_{t'} L^q_{x'}} \\ \leq C R^{\frac{d}{q}+\frac{2}{p}} \left\| P_1 \left(U_\varepsilon^{\frac{\beta(q)}{2}} f \right) \right\|_{L^2_{x'}} &\leq C \left\| P_R \left(U_\varepsilon^{\frac{\beta(q)}{2}} f \right) \right\|_{L^2} \end{aligned}$$

for any admissible pair (p, q). Similarly, the bound (4.5.13) implies that for admissible pairs (p, q) and (p_1, q_1) , we have

$$\left\| P_R \left(\int_{s<t} e^{i(t-s)H_\varepsilon} F(s) \mathrm{d}s \right) \right\|_{L^p_t L^q_x} \leq C^2 \left\| P_R \left(U_\varepsilon^{\delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)} F \right) \right\|_{L^{p_1'} L^{q_1'}}.$$

Hence, given an admissible pair (p, q), we compute

$$\begin{aligned} \left\| e^{itH_\varepsilon} f \right\|_{L^p \dot{B}^s_{q,2}} &\leq \left\| R^s \| P_R \left(e^{itH_\varepsilon} f \right) \|_{L^p L^q} \right\|_{l^2} \\ &\leq C \left\| R^s \left\| P_R \left(U_\varepsilon^{\delta(\frac{1}{2}-\frac{1}{q})} f \right) \right\|_{L^2} \right\|_{l^2} \\ &\leq C \left\| U_\varepsilon^{\delta(\frac{1}{2}-\frac{1}{q})} f \right\|_{\dot{B}^s_{2,2}}, \end{aligned}$$

where we have used Minkowski inequality in the first and third inequality and (4.5.11) in the second. Similarly, we proceed for (4.5.12). Indeed,

$$\begin{aligned}\left\|\int_{\mathbf{R}} e^{i(t-s)H_\varepsilon}F(s)\mathrm{d}s\right\|_{L^p\dot{B}^s_{q,2}} &= \left\|R^s\|P_R\left(\int_{\mathbf{R}} e^{i(t-s)H_\varepsilon}F(s)\mathrm{d}s\right)\|_{L^pL^q}\right\|_{l^2}\\ &\leq C\left\|R^s\left\|P_R\left(U_\varepsilon^{\delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)}F\right)\right\|_{L^{p_1'}L^{q_1'}}\right\|_{l^2}\\ &\leq C\|U_\varepsilon^{\delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)}F\|_{L^{p_1'}\dot{B}^s_{q_1',2}},\end{aligned}$$

□

The final estimates follows upon observing that the presence of the operator U_ε may be exploited to gain a factor ε as shown in (4.5.10). Further, for our purpose we need the Strichartz estimates to hold in non-homogeneous Besov spaces.

Proposition 4.5.10. *Fix $\varepsilon > 0$, fix $\frac{d-2}{2} > \delta > 0$ and $s \in \mathbf{R}$. Let $d \geq 2$. There exists a constant $C > 0$ independent from T, ε such that for any (p, q) admissible pair, the following hold true,*

$$\|e^{itH_\varepsilon}f\|_{L^p_tB^{-s-s_0}_{q,2}} \leq C\varepsilon^{\delta(\frac{1}{2}-\frac{1}{q})}\|f\|_{B^{-s}_{2,2}}, \tag{4.5.16}$$

where $s_0 = \delta\left(\frac{1}{2}-\frac{1}{q}\right)$.
Moreover for any (p_1, q_1) admissible, we have

$$\left\|\int_{s<t} e^{i(t-s)H_\varepsilon}F(s)\mathrm{d}s\right\|_{L^p_tB^{-s-s_1}_{q,2}} \leq C\varepsilon^{s_1}\|F\|_{L^{p_1'}B^{-s}_{q_1',2}}, \tag{4.5.17}$$

where $s_1 = \delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)$

Proof. For $\alpha \geq 0$, one has the following bound for the Fourier multiplier U^α_ε,

$$\|U^\alpha_\varepsilon f\|_{\dot{B}^s_{q,2}} \leq C\varepsilon^\alpha\|f\|_{\dot{B}^{s+\alpha}_{q,2}}.$$

Hence, we conclude

$$\|e^{itH_\varepsilon}f\|_{L^p_t\dot{B}^0_{q,2}} \leq C\varepsilon^{\delta(\frac{1}{2}-\frac{1}{q})}\|f\|_{B^{\delta(\frac{1}{2}-\frac{1}{q})}_{2,2}} \tag{4.5.18}$$

Moreover for any (p_1, q_1) admissible, we have

$$\left\|\int_{s<t} e^{i(t-s)H_\varepsilon}F(s)\mathrm{d}s\right\|_{L^p_t\dot{B}^0_{q,2}} \leq C\varepsilon^{\delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)}\|F\|_{L^{p_1'}\dot{B}^{\delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)}_{q_1',2}}. \tag{4.5.19}$$

For $s > 0$, one has that $B^s_{q,r}$ is continuously embedded in $\dot{B}^s_{q,r}$ if q is finite with

$$\|f\|_{\dot{B}^s_{q,r}} \leq \frac{C}{s}\|f\|_{B^s_{q,r}},$$

while for $s<0$ the space $\dot{B}^s_{q,r}$ is continuously embedded in $B^s_{q,r}$ with

$$\|f\|_{B^s_{q,r}} \leq \frac{C}{|s|}\|f\|_{\dot{B}^s_{q,r}}.$$

Using these embeddings and applying the estimates to $(1-\Delta)^{-\frac{\tilde{s}}{2}}f$ with $\tilde{s}=s+\delta\left(\frac{1}{2}-\frac{1}{q}\right)$, we obtain from (4.5.18) that

$$\|e^{itH_\varepsilon}f\|_{L^p_t B^{-s-\delta\left(\frac{1}{2}-\frac{1}{q}\right)}_{q,2}} \leq C\varepsilon^{\delta\left(\frac{1}{2}-\frac{1}{q}\right)}\|f\|_{B^{-s}_{2,2}}.$$

Similarly, from (4.5.19), we conclude

$$\left\|\int_{s<t} e^{i(t-s)H_\varepsilon}F(s)\mathrm{d}s\right\|_{L^p_t B^0_{q,2}} \leq C\varepsilon^{\delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)}\|F\|_{L^{p_1'}B^{\delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)}_{q_1',2}}. \tag{4.5.20}$$

Applying (4.5.20) to $(1-\Delta)^{-\frac{\tilde{s}}{2}}F$ with $\tilde{s}=s+\delta\left(1-\frac{1}{q}-\frac{1}{q_1}\right)$ yields (4.5.17) $\square$

CHAPTER 5

Scaling limit for the QHD system and applications to vortex dynamics

Abstract

This chapter studies the scaling limit for the QHD system. The characteristic length scale of QHD system which is determined by the healing length is assumed to be infinitesimally small. The scaling limit is reminiscent to the incompressible limit. Section 5.1 proves that in the finite energy framework the convergence provides the trivial solution. We discuss the same limit in the presence of vortices by rephrasing [28] in the hydrodynamic framework in Section 5.2.

The characteristic length scale for the QHD system (2.0.1) over which the density ρ varies is given by the healing length ξ. The healing length, typically very small, characterises the core size of a quantized vortex being radial and of radius $\mathcal{O}(\xi)$, see Proposition 2.5.2. This motivates to consider equilibrium states at infinitesimally small healing length $\xi \to 0$, namely the regime in which the size of the vortex core shrinks to 0. If we denote $\varepsilon := \xi$ and rescale the spatial variable by $\frac{1}{\varepsilon}$, we obtain that the scaled stationary vortex solution of degree n given by Proposition 2.5.2 reads

$$\sqrt{\rho_\varepsilon} = f_n^\varepsilon(|x|), \qquad \Lambda_\varepsilon = n f_n^\varepsilon(|x|)\frac{x^\perp}{|x|^2}, \tag{5.0.1}$$

where $f_n^\varepsilon(|x|) = f_n(\frac{|x|}{\varepsilon})$ with f_n solution of (1.3.2). Taking into account the properties of solutions to (1.3.2), see also [103], we find that $f_{\varepsilon,n} \sim \left(\frac{|x|}{\varepsilon}\right)^n$ for $|x| << \varepsilon$, while for $|x| >> \varepsilon$, we have $1 - \frac{\varepsilon^2 n^2}{|x|^2} \leq f_{n,\varepsilon}(|x|) \leq 1$. For a single vortex, we hence recover that as $\varepsilon \to 0$,

$$\sqrt{\rho_\varepsilon} \to 1, \qquad \Lambda_\varepsilon \to n\frac{x^\perp}{|x|^2},$$

Moreover, this leads to

$$\operatorname{div} J_\varepsilon \to 0 \quad \text{in} \quad \mathcal{D}', \qquad \nabla \wedge J_\varepsilon \to 2\pi n \delta_0 \quad \text{in} \quad \mathcal{D}'.$$

One is hence heuristically led to an incompressible dynamics with highly concentrated vorticity. Formally, $\rho_\varepsilon \to 1$ and the weak limit of J_ε, that can be identified with the weak limit of Λ_ε, defines a velocity field satisfying

$$\operatorname{div} u = 0, \qquad \operatorname{curl} u = 2\pi n\delta.$$

An incompressible target system for the scaling limit is consistent with the physical intuition given in the Introduction of this thesis, see Section 0.1. The quantum pressure is relevant to the system if the density varies on length scales of order $\mathcal{O}(\varepsilon)$. Thus, heuristically by scaling the space variable by $\frac{1}{\varepsilon}$ one expects that the dynamics is described by the compressible Euler equation, as can easily be seen by comparing

$$P \sim \frac{1}{\gamma}\rho^{\gamma}, \qquad \text{and} \qquad P_{quantum} = \rho\nabla^2 \log\rho,$$

where we used (2.0.3) for the Bohm potential. As discussed in Chapter 4, the presence of the quantum pressure modifies the dispersion relation and, because of quantum effects, leads to the celebrated Bogoliubov dispersion relation and heuristically explains that this modification occurs at a threshold $\frac{1}{\varepsilon}$. For the scaled equations (5.0.2), this corresponds to density changes on length-scales of order $\mathcal{O}(\varepsilon)$. Rescaling further time and the current in a suitable way formally leads to a low Mach number regime. More precisely, we perform the typical parabolic scaling, $t' = \frac{t}{\varepsilon^2}$ and $x' = \frac{x}{\varepsilon}$. Hence, the hydrodynamic unknowns scale as

$$\sqrt{\rho_\varepsilon}(t',x') = \sqrt{\rho}\left(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\right), \qquad \Lambda_\varepsilon(t',x') = \frac{1}{\varepsilon}\Lambda\left(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\right),$$

leading us to investigate the scaling limit $\varepsilon \to 0$ for the rescaled QHD system

$$\begin{cases} \partial_t\rho_\varepsilon + \operatorname{div} J_\varepsilon = 0 \\ \partial_t J_\varepsilon + \operatorname{div}(\Lambda_\varepsilon \otimes \Lambda_\varepsilon) + \dfrac{1}{\varepsilon^2}\nabla p(\rho_\varepsilon) = \dfrac{1}{2}\rho_\varepsilon\nabla\left(\dfrac{\Delta\sqrt{\rho_\varepsilon}}{\sqrt{\rho_\varepsilon}}\right). \end{cases} \tag{5.0.2}$$

The scaled energy functional reads

$$\mathcal{E}_\varepsilon(\sqrt{\rho_\varepsilon},\Lambda_\varepsilon) = \int_{\mathbf{R}^d} \frac{1}{2}|\nabla\sqrt{\rho_\varepsilon}|^2 + \frac{1}{2}|\Lambda_\varepsilon|^2 + \frac{1}{\varepsilon^2\gamma(\gamma-1)}\left(\rho_\varepsilon^\gamma - 1 - \gamma(\rho_\varepsilon - 1)\right)\mathrm{d}x, \tag{5.0.3}$$

and corresponds to the celebrated Ginzburg-Landau energy for $\gamma = 2$. Formally, as $\varepsilon \to 0$, the target system ($d = 2$) is given by the incompressible Euler equation

$$\partial_t u + u\cdot\nabla u + \nabla p = 0, \qquad \operatorname{div} u = 0. \tag{5.0.4}$$

We are going to consider two distinguished scenarios: firstly we consider finite energy weak solutions to the primitive system (2.0.1), secondly and more interestingly we address the scaling limit for vortex solutions of (5.0.2) which carry infinite energy. In the former case, we show that any sequence of finite energy weak solutions consistent with a scaled wave-function dynamics converges to the trivial solution $\rho = 1$ and $u = 0$. This results holds true in $\mathbf{R}^d$ for $d = 2,3$. In view of the considerations made in Section 2.5.2, the assumption of finite energy rules out the presence of vortices. The physical intuition suggests that since rotational motion is only carried by quantized vortices appearing in the vacuum region, no dynamics should emerge in the given asymptotic regime. The solution of the target system is divergence free, irrotational and of finite energy, hence the trivial one. This heuristics is made rigorous in Section 5.1. In Section 5.2, we aim to consider a sequence of solutions to (2.0.1) exhibiting vortices. We observe that the energy for the stationary solution (5.0.1) to (5.0.2) computed on a sufficiently large ball $B_R(0) \subset \mathbf{R}^2$ reads

$$\mathcal{E}_\varepsilon(\sqrt{\rho_\varepsilon}, \Lambda_\varepsilon, B_R) = 2\pi n^2 |\log \varepsilon| + 2\pi n^2 |\log R| + C,$$

thus is diverging logarithmically. In order to compute the energy functional on the plane, we renormalize it by means of (2.2.2). In this context, we rephrase the result in [28] in hydrodynamic variables. Our motivation is to illustrate the analogy and differences of the vortex dynamics for (2.0.1) in the ε-limit compared to the dynamics of highly concentrated vorticity in an ideal and incompressible fluid. It is well-known that in this context the incompressible Euler equation reduces to the singular dynamics described by the Kirchhoff law (5.2.4) for point vortices. The mathematical model for point vortices has been introduced in [140]. The motion of a two-dimensional ideal incompressible fluid in which the vorticity is given by a diffuse part and a sum of Dirac δ is described by the vortex wave system [139, 124, 65]. For the scaled Gross-Pitaevskii equation (5.2.5), the main result in [28] states that for small $\varepsilon > 0$ the evolution of a sufficiently well-prepared multi-vortex configuration through (5.2.5) stays close to the multi-vortex configuration that is given by the evolution of the initial configuration through the Kirchhoff-Onsager ODE system, at least up to the first collision.

5.1 Scaling limit for finite energy weak solutions

This section presents the analysis of the scaling limit in the class of finite energy weak solutions provided by Theorem 2.1.3, namely solutions that are consistent with a underlying wave-function dynamics. As pointed out, the dynamics is asymptotically governed by an ideal incompressible fluid. Further, we consider finite energy weak solutions for the primitive system that satisfy the generalized irrotationality condition

$$\nabla \wedge J_\varepsilon = 2\nabla\sqrt{\rho_\varepsilon} \wedge \Lambda_\varepsilon,$$

that on its turn will entail $\nabla \wedge u = 0$ in the ε-limit. We recall the scaled energy functional is defined in (5.0.3).

Theorem 5.1.1. *Let $d = 2, 3$ and let $(\rho_\varepsilon, J_\varepsilon)$ be a finite energy weak solution consistent with a wave-function dynamics and such that there exists $C > 0$ with*

$$\mathcal{E}_\varepsilon(\sqrt{\rho_\varepsilon}, \Lambda_\varepsilon) \leq C,$$

for a.e. $t \in [0,T]$*. Then* $(\rho_\varepsilon, J_\varepsilon)$ *converges to* $\rho = 1$ *and* $J = 0$ *as* $\varepsilon \to 0$*. More precisely,*

$$\begin{aligned} \rho_\varepsilon - 1 \to 0 \quad &in \quad L^\infty(0,T;L^2(\mathbf{R}^d)); \\ J_\varepsilon \rightharpoonup^* 0 \quad &in \quad L^\infty(0,T;L^p_{loc}(\mathbf{R}^d)); \\ \Lambda_\varepsilon \rightharpoonup^* 0 \quad &in \quad L^\infty(0,T;L^2(\mathbf{R}^d)). \end{aligned}$$

The proof of the Theorem relies crucially on the assumptions that the sequence of solutions $(\rho_\varepsilon, J_\varepsilon)$ is consistent with the respective wave-function dynamics. This consistency allows us to exploit the continuity of the vorticity provided by Lemma 2.2.7 and to infer that the vorticity vanishes in the ε-limit. The incompressible limit for finite energy weak solutions consistent with the wave-function dynamics and $\gamma = 2$ has also been investigated in [128]. For well-prepared solutions it is shown that the limit system is governed by the incompressible Euler equations. We stress that in virtue of Theorem 5.1.1 the only limit solution that can be obtained is the trivial one $\rho = 1$ and $u = 0$. We remark that at present, we are not able to derive the same result for a sequence of finite energy weak solutions to (5.0.2) that is not necessarily consistent with the wave-function dynamics. The main obstruction is that Lemma 2.2.7 crucially relies on the consistency of the hydrodynamic variables with a wave-function ψ. Without this information, it is not clear whether

$$\nabla \wedge J_\varepsilon = 2\nabla\sqrt{\rho_\varepsilon} \wedge \Lambda_\varepsilon \rightharpoonup^* 0 \qquad \text{in} \quad L^\infty([0,T];L^1(\mathbf{R}^d)).$$

The uniform bound in $L^\infty L^1$ only ensures that up to passing to subsequences $\nabla \wedge J_\varepsilon$ converges to a measure μ.

If the system (2.0.1) is considered as model for capillary fluids in a general framework not limited to quantum fluids, it is meaningful to study finite energy weak solutions without the constraint of the generalized irrotationality condition. In this setting, the analysis of the scaling limit for finite energy solutions is conceivably more involved than the one presented in Chapter 4: the analysis of Theorem 4.1.2 relies on the uniform bounds deriving from the dissipation. Moreover, the global existence of weak solution to the incompressible Euler equations for $d = 3$ is open so that local strong solutions for the target system need to be considered. This analysis has been carried out in [72] for the problem posed on $\mathbf{T}^d$, where the authors use a relative entropy method to show convergence to local strong solutions of the incompressile Euler equation for ill-prepared data under suitable regularity assumptions.

We summarize needed uniform bounds in the next Lemma that are implied by Theorem 2.1.3.

Lemma 5.1.2. *Let* $T > 0$ *and* $\{\psi_\varepsilon^0\} \subset \mathbb{E}_2$ *be such that* $\mathcal{E}_2(\psi_\varepsilon^0)$ *is uniformly bounded in* ε*. Let* $(\rho_\varepsilon, J_\varepsilon)$ *be the finite energy weak solution to* (5.0.2) *on* $[0,T) \times \mathbf{R}^d$ *provided by Theorem 2.1.3. Then* $\mathcal{E}_\varepsilon(\sqrt{\rho_\varepsilon}(t), \Lambda_\varepsilon(t)) = \mathcal{E}_2(\psi_\varepsilon^0)$ *for all times and the following bounds hold uniformly in* ε*,*

$$\begin{aligned} &\sqrt{\rho_\varepsilon} - 1 \in L^\infty([0,T];H^1(\mathbf{R}^d)), \quad \Lambda_\varepsilon \in L^\infty([0,T];L^2(\mathbf{R}^d)), \\ &\rho_\varepsilon - 1 \in L^\infty([0,T];L^2(\mathbf{R}^d) \cap L^\gamma_2(\mathbf{R}^d)) \quad J_\varepsilon \in L^\infty([0,T];L^2(\mathbf{R}^d) + L^p(\mathbf{R}^d)), \end{aligned}$$

where $1 \leq p < 2$ for $d = 2$ and $1 \leq p \leq \frac{3}{2}$ for $d = 3$. In particular,

$$\|\sqrt{\rho_\varepsilon} - 1\|_{L^\infty([0,T];L^2(\mathbf{R}^d))} \leq C\varepsilon^\alpha,$$

for some $\alpha > 0$. The vorticity enjoys the uniform bound

$$\nabla \wedge J_\varepsilon = 2\nabla\sqrt{\rho_\varepsilon} \wedge \Lambda_\varepsilon \in L^\infty([0,T];L^1(\mathbf{R}^d)).$$

Proof. The bounds are immediate consequences of the uniform energy bound and Lemma 2.2.3. We recall that $F(\rho_\varepsilon) \in L^1(\mathbf{R}^d)$ is equivalent to $\rho_\varepsilon - 1 \in L_2^\gamma(\mathbf{R}^d)$ that together with the bound on $\nabla\sqrt{\rho_\varepsilon}$ in $L^\infty L^2$ yields $\sqrt{\rho_\varepsilon} - 1 \in L^\infty([0,T];H^1(\mathbf{R}^d))$ uniformly, see also Lemma 2.2.3. Proceeding as in the proof of Lemma 4.2.1, we recover that there exists $\alpha > 0$ such that

$$\|\sqrt{\rho_\varepsilon} - 1\|_{L^\infty([0,T];L^2(\mathbf{R}^d)} \leq C\varepsilon^\alpha.$$

The uniform $L^\infty L^1$-bound for the vorticity follows from Hölder inequality. □

Next, we show that the vorticity $\nabla \wedge J_\varepsilon$ converges to 0 in distributional sense.

Proposition 5.1.3. *Let $(\rho_\varepsilon, J_\varepsilon)$ be a finite energy weak solution to* (5.0.2) *provided by Theorem 2.1.3 such that there exists $C > 0$ independent from ε such that for all $t \in [0,T]$ one has $\mathcal{E}_\varepsilon(\sqrt{\rho_\varepsilon}, \Lambda_\varepsilon) \leq C$. Then*

$$\nabla_x \wedge J_\varepsilon \to 0 \quad in \quad \mathcal{D}'((0,T) \times \mathbf{R}^d).$$

For $d = 2$, the result of Proposition 5.1.3 can also be infered from the following quantitative result provided by [106], see also Appendix B in [105]. If $\Omega \subset \mathbf{R}^2$ is a bounded domain, $0 < \varepsilon \leq 1$ and $\psi \in H^1(\Omega, \mathbf{C})$ such that $\mathcal{E}_\varepsilon(\psi) < \pi|\log\varepsilon|$, then

$$\|\mathrm{Jac}(\psi)\|_{\dot{W}^{-1,1}(\Omega)} \leq \varepsilon C\mathcal{E}_\varepsilon(\psi) exp(\frac{1}{\pi}\mathcal{E}_\varepsilon(\psi)). \tag{5.1.1}$$

Since $2\mathrm{Jac}(\psi) = \nabla \wedge J$, this quantifies the decay of the vorticity in the ε-limit. The energy bound ensures that the wave function exhibits no vortices. The motivation for the choice of the norm $\dot{W}^{-1,1}$ will become clear in Section 5.2. Here, we provide an alternative proof that yields convergence in distributional sense while the Jacobian estimate is tailored for a fine analysis of the concentration of Jacobian, i.e. vorticity in the hydrodynamic framework, in the presence of vortices.

Proof. Since provided by Theorem 2.1.3, the solution $(\rho_\varepsilon, J_\varepsilon)$ is generated by a sequence of solutions to the scaled nonlinear Schrödinger equation (5.2.5) that satisfies $\{\psi_\varepsilon\} \subset C(\mathbf{R}; \mathbb{E}_2)$. Moreover, we infer that there exists $\psi \in L^\infty(0,T;H^1_{loc}(\mathbf{R}^d))$ such that up to passing to a subsequence ψ_ε converges weakly-$*$ to ψ. From Lemma 2.2.7, it follows that there exists a subsequence J_{ε_k} that converges weakly-$*$ to $J = \mathrm{Im}(\overline{\psi}\nabla\psi)$ in $L^\infty(0,T;L^p_{loc}(\mathbf{R}^d))$ with $1 < p < 2$ for $d = 2$ and $1 < p \leq \frac{3}{2}$ for $d = 3$. Moreover, from the uniform energy bound one has that up to subsequences $\nabla\sqrt{\rho_\varepsilon}$ converges weakly-$*$ to 0 in $L^\infty(0,T;L^2(\mathbf{R}^d))$. Indeed, since

$\sqrt{\rho_\varepsilon} - 1 \in L^\infty([0,T]; H^1(\mathbf{R}^d))$ uniformly and $\sqrt{\rho_\varepsilon} - 1 \to 0$ strongly $L^\infty([0,T]; L^2(\mathbf{R}^d))$, the weak $L^\infty L^2$-limit of $\nabla\sqrt{\rho_\varepsilon}$ equals 0. There exists $\Lambda \in L^\infty(0,T; L^2(\mathbf{R}^d))$ such that again up to subsequences Λ_ε converges weakly-$*$ to Λ. By applying Lemma 2.2.7 we get that for all $\eta \in \mathcal{D}((0,T)\times\mathbf{R}^d)$,

$$\int_{(0,T)\times\mathbf{R}^d} (\nabla \wedge J_\varepsilon)\eta \to \int_{(0,T)\times\mathbf{R}^d} (\nabla\wedge J)\eta$$

Since J is induced by a wave-function $\psi \in H^1_{loc}(\mathbf{R}^d)$, it follows from Lemma 2.2.3 that

$$\nabla \wedge J = \nabla\wedge \mathrm{Im}(\overline{\psi}\nabla\psi) = 2\nabla\sqrt{\rho}\wedge\Lambda = 0, \quad a.e.\,\text{in}\,\mathbf{R}^d,$$

due to $\nabla\sqrt{\rho} = 0$ in the ε-limit. Hence, we have proven that

$$\nabla\wedge J_\varepsilon = 2\nabla\sqrt{\rho_\varepsilon}\wedge\Lambda_\varepsilon \to 0 \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathbf{R}^d).$$

□

We deduce that the current J_ε vanishes in the limit as $\varepsilon \to 0$.

Corollary 5.1.4. *Under the assumption of Proposition 5.1.3,*

$$J_\varepsilon \rightharpoonup 0 \quad in \quad L^p_{loc}(\mathbf{R}^d), \qquad \Lambda_\varepsilon \rightharpoonup 0 \quad in \quad L^2(\mathbf{R}^d).$$

Proof. Since $J_\varepsilon \in L^p(\mathbf{R}^d) + L^2(\mathbf{R}^d)$ for some $1 \le p < \frac{3}{2}$, the current J_ε defines a tempered distribution. We have that $\nabla\wedge J_\varepsilon \to 0$ in $\mathcal{S}'$ from Proposition 5.1.3. Moreover, for any $\eta \in \mathcal{D}((0,T)\times\mathbf{R}^d)$,

$$\int_{(0,T)\times\mathbf{R}^d} \mathrm{div}\, J_\varepsilon \eta \mathrm{d}x\mathrm{d}t = \varepsilon\int_{(0,T)\times\mathbf{R}^d} \frac{\partial_t(\rho_\varepsilon - 1)}{\varepsilon}\eta\mathrm{d}x\mathrm{d}t \to 0.$$

By density, also $\mathrm{div}\, J = 0$ in $\mathcal{S}'$. It follows that J is a tempered distribution such that $\mathrm{div}\, J = \nabla\wedge J = 0$. Since $J = \Lambda$ is square integrable on $\mathbf{R}^d$ it follows that $J = 0$, see e.g. Chapter 1 in [55]. □

We are now in position to show Theorem 5.1.1.

Proof of Theorem 5.1.1. We check that $\rho_\varepsilon - 1$ converges strongly to 0 in $L^\infty([0,T]; L^2(\mathbf{R}^d))$. With the uniform bounds of Lemma 5.1.2 at hand, we proceed as in the proof of Lemma 4.2.1 to infer the strong convergence result. Corollary 5.1.4 yields the statement of Theorem 5.1.1. □

5.2 Applications to vortex dynamics

This section provides the hydrodynamic rephrasing of the result in [28]. For that purpose, we briefly recall the point vortex system for an ideal incompressible fluid [140]. Subsequently, we discuss the analysis of vortices for the QHD sytem (2.0.1). For $d = 2$, the incompressible Euler equation reads

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \qquad \operatorname{div} u = 0, \tag{5.2.1}$$

where the unknown u describes the velocity field and p the pressure. We define the vorticity as $\omega = \nabla \wedge u$. In $d = 2$, the velocity field u and the vorticity ω are related through the Biot-Savart law,

$$v = \left(\nabla^{\perp} G\right) * \omega, \qquad G(x) = -\frac{1}{2\pi} \log |x|, \tag{5.2.2}$$

where G denotes the Green function. The velocity field u can hence be recovered from ω provided that the vorticity satisfies suitable regularity and integrability properties. If u is solution to (5.2.1), then ω formally satisfies the transport equation

$$\partial_t \omega + u \cdot \nabla \omega = 0. \tag{5.2.3}$$

At least in the smooth setting ω is constant along the trajectories $\omega(t, \Phi_t(x)) = \omega_0(x)$ where Φ is the flow associated to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi_t(x) = u(t, \Phi_t(x)), \qquad \Phi_0(x) = x.$$

There is a vast literature also in the context of rough vector fields initiated by the seminal paper [69]. This description gives rise to a variety of conserved quantities associated to (5.2.3) that turn out to be crucial for the purpose of the localization of vorticity. We recall the well-known existence and uniqueness result by Yudovich.

Theorem 5.2.1 ([110]). *Let $\omega_0 \in L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$. There exists a unique global weak solution (u, ω) of* (5.2.1) *complemented with $\omega(0, x) = \omega_0(x)$. Moreover, $\omega \in L^\infty(\mathbf{R}_+; L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2))$, and u, ω are related by* (5.2.2)*. Further, ω is a weak solution to* (5.2.3).

We also recall the well-known result from Delort [67].

Theorem 5.2.2 ([67]). *Let ω_0 is a positive Radon measure such that $\omega_0 \in H^{-1}(\mathbf{R}^2)$, then there exists a weak solution u to* (5.2.1).

In the present context, our motivation is the investigation of the dynamics of point vortices that has intrigued scientists since Helmholtz [102] suggested that in an ideal incompresible fluid vorticity that is highly concentrated in a collection of points stays concentrated for finite time. Since $\delta_0 \notin H^{-1}(\mathbf{R}^2)$, none of the previous results applies. The physical intuition leads to the idea that a single vortex describes a stationary solution in the absence of interaction with the boundary and a suitable symmetric approximation formally shows that if the initial data is given by a sum of Dirac δ located in $\mathbf{a} = (a_1, ..., a_k)$ with intensities $\mathbf{n} = (n_1, ..., n_k)$ and mean $n = \sum_{i=1}^k n_i$, i.e.

$$\omega_0 = \sum_{i=1}^{k} n_i \delta_{a_i(0)},$$

then the dynamics is formally governed up to the first collision time by the Kirchhoff [119] law

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} a_i(t) = \sum_{i=1}^{k} n_i \frac{(a_i(t)-a_j(t))^\perp}{|a_i(t)-a_j(t)|^2}, & \text{for} \quad i = 1, ..., N, \\ a_i(0) = a_i^0, \quad \text{for} \quad i = 1, ..., k \end{cases} \tag{5.2.4}$$

System (5.2.4) has been subject to a remarkable number of papers, we refer the reader to [140] and reference therein. The Kirchhoff law is a Hamiltonian system where the Hamiltonian is commonly referred to as Kirchhoff-Onsager functional and reads,

$$H(\mathbf{a}, \mathbf{n}) = -\sum_{i \neq j} n_i n_k \log |a_i - a_j|.$$

System (5.2.4) is known to have global solutions if all n_i have the same sign, collisions may occur otherwise. In particular, in [140] an explicit example of three point vortices that collapse in a self-similar way is given. The analysis of (5.2.4) substantially relies on its conserved quantities. While, as mentioned, a link between (5.2.1) and (5.2.4) had already been suspected since the days of Helmholtz, it has systematically been investigated in [140]. The authors show that if the vorticity is initially sharply concentrated at vortex blobs around points $\mathbf{a} = (a_1, ..., a_k)$, then it stays concentrated up to a finite time. The localization property of the vorticity turns out to present the key difficulty for the proof of the asymptotic dynamics. The strategy in [140] relies on conserved quantities for (5.2.3). Once the localization property is obtained, the authors prove that the dynamics is asymptotically described by the Kirchhoff law (5.2.4). The strategy requires that the blobs are well separated at initial time on scales of order $\mathcal{O}(1)$ and proves that on time scales of order $\mathcal{O}(1)$ the vorticity remains sharply concentrated. A series of improvements has been made subsequently. We mention in particular existence and uniqueness results for the vortex-wave system describing the evolution of a two-dimensional incompressible fluid with vorticity given by a sum of Dirac δ and a diffuse part [139, 124, 65]. Further, we indicate a very recent result [54] that is of interest in the present context as it uses weak concentration measures that are reminiscent of the techniques used for the analysis of Ginzburg-Landau vortices [63]. The stability of vortex patches has been investigated in [171]. The authors derive a L^1-stability criterion for circular vortex patches, where the L^1-stability is of Lyapunov type. The L^1-distance between a patch of arbitrary geometry and a circular one of the same total mass is controlled. In [164], stability in L^1 of circular vortex patches is established among the class of all bounded vortex patches of equal strength. More precisely,

Theorem 5.2.3 ([164])**.** *Let $B = B_r(0)$. Then for any bounded set $A \subset \mathbf{R}^2$, we have that*

$$\|\omega_{B_R} - \omega_{A(t)}\|_{L^1}^2 \leq 4\pi \sup_{\Omega_0 \Delta B_R} \left| |x|^2 - r^2 \right| \|\omega_{B_R} - \omega_{A(t)}\|_{L^1},$$

for all $t > 0$.

Given a configuration $(\mathbf{a}, \mathbf{n})$ such that their minimum mutual distance is of order one, and vortex patches of the form

$$\omega_i = \frac{n_i}{\varepsilon^2} \mathbf{1}_{B_\varepsilon}(a_i),$$

a direct computation shows that for the velocity u given by the Biot-Savart law the kinetic energy computed on a ball containing all

$$E_{Euler}(B_R) = \sum_{i=1}^{k} n_i^2 |\log \varepsilon|^2 + H(\mathbf{a}, \mathbf{n}) + C(d^2, \log R) + C,$$

where $H(\mathbf{a}, \mathbf{n})$ is the Kirchhoff-Onsager functional and C an absolute constant. We stress that the divergent term (in ε) does not depend on the position of the vortices but only on the degree.

Let us turn to the analysis of quantized vortices. Extensive literature has been devoted to the study of vortices in the limit for the wave-function dynamics described the scaled Gross-Pitaevskii equation

$$i\partial_t \psi_\varepsilon = -\frac{1}{2}\Delta\psi_\varepsilon + \frac{1}{\varepsilon^2}(|\psi_\varepsilon|^2 - 1)\psi_\varepsilon, \tag{5.2.5}$$

with Ginzburg-Landau energy,

$$\mathcal{E}_\varepsilon = \int_{\mathbf{R}^d} \frac{1}{2}|\nabla\psi_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(|\psi_\varepsilon|^2 - 1)^2 \mathrm{d}x. \tag{5.2.6}$$

It is well-known that for suitable initial data the vortex dynamics is governed by the Kirchhoff-Onsager ODE system (5.2.4). This was predicted in [78] and numerically investigated in [150]. Typically the initial data is given by small perturbations of the scaled multi-vortex configurations (1.3.3). First rigorous results appeared for the study of bounded domains [62, 63, 133]. The problem has been addressed in [28] on the plane. Based on (1.3.3), given $(a_1, ..., a_k) \in R^{2k}$ and $n_i = \pm 1$, we introduce

$$\psi_\varepsilon^*(a_i, n_i)(x) := \prod_{i=1}^{k} f_{\varepsilon, n_i}(|z - a_i|)\left(\frac{z - a_i}{|z - a_i}|\right)^{n_i}, \tag{5.2.7}$$

where f_{ε,n_i} as defined in (5.0.1) and set again $n = \sum_{i=1}^{k} n_i$. The energy of (5.2.7) computed on a large ball B_R containing all a_i is given by

$$\mathcal{E}(\psi_\varepsilon(a_i, n_i)) = \pi \sum_{i=1}^{k} n_i^2 |\log\varepsilon| + H(a_i, d_i) + C(d^2, |\log R|),$$

where H is the Kirchhoff-Onsager functional and C an absolute constant. The configuration (5.2.7) turns out to be almost energy minimizing in the class of functions with prescribed total degree equal to n. The energy of Ginzburg-Landau vortices has been intensively investigated, see for instance [25] and [28] and references therein. The assumptions $n_i = \pm 1$ is a necessary condition for the configuration to be almost energy minimizing since the energy depends quadratically on the degrees of the vortices. In hydrodynamic variables, the vortex configuration (5.2.7) reads,

$$\sqrt{\rho_\varepsilon^*} = \prod_{j=1}^{k} f_{\varepsilon, m_j}(|x - a_j|), \qquad \Lambda_\varepsilon^* = \left(\prod_{j=1}^{n} f_{\varepsilon, n_j}(|x - a_j|)\right) \sum_{i}^{k} n_j \frac{|x - a_j|^\perp}{|x - a_j|^2}. \tag{5.2.8}$$

Exploiting the properties of f_{ε,n_i} being the scaled solution of (1.3.2), we notice that the size of the vortex cores is of order $\mathcal{O}(\varepsilon^2)$ for such a configuration, while

$$\Lambda_\varepsilon^* \to \sum_i^k n_j \frac{|x-a_j|^\perp}{|x-a_j|^2}.$$

One computes that the vorticity for $(\sqrt{\rho_\varepsilon^*}, \Lambda_\varepsilon^*)$ concentrates in the sense that

$$\nabla \wedge J_\varepsilon^* \to 2\pi \sum_{i=1}^k n_i \delta_{a_i}.$$

The vortex configuration (5.2.8) therefore approximates circular vortex patches of size ε^2 with intensity n_i that weakly concentrate as a sum of weighted Dirac δ. The main result in [28] states that if the initial data is sufficiently close to a configuration of the type (5.2.7) or (5.2.8) respectively then the vorticity remains sharply concentrated. More precisely, up to first collision times, the vorticity remains sharply concentrated close to a vortex configuration of type (5.2.7) where (a_i, d_i) is given by the solution of (5.2.4) with initial data (a_i^0, d_i^0). We give the precise statement of the Theorem rephrased in hydrodynamic variables. We define $A_n = B(2^n) \backslash B(2^{n-1}) \subset \mathbf{R}^2$ and recall the Definition of well-preparedness [28].

Definition 5.2.4. *Let $a_1, ..., a_N$ be N points in $\mathbf{R}^2$, let $d_i = \pm 1$ for $i = 1, ..., N$. We say that a family $\{\psi_\varepsilon\}_{0<\varepsilon<1}$ of maps in $\mathcal{V}_*(\mathbf{R}^2) + H^1(\mathbf{R}^2)$ is well-prepared w.r.t. the vortex configuration (a_i, d_i) if and only if there exits $R = 2^{n_0} > \max\{|a_i|\}$ and $K_0 > 0$ such that the associated hydrodynamic variables $(\sqrt{\rho_\varepsilon}, \Lambda_\varepsilon)$ satisfy*

$$\lim_{\varepsilon \to 0} \left\| \nabla \wedge J_\varepsilon - 2\pi \sum_{i=1}^N d_i \delta_{a_i} \right\|_{[C_c^{0,1}(B(R))]^*} = 0, \tag{5.2.9}$$

$$\sup_{0<\varepsilon<1} \mathcal{E}_\varepsilon(\sqrt{\rho_\varepsilon}, \Lambda_\varepsilon, A_n) \leq K_0, \quad \forall n \geq n_0, \tag{5.2.10}$$

and denoting $\mathcal{E}_{\varepsilon,\Lambda_d}(\cdot,\cdot)$ the scaled renormalized energy functional (2.2.2) w.r.t. to $\Lambda_d = \left(\frac{x^\perp}{|x|^2}\right)^d$, it holds

$$\lim_{\varepsilon \to 0} \left[\mathcal{E}_{\varepsilon,\Lambda_d}(\sqrt{\rho_\varepsilon}, \Lambda_\varepsilon) - \mathcal{E}_{\varepsilon,\Lambda_d}(\sqrt{\rho_\varepsilon}^*, \Lambda_\varepsilon^*(a_i, n_i))\right] = 0 \tag{5.2.11}$$

The assumption (5.2.9) describes the assumption that the vorticity is concentrated close to the sum of Dirac deltas characterized by the vortex configuration (a, d). Inequality (5.2.10) implements the requirement that no vortex is located at spatial infinity. Finally, in view of condition (5.2.11) on the difference of the renormalized energies, the solution $(\sqrt{\rho_\varepsilon}, \Lambda_\varepsilon)$ is sufficiently well approximated by (5.2.8) with vortex configuration (a, d) in the sense that the excess energy is small.

In hydrodynamic terms, Theorem 1 in [28] then reads as follows.

Theorem 5.2.5. *Let $\{\psi_\varepsilon^0\}_{0<\varepsilon<1}$ be well-prepared with respect to the configuration (a^0, n_i). Define $\sqrt{\rho_\varepsilon^0} = |\psi_\varepsilon^0|$ and $J_\varepsilon^0 = Im(\overline{\psi_\varepsilon^0}\nabla\psi_\varepsilon^0)$ and denote by $(\sqrt{\rho_\varepsilon}, \Lambda_\varepsilon)$ the solution to* (5.0.2) *provided by Theorem 2.1.4. Let $\{a_i(t)\}_{i=1,...,N}$ denote the solution of the point-vortex system* (5.2.4) *with initial data $(a_i^0)_{i=1,..,N}$ and let (T_*, T^*) denote its maximal interval of existence. Then, for every $T_* \leq t \leq T^*$, the solution $(\sqrt{\rho_\varepsilon}, \Lambda_\varepsilon)$ is well-prepared.*

The Theorem should be read as follows. Fixed $\mathbf{a} = (a_1, ...a_k)$ and $n_i = \pm 1$, let us assume that the initial data is well-prepared with respect to the vortex configuration described by (5.2.8). Then up a to a finite time depending on the first collision, the hydrodynamic solution $(\rho_\varepsilon, \Lambda_\varepsilon)$ exhibits a vorticity that is highly characterised in the sense that it is concentrated close to the multi-vortex configuration (5.2.8) where the vortex cores are the evolution of the initial vortex cores by the Kirchhoff law - in analogy to what happens for ideal incompressible fluids. This results remains valid up to the first collision time that may occur in finite time [140]. In [28] a quantitative analysis for small $\varepsilon > 0$ is provided while previous results only characterise the ε-limit. This is due to an improvement of refined Jacobian estimates introduced in [107] measuring the concentration of vorticity. To consider the problem on an unbounded domain entails several mathematical obstacles such as the divergence of the energy and the need of a renormalized energy functional as considered in [32], see also Section 1.3 and the respective version (2.1.1) for (2.0.1). As it emerged in Section 2.5.2 the suitable notion of degree is highly not trivial, see e.g. [28] and [6]. The proof in the framework of the wave-function dynamics described by (5.2.5) in [28] is based on several crucial facts. The solution given by (1.3.3) is almost energy minimizing if initially well-prepared in the sense of Definition 5.2.4. Highly accurate Jacobian estimates provide a characterization of the solution for small $\varepsilon > 0$ meaning that the vorticity is concentrated in a suitable sense close to a sum of Dirac δ. This allows one to control the distance of ψ_ε to an almost energy minimizing vortex configuration of the type (5.2.7) by means of a relative energy functional. However, we stress that the analysis in the context of (5.2.5) requires conceivably more sophisticated techniques. Once the localization property is obtained, one considers the evolution of the vorticity described by the equation

$$\partial_t (\nabla \wedge J_\varepsilon) = -\nabla \wedge \operatorname{div}(\Lambda_\varepsilon \otimes \Lambda_\varepsilon) - \nabla \wedge \operatorname{div}(\nabla\sqrt{\rho_\varepsilon} \otimes \nabla\sqrt{\rho_\varepsilon}), \tag{5.2.12}$$

to be seen as analogue of (5.2.3) - even though due to the compressibility of system (2.0.1), equation (5.2.12) does not describe a transport equation. The strong concentration property together with the mentioned relative entropy functional can be exploited to derive the approximation by the ODE system in the ε-limit.
In the context of (5.0.2) and (5.2.5), we deal with a compressible system; difficulties can mathematically be associated to the fact that while the Biot-Savart law (5.2.2) (formally) relates velocity and vorticity while the same is not true in the quantum setting. On the level of the wave-function dynamics the Jacobian corresponding to half of the vorticity does not relate to the velocity field, thought of as the phase gradient in the ε-limit, nicely. Mathematically, this can be also put in the context of lifting problems. This is even less clear in the context of the

generalized irrotationality condition $\nabla \wedge J_\varepsilon = 2\nabla\sqrt{\rho_\varepsilon} \wedge \Lambda_\varepsilon$. On the other hand, the dynamics of (5.0.2) and (5.2.5) provides information that is not available in the classical setting. For instance, the quantized vortices have a radial-symmetric geometry and a determined core size that is prescribed by the healing length. The scaling parameter ε arises naturally. To complete the analogy with classical fluid dynamics, we notice that the stronger concentration estimates for the vorticity of (5.2.5) compared to (5.2.1) suggest that the dynamics of a quantum fluid allows one to better localize concentrated vorticity. While the stability estimates of Theorem 5.2.3 and the one in [171] are posed in L^1, the vorticity for the QHD system concentrates in $W^{-1,1}$. We mention the recent result [54] where the authors rely on weak concentration measure to analyse the vortex dynamics for the incompressible Euler equation. In this respect, we also point out that for $d = 3$ in [105] leap-frogging was proved for vortex rings solution to (5.2.5) while to the best of our knowledge the analogue for the incompressible Euler equations is open. It is therefore natural to pose the question if the results for (5.2.5) can be recovered for (5.0.2) without relying on the wave-function dynamics. For instance, it would be of great interest to study the scaling limit for (5.0.2) for $d = 2$ only relying on the hydrodynamic variables $(\sqrt{\rho}, \Lambda)$. To that end, it occurs to further study to which extent the properties of the finite energy weak solutions to (2.0.1) including the generalized irrotationality condition $\nabla \wedge J = 2\nabla\sqrt{\rho} \wedge \Lambda$ can compensate for the properties entailed by the wave-function dynamics.

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