Gran Sasso Science Institute

PHD PROGRAMME IN MATHEMATICS IN NATURAL, SOCIAL AND LIFE SCIENCES

November 5, 2021

ERGODIC BEHAVIOR OF CONTROL SYSTEMS AND FIRST-ORDER MEAN FIELD GAMES

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Ai miei genitori.

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Resumé

Les travaux de cette thèse concernent l'analyse de systèmes de jeu à champ moyen (MFG) du premier ordre avec contrôle de l'accélération et l'étude du comportement en temps moyen long de systèmes de contrôle de type sous-riemannien.

Plus précisément, dans la première partie nous commençons par étudier le caractère bien posé du système MFG associé à un problème de commande à équation linéaire en espace et en état de commande. En particulier, nous prouvons l'existence et l'unicité des solutions généralisées et nous étudions également leur régularité. Ensuite, nous nous concentrons sur le système MFG avec contrôle de l'accélération, un cas particulier de celui décrit ci-dessus, et nous étudions le comportement en temps moyen long des solutions en montrant la convergence vers une constante ergodique. Un tel système MFG est donné par

$$\begin{cases} -\partial_t u^T(t,x,v) + \frac{1}{2}|D_v u^T(t,x,v)|^2 - \langle D_x u^T(t,x,v), v\rangle \\ \qquad\qquad = F(x,v,m_t^T), & \text{dans } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ \partial_t m_t^T - \langle v, D_x m_t^T\rangle - \text{div}\left(m_t^T D_v u^T(t,x,v)\right) = 0, & \text{dans } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^T(T,x,v) = g(x,v,m_T^T), \quad m_0^T(x,v) = m_0(x,v) & \text{dans } \mathbb{T}^d\times\mathbb{R}^d. \end{cases}$$

Ici, comme pour l'analyse précédente, le principal problème est le manque de convexité et de coercivité stricte du Hamiltonien par rapport à la variable de quantité de mouvement. Cela conduit par exemple à la non-existence de solutions de viscosité continue aux équations ergodiques de Hamilton-Jacobi et, par conséquent, ce permet pas de définir le système MFG ergodique au sens classique. Nous concluons cette première partie en établissant un lien entre le système MFG avec contrôle de l'accélération et le système MFG classique. Pour ce faire, nous étudions le problème de perturbation singulière pour le système d'accélération MFG, c'est-à-dire que nous analysons le comportement des solutions aux systèmes de jeu à champ moyen dont le coût d'accélération devient nul. Encore une fois, nous résolvons le problème en utilisant des techniques de calcul des variations en raison du problème résultant du manque de convexité et de coercivité strictes du Hamiltonien par rapport à la variable de quantité de mouvement.

Dans la deuxième partie, nous nous concentrons sur les systèmes de contrôle affine sans dérive (de type sous-riemannien), c'est-à-dire les dynamiques contrôlées de la forme

$$\dot{\gamma}(t) = \sum_{i=1}^{m} u_i f_i(\gamma(t))$$

où les f_i sont $m \in \{1, \cdots, d\}$ champs de vecteurs définis sur $\mathbb{R}^d$, et les contrôles u_i sont des fonctions mesurables sur $\mathbb{R}^m$. A la différence du cas de l'accélération, nous montrons qu'il existe une constante critique et que l'équation ergodique de Hamilton-Jacobi associée à une telle constante qui possède des solutions de viscosité continues.

Pour cela nous faisons appel à la géométrie sous-riemannienne sur l'espace d'état. Toujours en utilisant les propriétés de cette géométrie, nous définissons le semi-groupe de Lax-Oleinink et nous prouvons l'existence d'un point fixe de ce semi-groupe. Nous concluons cette partie, et donc cette thèse, en étendant la célèbre théorie d'Aubry-Mather au cas du système de contrôle sous-riemannien. Nous montrons d'abord une formule de représentation variationnelle de la constante critique et, à partir de celle-ci, nous définissons l'ensemble de Mather et l'ensemble d'Aubry. En utilisant une approche dynamique, nous étudions les propriétés analytiques et topologiques de tels ensembles comme, par exemple, la différentiabilité horizontale de la solution critique en tout point se trouvant dans l'un des deux ensembles. Enfin, nous appliquons ces résultats pour étudier le caractère bien posé du système MFG ergodique associé à de tels systèmes de contrôle.

Abstract

The work in this thesis concerns the analysis of first-order mean field game (MFG) systems with control of acceleration and the study of the long time-average behavior of control systems of sub-Riemannian type.

More precisely, in the first part we begin by studying the well-posedness of the MFG system associated with a control problem with linear state equation. In particular, via a relaxed approach, we prove the existence and the uniqueness of mild solutions and we also study their regularity. Then, we focus on the MFG system with control of the acceleration, a particular case of the one above, and we investigate the long time-average behavior of solutions showing the convergence to the critical constant. Such MFG system is given by

$$\begin{cases} -\partial_t u^T(t,x,v) + \frac{1}{2}|D_v u^T(t,x,v)|^2 - \langle D_x u^T(t,x,v), v\rangle \\ \qquad\qquad = F(x,v,m_t^T), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ \partial_t m_t^T - \langle v, D_x m_t^T\rangle - \operatorname{div}\left(m_t^T D_v u^T(t,x,v)\right) = 0, & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^T(T,x,v) = g(x,v,m_T^T), \quad m_0^T(x,v) = m_0(x,v) & \text{in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases}$$

Here, as for the previous analysis, the main issues are the lack of strict convexity and coercivity of the Hamiltonian with respect to the momentum variable. Indeed, for instance, when studying the asymptotic behavior of the control system this lead us to a non existence result of continuous viscosity solutions to the ergodic Hamilton-Jacobi equation. Consequently, it does not allowed us to the define the ergodic MFG system as one would expect. We conclude this first part establishing a connection between the MFG system with control of acceleration and the classical one. To do so, we study the singular perturbation problem for MFG system of acceleration, that is, we analyze the behavior of solutions to the system when the acceleration cost goes to zero. Again, we solve the problem by using variation techniques due to the problems arising from the lack of strict convexity and coercivity of the Hamiltonian with respect to the momentum variable.

In the second part, we concentrate the attention to drift-less affine control systems (sub-Riemannian type), i.e., controlled dynamics of the form

$$\dot{\gamma}(t) = \sum_{i=1}^{m} u_i f_i(\gamma(t))$$

where f_i are $m \in \{1, \cdots, d\}$ vector fields defined on $\mathbb{R}^d$, and the controls u_i are measurable functions on $\mathbb{R}^m$. Differently from the case of acceleration, we prove that there exists a critical constant and the ergodic Hamilton-Jacobi equation associated with such a constant has continuous viscosity solutions. This is possible appealing to the properties of the sub-Riemannian geometry on the state space. Still using the properties of

this geometry we finally define the Lax-Oleinink semigroup and we prove the existence of a fixed point of such semigroup. We conclude this part, and thus this thesis, extending the celebrated Aubry-Mather Theory to the case of sub-Riemannian control system. We first show a variational representation formula for the critical constant and from this we define the Aubry set. By using a dynamical approach we study the analytical and topological properties of such sets as, for instance, horizontal differentiability of the critical solution at any points lying in such a set.

Contents

Introduction

Before discussing the state of the art and introducing the topics covered in this thesis, we proceed to motivate why we get interested in these problems.

The work is divided in two parts: the first is devoted to the systematic study of Mean Field Game (MFG) systems with control on the acceleration; in the second we address the problem of long time-average behavior of solutions to Hamilton-Jacobi equations associated with sub-Riemannian control systems and Aurby-Mathery Theory for these latter. We will explain below how these two topics are related.

Let us start describing the first. In the recent years there have been an increasing attention to the study of multi-agent systems with control on the acceleration, i.e., models of interacting individuals in which each player wants to control their acceleration instead of the classical control of the velocity. For this reason in Chapter 2 we study the well-posedness of MFG systems associated with such control problems. By using a relaxed notion of Nash equilibrium (MFG equilibrium) we provide existence, uniqueness and regularity results for the so-called mild solutions. We conclude by investigating the connections between these solutions and the PDEs system. Then, the aim of Chapter 3 is to study the long time-average behavior of solutions to the MFG system studied in Chapter 2 as the time horizon goes to infinity. The main issue for this is the lack of small time controllability that prevents to define the associated ergodic MFG system in the standard way. We conclude this first part addressing the problem of singular perturbation for "pure" control systems and for MFG with control of acceleration in Chapter 4. In particular, solving this problem we found a relation between MFG of acceleration and the classical system.

At this point, the difficulties in Chapter 3 lead us to the following question: are these issues common to more general control systems than the control of acceleration? To address this problem, we start with a general drift-less control system and in Chapter 5 we address the problem of the long time behavior of solutions to Hamilton-Jacobi equations. Note that, the case of control of acceleration does not fit into this class of systems since it has a linear non-zero drift. However, what we immediately realize is that sub-Riemannian control systems are locally small time controllable. So, by using new ideas which relies on the different geometry on the state space we prove the existence of a critical constant and of a critical viscosity solution to the ergodic Hamilton-Jacobi equation. Moreover, we study the well-posedness of the Lax-Oleinik semigroup and we prove the existence of a fixed-point. Finally, in Chapter 6 we extend the well-known Aubry-Mather theory for Tonelli Hamiltonian systems to the sub-Riemannian ones.

More details on the results and on the difficulties to achieve them are given in the following sections.

Mean field games

Since MFG is the common subject of the first part of this thesis, we introduce here the argument and describe the state of the art.

Game theory is a branch of mathematics which aims to describe the behavior of a group of interacting agents. Fix, for instance, this number to $N \in \mathbb{N}$. Each player satisfy a certain dynamics that depends on the interaction with the other agents and they choose their strategy in order to minimize/maximize a certain cost functional. A fundamental tool in the analysis of these models is the notion of *Nash equilibria*, introduce by Nash in [62]. Roughly speaking, a strategy is called a Nash equilibrium if each agent is not interested to be the unique who changes strategy.

However, the study of the N-players games lead to several issues as N becomes large and, in this case, we are interested in describing the behavior of Nash equilibria as $N \to \infty$. In order to overcame these difficulties, MFG system has been introduced by J.M. Lasry and P.L. Lions in [54, 55, 56] and a similar analysis was also developed, in the same years but independently, by P. Caines, M. Huang and R. Malhamé in [47, 48]. At the macroscopic level the model turns out to be described by a systems of PDEs: an Hamilton-Jacobi equation which describes the single agent's strategy and a Kolmogorov Fokker-Planck equation (continuity equation) which explains how the distribution of players evolves in time according to the optimal strategy provided by the first equation. Classically, the mean-field interaction term that coupled the two equation is given by a function of space and measure and, moreover, the drift appearing in the continuity equation depends on the value function satisfying the Hamilton-Jacobi equation. Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be an Hamiltonian function, let $F : \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ be the coupling function describing the interaction of the agents, let $\mu_0 \in \mathscr{P}(\mathbb{R}^d)$ be the initial distribution of players in space and let $G : \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ be the terminal costs. Then, the simplest for of this system is the following

$$\begin{cases} -\partial_t u(t,x) + H(x, D_x u(t,x)) = F(x, m_t), & (t,x) \in [0,T] \times \mathbb{R}^d \\ \partial_t m_t - \operatorname{div}\big(m_t D_p H(x, D_x u(t,x))\big) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ m_0 = \mu_0, \quad u(T,x) = G(x, m_T), & x \in \mathbb{R}^d. \end{cases}$$

Let us describe heuristically the meaning of such a system. To do so, let $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be the Lagrangian associated with the Hamiltonian H by taking the Legendre Transform. Then, each player choose is own strategy in order to minimize the cost functional of the form

$$\int_t^T \big(L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m_s)\big)\, ds + G(\gamma(T), m_T)$$

where the control system is of the following simple form

$$\dot\gamma(s) = u(s), \quad (s \in [0,T]).$$

There is by now an extensive literature concerning MFG system of the above form concerning problems as existence, uniqueness and regularity of solutions depending on the assumptions on F. For an overview on the subject we refer the reader to [16, 35, 36, 46, 28], which is however far from being complete.

So far, most of the literature concerns the analysis of the above system describing models in which the agents has control only of their velocity. However, in many applications, see for instance [39], one might be interested in studying systems in which players needs to have control on their acceleration. In this case, proceeding heuristically as before, we have that each agent choose is strategy in order to minimize a cost functional of the form

$$\int_t^T \big(L(\gamma(s), \dot\gamma(s), \ddot\gamma(s)) + F(\gamma(s), \dot\gamma(s), m_s)\big)\, ds + G(\gamma(T), m_T)$$

where the control system now has the form

$$\begin{cases} \dot\gamma(t) = v(t), \\ \dot v(t) = u(t). \end{cases}$$

Hence, the PDEs system is given by

$$\begin{cases} -\partial_t u(t,x,v) + H(x, v, D_x u(t,x,v), D_v u(t,x,v)) = F(x, v, m_t), & (t,x,v) \in [0,T] \times \mathbb{R}^{2d} \\ \partial_t m_t - \mathrm{div}_{x,v}\big(m_t D_p H(x, v, D_x u(t,x,v), D_v u(t,x,v))\big) = 0, & (t,x,v) \in [0,T] \times \mathbb{R}^{2d} \\ m_0 = \mu_0, \quad u(T,x,v) = G(x, v, m_T), & (x,v) \in \mathbb{R}^{2d}. \end{cases}$$

Note that, now the state space is not $\mathbb{R}^d$ but $\mathbb{R}^d \times \mathbb{R}^d$ which takes into account not only the position but also the dependence of the strategy on the velocity $v \in \mathbb{R}^d$. Consequently, we also have that for any $t \in [0,T]$ the distribution m_t is a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$.

This is what motivated us at beginning of this project to analyze this problem and it is what Chapter 2 concerns, i.e., the study of the existence, uniqueness and regularity of solutions to the MFG system with control of acceleration. In particular, we consider the acceleration model as embedded into a more general setting which is the case of linear state equation linear, that is, a dynamics of the form

$$\dot\gamma(t) = A\gamma(t) + Bu(t)$$

for some constant matrices A and B.

MFG for linear control systems

Fixed a time horizon $T > 0$, we consider players having the following dynamics the whole space $\mathbb{R}^d$

$$\dot\gamma(t) = A\gamma(t) + Bu(t), \quad \forall\, t \in [0,T] \tag{1}$$

where A and B are real matrices and u is a measurable control function. Each player aims to minimize a cost functional of the form

$$\int_0^T L(\gamma(s), u(s), m_s)\, ds + G(\gamma(T), m_T), \tag{2}$$

where, for each time $t \in [0,T]$, the probability measure m_t on $\mathbb{R}^d$ represents their distribution. In this framework the MFG system reads as

$$\begin{cases} -\partial_t V(t,x) + H(x, D_x V(t,x), m_t) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ \partial_t m_t + \mathrm{div}\Big(m_t D_p H(x, D_x V(t,x), m_t)\Big) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ m_0 = m_0, \quad V(T,x) = G(x, m_T), \ \forall\, x \in \mathbb{R}^d \end{cases} \tag{3}$$

where the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$H(x, p, m) = \sup_{u \in \mathbb{R}^k} \big\{ -\langle Ax + Bu, p\rangle - L(x, u, m)\big\}.$$

One can immediately observe that if the Lagrangian L is of Tonelli type (strictly convex and coercive w.r.t. control variable) then H fails to be Tonelli. Hence, using the standard approach in MFG via fixed-point methods would lead us to several issues.

Therefore, in order to overcome this issue we solve the problem via the Lagrangian approach (see, for instance, [18] and [60]). That is, we define the metric space

$$\Gamma_T = \Big\{\gamma \in \mathrm{AC}([0, T]) : \gamma(t) \text{ satisfy } (1),\ \gamma(0) \in \mathbb{R}^d\Big\},$$

endowed with uniform metric $\| \cdot \|_\infty$ and we consider Borel probability measures η supported on Γ_T. Then, we restrict the attention to probability distributions on $\mathbb{R}^d$ of the form $m_t = e_t \sharp \eta$ where $e_t : \Gamma_T \to \mathbb{R}^d$ denotes the evaluation map and $\sharp$ stands for the push-forward operator. This correspond to consider only flow of measures concentrated on trajectories satisfying (1).

Let us describe the results of this work. The first problem we deal with is the definition of MFG equilibria for this class of problems. So, given an initial distribution of players $m_0 \in \mathscr{P}(\mathbb{R}^d)$ we say that $\eta \in \mathscr{P}(\Gamma_T)$ is a MFG equilibrium if it is supported on minimizing curves of (2), with starting point in $\mathrm{spt}(m_0)$. Then, we prove that such equilibria exist (Theorem 2.13) and having this at our disposal we give the definition of mild solutions, $(V, m) \in C([0, T] \times \mathbb{R}^d) \times C([0, T]; \mathscr{P}(\mathbb{R}^d))$, of our MFG problem. For these, we study the existence, the uniqueness and the regularity. In particular, we show that $\{e_t \sharp \eta\}_{t\in[0,T]}$ is $\frac{1}{2}$-Hölder continuous in time (Theorem 2.17) and, consequently, the value function V is locally semiconcave on $[0, T] \times \mathbb{R}^d$ linearly in space and with fractional semiconcave modulus in time (Theorem 2.18). Moreover, by standard tools of optimal control theory we get that V is locally Lipschitz continuous (Theorem 2.20).

Under an extra growth assumption on the Lagrangian, we also show that there exists a MFG equilibrium such that the flow of measures $\{e_t \sharp \eta\}_{t\in[0,T]}$ is Lipschitz continuous in time. This yields to linear semiconcavity estimates for the value function V both in space and in time. In conclusion, we show that the notion of mild solution is strictly related with the classical definition of weak solutions for the MFG system. Indeed, we prove that they coincide, in the sense that: a mild solution is a weak solution and vice versa (Theorem 2.30).

After this work was submitted, similar results were obtained in [1] for the special case of mean field games with control on acceleration.

Ergodic behavior of MFG of acceleration

In this Chapter we focus the attention on a special case of system (2.21). Indeed, we consider the case of control of acceleration which can be written as

$$\begin{bmatrix}\gamma(t)\\ \dot\gamma(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\gamma(t)\\ \dot\gamma(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}0\\ u(t)\end{bmatrix}, \quad t \geq 0.$$

In this case, we have that the MFG system is given by

$$\begin{cases} -\partial_t u^T(t,x,v) + \frac{1}{2}|D_v u^T(t,x,v)|^2 - \langle D_x u^T(t,x,v), v\rangle \\ \qquad = F(x,v,m_t^T), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ \partial_t m_t^T - \langle v, D_x m_t^T\rangle - \operatorname{div}\Big(m_t^T D_v u^T(t,x,v)\Big) = 0, & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^T(T,x,v) = g(x,v,m_T^T), \quad m_0^T(x,v) = m_0(x,v) & \text{in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases} \tag{4}$$

During the last years, the question of the long time behavior of solutions of (standard) MFG systems has attracted a lot of attention. Results describing the long-time average of solutions were obtained in several context: see [30, 31], for second order systems on $\mathbb{T}^d$, and [29, 19, 20], for first order systems on $\mathbb{T}^d$, $\mathbb{R}^d$ and for state constraint case respectively. Recently, Cardaliaguet and Porretta studied the long time behavior of solutions for the so-called Master equation associated with a second order MFG system, see [34]. In view of the results obtained in these works one would expect the limit of u^T/T to be described by the following ergodic system

$$\begin{cases} \frac{1}{2}|D_v u(x,v)|^2 - \langle D_x u(x,v), v\rangle = F(x,v,m), & (x,v)\in\mathbb{T}^d\times\mathbb{R}^d \\ -\langle v, D_x m\rangle - \operatorname{div}\Big(mD_v u(x,v)\Big) = 0, & (x,v)\in\mathbb{T}^d\times\mathbb{R}^d \\ \int_{\mathbb{T}^d\times\mathbb{R}^d} m(dx,dv) = 1. \end{cases} \tag{5}$$

The main issue of this work is that this ergodic system makes no sense. Indeed, as we explain below, even for problems without mean field interaction, we cannot expect to have a solution to the corresponding ergodic Hamilton-Jacobi equation (the first equation in (5)). As the drift of the continuity equation (the second equation in (5)) is given in terms of solution to the ergodic Hamilton-Jacobi equation, there is no hope to formulate the problem in this way. As far as we know, this is the first time this kind of problem is faced in the literature.

To overcome the issue just described, we first study the ergodic Hamilton-Jacobi equation without mean field interaction. More precisely, in the first part we investigate the existence of the limit, as T tends to infinity, of $u^T(0,\cdot,\cdot)/T$, where now u^T solves the Hamilton-Jacobi equation (without mean field interaction)

$$\begin{cases} -\partial_t u^T(t,x,v) + \frac{1}{2}|D_v u^T(t,x,v)|^2 - \langle D_x u^T(t,x,v), v\rangle = F(x,v), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^T(T,x,v) = 0 \text{ in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases}$$

Here $F:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is periodic in space (the first variable) and coercive in velocity (the second one). Following the seminal paper [57], it is known that the existence of the limit of u^T/T is related with the existence of a corrector, namely to a solution of the ergodic Hamilton-Jacobi equation:

$$-\langle D_x u(x,v), v\rangle + \frac{1}{2}|D_v u(x,v)|^2 = F(x,v) + \bar{c}, \quad (x,v)\in\times\mathbb{T}^d\times\mathbb{R}^d,$$

for some constant $\bar{c}$. However, we stress again the fact that due to the lack of coercivity and due to the lack of small time controllability of our model, we do not expect the existence of a continuous viscosity solutions of the ergodic equation (see, however, this reference [40] on this point). This problem has been overcome in several other frameworks: we can quote for instance [63, 33, 68, 27, 13, 9, 6, 10, 17, 45, 44], for related problems see also [5, 53] and the references therein. Following techniques developed in [10] we prove in the first part of Theorem 3.2 that the limit of u^T/T exists and is equal to a constant. However, this convergence result does not suffice to handle our MFG system of acceleration: indeed, we also need to understand, when the map F also depends on the extra time dependent parameter $\{m_t\}_{t\geq 0}$, how this ergodic constant depends on this. For doing so, we follow ideas from weak-KAM theory (see for instance [41]) and characterize the ergodic constant in terms of closed probability measures: namely, we prove in the second part of Theorem 3.2 that, for any $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$,

$$\lim_{T\to+\infty} \frac{u^T(0,x,v)}{T} = \inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v)\right)\ \mu(dx,dv,dw)$$

where $\mathcal{C}$ is the set of Borel probability measures μ on $\mathbb{T}^d \times \mathbb{R}^d$ with suitable finite moments and which are closed in the sense that, for any test function $\varphi \in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)$,

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\langle D_x\varphi(x,v), v\rangle + \langle D_v\varphi(x,v), w\rangle\Big)\ \eta(dx,dv,dw) = 0,$$

(see also Definition 3.1).

We now come back to our MFG of acceleration (4). In view of the characterization of the ergodic constant for the Hamilton-Jacobi equation without mean field interaction, it is natural to describe an equilibrium for the ergodic MFG problem with acceleration as a fixed-point problem on the Wasserstein space. We say that $(\bar\lambda,\bar\mu) \in \mathbb{R}\times\mathcal{C}$ is a solution of the ergodic MFG problem of acceleration if

$$\begin{aligned}\bar\lambda =& \inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\bar\mu)\right)\ \mu(dx,dv,dw)\\ =& \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\bar\mu)\right)\ \bar\mu(dx,dv,dw),\end{aligned}$$

where $\pi : \mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{T}^d\times\mathbb{R}^d$ is the canonical projection onto the first two variables. We show that such an ergodic MFG problem with acceleration has a solution and that the associated ergodic constant $\bar\lambda$ is unique under the following monotonicity condition (first introduced in [54, 55]): there exists a constant $M_F > 0$ such that for any m_1, $m_2 \in \mathscr{P}(\mathbb{T}^d\times\mathbb{R}^d)$

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}^d} \big(F(x,v,m_1) - F(x,v,m_2)\big)\ \big(m_1(dx,dv) - m_2(dx,dv)\big)\\ &\qquad \geq\ M_F \int_{\mathbb{T}^d\times\mathbb{R}^d} \big(F(x,v,m_1) - F(x,v,m_2)\big)^2\ dxdv,\end{aligned}$$

see (1) in Theorem 3.5. The main result of this Chapter is the fact that, if (u^T, m^T) solves the MFG system of acceleration (4), then $u^T(0,x,v)/T$ converges, as T tends to infinity, to the unique ergodic constant $\bar\lambda$ of the ergodic MFG problem, see (2) in Theorem 3.5. The main technical step for this is to rewrite the MFG system in terms of time-dependent closed measure (a kind of occupation measure in this set-up), see Theorem 3.25, and to understand the long-time average of these measures.

Singular perturbation problem

Here we address the singular perturbation problem for control systems of acceleration and of MFG systems with control on the acceleration. The main goal of this analysis is the behavior of such MFG system when the acceleration costs goes to zero. So, the study of the singular problem without mean-field interaction is used to understand the expected behavior of the system. Hence, we first study the limit behavior of the solutions to the Hamilton-Jacobi equation

$$\begin{cases} -\partial_t u^\varepsilon + \frac{1}{2\,\varepsilon}|D_v u^\varepsilon|^2 - \langle D_x u^\varepsilon, v\rangle - L_0(x,v) = 0, & (t,x,v) \in [0,T] \times \mathbb{R}^{2d} \\ u^\varepsilon(T,x,v) = g(x), & (x,v) \in \mathbb{R}^{2d}. \end{cases}$$

as $\varepsilon \to 0$. As already pointed out in the previous Chapters, the Hamiltonian

$$H(x,v,p_x,p_v) = \frac{1}{2\,\varepsilon}|p_v|^2 - \langle p_x, v\rangle - L_0(x,v)$$

fails to be strictly convex and coercive w.r.t. momentum variables. So, also in this case, we solve the problem by using variational technics observing that the value function u^ε can be represented as

$$u^\varepsilon(t,x,v) = \inf_{\substack{\gamma(t)=x \\ \dot\gamma(t)=v}} \left\{ \int_t^T \left(\frac{\varepsilon}{2}|\ddot\gamma(s)|^2 + L_0(\gamma(s),\dot\gamma(s)) \right)\, ds + g(\gamma(T)) \right\}.$$

However, since this represents the test bench for the study of the singular perturbation problem for MFG with control of acceleration we immediately focus the attention on the latter describing it in details. The system we consider here is given by

$$\begin{cases} -\partial_t u^\varepsilon + \frac{1}{2\,\varepsilon}|D_v u^\varepsilon|^2 - \langle D_x u^\varepsilon, v\rangle - L_0(x,v,m_t^\varepsilon) = 0, & (t,x,v) \in [0,T] \times \mathbb{R}^{2d} \\ \partial_t \mu_t^\varepsilon - \langle D_x \mu_t^\varepsilon, v\rangle - \frac{1}{\varepsilon}\operatorname{div}_v(\mu_t^\varepsilon D_v u^\varepsilon) = 0, & (t,x,v) \in [0,T] \times \mathbb{R}^{2d} \\ \mu_0^\varepsilon = \mu_0, \quad u^\varepsilon(T,x,v) = g(x,m_T^\varepsilon), & (x,v) \in \mathbb{R}^{2d} \end{cases} \tag{6}$$

where $u^\varepsilon : [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}$ is the value function, $\mu \in C([0,T];\mathscr{P}_1(\mathbb{R}^{2d}))$ is the joint distribution of position and velocity of a typical agent and $m_t^\varepsilon = \pi_1 \sharp \mu_t^\varepsilon$ with $\pi_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ the projection map onto the first variable. MFG systems with control on the acceleration describes in general models in which the interacting agents controls their acceleration. Here, the systems we are interested in are those in which the acceleration cost vanishes, as it can be easily observed from (6).

The Lagrangian L_0 appearing in the system is assumed to be smooth in space and in velocity variables and to have Tonelli type dependence on v. We refer to **(M3)** below for what concern the dependence of L_0 on the measure variable. One can immediately recognize that the underlying minimization problem associated with the above PDEs systems has the following form

$$\inf_{\substack{\gamma(t)=x \\ \dot\gamma(t)=v}} \left\{ \int_t^T \left(\frac{\varepsilon}{2}|\ddot\gamma(s)|^2 + L_0(\gamma(s),\dot\gamma(s),m_s^\varepsilon) \right)\, ds + g(\gamma(T),m_T^\varepsilon) \right\}$$

for any initial position and velocity $(x,v) \in \mathbb{R}^{2d}$.

The singular perturbation problem has been widely studied for control problems and, more recently, for differential games. For an overview on the subject, which is far from being complete, we refer the reader to [10, 6, 11], and references therein. For these kind of problems, the general structure is to consider a classical controlled dynamic coupled with one that depends on a small parameter $\varepsilon > 0$. Then, as $\varepsilon \to 0$ the limit system turns out to be defined only on $\mathbb{R}^d$ where the unperturbed system is defined. Some type of perturbation problems in MFG have been studied, recently, in

[29, 19, 20, 32] where the authors study the long time-average behaviour of solutions to first order MFG system and in [37, 58] where the authors study the homogenisation problem for second order MFG system. Note that, in homogenisation the structure of the MFG system might be lost in the limit (as proved in [37]) which is not the case here, as we will show in Theorem 4.3.

Indeed, going back to the MFG system (6) we prove that $(u^\varepsilon, m^\varepsilon)$, where we recall that m^ε_t is the space marginal of the solution μ^ε_t for any $t \in [0,T]$, converges (up to subsequence) to a solution (u^0, m^0) to the classical MFG system

$$\begin{cases} (i)\ -\partial_t u^0(t,x) + H_0(x, D_x u^0(t,x), m^0_t) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ (ii)\ \partial_t m^0_t - \mathrm{div}\left(m^0_t D_p H_0(x, D_x u^0(t,x), m^0_t)\right) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ m^0_0 = m_0,\ u^0(T,x) = g(x, m^0_T), & x \in \mathbb{R}^d \end{cases} \tag{7}$$

where $H_0 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the Hamiltonian associated with the Lagrangian L_0. As observed so far, we can see from (7) that the limit $\varepsilon \to 0$ leads to the elimination of the velocity as state variable, whose dynamics was controlled via the perturbation ε. At this point, we again want to stress the fact that the Lagrangian L_0 in (6) depends only on the space marginal of the measure μ^ε. First, this comes from the elimination of the velocity as state variable and so, also for in the analysis of the measure μ^ε, the limit does no see the behavior of the second marginal. Moreover, we are interested in connecting the MFG system of acceleration with the classical one which we know depends only on a flow of probability measures in space which describes the motion of the agents.

Let us briefly explain the method of proof. We first show that u^ε is equibounded and m^ε is tight (see Lemma 4.11 and Theorem 4.14). Thus, as a first consequence we get that, up to a subsequence, there exists $m^0 \in C([0,T]; \mathscr{P}_1(\mathbb{R}^d)$ such that $m^\varepsilon \to m^0$ in $C([0,T]; \mathscr{P}_1(\mathbb{R}^d))$. Then, we proceed with the analysis of the value function u^ε: we show that $u^\varepsilon(t, \cdot, v)$ is equi-Lipschitz continuous, $u^\varepsilon(\cdot, x, v)$ is equicontinuous and $u^\varepsilon(t, x, \cdot)$ has decreasing oscillation w.r.t. ε (see Lemma 4.16 and Proposition 4.17). We finally address the locally uniform convergence of u^ε, showing that there exists a subsequence $\varepsilon_k \downarrow 0$ such that $(u^{\varepsilon_k}, m^{\varepsilon_k})$ converges to a solution (u^0, m^0) of (7) (see Theorem 4.19, Proposition 4.20 and Corollary 4.22). The main issues in proving the above results are due to the lack of strict convexity and the lack of superlinearity of the Hamiltonian in system (6). In particular, these and the fact that Lagrangian L_0 is non-autonomous motivated us to use a variational approach instead of a PDEs approach since the latter creates series difficulties in estimating uniformly the gradient of u^ε w.r.t. velocity variable. We recall that such gradient plays a key role in understanding the limit state space since it capture the behaviour of the velocity as state variable in $\mathbb{R}^d \times \mathbb{R}^d$.

Ergodic behavior of sub-Riemannian control systems

In recent years, increasing attention has been devoted to control systems of the form

$$\dot{\gamma}(t) = \sum_{i=1}^{m} u_i f_i(\gamma(t)) \tag{8}$$

where f_i are $m \in \{1, \cdots, d\}$ vector fields defined on $\mathbb{R}^d$, with sublinear growth, and controls u_i are measurable functions on $\mathbb{R}^m$. The main assumption on the model is the so-called Chow condition (also known as Hördmander condition in PDE), i.e., the fact that iterated Lie brackets of $f_1, \ldots, f_m$ generate the whole tangent space at any point. Indeed, this condition implies that the system is controllable, that is, given any two points in the state space one can find a control that generates a path which joins the two points. Such systems are naturally associated with a new metric on the state space—the sub-Riemannian metric—which in general fails to be equivalent to the classical Euclidean metric, see for instance [2, 38, 65, 61].

Given a Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, an initial position $x \in \mathbb{R}^d$, and a time horizon $T > 0$ we consider the problem of minimizing the functional

$$u \mapsto \int_0^T L(\gamma_u^x(s), u(s))\ ds$$

over the space of all measurable controls $u : [0, T] \to \mathbb{R}^m$, where γ_u^x denotes the solution of (11) such that $\gamma(0) = x$. The first part of this work is devoted to the analysis of the long-time average behavior of the value function of the above problem as $T \to \infty$, that is, the existence of the limit of $V_T(x)/T$ as $T \to \infty$ where

$$V_T(x) = \inf_u \int_0^T L(\gamma_u^x(t), u(t))\ dt.$$

In particular, we prove that such a limit exists locally uniformly and is independent of the initial position $x \in \mathbb{R}^d$, that is,

$$\lim_{T\to\infty} \frac{1}{T} V_T(x) = \alpha(L)\,. \tag{9}$$

Following [41], it is known that the existence of the limit in (9) is related to the existence of a critical constant $c \in \mathbb{R}$ and of a viscosity solution χ to the ergodic Hamilton-Jacobi equation

$$H(x, D\chi(x)) = c \quad (x \in \mathbb{R}^d) \tag{10}$$

where

$$H(x,p) = \sup_{u\in\mathbb{R}^m} \left\{ \sum_{i=1}^m u_i\ \langle p, f_i(x)\rangle - L(x,u) \right\}.$$

The existence of the critical constant for equation (12), in a certain sub-Riemannian setting, was obtained in [3] by a technique based on optimal transport. The analysis in [3] covers compact manifolds and families of 3-generating vector fields (i.e., a step-2 Lie algebra).

Our analysis, unlike [3], is performed on a noncompact state space equipped with a general bracket-generating distribution. The lack of compactness is a major difficulty that we overcome by condition **(L3)** below, which ensures the existence of a compact attractor for all minimizing trajectories. We observe that an assumption of the same type was used, in [19], to study the long-time behavior of first order Mean Field Games systems on Euclidean space and, in [51], to investigate the limit behavior of discounted Hamilton-Jacobi equations on the whole space.

By analyzing the limit behavior of the discounted Hamilton-Jacobi equation associated with (12), we deduce that the ergodic equation admits solutions for $c = -\alpha(L)$ (Theorem 5.13). Then we construct a specific solution of such an equation which coincides with its Lax-Oleinik evolution. Our interest in such a solution is motivated by the fact that we need it to derive a further characterization of the ergodic constant as the minimum of the Lagrangian action on closed measures. As we will show in the following Chapter, this is a crucial step to investigate the related Mather and Aubry sets, on which ergodic solutions have important regularity properties.

We recall that for Tonelli, or even more general, Hamiltonians on a compact or a non-compact manifold, the existence of solutions to the ergodic equation (12) has a long history going back to the seminal paper [57]. Among the many papers that have been published on the subject, when the state space is compact and the Hamiltonian is Tonelli we refer, for instance, to [43, 41] and references therein. If the state space fails to be compact and the Hamiltonian is Tonelli or quasi-Tonelli we refer, for instance, to [15], [14], [42], [50], [49], [51].

However, when the Hamiltonian is not coercive the problem of finding solutions to (12) is open. This issue has been addressed in specific frameworks: we quote for instance [33], [68], [27], [13], for the ergodic problem associated with the so-called G-equation or other noncoercive Hamiltonians. Moreover, we refer to [5, 6] [53], [63] for more on second order differential games.

We want to point out that some of the results of this work are specific to affine-control systems without drift. Indeed, in the presence of a drift, the existence of a continuous viscosity solution to (12) with a noncoercive Hamiltonian remains a challenging problem. We also mention systems with control on the acceleration (see, for instance, [32]) for which it has been proved that, due to the lack of small time local controllability, there are no continuous viscosity solutions to the associated ergodic Hamilton-Jacobi equation.

Aubry-Mather theory for sub-Riemannian control systems

In Chapter 5, we have studied the asymptotic behaviour as $T \to +\infty$ of the value function

$$V_T(x) = \inf_{u(\cdot)} \int_0^T L(\gamma_u^x(t), u(t))\, dt \qquad (x \in \mathbb{R}^d)$$

where L is a Tonelli Lagrangian, controls $u : [0,T] \to \mathbb{R}^m$ $(1 \le m \le d)$ are square integrable functions, γ_u^x is the solution of the sub-Riemannian state equation

$$\begin{cases} \dot{\gamma}(t) = \sum_{i=1}^m u_i(t) f_i(\gamma(t)) & \text{a.e. } t \in [0,T] \\ \gamma(0) = x\,, \end{cases} \tag{11}$$

and $\{f_1, \dots, f_m\}$ are linearly independent smooth vector fields satisfying the so-called Lie algebra rank condition. Observe that the above assumptions ensure that system (11) is small time locally controllable. By using such a property and assuming the existence of a compact attractor for the optimal trajectories of (11), we proved that $V_T(x)/T$ converges to a constant—$\alpha(L)$, the critical constant of L—as $T \to +\infty$, uniformly on all bounded subsets of $\mathbb{R}^d$.

As is well known, the convergence of the above time averages entails the (locally uniform) convergence as $\lambda \downarrow 0$ of the Abel means $\{\lambda v_\lambda\}_{\lambda>0}$, where

$$v_\lambda(x) = \inf_{u(\cdot)} \int_0^\infty e^{-\lambda t} L(\gamma_u^x(s), u(s))\, ds \qquad (x \in \mathbb{R}^d)\,.$$

This fact in turn allows to construct a corrector χ, that is, a continuous viscosity solution of the so-called ergodic Hamilton-Jacobi equation

$$\alpha(L) + H(x, D\chi(x)) = 0 \qquad (x \in \mathbb{R}^d)\,, \tag{12}$$

where H is defined by

$$H(x,p) = \sup_{u\in\mathbb{R}^m} \left\{ \sum_{i=1}^{m} u_i \langle p_i, f_i(x) \rangle - L(x,u) \right\} \qquad (x,p) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{13}$$

The above analysis is by now classical in the Tonelli case, that is, when both $L(x, v)$ and $H(x, p)$ are smooth functions, strictly convex and superlinear in v and p, respectively. Moreover, in such settings, the critical constant has a powerful variational interpretation in terms of probability measures minimizing the Lagrangian action on the tangent bundle. This connection is well explained by the celebrated Aubry-Mather and weak KAM theories (see, for instance, [41, 66, 59] and the references therein).

However, it is easy to see that the Hamiltonian in (13) fails to be Tonelli, in general. So, the classical weak KAM theory does not apply to minimization problems for sub-Riemannian control systems which are, on the other hand, quite relevant for both theory ([2]) and applications ([52]). Introducing new ideas and techniques to make this extension possible is the purpose of this work.

To be more precise, we point out that the underlying geometry on the state space, namely the sub-Riemannian structure induced by the family of vector fields $\{f_i\}_{i=1,\dots,m}$ on $\mathbb{R}^d$ (see for instance [2, 38, 65] and references therein), plays a crucial role in our approach. Moreover, in order to improve the natural regularity of correctors—which would just be Hölder continuous, see [23]—we restrict the analysis to the class of sub-Riemannian systems that admit no singular minimizing controls different from zero. Then, owing to [25], we know that correctors are locally semiconcave, hence locally Lipschitz, on $\mathbb{R}^d$. Finally, in order to deal with unbounded state and control spaces, we assume the existence of a compact attractor for all optimal trajectories as is customary in this kind of situations.

We now proceed to describe the main results of this Chapter. First, extending the classical notion of closed measures on the tangent bundle (see, e.g., [43]), we introduce the class $\mathcal{C}_F$ of closed probability measures adapted to the sub-Riemannian structure and we show that the critical constant $\alpha(L)$ is the minimum of the Lagrangian action on $\mathcal{C}_F$ (Theorem 6.8). In this context, it is worth noting that closed measures are naturally supported on the distribution associated with $\{f_i\}_{i=1,\dots,m}$, which in our case reduces to $\mathbb{R}^d \times \mathbb{R}^m$.

Then, we introduce and study the Aubry set $\mathcal{A}$ from a dynamical and topological point of view, proving that $\mathcal{A}$ is a nonempty compact subset of $\mathbb{R}^d$ (Theorem 6.22), invariant for the class of calibrated curves for Peierl's barrier (6.26) (Proposition 6.28). Moreover, we show that any critical solution to (12) is differentiable along the range of the vector fields $\{f_i\}_{i=1,\cdots,m}$ at any point $x \in \mathcal{A}$ (see Theorem 6.27 establishing horizontal differentiability).

Papers extracted: We conclude this introduction quoting the papers which has been extracted from the work in this thesis.

1. P. Cannarsa, C. Mendico, *Mild and weak solutions of mean field game problems for linear control systems*, Minimax Theory Appl. 5, No. 2, 221–250 (2020).

2. P. Cardaliaguet, C. Mendico, *Ergodic behavior of control and mean field games problems depending on acceleration*, Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods 203, 41 p. (2021).

3. C. Mendico, *Singular perturbation problem for mean field game of acceleration*, Arxiv:2107.08479, (submitted).

4. P. Cannarsa, C. Mendico, *Asymptotic analysis for Hamilton-Jacobi equations associated with sub-Riemannian control systems*, Arxiv:2012.09099, (submitted).

5. P. Cannarsa, C. Mendico, *On the Aubry set for sub-Riemannian control systems*, (forthcoming).

Chapter 1

Preliminaries

In this chapter we collect some preliminary definitions and results that we are going to use throughout this thesis. In particular, they concern:

1. Wasserstein spaces and Wasserstein distance, for which we refer to [67, 7] for more details.

2. Sub-Riemannian geometry and sub-Riemannian control systems on $\mathbb{R}^d$, see [61, 65, 2].

3. Weak KAM theory for Tonelli Hamiltonian systems for which we refer to [41, 42, 66, 59].

1.1 Measure Theory

Let (X, d) be a metric space (in the work, we use $X = \mathbb{R}^d$ or $X = \mathbb{R}^d \times \mathbb{R}^m$). Denote by $\mathscr{B}(X)$ the Borel σ-algebra on X and by $\mathscr{P}(X)$ the space of Borel probability measures on X. The support of a measure $\mu \in \mathscr{P}(X)$, denoted by $\mathrm{spt}(\mu)$, is the closed set defined by

$$\mathrm{spt}(\mu) := \Big\{x \in X : \mu(V_x) > 0 \text{ for each open neighborhood } V_x \text{ of } x\Big\}.$$

We say that a sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathscr{P}(X)$ is weakly-$*$ convergent to $\mu \in \mathscr{P}(X)$, denoted by $\mu_k \xrightarrow{w^*} \mu$, if

$$\lim_{n\to\infty} \int_X f(x)\, d\mu_n(x) = \int_X f(x)\, d\mu(x), \quad \forall f \in C_b(X).$$

There exists an interesting link between the weak-$*$ convergence and the convergence of the support of the measures, see [7, Proposition 5.1.8]. Indeed, if $\{\mu_j\}_{j\in\mathbb{N}} \subset \mathscr{P}(X)$ weakly-$*$ converges to $\mu \in \mathscr{P}(X)$ then

$$\forall\ x \in \mathrm{spt}(\mu) \quad \exists x_j \in \mathrm{spt}(\mu_j) : \quad \lim_{j\to\infty} x_j = x. \tag{1.1}$$

For $p \in [1, +\infty)$, the Wasserstein space of order p is defined as

$$\mathscr{P}_p(X) := \left\{m \in \mathscr{P}(X) : \int_X d(x_0, x)^p\, dm(x) < +\infty\right\},$$

for some (and thus all) $x_0 \in X$. Given any two measures m and m' in $\mathscr{P}_p(X)$, define

$$\Pi(m, m') := \Big\{\lambda \in \mathscr{P}(X \times X) : \lambda(A \times X) = m(A),\ \lambda(X \times A) = m'(A),\ \forall A \in \mathcal{B}(X)\Big\}. \tag{1.2}$$

The Wasserstein distance of order p between m and m' is defined by

$$d_p(m,m') = \inf_{\lambda\in\Pi(m,m')} \left(\int_{X\times X} d(x,y)^p \, d\lambda(x,y)\right)^{1/p}.$$

The distance d_1 is also commonly called the Kantorovich-Rubinstein distance and can be characterized by a useful duality formula (see, for instance, [67]) as follows

$$d_1(m,m') = \sup\left\{\int_X f(x)\,dm(x) - \int_X f(x)\,dm'(x) \mid f : X \to \mathbb{R} \text{ is 1-Lipschitz}\right\}, \tag{1.3}$$

for all m, $m' \in \mathscr{P}_1(X)$.

Let $\mathcal{K}$ be a subset of $\mathscr{P}(X)$. We say that the set $\mathcal{K}$ has uniformly integrable p-moment with respect some (and thus any) $\bar{x} \in X$ if and only if

$$\lim_{i\to\infty} \int_{X\setminus B_i(\bar{x})} d(x,\bar{x})^p \;\mu(dx) = 0, \quad \text{uniformly with respect to } \mu \in \mathcal{K}.$$

Remark 1.1. Notice that, if

$$0 < p < p_1, \quad \text{and} \quad \sup_{\mu\in\mathcal{K}} \int_X d(x,\bar{x})^{p_1} \;\mu(dx) < +\infty,$$

then $\mathcal{K}$ has uniformly integrable p-moment.

Theorem 1.2 (Compactness and convergence). *A set $\mathcal{K} \subset \mathscr{P}_p(X)$ is relatively compact if and only if it is p-uniformly integrable and tight. Moreover, for a given sequence $\{\mu_i\}_{i\in\mathbb{N}} \subset \mathscr{P}_p(X)$ we have that*

$$\lim_{i\to\infty} d_p(\mu_i,\mu) = 0$$

if and only if μ_i narrowly converge to μ and $\{\mu_i\}_{i\in\mathbb{N}}$ has uniformly integral p-moment.

Theorem 1.3. *Let $r \geq p > 0$ and let $\mathcal{K} \subset \mathscr{P}_p(X)$ be such that*

$$\sup_{\mu\in\mathcal{K}} \int_X |x|^r \;\mu(dx) < \infty.$$

Then the set $\mathcal{K}$ is tight. If, moreover, $r > p$ then $\mathcal{K}$ is relatively compact for the d_p distance.

Let X_1, X_2 be metric spaces, let $\mu \in \mathscr{P}(X_1)$ and let $f : X_1 \to X_2$ be a μ measurable map. Then, we denote by $f\sharp\mu \in \mathscr{P}(X_2)$ the push-forward of μ through f defined by

$$f\sharp\mu(B) := \mu(f^{-1}(B)), \quad \forall\; B \in \mathscr{B}(X_2).$$

More generally, in integral form, it reads as

$$\int_{X_1} \varphi(f(x)) \;\mu(dx) = \int_{X_2} \varphi(y) \;f\sharp\mu(dy).$$

We conclude this introductory section recalling the so-called disintegration theorem.

Theorem 1.4 (Disintegration Theorem). *Let X and Y be Radon separable metric spaces, let μ be a Borel probability measure on X and let $\pi : X \to Y$ be Borel map. Define $\nu = \pi\sharp\mu \in \mathscr{P}(Y)$. Then there exists a μ-a.e. uniquely determined Borel measurable family of probability measures $\{\nu_y\}_{y\in Y} \subset \mathscr{P}(X)$ such that*

$$\nu_y(X\backslash\pi^{-1}(y)) = 0, \qquad \text{for } \mu - a.e.\ y \in Y,$$

and

$$\int_X f(x)\mu(dx) = \int_Y \left(\int_{\pi^{-1}(y)} f(x)\nu_y(dx) \right) \nu(dy)$$

for every Borel map $f : X \to [0, +\infty]$.

1.2 Sub-Riemannian control

A class of nonholonomic drift-less systems on $\mathbb{R}^d$ is a control system of the form

$$\dot{\gamma}(t) = \sum_{i=1}^{m} f_i(\gamma(t))u_i(t), \quad t \in [0, +\infty) \tag{1.4}$$

Such a system induces a distance on $\mathbb{R}^d$ in the following way. First, we define the sub-Riemannian metric to be the function $g : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ given by

$$g(x, v) = \inf \left\{ \sum_{i=1}^{m} u_i^2 : v = \sum_{i=1}^{m} f_i(x)u_i \right\}.$$

If $v \in \text{span}\{f_1(x), \dots, f_m(x)\}$ then the infimum is attained at a unique value $u_x \in \mathbb{R}^m$ and $g(x, v) = |u_x|^2$. Then, since $g(\gamma(t), \dot{\gamma}(t))$ is measurable, being the composition of the lower semicontinuous function g with a measurable function, we can define the length of an absolutely continuous curve $\gamma : [0, 1] \to \mathbb{R}^d$ as

$$\text{length}(\gamma) = \int_0^1 \sqrt{g(\gamma(t), \dot{\gamma}(t))}\ dt.$$

In conclusion, one defines the sub-Riemannian distance as

$$d_{\text{SR}}(x, y) = \inf_{(\gamma,u)\in\Gamma_{0,1}^{x\to y}} \text{length}(\gamma)$$

where $\Gamma_{0,1}^{x\to y}$ denotes the set of all trajectory-control pairs such that $u \in L^2(0, 1; \mathbb{R}^m)$, γ solves (1.4) for such a control u, $\gamma(0) = x$ and $\gamma(1) = y$. Following [25] it is possible to represent the sub-Riemannian distance as follows

$$d_{\text{SR}}(x, y) = \inf \left\{ T > 0 : \exists\ (\gamma, u) \in \Gamma_{0,T}^{x\to y},\ |u(t)| \le 1 \text{ a.e. } t \in [0, T] \right\} \tag{1.5}$$

for any x, $y \in \mathbb{R}^d$. Moreover, again from [25] the sub-Riemannian distance can be characterised in terms of the sub-Riemannian energy: setting

$$e_{\text{SR}}(x, y) = \inf_{(\gamma,u)\in\Gamma_{0,1}^{x\to y}} \int_0^1 g(\gamma(t), \dot{\gamma}(t))\ dt,$$

one can prove that

$$d_{\text{SR}}(x, y) = \sqrt{e_{\text{SR}}(x, y)} \tag{1.6}$$

(see, for instance, [25, Lemma 11]).

Among the many properties of these systems we are interested in the controllability. For such a system, controllability can be obtained by using the Lie algebra generated by $f_1, \ldots, f_m$, which is defined as follows. Set

$$\Delta^1 = \text{span}\{f_1, \ldots, f_m\}$$

and, for any integer $s \geq 1$,

$$\Delta^{s+1} = \Delta^s + [\Delta^1, \Delta^s]$$

where $[\Delta^1, \Delta^s] := \text{span}\{[X, Y] : X \in \Delta^1, Y \in \Delta^s\}$. The Lie algebra generated by $f_1, \ldots, f_m$ is defined as

$$\text{Lie}(f_1, \ldots, f_m) = \bigcup_{s \geq 1} \Delta^s.$$

We say that system (1.4) satisfies *Chow's condition* if $\text{Lie}(f_1, \ldots, f_m)(x) = \mathbb{R}^d$ for any $x \in \mathbb{R}^d$, where $\text{Lie}(f_1, \ldots, f_m)(x) = \{X(x) : X \in \text{Lie}(f_1, \ldots, f_m)\}$. Equivalently, for any $x \in \mathbb{R}^d$ there exists an integer $r \geq 1$ such that $\Delta^r(x) = \mathbb{R}^d$. The minimum integer with such a property is called the degree of nonholonomy at x and will be denoted by $r(x)$. Chow's condition is also known as the *Lie algebra rank condition* (LARC) in control theory and as the *Hörmander condition* in the context of PDEs.

Example 1.5. *The following are two well-known examples of sub-Riemannian systems for which Chow's condition holds true.*

(i) **Heisenberg group:** *We consider the system in* $\mathbb{R}^3$

$$\begin{cases} \dot{x}(t) & = u(t), \\ \dot{y}(t) & = v(t), \\ \dot{z}(t) & = u(t)y(t) - v(t)x(t) \end{cases}$$

In this case, the matrix of the system is given by

$$A(x, y, z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y & -x \end{bmatrix}$$

and the columns of such matrix satisfy the Hördmander condition: $X_1 = (1, 0, y)$, $X_2 = (0, 1, -x)$ *and* $[X_1, X_2] = (0, 0, 2)$ *generate* $\mathbb{R}^3$.

(ii) **Grushin type systems:** *Consider a control system of the form*

$$\begin{cases} \dot{x}(t) & = u(t), \\ \dot{y}(t) & = \varphi(x(t))v(t) \end{cases}$$

for a nonzero continuous function $\varphi(x)$ *with sub-linear growth. The classical Grushin system in* $\mathbb{R}^2$ *is obtained taking* $\varphi(x) = x$. *Then, the dynamics is given by the matrix*

$$A(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$$

whose columns satisfy the Hördmander condition: $X_1 = (1, 0)$ *and* $[X_1, X_2] = (0, 1)$ *generates* $\mathbb{R}^2$. □

Theorem 1.6 (**Chow-Rashevsky theorem, [38, Theorem 3.1.8]**). *If system* (1.4) *satisfies Chow's condition, then any two points in* $\mathbb{R}^d$ *can be joined by a trajectory satisfying* (1.4).

Besides controllability, another important consequence of Chow's condition is the well-known Ball-Box Theorem, see for instance [2, Theorem 10.67]. Of particular interest to us is a corollary of such a theorem which gives Hölder equivalence between the Euclidean distance and the sub-Riemannian one. First, we observe that for any $x \in \mathbb{R}^d$ a continuity argument ensures the existence of a neighborhood U_x of x such that

$$\Delta^{r(x)}(y) = \mathbb{R}^d, \quad \forall\, y \in U_x. \tag{1.7}$$

Thus, given a compact set $\mathcal{K}$ there exists a finite cover given by $\{U_{x_i}\}_{i=1,\dots,N}$ and a set of integers $\{r(x_i)\}_{i=1,\dots,N}$ such that (1.7) holds on U_{x_i} with $r(x) = r(x_i)$. Taking

$$r = \max_{i=1,\dots,N} r(x_i)$$

we obtain

$$\Delta^{r}(y) = \mathbb{R}^d, \quad \forall\, y \in \mathcal{K}. \tag{1.8}$$

We call degree of nonholonomy of $\mathcal{K}$ the minimum integer such that (1.8) holds true and we denote it by $r(\mathcal{K})$. Moreover, we recall that a family of vector fields $\{f_i\}_{i=1,\dots,m}$ is an equi-regular distribution on $\mathbb{R}^d$ if there exists $r_0 \geq 1$ such that $\Delta^{r_0}(x) = \mathbb{R}^d$ for any $x \in \mathbb{R}^d$.

Corollary 1.7. *For any compact set* $\mathcal{K} \subset \mathbb{R}^d$ *there exist two constants* $\tilde{c}_1$, $\tilde{c}_2 > 0$ *such that*

$$\widetilde{c}_1|x-y| \leq d_{\mathrm{SR}}(x,y) \leq \widetilde{c}_2|x-y|^{\frac{1}{r(\mathcal{K})}}, \quad \forall\, x, y \in \mathcal{K}. \tag{1.9}$$

Furthermore, we recall that the topology induced by $(\mathbb{R}^d, d_{\mathrm{SR}})$ coincides with the topology induced by the Euclidean distance on $\mathbb{R}^d$ ([2, Theorem 3.31]). In particular, from this result, we obtain that a set is compact in $(\mathbb{R}^d, d_{\mathrm{SR}})$ if and only if it is compact in $\mathbb{R}^d$ w.r.t. Euclidean distance.

We conclude this preliminary part with a brief introduction to singular controls. Let $x_0 \in \mathbb{R}^d$ and fix $t > 0$. The end-point mapping associated with system (1.4) is the function

$$E^{x_0,t} : L^2(0,t;\mathbb{R}^m) \to \mathbb{R}^d$$

defined as

$$E^{x_0,t}(u) = \gamma(t)$$

where γ is a solution of (1.4) associated with u such that $\gamma(0) = x_0$. Under the assumption that the vector field f_i has sub-linear growth for any $i = 1, \dots m$ it is known that $E^{x_0,t}$ is of class C^1 on $L^2(0,t;\mathbb{R}^m)$. Then, we say that a control $\bar{u} \in L^2(0,t;\mathbb{R}^m)$ is singular for $E^{x_0,t}$ if $dE^{x_0,t}(\bar{u})$ is not surjective. Moreover, defining the function $H_0 : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ as

$$H_0(x,p,u) = \sum_{i=1}^{m} u_i \langle p, f_i(x) \rangle$$

we have the following well-known characterization of singular controls.

Theorem 1.8. *A control* $u \in L^2(0,t;\mathbb{R}^m)$ *is singular for* $E^{x_0,t}$ *if and only if there exists an absolutely continuous arc* $p : [0,t] \to \mathbb{R}^d \backslash \{0\}$ *such that*

$$\begin{cases} \dot{\gamma}(s) & = \; D_p H_0(\gamma(s), p(s), u(s)) \\ -\dot{p}(s) & = \; D_x H_0(\gamma(s), p(s), u(s)) \end{cases}$$

with $\gamma(0) = x_0$ *and*

$$D_u H_0(\gamma(s), p(s), u(s)) = 0, \quad \text{for a.e. } s \in [0,t],$$

that is,

$$\langle f_i(\gamma(s)), p(s) \rangle = 0$$

for any $s \in [0,t]$.

1.3 Weak-KAM Theory

Definition 1.9 (Tonelli Lagrangians)**.** *A function* $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is called a Tonelli Lagrangian if it belongs to* C^2 *and it satisfies the following.*

(i) For each $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$*, the Hessian* $D^2_{vv}L(x,v)$ *is positive definite.*

(ii) For each $A > 0$ *there exists* $B(A) \in \mathbb{R}$ *such that*

$$L(x,v) > A|v| + B(A), \quad \forall (x,v) \in \mathbb{R}^d \times \mathbb{R}^d.$$

(iii) For each $R > 0$

$$A(R) := \sup \Big\{ L(x,v) : |v| \leq R \Big\} < +\infty.$$

Define the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ associated with L by

$$H(x,p) = \sup_{v \in \mathbb{R}^n} \Big\{ \big\langle p, v \big\rangle - L(x,v) \Big\}, \quad \forall (x,p) \in \mathbb{R}^d \times \mathbb{R}^d.$$

It is straightforward to check that if L is a Tonelli Lagrangian, then H defined above also satisfies (i), (ii), and (iii) in Definition 1.9. Such a function H is called a Tonelli Hamiltonian. Moreover, if L is a *reversible* Lagrangian, i.e., $L(x,v) = L(x,-v)$ for all $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, then $H(x,p) = H(x,-p)$ for all $(x,p) \in \mathbb{R}^d \times \mathbb{R}^d$.

Let us recall definitions of weak KAM solutions and viscosity solutions of the Hamilton-Jacobi equation

$$H(x, Du) = c, \quad x \in \mathbb{R}^d, \tag{1.10}$$

where c is a real constant.

Definition 1.10 (Weak KAM solutions)**.** *A function* $u \in C(\mathbb{R}^d)$ *is called a backward (resp. forward) weak KAM solution of equation* (1.10) *if the following holds.*

(i) For each continuous piecewise C^1 *curve* $\gamma : [t_1, t_2] \to \mathbb{R}^d$*, we have that*

$$u(\gamma(t_2)) - u(\gamma(t_1)) \leq \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s))ds + c(t_2 - t_1);$$

(ii) For each $x \in \mathbb{R}^d$, there exists a C^1 curve $\gamma : (-\infty, 0] \to \mathbb{R}^d$ (resp. $\gamma : [0, +\infty) \to \mathbb{R}^d$) with $\gamma(0) = x$ such that

$$u(x) - u(\gamma(t)) = \int_t^0 L(\gamma(s), \dot{\gamma}(s))ds - ct, \quad \forall t < 0$$

(resp. $u(\gamma(t)) - u(x) = \int_0^t L(\gamma(s), \dot{\gamma}(s))ds + ct, \quad \forall t > 0$).

Remark 1.11. A function u on $\mathbb{R}^d$ is said to be dominated by $L + c$, denoted by $u \prec L + c$, if u satisfies condition (i) of Definition 1.10. A curve γ is said to be (u, L, c)-calibrated if it satisfies condition (ii) of Definition 1.10.

Definition 1.12 (Viscosity solutions)**.** *Let $V \subset \mathbb{R}^d$ be an open set.*

(i) A function $u : V \to \mathbb{R}$ is called a viscosity subsolution of equation (1.10), if for every C^1 function $\varphi : V \to \mathbb{R}$ and every point $x_0 \in V$ such that $u - \varphi$ has a local maximum at x_0, we have that

$$H(x_0, D\varphi(x_0)) \leq c;$$

(ii) A function $u : V \to \mathbb{R}$ is called a viscosity supersolution of equation (1.10), if for every C^1 function $\psi : V \to \mathbb{R}$ and every point $y_0 \in V$ such that $u - \psi$ has a local minimum at y_0, we have that

$$H(y_0, D\psi(y_0)) \geq c;$$

(iii) A function $u : V \to \mathbb{R}$ is called a viscosity solution of equation (1.10) if it is both a viscosity subsolution and a viscosity supersolution.

Definition 1.13 (Mañé critical value)**.** *The Mañé critical value of a Tonelli Hamiltonian H is defined by*

$$c(H) := \inf \left\{ c \in \mathbb{R} : \ \exists \ u \in C(\mathbb{R}^d) \text{ viscosity sol. of } H(x, Du) = c \right\}.$$

See [42, Theorem 1.1] for the following weak KAM theorem for noncompact state spaces.

Theorem 1.14 (Weak KAM theorem)**.** *Let H be a Tonelli Hamiltonian. Then, there exists a global viscosity solution of equation*

$$H(x, Du) = c(H), \quad x \in \mathbb{R}^d.$$

In [42], viscosity solutions are shown to coincide with backward weak KAM solutions. Observe that, as $\mathbb{R}^d$ can be seen as a covering of the torus $\mathbb{T}^d$, Mañé's critical value can be characterized as follows:

$$c(H) = \inf_{u \in C^\infty(\mathbb{R}^d)} \sup_{x \in \mathbb{R}^d} H(x, Du(x)).$$

We conclude this section by recalling the notion of Mather set and the role such a set plays for the regularity of viscosity solutions. Let L be a Tonelli Lagrangian. As is well known, the associated Euler-Lagrange equation, i.e.,

$$\frac{d}{dt} D_v L(x, \dot{x}) = D_x L(x, \dot{x}), \tag{1.11}$$

generates a flow of diffeomorphisms $\phi_t^L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$, with $t \in \mathbb{R}$, defined by

$$\phi_t^L(x_0, v_0) = (x(t), \dot{x}(t)),$$

where $x : \mathbb{R} \to \mathbb{R}^d$ is the maximal solution of (1.11) with initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$. It should be noted that, for any Tonelli Lagrangian, the flow ϕ_t^L is complete, see for instance [42].

We recall that a Borel probability measure μ on $\mathbb{R}^d \times \mathbb{R}^d$ is called ϕ_t^L-invariant, if

$$\mu(B) = \mu(\phi_t^L(B)), \quad \forall t \in \mathbb{R}, \quad \forall B \in \mathscr{B}(\mathbb{R}^d \times \mathbb{R}^d),$$

or, equivalently,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(\phi_t^L(x,v))\ \mu(dx,dv) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v)\ \mu(dx,dv),\ \forall f \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d).$$

We denote by $\mathcal{M}_L$ the class of all ϕ_t^L-invariant probability measures.

Definition 1.15 (Mather measures [59])**.** *A probability measure $\mu \in \mathcal{M}_L$ is called a Mather measure for L, if it satisfies*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} L(x,v)\ \mu(dx,dv) = \inf_{\nu \in \mathcal{M}_L} \int_{\mathbb{R}^d \times \mathbb{R}^d} L(x,v)\ \nu(dx,dv).$$

In [41], it was proved that

$$c(H) = -\inf_{\nu \in \mathcal{M}_L} \int_{\mathbb{R}^d \times \mathbb{R}^d} L(x,v)\ \nu(dx,dv).$$

Denote by $\mathcal{M}_L^*$ the set of all Mather measures. Observe that, if L (resp. H) is a reversible Lagrangian (resp. reversible Hamiltonian), then

$$-c(H) = \inf_{x \in \mathbb{R}^d} L(x,0).$$

The Mather set is the subset $\widetilde{\mathcal{M}}_0 \subset \mathbb{R}^d \times \mathbb{R}^d$ defined by

$$\widetilde{\mathcal{M}}_0 = \overline{\bigcup_{\mu \in \mathcal{M}_L^*} \operatorname{spt}(\mu)}.$$

We call $\mathcal{M}_0 = \pi_1(\mathcal{M}_0) \subset \mathbb{R}^d$ the projected Mather set. See [41, Theorem 4.12.3] for the following result.

Theorem 1.16. *If u is dominated by $L + c(H)$, then it is differentiable at every point of the projected Mather set $\mathcal{M}_0$. Moreover, if $(x,v) \in \mathcal{M}_0$, then*

$$Du(x) = D_v L(x,v)$$

and the map $M_0 \to \mathbb{R}^d \times \mathbb{R}^d$, defined by $x \mapsto (x, Du(x))$, is locally Lipschitz with a Lipschitz constant which is independent of u.

Part I

Mean field control of acceleration

Chapter 2

Mild and weak solutions of Mean Field Games problems for linear control systems

2.1 Setting of the Mean Field Games problem

2.1.1 Assumptions

Let us consider a Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^k \times \mathscr{P}_1(\mathbb{R}^d) \to \mathbb{R}$ and a function $G : \mathbb{R}^d \times \mathscr{P}_1(\mathbb{R}^d) \to \mathbb{R}$ satisfying the following.

(L1) For any $m \in \mathscr{P}_1(\mathbb{R}^d)$, the map $(x,u) \mapsto L(x,u,m)$ is of class $C^2(\mathbb{R}^d \times \mathbb{R}^k)$ and the map $m \mapsto L(x,u,m)$, from $\mathscr{P}_1(\mathbb{R}^d)$ to $\mathbb{R}$, is Lipschitz continuous with respect to the d_1 distance, i.e.

$$Q_L := \sup_{\substack{(x,u)\in\mathbb{R}^d\times\mathbb{R}^k \\ m_1,\ m_2\in\mathscr{P}_1(\mathbb{R}^d) \\ m_1\neq m_2}} \frac{|L(x,u,m_1) - L(x,u,m_2)|}{d_1(m_1,m_2)} < +\infty.$$

(L2) The map $(x,m) \mapsto G(x,m)$ is of class $C_b(\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d))$ and for every $m \in \mathscr{P}_1(\mathbb{R}^d)$ the map $x \to G(x,m)$ belongs to $C^1_b(\mathbb{R}^d)$.

(L3) *(i)* There exist a constant C_0 such that

$$\frac{\mathrm{Id}}{C_0} \leq D_{uu}L(x,u,m) \leq C_0\mathrm{Id}, \quad \forall\ (x,u,m) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathscr{P}_1(\mathbb{R}^d).$$

(ii) There exists a constant $C_1 \geq 0$ such that for any $(x,u,m) \in \mathbb{R}^d\times\mathbb{R}^k\times\mathscr{P}_1(\mathbb{R}^d)$

$$\|D^2_{xu}L(x,u,m)\| \leq C_1(1+|u|).$$

(iii) There exists a constant $C_2 \geq 0$ such that for any $(x,u,m) \in \mathbb{R}^d\times\mathbb{R}^k\times\mathscr{P}_1(\mathbb{R}^d)$

$$|L(x,0,m)| + |D_xL(x,0,m)| + |D_u(x,0,m)| \leq C_2.$$

Remark 2.1. Note that, from **(L3)**, it is not difficult to check that there exist $c_0 \geq 0$ and $c_1 \geq 0$ such that

$$c_0|u|^2 - c_1 \leq L(x,u,m) \leq c_1 + \frac{1}{c_0}|u|^2 \quad \forall\ (x,u,m) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathscr{P}_1(\mathbb{R}^d).$$

Fix a time horizon $T > 0$. Let A and B be real matrices, $d \times d$ and $d \times k$, respectively, and consider the control system defined by

$$\dot\gamma(t) = A\gamma(t) + Bu(t), \quad t \in [0, T] \tag{2.1}$$

where $u : [0, T] \to \mathbb{R}^k$ is a summable function. For all $x \in \mathbb{R}^d$ we denote by $\gamma(\cdot\, ; x, u)$ the solution of the differential equation (2.1) such that $\gamma(0) = x$ and define the metric space

$$\Gamma_T = \left\{ \gamma(\cdot; x, u) : x \in \mathbb{R}^d,\ u \in L^1(0, T; \mathbb{R}^k) \right\} \subset \mathrm{AC}([0, T]; \mathbb{R}^d)$$

endowed with the uniform norm, denoted by $\|\cdot\|_\infty$. Moreover, set

$$\Gamma_T(x) = \{\gamma \in \Gamma_T : \gamma(0) = x\}.$$

For any $x \in \mathbb{R}^d$, any $u \in L^1(0, T)$ and any flow of probability measures $m \in C([0, T]; \mathscr{P}_1(\mathbb{R}^d))$ define the functional

$$J(x, u, \{m_t\}_t) = \int_0^T L(\gamma(t, x, u), u(t), m_t)\ dt + G(\gamma(T, x, u), m_T),$$

and the associated optimal control problem

$$\inf_{u \in L^2(0,T;\ \mathbb{R}^k)} J(x, u, \{m_t\}_t). \tag{2.2}$$

Notice that the restriction to controls $u \in L^2(0, T; \mathbb{R}^k)$ is due to the structure assumptions we imposed on L.

We proceed now to prove some estimates on the optimal controls and the associated optimal trajectories.

Proposition 2.2. *Assume* **(L1)** – **(L3)***. Then, there exists a real positive constant K such that for any $x \in \mathbb{R}^d$, any $m \in C([0, T]; \mathscr{P}_1(\mathbb{R}^d))$ and any optimal control u^* of (2.2), we have that*

$$\|u^*\|_2 \leq K.$$

Proof. By Remark 2.1 and the optimality of u^* we deduce that

$$c_1 T + \|G\|_\infty \geq J(x, 0, \{m_t\}_t) \geq J(x, u^*, \{m_t\}_t) \geq c_0 \int_0^T |u^*(t)|^2 dt - c_1 T - \|G\|_\infty.$$

Therefore, from the above inequalities we deduce that

$$\|u^*\|_2^2 = \int_0^T |u^*(t)|^2 dt \leq \frac{2}{c_0}\left(c_1 T + \|G\|_\infty\right) =: K^2.$$

Thus, the proof is complete. □

Corollary 2.3. *Assume* **(L1)** – **(L3)***. Then, there exists a constant $\widetilde{C_1} \geq 0$ such that for any $x \in \mathbb{R}^d$, any $m \in C([0, T]; \mathscr{P}_1(\mathbb{R}^d))$ and any optimal control u^* of (2.2) we have that*

$$\|\gamma^*\|_\infty \leq \widetilde{C_1}(1 + |x|)$$

where γ^* is the trajectory associated with u^*.

Proof. Since γ^* is a solution of (2.1) associated with u^*, we know that

$$\gamma^*(t) = e^{tA}x + \int_0^t e^{(t-s)A}Bu^*(s)\, ds.$$

Hence,

$$|\gamma^*(t)| \le e^{T\|A\|}\left(|x| + \|B\|\int_0^t |u^*(s)|\, ds\right)$$

and by Hölder's inequality

$$|\gamma^*(t)| \le e^{T\|A\|}\left(|x| + \|B\|T^{\frac{1}{2}}\|u^*\|_2\right). \qquad \square$$

Lemma 2.4. *Assume* **(L1)** – **(L3)**. *Then, there exists a constant* $\widetilde{C}_2 > 0$ *such that for any* $x \in \mathbb{R}^d$, *any* $m \in C([0,T];\mathscr{P}_1(\mathbb{R}^d))$ *and any* u^* *optimal control for* (2.2) *we have that*

$$\|\dot{\gamma}^*\|_2 \le \widetilde{C}_2(1 + |x|)$$

where γ^* *is the trajectory associated with* u^*. *Moreover, the family of minimizing trajectories* $\Gamma^*(x)$ *is uniformly Hölder continuous.*

Proof. From Proposition 2.2 and Corollary 2.3 there holds

$$\begin{aligned}
\|\dot{\gamma}^*\|_2 = \|A\gamma^*(t) + Bu^*(t)\|_2 &\le \|A\|^{\frac{1}{2}}\|\gamma^*\|_2 + \|B\|^{\frac{1}{2}}\|u^*\|_2 \\
\le &\|A\|^{\frac{1}{2}}\left(\int_0^T |\gamma^*(t)|^2 dt\right)^{\frac{1}{2}} + \|B\|^{\frac{1}{2}}K \\
\le &\|A\|^{\frac{1}{2}}T^{\frac{1}{2}}\tilde{C}(1 + |x|) + \|B\|^{\frac{1}{2}}K.
\end{aligned}$$

Thus, for any t, $s \in [0,T]$ such that $s \le t$ we get

$$\begin{aligned}
&|\gamma^*(t) - \gamma^*(s)| \le \int_s^t |\dot{\gamma}^*(\tau)|\, d\tau \\
&\le \|\dot{\gamma}^*\|_2|t-s|^{\frac{1}{2}} \le \left(\|A\|^{\frac{1}{2}}T^{\frac{1}{2}}\tilde{C}(1 + |x|) + \|B\|^{\frac{1}{2}}K\right)|t-s|^{\frac{1}{2}}
\end{aligned}$$

Which completes the proof. $\square$

In order to express our MFG problem in terms of the Lagrangian formulation we are going to give a special structure to the continuous flow of probability measures $\{m_t\}_{t\in[0,T]}$. Let $\alpha > 1$, let m_0 be a Borel probability measure in $\mathscr{P}_\alpha(\mathbb{R}^d)$, and denote by $[m_0]_\alpha$ the α-moment of m_0, i.e.,

$$[m_0]_\alpha = \int_{\mathbb{R}^d} |x|^\alpha\, m_0(dx). \tag{2.3}$$

Let R be a real constant such that $R \ge [m_0]_\alpha$ and define the following space of probability measures on Γ_T

$$\mathcal{P}_{m_0}(\Gamma_T, R) = \left\{\eta \in \mathcal{P}(\Gamma_T) : \int_{\Gamma_T} \|\dot{\gamma}\|_2^\alpha\, \eta(d\gamma) \le R,\ e_0\sharp\eta = m_0\right\}$$

where $e_t(\gamma) = \gamma(t)$ is the evaluation map. Note that the sets $\mathscr{P}_{m_0}(\Gamma_T, R)$ are compact subsets of $\mathscr{P}(\Gamma_T)$ with respect to d_1 distance. Indeed, for any $r > 0$ define the following sets

$$\mathcal{C}_r = \{\gamma \in \Gamma_T : |\gamma(0)| \le r,\ \|\dot{\gamma}\|_2 \le r\},$$

which are compact by Ascoli-Arzela Theorem. Observe, also, that by definition

$$\mathcal{C}_r^c \subset \{\gamma \in \Gamma_T : \|\dot{\gamma}\|_2 > r\} \cup \{\gamma \in \Gamma_T : |\gamma(0)| > r\}.$$

Thus, given $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ we have that

$$\eta(\{\gamma \in \Gamma_T : |\gamma(0)| > r\}) = m_0(B_r^c)$$

which goes to zero as $r \to +\infty$. Moreover, by Bienaymé-Tchebychev inequality we obtain

$$\eta(\{\gamma \in \Gamma_T : \|\dot{\gamma}\|_2 > r\}) \leq \frac{R}{r^\alpha}.$$

Therefore, we get

$$\eta(\mathcal{C}_r^c) \leq \frac{R}{r^\alpha} + m_0(B_r^c)$$

which in turn yields the compactness of $\mathscr{P}_{m_0}(\Gamma_T, R)$.

Remark 2.5. There exist at least one constant $R \geq [m_0]_\alpha$ such that the set $\mathscr{P}_{m_0}(\Gamma_T, R)$ is non-empty. Indeed, fixed a Borel probability measure $m_0 \in \mathscr{P}_\alpha(\mathbb{R}^d)$, consider the map $p : \mathbb{R}^d \to \Gamma_T$ such that

$$x \mapsto p[x](t) := e^{tA}x, \quad \forall t \in [0, T]$$

and define the measure $\eta = p[\cdot]\sharp m_0 \in \mathscr{P}(\Gamma_T)$. Note that, for any $x \in \mathbb{R}^d$ the curve $e^{tA}x$ is an admissible curve associated with the control $u \equiv 0$.

Then, the following holds:

1. for any bounded continuous function f on $\mathbb{R}^d$, we have that $e_0\sharp\eta = m_0$. Indeed,

$$\begin{aligned}&\int_{\mathbb{R}^d} f(x)\ e_0\sharp\eta(dx) = \int_{\Gamma_T} f(\gamma(0))\ \eta(d\gamma)\\ &= \int_{\Gamma_T} f(\gamma(0))\ p\sharp m_0(d\gamma) = \int_{\mathbb{R}^d} f(p[x](0))\ m_0(dx)\\ &= \int_{\mathbb{R}^d} f(x)\ m_0(dx);\end{aligned}$$

2. the α-moment of η is bounded:

$$\begin{aligned}&\int_{\Gamma_T} \|\dot{\gamma}\|_2^\alpha\ \eta(d\gamma) = \int_{\mathbb{R}^d} \|\dot{p}[x]\|_2^\alpha\ m_0(dx)\\ &\leq \left(\|A\|e^{T\|A\|}\right)^\alpha \int_{\mathbb{R}^d} |x|^\alpha\ m_0(dx) \leq \left(\|A\|e^{T\|A\|}\right)^\alpha [m_0]_\alpha.\end{aligned}$$

Therefore, taking $R \geq \left(\|A\|e^{T\|A\|}\right)^\alpha [m_0]_\alpha$ we have that $\eta \in \mathcal{P}_{m_0}(\Gamma_T, R)$. □

2.1.2 Definitions and first properties

For any $x \in \mathbb{R}^d$, any $u \in L^1(0,T)$ and any $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$, define the functional

$$J_\eta(x, u) = \int_0^T L(\gamma(t, x, u), u(t), e_t\sharp\eta)\ dt + G(\gamma(T, x, u), e_T\sharp\eta)$$

and the associated optimal control problem

$$\inf_{u\in L^2(0,T;\ \mathbb{R}^k)} J_\eta(x,u). \tag{2.4}$$

We denote by $\Gamma^*_\eta(x)$ the set of curves associated with an optimal control u^* (2.4), i.e.

$$\Gamma^*_\eta(x) = \left\{\gamma(\cdot\ ; x, u^*) : J_\eta(x,u^*) = \inf_{u\in L^2(0,T;\mathbb{R}^k)} J_\eta(x,u)\right\}.$$

Definition 2.6 (**Mean Field Games equilibrium).** *Given $m_0 \in \mathscr{P}_\alpha(\mathbb{R}^d)$, we say that $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ is a Mean Field Games equilibrium for m_0 if*

$$\mathrm{spt}(\eta) \subset \bigcup_{x\in\mathbb{R}^d} \Gamma^*_\eta(x).$$

Proposition 2.7. *Assume* **(L1)** – **(L3)**.

1. *For any $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ we have that*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|^\alpha\ e_t\sharp\eta(dx) \leq R. \tag{2.5}$$

   Consequently, the family of measures $\{e_t\sharp\eta\}_{t\in[0,T]}$ is tight.

2. *For any $\{\eta_i\}_{i\in\mathbb{N}} \subset \mathscr{P}_{m_0}(\Gamma_T, R)$ and $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ such that $\eta_i \rightharpoonup^* \eta$ we have that*

$$d_1(e_t\sharp\eta_i, e_t\sharp\eta) \to 0$$

   for every $t \in [0,T]$.

3. *For any $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ we have that the map $t \in [0,T] \mapsto e_t\sharp\eta$ is continuous.*

Proof. We are going to prove only the point (1), see [18, Lemma 3.2] for a proof of (2) and (3).

Given $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ we have that

$$\int_{\mathbb{R}^d} |x|^\alpha\ e_t\sharp\eta(dx) = \int_{\Gamma_T} |\gamma(t)|^\alpha\ \eta(d\gamma) \leq \int_{\Gamma_T} \|\gamma\|^\alpha_\infty\ \eta(d\gamma) \leq C_0,$$

where the last inequality holds by definition of $\mathscr{P}_{m_0}(\Gamma_T, R)$. So, by Theorem 1.3 the family of measures $\{e_t\sharp\eta\}_{t\in[0,T]}$ is tight in $\mathscr{P}_\alpha(\mathbb{R}^d)$ with respect to the d_1 distance since by assumption $\alpha > 1$. □

Remark 2.8. Note that, in (2.5) the constant R in independent of $t \in [0,T]$ and η. Indeed, as explained so far it is fixed a priori such that $R \geq [m_0]_\alpha$.

2.2 Mean Field Games equilibria: Existence and Uniqueness

At this point, it is not difficult to prove that for any given $\alpha > 0$ and any given initial measure $m_0 \in \mathscr{P}_\alpha(\mathbb{R}^d)$ there exists $R_0 \geq 0$ such that for any $R \geq R_0$ there exists at least one Mean Field Games equilibrium $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ and that, under a classical monotonicity assumption, such an equilibrium is unique.

For the sake of completeness, we give below the key ideas and steps to prove the existence of a Mean Field Games equilibrium, following the appoach in [18].

Given $m_0 \in \mathscr{P}_\alpha(\mathbb{R}^d)$ and given $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ we recall that by Theorem 1.4 there exists a unique Borel measurable family of probability measures $\{\eta_x\}_{x\in\mathbb{R}^d}$ on Γ_T such that

$$\eta(d\gamma) = \int_{\mathbb{R}^d} \eta_x(d\gamma)\; m_0(dx)$$
$$\mathrm{spt}(\eta_x) \subset \;\Gamma_T(x), \quad m_0 - \text{a.e.},\; x \in \mathbb{R}^d.$$

Define the set-valued map

$$E: \; \big(\mathscr{P}_{m_0}(\Gamma_T, R), d_1\big) \rightrightarrows \big(\mathscr{P}_{m_0}(\Gamma_T, R), d_1\big)$$

that associates with any $\eta \in \mathcal{P}_{m_0}(\Gamma_T, R)$ the set

$$E(\eta) = \Big\{\nu \in \mathcal{P}_{m_0}(\Gamma_T, R) : \; \mathrm{spt}(\nu_x) \subset \; \Gamma^*_\eta(x),\; m_0 - \text{a.e.}\Big\}.$$

It is easy to realize that a given $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ is a Mean Field Games equilibrium if and only if η is a fixed point of the above set-valued map, that is, $\eta \in E(\eta)$. Therefore, in order to prove the existence of Mean Field Games equilibria, we appeal to Kakutani-Fan-Glicksberg's fixed point theorem, see for instance [4, Corollary 17.55], which provides conditions under which the set-valued map E has a fixed point.

We check the validity of such conditions in the following Lemmas.

Lemma 2.9. *Assume* **(L1)** – **(L3)**. *Let $R \geq [m_0]_\alpha$. For any $x_i \to x$ in $\mathbb{R}^d$, any $\eta_i \rightharpoonup^* \eta$ in $\mathcal{P}_{m_0}(\Gamma_T, R)$ and any $\gamma_i \in \Gamma^*_{\eta_i}(x_i)$ such that $\gamma_i \to \gamma$ in Γ_T we have that $\gamma \in \Gamma^*_\eta(x)$.*

Proof. Since $\gamma_i \in \Gamma^*_{\eta_i}(x_i)$ we know that there exists a sequence of optimal controls $u_i \in L^2(0,T)$ such that $\gamma_i(\cdot) = \gamma_i(\cdot, x_i, u_i)$ for every $t \in [0,T]$. Moreover, from Proposition 2.2 we get that $\|u_i\|_2 \leq K$. Therefore, up to a subsequence, we obtain that there exists $\bar{u} \in L^2(0,T)$ such that $u_i \rightharpoonup \bar{u}$ in L^2. Hence, we are reduced to prove that

1. $\bar{\gamma}(\cdot) = \gamma(\cdot, x, \bar{u})$;
2. $J_\eta(x, \bar{u}) \leq J_\eta(x, u)$ for every $u \in L^2([0,T])$,

<u>Point 1:</u>

By definition of γ_i, we know that

$$\gamma_i(t) = e^{At}x + \int_0^t e^{A(t-s)}Bu_i(s)\; ds.$$

Let v be a vector on $\mathbb{R}^d$, then

$$\begin{aligned}\langle v, \gamma_i(t)\rangle &= \langle v, e^{At}x\rangle + \int_0^t \langle v, e^{A(t-s)}Bu_i(s)\rangle\; ds \\ &= \langle v, e^{At}x\rangle + \int_0^t \langle (e^{A(t-s)}B)^* v, u_i(s)\rangle\; ds.\end{aligned}$$

Thus, letting $i \to \infty$ by the weak L^2 convergence of u_i we obtain that

$$\langle v, \bar{\gamma}(t)\rangle = \langle v, e^{At}x\rangle + \int_0^t \langle v, e^{A(t-s)}B\bar{u}(s)\rangle \, ds.$$

This concludes the proof of point 1.

Point 2:

We now prove that

$$J_\eta(x, \bar{u}) \leq \liminf_{i\to\infty} J_{\eta_i}(x_i, u_i).$$

By **(L2)** and the convergence of γ_i in Γ_T and that of η_i, it follows that

$$G(\gamma_i(T), e_T\sharp\eta_i) \to G(\gamma(T), e_T\sharp\eta).$$

Therefore, it suffices to prove that

$$\int_0^T L(\bar{\gamma}(t), \bar{u}(t), e_t\sharp\eta)\, dt \leq \liminf_{i\to\infty} \int_0^T L(\gamma_i(t), u_i(t), e_t\sharp\eta_i)\, dt.$$

Now,

$$\begin{aligned}
&\int_0^T \big(L(\bar{\gamma}(t), \bar{u}(t), e_t\sharp\eta) - L(\gamma_i(t), u_i(t), e_t\sharp\eta)\big)\, dt \\
&= \underbrace{\int_0^T \big(L(\bar{\gamma}(t), \bar{u}(t), e_t\sharp\eta) - L(\bar{\gamma}(t), u_i(t), e_t\sharp\eta)\big)\, dt}_{\mathbf{A}} \\
&+ \underbrace{\int_0^T \big(L(\bar{\gamma}(t), u_i(t), e_t\sharp\eta) - L(\gamma_i(t), u_i(t), e_t\sharp\eta_i)\big)\, dt\,.}_{\mathbf{B}}
\end{aligned}$$

By assumption **(L3)** (iii) and Lipschitz condition **(L1)** we have that $\mathbf{B} \to 0$ as $i \to 0$. Thus, we have to prove now that the functional

$$\Lambda(u) = \int_0^T L(\bar{\gamma}(t), u(t), e_t\sharp\eta)\, dt$$

is weakly lower semicontinuous with respect to the L^2 topology. Define, for every $\lambda \in \mathbb{R}$,

$$X_\lambda = \{u \in L^2(0,T) : \Lambda(u) \leq \lambda\}.$$

By assumption **(L3)** on convexity of the Lagrangian L with respect to controls, we get that the sets X_λ are convex. Furthermore, such sets are closed in the strong L^2 topology. Indeed, if $\{u_i\}_{i\in\mathbb{N}} \subset X_\lambda$ is such that $u_i \to u_\infty$ in L^2 then $u_i \to u_\infty$ a.e. up to a subsequence. Thus, by the continuity of L we have that $L(\gamma(t), u_i(t), e_t\sharp\eta) \to L(\gamma(t), u_\infty(t), e_t\sharp\eta)$ a.e. and by Remark 2.1 the Lagrangian L is bounded from below. Therefore, by Fatou's Lemma we obtain that $u_\infty \in X_\lambda$. Hence, since the sets X_λ are convex and strongly closed it implies that they are closed also in the L^2 weak topology. □

Corollary 2.10. *Assume* **(L1)** – **(L3)**. *Then, the set-valued map*

$$\begin{aligned}
\phi : (\mathbb{R}^d, |\cdot|) &\rightrightarrows (\Gamma_T, \|\cdot\|_\infty) \\
x &\mapsto \Gamma_\eta^*(x)
\end{aligned}$$

has closed graph.

Lemma 2.11. *Assume* **(L1)** – **(L3)**. *Then, there exists a constant* $R(\alpha, [m_0]_\alpha) > 0$ *such that if* $R \geq R(\alpha, [m_0]_\alpha)$ *then* $E(\eta)$ *is non-empty. Moreover,* $E(\eta)$ *is convex and compact.*

Proof. We, first, prove that given $m_0 \in \mathscr{P}_\alpha(\mathbb{R}^d)$ for any $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ the set $E(\eta)$ is non empty for some constant $R \geq [m_0]_\alpha$. Indeed, we have that by Corollary 2.10 and [21, Proposition 9.5] the set-valued map $x \rightrightarrows \Gamma^*_\eta(x)$ is measurable with closed values. Thus, by [26, Theorem A 5.2], there exists a measurable selection $\tilde{\gamma}_x \in \Gamma^*_\eta(x)$, that is $\tilde{\gamma}_x(t) = \tilde{\gamma}(t, x, u^*)$ for some $u^* \in L^2(0,T)$ solution of (2.4) associated with η. Define, now, the measure $\tilde{\eta}$ as follows

$$\tilde{\eta}(A) = \int_{\mathbb{R}^d} \delta_{\tilde{\gamma}_x}(A)\ m_0(dx) \quad \text{for any } A \in \mathcal{B}(\Gamma_T).$$

Thus, we need to prove that $\tilde{\eta} \in \mathscr{P}_{m_0}(\Gamma_T, R)$. Indeed, $e_0 \sharp \tilde{\eta} = m_0$ by definition and

$$\int_{\Gamma_T} \|\dot{\gamma}\|_2^\alpha\ \tilde{\eta}(d\gamma) = \int_{\mathbb{R}^d} \|\dot{\tilde{\gamma}}_x\|_2^\alpha\ m_0(dx) \leq \int_{\mathbb{R}^d} \tilde{C_2}^\alpha\ (1+|x|)^\alpha\ \ m_0(dx),$$

where the last inequality holds by Lemma 2.4. Therefore, we deduce that

$$\int_{\Gamma_T} \|\dot{\gamma}\|_2^\alpha\ \tilde{\eta}(d\gamma) \leq \tilde{C}_2^\alpha \left(\int_{\mathbb{R}^d} |x|^\alpha\ m_0(dx) + 1 \right) \leq \tilde{C}_2^\alpha([m_0]_\alpha + 1).$$

Hence, taking $R \geq R(\alpha, [m_0]_\alpha)$, where

$$R(\alpha, [m_0]_\alpha) := \tilde{C}_2^\alpha([m_0]_\alpha + 1)$$

we obtain that $\tilde{\eta} \in \mathscr{P}_{m_0}(\Gamma_T, R)$. Consequently, that $E(\eta)$ is non-empty. The proof of convexity is a straightforward application of [18, Lemma 3.5]. In conclusion, for any $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ the sets $E(\eta)$ are compact, with respect to the d_1 distance, since $E(\eta) \subset \mathscr{P}_{m_0}(\Gamma_T, R)$ which is compact. □

Lemma 2.12. *Assume* **(L1)** – **(L3)**. *Then, for any* $R \geq R(\alpha, [m_0]_\alpha)$, *the set-valued map*

$$E : \left(\mathscr{P}_{m_0}(\Gamma_T, R), d_1\right) \rightrightarrows \left(\mathscr{P}_{m_0}(\Gamma_T, R), d_1\right)$$
$$\eta \mapsto E(\eta)$$

has closed graph.

Proof. The proof of this Lemma is a straightforward application of [18, Lemma 3.6]. □

Theorem 2.13 (**Existence of Mean Field Games equilibria**). *Assume* **(L1)** – **(L3)**. *Let* $R \geq R(\alpha, [m_0]_\alpha)$, *where* $R(\alpha, [m_0]_\alpha)$ *is defined as in Lemma 2.11. Then, the set-valued map* E *has a fixed point.*

Proof. By the above lemmas the assumptions of Kakutani-Fan-Glicksberg's fixed point theorem (see, for instance, [4, Corollary 17.55]) are satisfied and therefore, there exists a fixed point of the map E, that is $\bar{\eta} \in E(\bar{\eta})$ and $\bar{\eta}$ is a Mean Field Games equilibrium. □

At this point, for $\alpha > 1$ fix $m_0 \in \mathscr{P}_\alpha(\mathbb{R}^d)$ and $R \geq R(\alpha, [m_0]_\alpha)$, where $R(\alpha, [m_0]_\alpha)$ is defined as in Lemma 2.11. Thus, by Theorem 2.13 we have that there exists at least one Mean Field Games equilibrium $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$.

From now on, we denote by $\gamma(s; t, x, u)$ the solution to the following control system

$$\begin{cases} \dot{\gamma}(s) = A\gamma(s) + Bu(s), & s \in [t, T] \\ \gamma(t) = x, \end{cases} \tag{2.6}$$

where $u : [t, T] \to \mathbb{R}^k$ belongs to $L^2(t, T; \mathbb{R}^k)$. Moreover, we introduce the following notation

$$m_t^\eta = e_t \sharp \eta, \tag{2.7}$$

for any $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$.

Definition 2.14 (Mild solutions of Mean Field Games problem). *We say that $(V, m) \in C([0, T] \times \mathbb{R}^d) \times C([0, T], \mathcal{P}_\alpha(\mathbb{R}^d))$ is a mild solution for the Mean Field Games problem if there exists a Mean Field Games equilibrium $\eta \in \mathcal{P}_{m_0}(\Gamma_T)$ such that*

(i) $m_t = m_t^\eta$ for all $t \in [0, T]$;

(ii) V can be represented as the value function of the optimal control problem (2.4), that is

$$V(t, x) = \inf_{u \in L^2(0,T;\ \mathbb{R}^k)} \left\{ \int_t^T L(\gamma(s; t, x, u), u(s), m_s^\eta)\ ds + G(\gamma(T; t, x, u), m_T^\eta) \right\} \tag{2.8}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

Note that the above definition is well-posed since we have proved so far that there exists at least one Mean Field Games equilibrium and the map

$$\begin{aligned} [0, T] &\to \mathscr{P}_\alpha(\mathbb{R}^d) \\ t &\mapsto e_t \sharp \eta \end{aligned}$$

is continuous with respect to d_1. Moreover, for the same reasons we know that there exists at least one mild solution of the Mean Field Games problem.

In order to study the uniqueness of mild solutions, we focus the attention on a particular Lagrangian function, that is

$$L(x, u, m) := \ell(x, u) + F(x, m), \tag{2.9}$$

where ℓ and F satisfy the assumptions **(L1)**–**(L3)**.

Definition 2.15 (Monotonicity). *We say that $\Psi : \mathbb{R}^d \times \mathscr{P}_1(\mathbb{R}^d) \to \mathbb{R}$ is monotone if*

$$\int_{\mathbb{R}^d} \Big(\Psi(x, m_1) - \Psi(x, m_2) \Big)\ (m_1 - m_2)(dx) \geq 0, \tag{2.10}$$

for all $m_1, m_2 \in \mathscr{P}_1(\mathbb{R}^d)$.

We say that Ψ is strictly monotone if (2.10) holds true and

$$\int_{\mathbb{R}^d} \Big(\Psi(x, m_1) - \Psi(x, m_2) \Big)\ (m_1 - m_2)(dx) = 0 \iff F(x, m_1) = F(x, m_2)$$

for any $x \in \mathbb{R}^d$.

Theorem 2.16 (**Uniqueness of mild solutions**). *Assume* **(L1)** – **(L3)**. *Let F and G be strictly monotone. Then, for any Mean Field Games equilibria η_1 and η_2 in $\mathscr{P}_{m_0}(\Gamma_T, R)$ we have that the associated functionals J_{η_1} and J_{η_2} are equal.*

Consequently, if (V_1, m_1) and (V_2, m_2) are two mild solutions associated with the Mean Field Games equilibria η_1 and η_2, then $V_1 = V_2$.

We omit the proof of the Theorem 2.16 which is similar to the one of [18, Theorem 4.1].

2.3 Further regularity of mild solutions

Throughout this section, given $\alpha > 1$ fix $m_0 \in \mathscr{P}_\alpha(\mathbb{R}^d)$ and $R \geq R(\alpha, [m_0]_\alpha)$, where $R(\alpha, [m_0]_\alpha)$ in defined as in Lemma 2.11. At this point, we know that under assumptions **(L1)**–**(L3)** by Theorem 2.13 there exists at least one Mean Field Games equilibrium $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$. Furthermore, if the Lagrangian L is of the form (2.9), the coupling function F and the terminal costs G satisfy the strict monotonicity assumption, see Definition 2.15, then the mild solution is unique. For this reasons, from now on we fix $R \geq R(\alpha, [m_0]_\alpha)$.

Now, we are going to prove that any Mean Field Games equilibrium generates a family of probability measures $\{m_t^\eta\}_{t\in[0,T]}$ which is $\frac{1}{2}$-Hölder continuous in time. Consequently, any mild solution (V, m^η) is such that the value function V is locally Lipschitz continuous and locally fractionally semiconcave on $[0, T] \times \mathbb{R}^d$. Moreover, we will prove that there exists at least one Mean Field Games equilibrium $\eta \in \mathscr{P}_\alpha(\Gamma_T, R)$ such that $t \to m_t^\eta$ is Lipschitz continuous.

Given the control system (2.1), the Hamiltonian associated with the Lagrangian function L is defined as

$$H(x, p, m) = \sup_{u\in\mathbb{R}^k} \Big\{ -\langle p, Ax + Bu\rangle - L(x, u, m)\Big\}.$$

The Hamiltonian H can be explicitly written as follows

$$H(x, p, m) = -\langle p, Ax\rangle - L^*(x, -B^\star p, m), \quad \forall\, (x, p, m) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathscr{P}_\alpha(\mathbb{R}^d) \tag{2.11}$$

where L^* denotes the Legendre Transform of L, i.e.,

$$L^*(x, p, m) = \sup_{u\in\mathbb{R}^k} \Big\{ -\langle p, u\rangle - L(x, u, m)\Big\}.$$

Moreover, it is easy to check that there exists a constant $c_2 \geq 0$ such that for any $(x, p, m) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathscr{P}_\alpha(\mathbb{R}^d)$

$$|D_pH(x, p, m)| \leq c_2(1 + |x| + |p|) \tag{2.12}$$

and, from (i) in **(L1)** one can also deduce that there exists a constant $c_H \geq 0$ such that

$$|D_pH(x, p, m) - D_pH(y, q, m)| \leq c_H(|p - q| + |x - y|) \tag{2.13}$$

for any $(x, y) \in \mathbb{R}^{2d}$, $(p, q) \in \mathbb{R}^{2k}$ and $m \in \mathscr{P}_\alpha(\mathbb{R}^d)$.

2.3.1 Local Lipschitz continuity and local fractional semiconcavity of the Value function

Let (V, m^η) a mild solution of the Mean Field Games problem associated with an equilibrium $\eta \in \mathscr{P}_\alpha(\Gamma_T, R)$. In this section, we prove that the flow of measures $\{m_t^\eta\}_{t\in[0,T]}$ are Hölder continuous and consequently, that the associated value function is locally semiconcave on $[0,T]\times\mathbb{R}^d$, linearly in space and with a fractional modulus of semiconcavity in time. Moreover, we show that the value function V is locally Lipschitz continuous on $[0,T]\times\mathbb{R}^d$. We conclude this section proving that, under some extra assumptions on the data, there exists at least one equilibrium $\eta \in \mathscr{P}_\alpha(\Gamma_T, R)$ such that $\{m_t^\eta\}_{t\in[0,T]}$ is Lipschitz continuous in time.

We recall that V is defined as the value function

$$V(t,x) = \inf_{u\in L^2(0,T;\ \mathbb{R}^k)} \left\{ \int_t^T L(\gamma(s;t,x,u), u(s), m_s^\eta)\ ds + G(\gamma(T;t,x,u), m_T^\eta) \right\}.$$

Theorem 2.17 (**Hölder continuity of equilibria).** *Assume* **(L1)** – **(L3)**. *Then, given any Mean Field Games equilibrium η, the map $t \to m_t^\eta$ is $\frac{1}{2}$-Hölder continuous in time.*

Proof. By definition of d_1, we have that

$$\begin{aligned} d_1(m_t^\eta, m_s^\eta) &= \inf_{\varphi\in \mathrm{Lip}_1(\mathbb{R}^d)} \int_{\mathbb{R}^d} \varphi(x)(m_t^\eta - m_s^\eta)(dx) \\ &= \inf_{\varphi\in \mathrm{Lip}_1(\mathbb{R}^d)} \int_{\Gamma_T} (\varphi(\gamma(t)) - \varphi(\gamma(s))\eta(d\gamma) \le \int_{\Gamma_T} |\gamma(t)-\gamma(s)|\eta(d\gamma), \end{aligned}$$

where $\mathrm{Lip}_1(\mathbb{R}^d)$ is the set of Lipschitz continuous functions such that the Lipschitz constant is equal to 1.

We recall that, since η is a Mean Field Games equilibrium then it is supported on the set of all minimizing curves of problem (2.4). Therefore, by Lemma 2.4 and recalling that $x = \gamma(0)$ we obtain

$$\begin{aligned} d_1(m_t^\eta, m_s^\eta) &\le \int_{\Gamma_T} |\gamma(t)-\gamma(s)|\eta(d\gamma) \\ &\le |t-s|^{\frac{1}{2}} \int_{\Gamma_T} \left(\|A\|^{\frac{1}{2}} T^{\frac{1}{2}} \tilde{C}\big(1+|x|\big) + \|B\|^{\frac{1}{2}} K \right)\ \eta(d\gamma) = \kappa([m_0]_\alpha])|t-s|^{\frac{1}{2}}, \end{aligned}$$

where the constant κ depends on the moment of m_0 which we know is bounded by construction. Thus, the proof is complete. □

In order to prove the semiconcavity of the value function V, we need to add the following assumption on the Lagrangian L and terminal cost G:

(L4) There exists two constants $w_L \ge 0$ and $w_G \ge 0$ such that for any $\lambda \in [0,1]$, any radius $R > 0$, any $u \in \mathbb{R}^k$, any x_0, $x_1 \in B_R$, and any $m \in \mathscr{P}_1(\mathbb{R}^d)$ such that

$$\begin{aligned} &\lambda L(x_0,u,m) + (1-\lambda)L(x_1,u,m) - L(\lambda x_0 + (1-\lambda)x_1, u, m) \\ &\le w_L \lambda(1-\lambda)|x_0-x_1|^2, \end{aligned}$$

and

$$\begin{aligned} &\lambda G(x_0,m) + (1-\lambda)G(x_1,m) - G(\lambda x_0 + (1-\lambda)x_1, m) \\ &\le w_G \lambda(1-\lambda)|x_0-x_1|^2. \end{aligned}$$

Theorem 2.18 (**Local fractional semiconcavity of** V**).** *Assume* **(L1)** – **(L3)**. *Let* R *be a positive radius. Then, there exists a constant* $\Lambda \geq 0$ *such that for any* $(t,x) \in [0,T] \times \overline{B}_R$*, any* $(h,\delta) \in \mathbb{R} \times \mathbb{R}$ *such that* $(x+h, t+\delta) \in [0,T] \times \overline{B}_R$ *and* $(x-h, t-\delta) \in [0,T] \times \overline{B}_R$ *we have that*

$$V(t+\delta, x+h) + V(t-\delta, x-h) - 2V(t,x) \leq \Lambda \left(|h|^2 + |\delta|^{\frac{3}{2}}\right).$$

Proof. We first prove that the value function V is locally semiconcave in space uniformly in time and then, that it is locally semiconcave in space and time.

Let $R > 0$ be a positive radius and fix $(t,x) \in [0,T] \times \overline{B}_R$. Let $h \in \mathbb{R}^d$ be such that $x+h$, $x-h \in \overline{B}_R$ and let $u^* \in L^2$ be an optimal control for $(t,x) \in [0,T] \times \overline{B}_R$. Then, define the following curves

$$\begin{aligned} \gamma(s) =& \gamma(s; t, x, u^*), \quad s \in [t,T] \\ \gamma_+(s) =& \gamma(s; t, x+h, u^*), \quad s \in [t,T] \\ \gamma_-(s) =& \gamma(s; t, x-h, u^*), \quad s \in [t,T]. \end{aligned}$$

Thus, we have that

$$\begin{aligned} & V(t, x+h) + V(t, x-h) - 2V(t,x) \\ \leq & \int_t^T \Big(L(\gamma_+(s), u^*(s), m_s^\eta) + L(\gamma_-(s), u^*(s), m_s^\eta) - 2L(\gamma(s), u^*(s), m_s^\eta) \Big)\, ds \qquad (2.14) \\ & + G(\gamma_+(T), m_T^\eta) + G(\gamma_-(T), m_T^\eta) - 2G(\gamma(T), m_T^\eta). \end{aligned}$$

Consider, first, the expression involving only the terminal costs:

$$\begin{aligned} & G(\gamma_+(T), m_T^\eta) + G(\gamma_-(T), m_T^\eta) - 2G(\gamma(T), m_T^\eta) \\ = & G(\gamma_+(T), m_T^\eta) + G(\gamma_-(T), m_T^\eta) - 2G\left(\frac{\gamma_+(T) + \gamma_-(T)}{2}, m_T^\eta\right) \\ & + 2G\left(\frac{\gamma_+(T) + \gamma_-(T)}{2}, m_T^\eta\right) - 2G(\gamma(T), m_T^\eta). \end{aligned}$$

By **(L1)** and **(L4)** we deduce that

$$\begin{aligned} & G(\gamma_+(T), m_T^\eta) + G\left(\gamma_-(T), m_T^\eta\right) - 2G\left(\frac{\gamma_+(T) + \gamma_-(T)}{2}, m_T^\eta\right) \\ & \qquad \leq w_G |\gamma_+(T) - \gamma_-(T)|^2, 2G\left(\frac{\gamma_+(T) + \gamma_-(T)}{2}, m_T^\eta\right) - 2G\left(\gamma(T), m_T^\eta\right) \\ & \qquad\qquad \leq \|G\|_\infty |\gamma_+(T) + \gamma_-(T) - 2\gamma(T)|. \end{aligned}$$

From the definition of γ, γ_+ and γ_- we have that these curves are solutions of (2.1). Therefore, we get that there exists a real positive constant W such that

$$\begin{aligned} & |\gamma_+(T) - \gamma_-(T)|^2 \leq W|h|^2, \\ & |\gamma_+(T) + \gamma_-(T) - 2\gamma(T)| \leq W|h|^2. \end{aligned}$$

Hence, we get

$$G(\gamma_+(T), m_T^\eta) + G(\gamma_-(T), m_T^\eta) - 2G(\gamma(T), m_T^\eta) \leq W\left(w_G + \|G\|_\infty\right)|h|^2.$$

By almost similar arguments, one can prove that also the integral term in (2.14) is bounded by a constant times $|h|^2$. This proves that V is locally semiconcave in space uniformly in time.

We proceed to show that V is locally semiconcave on $[0,T]\times\mathbb{R}^d$. Fix $(t,x)\in[0,T]\times\overline{B}_R$ and let $h\in\mathbb{R}^d$, $\delta\in\mathbb{R}$ be such that $x+h$, $x-h\in\overline{B}_R$ and $0<t-\delta<t+\delta<T$. Let u^* be an optimal control for (t,x) and define the control

$$\bar{u}(s)=u^*\left(\frac{t+\delta+s}{2}\right),\quad s\in[t-\delta,t+\delta].$$

By the Dynamic Programming Principle we get

$$\begin{aligned}&V(t+\delta,x+h)+V(t-\delta,x-h)-2V(t,x)\\ \leq\ &\underbrace{V(t+\delta,x+h)+V(t+\delta,\gamma(t+\delta;t-\delta,x-h,\bar{u}))-2V(t+\delta,\gamma(t+\delta;t,x,u^*))}_{I}\\ &+\underbrace{\int_{t-\delta}^{t+\delta}L(\gamma(s;t-\delta,x-h,u^*),\bar{u}(s),m_s^\eta)\,ds-2\int_t^{t+\delta}L(\gamma(s;t,x,u^*),u^*(s),m_s^\eta)\,ds}_{II}.\end{aligned}$$

Thus, by the first parte of the proof term I is bounded by a constant times $|h|^2+|\delta|^2$. Now, we have to estimate term II. Let us denote, for simplicity, by γ^- the curve $\gamma(\cdot\,;t-\delta,x-h,u^*)$. Then, by assumption **(L1)** we have that there exists a constant $D\geq 0$ such that

$$\begin{aligned}II&=2\int_t^{t+\delta}\Big(L(\gamma^-(2s-t-\delta),u^*(s),m^\eta_{2s-t-\delta})-L(\gamma(s),u^*(s),m^\eta_s)\Big)\,ds\\ &\leq D\int_t^{t+\delta}\Big(|\gamma^-(2s-t-\delta)-\gamma(s)|+d_1(m^\eta_{2s-t-\delta},m^\eta_s)\Big)\,ds\end{aligned}\tag{2.15}$$

Since η is a Mean Field Games equilibrium we know by Theorem 2.17 that $\{m_t^\eta\}_{t\in[0,T]}$ is $\frac{1}{2}$-Hölder continuous in time with respect to the d_1 distance. Therefore,

$$\int_t^{t+\delta}d_1(m^\eta_{2s-t-\delta},m^\eta_s)\,ds\leq\kappa([m_0]_\alpha)\int_t^{t+\delta}|s-t-\delta|^{\frac{1}{2}}\,ds\leq\frac{2}{3}\kappa([m_0]_1)|\delta|^{\frac{3}{2}}.\tag{2.16}$$

Now, we have to estimate the distance between the curves γ_- and γ. For that, we recall that since γ_- and γ are solutions of (2.1) we know that

$$\begin{aligned}\gamma^-(2s-t-\delta)=&e^{(s-t+\delta)A}(x-h)+\int_{t-\delta}^{2s-t-\delta}e^{(\tau-t+\delta)A}B\bar{u}(\tau)\,d\tau,\\ \gamma(s)=&e^{(s-t)A}x+\int_t^s e^{(\tau-t)A}Bu^*(\tau)\,d\tau.\end{aligned}$$

From [26, Theorem 7.4.6], without loss of generality, we can assume that u^* belongs to L^∞ and consequently, $\bar{u}\in L^\infty$. Thus, we obtain that for any $s\in[t,t+\delta]$

$$\begin{aligned}&|\gamma^-(2s-t-\delta)-\gamma_-(s)|\\ &\qquad\leq\ e^{T\|A\|}|h|+2se^{T\|A\|}\|B\|\|\bar{u}\|_\infty+(s-t)e^{T\|A\|}\|B\|\|u^*\|_\infty.\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}&\int_t^{t+\delta}\Big(e^{T\|A\|}|h|+2se^{T\|A\|}\|B\|\|\bar{u}\|_\infty+(s-t)e^{T\|A\|}\|B\|\|u^*\|_\infty\Big)\,ds\\ &\leq\delta e^{T\|A\|}|h|+\Big(2e^{T\|A\|}\|B\|\|\bar{u}\|_\infty+e^{T\|A\|}\|B\|\|u^*\|_\infty\Big)\delta^2.\end{aligned}\tag{2.17}$$

Hence, combining (2.16) and (2.17) with (2.15) yields the conclusion. □

Remark 2.19. We note that Theorem 2.18 guarantees that the function $x \mapsto V(t,x)$ is linearly semiconcave, locally uniformly in time.

The proof of the following theorem is given in Section 2.5.1 since the techniques we have used to prove it are classical in optimal control theory.

Theorem 2.20. *Assume* **(L1)** – **(L3)**. *V is locally Lipschitz continuous on* $[0,T]\times\mathbb{R}^d$.

2.3.2 Lipschitz regularity of Mean Field Games equilibrium

Define the following class of curves on $\mathscr{P}_\alpha(\mathbb{R}^d)$

$$\mathrm{Lip}(\mathscr{P}_\alpha) = \left\{ m \in C([0,T];\mathscr{P}_\alpha(\mathbb{R}^d)) : \sup_{\substack{t\neq s\\ t,s\in[0,T]}} \frac{d_1(m_t,m_s)}{|t-s|} < \infty \right\},$$

and set

$$\mathscr{P}_{m_0}^{\mathrm{Lip}(\mathscr{P}_\alpha)}(\Gamma_T) = \Big\{ \eta \in \mathscr{P}_{m_0}(\Gamma_T,R) : m^\eta \in \mathrm{Lip}(\mathscr{P}_1) \Big\}.$$

Remark 2.21. The set $\mathscr{P}_{m_0}^{\mathrm{Lip}(\mathscr{P}_\alpha)}(\Gamma_T)$ is non-empty. Following the construction we have done in Remark 2.5, let $p:\mathbb{R}^d \to \Gamma_T$ be defined as

$$x \mapsto p[x](t) := e^{tA}x, \forall\, t\in[0,T]$$

and define $\eta = p\sharp m_0$. Therefore, by Remark 2.5, we only need to prove that $m^\eta \in \mathrm{Lip}(\mathscr{P}_\alpha)$.

Indeed,

$$\begin{aligned} d_1(m^\eta_{t_1}, m^\eta_{t_2}) &= \sup_{\phi\in 1-\mathrm{Lip}} \int_{\mathbb{R}^d} \phi(x)\big(m^\eta_{t_1}(dx) - m^\eta_{t_2}(dx)\big) \\ &= \sup_{\phi\in 1-\mathrm{Lip}} \int_{\Gamma_T} \big(\phi(\gamma(t_1)) - \phi(\gamma(t_2))\big)\eta(d\gamma) \\ &= \sup_{\phi\in 1-\mathrm{Lip}} \int_{\Gamma_T} \big(\phi(\gamma(t_1)) - \phi(\gamma(t_2))\big)p\sharp m_0(d\gamma) \\ &= \sup_{\phi\in 1-\mathrm{Lip}} \int_{\mathbb{R}^d} \big(\phi(p[x](t_1)) - \phi(p[x](t_2))\big)m_0(dx) \\ &= \sup_{\phi\in 1-\mathrm{Lip}} \int_{\mathbb{R}^d} \big(\phi(e^{At_1}x) - \phi(e^{At_2}x)\big)m_0(dx) \\ &\leq \int_{\mathbb{R}^d} \big|e^{At_1}x - e^{At_2}x\big| m_0(dx). \end{aligned}$$

Since the function $t\mapsto e^{At}x$ is Lipschitz continuous in any compact subintervals of $\mathbb{R}$ we get the conclusion. □

Proposition 2.22. *Assume* **(L1)** – **(L3)**. *Let $x\in\mathbb{R}^d$ and $\eta \in \mathscr{P}_{m_0}^{Lip(\mathscr{P}_1)}(\Gamma_T)$. Let u^* be an optimal control for* (2.4) *and let γ^* be the minimizing curve generated by u^*. Then, there exists a real positive constant Q_1 such that*

$$\|\dot\gamma^*\|_\infty \leq Q_1(1+|x|).$$

Proof. From the same reasoning in [26, Theorem 7.4.6] one can prove that

$$\|u^*\|_\infty \le Q_0(1+|x|). \tag{2.18}$$

Hence, from the state equation we obtain

$$\|\dot\gamma^*(t)\|_\infty \le \|A\|\|\gamma^*\|_\infty + \|B\|\|u^*\|_\infty.$$

Thus, by Corollary 2.3 and by (2.18) we get

$$\|\dot\gamma^*(t)\|_\infty \le \max\{\|A\|\widetilde{C}_1, \|B\|Q_0\}(1+|x|). \qquad \square$$

Remark 2.23. We observe that the above result is a generalization of [24, Proposition 5.6] where we assumed that there exist two constants c_3, c_4 such that for any $(x,p,m) \in \mathbb{R}^d \times \mathbb{R}^k \times \mathscr{P}_\alpha(\mathbb{R}^d)$

$$\langle D_x H(x,p,m), p\rangle \ge c_3|p|^2 - c_4.$$

We recall the definition of the set-valued map E given in the Section 3, that is,

$$E : \big(\mathscr{P}_{m_0}(\Gamma_T, R), d_1\big) \rightrightarrows \big(\mathscr{P}_{m_0}(\Gamma_T, R), d_1\big)$$

with

$$E(\eta) = \big\{\nu \in \mathcal{P}_{m_0}(\Gamma_T, R) : \ \mathrm{spt}(\nu_x) \subset \ \Gamma^*_\eta(x), \ m_0 - \text{a.e.}\big\}.$$

Lemma 2.24. *Assume* **(L1)** – **(L3)***. Then,* $E(\mathscr{P}_{m_0}^{Lip(\mathscr{P}_\alpha)}) \subset \mathscr{P}_{m_0}^{Lip(\mathscr{P}_\alpha)}$.

Proof. Fix $\eta \in \mathscr{P}_{m_0}^{\mathrm{Lip}(\mathscr{P}_\alpha)}$ and let μ be a Borel probability measure in $E(\eta)$. We need to prove that for any t_1, $t_2 \in [0,T]$, with $t_1 < t_2$

$$\sup_{\substack{t\neq s \\ t,s\in[0,T]}} \frac{d_1(m_t, m_s)}{|t-s|} < \infty.$$

Hence

$$\begin{aligned} d_1(m^\mu_{t_1}, m^\mu_{t_2}) =& \sup_{\phi\in \mathrm{Lip}_1(\mathbb{R}^d)} \int_{\mathbb{R}^d} \phi(x)\big(m^\mu_{t_1}(dx) - m^\mu_{t_1}(dx)\big) \\ =& \sup_{\phi\in \mathrm{Lip}_1(\mathbb{R}^d)} \int_{\Gamma_T} \big(\phi(\gamma(t_1)) - \phi(\gamma(t_2)\big)\ \mu(d\gamma) \\ \le& \int_{\Gamma_T} \big|\gamma(t_1) - \gamma(t_2)\big|\ \mu(d\gamma) \le |t_1 - t_2| \int_{\Gamma_T} \|\dot\gamma\|_\infty|\ \mu(d\gamma) \\ \le& |t_1 - t_2| \int_{\Gamma_T} Q_1(1+|x|)\ \mu(d\gamma), \end{aligned}$$

where the last inequality follows by Proposition 2.22. Therefore, recalling that $x = \gamma(0)$ and μ belongs to $\mathscr{P}_{m_0}(\Gamma_T, R)$, we obtain the conclusion. $\square$

Theorem 2.25 **(Existence of Lipschitz Mean Field Games equilibria).** *Assume* **(L1)** – **(L3)***. Then, there exist at least one Mean Field Games equilibrium such that the associated family of measures* $\{m^\eta_t\}_{t\in[0,T]}$ *belongs to* $Lip(\mathscr{P}_\alpha)$.

Proof. It is sufficient to prove that the set-valued map $E : \mathscr{P}_{m_0}^{\mathrm{Lip}(\mathscr{P}_\alpha)}(\Gamma_T) \rightrightarrows \mathscr{P}_{m_0}^{\mathrm{Lip}(\mathscr{P}_\alpha)}$ has a fix point and in order to prove it we want to use Kakutani's fixed point theorem.

We recall that by Lemma 2.12 we have that the map E has closed graph and so also the restriction of E on $\mathscr{P}_{m_0}^{\mathrm{Lip}(\mathscr{P}_\alpha)}$. Moreover, since $\mathscr{P}_{m_0}^{\mathrm{Lip}(\mathscr{P}_\alpha)} \subset \mathscr{P}_{m_0}(\Gamma_T, R)$ we have that $\mathscr{P}_{m_0}^{\mathrm{Lip}(\mathscr{P}_\alpha)}$ is compact.Therefore, all the assumptions of Kakutani's fixed point theorem are satisfied and this concludes the proof. □

Corollary 2.26. *Assume* **(L1)** – **(L3)**. *Let $\eta \in \mathscr{P}_\alpha(\Gamma_T, R)$ be a Lipschitz Mean Field Games equilibrium and let (V, m^η) be a mild solution associated with η. Then, the value function V is locally semiconcave on $[0,T] \times \mathbb{R}^d$ with a linear modulus of semiconcavity.*

2.4 Mean Field Games: PDEs system

2.4.1 Optimal syntesis

In order to deduce the PDE system for our Mean Field Games problem, we have to derive first some optimality conditions for the following problem:

$$J(x,u) = \inf_{\gamma \in \Gamma_T(x)} \left\{ g(\gamma(T)) + \int_0^T L(t, \gamma(t), u(t))\, dt \right\}. \tag{OC}$$

As usual, let V be the value function of the above **(OC)** problem.

Proposition 2.27. *Assume* **(L1)** – **(L3)**. *Let $(t_0, x_0) \in [0,T] \times \overline{B}_R$ and let p_0 be a point in $D_x^* V(t_0, x_0)$. Then, there exists a pair of curves $(\overline{\gamma}, \overline{p})$ solving the Hamiltonian system*

$$\begin{cases} \dot{\overline{\gamma}}(t) = -D_p H(t, \overline{\gamma}(t), \overline{p}(t)), & \overline{\gamma}(t_0) = x_0 \\ \dot{\overline{p}}(t) = D_x H(t, \overline{\gamma}(t), \overline{p}(t)), & \overline{p}(t_0) = p_0 \end{cases}$$

such that $\overline{\gamma}$ is a minimizer of $V(t_0, x_0)$. In particular, if $V(t_0, \cdot)$ is differentiable at x_0 then $\overline{\gamma}$ is the unique minimizer of $V(t_0, x_0)$.

Proof. Let p_0 be a point in $D_x^* V(t_0, x_0)$ such that $(t_0, x_0) \in [0,T] \times \overline{B}_R$. By definition of reachable gradient, there exists a sequence $\{x_k\}_{k \in \mathbb{N}}$ such that

$$\begin{aligned} x_k &\to x_0 \\ -p_0 &= \lim_{k \to \infty} D_x V(t_0, x_k). \end{aligned}$$

Let $\bar{u}_k$ and $\bar{\gamma}_k$ be, respectively, an optimal control and an optimal trajectory with starting point (t_0, x_k). By the maximum principle, we have that there exists an absolutely continuous arc $\bar{p}_k$ such that

$$\begin{cases} -\dot{\bar{p}}_k(t) = A^* \bar{p}_k(t) + D_x L(t, \bar{\gamma}_k(t), \bar{u}_k(t)) \\ \bar{p}_k(T) = Dg(\bar{\gamma}_k(T)). \end{cases} \tag{2.19}$$

By the maximum principle we know that

$$\begin{cases} \dot{\bar{\gamma}}_k(t) = -D_p H(t, \bar{\gamma}_k(t), \bar{p}_k(t)) \\ \dot{\bar{p}}_k(t) = D_x H(t, \bar{\gamma}_k(t), \bar{p}_k(t)). \end{cases} \tag{2.20}$$

Since the sequence $\{x_k\}_{k\in\mathbb{N}}$ is convergent, by Corollary 2.3 and Proposition 2.22 we obtain that $\{\gamma_k\}_{k\in\mathbb{N}}$ is equibounded and equicontinuous.

Moreover, by (2.19) we have that for any $t \geq t_0$

$$\bar{p}_k(t) = e^{(T-t)A^*} Dg(\bar{\gamma}_k(T)) + \int_t^T e^{(s-t)A^*} D_x L(s, \bar{\gamma}_k(s), \bar{u}_k(s)) \, ds.$$

Thus, it easily follows that also the sequence of dual arcs $\{\bar{p}_k\}_{k\in\mathbb{N}}$ is equibounded and equicontinuous. Therefore, there exist an absolutely continuous arc $\bar{p}$ and a curve $\bar{\gamma}$ such that $\bar{p}_k \to \bar{p}$ and $\bar{\gamma}_k \to \bar{\gamma}$, uniformly as $k \to \infty$.

From **(L3)**, we have that there exists a constant $\kappa \geq 0$ such that

$$|D_x L(t, x, u)| \leq \kappa(1 + |u|^2).$$

Moreover, since $x \in \overline{B}_R$ we deduce by [26, Theorem 7.4.6] that there exists a constant $\tilde{\kappa} \geq 0$ such that $\|u_k\|_\infty \leq \tilde{\kappa}$. Consequently, we obtain that $D_x L(t, \bar{\gamma}_k(t), \bar{u}_k(t))$ weakly converges in $L^2(0, T; \mathbb{R}^d)$ to $D_x L(t, \bar{\gamma}(t), \bar{u})$ as $k \to \infty$.

Therefore, passing to the limit in (2.19) we get that $\bar{p}$ is a solution of the limit equation and by the maximum principle the pair $(\bar{\gamma}, \bar{p})$ solves system (2.20). In conclusion, as $k \to \infty$ in the value function we obtain that the curve $\bar{\gamma}$ is a minimizer for (t_0, x_0). □

2.4.2 Weak solutions

In this section, we consider the case of splitted Langrangian, that is L is of the form (2.9).

We recall that, given the control system (2.1), the Hamiltonian associated with the Lagrangian function L is defined as

$$H(x, p) = \sup_{u \in \mathbb{R}^k} \Big\{ - \langle p, Ax + Bu \rangle - \ell(x, u) \Big\}.$$

For $\alpha > 1$, let $m_0 \in \mathscr{P}_\alpha(\mathbb{R}^d)$ be a Borel probability measure and introduce the following Mean Field Games PDEs system

$$\begin{cases} -\partial_t V(t,x) + H(x, D_x V(t,x)) = F(x, m_t), & (t,x) \in [0,T] \times \mathbb{R}^d \\ \partial_t m_t + \operatorname{div}\Big(m_t D_p H(x, D_x V(t,x))\Big) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ m_0 = m_0, \quad V(T,x) = G(x, m_T), \ \forall \, x \in \mathbb{R}^d. \end{cases} \tag{2.21}$$

Definition 2.28 (Weak solutions). *We say that $(V, m) \in W^{1,\infty}([0,T] \times \mathbb{R}^d) \times C([0,T], \mathcal{P}_\alpha(\mathbb{R}^d))$ is a weak solution of the Mean Field Games PDEs system if:*

(i) m is a solution in the sense of distribution of the continuity equation, i.e. for any test function $\varphi \in C^1_c([0,T) \times \mathbb{R}^d)$ we have that

$$- \int_{\mathbb{R}^d} \varphi(0,x) \, m_0(dx) = \int_0^T \int_{\mathbb{R}^d} \Big(\partial_t \varphi(t,x) - \langle D_x \varphi(t,x), D_p H(x, D_x V(t,x))\rangle\Big) \, m_t(dx).$$

(ii) $V(t,\cdot)$ is differentiable on $\operatorname{spt}(m_t)$ and V is a continuous viscosity solution of Hamilton-Jacobi equation.

***Remark* 2.29.** We recall that by classical optimal control theory, see for instance [26], the following holds:

1. from the maximum principle one can deduce that any minimizer γ of problem (2.4) has the same regularity of the data, thus in this case we obtain that $\gamma \in C^2$;
2. given a Mean Field Games equilibrium η we have that for any $x \in \mathrm{spt}(m_t^\eta)$ the value function V is differentiable since the value function of an optimal control problem with a strictly convex Hamiltonian (with respect to p) is known to be differentiable in the interior of any optimal trajectory, see for instance [26, Theorem 6.4.7] and [22, Proposition 4.4].

Theorem 2.30 (Equivalence between mild and weak solutions). *Assume* **(L1)**—**(L4)***. Fix $\alpha > 1$ and let $m_0 \in \mathscr{P}_\alpha(\mathbb{R}^d)$ be an absolutely continuous with respect the Lebesgue measure and with compact support. Then, $(V, m) \in C([0,T] \times \mathbb{R}^d) \times C([0,T], \mathcal{P}_\alpha(\mathbb{R}^d))$ is a mild solution of the Mean Field Games problem if and only if it is a weak solution of system (2.21).*

Proof. First, we show that any mild solutions (V, m^η) is a weak solution.

Let V be the value function defined as in Definition 2.14, in expression (2.8). Then, it is well-known that it is a continuous viscosity solution of the Hamilton-Jacobi equation in system (2.21) and satisfies the terminal condition. Hence, we are left to prove that m^η is a solution of the continuity equation in system (2.21) in the sense of distributions.

Indeed, for any $\varphi \in C_c^1([0,T) \times \mathbb{R}^d)$, we have that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} \varphi(t,x)\ m_t^\eta(dx) &= \frac{d}{dt}\int_{\Gamma_T} \varphi(t,\gamma(t))\ \eta(d\gamma)\\ &= \int_{\Gamma_T} \Big(\partial_t\varphi(t,\gamma(t)) + \langle D_x\varphi(t,\gamma(t)), \dot\gamma(t)\rangle\Big)\ \eta(d\gamma),\end{aligned}$$

where the last integral is well-posed by point (1) in Remark 2.29. Since $\eta \in \mathscr{P}_{m_0}(\Gamma_T, R)$ is a Lipschitz Mean Field Games equilibrium we know that η is supported on the minimizers of problem (2.4). So, from the Maximum Principle we know that

$$\dot\gamma(t) = -D_pH\big(\gamma(t), D_xV(t,\gamma(t))\big).$$

Therefore,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^d} \varphi(t,x)\ m_t^\eta(dx)\\ &= \int_{\Gamma_T} \Big(\partial_t\varphi(t,\gamma(t)) + \langle D_x\varphi(t,\gamma(t)), \dot\gamma(t)\rangle\Big)\ \eta(d\gamma)\\ &= \int_{\Gamma_T} \Big(\partial_t\varphi(t,\gamma(t)) - \langle D_x\varphi(t,\gamma(t)), D_pH(\gamma(t), D_xV(t,\gamma(t)))\rangle\Big)\ \eta(d\gamma)\\ &= \int_{\mathbb{R}^d} \Big(\partial_t\varphi(t,x) - \langle D_x\varphi(t,x), D_pH(x, D_xV(t,x))\rangle\Big)\ m_t^\eta(dx),\end{aligned}$$

where the last integral in above series of equality is well-posed by point (2) in Remark 2.29. The conclusion follows by integrating the above equalities over $[0,T]$.

Now, let (V, m) be a weak solution of Mean Field Games system. Since V is a viscosity solution of the Hamilton-Jacobi equation we know that it can be represented by the formula (2.8) in Definition 2.14. Hence, we only have to prove that there exists a Mean Field Games equilibrium η such that $m_t = e_t\sharp\eta$.

Since m is a solution of the continuity equation in the sense of distributions, by the superposition principle [7, Theorem 8.2.1] we know that there exists a probability measure $\mu \in \mathscr{P}(\Gamma_T)$ such that $m_t = e_t\sharp\mu$ and μ-a.e. is a solution of the following equation

$$\dot{\gamma}(t) = -D_pH(\gamma(t), D_xV(t,\gamma(t))), \quad t \in [0,T]. \tag{2.22}$$

As $m_0 = e_0\sharp\mu$, by Theorem 1.4 there exists a family of Borel probability measures μ_x, for any $x \in \mathrm{spt}(m_0)$, such that

$$\mu(d\gamma) = \int_{\mathbb{R}^d} \mu_x(d\gamma)m_0(x)\; dx.$$

Since m_0 is absolutely continuous with compact support and the value function V is locally Lipschitz continuous, it follows that m_0-a.e. and μ_x-a.e. γ is a solution of (2.22) such that $\gamma(0) = x$. Therefore, by the optimal synthesis explained above, such a curve γ is a minimizer of the underlying optimal control problem and from (2.13) it is also the unique solution of (2.22). Hence, the measures μ_x are supported on minimizing curves of the optimal control problem. Consequently, μ is a Mean Field Games equilibrium for m_0.

□

The following result is an immediate consequence of Theorem 2.30 and Theorem 2.16.

Corollary 2.31. *Assume* **(L1)**—**(L4)**. *Assume that F is strictly monotone, in the sense of definition Definition 2.15. Let η_1, $\eta_2 \in \mathscr{P}_{m_0}(\Gamma_T, R)$ be two Lipschitz Mean Field Games equilibria and let (V_1, m^{η_1}), (V_2, m^{η_2}) be, respectively, the weak solutions of system (2.21). Then, $V_1 \equiv V_2$.*

2.5 Appendix

2.5.1 Proof of Theorem 2.20

We divide the proof in two steps: first, we prove that V is locally Lipschitz in space and then, we prove that it is locally Lipschitz in both the variables.

Let R be a positive radius and denote by B_R the ball of radius R centered in the origin on $\mathbb{R}^d$. Fix $x \in \overline{B}_R$ and $h \in \mathbb{R}^d$ such that $x + h \in \overline{B}_R$. Then, given an optimal control u^* associated with $(t,x) \in [0,T] \times \overline{B}_R$ we get that

$$\begin{aligned} & V(t,x+h) - V(t,x) \\ \leq \int_t^T & \left(L(\gamma(s;t,x+h,u^*),u^*(s),m_s^\eta) - L(\gamma(s;t,x,u^*),u^*(s),m_s^\eta)\right)\,ds \\ & +G(\gamma(T;t,x+h,u^*),m_T^\eta) - G(\gamma(T;t,x,u^*),m_T^\eta). \end{aligned} \tag{2.23}$$

Thus, we have to estimate the distance between two admissible paths: the one starting in (t,x) and the other one starting in $(t, x+h)$. Recall that

$$\gamma(s;t,x,u) = e^{(s-t)A}x + \int_t^s e^{(\tau-t)A}Bu^*(\tau)\; d\tau, \quad \forall\; s \in [t,T]$$

to obtain

$$|\gamma(s;t,x+h,u^*)-\gamma(s;t,x,u^*)| \le e^{T\|A\|}|h|, \quad \forall\ s\in[t,T].$$

Therefore, by assumption **(L2)** we get

$$G(\gamma(T;t,x+h,u^*),m_T^\eta)-G(\gamma(T;t,x,u^*),m_T^\eta) \le \|G\|_\infty e^{T\|A\|}|h|.$$

So, we just have to bound the integral term in (2.23). By assumption **(L3)**, we have that

$$\begin{aligned}
\int_t^T &\left(L(\gamma(s;t,x+h,u^*),u^*(s),m_s^\eta)-L(\gamma(s;t,x,u^*),u^*(s),m_s^\eta)\right)\,ds\\
&= \int_t^T\int_0^1 \langle D_xL(\lambda\gamma(s;t,x+h,u^*)\\
&\quad +(1-\lambda)\gamma(s;t,x,u^*),u^*(s),m_s^\eta,\gamma(s;t,x+h,u^*)-\gamma(s;t,x,u^*)\rangle\,ds\\
&\le \int_t^T\int_0^1 \left|D_xL(\lambda\gamma(s;t,x+h,u^*)+(1-\lambda)\gamma(s;t,x,u^*)\right|\\
&\qquad \left|u^*(s),m_s^\eta,\gamma(s;t,x+h,u^*)-\gamma(s;t,x,u^*)\right|\,ds\\
&\le \int_t^T\int_0^1 c_2\big(1+|u^*(s)|\big)\left|\gamma(s;t,x+h,u^*)-\gamma(s;t,x,u^*)\right|\,ds\\
&\le\ Tc_2e^{T\|A\|}|h|+c_2\sqrt{T}\|u^*\|_2|h| = \left(c_2Te^{T\|A\|}+c_2\sqrt{T}K\right)|h|,
\end{aligned}$$

where $\|u^*\|_2\le K$ by Proposition 2.2. Then, we conclude that

$$V(t,x+h)-V(t,x) \le \left(c_2Te^{T\|A\|}+c_2\sqrt{T}K+\|G\|_\infty e^{T\|A\|}\right)|h|.$$

By similar considerations, one can easily prove that the reverse inequality also holds true. Therefore, we have that V is locally Lipschitz in space.

We now prove that V is locally Lipschitz in space and time on $[0,T]\times\overline{B}_R$ for any $R>0$. Fix $t\in[0,T]$, $x\in\overline{B}_R$ and let $\delta\in\mathbb{R}$ be such that $t+\delta\in[0,T]$.

We recall that, by the Dynamic Programming Principle we know that

$$V(t,x)=\inf_{u\in L^2}\left\{V(t+\delta,\gamma(t+\delta;t,x,u))+\int_t^{t+\delta} L(\gamma(s;t,x,u),u(s),m_s^\eta)\,ds\right\}. \quad (2.24)$$

Moreover, by [26, Theorem 7.4.6] we know that, under the assumptions **(L1)**–**(L4)**, for any $\eta\in\mathscr{P}_{m_0}(\Gamma_T)$ and any $x\in\mathbb{R}^d$, problem (2.4) is equivalent to the following one

$$\inf_{u\in L^\infty(0,T;\mathbb{R}^k)} J_\eta(x,u).$$

Thus, we can minimize over the set of bounded controls. Let the control $u^*\in L^\infty$ be optimal for $V(t,x)$. By (2.24) we deduce that for any $\epsilon\ge 0$

$$V(t,x)+\epsilon \ge \int_t^{t+\delta} L(\gamma(s;t,x,u^*),u^*(s),m_s^\eta)\,ds+V(t+\delta,\gamma(t+\delta;t,x,u^*)).$$

Hence, we have that

$$
\begin{aligned}
& V(t+\delta, x)-V(t, x) \\
\leq\ & V(t+\delta, x)-V(t+\delta, \gamma(t+\delta; t, x, u^*)) \\
& -\int_t^{t+\delta} L(\gamma(s; t, x, u^*), u^*(s), m_s^\eta)\, ds+\epsilon \\
\leq\ & \left(c_2 T e^{T\|A\|}+c_2 \sqrt{T} K+\|G\|_\infty e^{T\|A\|}\right)|x-\gamma(t+\delta; t, x, u^*)| \\
& +\delta\left(c_1+\frac{1}{c_0}\|u^*\|_\infty\right),
\end{aligned} \tag{2.25}
$$

where the last inequality holds true by the first step of the proof and assumption **(L3)**. Moreover, since the curve $\gamma(\cdot; t, x, u^*)$ is Lipschitz continuous in time, we know that the first term of the right-hand side is bounded by a constant times δ. Thus, the proof of first estimate is complete.

On the other hand, again by (2.24) we know that taking $u \equiv 0$ we have that

$$
V(t, x) \leq V(t+\delta, \gamma(t+\delta; t, x, 0))+\int_t^{t+\delta} L(\gamma(s; t, x, 0), 0, m_s^\eta)\, ds.
$$

Therefore, adding and subtracting the term $V(t+\delta, x)$ we get that

$$
\begin{aligned}
& V(t, x)-V(t+\delta, x) \\
& \qquad \leq V(t+\delta, \gamma(t+\delta; t, x, 0))-V(t+\delta, x)+\int_t^{t+\delta} L(\gamma(s; t, x, 0), 0, m_s^\eta)\, ds.
\end{aligned}
$$

Hence, by the same considerations as in (2.25) we get the result. □

Chapter 3

Ergodic behavior of control and mean field games problems depending on acceleration

3.1 Setting and assumptions of the problems

3.1.1 Calculus of variation with acceleration

In our first main result we study the large time average of an optimal control problem of acceleration. Let $L : \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be the Lagrangian function defined as

$$L(x, v, w) = \frac{1}{2}|w|^2 + F(x, v)$$

where $F : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following assumptions:

(**F1**) F is globally continuous with respect to both variables;

(**F2**) there exists $\alpha > 1$ and there exists a constant $c_F \geq 1$ such that for any $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\frac{1}{c_F}|v|^\alpha - c_F \leq F(x, v) \leq c_F(1 + |v|^\alpha) \tag{3.1}$$

and, without loss of generality, we assume $F(x, v) \geq 0$ for an $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$;

(**F3**) there exists a constant $C_F \geq 0$ such that

$$|D_x F(x, v)| + |D_v F(x, v)| \leq C_F(1 + |v|^\alpha).$$

Let Γ be the set C^1 curves $\gamma : [0, +\infty) \to \mathbb{T}^d$ (endowed with the local uniform convergence of the curve and its derivative) and for $(t, x, v) \in [0, T] \times \mathbb{T}^d \times \mathbb{R}^d$ let $\Gamma_t(x, v)$ be the subset of Γ such that $\gamma(t) = x$ and $\dot{\gamma}(t) = v$. Define the functional $J^{t,T} : \Gamma \to \mathbb{R}$ as

$$J^{t,T}(\gamma) = \int_t^T \left(\frac{1}{2}|\ddot{\gamma}(s)|^2 + F(\gamma(s), \dot{\gamma}(s))\right) ds, \quad \text{if } \gamma \in H^2(0, T; \mathbb{T}^d), \tag{3.2}$$

and $J^{t,T}(\gamma) = +\infty$ if $\gamma \notin H^2(0, T; \mathbb{T}^d)$, and let $V^T(t, x, v)$ denote the value function associated with the functional $J^{t,T}$, i.e.

$$V^T(t, x, v) = \inf_{\gamma \in \Gamma_t(x,v)} J^{t,T}(\gamma). \tag{3.3}$$

Let H be the Hamiltonian associated with the Lagrangian L, that is for any $(x, v, p_v) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d$,

$$H(x, v, p_v) = \frac{1}{2}|p_v|^2 - F(x, v),$$

where $p_v \in \mathbb{R}^d$ denotes the momentum variable associated with $v \in \mathbb{R}^d$. Then, it is not difficult to see that the value function V^T is a continuous viscosity solution of the following Hamilton-Jacobi equation on $[0, T] \times \mathbb{T}^d \times \mathbb{R}^d$:

$$\begin{cases} -\partial_t V^T(t, x, v) - \langle D_x V^T(t, x, v), v \rangle + \frac{1}{2}|D_v V^T(t, x, v)|^2 = F(x, v), \\ V^T(T, x, v) = 0 \quad \text{in } \mathbb{T}^d \times \mathbb{R}^d. \end{cases}$$

Our aim is to characterize the behavior of $V^T(0, \cdot, \cdot)$ as $T \to +\infty$. To state the result, we need the notion of closed measure, which requires another notation: we set

$$\begin{aligned} &\mathscr{P}_{\alpha,2}(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d) \\ &\quad = \left\{ \mu \in \mathscr{P}(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d) : \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(|w|^2 + |v|^\alpha \right) \mu(dx, dv, dw) < +\infty \right\} \end{aligned}$$

endowed with the weak-* convergence.

Definition 3.1 (**Closed measure).** *Let $\eta \in \mathscr{P}_{\alpha,2}(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$. We say that η is a closed measure if for any test function $\varphi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ the following holds*

$$\int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \Big(\langle D_x \varphi(x, v), v \rangle + \langle D_v \varphi(x, v), w \rangle \Big) \, \eta(dx, dv, dw) = 0.$$

We denote by $\mathcal{C}$ the set of closed measures.

Theorem 3.2 (**Main result 1).** *Assume that F satisfies assumptions* **(F1)** *and* **(F2)**. *Then, the following limits exist:*

$$\lim_{T \to +\infty} \frac{1}{T} V^T(0, x, v) = \lim_{T \to +\infty} \inf_{\gamma \in \Gamma_0(x,v)} \frac{1}{T} J^T(\gamma)$$

and are independent of $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$. Moreover, if F satisfies also **(F3)** *then*

$$\lim_{T \to \infty} \frac{1}{T} V^T(0, x, v) = \inf_{\mu \in \mathcal{C}} \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x, v) \right) \mu(dx, dv, dw).$$

Remark 3.3.

1. If we denote by $\bar{\lambda}$ the above limits, the convergence of

$$V^T(0, x, v) - \lambda T$$

is a completely open problem in this context. This is related to the lack of solution of the ergodic HJ equation.

2. The (strong) structure condition on L and the fact that the problem is periodic in the x variable can probably be relaxed: this would require however more refined and technical estimates and we have chosen to work in this simpler framework.

3.1.2 Mean Field Games of acceleration

In our second main result, we consider a mean field game problem of acceleration. The Lagrangian function $L : \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathscr{P}_1(\mathbb{T}^d \times \mathbb{R}^d) \to \mathbb{R}$ now takes the form

$$L(x, v, w, m) = \frac{1}{2}|w|^2 + F(x, v, m)$$

where $F : \mathbb{T}^d \times \mathbb{R}^d \times \mathscr{P}_1(\mathbb{T}^d \times \mathbb{R}^d) \to \mathbb{R}$ satisfies the following assumptions:

(**F1'**) F is globally continuous with respect to all the variables;

(**F2'**) there exists $\alpha > 1$ and a constant $c_F \geq 1$ such that for any $(x, v, m) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathscr{P}_1(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$

$$\frac{1}{c_F}|v|^\alpha - c_F \leq F(x, v, m) \leq c_F(1 + |v|^\alpha)$$

and, without loss of generality, we assume $F(x, v, m) \geq 0$ for any $(x, v, m) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathscr{P}_1(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$;

(**F3'**) there exists a constant $C_F \geq 0$ such that, for any $(x, v, m) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathscr{P}_1(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$,

$$|D_x F(x, v, m)| + |D_v F(x, v, m)| \leq C_F(1 + |v|^\alpha).$$

We consider the time-dependent MFG system on $[0, T] \times \mathbb{T}^d \times \mathbb{R}^d$

$$\begin{cases} -\partial_t u^T(t, x, v) - \langle D_x u^T(t, x, v), v\rangle + \frac{1}{2}|D_v u^T(t, x, v)|^2 = F(x, v, m_t^T), \\ \partial_t m_t^T - \langle v, D_x m_t^T\rangle - \operatorname{div}\left(m_t^T D_v u^T(t, x, v)\right) = 0, \\ u^T(T, x, v) = g(x, v, m_T^T), \text{ in } \mathbb{T}^d \times \mathbb{R}^d, \quad m_0^T = m_0 \in \mathscr{P}(\mathbb{T}^d \times \mathbb{R}^d). \end{cases} \tag{3.4}$$

where the terminal condition of the Hamilton-Jacobi equation satisfies the following:

(G1) $(x, v) \mapsto g(x, v, m)$ belongs to $C_b^1(\mathbb{T}^d \times \mathbb{R}^d)$ for any $m \in \mathscr{P}(\mathbb{T}^d \times \mathbb{R}^d)$ (without loss of generality we assume $g(x, v, m) \geq 0$) and $m \mapsto g(x, v, m)$ is Lipschitz continuous with respect to the d_1 distance, uniformly in $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$.

We recall that (u^T, m^T) is a solution of (3.4) if u^T is a viscosity solution of the first equation and m^T is a solution in the sense of distributions of the second equation.

Our aim is to understand the averaged limit of u^T as $T \to +\infty$. For this we define the ergodic MFG problem, inspired by the characterization of the limit in Theorem 3.2. Let us recall that the notion of closed measure was introduced in Definition 3.1 and that $\mathcal{C}$ denotes the set of closed measures.

Definition 3.4 (**Solution of the ergodic MFG problem**). *We say that $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R} \times \mathcal{C}$ is a solution of the ergodic MFG problem if*

$$\begin{aligned} \bar{\lambda} = & \inf_{\mu \in \mathcal{C}} \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x, v, \pi \sharp \bar{\mu})\right) \mu(dx, dv, dw) \\ = & \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x, v, \pi \sharp \bar{\mu})\right) \bar{\mu}(dx, dv, dw). \end{aligned} \tag{3.5}$$

Theorem 3.5 (**Main result 2**). *Assume that F and G satisfy* **(F1')**, **(F2')** *and* **(G1)**.

1. *There exists at least one solution* $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R} \times \mathcal{C}$ *of the ergodic MFG problem* (3.5). *Moreover, if* F *satisfies the following monotonicity assumption: there exists* $M_F > 0$ *such that for* m_1, $m_2 \in \mathscr{P}(\mathbb{T}^d \times \mathbb{R}^d)$

$$\begin{aligned} \int_{\mathbb{T}^d \times \mathbb{R}^d} \big(F(x,v,m_1) - F(x,v,m_2)\big) \, (m_1(dx,dv) - m_2(dx,dv)) \\ \geq M_F \int_{\mathbb{T}^d \times \mathbb{R}^d} \big(F(x,v,m_1) - F(x,v,m_2)\big)^2 \, dxdv, \end{aligned} \tag{3.6}$$

then the ergodic constant is unique: If $(\bar{\lambda}_1, \bar{\mu}_1)$ *and* $(\bar{\lambda}_2, \bar{\mu}_2)$ *are two solutions of the ergodic MFG problem, then* $\bar{\lambda}_1 = \bar{\lambda}_2$.

2. *Assume in addition that* $\alpha = 2$, *that* **(F3')** *and* (3.6) *hold and that the initial distribution* m_0 *is in* $\mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$. *Let* (u^T, m^T) *be a solution of the MFG system (3.4) and let* $(\bar{\lambda}, \bar{\mu})$ *be a solution of the ergodic MFG problem (3.5). Then* $T^{-1}u^T(0,\cdot,\cdot)$ *converges locally uniformly to* $\bar{\lambda}$ *and we have*

$$\lim_{T \to +\infty} \frac{1}{T} \int_{\mathbb{T}^d \times \mathbb{R}^d} u^T(0,x,v) \, m_0(dx,dv) = \bar{\lambda}.$$

3.2 Ergodic behavior of control of acceleration

3.2.1 Existence of the limit

Before proving the main result of this section, Proposition 3.13, we need a few preliminary lemmas.

Lemma 3.6. *Assume that* F *satisfies* **(F1)** *and* **(F2)**. *Then, for any* $(x,v) \in \mathbb{T}^d \times B_R$, *with* $R \geq 0$, *and for any* $T > 0$, *we have*

$$\frac{1}{T} V^T(0,x,v) \leq c_F(1 + R^\alpha).$$

Remark 3.7. The result also holds when $F = F(t,x,v)$ depends also on time, provided that F is continuous and satisfies **(F2)** with a constant c_F independent of t.

Proof. Define the curve $\xi(t) = x + tv$, for $t \in [0,T]$. Then, by definition of the value function V^T, we have

$$V^T(0,x,v) \leq J^T(\xi) = \int_0^T F(x+tv, v) \, dt \leq T c_F(1 + R^\alpha).$$

□

Lemma 3.8. *Assume that* F *satisfies* **(F1)** *and* **(F2)**. *Let* $\theta \geq 1$, (x_0, v_0) *and* (x,v) *be in* $\mathbb{T}^d \times B_R$ *for some* $R \geq 1$. *Then, there exists a constant* $C_2 \geq 0$ *(depending only the constants* α *and* c_F *in* **(F2)**) *and a curve* $\sigma : [0,\theta] \to \mathbb{R}^d$ *such that* $\sigma(0) = x_0$, $\dot{\sigma}(0) = v_0$ *and* $\sigma(\theta) = x$, $\dot{\sigma}(\theta) = v$ *and*

$$J^\theta(\sigma) \leq C_2(R^2\theta^{-1} + R^\alpha \theta). \tag{3.7}$$

Remark 3.9. The result also holds when $F = F(t,x,v)$ depends also on time, provided that F is continuous and satisfies **(F2)** with a constant c_F independent of t.

Proof. Define the following parametric curve

$$\sigma(t) = x_0 + v_0 t + Bt^2 + Ct^3, \quad t \in [0, \theta].$$

Choosing

$$\begin{cases} B = 3(x - x_0) - \theta v - 2\theta v_0)\theta^{-2} \\ C = (-2(x - x_0) + \theta(v + v_0))\theta^{-3}, \end{cases}$$

we have that $\sigma(0) = x_0$, $\dot\sigma(0) = v_0$ and $\sigma(1) = x$, $\dot\sigma(1) = v$.

By definition of the functional J^θ we get

$$\begin{aligned} J^\theta(\sigma) &= \int_0^\theta \left(\frac{1}{2}|\ddot\sigma(t)|^2 + F(\sigma(t), \dot\sigma(t))\right) dt \\ &\leq \int_0^\theta \left(\frac{1}{2}|2B + 6Ct|^2 + c_F(1 + |v_0 + 2tB + 3t^2 C|^\alpha)\right) dt \\ &\leq C_2(R^2\theta^{-1} + R^\alpha\theta), \end{aligned}$$

for some constant C_2 depending on the constants α and c_F in **(F2)** only. □

Lemma 3.10. *Let $T \geq 2$ and $(x, v) \in \mathbb{T}^d \times B_{R_0}$ for some $R_0 \geq c_F^{\frac{2}{\alpha}}$. Let $\gamma \in \Gamma(x, v)$ be optimal for $V^T(0, x, v)$. Then for any $\lambda \geq 2$ there exists $\tilde\gamma \in \Gamma(x, v)$ with $\tilde\gamma(T) = x$, $\dot{\tilde\gamma}(T) = v$ and*

$$J^T(\tilde\gamma) \leq J^T(\gamma) + C_3(\lambda^2 R_0^2 + R_0^\alpha \lambda^{-\alpha} T),$$

where the constant C_3 depends on α and c_F only.

Remark 3.11. The result also holds when $F = F(t, x, v)$ depends also on time, provided that F is continuous and satisfies **(F2)** with a constant c_F independent of t. In addition, by the construction in the proof, there exists $\tau > 0$ such that $\tilde\gamma = \gamma$ on $[0, \tau]$ and

$$\int_\tau^T (\frac{1}{2}|\ddot{\tilde\gamma}(t)|^2 + c_F(1 + |\dot{\tilde\gamma}(t)|^\alpha))dt \leq C_3(\lambda^2 R_0^2 + R_0^\alpha \lambda^{-\alpha} T).$$

Finally, the map which associates $\tilde\gamma$ and τ to γ is measurable.

Proof. Let

$$\tau := \begin{cases} \sup\{t \geq 0, \ |\dot\gamma(t)| \leq \lambda R_0\} & \text{if } |\gamma(T-1)| > \lambda R_0, \\ T - 1 & \text{otherwise.} \end{cases}$$

If $\tau \geq T - 2$, we set

$$\tilde\gamma(t) = \begin{cases} \gamma(t) & \text{for } t \in [0, \tau], \\ \sigma(t - \tau) & \text{for } t \in [\tau, T], \end{cases}$$

where σ is the map built in Lemma 3.8 with $\theta = T - \tau$, $\sigma(0) = \gamma(\tau)$, $\dot\sigma(0) = \dot\gamma(\tau)$, $\sigma(T - \tau) = x$, $\dot\sigma(T - \tau) = v$. If $\tau < T - 2$, then we set

$$\tilde\gamma(t) = \begin{cases} \gamma(t) & \text{for } t \in [0, \tau], \\ \sigma_1(t - \tau) & \text{for } t \in [\tau, \tau + 1], \\ \sigma_2(t - \tau - 1) & \text{for } t \in [\tau + 1, T], \end{cases}$$

where σ_1 and σ_2 are the map built in Lemma 3.8 with $\theta = 1$, $\sigma_1(0) = \gamma(\tau)$, $\dot\sigma_1(0) = \dot\gamma(\tau)$, $\sigma_1(1) = x$, $\dot\sigma(1) = v$ and $\theta = T - \tau - 1$ and $\sigma_2(0) = \sigma_2(T - \tau - 1) = x$ and $\dot\sigma_2(0) = \dot\sigma_2(T - \tau - 1) = v$ respectively. Note that $\tilde\gamma(T) = x$ and $\dot{\tilde\gamma}(T) = v$.

In order to estimate $J^T(\tilde{\gamma})$, we first show that τ cannot be too small: namely we claim that

$$\tau \geq T\left(1 - \frac{c_F(1+R_0^\alpha)}{\frac{1}{c_F}(\lambda R_0)^\alpha - c_F}\right) - 1. \tag{3.8}$$

Indeed, let us first recall that by Lemma 3.6 we have

$$J^T(\gamma) \leq c_F(1+R_0^\alpha)T.$$

On the other hand, by assumption **(F2)** and the fact that $|\dot{\gamma}(t)| > \lambda R_0$ on $[\tau, T-1]$ and that $F \geq 0$, we also have that

$$\begin{aligned} J^T(\gamma) &= \int_0^T \left(\frac{1}{2}|\ddot{\gamma}(t)|^2 + F(\gamma(t), \dot{\gamma}(t))\right)\, dt \\ &\geq \int_\tau^{T-1} \left(\frac{1}{c_F}|\dot{\gamma}(t)|^\alpha - c_F\right)\, dt \geq (T-\tau-1)\left(\frac{1}{c_F}(\lambda R_0)^\alpha - c_F\right). \end{aligned}$$

So (3.8) holds for $R_0 \geq c_F^{2/\alpha}$.

We estimate $J^T(\tilde{\gamma})$ in the case $\tau < T-2$, the other case being similar and easier. Note that $|\dot{\gamma}(\tau)| \leq \lambda R_0$. By Lemma 3.8 and the fact that $F \geq 0$, we have

$$\begin{aligned} J^T(\tilde{\gamma}) &= \int_0^\tau (\frac{1}{2}|\ddot{\gamma}(t)|^2 + F(\gamma(t), \dot{\gamma}(t)))dt + \int_0^1 (\frac{1}{2}|\ddot{\sigma}_1(t)|^2 + F(\sigma_1(t), \dot{\sigma}_1(t)))dt \\ &\quad + \int_0^{T-\tau-1} (\frac{1}{2}|\ddot{\sigma}_2(t)|^2 + F(\sigma_2(t), \dot{\sigma}_2(t)))dt \\ &\leq J^T(\gamma) + C_2((\lambda R_0)^2 + (\lambda R_0)^\alpha + R_0^2(T-\tau-1)^{-1} + R_0^\alpha(T-\tau-1)). \end{aligned}$$

In view of (3.8) this implies that

$$J^T(\tilde{\gamma}) \leq J^T(\gamma) + C_3(\lambda^2 R_0^2 + R_0^\alpha \lambda^{-\alpha} T),$$

for a constant C_3 depending on α and c_F only. □

Next we prove that the $(x,v) \mapsto V^T(0,x,v)$ have locally uniformly bounded oscillations.

Lemma 3.12. *There exists a constant $M_1(R) \geq 0$ such that for any (x,v) and (x_0,v_0) in $\mathbb{T}^d \times \overline{B}_R$ we have that*

$$V^T(0,x,v) - V^T(0,x_0,v_0) \leq M_1(R).$$

Proof. Let γ^* be a minimizer for $V^T(0,x_0,v_0)$ and let $\sigma : [0,1] \to \mathbb{T}^d$ be such that $\sigma(0) = x$, $\dot{\sigma}(0) = v$ and $\sigma(1) = x_0$, $\dot{\sigma}(1) = v_0$ as in Lemma 3.8 for $\theta = 1$. Define

$$\tilde{\gamma}(t) = \begin{cases} \sigma(t), & t \in [0,1] \\ \gamma^*(t-1), & t \in [1,T]. \end{cases}$$

Then $\tilde{\gamma} \in \Gamma_0(x,v)$ and, by Lemma 3.8 and the assumption that $F \geq 0$, we have that

$$V^T(0,x,v) - V^T(0,x_0,v_0) \leq \int_0^1 \left(\frac{1}{2}|\ddot{\sigma}(t)|^2 + F(\sigma(t),\dot{\sigma}(t))\right) \, dt$$
$$+ \int_1^T \left(\frac{1}{2}|\ddot{\gamma}^*(t-1)|^2 + F(\gamma^*(t-1),\dot{\gamma}^*(t-1))\right) \, dt - V^T(0,x_0,v_0)$$
$$\leq 2C_2R^2 + \int_0^{T-1} \left(\frac{1}{2}|\ddot{\gamma}^*(t)|^2 + F(\gamma^*(t),\dot{\gamma}^*(t))\right) \, dt - V^T(0,x_0,v_0)$$
$$\leq 2C_2R^2 - \int_{T-1}^{T} \left(\frac{1}{2}|\ddot{\gamma}^*(t)|^2 + F(\gamma^*(t),\dot{\gamma}^*(t))\right) \, dt \;\leq\; 2C_2R^2,$$

which is the claim. □

Proposition 3.13 (**Existence of the limit).** *Assume that F satisfies* **(F1)** *and* **(F2)**. *Then, for any $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, the following limits exist:*

$$\lim_{T\to+\infty} \frac{1}{T} V^T(0,x,v) = \lim_{T\to+\infty} \frac{1}{T} \inf_{\gamma\in\Gamma_0(x,v)} J^T(\gamma).$$

In addition the convergence is locally uniform in (x,v) and the limit is independent of (x,v).

Proof. Fix $R_0 \geq c_F^{2/\alpha}$ such that $|v| \leq R_0$. Let $\{T_n\}_{n\in\mathbb{N}}$ and let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of minimizers for $V^{T_n}(0,x,v)$ such that $T_n \to \infty$ as $n \to \infty$ and

$$\liminf_{T\to\infty} \frac{1}{T} V^T(0,x,v) = \lim_{n\to\infty} \frac{1}{T_n} J^{T_n}(\gamma_n).$$

For $\lambda \geq 2$, let us define $\tilde{\gamma}_n$ is in Lemma 3.10. Then we know that $\tilde{\gamma}_n(T) = x$, $\dot{\tilde{\gamma}}_n(T) = v$ and

$$J^{T_n}(\tilde{\gamma}_n) \leq J^{T_n}(\gamma_n) + C_3(\lambda^2 R_0^2 + R_0^\alpha \lambda^{-\alpha} T_n). \tag{3.9}$$

Let us define $\hat{\gamma}_n$ as the periodic extension of the curve $\tilde{\gamma}_n$, i.e. $\hat{\gamma}_n$ is T_n-periodic and it is equal to $\tilde{\gamma}_n$ on $[0,T_n]$. Then, taking $\hat{\gamma}_n$ as competitors for J^T we obtain that

$$\limsup_{T\to\infty} \inf_{\gamma\in\Gamma_0(x,v)} \frac{1}{T} J^T(\gamma) \leq \limsup_{T\to\infty} \frac{1}{T} J^T(\hat{\gamma}_n)$$
$$= \frac{1}{T_n} J^{T_n}(\tilde{\gamma}_n) \leq \left(\frac{1}{T_n} J^{T_n}(\gamma_n) + C_3(\lambda^2 R_0^2 T_n^{-1} + R_0^\alpha \lambda^{-\alpha})\right),$$

where the equality holds true since we are taking the limit of a periodic function and the last inequality holds by (3.9).

We get the conclusion letting $n \to \infty$ and then $\lambda \to \infty$, indeed: as $n \to \infty$ we deduce that

$$\limsup_{T\to\infty} \inf_{\gamma\in\Gamma_0(x,v)} \frac{1}{T} J^T(\gamma) \leq \lim_n \frac{1}{T_n} J^{T_n}(\gamma_n) + C_3 R_0^\alpha \lambda^{-\alpha}$$
$$= \liminf_{T\to+\infty} \inf_{\gamma\in\Gamma_0(x,v)} \frac{1}{T} J^T(\gamma) + C_3 R_0^\alpha \lambda^{-\alpha}$$

and then, taking the limit as $\lambda \to \infty$ we get

$$\limsup_{T\to\infty} \inf_{\gamma\in\Gamma_0(x,v)} \frac{1}{T} J^T(\gamma) \leq \liminf \inf_{\gamma\in\Gamma_0(x,v)} \frac{1}{T} J^T(\gamma).$$

As the $(V^T(0,\cdot,\cdot))$ have locally bounded oscillation (Lemma 3.12), the above convergence is locally uniform and the limit does not depend on (x,v). □

3.2.2 Characterization of the ergodic limit

In this part we characterize the limit given in Proposition 3.13 in term of closed measures. The proof of the main result, Proposition 3.22, where this characterization is stated, is technical and requires several steps. Here are the main ideas of the proof. By using standard results on occupational measures, one can obtain in a relatively easy way that

$$\lambda := \lim_{T\to\infty} \inf_{\gamma\in\Gamma_0(x_0,v_0)} \frac{1}{T} J^T(\gamma) \\ \geq \inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v)\right)\, \mu(dx,dv,dw),$$

where $\mathcal{C}$ denotes the set of closed probability measures (see Definition 3.1). The difficult part of the proof is the opposite inequality. The first step for this is a min-max formula (Theorem 3.15) which gives, by using the characterization of closed measures, that

$$\inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v)\right)\, \mu(dx,dv,dw) \\ = \sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \left\{-\frac{1}{2}|D_v\varphi(x,v)|^2 - \langle D_x\varphi(x,v), v\rangle + F(x,v)\right\}.$$

In order to exploit this inequality, one just needs to find a map $\varphi \in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)$ for which

$$-\frac{1}{2}|D_v\varphi(x,v)|^2 - \langle D_x\varphi(x,v), v\rangle + F(x,v)$$

is almost equal to λ. This is not easy because the corrector of our ergodic problem does not seem to exist (at least in the usual sense) because of the lack of controllability and, if it existed, it certainly would not be smooth with a compact support. The standard idea in this set-up is to use instead the approximate corrector, i.e., the solution V_δ to

$$\delta V_\delta(x,v) + \frac{1}{2}|D_v V_\delta(x,v)|^2 + \langle D_x V_\delta(x,v), v\rangle = F(x,v) \qquad \text{in } \mathbb{T}^d\times\mathbb{R}^d.$$

However, this approximate corrector has not a compact support either (it is even coercive, see Proposition 3.16) and δV_δ does not converge uniformly to $-\lambda$, but only locally uniformly. We overcome these issues by an extra approximation argument (Lemma 3.18).

Let us first explain why closed measures pop up naturally in our problem. To see this, let $(x_0,v_0)\in\mathbb{T}^d\times\mathbb{R}^d$ be an initial position and let $\gamma^T_{(x_0,v_0)}$ be an optimal trajectory for $V^T(0,x_0,v_0)$. We define the family of Borel probability measures $\{\mu_T\}_{T>0}$ as follows: for any function $\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(x,v,w)\, \mu^T(dx,dv,dw) \\ = \frac{1}{T}\int_0^T \varphi(\gamma^T_{(x_0,v_0)}(t), \dot\gamma^T_{(x_0,v_0)}(t), \ddot\gamma^T_{(x_0,v_0)}(t))\, dt. \quad (3.10)$$

Lemma 3.14. *Assume that F satisfies* **(F1)** *and* **(F2)**. *Let the family of probability measures $\{\mu^T\}_{T>0}$ be defined by (3.10). Then, $\{\mu^T\}_{T>0}$ is tight and there exists a closed measure μ^* such that, up to a subsequence, $\mu^T \rightharpoonup^* \mu^*$ as $T\to+\infty$.*

Proof. We first prove that $\{\mu_T\}_{T>0}$ its a tight family of probability measures. Indeed, by assumption **(F2)** for $(x_0, v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ we know that

$$\begin{aligned}\frac{1}{T}V^T(0,x_0,v_0) =& \frac{1}{T}\int_0^T \left(\frac{1}{2}|\ddot{\gamma}^T_{(x_0,v_0)}(t)|^2 + F(\gamma^T_{(x_0,v_0)}(t), \dot{\gamma}^T_{(x_0,v_0)}(t))\right)\, dt \\ =& \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v)\right)\, \mu^T(dx,dv,dw) \\ \geq& \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + \frac{1}{c_F}|v|^\alpha - c_F\right)\, \mu^T(dx,dv,dw).\end{aligned}$$

On the other hand, by Lemma 3.6 we have that

$$\frac{1}{T}V^T(0,x_0,v_0) \leq C_1$$

where C_1 only depends on the initial point (x_0, v_0). Therefore, we obtain that

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + \frac{1}{c_F}|v|^\alpha\right)\, \mu^T(dx,dv,dw) \leq C_1$$

which implies that $\{\mu^T\}_{T>0}$ is tight. By Prokhorov theorem there exists a measure $\mu^* \in \mathscr{P}(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ such that up to a subsequence $\mu^T \rightharpoonup^* \mu^*$ as $T \to +\infty$.

We now show that the measure μ^* is closed in the sense of Definition 3.1. Let $\varphi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be a test function and let $R \geq 0$ be such that $\varphi(x,v) = 0$ for any $(x,v) \in \mathbb{T}^d \times B_R^c$. Moreover, define

$$\tau^* = \begin{cases} \sup\{t \in [0,T] : |\dot{\gamma}^T_{(x_0,v_0)}(t)| \leq R\}, & \text{if } |\dot{\gamma}_{(x_0,v_0)}(T)| > R \\ T, & \text{if } |\dot{\gamma}_{(x_0,v_0)}(T)| \leq R \end{cases}$$

and let $\sigma^* : [\tau^*, \tau^*+1] \to \mathbb{T}^d$ be as in Lemma 3.8 such that $\sigma^*(\tau^*) = \gamma^T_{(x_0,v_0)}(\tau^*)$, $\dot{\sigma}^*(\tau^*) = \dot{\gamma}^T_{(x_0,v_0)}(\tau^*)$ and $\sigma^*(\tau^*+1) = x_0$, $\dot{\sigma}^*(\tau^*+1) = v_0$. Moreover, define

$$\tilde{\gamma}(t) = \begin{cases} \gamma^T_{(x_0,v_0)}(t), & t \in [0,\tau^*] \\ \sigma^*(t), & t \in (\tau^*, \tau^*+1]. \end{cases}$$

Then we get

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\langle D_x\varphi(x,v), v\rangle + \langle D_v\varphi(x,v), w\rangle\Big)\, d\mu^T(x,v,w) \\ &= \frac{1}{T}\int_0^T \Big(\langle D_x\varphi(\gamma^T_{(x_0,v_0)}(t), \dot{\gamma}^T_{(x_0,v_0)}(t)), \dot{\gamma}^T_{(x_0,v_0)}(t)\rangle \\ &\quad + \langle D_v\varphi(\gamma^T_{(x_0,v_0)}(t), \dot{\gamma}^T_{(x_0,v_0)}(t)), \ddot{\gamma}^T_{(x_0,v_0)}(t)\rangle\Big)\, dt \\ &= \frac{1}{T}\int_0^{\tau^*+1} \Big(\langle D_x\varphi(\tilde{\gamma}^T(t), \dot{\tilde{\gamma}}^T(t)), \dot{\tilde{\gamma}}^T(t)\rangle + \langle D_v\varphi(\tilde{\gamma}^T(t), \dot{\tilde{\gamma}}^T(t)), \ddot{\tilde{\gamma}}^T(t)\rangle\Big)\, dt \\ &- \frac{1}{T}\int_{\tau^*}^{\tau^*+1} \Big(\langle D_x\varphi(\sigma^*(t), \dot{\sigma}^*(t)), \dot{\sigma}^*(t)\rangle + \langle D_v\varphi(\sigma^*(t), \dot{\sigma}^*(t)), \ddot{\sigma}^*(t)\rangle\Big)\, dt \\ &+ \int_{\tau^*}^T \Big(\langle D_x\varphi(\gamma^T_{(x_0,v_0)}(t), \dot{\gamma}^T_{(x_0,v_0)}(t)), \dot{\gamma}^T_{(x_0,v_0)}(t)\rangle \\ &\quad + \langle D_v\varphi(\gamma^T_{(x_0,v_0)}(t), \dot{\gamma}^T_{(x_0,v_0)}(t)), \ddot{\gamma}^T_{(x_0,v_0)}(t)\rangle\Big)\, dt\end{aligned}$$

One can immediately observe that by construction the last integral is 0 (since φ has a support in $\mathbb{T}^d \times B_R$) and by the definition of $\tilde{\gamma}$ one also has that the first one is 0. The behavior of the second is also immediate because, as φ is bounded,

$$\begin{aligned}&\frac{1}{T}\int_{\tau^*}^{\tau^*+1}\Big(\langle D_x\varphi(\sigma^*(t),\dot{\sigma}^*(t)),\dot{\sigma}^*(t)\rangle+\langle D_v\varphi(\sigma^*(t),\dot{\sigma}^*(t)),\ddot{\sigma}^*(t)\rangle\Big)\,dt\\&=\frac{1}{T}(\varphi(\sigma^*(\tau^*+1),\dot{\sigma}^*(\tau^*+1))-\varphi(\sigma^*(\tau^*),\sigma^*(\tau^*))\to 0,\quad \text{as } T\to+\infty.\end{aligned}$$

The proof is thus complete. □

The next step consists in formulating in two different ways the expected limit of Proposition 3.13.

Theorem 3.15 (Minmax formula). *Assume that F satisfies* **(F1)** *and* **(F2)**. *Then, the following equality holds true:*

$$\begin{aligned}&\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)\right)\,\mu(dx,dv,dw)\\&=\sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\left\{-\frac{1}{2}|D_v\varphi(x,v)|^2-\langle D_x\varphi(x,v),v\rangle+F(x,v)\right\}.\end{aligned}\tag{3.11}$$

Proof. By definition of a closed measure we can write

$$\begin{aligned}&\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)\right)\,\mu(dx,dv,dw)\\&=\inf_{\mu\in\mathscr{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2+F(x,v)\\&\qquad-\langle D_x\varphi(x,v),v\rangle-\langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw).\end{aligned}$$

Our aim is to use the min-max Theorem (see Theorem 3.34 below). We use for this the notation introduced in Appendix A and set $\mathbb{A}=C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)$, $\mathbb{B}=\mathscr{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ and for any $(\varphi,\mu)\in\mathbb{A}\times\mathbb{B}$

$$\begin{aligned}&\mathcal{L}(\varphi,\mu)\\&:=\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)-\langle D_x\varphi(x,v),v\rangle-\langle D_v\varphi(x,v),w\rangle\right)\,\mu(dx,dv,dw).\end{aligned}$$

Let us choose $\varphi^*(x,v)=0$ and

$$\begin{aligned}c^*=1+&\inf_{\mu\in\mathscr{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2+F(x,v)\\&-\ \langle D_x\varphi(x,v),v\rangle-\langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw).\end{aligned}$$

Note that c^* is finite (since it is bounded below by assumption (3.1) and bounded above for $\mu=\delta_{(x_0,0,0)}$ for any $x_0\in\mathbb{T}^d$). In addition, the set $\mathbb{B}^*=\{\mu\in\mathbb{B}:\mathcal{L}(\varphi^*,\mu)\le c^*\}$ is nonempty and tight, and thus compact, in $\mathscr{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ for the weak-$*$ convergence. Finally, we have

$$\begin{aligned}c^*\ge 1+&\sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\inf_{\mu\in\mathscr{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2+F(x,v)\\&-\ \langle D_x\varphi(x,v),v\rangle-\langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw).\end{aligned}$$

Therefore, Theorem 3.34 states that

$$\inf_{\mu\in\mathscr{P}_{2,\alpha}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2+F(x,v)$$
$$-\langle D_x\varphi(x,v),v\rangle-\langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw)$$
$$=\sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\inf_{\mu\in\mathscr{P}_2(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{1}{2}|w|^2+F(x,v)$$
$$-\langle D_x\varphi(x,v),v\rangle-\langle D_v\varphi(x,v),w\rangle\Big)\,\mu(dx,dv,dw)$$
$$=\sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\inf_{(x,v,w)\in\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\Big\{\frac{1}{2}|w|^2+F(x,v)-\langle D_x\varphi(x,v),v\rangle$$
$$-\langle D_v\varphi(x,v),w\rangle\Big\}$$
$$=\sup_{\varphi\in C_c^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\Big\{-\frac{1}{2}|D_v\varphi(x,v)|^2-\langle D_x\varphi(x,v),v\rangle+F(x,v)\Big\}.$$

This complete the proof. □

Next we introduce and study the discounted problem associated with (3.2). For any $\delta>0$ and any $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$ we define $J_\delta:\Gamma\to\mathbb{R}\cup\{+\infty\}$ as

$$J_\delta(\gamma)=\int_0^{+\infty}e^{-\delta t}\left(\frac{1}{2}|\ddot\gamma(t)|^2+F(\gamma(t),\dot\gamma(t))\right)dt$$

if $\dot\gamma$ is absolutely continuous with $\int_0^{+\infty}e^{-\delta t}\left(\frac{1}{2}|\ddot\gamma(t)|^2+|\dot\gamma(t))|^\alpha\right)dt<+\infty$, and $J_\delta(\gamma)=+\infty$ otherwise. We define the associated value function (the approximate corrector)

$$V_\delta(x,v)=\inf_{\gamma\in\Gamma_0(x,v)}J_\delta(\gamma). \tag{3.12}$$

We recall that V_δ is the unique continuous viscosity solution with a polynomial growth of the following Hamilton-Jacobi equation

$$\delta V_\delta(x,v)+\frac{1}{2}|D_vV_\delta(x,v)|^2+\langle D_xV_\delta(x,v),v\rangle=F(x,v). \tag{3.13}$$

As the convergence of $V^T(0,\cdot,\cdot)/T$ is locally uniform (by Lemma 3.12), we can apply the Abelian-Tauberian Theorem of [63] and we have that for any $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$

$$\lim_{\delta\to0^+}\delta V_\delta(x,v)=\lim_{T\to\infty}\frac{1}{T}V^T(0,x,v)=:\lambda. \tag{3.14}$$

In the proof of the main result of this section (Proposition 3.22) we will have to smoothen the map V^δ. This involves some local regularity properties of V^δ, which is the aim of the next result.

Proposition 3.16. *Assume that F satisfies* **(F1)** – **(F3)**. *Then, we have:*

(i) $\{\delta V_\delta(x,v)\}_{\delta>0}$ is locally uniformly bounded;

(ii) $\{V_\delta(x,v)\}_{\delta>0}$ has locally uniformly bounded oscillation, i.e. there exists a constant $M(R)\geq0$ such that for any $(x_0,v_0),(x,v)\in\mathbb{T}^d\times\overline{B}_R$

$$V_\delta(x,v)-V_\delta(x_0,v_0)\leq M(R).$$

(iii) *there exists a constant* $\tilde{C} \geq 0$ *such that for any* $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$

$$\tilde{C}^{-1}|v|^{\alpha} - \tilde{C}\delta^{-1} \leq V_{\delta}(x,v) \leq c_F \delta^{-1}(|v|^{\alpha} + 1); \tag{3.15}$$

(iv) *the map* $x \mapsto V_{\delta}(x, v)$ *is locally Lipschitz continuous and there exists a constant* $C_{\delta} \geq 0$ *such that for a.e.* $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ *the following holds:*

$$|D_x V_{\delta}(x,v)| \leq C_{\delta}(1 + |v|^{\alpha}). \tag{3.16}$$

Proof. *(i)* Fix $(x, v) \in \mathbb{T}^d \times \overline{B}_R$ and define a competitor $\gamma : [0, +\infty] \to \mathbb{T}^d$ such that $\gamma(t) = x + tv$. By definition and (3.1) we get

$$\delta V_{\delta}(x,v) \leq \delta \int_0^{\infty} e^{-\delta t} F(\gamma(t), \dot{\gamma}(t))\, ds \leq c_F(1 + |v|^{\alpha}) \ \leq \ c_F(1 + R^{\alpha}).$$

On the other hand, we have by **(F2)** that $F \geq 0$ and thus $V_{\delta} \geq 0$, which completes the proof of (i).

(ii) Let (x_0, v_0), $(x, v) \in \mathbb{T}^d \times \overline{B}_R$ be fixed points, let γ^* be a minimizer for $V_{\delta}(x_0, v_0)$ and let σ be defined as in Lemma 3.8 such that $\sigma(0) = x$, $\dot{\sigma}(0) = v$ and $\sigma(1) = x_0$, $\dot{\sigma}(1) = v_0$. We define a new curve $\gamma : [0, +\infty) \to \mathbb{T}^d$ as follows

$$\gamma(t) = \begin{cases} \sigma(t), & t \in [0,1] \\ \gamma^*(t-1), & t \in (1, +\infty). \end{cases}$$

Then

$$\begin{aligned} &V_{\delta}(x,v) - V_{\delta}(x_0, v_0) \\ \leq &\int_0^1 e^{-\lambda t}\left(\frac{1}{2}|\ddot{\gamma}(t)|^2 + F(\gamma(t), \dot{\gamma}(t))\right)\, dt \\ + &\int_1^{+\infty} e^{-\lambda t}\left(\frac{1}{2}|\ddot{\gamma}(t)|^2 + F(\gamma(t), \dot{\gamma}(t))\right)\, dt \\ &-V_{\lambda}(x_0, v_0). \end{aligned} \tag{3.17}$$

By a change of variable, we have that

$$\begin{aligned} &\int_1^{+\infty} e^{-\delta t}\left(\frac{1}{2}|\ddot{\gamma}^*(t)|^2 + F(\gamma^*(t), \dot{\gamma}^*(t))\right)\, dt \\ =&e^{-\delta}\int_0^{\infty} e^{-\delta s}\left(\frac{1}{2}|\ddot{\gamma}^*(s)|^2 + F(\gamma^*(s), \dot{\gamma}^*(s))\right)\, ds = e^{-\delta} V_{\delta}(x_0, v_0). \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} \left|\int_1^{+\infty} e^{-\delta t}\left(\frac{1}{2}|\ddot{\gamma}(t)|^2 + F(\gamma(t), \dot{\gamma}(t))\right)\, dt - V_{\delta}(x_0, v_0)\right| \\ \leq \left|e^{-\delta} - 1\right| V_{\delta}(x_0, v_0) \\ \leq \ \delta |V_{\delta}(x_0, v_0)| \leq \ c_F(1 + R^{\alpha}), \end{aligned} \tag{3.18}$$

where the last inequality holds true by (i). Moreover, by construction of σ in Lemma 3.8 we have that

$$\int_0^1 e^{-\delta t}\left(\frac{1}{2}|\ddot{\sigma}(t)|^2 + F(\sigma(t), \dot{\sigma}(t))\right)\, dt \leq \ J^1(\sigma) \leq C_2(R^2 + R^{\alpha}). \tag{3.19}$$

Combining together inequality (3.18) and (3.19) in (3.17) we get (ii):

$$V_{\delta}(x,v) - V_{\delta}(x_0, v_0) \leq c_F(1 + R^{\alpha}) + C_2(R^2 + R^{\alpha}) =: M(R).$$

(iii) For some constants M_1 and M_2 we have that the map $Z : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $Z(x,v) = M_1^{-1}|v|^\alpha - M_2\delta^{-1}$ is a subsolution of (3.13), indeed

$$\begin{aligned}\delta Z(x,v) + \frac{1}{2}|D_v Z(x,v)|^2 + \langle D_x Z(x,v), v\rangle - F(x,v) \\ \leq \delta M_1^{-1}|v|^\alpha - M_2 + \frac{1}{2}M_1^{-2}\alpha^2|v|^{2(\alpha-1)} - c_F^{-1}|v|^\alpha + c_F.\end{aligned}$$

As $2(\alpha-1) \leq \alpha$, since $\alpha \in (1,2]$, we get, for M_1 and M_2 large enough,

$$\delta Z(x,v) + \frac{1}{2}|D_v Z(x,v)|^2 + \langle D_x Z(x,v), v\rangle - F(x,v) \leq 0.$$

By comparison we obtain $V_\delta \geq Z$, which proves the first inequality in (3.15).

In the same way, considering the map $Z(x,v) = c_F\delta^{-1}(|v|^\alpha + 1)$, we have

$$\begin{aligned}\delta Z(x,v) + \frac{1}{2}|D_v Z(x,v)|^2 + \langle D_x Z(x,v), v\rangle - F(x,v) \\ \geq c_F(|v|^\alpha + 1) + \frac{1}{2}\delta^{-2}(c_F\alpha)^2|v|^{2(\alpha-1)} - c_F|v|^\alpha - c_F \ \geq \ 0,\end{aligned}$$

so that Z is a supersolution. By comparison we conclude that the second inequality in (3.15) holds.

(iv) Let γ^* be optimal for $V_\delta(x,v)$ and let $h \in \mathbb{R}^d$. Then

$$\begin{aligned}V_\delta(x+h,v) \leq \int_0^{+\infty} e^{-\delta t}\left(\frac{1}{2}|\ddot\gamma^*(t)|^2 + F(\gamma^*(t)+h, \dot\gamma^*(t))\right)\, dt \\ \leq \ V_\delta(x,v) + \int_0^{+\infty} e^{-\delta t}\left(F(\gamma^*(t)+h, \dot\gamma^*(t)) - F(\gamma^*(t), \dot\gamma^*(t))\right)\, dt \qquad (3.20) \\ \leq \ V_\delta(x,v) + \int_0^{+\infty} e^{-\delta t} c_F(1 + |\dot\gamma^*(t)|^\alpha)|h|\, dt,\end{aligned}$$

where the last inequality holds true by assumption **(F3)**. Moreover, by (3.15) we deduce that there exists a constant $C_\delta \geq 0$ such that

$$\int_0^{+\infty} e^{-\delta t}(c_F^{-1}|\dot\gamma^*(t)|^\alpha - c_F)\, dt \leq V_\delta(x,v) \leq C_\delta(1+|v|^\alpha).$$

Therefore, by (3.20) we deduce that

$$V_\delta(x+h,v) - V_\delta(x,v) \leq C_\delta(1+|v|^\alpha)|h|,$$

which implies that V_δ is locally Lipschitz continuous in space and proves (iv). □

We now strengthen a little the convergence in (3.14):

Proposition 3.17. *Assume that F satisfies* **(F1)**—**(F3)**. *Then*

$$\lambda = \lim_{\delta\to 0^+} \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \delta V_\delta(x,v),$$

with λ defined in (3.14).

Proof. First we note that, by (i) in Proposition 3.16, the convergence in (3.14) is locally uniform. Fix $R \geq 0$ such that

$$c_F^{-1} R^\alpha - c_F > \lambda. \tag{3.21}$$

Then, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for any $\delta \in (0, \delta_\varepsilon)$ we have that

$$\inf_{(x,v)\in\mathbb{T}^d\times B_R} \delta V_\delta(x,v) \geq \lambda - \varepsilon\,. \tag{3.22}$$

Fix $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ and let γ^*_δ be a minimizer for $V_\delta(x, v)$. We define

$$\tau_\delta = \begin{cases} \inf\{t \in [0,+\infty] : |\dot\gamma^*_\delta(t)| \leq R\}, & \text{if } \{t \in [0,+\infty] : |\dot\gamma^*_\delta(t)| \leq R\} \neq \emptyset \\ +\infty, & \text{if } \{t \in [0,+\infty] : |\dot\gamma^*_\delta(t)| \leq R\} = \emptyset. \end{cases}$$

By Dynamic Programming Principle we get

$$V_\delta(x,v) = \int_0^{\tau_\delta} e^{-\delta t}\left(\frac{1}{2}|\ddot\gamma^*_\delta(t)|^2 + F(\gamma^*_\delta(t), \dot\gamma^*_\delta(t))\right)\, dt + e^{-\delta\tau_\delta} V_\delta(\gamma^*_\delta(\tau_\delta), \dot\gamma^*_\delta(\tau_\delta))$$

and by assumption (3.1) and definition of τ_δ we deduce that

$$\delta V_\delta(x,v) \geq (c_F^{-1}R^\alpha - c_F)(1 - e^{-\delta\tau_\delta}) + e^{-\delta\tau_\delta}\delta V_\delta(\gamma^*_\delta(\tau_\delta), \dot\gamma^*_\delta(\tau_\delta)). \tag{3.23}$$

If τ_δ is finite, we have that $|\dot\gamma^*_\delta(\tau_\delta)|$ is bounded by R and thus, by (3.21) and (3.22) we deduce that for any $\delta \in (0, \delta_\varepsilon)$

$$\delta V_\delta(x,v) \geq \lambda(1 - e^{-\delta\tau_\delta}) + e^{-\delta\tau_\delta}(\lambda - \varepsilon) \geq \lambda - \varepsilon\,.$$

By (3.21) and (3.23) the same inequality also holds if $\tau_\delta = +\infty$. Hence, we obtain that

$$\lim_{\delta\to 0^+} \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \delta V_\delta(x,v) \geq \lambda - \varepsilon\,.$$

By (3.14) we infer that

$$\lambda = \lim_{\delta\to 0^+} \delta V_\delta(0,0) \geq \lim_{\delta\to 0^+} \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \delta V_\delta(x,v) \geq \lambda - \varepsilon,$$

which implies the desired result since ε is arbitrary. □

As V_δ is coercive, we cannot use it directly as a test function to test the fact that a measure is closed. To overcome this issue we approximate V_δ by family of Lipschitz maps (V_δ^R).

Lemma 3.18 (**Approximate problem 1**)**.** *Assume that F satisfies assumption* **(F1)**—**(F3)***. Let $R > 0$ and define $F_R(x, v) = \min\{F(x, v), R\}$ for any $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$. Let V_δ^R be the unique continuous and bounded viscosity solution to*

$$\delta V_\delta^R(x,v) + \frac{1}{2}|D_v V_\delta^R(x,v)|^2 + \langle D_x V_\delta^R(x,v), v\rangle = F_R(x,v), \quad (x,v) \in \mathbb{T}^d \times \mathbb{R}^d. \tag{3.24}$$

Then, the following holds:

(i) V_δ^R is globally Lipschitz continuous;

(ii) there are two positive constants $\tilde c_{1,\delta}$ and $\tilde c_{2,\delta}$ such that

$$\delta V_\delta^R(x,v) \geq \tilde c_{1,\delta}\big(1 + \min\{|v|^\alpha, R\}\big) - \tilde c_{2,\delta} \tag{3.25}$$

for any $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$;

(iii) there is a constant $\tilde{C}_\delta \geq 0$ such that

$$|D_x V_\delta^R(x,v)| \leq \tilde{C}_\delta\big(1+\min\{|v|^\alpha, R\}\big) \tag{3.26}$$

for a.e. $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$;

(iv) V_δ^R converge, as $R \to +\infty$, uniformly on compact subsets of $\mathbb{T}^d \times \mathbb{R}^d$ to the map V_δ defined in (3.12).

The proofs of (i) and (iv) are direct consequences of optimal control theory while the proofs of (3.25) and (3.26) follow the same argument as for (3.15) and (3.16), respectively and we omit these proofs.

Lemma 3.19. *Assume that F satisfies* **(F1)** – **(F3)**. *Let F_R and V_δ^R be defined in Lemma 3.18. Then we have that*

$$\inf_{\mu \in \mathcal{C}} \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v)\right) \mu(dx,dv,dw) \geq \inf_{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d} \delta V_\delta^R(x,v). \tag{3.27}$$

Remark 3.20. Note that we can allow for a larger class of test functions in Definition 3.1, i.e. $\varphi \in W^{1,\infty}(\mathbb{T}^d \times \mathbb{R}^d) \cap C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$. Indeed, let $\varphi \in W^{1,\infty}(\mathbb{T}^d \times \mathbb{R}^d) \cap C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ and for $R > 1$ let $\xi_R \in C_c^\infty(\mathbb{R}^d)$ be such that $\xi_R(x,v) = 1$ for $(x,v) \in \mathbb{T}^d \times B_R$, $\xi_R(x,v) = 0$ for $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d \backslash B_{2R}$, $0 \leq \xi_R(x,v) \leq 1$ for $\mathbb{T}^d \times B_{2R} \backslash B_R$ and there exists a constant $M \geq 0$ such that $|D\xi_R(x,v)| \leq MR^{-1}$ for any $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$. Set $\varphi_R = \varphi\xi_R$. Then, we have that $\varphi_R \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, $D\varphi_R$ is uniformly bounded and converges locally uniformly to $D\varphi$. For $\mu \in \mathcal{C}$ we have:

$$\int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \Big(\langle D_x\varphi_R(x,v), v\rangle + \langle D_v\varphi_R(x,v), w\rangle\Big)\, \mu(dx,dv,dw) = 0. \tag{3.28}$$

Since $\mu \in \mathscr{P}_{2,\alpha}(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$, we can pass to the limit in (3.28) as $R \to +\infty$ by dominate convergence. This proves that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \Big(\langle D_x\varphi(x,v), v\rangle + \langle D_v\varphi(x,v), w\rangle\Big)\, \mu(dx,dv,dw) = 0$$

for $\varphi \in W^{1,\infty}(\mathbb{T}^d \times \mathbb{R}^d) \cap C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$. □

Proof. Let $\xi^{1,\varepsilon} \in C_c^\infty(\mathbb{R}^d)$ be such that $\mathrm{spt}(\xi^{1,\varepsilon}) \subset B_\varepsilon$, $\xi^{1,\varepsilon}(x) \geq 0$ and $\int_{B_\varepsilon} \xi^{1,\varepsilon}(x)\, dx = 1$, and define $V_\delta^{R,\varepsilon}(x,v) = V_\delta^R \star_x \xi^{1,\varepsilon}(x,v)$ where the mollification only holds in x. Then V_δ^ε satisfies the following inequality in the viscosity sense

$$\begin{aligned} \delta V_\delta^\varepsilon(x,v) + \frac{1}{2}|D_v V_\delta^\varepsilon(x,v)|^2 + \langle D_x V_\delta^\varepsilon(x,v), v\rangle \\ \leq F_R \star \xi^{1,\varepsilon}(x,v) \leq F_R(x,v) + C_F\, \varepsilon(1+\min\{|v|^\alpha, R\}) \end{aligned}$$

where the last inequality holds true by **(F3)** and the definition of F_R. Now, let $\xi^{2,\varepsilon} \in C_c^\infty(\mathbb{R}^d)$ be such that $\mathrm{spt}(\xi^{2,\varepsilon}) \subset B_\varepsilon$, $\xi^{2,\varepsilon}(v) \geq 0$ and $\int_{B_\varepsilon} \xi^{2,\varepsilon}(v)\, dv = 1$ and define $\varphi_R^{\varepsilon,\delta}(x,v) = \xi^{2,\varepsilon} \star_v V_\delta^{R,\varepsilon}(x,v)$ (where the the mollification now only holds in v). Then, by (3.26) we have that

$$\begin{aligned} |\xi^{2,\varepsilon} \star_v (\langle D_x V_\delta^{R,\varepsilon}(x,\cdot), \cdot\rangle)(v) - \langle D_x\varphi_R^{\varepsilon,\delta}(x,v), v\rangle| \\ \leq \varepsilon\, \|D_x V_\delta^{R,\varepsilon}\|_{L^\infty(B_\varepsilon(x,v))} \leq C_\delta\, \varepsilon(1+\min\{|v|^\alpha, R\}), \end{aligned}$$

which implies that

$$
\begin{aligned}
&\delta\varphi_R^{\varepsilon,\delta}(x,v) + \frac{1}{2}|D_v\varphi_R^{\varepsilon,\delta}(x,v)|^2 + \langle D_x\varphi_R^{\varepsilon,\delta}(x,v), v\rangle \\
&\qquad \leq\ \delta\varphi_R^{\varepsilon,\delta}(x,v) + \frac{1}{2}|D_v\varphi_R^{\varepsilon,\delta}(x,v)|^2 \\
&\qquad + \xi^{2,\varepsilon} \star_v \langle D_x V_\delta^{R,\varepsilon}(x,v), v\rangle + C_\delta\,\varepsilon(1+\min\{|v|^\alpha, R\}) \\
&\qquad \leq\ F_R \star \xi^{2,\varepsilon}(x,v) + C_\delta\,\varepsilon(1+\min\{|v|^\alpha, R\}) \\
&\qquad \leq F_R(x,v) + C_{1,\delta}\,\varepsilon(1+\min\{|v|^\alpha, R\})
\end{aligned}
$$

where the last inequality holds true by assumption **(F3)**. Thus, so far we have proved that for any $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$

$$
\begin{aligned}
&\delta\varphi_R^{\varepsilon,\delta}(x,v) + \frac{1}{2}|D_v\varphi_R^{\varepsilon,\delta}(x,v)|^2 + \langle D_x\varphi_R^{\varepsilon,\delta}(x,v), v\rangle \\
&\qquad \leq\ F_R(x,v) + C_{1,\delta}\,\varepsilon(1+\min\{|v|^\alpha, R\}).
\end{aligned} \tag{3.29}
$$

Moreover, in view of (3.25) we deduce that there exists a constant $C_{2,\delta} \geq 0$ such that for any $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$ we have that

$$
\delta\varphi_R^{\varepsilon,\delta}(x,v) \geq C_{2,\delta}^{-1}\min\{|v|^\alpha, R\} - C_{2,\delta}. \tag{3.30}
$$

We claim that for $\varepsilon > 0$ small enough, the following holds:

$$
\begin{aligned}
&\inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v)\right)\, d\mu(x,v,w) \\
&\qquad \geq \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \left(\delta\varphi_R^{\varepsilon,\delta}(x,v) - C_{1,\delta}\,\varepsilon\left(1+\min\{|v|^\alpha, R\}\right)\right).
\end{aligned} \tag{3.31}
$$

By Remark 3.20 above, we can test the fact that a measure is closed by smooth and globally Lipschitz continuous maps. Let $\mathcal{E}(\mathbb{T}^d \times \mathbb{R}^d)$ be such a set. Then

$$
\begin{aligned}
&\inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v)\right)\, \mu(dx,dv,dw) \\
&\quad = \inf_{\mu\in\mathscr{P}_{\alpha,2}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)}\ \sup_{\psi\in\mathcal{E}(\mathbb{T}^d\times\mathbb{R}^d)} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \bigg(\frac{1}{2}|w|^2 + F_R(x,v) \\
&\qquad - \langle D_x\psi(x,v), v\rangle - \langle D_v\psi(x,v), w\rangle\bigg)\, \mu(dx,dv,dw) \\
&\quad \geq \sup_{\psi\in\mathcal{E}(\mathbb{T}^d\times\mathbb{R}^d)}\ \inf_{\mu\in\mathscr{P}_{\alpha,2}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \bigg(\frac{1}{2}|w|^2 + F_R(x,v) \\
&\qquad - \langle D_x\psi(x,v), v\rangle - \langle D_v\psi(x,v), w\rangle\bigg)\, \mu(dx,dv,dw) \\
&\quad \geq \inf_{\mu\in\mathscr{P}_{\alpha,2}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \bigg(\frac{1}{2}|w|^2 + F_R(x,v) \\
&\qquad - \langle D_x\varphi_R^{\varepsilon,\delta}(x,v), v\rangle - \langle D_v\varphi_R^{\varepsilon,\delta}(x,v), w\rangle\bigg)\, \mu(dx,dv,dw) \\
&\quad = \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \left\{-\frac{1}{2}|D_v\varphi_R^{\varepsilon,\delta}(x,v)|^2 + F_R(x,v) - \langle D_x\varphi_R^{\varepsilon,\delta}(x,v), v\rangle\right\},
\end{aligned}
$$

which proves (3.31) thanks to (3.29). Recalling (3.30), the right hand side of (3.31) is coercive in v uniformly in ε for ε small. As in addition $\varphi_R^{\varepsilon,\delta}$ converges locally uniformly to V_δ^R as $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon\to 0} \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \left(\delta\varphi_R^{\varepsilon,\delta}(x,v) - C_{2,\delta}\,\varepsilon\left(1+\min\{|v|^\alpha, R\}\right)\right)$$
$$= \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \delta V_\delta^R(x,v).$$

So we can let $\varepsilon \to 0$ in (3.31) to obtain the result. □

In the next step, we let $R \to +\infty$ in (3.27):

Lemma 3.21. *Assume that F satisfies* **(F1)** – **(F3)***. Let V_δ be defined in* (3.12)*. Then*

$$\inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v)\right)\, \mu(dx,dv,dw) \geq \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \delta V_\delta(x,v). \tag{3.32}$$

Proof. We first consider the left-hand side of (3.27), for which we obviously have, by the definition of F_R in Lemma 3.18,

$$\begin{aligned}\inf_{\mu\in\mathcal{C}} &\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v)\right)\, \mu(dx,dv,dw)\\ &\leq \inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v)\right)\, \mu(dx,dv,dw).\end{aligned} \tag{3.33}$$

As for the right hand side of (3.27), we note that, if $(x_R, v_R) \in \mathbb{T}^d \times \mathbb{R}^d$ satisfies

$$V_\delta^R(x_R, v_R) \leq \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} V_\delta^R(x,v) + R^{-1},$$

then, as $V_\delta^R \leq V_\delta$ and (3.25) holds, we have

$$\tilde{c}_{1,\delta}\big(1+\min\{|v_R|^\alpha, R\}\big) - \tilde{c}_{2,\delta} \leq \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} V_\delta(x,v) + R^{-1}.$$

This proves that v_R remains bounded in R and we can find a subsequence of (x_R, v_R), denoted in the same way, which converges to some $(\bar{x}, \bar{v}) \in \mathbb{T}^d \times \mathbb{R}^d$ as $R \to +\infty$. Then by local uniform convergence of V_δ^R to V_δ, we obtain that

$$\inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} V_\delta(x,v) \leq V_\delta(\bar{x},\bar{v}) = \lim_{R\to+\infty} V_\delta^R(x_R, v_R) = \lim_{R\to+\infty} \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} V_\delta^R(x,v). \tag{3.34}$$

Passing to the limit as $R \to +\infty$ in (3.27) proves the Lemma thanks to (3.33) and (3.34). □

We are now ready to prove the main result of this section.

Proposition 3.22 (**Characterization with closed measures**)**.** *Assume that F satisfies* **(F1)** — **(F3)***. For any $(x_0, v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ we have that*

$$\lim_{T\to\infty} \frac{1}{T} V^T(0, x_0, v_0) = \inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v)\right)\, \mu(dx,dv,dw).$$

Proof. Let $\gamma^T_{(x_0,v_0)}$ be a minimum for the problem

$$\inf_{\gamma\in\Gamma_0(x_0,v_0)} J^T(\gamma).$$

Let us define the probability measures μ_T by

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,v,w)\;d\mu^T(x,v,w)=\frac{1}{T}\int_0^T\varphi(\gamma^T_{(x_0,v_0)}(t),\dot\gamma^T_{(x_0,v_0)}(t),\ddot\gamma^T_{(x_0,v_0)}(t))\;dt$$

for any $\varphi\in\mathcal{C}^\infty_c(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$. By Lemma 3.14, the (μ^T) converge, up to a subsequence (T_n), weak-$*$ to a closed measure μ^*. Therefore

$$\begin{aligned}
&\lim_{T\to\infty}\inf_{\gamma\in\Gamma_0(x_0,v_0)}\frac{1}{T}J^T(\gamma)\\
&\qquad=\lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\left(\frac{1}{2}|\ddot\gamma^{T_n}_{(x_0,v_0)}(t)|^2+F(\gamma^{T_n}_{(x_0,v_0)}(t),\dot\gamma^{T_n}_{(x_0,v_0)}(t))\right)\;dt\\
&\qquad=\lim_{n\to\infty}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)\right)\;\mu^{T_n}(dx,dv,dw)\\
&\qquad\qquad\geq\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)\right)\;\mu^*(dx,dv,dw).
\end{aligned}$$

Thus, taking the infimum over the set of closed measures $\mathcal{C}$ we obtain that

$$\lim_{T\to\infty}\inf_{\gamma\in\Gamma_0(x_0,v_0)}\frac{1}{T}J^T(\gamma)\geq\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)\right)\;d\mu(x,v,w).$$

To obtain the opposite inequality, we note that, by (3.32) (which holds for any $\delta>0$) and Proposition 3.17, we have

$$\begin{aligned}
&\inf_{\mu\in\mathcal{C}}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v)\right)\;\mu(dx,dv,dw)\\
&\qquad\qquad\geq\lim_{\delta\to0^+}\inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\delta V_\delta(x,v)=\lambda,
\end{aligned}$$

where λ defined in (3.14). Then we can conclude thanks to (3.14). □

Proof of Theorem 3.2. The existence of the limit and the fact that it does not depend on (x,v) is the main statement of Proposition 3.13 while the characterization of this limit is given by Proposition 3.22. □

3.3 Asymptotic behavior of MFG with acceleration

We now turn to MFG problems of acceleration. In order to study the asymptotic behavior of these problems, we first need to describe the expected limit: the ergodic MFG problems of acceleration. The difficulty here is that, as explained in the previous part, we do not expect the existence of a corrector and therefore the ergodic MFG problem cannot be phrased in these terms. We overcome this issue by using the characterization of the ergodic limit given by Theorem 3.2 in terms of closed measures. This suggests the definition of equilibria for ergodic MFG of acceleration (Definition 3.4). We prove the existence and the uniqueness of a solution in Proposition 3.23. In order to pass to the limit in the time-dependent MFG system of acceleration, we first need to rephrase the solution of this system in terms of closed measures (more precisely in terms of the so-called $T-$closed measures, see Definition 3.24). This is the aim of the second part of the section (Theorem 3.25). Thanks to this characterization, we are then able to conclude on the long time average and complete the proof of Definition 3.4.

3.3.1 Ergodic MFG with acceleration

Following Definition 3.1 we recall that $\mathcal{C} \subset \mathscr{P}_{\alpha,2}(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ denotes the set of closed measures, i.e. $\mu \in \mathcal{C}$ if it satisfies for any test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ the following condition:

$$\int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \Big(\langle D_x\varphi(x,v), v\rangle + \langle D_v\varphi(x,v), w\rangle \Big)\ \mu(dx,dv,dw) = 0.$$

The candidate limit problem that we are going to study is the following fixed point problem: we look for a measure $\mu \in \mathcal{C}$ such that

$$\mu \in \operatorname*{argmin}_{\eta\in\mathcal{C}} \left\{ \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu) \right)\ \eta(dx,dv,dw) \right\} \tag{3.35}$$

where $\pi : \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d$, defined as $\pi(x,v,w) = (x,v)$, is the projection function.

Proposition 3.23. *Assume that F satisfies* **(F1')** *and* **(F2')**. *Then, there exists at least one solution $(\bar\lambda, \bar\mu) \in \mathbb{R} \times \mathcal{C}$ of the ergodic MFG problem.*

Moreover, if F satisfies the monotonicity assumption (3.6) and if $(\bar\lambda_1, \bar\mu_1)$ and $(\bar\lambda_2, \bar\mu_2)$ are two solutions of the ergodic MFG problem, then $\bar\lambda_1 = \bar\lambda_2$.

Proof. Let $\mathcal{K}$ be the set of probability measures $\mu \in \mathcal{C}$ such that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} (\frac{1}{2}|w|^2 + c_F^{-1}|v|^\alpha)\ \mu(dx,dv,dw) \leq 2c_F,$$

where α and c_F are given by assumption **(F2')**. We endow $\mathcal{K}$ with the d_1 distance and define, for any $\mu \in \mathcal{K}$, the set $\Psi(\mu)$ as the set of minimizers $\bar\eta \in \mathcal{C}$ of the map defined on $\mathcal{C}$

$$\eta \to \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu) \right)\ \eta(dx,dv,dw) \tag{3.36}$$

We also denote by $\Lambda(\mu)$ the value of this minimum. First, we show that the set-valued map Ψ is well-defined from $\mathcal{K}$ into $\mathcal{K}$. Indeed, if $\mu \in \mathcal{K}$ and $\bar\eta \in \mathcal{C}$ is any minimum of (3.36), we have by assumption **(F2')** (setting $\tilde\eta = \delta_{(x_0,0,0)} \in \mathcal{C}$ for an arbitrary point $x_0 \in \mathbb{T}^d$):

$$\begin{aligned}
&\int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} (\frac{1}{2}|w|^2 + c_F^{-1}|v|^\alpha - c_F)\ \bar\eta(dx,dv,dw) \\
&\qquad \leq \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} (\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu))\ \bar\eta(dx,dv,dw) \\
&\qquad\qquad \leq \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} (\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu))\ \tilde\eta(dx,dv,dw)\ \leq\ c_F.
\end{aligned}$$

So $\bar\eta$ belongs to $\mathcal{K}$. Moreover, we observe that a solution of the ergodic MFG problem exists if the set-valued map Ψ has a fixed-point and we prove that this is the case using the Kakutani fixed-point theorem. Since $\alpha > 1$, by the above considerations, we know that the space $\mathcal{K}$ is compact with respect to the d_1 distance. Thus, for any $\mu \in \mathcal{K}$, the set $\Psi(\mu)$ is convex and compact. It remains to check that Ψ has closed graph. Fix a sequence $\{\mu_j\}_{j\in\mathbb{N}} \subset \mathcal{K}$ and a sequence $\{\eta_j\}_{j\in\mathbb{N}} \subset \mathcal{K}$ such that

$$\mu_j \rightharpoonup^{d_1} \mu, \quad \eta_j \rightharpoonup^{d_1} \bar\eta, \quad \text{and} \quad \eta_j \in \Psi(\mu_j)\ \forall j \in \mathbb{N}.$$

Let us show that $\bar\eta \in \Psi(\mu)$. Note that $\bar\eta \in \mathcal{C}$. It remains to check that $\bar\eta$ minimizes (3.36). By standard lower-semi continuity arguments and continuity of F, we have:

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu))\ \bar\eta(dx,dv,dw)\\ &\leq \liminf_j \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu_j))\ \eta_j(dx,dv,dw).\end{aligned} \tag{3.37}$$

We now check that the right-hand side is not larger that $\Lambda(\mu)$. Indeed, let $\tilde\eta$ belong to $\Psi(\mu)$ and fix $\eta > 0$. As $\tilde\eta$ belongs to $\mathcal{K}$ we can find $R > 0$ such that

$$\int_{(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)\setminus B_R} (\frac{1}{2}|w|^2 + c_F|v|^\alpha + c_F)\ \tilde\eta(dx,dv,dw) \leq \varepsilon\,.$$

As $\pi\sharp\mu_j$ converges to $\pi\sharp\mu$ for the d_1 distance, we have by assumption **(F1')** that, for j large enough,

$$\lim_{j\to+\infty} \sup_{(x,v)\in B_R} |F(x,v,\pi\sharp\mu_j) - F(x,v,\pi\sharp\mu)| \leq \varepsilon\,.$$

So, by optimality of η_j and the estimates above,

$$\begin{aligned}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu_j))\ \eta_j(dx,dv,dw) &= \Lambda(\mu_j)\\ \leq \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (\frac{1}{2}|w|^2 &+ F(x,v,\pi\sharp\mu_j))\ \tilde\eta(dx,dv,dw)\\ \leq \int_{B_R} (\frac{1}{2}|w|^2 &+ F(x,v,\pi\sharp\mu_j))\ \tilde\eta(dx,dv,dw)\\ + \int_{B_R^c} (\frac{1}{2}|w|^2 &+ c_F|v|^\alpha + c_F)\ \tilde\eta(dx,dv,dw)\\ \leq \int_{B_R} (\frac{1}{2}|w|^2 &+ F(x,v,\pi\sharp\mu))\ \tilde\eta(dx,dv,dw) + 2\,\varepsilon \leq \Lambda(\mu) + 2\,\varepsilon\,.\end{aligned}$$

Coming back to (3.37), this shows that

$$\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\mu))\ \bar\eta(dx,dv,dw) \leq \Lambda(\mu),$$

and therefore that $\bar\eta$ belongs to $\Psi(\mu)$. Therefore, applying Kakutani fixed-point theorem we have that there exists a fixed point $\bar\eta$ of Ψ and this is a solution of the ergodic MFG problem.

Now, we prove that under the monotonicity assumption (3.6) the critical value is unique. Let $(\bar\lambda_1, \bar\mu_1)$ and $(\bar\lambda_2, \bar\mu_2)$ be two solutions of the ergodic MFG problem. Then, by definition we have that, for $i = 1$ or $i = 2$,

$$\begin{aligned}\bar\lambda_i &= \inf_{\mu\in\mathcal{C}} \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\bar\mu_i)\right)\ \mu(dx,dv,dw)\\ &= \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\bar\mu_i)\right)\ \bar\mu_i(dx,dv,dw).\end{aligned} \tag{3.38}$$

Thus, exchanging the role of $\bar\mu_1$ and $\bar\mu_2$ as competitor for $\bar\lambda_1$ and $\bar\lambda_2$, respectively, we get

$$\bar\lambda_1 \leq \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\bar\mu_1)\right)\ \bar\mu_2(dx,dv,dw) \tag{3.39}$$

and

$$\bar{\lambda}_2 \leq \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\pi\sharp\bar{\mu}_2)\right) \bar{\mu}_1(dx,dv,dw). \tag{3.40}$$

We first take the difference between (3.39) and (3.38) for $i = 2$ and we get

$$\bar{\lambda}_1 - \bar{\lambda}_2 \leq \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \big(F(x,v,\pi\sharp\bar{\mu}_1) - F(x,v,\pi\sharp\bar{\mu}_2)\big)\, d\bar{\mu}_2(dx,dv,dw).$$

Taking the difference between (3.39) for $i = 1$ and (3.40) we get

$$\bar{\lambda}_1 - \bar{\lambda}_2 \geq \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \big(F(x,v,\pi\sharp\bar{\mu}_1) - F(x,v,\pi\sharp\bar{\mu}_2)\big)\, d\bar{\mu}_1(dx,dv,dw).$$

Thus, taking the difference of the above expressions we deduce that

$$0 \geq \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \big(F(x,v,\pi\sharp\bar{\mu}_1) - F(x,v,\pi\sharp\bar{\mu}_2)\big)\, \big(\bar{\mu}_1(dx,dv,dw) - \bar{\mu}_2(dx,dv,dw)\big)$$

which implies by monotonicity assumption (3.6) that $F(x,v,\pi\sharp\bar{\mu}_1) = F(x,v,\pi\sharp\bar{\mu}_2)$. Coming back to (3.39), it follows that $\bar{\lambda}_1 = \bar{\lambda}_2$. □

3.3.2 Representation of the solution of the time-dependent MFG system

We now consider the time-dependent MFG system (3.4). We have shown in Chapter 2 that such system has a solution (u^T, m^T) and that the function u^T can be represented as

$$\begin{aligned} &u^T(t,x,v) \\ &= \inf_{\gamma\in\Gamma_t(x,v)} \left\{ \int_t^T \left(\frac{1}{2}|\ddot{\gamma}(s)|^2 + F(\gamma(s),\dot{\gamma}(s),m_s^T)\right) ds + g(\gamma(T),\dot{\gamma}(T),m_T^T) \right\}. \end{aligned} \tag{3.41}$$

In order to compare the solution of this time-dependent problem with the solution of the ergodic MFG problem, which is written in terms of closed measures, we need to rewrite the time-dependent problem in term of flows of Borel probability measures on $\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d$. The following definition mirrors the definition of closed measure in the ergodic setting:

Definition 3.24 (T-Closed measures). *Let T be a finite time horizon and let $m_0 \in \mathscr{P}_1(\mathbb{T}^d\times\mathbb{R}^d)$. If $\eta \in C([0,T];\mathscr{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d))$, we say that η is a T-closed measure associated with m_0 if for any test function $\varphi \in C_c^\infty([0,T]\times\mathbb{T}^d\times\mathbb{R}^d)$ the following holds*

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\partial_t\varphi(t,x,v) + \langle D_x\varphi(t,x,v),v\rangle + \langle D_v\varphi(t,x,v),w\rangle\Big)\, \eta_t(dx,dv,dw)dt \\ &= \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(T,x,v)\, \eta_T(dx,dv,dw) - \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(0,x,v)\, m_0(dx,dv). \end{aligned} \tag{3.42}$$

We denote by $\mathcal{C}^T(m_0)$ the set of T-closed measures associated with $m_0 \in \mathscr{P}_1(\mathbb{T}^d\times\mathbb{R}^d)$.

The goal of the subsection is to prove the following equality:

Theorem 3.25. *Assume that F satisfies* **(F1')**, **(F2')** *and g satisfies* **(G1)**. *Let $M \geq 0$ and assume that*

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} |v|^\alpha \; m_0(dx,dv) \leq M. \tag{3.43}$$

Let (u^T, m^T) be a solution to (3.4). Then

$$\begin{aligned}\inf_{\mu\in\mathcal{C}^T(m_0)} &\Big\{ \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right) \; \mu_t(dx,dv,dw)dt \\ &\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T) \; \mu_T(dx,dv,dw) \Big\} \\ &= \int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v) \; m_0(dx,dv).\end{aligned} \tag{3.44}$$

In addition, there exists a minimizer $\bar{\mu}^T \in \mathcal{C}^T(m_0)$ of the problem in the left-hand side of (3.44) such that $m_t^T = \pi\sharp\bar{\mu}_t^T$, where $\pi : \mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{T}^d\times\mathbb{R}^d$ is the canonical projection on the two first coordinates, i.e. such that $\pi(x,v,w) = (x,v)$.

The proof of Theorem 3.25 follows standard arguments but is slightly technical because the problem is stated in the whole space in velocity. The main problem is to regularize the map u^T in order to have a smooth function with a compact support which satisfies a suitable (approximate) Hamilton-Jacobi inequality. The first step towards this aim is the following Lemma:

Lemma 3.26 (Approximate problem 2). *Let $f : \mathbb{T}^d \times \mathbb{R}^d \times [0,T] \to \mathbb{R}$ be a continuous map with at most a polynomial growth and which is locally Lipschitz continuous in space locally uniformly in time and $g : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz continuous map with at most a polynomial growth. Let $R > 0$ and let ξ^R be a smooth cut–off function such that $\xi^R \geq 0$, $\xi^R(x,v) = 1$ if $(x,v) \in \mathbb{T}^d \times \overline{B}_R$, $0 \leq \xi^R(x,v) \leq 1$ if $(x,v) \in \mathbb{T}^d \times \overline{B}_{2R}\backslash\overline{B}_R$ and $\xi^R(x,v) = 0$ if $(x,v) \in \mathbb{T}^d \times \overline{B}_R^c$. Define $f_R : \mathbb{T}^d \times \mathbb{R}^d \times [0,T] \to \mathbb{R}$ and $g_R : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ as $f_R = \xi^R f$ and $g_R = \xi^R g$. Let u_R^T be the viscosity solution of the following problem*

$$\begin{cases} -\partial_t u_R^T(t,x,v) + \frac{1}{2}|D_v u_R^T(t,x,v)|^2 - \langle D_x u_R^T(t,x,v), v\rangle \\ \qquad\qquad\qquad\qquad = f_R(t,x,v), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^T(T,x,v) = g_R(x,v), & \text{in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases} \tag{3.45}$$

Then, the following hold:

1. *u_R^T has compact support;*
2. *u_R^T is Lipschitz continuous in space and velocity variable;*
3. *u_R^T converge, as $R \to +\infty$, locally uniformly to the solution u^T of the following problem*

$$\begin{cases} -\partial_t u^T(t,x,v) + \frac{1}{2}|D_v u^T(t,x,v)|^2 - \langle D_x u^T(t,x,v), v\rangle \\ \qquad\qquad = f(t,x,v), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^T(T,x,v) = g(x,v), & \text{in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases}$$

The proof of the Lemma follows standard argument in optimal control and we omit it. Next we prove Theorem 3.25 in the simpler case where F and g are replaced by F_R and g_R:

Proposition 3.27. *Assume that F satisfies* **(F1')** *and* **(F2')** *and g satisfies* **(G1)**. *Let (u^T, m^T) be a solution of system (3.4). For $R>0$, let ξ^R be a smooth cut–off function as in Lemma 3.26 and let us set $F_R = \xi^R F$ and $g_R = \xi^R g$. Let u^T_R be the continuous viscosity solution of the following problem*

$$\begin{cases} -\partial_t u^T_R(t,x,v) + \frac{1}{2}|D_v u^T_R(t,x,v)|^2 - \langle D_x u^T_R(t,x,v), v\rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad = F_R(x,v,m^T_t), & \text{in } [0,T]\times\mathbb{T}^d\times\mathbb{R}^d \\ u^R(T,x,v) = g_R(x,v,m^T_T), & \text{in } \mathbb{T}^d\times\mathbb{R}^d. \end{cases} \tag{3.46}$$

Then

$$\begin{aligned} \inf_{\mu\in\mathcal{C}^T(m_0)} \Big\{ \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{1}{2}|w|^2 + F_R(x,v,m^T_t)\Big)\ \mu_t(dx,dv,dw)dt \\ + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m^T_T)\ \mu_T(dx,dv,dw)\Big\} \\ = \int_{\mathbb{T}^d\times\mathbb{R}^d} u^T_R(0,x,v)\ m_0(dx,dv). \end{aligned}$$

Proof. We first prove that

$$\begin{aligned} \inf_{\mu\in\mathcal{C}^T(m_0)} \Big\{ \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{1}{2}|w|^2 + F_R(x,v,m^T_t)\Big)\ \mu_t(dx,dv,dw)dt \\ + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m^T_T)\ \mu_T(dx,dv,dw)\Big\} \\ \geq \int_{\mathbb{T}^d\times\mathbb{R}^d} u^T_R(0,x,v)\ m_0(dx,dv). \end{aligned} \tag{3.47}$$

We have that

$$\begin{aligned} &\inf_{\mu\in\mathcal{C}^T(m_0)} \Big\{ \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{1}{2}|w|^2 + F_R(x,v,m^T_t)\Big)\ \mu_t(dx,dv,dw)dt \\ &\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m^T_T)\ \mu_T(dx,dv,dw)\Big\} \\ &= \inf_{\mu\in C([0,T];\mathscr{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d))} \sup_{\varphi\in C^\infty_c([0,T]\times\mathbb{T}^d\times\mathbb{R}^d)} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{1}{2}|w|^2 + F_R(x,v,m^T_t) \\ &\qquad + \partial_t\varphi(t,x,v) + \langle D_x\varphi(t,x,v), v\rangle + \langle D_v\varphi(t,x,v), w\rangle\Big)\ \mu_t(dx,dv,dw)dt \\ &+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(g_R(x,v,m^T_T) - \varphi(T,x,v)\Big)\ \mu_T(dx,dv,dw) \\ &\qquad\qquad\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d} \varphi(0,x,v)\ m_0(dx,dv) \\ &\geq \sup_{\varphi\in C^\infty_c([0,T]\times\mathbb{T}^d\times\mathbb{R}^d)} \inf_{\mu\in C([0,T];\mathscr{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d))} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{1}{2}|w|^2 + F_R(x,v,m^T_t) \\ &\qquad + \partial_t\varphi(t,x,v) + \langle D_x\varphi(t,x,v), v\rangle + \langle D_v\varphi(t,x,v), w\rangle\Big)\ \mu_t(dx,dv,dw)dt \\ &+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(g_R(x,v,m^T_T) - \varphi(T,x,v)\Big)\ \mu_T(dx,dv,dw) \\ &\qquad\qquad\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d} \varphi(0,x,v)\ m_0(dx,dv). \end{aligned}$$

In the argument below, the constant c_R depends on R and on the data and may change from line to line. Let $\xi^{1,\varepsilon} = \xi^{1,\varepsilon}(x)$ be a smooth mollifier such that $\mathrm{spt}(\xi^{1,\varepsilon}) \subset B_\varepsilon$, $\xi^{1,\varepsilon}(x) \geq 0$ and $\int_{B_\varepsilon} \xi^{1,\varepsilon}(x)\, dx = 1$, and define $u_1^{\varepsilon,R} = u_R^T \star_x \xi^{1,\varepsilon}(t,x,v)$ (the convolution being in the x variable only). Let $R^{'} \geq R$ be such that $\mathrm{spt}(u_R^T)$, $\mathrm{spt}(F_R)$ and $\mathrm{spt}(g_R)$ are contained in $B_{R'}$. Then, we have that $u_1^{\varepsilon,R}$ satisfies the following inequality in the viscosity sense

$$\begin{aligned}
&- \partial_t u_1^{\varepsilon,R}(t,x,v) + \frac{1}{2}|D_v u_1^{\varepsilon,R}(t,x,v)|^2 - \langle D_x u_1^{\varepsilon,R}(t,x,v), v\rangle \leq F_R \star \xi^{1,\varepsilon}(t,x,v) \\
&\quad \leq\; F_R(x,v,m_t^T) + C_F\, \varepsilon (1+|v|^\alpha)\mathbf{1}_{(x,v)\in\mathbb{T}^d\times B_{R'}}.
\end{aligned}$$

Now, let $\xi^{2,\varepsilon} = \xi^{2,\varepsilon}(v)$ be a smooth mollifier such that $\mathrm{spt}(\xi^{2,\varepsilon}) \subset B_\varepsilon$, $\xi^{2,\varepsilon}(v) \geq 0$ and $\int_{B_\varepsilon} \xi^{2,\varepsilon}(v)\, dv = 1$ and define $u_2^{R,\varepsilon} = \xi^{2,\varepsilon} \star_v u_1^{R,\varepsilon}(t,x,v)$ (the convolution being now in the v variable only). Then, by the Lipschitz regularity of u_R^T stated in Lemma 3.26 we have that

$$|\xi^{2,\varepsilon} \star_v \langle D_x u_1^{R,\varepsilon}(t,x,\cdot), \cdot\rangle(v) - \langle D_x u_2^{R,\varepsilon}(t,x,v), v\rangle| \leq \varepsilon\, \|D_x u_1^{R,\varepsilon}\|_\infty \leq c_R\, \varepsilon\, \mathbf{1}_{(x,v)\in\mathbb{T}^d\times B_{R'}}.$$

Hence $u_2^{\varepsilon,R}$ satisfies in the viscosity sense:

$$\begin{aligned}
&- \partial_t u_2^{\varepsilon,R}(t,x,v) + \frac{1}{2}|D_v u_2^{\varepsilon,R}(t,x,v)|^2 - \langle D_x u_2^{\varepsilon,R}(t,x,v), v\rangle \\
&\quad \leq\; F_R(x,v,m_t^T) + c_R\, \varepsilon\, \mathbf{1}_{(x,v)\in\mathbb{T}^d\times B_{R'}}.
\end{aligned}$$

We finally regularize $u_2^{\varepsilon,R}$ in time. Let $\xi^{3,\varepsilon} = \xi^{3,\varepsilon}(t)$ be a smooth mollifier such that $\mathrm{spt}(\xi^{2,\varepsilon}) \subset B_\varepsilon$, $\xi^{2,\varepsilon}(t) \geq 0$ and $\int_{B_\varepsilon} \xi^{2,\varepsilon}(t)\, dt = 1$ and define $u_3^{R,\varepsilon} = \xi^{3,\varepsilon} \star_t u_2^{R,\varepsilon}(t,x,v)$ (convolution in time). Thus, $u_3^{R,\varepsilon}$, for any $(t,x,v) \in (-\infty, T-\varepsilon] \times \mathbb{T}^d \times \mathbb{R}^d$, satisfies (in the classical sense)

$$\begin{aligned}
&- \partial_t u_3^{R,\varepsilon}(t,x,v) + \frac{1}{2}|D_v u_3^{R,\varepsilon}(t,x,v)|^2 - \langle D_x u_3^{R,\varepsilon}(t,x,v), v\rangle \\
&\quad \leq \xi^{3,\varepsilon} \star_t F_R(x,v,m_\cdot^T)(t) + c_R\, \varepsilon\, \mathbf{1}_{(x,v)\in\mathbb{T}^d\times B_{R'}}.
\end{aligned}$$

By Theorem 2.25 in Chapter 2 we know that m^T is Lipschitz continuous in time with respect to the d_1 distance. Setting $\hat{u}_\varepsilon^R(t,x,v) = u_3^{R,\varepsilon}(t-\varepsilon,x,v)$, $\hat{u}_\varepsilon^R$ satisfies therefore

$$\begin{aligned}
&- \partial_t \hat{u}_\varepsilon^R(t,x,v) + \frac{1}{2}|D_v \hat{u}_\varepsilon^R(t,x,v)|^2 - \langle D_x \hat{u}_\varepsilon^R(t,x,v), v\rangle \\
&\quad \leq\; F_R(x,v,m_t^T) + c_R\, \varepsilon\, \mathbf{1}_{(x,v)\in\mathbb{T}^d\times B_{R'}}.
\end{aligned} \tag{3.48}$$

We note that $\hat{u}_\varepsilon^R$ is smooth and has a compact support and converges uniformly to u^R

as $\varepsilon \to 0$. Using $\hat{u}^R_\varepsilon$ as test function we get

$$
\begin{aligned}
&\sup_{\varphi \in C^\infty_c([0,T]\times\mathbb{T}^d\times\mathbb{R}^d)} \inf_{\mu \in C([0,T];\mathscr{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d))} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{1}{2}|w|^2 + F_R(x,v,m^T_t) \\
&\qquad + \partial_t\varphi(t,x,v) + \langle D_x\varphi(t,x,v), v\rangle + \langle D_v\varphi(t,x,v), w\rangle\Big)\ \mu_t(dx,dv,dw)dt \\
&+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(g_R(x,v,m^T_T) - \varphi(T,x,v)\Big)\ \mu_T(dx,dv,dw) + \int_{\mathbb{T}^d\times\mathbb{R}^d} \varphi(0,x,v)\ m_0(dx,dv) \\
&\geq \inf_{\mu \in C([0,T];\mathscr{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d))} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{1}{2}|w|^2 + F_R(x,v,m^T_t) + \partial_t\hat{u}^R_\varepsilon(t,x,v) \\
&\qquad\qquad + \langle D_x\hat{u}^R_\varepsilon(t,x,v), v\rangle + \langle D_v\hat{u}^R_\varepsilon(t,x,v), w\rangle\Big)\ \mu_t(dx,dv,dw)dt \\
&+ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \Big(g_R(x,v,m^T_T) - \hat{u}^R_\varepsilon(T,x,v)\Big)\ \mu_T(dx,dv,dw) \\
&\qquad\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d} \hat{u}^R_\varepsilon(0,x,v)\ m_0(dx,dv) \\
&= \inf_{(t,x,v)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}^d} \Big\{\frac{1}{2}|D_v\hat{u}^R_\varepsilon(t,x,v)|^2 + F_R(x,v,m^T_t) - \partial_t\hat{u}^R_\varepsilon(t,x,v) \\
&\qquad + \langle D_x\hat{u}^R_\varepsilon(t,x,v), v\rangle + \ g_R(x,v,m^T_T) - \hat{u}^R_\varepsilon(T,x,v)\Big\} + \int_{\mathbb{T}^d\times\mathbb{R}^d} \hat{u}^R_\varepsilon(0,x,v)\ m_0(dx,dv).
\end{aligned}
$$

By (3.48) we obtain that

$$
\begin{aligned}
&\inf_{(t,x,v)\in[0,T]\times\mathbb{T}^d\times\mathbb{R}^d} \Big\{\Big(\frac{1}{2}|D_v\hat{u}^R_\varepsilon(t,x,v)|^2 + F_R(x,v,m^T_t) + \partial_t\hat{u}^R_\varepsilon(t,x,v) + \langle D_x\hat{u}^R_\varepsilon(t,x,v), v\rangle\Big) \\
&\qquad + \ g_R(x,v,m^T_T) - \hat{u}^R_\varepsilon(T,x,v)\Big\} + \int_{\mathbb{T}^d\times\mathbb{R}^d} \hat{u}^R_\varepsilon(0,x,v)\ m_0(dx,dv) \\
&\geq -\,c_R\,\varepsilon + \inf_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d} \Big\{\ g_R(x,v,m^T_T) - \hat{u}^R_\varepsilon(T,x,v)\Big\} + \int_{\mathbb{T}^d\times\mathbb{R}^d} \hat{u}^R_\varepsilon(0,x,v)\ m_0(dx,dv).
\end{aligned}
$$

As $\varepsilon \to 0^+$ we obtain (3.47).

On the other hand, since u^T_R is a continuous viscosity solution we know that it can be represented as follows:

$$
u^T_R(0,x,v) = \inf_{\gamma\in\Gamma_0(x,v)} \left\{\int_0^T \Big(\frac{1}{2}|\ddot{\gamma}(t)|^2 + F_R(\gamma(t),\dot{\gamma}(t),m^T_t)\Big)\ dt + g_R(\gamma(T),\dot{\gamma}(T),m^T_T)\right\}. \tag{3.49}
$$

We define the measure $\nu \in C([0,T];\mathscr{P}_1(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d))$ as

$$
\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(x,v,w)\ \nu_t(dx,dv,dw) = \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \varphi(\gamma_{(x,v)}(t),\dot{\gamma}_{(x,v)}(t),\ddot{\gamma}_{(x,v)}(t))\ m_0(dx,dv),
$$

for any $\varphi \in C^\infty_c(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ and any $t\in[0,T]$, where $\gamma_{(x,v)}$ is a measurable selection of minimizers of problem (3.49), see Lemma 3.28. By the regularity of the minimizers it is not difficult to prove that $\nu \in \mathcal{C}^T(m_0)$. Moreover, integrating the equality

$$
\begin{aligned}
u^T_R(0,x,v) = &\int_0^T \Big(\frac{1}{2}|\ddot{\gamma}_{(x,v)}(t)|^2 + F_R(\gamma_{(x,v)}(t),\dot{\gamma}_{(x,v)}(t),m^T_t)\Big)\ dt \\
&+ g_R(\gamma_{(x,v)}(T),\dot{\gamma}_{(x,v)}(T),m^T_T)
\end{aligned}
$$

against the measure m_0 we deduce that

$$\begin{aligned}
&\int_{\mathbb{T}^d\times\mathbb{R}^d} u_R^T(0,x,v)\ m_0(dx,dv)\\
&=\int_{\mathbb{T}^d\times\mathbb{R}^d}\int_0^T \left(\frac{1}{2}|\ddot{\gamma}_{(x,v)}(t)|^2 + F_R(\gamma_{(x,v)}(t),\dot{\gamma}_{(x,v)}(t),m_t^T)\right)\ dt\ m_0(dx,dv)\\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d} g_R(\gamma_{(x,v)}(T),\dot{\gamma}_{(x,v)}(T),m_T^T)\ m_0(dx,dv)\\
&=\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\right)\ \nu_t(dx,dv,dw)dt\\
&+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\ \nu_T(dx,dv,dw)\\
&\geq \inf_{\mu\in\mathcal{C}^T(m_0)}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\right)\ \mu_t(dx,dv,dw)dt\\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\ \mu_T(dx,dv,dw).
\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.25. Using the notation of Proposition 3.27 we know that for any $R\geq 0$

$$\begin{aligned}
\inf_{\mu\in\mathcal{C}^T(m_0)}&\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\right)\ \mu_t(dx,dv,dw)dt\\
&+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\ \mu_T(dx,dv,dw)\\
&=\int_{\mathbb{T}^d\times\mathbb{R}^d} u_R^T(0,x,v)\ m_0(dx,dv).
\end{aligned}$$

Then, on the one hand it is easy to see, by standard optimal control arguments, that for any $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$ we have that $|u_R^T(0,x,v)|\leq \tilde{C}_1(1+|v|^\alpha)$. By Dominated Convergence Theorem we get

$$\lim_{R\to+\infty}\int_{\mathbb{T}^d\times\mathbb{R}^d} u_R^T(0,x,v)\ m_0(dx,dv) = \int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)\ m_0(dx,dv).$$

On the other hand, without loss of generality we can define a cut-off function ξ_R as in Proposition 3.27 such that F_R and g_R are non-decreasing in R. Thus

$$\begin{aligned}
\limsup_{R\to+\infty}\inf_{\mu\in\mathcal{C}^T(m_0)}&\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\right)\ \mu_t(dx,dv,dw)dt\\
&+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\ \mu_T(dx,dv,dw)\\
&\leq \inf_{\mu\in\mathcal{C}^T(m_0)}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\ \mu_t(dx,dv,dw)dt\\
&+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\ \mu_T(dx,dv,dw).
\end{aligned}$$

To prove the reverse inequality, let $\{R_j\}_{j\in\mathbb{N}}$ and $\{\mu_t^j\}_{j\in\mathbb{N}} \subset \mathcal{C}^T(m_0)$ be such that

$$\begin{aligned}
&\liminf_{R\to+\infty} \inf_{\mu\in\mathcal{C}^T(m_0)} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_R(x,v,m_t^T)\right) \mu_t(dx,dv,dw)dt \\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_R(x,v,m_T^T)\ \mu_T(dx,dv,dw) \\
&\qquad = \lim_{j\to+\infty} \inf_{\mu\in\mathcal{C}^T(m_0)} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_{R_j}(x,v,m_t^T)\right) \mu_t(dx,dv,dw)dt \\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_{R_j}(x,v,m_T^T)\ \mu_T(dx,dv,dw) \\
&\qquad = \lim_{j\to+\infty} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_{R_j}(x,v,m_t^T)\right) \mu_t^j(dx,dv,dw)dt \\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g_{R_j}(x,v,m_T^T)\ \mu_T^j(dx,dv,dw).
\end{aligned}$$

We claim that $\{\mu_t^j\}_{j\in\mathbb{N}}$ is tight. Indeed, the lower bound on F and g, there exists a constant $C \geq 0$ such that

$$\sup_j \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} |w|^2\ \mu_t^j(dx,dv,dw)dt \leq C \tag{3.50}$$

and thus it is enough to prove that the moment with respect to v is also bounded. In order to prove this bound, let $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\psi(0) = 0$ and such that $|D\psi(p)| \leq 1$. For $\varphi(t,x,v) = (T-t)\psi(v)$, we have, by the definition of a $T-$closed measure in (3.42),

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \big(-\psi(v) + (T-t)\langle D\psi(v), w\rangle\big)\ \mu_t^j(dx,dv,dw)dt \\
&= -T\int_{\mathbb{T}^d\times\mathbb{R}^d} \psi(v)\ m_0(dx,dv)
\end{aligned} \tag{3.51}$$

and by (3.50) and Cauchy-Schwarz inequality we get

$$\left|\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} (T-t)\langle D\psi(v), w\rangle\ \mu_t^j(dx,dv,dw)dt\right| \leq TC^{1/2}.$$

Thus, by (3.51) we obtain that

$$\left|\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \psi(v)\ \mu_t^j(dx,dv,dw)dt\right| \leq C,$$

for some new constant C. If we choose ψ_n such that $\psi_n(v)$ increases in n and converges to $|v|$, we get therefore

$$\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} |v|\ \mu_t^j(dx,dv,dw)dt \leq C.$$

This implies that $\{\mu_t^j\}_{j\in\mathbb{N}}$ is tight and, up to a subsequence still denoted by μ_t^j, converges to some $\bar{\mu} \in \mathcal{C}^T(m_0)$. Then, we have that

$$\begin{aligned}
&\inf_{\mu \in \mathcal{C}^T(m_0)} \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x, v, m_t^T)\right) \mu_t(dx, dv, dw)dt \\
&+ \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} g(x, v, m_T^T)\ \mu_T(dx, dv, dw) \\
&\leq \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x, v, m_t^T)\right) \bar{\mu}_t(dx, dv, dw)dt \\
&+ \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} g(x, v, m_T^T)\ \bar{\mu}_T(dx, dv, dw) \\
&\leq \lim_{j \to +\infty} \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_{R_j}(x, v, m_t^T)\right) \mu_t^j(dx, dv, dw)dt \\
&+ \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} g_{R_j}(x, v, m_T^T)\ \mu_T^j(dx, dv, dw) \\
&= \liminf_{j \to +\infty} \inf_{\mu \in \mathcal{C}^T(m_0)} \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F_{R_j}(x, v, m_t^T)\right) \mu_t(dx, dv, dw)dt \\
&+ \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} g_{R_j}(x, v, m_T^T)\ \mu_T(dx, dv, dw).
\end{aligned}$$

This completes the proof of equality (3.44).

It remain to check the existence of a minimizer $\bar{\mu}^T \in \mathcal{C}^T(m_0)$ of the problem in the left-hand side such that $m_t^T = \pi \sharp \bar{\mu}_t^T$. For this, let $\gamma_{(x,v)}$ denote the measurable selection of minimizers of $u^T(0, x, v)$ in (3.41) as in Lemma 3.28 below and define the measure

$$\bar{\mu}_t^T = \left((x, v) \to (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t), D_v u^T(t, \gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)))\right) \sharp m_0$$

for any $t \in [0, T]$. Note that by [1, Lemma 3.5] $\bar{\mu}_t^T$ is well-defined since $u(t, x, \cdot)$ is differentiable along the optimal trajectory $\gamma_{(x,v)}$ with

$$\ddot{\gamma}_{(x,v)}(t) = D_v u^T(t, \gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)), \quad t \in [0, T]$$

In particular, it is easy to see that $\bar{\mu}^T \in \mathcal{C}^T(m_0)$ and moreover, by [1, Proposition 4.2] we have that $m_t^T = \pi \sharp \bar{\mu}_t^T$ since $m_t^T = ((x, v) \to (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))) \sharp m_0$. By the representation formula of the value function we have that

$$\begin{aligned}
u^T(0, x, v) &= \int_0^T \left(\frac{1}{2}|\ddot{\gamma}_{(x,v)}(t)|^2 + F(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t), m_t^T)\right) dt + g(\gamma_{(x,v)}(T), \dot{\gamma}_{(x,v)}(T), m_T^T) \\
&= \int_0^T \left(\frac{1}{2}|D_v u^T(t, \gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))|^2 + F(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t), m_t^T)\right) dt \\
&\quad + g(\gamma_{(x,v)}(T), \dot{\gamma}_{(x,v)}(T), m_T^T).
\end{aligned}$$

Integrating both side against the measure m_0 and using the definition of $\bar{\mu}^T$, we obtain that $\bar{\mu}^T$ satisfies the equality in (3.44) and therefore is optimal. □

Lemma 3.28. *Assume that F satisfies* **(F1')** *and* **(F2')** *and g satisfies* **(G1)**. *For $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$ let $\Gamma^*(x, v) \subset \Gamma_0(x, v)$ be the set of minimizers of problem (3.41) for $t = 0$. Then, the set-valued map*

$$\Gamma^* : (\mathbb{T}^d \times \mathbb{R}^d, |\cdot|) \rightrightarrows (\Gamma, \|\cdot\|_\infty), \quad (x, v) \mapsto \Gamma^*(x, v)$$

has a measurable selection $\gamma_{(x,v)}$, i.e. $(x, v) \to \gamma_{(x,v)}$ is measurable and, for any $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, $\gamma_{(x,v)} \in \Gamma^*(x, v)$.

Proof. By using classical results from optimal control theory it is not difficult to see that Γ^* has a closed graph, see for instance Lemma 2.9 in Chapter 2. Therefore, by [21, Proposition 9.5] the set-valued map $(x, v) \rightrightarrows \Gamma^*(x, v)$ is measurable with closed values. This implies by [26, Theorem A 5.2] the existence of a measurable selection $\gamma_{(x,v)} \in \Gamma^*(x, v)$. □

3.3.3 Convergence of the solution of the time dependent MFG system

We now investigate the limit as the horizon $T \to +\infty$ of the time-dependent MFG problem. The main result of this subsection is the following proposition:

Proposition 3.29 (**Convergence of MFG solution**). *Assume that F satisfies* **(F1')**, **(F2')**, **(F3')** *with $\alpha = 2$ and the monotonicity condition (3.6), that g satisfies* **(G1)** *and that the initial distribution m_0 in (3.4) belongs to $\mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$. Let (u^T, m^T) be a solution of the MFG system (3.4) and let $(\bar{\lambda}, \bar{\mu})$ be the solution of the ergodic MFG problem (3.35). Then*

$$\lim_{T \to +\infty} \frac{1}{T} \int_{\mathbb{T}^d \times \mathbb{R}^d} u^T(0, x, v)\; m_0(dx, dv) = \bar{\lambda}.$$

Throughout the section, we assume that the assumption of Proposition 3.29 are in force. The proof of the proposition—given at the end of the subsection—is made at the level of the closed and $T-$closed measures. For this we first need to discuss how to manipulate them. The first lemma is a straightforward application of the definition of $T-$closed measures:

Lemma 3.30 (**Concatenation of T-closed measure**). *Let $T, T' > 0$, $m_0 \in \mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$, $\mu_1 \in \mathcal{C}^T(m_0)$ and $\mu_2 \in \mathcal{C}^{T'}(m_1)$ with $m_1 = \pi\sharp\mu_1(T)$. Then, the measure*

$$\mu_t := \begin{cases} \mu_1(t), & t \in [0, T] \\ \mu_2(t - T), & t \in (T, T + T'] \end{cases}$$

belongs to $\mathcal{C}^{T+T'}(m_0)$.

Next we explain how to link two measures by a $T-$closed measure:

Lemma 3.31 (**Linking two measures by a T-closed measure**). *Let m_0^1 and m_0^2 belong to $\mathcal{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$. Then, there exists $\mu^{m_0^1 \to m_0^2} \in \mathcal{C}^{T=1}(m_0^1)$ such that $m_0^2 = \pi\sharp\mu_1^{m_0^1 \to m_0^2}$ and*

$$\int_0^1 \int_{\mathbb{T}^d \times \mathbb{R}^{2d}} (\frac{1}{2}|w|^2 + c_F(1 + |v|^2))\; \mu_t^{m_0^1 \to m_0^2}(dx, dv, dw)dt \leq C_2(1 + M_2(m_0^1) + M_2(m_0^2)), \tag{3.52}$$

where $M_2(m) = \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 dm(x, v)$ (for $m \in \mathscr{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$) and where C_2 depends only on α and c_F.

Proof. Let $(x_0, v_0) \in \mathrm{spt}(m_0^1)$ and let $(x, v) \in \mathrm{spt}(m_0^2)$. Then, following the proof of Lemma 3.8, there exists a curve $\sigma_{(x_0,v_0)}^{(x,v)} : [0, 1] \to \mathbb{T}^d$ such that $\sigma_{(x_0,v_0)}^{(x,v)}(0) = x_0$, $\dot{\sigma}_{(x_0,v_0)}^{(x,v)}(0) = v_0$ and $\sigma_{(x_0,v_0)}^{(x,v)}(1) = y$, $\dot{\sigma}_{(x_0,v_0)}^{(x,v)}(1) = w$ with

$$\int_0^1 (\frac{1}{2}|\ddot{\sigma}_{(x_0,v_0)}^{(x,v)}(t)|^2 + c_F(1 + |\dot{\sigma}_{(x_0,v_0)}^{(x,v)}(t)|^2))dt \leq C_2(1 + |v|^2 + |v_0|^2). \tag{3.53}$$

Moreover, by construction, σ depends continuously on (x_0, v_0, x, v). Let $\lambda \in \Pi(m_0^1, m_0^2)$ be a transport plan between m_0^1 and m_0^2 (see (1.2)). We define the measure $\mu^{m_0^1 \to m_0^2} \in \mathcal{C}^1(m_0^1)$ by

$$\begin{aligned}
&\int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \varphi(x, v, w) \mu_t^{m_0^1 \to m_0^2}(dx, dv, dw) \\
&= \int_{(T^d \times \mathbb{R}^d)^2} \varphi(\sigma_{(x_0,v_0)}^{(x,v)}(t), \dot{\sigma}_{(x_0,v_0)}^{(x,v)}(t), \ddot{\sigma}_{(x_0,v_0)}^{(x,v)}(t)) \; \lambda(dx_0, dv_0, dx, dv)
\end{aligned}$$

for any $\varphi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$. Then, on easily checks that $m_0^2 = \pi \sharp \mu_1^{m_0^1 \to m_0^2}$ and that, by (3.53):

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{T}^d \times \mathbb{R}^{2d}} (\frac{1}{2}|w|^2 + c_F(1 + |v|^2)) \; \mu_t^{m_0^1 \to m_0^2}(dx, dv, dw) dt \\
&\qquad \leq C_2 \int_0^1 \int_{\mathbb{T}^d \times \mathbb{R}^{2d}} (1 + |v|^2 + |v_0|^2) \; \mu_t^{m_0^1 \to m_0^2}(dx, dv, dw) dt \\
&= C_2(1 + M_2(m_0^1) + M_2(m_0^2)).
\end{aligned}$$

□

Proposition 3.32 (Energy estimate). *Under the notation and assumption of Proposition 3.29, there exists a constant $C \geq 0$ (independent of T) such that*

$$\int_0^T \sup_{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d} \frac{|F(x, v, m_t^T) - F(x, v, \bar{m})|^{2d+2}}{(1 + |v|^2)^{2d}} dt \leq \; CT^{\frac{1}{2}}, \tag{3.54}$$

where $\bar{m} = \pi \sharp \bar{\mu}$, with $\pi(x, v, w) = (x, v)$.

Proof. The proof consists in building from $\bar{\mu}$ and μ^T competitors in problems (3.35) and (3.44) respectively. Let us recall that μ^T and $\bar{\mu}$ are minimizers for these respective problems.

We start with problem (3.44). Fix $T \geq 2$. We define the measure $\tilde{\mu}^T$ by

$$\tilde{\mu}_t^T = \begin{cases} \mu_t^{m_0 \to \bar{m}}, & t \in [0, 1] \\ \bar{\mu}, & t \in (1, T], \end{cases} \tag{3.55}$$

where $\mu^{m_0 \to \bar{m}}$ is the measure defined by Lemma 3.31. We know by Lemma 3.30 that $\tilde{\mu}^T$ belongs to $\mathcal{C}^T(m_0)$. So we can use $\tilde{\mu}^T$ as a competitor in problem (3.44) to get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x, v, m_t^T) \right) \; \mu_t^T(dx, dv, dw) dt \\
&\qquad + \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} g(x, v, m_T^T) \; \mu_T^T(dx, dv, dw) \\
&\leq \int_0^1 \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x, v, m_t^T) \right) \; \mu_t^{m_0 \to \bar{m}}(dx, dv, dw) dt \\
&\qquad + \int_1^T \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x, v, m_t^T) \right) \; \bar{\mu}(dx, dv, dw) dt \\
&\qquad + \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} g(x, v, m_T^T) \; \bar{\mu}(dx, dv, dw).
\end{aligned} \tag{3.56}$$

Next we build from μ^T a competitor for the minimization problem (3.35) for which $\bar{\mu}$ is a minimizer. In view of [1, Proposition 4.2] there exists a Borel measurable maps $(x,v) \to \gamma_{(x,v)}$ such that, for each $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, $\gamma_{(x,v)}$ is a minimizer for $u^T(0,x,v)$ in (3.44) and satisfies

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \varphi(x,v,w)\ \mu_t^T(dx,dv,dw)dt \\ &= \int_{\mathbb{T}^d\times\mathbb{R}^d} \int_0^T \varphi(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t), \ddot{\gamma}_{(x,v)}(t))dt m_0(dx,dv) \end{aligned} \tag{3.57}$$

for any test function $\varphi \in C_b^0(\mathbb{T}^d \times \mathbb{R}^{2d})$. By Lemma 3.10 and Remark 3.11, for any $\lambda \geq 2$, there exist Borel measurable maps $(x,v) \to \tilde{\gamma}_{(x,v)}$ and $(x,v) \to \tau_{(x,v)}$ such that

$$\tilde{\gamma}_{(x,v)}(0) = \tilde{\gamma}_{(x,v)}(T) = x,\ \dot{\tilde{\gamma}}_{(x,v)}(0) = \dot{\tilde{\gamma}}_{(x,v)}(T) = v \text{ and } \tilde{\gamma}_{(x,v)} = \gamma_{(x,v)} \text{ on } [0, \tau_{(x,v)}] \tag{3.58}$$

and

$$\int_{\tau_{(x,v)}}^T (\frac{1}{2}|\ddot{\tilde{\gamma}}_{(x,v)}(t)|^2 + c_F(1 + |\dot{\tilde{\gamma}}_{(x,v)}(t)|^2)dt \leq C_3(1+|v|)^2(\lambda^2 + \lambda^{-2}T). \tag{3.59}$$

Let us define $\hat{\mu}^T$ by

$$\begin{aligned} &\int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \varphi(x,v,w)\ \hat{\mu}^T(dx,dv,dw) \\ &= T^{-1} \int_{\mathbb{T}^d\times\mathbb{R}^d} \int_0^T \varphi(t, \tilde{\gamma}_{(x,v)}(t), \dot{\tilde{\gamma}}_{(x,v)}(t), \ddot{\tilde{\gamma}}_{(x,v)}(t))dt m_0(dx,dv) \end{aligned} \tag{3.60}$$

for any test function $\varphi \in C_b^0(\mathbb{T}^d \times \mathbb{R}^{2d})$. Note that, by (3.58), $\hat{\mu}^T$ belongs to $\mathcal{C}$. So using the closed measure $\hat{\mu}^T$ as a competitor in problem (3.35) we deduce that

$$\begin{aligned} &\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\ \bar{\mu}(dx,dv,dw) \\ &\qquad \leq \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\ \hat{\mu}^T(dx,dv,dw). \end{aligned} \tag{3.61}$$

Note that by the definition of $\hat{\mu}^T$ in (3.60) and by (3.58) and (3.59), we have

$$\begin{aligned} &T \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\ \hat{\mu}^T(dx,dv,dw) \\ &= \int_{\mathbb{T}^d\times\mathbb{R}^d} \int_0^T (\frac{1}{2}|\ddot{\tilde{\gamma}}_{(x,v)}(t)|^2 + F(\tilde{\gamma}_{(x,v)}(t), \dot{\tilde{\gamma}}_{(x,v)}(t), \bar{m}))dt\ m_0(dx,dv) \\ &\leq \int_{\mathbb{T}^d\times\mathbb{R}^d} \Big(\int_0^{\tau_{(x,v)}} (\frac{1}{2}|\ddot{\gamma}_{(x,v)}(t)|^2 + F(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t), \bar{m}))dt \\ &\qquad + \int_{\tau_{(x,v)}}^T (\frac{1}{2}|\ddot{\tilde{\gamma}}_{(x,v)}(t)|^2 + c_F(1+|\dot{\tilde{\gamma}}_{(x,v)}(t)|^2))dt \Big)\ m_0(dx,dv) \\ &\leq \int_{\mathbb{T}^d\times\mathbb{R}^d} \Big(\int_0^T (\frac{1}{2}|\ddot{\gamma}_{(x,v)}(t)|^2 + F(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t), \bar{m}))dt \\ &\qquad + C_3(1+|v|)^2(\lambda^2 + \lambda^{-2}T) \Big)\ m_0(dx,dv). \end{aligned}$$

Plugging this inequality into (3.61) and using the representation of μ^T in (3.57) then gives

$$\begin{aligned}
&\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\bar m)\right)\,\bar\mu(dx,dv,dw)\\
&\quad\le \int_{\mathbb{T}^d\times\mathbb{R}^{2d}}\left(\frac{1}{2}|w|^2+F(x,v,\bar m)\right)\,\hat\mu^T(dx,dv,dw)\\
&\quad\le T^{-1}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\bar m)\right)\,\mu^T_t(dx,dv,dw)dt\\
&\qquad\quad +2C_3(1+M_2(m_0))(\lambda^2T^{-1}+\lambda^{-2}),
\end{aligned}\tag{3.62}$$

where $M_2(m_0)=\int_{\mathbb{T}^d\times\mathbb{R}^d}|v|^2dm_0(x,v)$. Putting together (3.56) and (3.62) (multiplied by T) then implies that

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}(\frac{1}{2}|w|^2+F(x,v,m^T_t))d\mu^T_t(x,v,w)\\
&+\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}\left(\frac{1}{2}|w|^2+F(x,v,\bar m)\right)\,\bar\mu(dx,dv,dw)dt\\
&\le\int_0^1\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}\left(\frac{1}{2}|w|^2+F(x,v,m^T_t)\right)\,\mu_t^{m_0\to\bar m}(dx,dv,dw)dt\\
&+\int_1^T\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}\left(\frac{1}{2}|w|^2+F(x,v,m^T_t)\right)\,\bar\mu(dx,dv,dw)dt\\
&+\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}g(x,v,m^T_T)\,\bar\mu(dx,dv,dw)-\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}g(x,v,m^T_T)\,\mu^T_T(dx,dv,dw)\\
&+\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}\left(\frac{1}{2}|w|^2+F(x,v,\bar m)\right)\,\mu^T_t(dx,dv,dw)dt+2C_3(1+M_2(m_0))(\lambda^2+\lambda^{-\alpha}T).
\end{aligned}$$

Using (3.52) to bound the first term in the right-hand side (note that $\bar m$ belongs to $\mathscr{P}_{\alpha,2}(\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ with $\alpha=2$, so that $\bar m\in\mathscr{P}_2(\mathbb{T}^d\times\mathbb{R}^d)$) we obtain therefore

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}(F(x,v,m^T_t)-F(x,v,\bar m))\,(\mu^T_t(dx,dv,dw)-\bar\mu(dx,dv,dw))dt\\
&\qquad\le\ C_2(1+M_2(m_0)+M_2(\bar m))+2\|g\|_\infty+2C_3(1+M_2(m_0))(\lambda^2+\lambda^{-2}T).
\end{aligned}$$

We now use the strong monotonicity condition (3.6) and choose $\lambda=T^{1/4}$ to get

$$\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^{2d}}(F(x,v,m^T_t)-F(x,v,\bar m))^2dxdvdt\le\ CT^{\frac{1}{2}}$$

for a constant C independent of T. Recalling that F satisfies **(F3')**, we obtain (3.54) by the interpolation inequality Lemma 3.35 in the Appendix. □

Proof of Proposition 3.29. Throughout the proof, C denotes a constant independent of T and which may change from line to line. Let $\mu^T\in\mathcal{C}^T(m_0)$ be associated with a solution (u^T,m^T) of the MFG system (3.4) as in Theorem 3.25. By Theorem 3.25 we have that

$$\begin{aligned}
&\frac{1}{T}\int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)\ m_0(dx,dv)\\
&\quad= \frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,m_t^T)\right)\ \mu_t^T(dx,dv,dw)dt\\
&\qquad+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\ \mu_T^T(dx,dv,dw)\Big\} \qquad (3.63)\\
&\quad= \inf_{\mu\in\mathcal{C}^T(m_0)}\frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,m_t^T)\right)\ \mu_t(dx,dv,dw)dt\\
&\qquad+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\ \mu_T(dx,dv,dw)\Big\}.
\end{aligned}$$

We first claim that

$$\begin{aligned}
&\limsup_{T\to+\infty}\inf_{\mu\in\mathcal{C}^T(m_0)}\frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,m_t^T)\right)\ \mu_t(dx,dv,dw)dt\\
&\qquad+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\ \mu_T(dx,dv,dw)\Big\} \qquad (3.64)\\
&\quad\le \inf_{\tilde\mu\in\mathcal{C}}\Big\{\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\bar m)\right)\ \tilde\mu(dx,dv,dw)\Big\}.
\end{aligned}$$

In order to prove the claim, we first note that, by Young's inequality and Proposition 3.32, we have, for any $\mu\in\mathcal{C}^T(m_0)$,

$$\begin{aligned}
&\Big|\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\big(F(x,v,m_t^T)-F(x,v,\bar m)\big)\ \mu_t(dx,dv,dw)dt\Big|\\
&\le \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\sup_{(x',v')\in\mathbb{T}^d\times\mathbb{R}^d}\frac{|F(x',v',m_t^T)-F(x',v',\bar m)|}{(1+|v'|^2)^{\frac{d}{d+1}}}\ (1+|v|^2)^{\frac{d}{d+1}}\ \mu_t(dx,dv,dw)dt\\
&\le \frac{T^{\frac14}}{2d+2}\int_0^T \sup_{(x,v)\in\mathbb{T}^d\times\mathbb{R}^d}\frac{|F(x,v,m_t^T)-F(x,v,\bar m)|^{2d+2}}{(1+|v|^2)^{2d}}dt\\
&\qquad+\frac{(2d+1)T^{-\frac{1}{4(2d+1)}}}{2d+2}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}(1+|v|^2)^{\frac{2d}{(2d+1)}}\ \mu_t(dx,dv,dw)dt\\
&\le CT^{\frac34}+T^{-\frac{1}{4(2d+1)}}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}(1+|v|^2)\ \mu_t(dx,dv,dw)dt.
\end{aligned} \qquad (3.65)$$

As g is bounded, we have therefore, for any $\mu\in\mathcal{C}^T(m_0)$,

$$\begin{aligned}
&\frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,m_t^T)\right)\ \mu_t(dx,dv,dw)dt\\
&\qquad+\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\ \mu_T(dx,dv,dw)\Big\}\\
&\le \frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}\left(\frac{1}{2}|w|^2+F(x,v,\bar m)\right)\ \mu_t(dx,dv,dw)dt\\
&\qquad+T^{-\frac{1}{4(2d+1)}}\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d}(1+|v|^2)\ \mu_t(dx,dv,dw)dt\Big\}+CT^{-\frac14}+T^{-1}\|g\|_\infty.
\end{aligned} \qquad (3.66)$$

Given $\tilde{\mu} \in \mathcal{C}$, we know from Lemma 3.31 that there exists $\mu^{m_0 \to \pi\sharp\tilde{\mu}}$ such that

$$\begin{aligned} &\int_0^1 \int_{\mathbb{T}^d \times \mathbb{R}^{2d}} (\frac{1}{2}|w|^2 + c_F(1+|v|^2))\mu_t^{m_0 \to \pi\sharp\tilde{\mu}}(dx, dv, dw)dt \\ &\leq C_2(1 + M_2(m_0) + M_2(\pi\sharp\tilde{\mu})). \end{aligned} \tag{3.67}$$

Let us then define $\tilde{\mu}^T$ by

$$\tilde{\mu}_t^T = \begin{cases} \mu_t^{m_0 \to \pi\sharp\tilde{\mu}}, & t \in [0,1] \\ \tilde{\mu}, & t \in (1,T], \end{cases}$$

By Lemma 3.31, $\tilde{\mu}^T$ belongs to $\mathcal{C}^T(m_0)$ and we have, in view of (3.67),

$$\begin{aligned} T^{-1} \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right) \tilde{\mu}_t^T(dx,dv,dw)dt& \\ \leq C_2 T^{-1}(1 + M_2(m_0) + M_2(\pi\sharp\tilde{\mu}))& \\ + T^{-1}(T-1) \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right) \tilde{\mu}(dx,dv,dw)& \end{aligned}$$

while

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} (1+|v|^2)\ \tilde{\mu}_t^T(dx,dv,dw)dt \\ &\leq C_2(1 + M_2(m_0) + M_2(\pi\sharp\tilde{\mu})) + (T-1)M_2(\pi\sharp\tilde{\mu}). \end{aligned}$$

Therefore, coming back to (3.66) and using the $\tilde{\mu}^T$ built as above from the $\tilde{\mu} \in \mathcal{C}$ as competitors, we have

$$\begin{aligned} \inf_{\mu \in \mathcal{C}^T(m_0)} \frac{1}{T} \Big\{ \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right) \mu_t(dx,dv,dw)dt& \\ + \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} g(x,v,m_T^T)\ \mu_T(dx,dv,dw) \Big\}& \\ \leq \inf_{\tilde{\mu} \in \mathcal{C}} \Big\{ \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right) \tilde{\mu}(dx,dv,dw)& \\ + CT^{-\frac{1}{4(2d+1)}}(1 + M_2(m_0) + M_2(\pi\sharp\tilde{\mu})) \Big\} + CT^{-\frac{1}{4}} + T^{-1}\|g\|_\infty.& \end{aligned} \tag{3.68}$$

Since, by assumption **(F2')**,

$$\int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} F(x,v,\bar{m})\ \tilde{\mu}(dx,dv,dw) \geq c_F^{-1} M_2(\pi\sharp\tilde{\mu}) - c_F,$$

one easily checks that the limit of the right-hand side of (3.68) as $T \to +\infty$ is

$$\inf_{\tilde{\mu} \in \mathcal{C}} \Big\{ \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right) \tilde{\mu}(dx,dv,dw) \Big\}.$$

This proves our claim (3.64).

Next we claim that there exists a closed measure $\hat{\mu} \in \mathcal{C}$ such that

$$\begin{aligned}\liminf_{T\to+\infty} \frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t^T(dx,dv,dw)dt \\ + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T^T(dx,dv,dw)\Big\} \\ \geq \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\, \hat{\mu}(dx,dv,dw).\end{aligned} \tag{3.69}$$

For the proof of (3.69), we work with a subsequence of $T \to +\infty$ (still denoted by T) along which the lower limit in the left-hand side is achieved. Coming back to (3.65), we have

$$\begin{aligned}&\frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t^T(dx,dv,dw)dt \\ &\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T^T(dx,dv,dw)\Big\} \\ &\geq \frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\, \mu_t^T(dx,dv,dw)dt \\ &\qquad - T^{-\frac{1}{4(2d+1)}} \int_0^T (1 + M_2(\pi\sharp\mu_t^T))dt\Big\} - CT^{-\frac{1}{4}} - \|g\|_\infty T^{-1}.\end{aligned}$$

By the coercivity of F in assumption **(F2')**, we can absorb the second term in the right-hand side into the first one and obtain:

$$\begin{aligned}&\frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\, \mu_t^T(dx,dv,dw)dt \\ &\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\, \mu_T^T(dx,dv,dw)\Big\} \\ &\geq \frac{1}{T}(1 - C^{-1}T^{-\frac{1}{4(2d+1)}}) \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\, \mu_t^T(dx,dv,dw)dt \\ &\qquad - CT^{-\frac{1}{4(2d+1)}} - CT^{-\frac{1}{4}} - \|g\|_\infty T^{-1}.\end{aligned} \tag{3.70}$$

As in the proof of Proposition 3.32 (see (3.62)), for any $\lambda \geq 1$, we can find a closed measure $\hat{\mu}^T \in \mathcal{C}$ such that

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\, \hat{\mu}^T(dx,dv,dw) \\ &\qquad \leq T^{-1} \int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\, \mu_t^T(dx,dv,dw)dt \\ &\qquad + 2C_3(1 + M_2(m_0))(\lambda^2 T^{-1} + \lambda^{-2}).\end{aligned}$$

Plugging this inequality into (3.70) we find therefore

$$
\begin{aligned}
&\frac{1}{T}\Big\{\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,m_t^T)\right)\ \mu_t^T(dx,dv,dw)dt \\
&\qquad\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} g(x,v,m_T^T)\ \mu_T^T(dx,dv,dw)\Big\} \\
&\geq (1 - C^{-1}T^{-\frac{1}{4(2d+1)}}) \int_{\mathbb{T}^d\times\mathbb{R}^{2d}} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\ \hat{\mu}^T(dx,dv,dw) \\
&\qquad\qquad - 2C_3(1+M_2(m_0))(\lambda^2 T^{-1} + \lambda^{-2}) - CT^{-\frac{1}{4(2d+1)}}.
\end{aligned}
$$

By assumption **(F2')**, the functional in the right-hand side of the inequality is coercive for T large enough. So $\hat{\mu}^T$ weakly-$*$ converges (up to a subsequence) to a closed measure $\hat{\mu}$. Taking the lower-limit in the last inequality then implies (3.69).

Putting together (3.64) and (3.69), we find that $\hat{\mu}$ is a minimizer in the right-hand side of (3.64) and that the semi-limits and the inequalities in (3.64) and (3.69) are in fact limits and equalities. So coming back to (3.63) we find that

$$
\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{T} \int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)\ m_0(dx,dv) \\
&= \inf_{\tilde{\mu}\in\mathcal{C}} \Big\{ \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + F(x,v,\bar{m})\right)\ \tilde{\mu}(dx,dv,dw)\Big\}.
\end{aligned}
$$

The right-hand side of this equality is nothing than but $\bar{\lambda}$ since $(\bar{\lambda},\bar{\mu})$ is a solution to the the ergodic MFG problem with $\bar{m} = \pi\sharp\bar{\mu}$: this completes the proof of the proposition. □

To complete the proof of Theorem 3.5, we need estimates on the oscillation of u^T. This comes next:

Lemma 3.33. *For any $R \geq 1$ and $(x,v),(x',v') \in \mathbb{T}^d \times B_R$, we have*

$$
|u^T(0,x,v) - u^T(0,x',v')| \leq CR^2 T^{\frac{4d+3}{4(d+1)}},
$$

where C is independent of T and R.

Proof. Let $\gamma \in \Gamma(x,v)$ be optimal for $u^T(0,x,v)$ in (3.41). We define $\tilde{\gamma} \in \Gamma(x',v')$ by

$$
\tilde{\gamma}(t) = \begin{cases} \sigma(t) & \text{if } t \in [0,1] \\ \gamma(t-1) & \text{if } t \in [1,T]. \end{cases}
$$

where σ is as in Lemma 3.8 with $\sigma(0) = x'$, $\dot{\sigma}(0) = v'$, $\sigma(1) = x$, $\dot{\sigma}(1) = v$ and

$$
\int_0^1 \Big(\frac{1}{2}|\ddot{\sigma}(t)|^2 + F(\sigma(t),\dot{\sigma}(t),m_t^T)\Big)dt \leq 2C_2R^2.
$$

Note that, as the problem for u^T depends on time through (m_t^T), the cost associated with $\tilde{\gamma}$ could be quite far from the cost associated with γ. To overcome this issue, we use in a crucial way Proposition 3.32. Indeed, applying (3.54) in Proposition 3.32, we have

$$\int_0^T \left|F(\gamma(t),\dot\gamma(t),m_t^T)-F(\gamma(t),\dot\gamma(t),\bar m))\right|\ dt$$

$$\leq \int_0^T (1+|\dot\gamma(t)|^2)^{\frac{d}{d+1}} \sup_{(y,z)\in\mathbb{T}^d\times\mathbb{R}^d} \frac{|F(y,z,m_t^T)-F(y,z,\bar m)|}{(1+|v|^2)^{\frac{d}{d+1}}}dt$$

$$\leq \left(\int_0^T (1+|\dot\gamma(t)|^2)^{\frac{2d}{2d+1}}dt\right)^{\frac{2d+1}{2d+2}} \left(\int_0^T \sup_{(y,z)\in\mathbb{T}^d\times\mathbb{R}^d} \frac{|F(y,z,m_t^T)-F(y,z,\bar m)|^{2d+2}}{(1+|v|^2)^{2d}}dt\right)^{\frac{1}{2d+2}}$$

$$\leq CT^{\frac{1}{4(d+1)}}\left(\int_0^T (1+|\dot\gamma(t)|^2)dt\right)^{\frac{2d+1}{2d+2}}.$$

We have by assumption **(F2')** and Lemma 3.6 that

$$\int_0^T (c_F^{-1}|\dot\gamma(t)|^2-c_F)dt \leq u^T(0,x,v)\leq c_F T(1+|v|^2). \tag{3.71}$$

Therefore

$$\int_0^T \left|F(\gamma(t),\dot\gamma(t),m_t^T)-F(\gamma(t),\dot\gamma(t),\bar m))\right| \leq CT^{\frac{4d+3}{4(d+1)}}(1+R^2)^{\frac{2d+1}{2d+2}}. \tag{3.72}$$

For the very same reason we also have

$$\int_1^T \left|F(\gamma(t-1),\dot\gamma(t-1),m_t^T)-F(\gamma(t-1),\dot\gamma(t-1),\bar m))\right| \leq CT^{\frac{4d+3}{4(d+1)}}(1+R^2)^{\frac{2d+1}{2d+2}}, \tag{3.73}$$

because we only used the optimality of γ only in the estimate (3.71). So, by (3.72) and (3.73) we obtain

$$u^T(0,x',v')\leq \int_0^T\Big(\frac{1}{2}|\ddot{\tilde\gamma}(t)|^2+F(\tilde\gamma(t),\dot{\tilde\gamma}(t),m_t^T)\Big)dt$$

$$=\int_0^1\Big(\frac{1}{2}|\ddot\sigma(t)|^2+F(\sigma(t),\dot\sigma(t),m_t^T)\Big)dt+\int_0^{T-1}\Big(\frac{1}{2}|\ddot\gamma(t-1)|^2+F(\gamma(t-1),\dot\gamma(t-1),m_t^T)\Big)dt$$

$$\leq 2C_2R^2+\int_0^T\Big(\frac{1}{2}|\ddot\gamma(t)|^2+F(\gamma(t),\dot\gamma(t),\bar m)\Big)dt+CT^{\frac{4d+3}{4(d+1)}}(1+R^2)^{\frac{2d+1}{2d+2}}$$

$$\leq 2C_2R^2+\int_0^T\Big(\frac{1}{2}|\ddot\gamma(t)|^2+F(\gamma(t),\dot\gamma(t),m_t^T)\Big)dt+2CT^{\frac{4d+3}{4(d+1)}}(1+R^2)^{\frac{2d+1}{2d+2}}$$

$$\leq u^T(0,x,v)+2C_2R^2+2CT^{\frac{4d+3}{4(d+1)}}(1+R^2)^{\frac{2d+1}{2d+2}},$$

from which the result derives easily. □

Proof of Theorem 3.5. Proposition 3.23 states the existence of a solution for the ergodic MFG system and its uniqueness under assumption (3.6). From Proposition 3.29 we know that

$$\lim_{T\to+\infty}\frac{1}{T}\int_{\mathbb{T}^d\times\mathbb{R}^d} u^T(0,x,v)m_0(dx,dv)=\bar\lambda.$$

It remains to prove the local uniform convergence of u^T to $\bar\lambda$. Fix $R>0$ and $\varepsilon>0$. We have by Lemma 3.6 that

$$0\leq u^T(0,x,v)\leq c_F T(1+|v|^2). \tag{3.74}$$

As $m_0 \in \mathscr{P}_2(\mathbb{T}^d \times \mathbb{R}^d)$, there exists $R' \geq R$ such that

$$\int_{\mathbb{T}^d \times (\mathbb{R}^d \setminus B_{R'})} (1 + |v|^2) m_0(dx, dv) \leq \varepsilon \,. \tag{3.75}$$

Then, for any $(x_0, v_0) \in \mathbb{T}^d \times B_R$, we have, by Lemma 3.33, (3.74) and (3.75),

$$\begin{aligned}
|\frac{1}{T} u^T(0, x_0, v_0) - \bar{\lambda}\,| \leq & \left| \frac{1}{T} \int_{\mathbb{T}^d \times \mathbb{R}^d} u^T(0, x, v) m_0(dx, dv) - \bar{\lambda} \right| \\
& + \frac{1}{T} \int_{\mathbb{T}^d \times B_{R'}} \left| u^T(0, x, v) - u^T(0, x_0, v_0) \right| m_0(dx, dv) \\
& + \frac{1}{T} \int_{\mathbb{T}^d \times (\mathbb{R}^d \setminus B_{R'})} \left(|u^T(0, x, v)| + |u^T(0, x_0, v_0)| \right) m_0(dx, dv) \\
& \leq \left| \frac{1}{T} \int_{\mathbb{T}^d \times \mathbb{R}^d} u^T(0, x, v) m_0(dx, dv) - \bar{\lambda} \right| + C T^{-1} (R')^2 T^{\frac{4d+3}{4(d+1)}} + c_F \, \varepsilon (2 + R^2),
\end{aligned}$$

from which the local uniform convergence of $u^T(0, \cdot, \cdot)/T$ to $\bar{\lambda}$ can be obtained easily. □

3.4 Appendix

3.4.1 Von Neumann minmax theorem

Let $\mathbb{A}$, $\mathbb{B}$ be convex sets of some vector spaces and let us suppose that $\mathbb{B}$ is endowed with some Hausdorff topology. Let $\mathcal{L} : \mathbb{A} \times \mathbb{B} \to \mathbb{R}$ be a saddle function satisfying

1. $a \mapsto \mathcal{L}(a, b)$ is concave in $\mathbb{A}$ for every $b \in \mathbb{B}$,
2. $b \mapsto \mathcal{L}(a, b)$ is convex in $\mathbb{B}$ for every $a \in \mathbb{A}$.

It is always true that

$$\inf_{b \in \mathbb{B}} \sup_{a \in \mathbb{A}} \mathcal{L}(a, b) \geq \sup_{a \in \mathbb{A}} \inf_{b \in \mathbb{B}} \mathcal{L}(a, b).$$

Theorem 3.34 ([64]). *Assume that there exists $a^* \in \mathbb{A}$ and $c^* > \sup_{a \in \mathbb{A}} \inf_{b \in \mathbb{B}} \mathcal{L}(a, b)$ such that*

$$\mathbb{B}^* := \{ b \in \mathbb{B} : \mathcal{L}(a^*, b) \leq c^* \}$$

is not empty and compact in $\mathbb{B}$, and that $b \mapsto \mathcal{L}(a, b)$ is lower semicontinuous in $\mathbb{B}^*$ for every $a \in \mathbb{A}$.

Then

$$\min_{b \in \mathbb{B}} \sup_{a \in \mathbb{A}} \mathcal{L}(a, b) = \sup_{a \in \mathbb{A}} \inf_{b \in \mathbb{B}} \mathcal{L}(a, b).$$

3.4.2 An interpolation inequality

Lemma 3.35. *Assume that $f : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz continuous with*

$$|f(x, v)| + |D_x f(x, v)| + |D_v f(x, v)| \leq c_0 (1 + |v|^\alpha) \qquad \text{for a.e. } (x, v) \in \mathbb{T}^d \times \mathbb{R}^d \tag{3.76}$$

for some constants $c_0 > 0$ and $\alpha \in (1, 2]$. There exists a constants $C_d > 0$ (depending on dimension only) such that

$$\sup_{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d} \frac{|f(x, v)|^{2d+2}}{(1 + |v|^\alpha)^{2d}} \leq C_d c_0^{2d} \int_{\mathbb{T}^d \times \mathbb{R}^d} |f(x, v)|^2 dx dv.$$

Proof. Let $(x_0, v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ be such that $f(x_0, v_0) \neq 0$ and let $R = \frac{|f(x_0,v_0)|}{2c_0(3+2|v_0|^\alpha)}$. Note that, by our assumption on $|f|$ in (3.76), R is less than 1. Then, for any $(x, v) \in B_R(x_0, v_0)$, we have by assumption (3.76) that

$$|D_x f(x,v)|+|D_v f(x,v)| \leq c_0(1+(1+|v_0|)^\alpha) \leq c_0(1+2^{\alpha-1}+2^{\alpha-1}|v_0|^\alpha) \leq c_0(3+2|v_0|^\alpha),$$

(where we used the fact that $R \leq 1$ and that $(a + b)^\alpha \leq 2^{\alpha-1}(a^\alpha + b^\alpha)$ in the first inequality and the fact that $\alpha \leq 2$ in the second one). Therefore

$$|f(x, v)| \geq |f(x_0, v_0)| - c_0(3 + 2|v_0|^\alpha)R = \frac{|f(x_0, v_0)|}{2}.$$

Taking the square and integrating over $B_R(x_0, v_0)$ gives

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} |f(x,v)|^2 dxdv \geq |B_1|R^{2d}\frac{|f(x_0,v_0)|^2}{4} = |B_1|\frac{|f(x_0,v_0)|^{2d+2}}{2^{2d+2}c_0^{2d}(3+2|v_0|^\alpha)^{2d}} ,$$

which implies the result. □

Chapter 4

Singular limit problem for mean field control of acceleration

4.1 Assumptions and main results

In this following, we will use the same notation for similar objects for two problems. However, both the analysis are self contained and there are no intersections that might create ambiguity.

4.1.1 Control of acceleration

We begin with the analysis of the pure control problem of acceleration without mean field interaction.

Assume that the Lagrangian $L_0 : \mathbb{R}^{2d} \to \mathbb{R}$ satisfy the following.

(L1) $L_0 \in C^1(\mathbb{R}^{2d})$;

(L2) there exists $C_0 \geq 0$ such that for any $(x, v) \in \mathbb{R}^{2d}$

$$\frac{1}{C_0}|v|^2 - C_0 \leq L_0(x, v) \leq C_0(1 + |v|^2), \tag{4.1}$$

$$|D_x L_0(x, v)| \leq C_0\big(1 + |v|^2\big), \tag{4.2}$$

$$|D_v L_0(x, v)| \leq C_0\big(1 + |v|\big), \tag{4.3}$$

and, without loss of generality, we assume that $L_0(x, v) \geq 0$.

We consider the Hamilton-Jacobi equation

$$\begin{cases} -\partial_t u^\varepsilon(t, x, v) + \frac{1}{2\varepsilon}|D_v u^\varepsilon(t, x, v)|^2 - \langle D_x u^\varepsilon(t, x, v), v\rangle & \\ \qquad\qquad -L_0(x, v) = 0, & (t, x, v) \in [0, T] \times \mathbb{R}^{2d} \\ u^\varepsilon(T, x, v) = g(x), & (x, v) \in \mathbb{R}^{2d}. \end{cases} \tag{4.4}$$

and assume the following on the function $g : \mathbb{R}^d \to \mathbb{R}$.

(TC) $g(\cdot) \in C_b^1(\mathbb{R}^d)$ such that $C_0 \geq \max\big\{\frac{1}{2}, \frac{1}{2}\|Dg(\cdot)\|_{\infty,\mathbb{R}^d}\big\}$.

Let Γ be the set of C^1 curves $\gamma : [0, T] \to \mathbb{R}^d$, endowed with the local uniform convergence of the curve and its derivative, and given $(t, x, v) \in [0, T] \times \mathbb{R}^{2d}$ let $\Gamma_t(x, v)$

be the subset of Γ such that $\gamma(t) = x$, $\dot\gamma(t) = v$. Similarly, let $\Gamma_t(x)$ be the subset of Γ such that $\gamma(t) = x$. Define the functional $J^\varepsilon_{t,T} : \Gamma \to \mathbb{R}$

$$J^\varepsilon_{t,T}(\gamma) = \int_t^T \left(\frac{\varepsilon}{2}|\ddot\gamma(s)|^2 + L_0(\gamma(s), \dot\gamma(s))\right)\, ds + g(\gamma(T)), \quad \text{if } \gamma \in H^2(0,T;\mathbb{R}^d)$$

and set $J^\varepsilon_{t,T}(\gamma) = +\infty$ if $\gamma \notin H^2(0,T;\mathbb{R}^d)$. Then, we know that the solution u^ε of (4.4) can be represented as

$$u^\varepsilon(t,x,v) = \inf_{\gamma \in \Gamma_t(x,v)} J^\varepsilon_{t,T}(\gamma), \quad (t,x,v) \in [0,T] \times \mathbb{R}^{2d}. \tag{4.5}$$

Let $H_0 : \mathbb{R}^{2d} \to \mathbb{R}$ be the Hamiltonian associated with L_0, i.e.,

$$H_0(x,p) = \sup_{v \in \mathbb{R}^d} \big\{ -\langle p, v\rangle - L_0(x,v)\big\}.$$

Theorem 4.1 (**Main result 1**). *Assume* **(L1)**, **(L2)** *and* **(TC)**. *Let u^ε be a solution to (4.4). Then, there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \downarrow 0$ as $k \to \infty$, and a function $u^0 \in W^{1,\infty}_{loc}([0,T]\times\mathbb{R}^d)$ such that for any $R \geq 0$*

$$\lim_{k\to\infty} u^{\varepsilon_k}(t,x,v) = u^0(t,x), \quad \textit{uniformly on } [0,T] \times \overline{B}_R \times \overline{B}_R.$$

Moreover, u^0 satisfy

$$\begin{cases} -\partial_t u^0(t,x) + H_0(x, D_x u^0(t,x)) = 0, & (t,x) \in [0,T]\times\mathbb{R}^d \\ u^0(T,x) = g(x), & x \in \mathbb{R}^d \end{cases}$$

and, consequently, for any $(t,x) \in [0,T]\times\mathbb{R}^d$ we have that

$$u^0(t,x) = \inf_{\gamma\in\Gamma_t(x)} \left\{ \int_t^T L_0(\gamma(s), \dot\gamma(s))\, ds + g(\gamma(T))\right\}.$$

4.1.2 Mean field control of acceleration

We now list the main assumptions on the Lagrangian $L_0 : \mathbb{R}^{2d} \times \mathscr{P}_1(\mathbb{R}^d) \to \mathbb{R}$.

(M1) L_0 is continuous w.r.t. all variables and for any $m \in \mathscr{P}_1(\mathbb{R}^d)$ the map $(x,v) \mapsto L_0(x,v,m)$ belongs to $C^1(\mathbb{R}^d)$.

(M2) There exists $M_0 \geq 0$ such that for any $(x,v,m) \in \mathbb{R}^{2d} \times \mathscr{P}_1(\mathbb{R}^d)$

$$\frac{1}{M_0}|v|^2 - M_0 \leq L_0(x,v,m) \leq M_0(1+|v|^2), \tag{4.6}$$

$$|D_x L_0(x,v,m)| \leq M_0\big(1+|v|^2\big), \tag{4.7}$$

$$|D_v L_0(x,v,m)| \leq M_0\big(1+|v|\big), \tag{4.8}$$

and, without loss of generality, $L_0(x,v,m) \geq 0$ for any $(x,v,m) \in \mathbb{R}^{2d} \times \mathscr{P}_1(\mathbb{R}^d)$.

(M3) There exists two moduli $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ and $\omega_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|L_0(x,v,m_1) - L_0(x,v,m_2)| \leq \theta(|x|)\omega_0(d_1(m_1,m_2)),$$

for any $(x,v) \in \mathbb{R}^{2d}$ and m_1, $m_2 \in \mathscr{P}_1(\mathbb{R}^d)$.

Let $H_0 : \mathbb{R}^{2d} \times \mathscr{P}_1(\mathbb{R}^d) \to \mathbb{R}$ be the Hamiltonian associated with L_0, i.e.,

$$H_0(x, p, m) = \sup_{v \in \mathbb{R}^d} \big\{ -\langle p, v\rangle - L_0(x, v, m)\big\}.$$

We consider the following MFG system

$$\begin{cases} -\partial_t u^\varepsilon + \frac{1}{2\varepsilon}|D_v u^\varepsilon|^2 - \langle D_x u^\varepsilon, v\rangle - L_0(x, v, m_t^\varepsilon) = 0, & (t, x, v) \in [0, T] \times \mathbb{R}^{2d} \\ \partial_t \mu_t^\varepsilon - \langle D_x \mu_t^\varepsilon, v\rangle - \frac{1}{\varepsilon}\operatorname{div}_v(\mu_t^\varepsilon D_v u^\varepsilon) = 0, & (t, x, v) \in [0, T] \times \mathbb{R}^{2d} \\ \mu_0^\varepsilon = \mu_0, \quad u^\varepsilon(T, x, v) = g(x, m_T^\varepsilon), & (x, v) \in \mathbb{R}^{2d} \end{cases} \tag{4.9}$$

where $m_t^\varepsilon = \pi_1 \sharp \mu_t^\varepsilon$ and $\pi_1 : \mathbb{R}^{2d} \to \mathbb{R}^d$ denotes the projection onto the first factor, i.e., $\pi_1(x, v) = x$. We assume the following on the boundary data of the system:

(BC1) $\mu_0 \in \mathscr{P}(\mathbb{R}^{2d})$ is absolutely continuous w.r.t. Lebesgue measure, we still denote by μ_0 its density, and it has compact support.

(BC2) $g(\cdot, m)$ belongs to $C_b^1(\mathbb{R}^d)$ such that $M_0 \geq \max\{\frac{1}{2}, \frac{1}{2}\|Dg(\cdot, m)\|_{\infty, \mathbb{R}^d}\}$ and $g(x, \cdot)$ uniformly continuous w.r.t. space.

Let Γ be the set of C^1 curves $\gamma : [0, T] \to \mathbb{R}^d$, endowed with the local uniform convergence of the curve and its derivative, and given $(t, x, v) \in [0, T] \times \mathbb{R}^{2d}$ let $\Gamma_t(x, v)$ be the subset of Γ such that $\gamma(t) = x$, $\dot\gamma(t) = v$. Similarly, let $\Gamma_t(x)$ be the subset of Γ such that $\gamma(t) = x$. Define the functional $J_{t,T}^\varepsilon : \Gamma \to \mathbb{R}$

$$J_{t,T}^\varepsilon(\gamma) = \int_t^T \left(\frac{\varepsilon}{2}|\ddot\gamma(s)|^2 + L_0(\gamma(s), \dot\gamma(s), m_s^\varepsilon)\right) ds + g(\gamma(T), m_T^\varepsilon), \text{ if } \gamma \in H^2(0, T; \mathbb{R}^d)$$

and set $J_{t,T}^\varepsilon(\gamma) = +\infty$ if $\gamma \notin H^2(0, T; \mathbb{R}^d)$. Then, from Chapter 2 we know that there exist a solution $(u^\varepsilon, \mu^\varepsilon) \in W_{loc}^{1,\infty}([0, T] \times \mathbb{R}^{2d}) \times C([0, T]; \mathscr{P}_1(\mathbb{R}^{2d}))$ to system (4.9) such that

$$u^\varepsilon(t, x, v) = \inf_{\gamma \in \Gamma_t(x, v)} J_{t,T}^\varepsilon(\gamma) \tag{4.10}$$

and for any $t \in [0, T]$ the probability measure μ_t^ε is the image of μ_0 under the flow

$$\begin{cases} \dot\gamma(t) = v(t) \\ \dot v(t) = -\frac{1}{\varepsilon} D_v u^\varepsilon(t, \gamma(t), v(t)). \end{cases} \tag{4.11}$$

That is, u^ε solves the Hamilton-Jacobi equation in the viscosity sense and μ^ε solves the continuity equation in the sense of distributions.

Remark 4.2. Note that for a.e. $(x, v) \in \mathbb{R}^{2d}$ there exists a unique solution to system (4.11), which we will denote by $\gamma_{(x,v)}^\varepsilon$, such that $\gamma_{(x,v)}^\varepsilon(0) = x$ and $\dot\gamma_{(x,v)}^\varepsilon(0) = v$. Moreover, such a curve $\gamma_{(x,v)}^\varepsilon(\cdot)$ is optimal for $u^\varepsilon(t, x, v)$ satisfying $\gamma_{(x,v)}^\varepsilon(t) = x$ and $\dot\gamma_{(x,v)}^\varepsilon(t) = v$ as initial condition.

Theorem 4.3 **(Main result 2).** *Assume* **(M1)** – **(M3)** *and* **(BC)**. *Let $(u^\varepsilon, \mu^\varepsilon)$ be a solution to (4.9) and let $m_t^\varepsilon = \pi_1 \sharp \mu_t^\varepsilon$ for any $t \in [0, T]$. That is, there exists a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ with $\varepsilon_k \downarrow 0$, as $k \to \infty$, a function $u^0 \in W_{loc}^{1,\infty}([0, T] \times \mathbb{R}^d)$ and a flow of probability measures $\{m_t^0\}_{t \in [0,T]} \in C([0, T]; \mathscr{P}_1(\mathbb{R}^d))$ such that for any $R \geq 0$*

$$\lim_{k \to \infty} u^{\varepsilon_k}(t, x, v) = u^0(t, x), \quad \textit{uniformly on } [0, T] \times \overline{B}_R \times \overline{B}_R$$

and

$$\lim_{k \to \infty} m_t^{\varepsilon_k} = m_t^0, \quad \textit{in } C([0, T]; \mathscr{P}_1(\mathbb{R}^d)).$$

Moreover, the following holds.

(i) $(u^0, m^0) \in W^{1,\infty}_{loc}([0,T] \times \mathbb{R}^d) \times C([0,T]; \mathscr{P}_1(\mathbb{R}^d))$ *is a solution of*

$$\begin{cases} -\partial_t u^0(t,x) + H_0(x, D_x u^0(t,x), m_t^0) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ \partial_t m_t^0 - \operatorname{div}\left(m_t^0 D_p H_0(x, D_x u^0(t,x), m_t^0)\right) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ m_0^0 = m_0, \ u^0(T,x) = g(x, m_T^0), & x \in \mathbb{R}^d, \end{cases} \tag{4.12}$$

that is, u^0 solves the Hamilton-Jacobi equation in the viscosity sense and m^0 is a solution of the continuity equation in the sense of distributions.

(ii) *For any $t \in [0,T]$ the probability measure m_t^0 is the image of m_0 under the Euler flow associated with L_0.*

Remark 4.4. Let $(u^\varepsilon, \mu^\varepsilon)$ be a solution to (4.9). Assume that H_0 is of separated form, i.e., there exists a coupling function $F : \mathbb{R}^d \times \mathscr{P}_1(\mathbb{R}^d) \to \mathbb{R}$ such that

$$H_0(x,p,m) = H(x,p) - F(x,m), \quad \forall (x,p,m) \in \mathbb{R}^{2d} \times \mathscr{P}_1(\mathbb{R}^d).$$

Moreover, assume that F is continuous w.r.t. all variables, that the map $x \mapsto F(x,m)$ belongs to $C^1_b(\mathbb{R}^d)$ and that the functions F, g are monotone in the sense of Lasry-Lions, i.e.

$$\int_{\mathbb{R}^d} \left(F(x,m_1) - F(x,m_2)\right) (m_1(dx) - m_2(dx)) \geq 0, \quad \forall\, m_1, m_2 \in \mathscr{P}_1(\mathbb{R}^d)$$
$$\int_{\mathbb{R}^d} \left(g(x,m_1) - g(x,m_2)\right) (m_1(dx) - m_2(dx)) \geq 0, \quad \forall\, m_1, m_2 \in \mathscr{P}_1(\mathbb{R}^d).$$

Then, we know that that there exists a unique solution $(u^0, m^0) \in W^{1,\infty}_{loc}([0,T] \times \mathbb{R}^d) \times C([0,T]; \mathscr{P}_1(\mathbb{R}^d))$ of (4.12) and thus if $(u^\varepsilon, m^\varepsilon)$ is relatively compact then convergence of $(u^\varepsilon, m^\varepsilon)$ holds for the whole sequence.

4.2 Proof of the main result

4.2.1 Proof for the control of acceleration

We start our analysis by considering the case of control of acceleration without mean field interaction and first we will show that the value function u^ε is locally equibounded and locally equicontinuous.

Lemma 4.5. *Assume* **(L1)**, **(L2)** *and* **(TC)**. *Then we have that*

$$-C_0 T - \|g\|_{\infty,\mathbb{R}^d} \leq u^\varepsilon(t,x,v)| \leq C_0 T(1 + |v|^2) + \|g\|_{\infty,\mathbb{R}^d}$$

for any $(t,x,v) \in [0,T] \times \mathbb{R}^{2d}$ and any $\varepsilon > 0$.

Proof. On the one hand, by (4.5), (4.1) and **(TC)** we deduce that

$$u^\varepsilon(t,x,v) \geq -C_0 T - \|g\|_{\infty,\mathbb{R}^d}.$$

On the other hand, the functions

$$\zeta(t,x,v) = g(x) + C(1 + |v|^2)(T-t), \quad (t,x,v) \in [0,T] \times \mathbb{R}^{2d}$$

is a supersolution to the equation satisfied by u^ε for a suitable choice of the real constant $C \geq 0$. Indeed, we have that

$$\begin{aligned}
&-\partial_t\zeta(t,x,v) + \frac{1}{2\,\varepsilon}|D_v\zeta(t,x,v)|^2 - \langle D_x\zeta(t,x,v), v\rangle - L_0(x,v)\\
&\geq C(1+|v|^2) + 2\frac{(T-t)^2C^2}{\varepsilon}|v|^2 - \langle D_x g(x), v\rangle - C_0(1+|v|^2)\\
&\geq C(1+|v|^2) - \frac{1}{2}\|Dg(\cdot)\|_{\infty,\mathbb{R}^d} - \frac{1}{2}|v|^2 - C_0(1+|v|^2)
\end{aligned}$$

where the last inequality holds by Young's inequality. Thus, taking $C = 2C_0$ by **(TC)** we obtain

$$C_0(1+|v|^2) - \frac{1}{2}\|Dg(\cdot)\|_{\infty,\mathbb{R}^d} - \frac{1}{2}|v|^2 \geq 0. \qquad \square$$

An immediate consequence of Lemma 4.5 is the following uniform estimate on the velocity of minimizing trajectories for u^ε.

Corollary 4.6. *Assume* **(L1)**, **(L2)** *and* **(TC)**. *Let $(t,x,v) \in [0,T]\times\mathbb{R}^{2d}$ and let γ^ε be a minimizer for $u^\varepsilon(t,x,v)$. Then, there exists a constant $S \geq 0$ such that*

$$\int_t^T |\dot\gamma^\varepsilon(s)|^2\, ds \leq S(1+|v|^2)$$

where S is independent of ε, t, x and v.

Proof. From Lemma 4.5 we know that

$$u^\varepsilon(t,x,v) \leq C_0T\big(1+|v|^2\big) + \|g\|_{\infty,\mathbb{R}^d}, \quad \forall\ (t,x,v) \in [0,T]\times\mathbb{R}^{2d}.$$

On the other hand, let $(t,x,v) \in [0,T]\times\mathbb{R}^{2d}$ and let γ^ε be a minimizer for $u^\varepsilon(t,x,v)$. Then, we have that

$$\begin{aligned}
u^\varepsilon(t,x,v) &= \int_t^T \left(\frac{\varepsilon}{2}|\ddot\gamma^\varepsilon(s)|^2 + L_0(\gamma^\varepsilon(s),\dot\gamma^\varepsilon(s))\right)\, ds + g(\gamma^\varepsilon(T))\\
\geq \int_t^T L_0(\gamma^\varepsilon(s),\dot\gamma^\varepsilon(s))\, ds - \|g\|_{\infty,\mathbb{R}^d} &\geq \int_t^T \left(\frac{1}{C_0}|\dot\gamma^\varepsilon(s)|^2 - C_0\right)\, ds - \|g\|_{\infty,\mathbb{R}^d}.
\end{aligned}$$

Therefore, combining the above inequalities we get

$$\int_t^T |\dot\gamma^\varepsilon(s)|^2\, ds \leq 2C_0\big(\|g\|_{\infty,\mathbb{R}^d} + C_0T(1+|v|^2)\big) =: S(1+|v|^2). \qquad \square$$

We now provide uniform estimates estimates for the gradients of the value function u^ε w.r.t. time and space, and we also show that the gradient w.r.t. the velocity variable decrease linearly in ε.

Proposition 4.7. *Assume* **(L1)**, **(L2)** *and* **(TC)**. *Then, there exists a constant $C_1 \geq 0$ such that*

$$|\partial_t u^\varepsilon(t,x,v)| \leq C_1T(1+|v|^2), \tag{4.13}$$

$$|D_x u^\varepsilon(t,x,v)| \leq C_1T(1+|v|^2), \tag{4.14}$$

$$|D_v u^\varepsilon(t,x,v)|^2 \leq 2\,\varepsilon\, C_1(1+|v|^2+|v|^4) \tag{4.15}$$

for a.e. $(t,x,v) \in [0,T]\times\mathbb{R}^{2d}$.

Proof. We start by proving (4.13). By similar arguments to the one in Lemma 4.5 we deduce that

$$\|u^\varepsilon(t,\cdot,\cdot) - g(\cdot)\|_{\infty,\mathbb{R}^{2d}} \leq C_0(T-t)(1+|v|^2), \quad \forall\, t \in (0,T). \tag{4.16}$$

Moreover, the functions $f^{\pm} : [0,T] \times \mathbb{R}^{2d} \times \to \mathbb{R}$ defined by

$$f^{\pm}(t,x,v) = u^\varepsilon(t-h,x,v) \pm \left(\|u^\varepsilon(T-h,\cdot,\cdot) - g(\cdot)\|_{\infty,\mathbb{R}^{2d}} + h(T-t)(1+|v|^2)\right)$$

are, respectively, supersolution and subsolution to the equation satisfied by u^ε. Therefore, by Comparison Theorem [12, Theorem 3.7] we get

$$|u^\varepsilon(t,x,v) - u^\varepsilon(t-h,x,v)| \leq \|u^\varepsilon(T-h,\cdot,\cdot) - g(\cdot)\|_{\infty,\mathbb{R}^d \times \overline{B}_R} + h(T-t)$$

which yields to (4.13).

Next we show (4.14). Let $(t,x,v) \in [0,T] \times \mathbb{R}^{2d}$ and let γ^ε be a minimizer for $u^\varepsilon(t,x,v)$. Then, by (4.2) we get

$$\begin{aligned}
u^\varepsilon(t,x+h,v) \leq & \int_t^T \left(\frac{\varepsilon}{2}|\ddot{\gamma}^\varepsilon(s)|^2 + L_0(\gamma^\varepsilon(s)+h, \dot{\gamma}^\varepsilon(s))\right)\, ds + g(\gamma^\varepsilon(T)+h) \\
& = u^\varepsilon(t,x,v) + \int_t^T \left(L_0(\gamma^*(s)+h, \dot{\gamma}^*(s)) - L_0(\gamma^*(s), \dot{\gamma}^*(s))\right)\, ds \\
& + g(\gamma^\varepsilon(T)+h) - g(\gamma^\varepsilon(T)) \\
& \leq u^\varepsilon(t,x,v) + \int_t^T C_0|h|(1+|\dot{\gamma}^*(s)|^2)\, ds + \|Dg(\cdot)\|_\infty |h|.
\end{aligned}$$

Moreover, from (4.1) we obtain

$$\int_t^T \left(-C_0 + \frac{1}{C_0}|\dot{\gamma}^*(s)|^2\right)\, ds \leq u^\varepsilon(t,x,v) \leq C_0 T(1+|v|^2) + \|g\|_{\infty,\mathbb{R}^d}$$

which in turn completes the proof of (4.14).

We finally proceed with the proof of (4.15). To do so, we first show the result assuming that u^ε belongs to $C^1([0,T] \times \mathbb{R}^{2d})$ and then we treat the general case with an approximation argument.

Now, assuming that $u^\varepsilon \in C^1([0,T] \times \mathbb{R}^{2d})$ we have that

$$-\partial_t u^\varepsilon + \frac{1}{2\,\varepsilon}|D_v u^\varepsilon|^2 - \langle D_x u^\varepsilon, v\rangle - L_0(x,v) = 0$$

in the classical sense. Thus, by (4.1) we obtain

$$\begin{aligned}
\frac{1}{2\,\varepsilon}|D_v u^\varepsilon|^2 & \leq |\partial_t u^\varepsilon| + |D_x u^\varepsilon||v| - C_0(|v|^2+1) \\
& \leq |\partial_t u^\varepsilon| + \frac{1}{2}|D_x u^\varepsilon|^2 + \frac{1}{2}|v|^2 - C_0(|v|^2+1).
\end{aligned} \tag{4.17}$$

Hence, combining (4.17) with (4.13) and (4.14) we get (4.15).

Let us consider now the general case. Take $\delta > 0$ and let $\xi^{1,\delta} \in C_c^\infty([0,T])$ be a smooth mollifier w.r.t. time. Then define the function

$$u^\varepsilon_{1,\delta}(t,x,v) = u^\varepsilon \star \xi^{1,\delta}(t,x,v), \quad (t,x,v) \in [0,T] \times \mathbb{R}^{2d}.$$

Then, $u^\varepsilon_{1,\delta}$ satisfy the following inequality in the viscosity sense

$$-\partial_t u^\varepsilon_{1,\delta} + \frac{1}{2\,\varepsilon}|D_v u^\varepsilon_{1,\delta}|^2 - \langle D_x u^\varepsilon_{1,\delta}, v\rangle \leq L_0(x,v).$$

Let $\xi^{2,\delta} \in C^\infty_c(\mathbb{R}^d)$ be a smooth mollifier w.r.t. space and define the function

$$u^\varepsilon_{2,\delta}(t,x,v) = u^\varepsilon_{1,\delta} \star \xi^{2,\delta}(t,x,v), \quad (t,x,v) \in [0,T]\times\mathbb{R}^{2d}.$$

Then, we have that $u^\varepsilon_{2\,\delta}$ satisfy the following inequality in the viscosity sense

$$-\partial_t u^\varepsilon_{2,\delta} + \frac{1}{2\,\varepsilon}|D_v u^\varepsilon_{2,\delta}|^2 - \langle D_x u^\varepsilon_{2,\delta}, v\rangle \leq L_0 \star \xi^{2,\delta}(x,v) \leq L_0(x,v) + C_0\delta(1+|v|^2).$$

Let $\xi^{3,\delta} \in C^\infty_c(\mathbb{R}^d)$ be a smooth mollifier w.r.t. velocity variable and define the function

$$u^\varepsilon_{3,\delta}(t,x,v) = u^\varepsilon_{2,\delta} \star \xi^{3,\delta}(t,x,v), \quad (t,x,v) \in [0,T]\times\mathbb{R}^{2d}.$$

Then, by Jensen's inequality we deduce that $u^\varepsilon_{3,\delta}$ satisfy

$$\begin{aligned} &-\partial_t u^\varepsilon_{3,\delta} + \frac{1}{2\,\varepsilon}|D_v u^\varepsilon_{3,\delta}|^2 - \langle D_x u^\varepsilon_{3,\delta}, v\rangle \leq L_0 \star \xi^{3,\delta}(x,v) + C_\varepsilon\delta(1+|v|^2) \\ &\leq L_0(x,v) + C_{0,\varepsilon}\delta(1+|v|^2) \end{aligned} \tag{4.18}$$

in the classical sense. Therefore, applying the argument in (4.17) to the function $u^\varepsilon_{3,\delta}$, which solves (4.18), we get the result as $\delta \downarrow 0$. ☐

Now, define the function $u^0 : [0,T]\times\mathbb{R}^{2d}$ as

$$u^0(t,x) = \inf_{\gamma\in\Gamma_t(x)} \left\{ \int_t^T L_0(\gamma(s), \dot\gamma(s))\, ds + g(\gamma(T)) \right\}. \tag{4.19}$$

Then, by standard arguments in control theory it is easy to prove the following result.

Lemma 4.8. *Assume* **(L1)**, **(L2)** *and* **(TC)**. *Let $(t,x) \in [0,T]\times\mathbb{R}^d$ and let $\gamma^0 \in \Gamma_t(x)$ be a minimizer for $u^0(t,x)$. Then, we have that*

$$\int_t^T |\dot\gamma^0(s)|^2\, ds \leq C_T$$

for some constant $C_T \geq 0$.

We are now in the position to prove Theorem 4.1. To do so, we first show in Proposition 4.9 that u^ε locally uniformly converges to u^0 and then in Proposition 4.10 we prove that any minimizers of u^ε converges to a minimizer of u^0 at any point of differentiability of u^0.

Proposition 4.9. *Assume* **(L1)**, **(L2)** *and* **(TC)**. *Then, there exists a sequence $\varepsilon_k \to 0$ such that u^{ε_k} locally uniformly converges to u^0.*

Proof. It is enough to show that u^ε converges to u^0 pointwise. Indeed, if this holds then by Lemma 4.5 and by Proposition 4.7 we can apply Ascoli-Arzela Theorem to obtain that there exists $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that u^{ε_k} converges to u^0 locally uniformly.

Let $R \geq 0$, let $(t, x, v) \in [0, T] \times \mathbb{R}^d \times \overline{B}_R$ and let γ^ε be a minimizer for $u^\varepsilon(t, x, v)$. Then, we have that

$$
\begin{aligned}
u^\varepsilon(t, x, v) &= \int_t^T \left(\frac{\varepsilon}{2}|\ddot{\gamma}^\varepsilon(s)|^2 + L_0(\gamma^\varepsilon(s), \dot{\gamma}^\varepsilon(s))\right)\, ds + g(\gamma^\varepsilon(T)) \\
&\geq \int_t^T L_0(\gamma^\varepsilon(s), \dot{\gamma}^\varepsilon(s))\, ds + g(\gamma^\varepsilon(T)) \\
&\geq \inf_{\gamma \in \Gamma_t(x)} \int_t^T L_0(\gamma(s), \dot{\gamma}(s))\, ds + g(\gamma(T)) = u^0(t, x).
\end{aligned}
$$

On the other hand, for any $R \geq 0$ et $(t, x, v) \in [0, T] \times \mathbb{R}^d \times \overline{B}_R$ and let $\gamma^0 \in \Gamma_t(x)$ be a minimizer for $u^0(t, x)$. If $\dot{\gamma}^0(t) = v$, by the Euler equation and the regularity of L_0 we have that $\gamma \in C^2([0, T])$ and thus we can use γ^0 to estimate $u^\varepsilon(t, x, v)$ from above. So, we get

$$
u^\varepsilon(t, x, v) \leq \int_t^T \left(\frac{\varepsilon}{2}|\ddot{\gamma}^0(s)|^2 + L_0(\gamma^0(s), \dot{\gamma}^0(s))\right)\, ds + g(\gamma^0(T)) \leq u^0(t, x) + o(1). \tag{4.20}
$$

If this is not the case, we observe that

$$
u^\varepsilon(t, x, v) = \ u^\varepsilon(t, x, v) - u^\varepsilon(t, x, \dot{\gamma}^0(t)) + u^\varepsilon(t, x, \dot{\gamma}^0(t)) \leq o(1) + u^\varepsilon(t, x, \dot{\gamma}^0(t))
$$

where the last inequality holds by (4.15). Thus, in order to conclude it is enough to estimate $u^\varepsilon(t, x, \dot{\gamma}^0(t))$ as in (4.20). Therefore, we obtain

$$
u^0(t, x) \leq u^\varepsilon(t, x, v) \leq u^0(t, x) + o(1)
$$

which implies that u^ε converges to (4.19) pointwise. □

Proposition 4.10. *Assume* **(L1)**, **(L2)** *and* **(TC)**. *Let $(t, x, v) \in [0, T] \times \mathbb{R}^{2d}$ be a point of differentiability for $u^0(t, x)$ and let γ^ε be a minimizer for $u^\varepsilon(t, x, v)$. Then, γ^ε uniformly converges to a curve $\gamma^0 \in AC([0, T]; \mathbb{R}^d)$ and γ^0 minimize u^0 at (t, x).*

Proof. Let us start by proving that γ^ε uniformly converges, up to a subsequence. By Corollary 4.6 we know that

$$
\int_t^T |\dot{\gamma}^\varepsilon(s)|^2\, ds \leq S(1 + |v|^2).
$$

Thus, for any $s \in [t, T]$, by Hölder's inequality we have that

$$
|\gamma^\varepsilon(s)| \leq |x| + \sqrt{T}\sqrt{S}(1 + |v|^2)^{\frac{1}{2}}.
$$

Therefore, γ^ε is bounded in $H^1(0, T; \mathbb{R}^d)$ which implies that by Ascoli-Arzela Theorem there exists a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ and a curve $\gamma^0 \in \mathrm{AC}([0, T]; \mathbb{R}^d)$ such that γ^{ε_k} converges uniformly to γ^0.

We proceed now that γ^0 is a minimizer for $u^0(t, x)$. First, we observe that

$$
\begin{aligned}
&\liminf_{\varepsilon \to 0} \int_t^T \left(\frac{\varepsilon}{2}|\ddot{\gamma}^\varepsilon(s)|^2 + L_0(\gamma^\varepsilon(s), \dot{\gamma}^\varepsilon(s))\right)\, ds + g(\gamma^\varepsilon(T)) \\
\geq\ &\liminf_{\varepsilon \to 0} \int_t^T L_0(\gamma^\varepsilon(s), \dot{\gamma}^\varepsilon(s))\, ds + g(\gamma^\varepsilon(T)).
\end{aligned} \tag{4.21}
$$

Since γ^ε is uniformly bounded in $H^1(0,T)$, by lower-semicontinuity of the functional we deduce that

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\int_t^T L_0(\gamma^\varepsilon(s),\dot\gamma^\varepsilon(s))\,ds + g(\gamma^\varepsilon(T))\\ &\geq \int_t^T L_0(\gamma^0(s),\dot\gamma^0(s))\,ds + g(\gamma^0(T)).\end{aligned}$$

Moreover, given $(t,x,v)\in[0,T]\times\mathbb{R}^d\times\overline{B}_R$, for any $R\geq 0$, by Proposition 4.9 we have that

$$u^\varepsilon(t,x,v)\leq u^0(t,x)+o(1)$$

and by definition

$$u^\varepsilon(t,x,v)=\int_t^T\left(\frac{\varepsilon}{2}|\ddot\gamma^\varepsilon(s)|^2+L_0(\gamma^\varepsilon(s),\dot\gamma^\varepsilon(s))\right)\,ds+g(\gamma^\varepsilon(T)).$$

Hence, we have that

$$o(1)+u^0(t,x)\geq\int_t^T\left(\frac{\varepsilon}{2}|\ddot\gamma^\varepsilon(s)|^2+L_0(\gamma^\varepsilon(s),\dot\gamma^\varepsilon(s))\right)\,ds+g(\gamma^\varepsilon(T))$$

which implies the result passing to the limit as $\varepsilon\downarrow 0$ in (4.21) since

$$u^0(t,x)\geq\int_t^T L_0(\gamma^0(s),\dot\gamma^0(s))\,ds+g(\gamma^0(T)).\qquad\square$$

Proof of Theorem 4.1. The result follows by Proposition 4.9 and Proposition 4.10. $\square$

4.2.2 Proof for mean field game of acceleration

In order to prove the main result we proceed by steps analyzing the behavior of the value function u^ε and of the flow of probability measures m^ε separately. First, we show that u^ε is equibounded and we prove that, up to a subsequence, m^ε converges to a flow of probability measure in $C([0,T];\mathscr{P}_1(\mathbb{R}^d))$. Then, we address the convergence of the value function, up to a subsequence, to a solution of a suitable Hamilton-Jacobi equation and we study the limit of its minimizing trajectories. Finally, we are able to characterize the limit flow of measures as solution of a suitable continuity equation which coupled with the Hamilton-Jacobi equation, previously found, define the MFG system (4.12).

Lemma 4.11. *Assume* **(M1)** – **(M3)** *and* **(BC)**. *Then we have that*

$$-TM_0-\|g(\cdot,m_T^\varepsilon)\|_{\infty,\mathbb{R}^d}\leq u^\varepsilon(t,x,v)\leq M_0T(1+|v|^2)+\|g(\cdot,m_T^\varepsilon)\|_{\infty,\mathbb{R}^d},$$

for any $(t,x,v)\in[0,T]\times\mathbb{R}^{2d}$ *and for any* $\varepsilon>0$.

Proof. First, since u^ε satisfy (4.10), from (4.6) and **(BC)** follows that for any $(t,x,v)\in[0,T]\times\mathbb{R}^{2d}$ there holds

$$u^\varepsilon(t,x,v)\geq -C_0T-\|g(\cdot,m_T^\varepsilon)\|_{\infty,\mathbb{R}^d}.$$

On the other hand, the function

$$\zeta(t,x,v)=g(x,m_T^\varepsilon)+C(1+|v|^2)(T-t),\quad (t,x,v)\in[0,T]\times\mathbb{R}^{2d}$$

is a supersolution to the equation satisfied by u^ε for a suitable choice of the real constant $C \geq 0$. Indeed, we have that

$$
\begin{aligned}
& -\partial_t \zeta(t,x,v) + \frac{1}{2\,\varepsilon}|D_v\zeta(t,x,v)|^2 - \langle D_x\zeta(t,x,v), v\rangle - L_0(x,v) \\
& \geq C(1+|v|^2) + 2\frac{(T-t)^2C^2}{\varepsilon}|v|^2 - \langle D_x g(x, m_T^\varepsilon), v\rangle - M_0(1+|v|^2) \\
& \geq C(1+|v|^2) - \frac{1}{2}\|Dg(\cdot, m_T^\varepsilon)\|_{\infty,\mathbb{R}^d} - \frac{1}{2}|v|^2 - M_0(1+|v|^2)
\end{aligned}
$$

where the last inequality holds by Young's inequality. Thus, taking $C = 2M_0$ by **(BC)** we obtain

$$
M_0(1+|v|^2) - \frac{1}{2}\|Dg(\cdot, m_T^\varepsilon)\|_{\infty,\mathbb{R}^d} - \frac{1}{2}|v|^2 \geq 0. \qquad \square
$$

Corollary 4.12. *Assume* **(M1)** – **(M3)** *and* **(BC)**. *Let $(t,x,v) \in [0,T] \times \mathbb{R}^{2d}$ and let γ^ε be a minimizer for $u^\varepsilon(t,x,v)$. Then, there exists a constant $Q_1 \geq 0$ such that*

$$
\int_t^T |\dot\gamma^\varepsilon(s)|^2 \, ds \leq Q_1(1+|v|^2), \quad \forall\ \varepsilon > 0.
$$

where Q_1 is independent of ε, t, x and v.

Proof. On the one hand, from Lemma 4.11 we know that

$$
u^\varepsilon(t,x,v) \leq M_0 T\big(1+|v|^2\big) + \|g(\cdot, m_T^\varepsilon)\|_{\infty,\mathbb{R}^d}, \quad \forall\ (t,x,v) \in [0,T] \times \mathbb{R}^{2d}.
$$

On the other hand, let $(t,x,v) \in [0,T] \times \mathbb{R}^{2d}$ and let γ^ε be a minimizer for $u^\varepsilon(t,x,v)$. Then, by (4.6) we have that

$$
\begin{aligned}
u^\varepsilon(t,x,v) = & \int_t^T \left(\frac{\varepsilon}{2}|\ddot\gamma^\varepsilon(s)|^2 + L_0(\gamma^\varepsilon(s), \dot\gamma^\varepsilon(s), m_s^\varepsilon)\right) \, ds + g(\gamma^\varepsilon(T), m_T^\varepsilon) \\
\geq & \int_t^T L_0(\gamma^\varepsilon(s), \dot\gamma^\varepsilon(s), m_s^\varepsilon) \, ds - \|g(\cdot, m_T^\varepsilon)\|_{\infty,\mathbb{R}^d} \\
\geq & \int_t^T \left(\frac{1}{M_0}|\dot\gamma^\varepsilon(s)|^2 - M_0\right) \, ds - \|g(\cdot, m_T^\varepsilon)\|_{\infty,\mathbb{R}^d}.
\end{aligned}
$$

Therefore, combining the above inequalities we get

$$
\int_t^T |\dot\gamma^\varepsilon(s)|^2 \, ds \leq 2M_0\big(\|g(\cdot, m_T^\varepsilon)\|_{\infty,\mathbb{R}^d} + M_0 T(1+|v|^2)\big) =: Q_1(1+|v|^2)
$$

where Q_1 depends only on M_0, T and $\|g(\cdot, m_T^\varepsilon)\|_{\infty,\mathbb{R}^d}$ which is bounded uniformly in m_T^ε. $\square$

Corollary 4.13. *Assume* **(M1)** – **(M3)** *and* **(BC)**. *Then, there exists a constant $Q_2 \geq 0$ such that for any s_1, $s_2 \in [0,T]$ with $s_1 \leq s_2$ there holds*

$$
d_1(m_{s_2}^\varepsilon, m_{s_1}^\varepsilon) \leq Q_2|s_1 - s_2|^{\frac{1}{2}}, \quad \forall\ \varepsilon > 0
$$

where Q_2 is independent of ε.

Proof. We first recall that for any $t \in [0,T]$ we know that $m_t^\varepsilon = \pi_1 \sharp \mu_t^\varepsilon$ where μ_t^ε is the image of μ_0 under the flow (4.11) whose space marginal we denote by $\gamma_{(x,v)}^\varepsilon$ for $(x,v) \in \mathbb{R}^{2d}$.

Let $s_1,\ s_2 \in [0,T]$ be such that $s_1 \leq s_2$. Then, by (1.3) we have that

$$d_1(m_{s_1}^\varepsilon, m_{s_2}^\varepsilon) \leq \int_{\mathbb{R}^d} |\gamma_{(x,v)}^\varepsilon(s_1) - \gamma_{(x,v)}^\varepsilon(s_2)|\ \mu_0(dx,dv)$$

and thus, appealing to Corollary 4.12 and the Hölder inequality we obtain

$$d_1(m_{s_1}^\varepsilon, m_{s_2}^\varepsilon) \leq |s_1 - s_2|^{\frac{1}{2}} \left(\int_{\mathbb{R}^{2d}} Q_1(1+|v|^2)\ \mu_0(dx,dv) \right)^{\frac{1}{2}}.$$

So, since μ_0 has compact support we get the result setting

$$Q_2 = \left(\int_{\mathbb{R}^{2d}} Q_1(1+|v|^2)\ \mu_0(dx,dv) \right)^{\frac{1}{2}}. \qquad \square$$

We are now ready to prove that the flow of probability measures m^ε converges in $C([0,T];\mathscr{P}_1(\mathbb{R}^d))$, up to a subsequence. First, we recall that for any $t \in [0,T]$ the measure m_t^ε is the space marginal of μ_t^ε which is given by the push-forward of the initial distribution μ_0 under the optimal flow (4.11), that is

$$\begin{cases} \dot{\gamma}_{(x,v)}(t) = v(t), & \gamma_{(x,v)}(0) = x \\ \dot{v}(t) = -\frac{1}{\varepsilon} D_v u^\varepsilon(t, \gamma_{(x,v)}(t), v(t)), & v(0) = v. \end{cases}$$

Theorem 4.14. *Assume* **(M1)** – **(M3)** *and* **(BC)**. *Then, the flow of measures* $\{m_t^\varepsilon\}_{t\in[0,T]}$ *is tight and there exists a sequence* $\{\varepsilon_k\}_{k\in\mathbb{N}}$ *such that* m^{ε_k} *converges to some probability measure* m^0 *in* $C([0,T];\mathscr{P}_1(\mathbb{R}^d))$.

Proof. Since $m_t^\varepsilon = \pi_1 \sharp \mu_t^\varepsilon$, for any $t \in [0,T]$, where μ_t^ε is given by push-forward of μ_0 under the flow (4.11), we know that

$$\int_{\mathbb{R}^d} |x|^2\ m_t^\varepsilon(dx) = \int_{\mathbb{R}^{2d}} |\gamma_{(x,v)}^\varepsilon(t)|^2\ \mu_0(dx,dv).$$

So, we are interested in estimating the curve $\gamma_{(x,v)}^\varepsilon$ for any (x,v), uniformly in $\varepsilon > 0$. In order to get it, from Corollary 4.12 we immediately deduce that

$$|\gamma_{(x,v)}^\varepsilon(s)| \leq |x| + \sqrt{T}\sqrt{Q_1}(1+|v|^2)^{\frac{1}{2}}, \quad \forall\ s \in [0,T].$$

Hence, for any $t \geq 0$ we have that

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^2\ m_t^\varepsilon(dx) &= \int_{\mathbb{R}^d} |\gamma_{(x,v)}^\varepsilon(t)|^2\ \mu_0(dx,dv) \\ &\leq \int_{\mathbb{R}^{2d}} C_0\big(|x|^2 + TQ_1(1+|v|^2)\big)\ \mu_0(dx,dv) \end{aligned}$$

for some constant $C_0 \geq 0$. Thus, since μ_0 has compact support we deduce that $\{m_t^\varepsilon\}_{t\in[0,T]}$ has bounded second-order momentum, uniformly in $\varepsilon > 0$ and, consequently, $\{m_t^\varepsilon\}_{t\in[0,T]}$ is tight. Therefore, by Prokhorov Theorem and Ascoli-Arzela Theorem there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and measure $m^0 \in C([0,T];\mathscr{P}_1(\mathbb{R}^d))$ such that $m^{\varepsilon_k} \to m^0$ in $C([0,T];\mathscr{P}_1(\mathbb{R}^d))$. $\square$

Next, we turn to the convergence of u^ε. Before proving it, we need preliminary estimates on the oscillation of the value function w.r.t. velocity variable and then w.r.t. time and space variable. In particular, we will show that the function $u^\varepsilon(t,x,\cdot)$ has uniformly decreasing oscillation which will allowed us to conclude that the limit function does not depend on v.

Lemma 4.15. *Assume* **(M1)** – **(M3)** *and* **(BC)**. *Let $R \geq 0$ and let (x, v_0), $(x, v) \in \mathbb{R}^d \times \overline{B}_R$. Then, there exists $C_R \geq 0$ and a parametric curve $\sigma : [0, \sqrt{\varepsilon}] \to \mathbb{R}^d$ such that*

$$\sigma(0) = \sigma(\sqrt{\varepsilon}) = x, \quad \dot{\sigma}(0) = v_0, \quad \dot{\sigma}(\sqrt{\varepsilon}) = v$$

and

$$\frac{1}{\sqrt{\varepsilon}} \int_0^{\sqrt{\varepsilon}} \left(\frac{\varepsilon}{2} |\ddot{\sigma}(s)|^2 + L_0(\sigma(s), \dot{\sigma}(s), m_s^\varepsilon) \right) \, ds \leq C_R$$

where C_R is independent of ε, x, v and v_0.

Proof. Let $R \geq 0$ and let (x, v_0), $(x, v) \in \mathbb{R}^d \times \overline{B}_R$. Define the curve $\sigma : [0, \sqrt{\varepsilon}] \to \mathbb{R}^d$ by

$$\sigma(t) = x + v_0 t + Bt^2 + At^3$$

with A, $B \in \mathbb{R}$ satisfying the following conditions

$$\sigma(0) = \sigma(\sqrt{\varepsilon}) = x, \quad \dot{\sigma}(0) = v_0, \quad \dot{\sigma}(\sqrt{\varepsilon}) = v.$$

Thus, we obtain

$$\begin{cases} B = & -(2v_0 + v)\, \varepsilon^{-\frac{1}{2}} \\ A = & (v + v_0)\, \varepsilon^{-1}\,. \end{cases}$$

Hence, we get

$$\begin{aligned} &\int_0^{\sqrt{\varepsilon}} \left(\frac{\varepsilon}{2} |\ddot{\sigma}(s)|^2 + L_0(\sigma(s), \dot{\sigma}(s), m_s^\varepsilon) \right) \, ds \\ &\qquad \leq \int_0^{\sqrt{\varepsilon}} \left(\frac{\varepsilon}{2} |2B + 6At|^2 + M_0(1 + |v + 2tB + 3t^2 A|^2) \right) \, ds \leq \widehat{C} \sqrt{\varepsilon} R^2 \end{aligned}$$

for some positive constant $\widehat{C}$ and the proof is thus complete. □

Lemma 4.16. *Assume* **(M1)**, **(M2)** *and* **(BC)**. *Let $R \geq 0$, let $T > 1$ and $\varepsilon > 0$. Then, there exists $\widehat{C}_R(\varepsilon) \geq 0$ such that for any $t \in [0,T]$, any $x \in \mathbb{R}^d$, and any v, w in $\overline{B}_R$ there holds*

$$|u^\varepsilon(t,x,v) - u^\varepsilon(t,x,w)| \leq \widehat{C}_R(\varepsilon)$$

and $\widehat{C}_R(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$.

Proof. Fix $R \geq 0$ and take (x,v), $(x,w) \in \mathbb{R}^d \times \overline{B}_R$. Let γ^ε be a minimizer for $u^\varepsilon(t,x,v)$ and define the curve

$$\widehat{\gamma}(s) = \begin{cases} \sigma(s-t), & s \in [t, t+\sqrt{\varepsilon}] \\ \gamma^\varepsilon(s - \sqrt{\varepsilon}), & s \in [t + \sqrt{\varepsilon}, T] \end{cases}$$

where $\sigma : [0, \sqrt{\varepsilon}] \to \mathbb{R}^{2d}$ connects, in the sense of Lemma 4.15, (x, w) with (x, v). Then, we obtain

$$
\begin{aligned}
u^{\varepsilon}(t, x, w) - u^{\varepsilon}(t, x, v) \leq & \int_{t}^{t+\sqrt{\varepsilon}} \left(\frac{\varepsilon}{2}|\ddot{\sigma}(s-t)|^2 + L_0(\sigma(s-t), \dot{\sigma}(s-t), m_s^{\varepsilon})\right) \, ds \\
&+ \int_{t+\sqrt{\varepsilon}}^{T} \left(\frac{\varepsilon}{2}|\ddot{\gamma}^{\varepsilon}(s-\sqrt{\varepsilon})|^2 + L_0(\gamma^{\varepsilon}(s-\sqrt{\varepsilon}), \dot{\gamma}^{\varepsilon}(s-\sqrt{\varepsilon}), m_s^{\varepsilon})\right) \, ds \\
&+ g(\gamma^{\varepsilon}(T-\sqrt{\varepsilon}), m_T^{\varepsilon}) - u^{\varepsilon}(t, x, v) \\
&= \int_{t}^{t+\sqrt{\varepsilon}} \left(\frac{\varepsilon}{2}|\ddot{\sigma}(s-t)|^2 + L_0(\sigma(s-t), \dot{\sigma}(s-t), m_{s-t}^{\varepsilon})\right) \, ds \\
&+ \int_{t+\sqrt{\varepsilon}}^{T} \left(\frac{\varepsilon}{2}|\ddot{\gamma}^{\varepsilon}(s-\sqrt{\varepsilon})|^2 + L_0(\gamma^{\varepsilon}(s-\sqrt{\varepsilon}), \dot{\gamma}^{\varepsilon}(s-\sqrt{\varepsilon}), m_{s-\sqrt{\varepsilon}}^{\varepsilon})\right) \, ds \\
&+ g(\gamma^{\varepsilon}(T), m_T^{\varepsilon}) + g(\gamma^{\varepsilon}(T-\sqrt{\varepsilon}), m_T^{\varepsilon}) - g(\gamma^{\varepsilon}(T), m_T^{\varepsilon}) - u^{\varepsilon}(t, x, v) \\
&+ \int_{t}^{t+\sqrt{\varepsilon}} \left(L_0(\sigma(s-t), \dot{\sigma}(s-t), m_s^{\varepsilon}) - L_0(\sigma(s-t), \dot{\sigma}(s-t), m_{s-t}^{\varepsilon})\right) \, ds \\
&+ \int_{t+\sqrt{\varepsilon}}^{T} \left(L_0(\gamma^{\varepsilon}(s-\sqrt{\varepsilon}), \dot{\gamma}^{\varepsilon}(s-\sqrt{\varepsilon}), m_s^{\varepsilon}) - L_0(\gamma^{\varepsilon}(s-\sqrt{\varepsilon}), \dot{\gamma}^{\varepsilon}(s-\sqrt{\varepsilon}), m_{s-\sqrt{\varepsilon}}^{\varepsilon}\right) \, ds.
\end{aligned}
$$

Now, from Lemma 4.15 we know that

$$
\int_{t}^{t+\sqrt{\varepsilon}} \left(\frac{\varepsilon}{2}|\ddot{\sigma}(s-t)|^2 + L_0(\sigma(s-t), \dot{\sigma}(s-t), m_{s-t}^{\varepsilon})\right) \, ds \leq C_R\sqrt{\varepsilon}, \tag{4.22}
$$

and, moreover, from the optimality of γ^{ε} we get

$$
\begin{aligned}
&\int_{t+\sqrt{\varepsilon}}^{T} \left(\frac{\varepsilon}{2}|\ddot{\gamma}^{\varepsilon}(s-\sqrt{\varepsilon})|^2 + L_0(\gamma^{\varepsilon}(s-\sqrt{\varepsilon}), \dot{\gamma}^{\varepsilon}(s-\sqrt{\varepsilon}), m_{s-\sqrt{\varepsilon}}^{\varepsilon})\right) \, ds - u^{\varepsilon}(t, x, v) \\
&\leq - \int_{T-\sqrt{\varepsilon}}^{T} \left(\frac{\varepsilon}{2}|\ddot{\gamma}^{\varepsilon}(s)|^2 + L_0(\gamma^{\varepsilon}(s), \dot{\gamma}^{\varepsilon}(s), m_s^{\varepsilon})\right) \, ds \leq 0.
\end{aligned} \tag{4.23}
$$

Then, as observed before from Corollary 4.12 we obtain that

$$
|\gamma^{\varepsilon}(s)| \leq |x| + \sqrt{T}\sqrt{Q_1}(1+|v|^2)^{\frac{1}{2}}, \quad \forall\, s \in [0, T]
$$

and we also know that the curve σ is bounded. Hence, by **(M3)** and Corollary 4.13 we deduce that there exists $P(\varepsilon) \geq 0$, with $P(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, such that

$$
\begin{aligned}
&\int_{t}^{t+\sqrt{\varepsilon}} \left(L_0(\sigma(s-t), \dot{\sigma}(s-t), m_s^{\varepsilon}) - L_0(\sigma(s-t), \dot{\sigma}(s-t), m_{s-t}^{\varepsilon})\right) \, ds \\
&+ \int_{t+\sqrt{\varepsilon}}^{T} \left(L_0(\gamma^{\varepsilon}(s-\sqrt{\varepsilon}), \dot{\gamma}^{\varepsilon}(s-\sqrt{\varepsilon}), m_s^{\varepsilon}) - L_0(\gamma^{\varepsilon}(s-\sqrt{\varepsilon}), \dot{\gamma}^{\varepsilon}(s-\sqrt{\varepsilon}), m_{s-\sqrt{\varepsilon}}^{\varepsilon}\right) \, ds \\
&+ g(\gamma^{\varepsilon}(T-\sqrt{\varepsilon}), m_T^{\varepsilon}) - g(\gamma^{\varepsilon}(T), m_T^{\varepsilon}) \leq P(\varepsilon)
\end{aligned} \tag{4.24}
$$

where we have used that the modulus θ in **(M3)** is bounded from the boundedness of γ^{ε} and σ. Therefore, combining (4.22), (4.23) and (4.24) we get the result. □

Proposition 4.17. *Assume* **(M1)** – **(M3)** *and* **(BC)**. *Then, for any* $R \geq 0$ *there exists a modulus* $\omega_R : \mathbb{R}_+ \to \mathbb{R}_+$ *and a constant* $C_1 \geq 0$, *independent of* R, *such that for any* $\varepsilon > 0$ *the following holds:*

$$|u^\varepsilon(t,x,v) - u^\varepsilon(s,x,v)| \leq \omega_R(|t-s|), \quad \forall\ (t,s,x,v) \in [0,T] \times [0,T] \times \overline{B}_R \times \overline{B}_R \tag{4.25}$$

$$|D_x u^\varepsilon(t,x,v)| \leq C_1 T(1+|v|^2), \quad a.e.\ (t,x,v) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d. \tag{4.26}$$

Proof. We begin by proving (4.26). Let $(t,x,v) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ and let γ^ε be a minimizer for $u^\varepsilon(t,x,v)$. Then, from (4.7) we get

$$\begin{aligned} u^\varepsilon(t,x+h,v) \leq& \int_t^T \left(\frac{\varepsilon}{2}|\ddot{\gamma}^\varepsilon(s)|^2 + L_0(\gamma^\varepsilon(s)+h, \dot{\gamma}^\varepsilon(s), m_s^\varepsilon)\right)\, ds + g(\gamma^\varepsilon(T)+h, m_T^\varepsilon) \\ &= u^\varepsilon(t,x,v) + \int_t^T \left(L_0(\gamma^\varepsilon(s)+h, \dot{\gamma}^\varepsilon(s), m_s^\varepsilon) - L_0(\gamma^\varepsilon(s), \dot{\gamma}^\varepsilon(s), m_s^\varepsilon)\right)\, ds \\ &+ g(\gamma^\varepsilon(T)+h, m_T^\varepsilon) - g(\gamma^\varepsilon(T), m_T^\varepsilon) \\ &\leq u^\varepsilon(t,x,v) + \int_t^T M_0|h|(1+|\dot{\gamma}^\varepsilon(s)|^2)\, ds + \|Dg(\cdot, m_T^\varepsilon)\|_{\infty,\mathbb{R}^d}|h|. \end{aligned}$$

By (4.6) and by Lemma 4.11 we obtain

$$\int_t^T \left(\frac{1}{M_0}|\dot{\gamma}^*(s)|^2 - M_0\right)\, ds \leq u^\varepsilon(t,x,v) \leq M_0 T(1+|v|^2) + \|g(\cdot, m_T^\varepsilon)\|_{\infty,\mathbb{R}^d}$$

which, in turns, yields to the conclusion.

Next, we proceed to show (4.25). Let $R \geq 0$ and take $(t,x,v) \in [0,T] \times \overline{B}_R \times \overline{B}_R$. Let γ^ε be a minimizer for $u^\varepsilon(t,x,v)$ and let $h \in [0, T-t]$. Then, we have that

$$\begin{aligned} & u^\varepsilon(t+h,x,v) \\ \leq& \int_{t+h}^T \left(\frac{\varepsilon}{2}|\ddot{\gamma}^\varepsilon(s-h)|^2 + L_0(\gamma^\varepsilon(s-h), \dot{\gamma}^\varepsilon(s-h), m_s^\varepsilon)\right)\, ds + g(\gamma(T-h), m_T^\varepsilon) \\ =& \int_{t+h}^T \left(\frac{\varepsilon}{2}|\ddot{\gamma}^\varepsilon(s-h)|^2 + L_0(\gamma^\varepsilon(s-h), \dot{\gamma}^\varepsilon(s-h), m_{s-h}^\varepsilon)\right)\, ds + g(\gamma^\varepsilon(T), m_T^\varepsilon) \\ +& \int_{t+h}^T \left(L_0(\gamma^\varepsilon(s-h), \dot{\gamma}^\varepsilon(s-h), m_s^\varepsilon) - L_0(\gamma^\varepsilon(s-h), \dot{\gamma}^\varepsilon(s-h), m_{s-h}^\varepsilon)\right)\, ds \\ +& g(\gamma^\varepsilon(T-h), m_T^\varepsilon) - g(\gamma^\varepsilon(T), m_T^\varepsilon) \\ \leq& u^\varepsilon(t,x,v) + \int_{t+h}^T \theta(|\gamma^\varepsilon(s-h)|)\omega_0(d_1(m_s^\varepsilon, m_{s-h}^\varepsilon))\, ds + \|Dg(\cdot, m_T^\varepsilon)\|_{\infty,\mathbb{R}^d}|h| \end{aligned}$$

where the last inequality holds by **(M3)**. Hence, from Corollary 4.12 we know that

$$|\gamma^\varepsilon(s)| \leq |x| + \sqrt{T}\sqrt{Q_1}(1+|v|^2)^{\frac{1}{2}}, \quad \forall\ s \in [0,T]$$

and thus $\theta(\cdot)$ turns out to be bounded. Therefore, appealing to Corollary 4.13 we obtain

$$u^\varepsilon(t+h,x,v) - u^\varepsilon(t,x,v) \leq T\theta(R)\omega_0(|h|^{\frac{1}{2}}) + \|Dg(\cdot, m_T^\varepsilon)\|_{\infty,\mathbb{R}^d}|h|. \tag{4.27}$$

On the other hand, let $R \geq 0$ and let $(t,x,v) \in [0,T]\times\overline{B}_R\times\overline{B}_R$. For $h \in [0, T-t]$, define the curve $\gamma : [t, t+h] \to \mathbb{R}^d$ by $\gamma(s) = x + (s-t)v$. Then, by Dynamic Programming Principle we deduce that

$$\begin{aligned} u^{\varepsilon}(t,x,v) \leq & \int_t^{t+h} \left(\frac{\varepsilon}{2}|\ddot{\gamma}(s)|^2 + L_0(\gamma(s),\dot{\gamma}(s),m_s^{\varepsilon})\right)\, ds \\ & + u^{\varepsilon}(t+h,\gamma(t+h),\dot{\gamma}(t+h)) \\ = & \int_t^{t+h} L_0(x+(s-t)v, v, m_s^{\varepsilon})\, ds + u^{\varepsilon}(t+h, x+hv, v) \\ \leq & M_0(1+R^2)|h| + u^{\varepsilon}(t+h,x,v) + C_1T(1+R^2)|h| \end{aligned} \tag{4.28}$$

where we applied (4.6) and (4.26) to get the last inequality. Therefore, combining (4.27) and (4.28) the proof is complete. □

Remark 4.18. Next, we study the behavior of the value function u^{ε} as $\varepsilon \to 0$ and before doing that we recall the following argument needed to get uniform convergence from point-wise convergence.

Assume that there exists a nonnegative function $\Theta(\delta_0, \varepsilon_0, R_0)$ such that

$$\Theta(\delta_0,\varepsilon_0,R_0) \to 0, \quad \text{as} \quad \varepsilon_0, \delta_0 \downarrow 0,$$

and assume that for any $|t_1 - t_2| + |x_1 - x_2| \leq \delta_0$, any $\varepsilon \leq \varepsilon_0$ and any $|x_i|$, $|v_i| \leq R_0$ $(i = 1,2)$ there holds

$$|u^{\varepsilon}(t_1,x_1,v_1) - u^{\varepsilon}(t_2,x_2,v_2)| \leq \Theta(\delta_0,\varepsilon_0,R_0).$$

Then: if u^{ε} converge point-wise then u^{ε} converges locally uniformly and the limit function does not depend on v. □

Let $m^0 \in C([0,T];\mathscr{P}_1(\mathbb{R}^d))$ be the flow of measures obtained in Theorem 4.14 as limit of the flow m^{ε_k} in $C([0,T];\mathscr{P}_1(\mathbb{R}^d))$ for some subsequence $\varepsilon_k \downarrow 0$. Define the function $u^0 : [0,T]\times\mathbb{R}^{2d} \to \mathbb{R}$ by

$$u^0(t,x) = \inf_{\gamma\in\Gamma_t(x)} \left\{ \int_t^T L_0(\gamma(s),\dot{\gamma}(s),m_s^0)\, ds + g(\gamma(T), m_T^0) \right\}. \tag{4.29}$$

We will prove now that for the subsequence ε_k the sequence of value functions u^{ε_k} locally uniformly converge to u^0.

Theorem 4.19. *Assume* **(M1)** – **(M3)** *and* **(BC)**. *Then, there exists a subsequence* $\varepsilon_k \downarrow 0$ *such that* u^{ε_k} *locally uniformly converges to* u^0.

Proof. We proceed to show first the point-wise convergence of u^{ε_k} to u^0, for some subsequence $\varepsilon_k \downarrow 0$, and then, using Remark 4.18, i.e., constructing such a modulus Θ, we deduce that the convergence is locally uniform.

From Theorem 4.14, let ε_k be the subsequence such that $m^{\varepsilon_k} \to m^0$ in $C([0,T];\mathscr{P}_1(\mathbb{R}^d))$ as $k \to \infty$. Let $R \geq 0$, let $(t,x,v) \in [0,T]\times\mathbb{R}^d\times\overline{B}_R$ and let γ^{ε_k} be a minimizer for

$u^{\varepsilon_k}(t,x,v)$. Then, we have that

$$
\begin{aligned}
& u^{\varepsilon_k}(t,x,v) \\
&= \int_t^T \left(\frac{\varepsilon_k}{2}|\ddot{\gamma}^{\varepsilon_k}(s)|^2 + L_0(\gamma^{\varepsilon_k}(s),\dot{\gamma}^{\varepsilon_k}(s),m_s^{\varepsilon_k})\right)\, ds + g(\gamma^{\varepsilon_k}(T),m_T^{\varepsilon_k}) \\
&\geq \int_t^T L_0(\gamma^{\varepsilon_k}(s),\dot{\gamma}^{\varepsilon_k}(s),m_s^{\varepsilon_k})\, ds + g(\gamma^{\varepsilon_k}(T),m_T^{\varepsilon_k}) \\
&\geq \inf_{\gamma\in\Gamma_t(x)}\left\{\int_t^T L_0(\gamma(s),\dot{\gamma}(s),m_s^0)\, ds + g(\gamma(T),m_T^0)\right\} \\
&+ g(\gamma^{\varepsilon_k}(T),m_T^{\varepsilon}) - g(\gamma^{\varepsilon_k}(T),m_T^0) \\
&+ \int_t^T \left(L_0(\gamma^{\varepsilon_k}(s),\dot{\gamma}^{\varepsilon_k}(s),m_s^{\varepsilon_k}) - L_0(\gamma^{\varepsilon_k}(s),\dot{\gamma}^{\varepsilon_k}(s),m_s^0)\right)\, ds \geq\ u^0(t,x) - o(1)
\end{aligned}
$$

where the last inequality holds by **(M1)** and the convergence of m^{ε_k} in $C([0,T];\mathscr{P}_1(\mathbb{R}^d))$.

On the other hand, let $R\geq 0$ and take $(t,x,v)\in[0,T]\times\mathbb{R}^d\times\overline{B}_R$. Let $\gamma^0\in\Gamma_t(x)$ be a solution of

$$
\inf_{\gamma\in\Gamma_t(x)}\left\{\int_t^T L_0(\gamma(s),\dot{\gamma}(s),m_s^0)\, ds + g(\gamma(T),m_T^0)\right\}.
$$

Next, we distinguish two cases: first, when $\dot{\gamma}^0(t)=v$ and then when $\dot{\gamma}^0(t)\neq v$. Indeed, if $\dot{\gamma}^0(t)=v$, by the Euler equation and the C^2-regularity of L_0 we have that $\gamma\in C^2([0,T])$. Hence, we can use γ^0 as a competitor for $u^{\varepsilon_k}(t,x,v)$ and we get

$$
\begin{aligned}
u^{\varepsilon_k}(t,x,v) &\leq \int_t^T \left(\frac{\varepsilon}{2}|\ddot{\gamma}^0(s)|^2 + L_0(\gamma^0(s),\dot{\gamma}^0(s),m_s^{\varepsilon_k})\right)\, ds + g(\gamma^0(T),m^{\varepsilon_k}(T)) \\
&\leq \int_t^T \left(\frac{\varepsilon}{2}|\ddot{\gamma}^0(s)|^2 + L_0(\gamma^0(s),\dot{\gamma}^0(s),m_s^0)\right)\, ds + g(\gamma^0(T),m^0(T)) \\
&+ \int_t^T \left(L_0(\gamma^0(s),\dot{\gamma}^0(s),m_s^{\varepsilon_k}) - L_0(\gamma^0(s),\dot{\gamma}^0(s),m_s^0)\right)\, ds \\
&+ g(\gamma^0(T),m^{\varepsilon_k}(T)) - g(\gamma^0(T),m^0(T)) \\
&\leq u^0(t,x) + o(1)
\end{aligned}
\tag{4.30}
$$

where the last inequality again follows from the convergence of m^{ε_k} in $C([0,T];\mathscr{P}_1(\mathbb{R}^d))$. If this is not the case, i.e., $\dot{\gamma}^0(t)\neq v$, from Lemma 4.16 we deduce that

$$
u^{\varepsilon_k}(t,x,v) =\ u^{\varepsilon_k}(t,x,v) - u^{\varepsilon_k}(t,x,\dot{\gamma}^0(t)) + u^{\varepsilon_k}(t,x,\dot{\gamma}^0(t)) \leq o(1) + u^{\varepsilon_k}(t,x,\dot{\gamma}^0(t)).
$$

Thus, in order to conclude it is enough to estimate $u^{\varepsilon_k}(t,x,\dot{\gamma}^0(t))$ as in (4.30). Therefore, we obtain

$$
u^0(t,x) - o(1) \leq u^{\varepsilon_k}(t,x,v) \leq u^0(t,x) + o(1)
$$

which implies that u^{ε_k} point-wise converges to u^0.

Finally, in order to conclude we need to show that the convergence is locally uniform. From (4.25), (4.26) and Lemma 4.16 we have that for any $R\geq 0$ and any (t_1,x_1,v_1), $(t_2,x_2,v_2)\in[0,T]\times\overline{B}_R\times\overline{B}_R$ there holds

$$
\begin{aligned}
&|u^{\varepsilon}(t_1,x_1,v_1) - u^{\varepsilon}(t_2,x_2,v_2)| \\
&\leq \omega_R(|t_1-t_2|) + C_1|x_1-x_2| + C_R\sqrt{\varepsilon}.
\end{aligned}
$$

Therefore, setting

$$\Theta(\delta_0, \varepsilon_0, R_0) = \omega_{R_0}(\delta_0) + C_1\delta_0 + C_{R_0}\sqrt{\varepsilon_0}$$

by Remark 4.18 we deduce that the convergence is locally uniform and the proof is thus complete. □

After proving the convergence of u^ε, we go back to the analysis of the flow of measures and in particular we will characterize it in terms of the limit function u^0. In order to do so, we study the convergence of minimizers for u^ε and appealing to such a result we will show that $m^0 \in C([0,T]; \mathscr{P}_1(\mathbb{R}^d))$ solves a continuity equation with vector field $D_pH_0(x, D_x u^0)$, in the sense of distribution.

Proposition 4.20. *Assume* **(M1)** – **(M3)** *and* **(BC)**. *Let* $(t,x,v) \in [0,T] \times \mathbb{R}^{2d}$ *be such that* u^0 *is differentiable at* (t,x) *and let* γ^ε *be a minimizer for* $u^\varepsilon(t,x,v)$. *Then,* γ^ε *uniformly converges to a curve* $\gamma^0 \in AC([0,T];\mathbb{R}^d)$ *and* γ^0 *is the unique minimizer for* $u^0(t,x)$ *in* (4.29).

Proof. Let us start by proving that γ^ε uniformly converges, up to a subsequence. By Corollary 4.12 we know that

$$\int_t^T |\dot{\gamma}^\varepsilon(s)|^2 \, ds \leq Q_1(1 + |v|^2).$$

Thus, for any $s \in [t,T]$, by Hölder inequality we have that

$$|\gamma^\varepsilon(s)| \leq |x| + \sqrt{T}\sqrt{Q_1}(1+|v|^2)^{\frac{1}{2}}.$$

Therefore, γ^ε is bounded in $H^1(0,T;\mathbb{R}^d)$ which implies that by Ascoli-Arzela Theorem there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and a curve $\gamma^0 \in \mathrm{AC}([0,T];\mathbb{R}^d)$ such that γ^{ε_k} converges uniformly to γ^0.

We show now that such a limit γ^0 is a minimizer for $u^0(t,x)$. First, we observe that

$$\liminf_{k\to\infty} \left[\int_t^T \left(\frac{\varepsilon_k}{2} |\ddot{\gamma}^{\varepsilon_k}(s)|^2 + L_0(\gamma^{\varepsilon_k}(s), \dot{\gamma}^{\varepsilon_k}(s), m_s^\varepsilon) \right) \, ds + g(\gamma^{\varepsilon_k}(T), m_T^{\varepsilon_k}) \right]$$
$$\geq \liminf_{k\to\infty} \left[\int_t^T L_0(\gamma^{\varepsilon_k}(s), \dot{\gamma}^{\varepsilon_k}(s), m_s^{\varepsilon_k}) \, ds + g(\gamma^{\varepsilon_k}(T), m_T^{\varepsilon_k}) \right].$$

Then, as observed at the beginning of this proof γ^ε is uniformly bounded in $H^1(0,T)$. So by lower-semicontinuity of L and Theorem 4.14 we deduce that

$$\liminf_{k\to\infty} \left[\int_t^T L_0(\gamma^{\varepsilon_k}(s), \dot{\gamma}^{\varepsilon_k}(s)) \, ds + g(\gamma^{\varepsilon_k}(T), m_T^{\varepsilon_k}) \right]$$
$$\geq \int_t^T L_0(\gamma^0(s), \dot{\gamma}^0(s), m_s^0) \, ds + g(\gamma^0(T), m_T^0). \tag{4.31}$$

Moreover, for any $R \geq 0$ taking $(t,x,v) \in [0,T] \times \mathbb{R}^d \times \overline{B}_R$, from Theorem 4.19 we obtain

$$u^{\varepsilon_k}(t,x,v) \leq u^0(t,x) + o(1)$$

and we recall that

$$u^{\varepsilon_k}(t,x,v) = \int_t^T \left(\frac{\varepsilon_k}{2} |\ddot{\gamma}^{\varepsilon_k}(s)|^2 + L_0(\gamma^{\varepsilon_k}(s), \dot{\gamma}^{\varepsilon_k}(s), m_s^{\varepsilon_k}) \right) \, ds + g(\gamma^{\varepsilon_k}(T), m_T^{\varepsilon_k}).$$

Hence, we get

$$o(1) + u^0(t,x) \geq \int_t^T \left(\frac{\varepsilon_k}{2} |\ddot{\gamma}^{\varepsilon_k}(s)|^2 + L_0(\gamma^{\varepsilon_k}(s), \dot{\gamma}^{\varepsilon_k}(s), m_s^{\varepsilon_k}) \right) \, ds + g(\gamma^{\varepsilon_k}(T), m_T^{\varepsilon_k}).$$

Therefore, passing to the limit as $\varepsilon \downarrow 0$ from (4.31) we obtain

$$u^0(t,x) \geq \int_t^T L_0(\gamma^0(s), \dot{\gamma}^0(s), m_s^0) \, ds + g(\gamma^0(T), m_T^0)$$

which proves that γ^0 is a minimizer for $u^0(t,x)$. Since u^0 is differentiable at $(t,x) \in \mathbb{R}^d$ there exists a unique minimizing trajectory and thus we have that the uniform convergence of γ^ε holds for the whole sequence. $\square$

Remark 4.21. *Since u^0 is locally Lipschitz continuous w.r.t. time and Lipschitz continuous w.r.t. space, we have that Proposition 4.20 holds for a.e. $(t,x) \in [0,T] \times \mathbb{R}^d$.*

Let u^0 be as in (4.29) and let $\gamma_t^0(\cdot)$ be the flow associated with the vector field

$$x \mapsto D_p H_0(x, D_x u^0(t,x), m_t^0),$$

that is,

$$\begin{cases} \dot{\gamma}_t^0(x) = D_p H_0(\gamma_t^0(x), D_x u^0(t, \gamma_t^0(x)), m_t^0), & t \in [0,T] \\ \gamma_0(x) = x. \end{cases}$$

Note that such a flow exists since the vector field $x \mapsto D_p H_0(x, D_x u^0(t,x), m_t^0)$ is Lipschitz continuous by the Lipschitz continuity of the value function u^0 and by the regularity of the Hamiltonian H_0. We also recall that the measure μ^ε is the image of μ_0 under the flow (4.11), which is optimal as observed in Remark 4.2 for $u^\varepsilon(0,x,v)$ for a.e. $(x,v) \in \mathbb{R}^{2d}$, and thus, for any function $\varphi \in C_c^\infty(\mathbb{R}^d)$ the measure m_t^ε is given by

$$\int_{\mathbb{R}^d} \varphi(x) \, m_t^\varepsilon(dx) = \int_{\mathbb{R}^{2d}} \varphi(\gamma_{(x,v)}^\varepsilon(t)) \, \mu_0(dx, dv). \tag{4.32}$$

We finally recall that by assumption μ_0 is absolutely continuous w.r.t. Lebesgue measure.

Corollary 4.22. *Assume* **(M1)** – **(M3)** *and* **(BC)***. Then, we have that*

$$m_t^0 = \gamma_t^0(\cdot) \sharp m_0, \quad \forall \, t \in [0,T]. \tag{4.33}$$

Moreover, $m^0 \in C([0,T]; \mathscr{P}_1(\mathbb{R}^d))$ solves

$$\begin{cases} \partial_t m_t^0 - \operatorname{div}\left(m_t^0 D_p H_0(x, D_x u^0(t,x), m_t^0) \right) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ m_0^0 = m_0, & x \in \mathbb{R}^d, \end{cases}$$

in the sense of distributions.

Proof. From Theorem 4.14 let $\varepsilon_k \downarrow 0$ be such that $m^{\varepsilon_k} \to m^0$ in $C([0,T]; \mathscr{P}_1(\mathbb{R}^d))$. Then, since μ_0 is absolutely continuous w.r.t. Lebesgue measure by Proposition 4.20 we have that

$$\gamma_{(x,v)}^{\varepsilon_k}(t) \to \gamma_t^0(x), \quad \mu_0\text{-a.e.} \, (x,v), \; \forall t \in [0,T].$$

Therefore, from (4.32), for $\varepsilon = \varepsilon_k$, as $k \to \infty$ we get

$$\int_{\mathbb{R}^d} \varphi(x) \, m_t^0(dx) = \int_{\mathbb{R}^d} \varphi(\gamma_t^0(x)) \, m_0(dx), \quad \forall \, t \in [0,T]$$

which proves (4.33). Moreover, again by Proposition 4.20 we have that γ_t^0 is a minimiser for $u^0(0,x)$ since it is the limit of $\gamma^\varepsilon_{(x,v)}$ which is optimal $u^\varepsilon(0,x,v)$ and we are taking (x,v) in a subset of full measure w.r.t. μ_0. Therefore, from the optimality of γ^0 we get

$$\begin{cases} \dot{\gamma}_t^0(x) = D_p H_0(\gamma_t^0(x), Du^0(t,\gamma_t^0(x)), m_t^0), & t \in (0,T] \\ \gamma_0^0(x) = x. \end{cases}$$

Hence, for any $\psi \in C_c^\infty([0,T) \times \mathbb{R}^d)$ we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^d} \psi(t,x)\ m_t^0(dx) = \frac{d}{dt}\int_{\mathbb{R}^d} \psi(t,\gamma_t^0(x))\ m_0(dx) \\ &= \int_{\mathbb{R}^d} \big(\partial_t\psi(t,\gamma_t^0(x)) + \langle D_x\psi(t,\gamma_t^0(x)), D_pH_0(\gamma_t^0(x), D_x\psi(t,\gamma_t^0(x)), m_t^0)\big)\ m_0(dx) \\ &= \int_{\mathbb{R}^d} \big(\partial_t\psi(t,x) + \langle D_x\psi(t,x), D_pH_0(x, D_x\psi(t,x), m_t^0)\big)\ m_t^0(dx) \end{aligned}$$

and integrating, in time, over $[0,T]$ we get the result. □

We are now ready to prove the main result.

Proof of Theorem 4.3. Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be such that $m^{\varepsilon_k} \to m^0$ in $C([0,T];\mathscr{P}_1(\mathbb{R}^d)$ and $u^{\varepsilon_k} \to u^0$ locally uniformly on $[0,T] \times \mathbb{R}^{2d}$. Then, appealing to Theorem 4.19 and Corollary 4.22 we deduce that (u^0, m^0) is a solution to the MFG system

$$\begin{cases} -\partial_t u^0(t,x) + H_0(x, D_xu^0(t,x), m_t^0) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ \partial_t m_t^0 - \text{div}\left(m_t^0 D_pH_0(x, D_xu^0(t,x), m_t^0)\right) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ m_0^0 = m_0,\ u^0(T,x) = g(x, m_T^0) & x \in \mathbb{R}^d \end{cases}$$

which completes the proof. □

Part II

Weak KAM Theory and Aubry-Mather theory for sub-Riemannian control systems

Chapter 5

Asymptotic analysis for Hamilton-Jacobi equations associated with sub-Riemannian control systems

5.1 Settings and assumptions

For $m \in \mathbb{N}$ and $i = 1, \ldots, m$, let

$$f_i : \mathbb{R}^d \to \mathbb{R}^d$$

and

$$u_i : [0, \infty) \to \mathbb{R}$$

be smooth vector fields and measurable controls, respectively, and consider the following controlled dynamics of sub-Riemannian type

$$\dot{\gamma}(t) = \sum_{i=1}^{m} f_i(\gamma(t))u_i(t) = F(\gamma(t))U(t), \quad t \in [0, +\infty) \tag{5.1}$$

where $F(x) = [f_1(x)| \ldots |f_m(x)]$ is an $d \times m$ real matrix and $U(t) = (u_1(t), \ldots, u_m(t))^{\star}$[1].

For any s_0, $s_1 \in \mathbb{R}$ such that $s_0 < s_1$ and x, $y \in \mathbb{R}^d$ we set

$$\begin{aligned}
\Gamma^{x\to}_{s_0,s_1} &= \{(\gamma, u) \in \mathrm{AC}([s_0, s_1]; \mathbb{R}^d) \times L^2(s_0, s_1; \mathbb{R}^m) : \dot{\gamma}(t) = F(\gamma(t))u(t),\ \gamma(s_0) = x\},\\
\Gamma^{\to y}_{s_0,s_1} &= \{(\gamma, u) \in \mathrm{AC}([s_0, s_1]; \mathbb{R}^d) \times L^2(s_0, s_1; \mathbb{R}^m) : \dot{\gamma}(t) = F(\gamma(t))u(t),\ \gamma(s_1) = y\},\\
\Gamma^{x\to y}_{s_0,s_1} &= \Gamma^{x\to}_{s_0,s_1} \cap \Gamma^{\to y}_{s_0,s_1}.
\end{aligned}$$

Throughout the Chapter we assume the vector fields f_i to satisfy the following:

(**F0**) there exists a constant $c_f \geq 1$ such that for any $i = 1, \ldots, m$

$$|f_i(x)| \leq c_f(1 + |x|), \quad \forall x \in \mathbb{R}^d. \tag{5.2}$$

In literature, under assumption **(F0)** the distribution $\{f_i\}_{i=1,\ldots,m}$ is called regular.

By (5.2) and Gronwall's inequality we get the following estimate for the trajectories of (5.1).

[1] $(u_1, \ldots, u_m)^{\star}$ denotes the transpose of $(u_1, \ldots, u_m)$

Lemma 5.1. *Let $x \in \mathbb{R}^d$, $t \geq 0$ and $(\gamma, u) \in \Gamma_{0,t}^{x\to}$. If $u \in L^\infty(0,t;\mathbb{R}^m)$ then we have that*

$$|\gamma_u(s)| \leq (|x| + c_f\|u\|_\infty s)e^{c_f\|u\|_\infty s}, \quad \forall\ s \in [0,t].$$

Moreover, still from (5.2) we obtain the following.

Lemma 5.2. *Let $x \in \mathbb{R}^d$, $t \geq 0$ and $(\gamma, u) \in \Gamma_{0,t}^{x\to}$. Then there exists a constant $\kappa(\|u\|_2, t) \geq 0$ such that*

$$|\gamma(s)| \leq \kappa(\|u\|_2, t)(1 + |x|), \quad \forall\ s \in [0,t] \tag{5.3}$$

and

$$|\gamma(t_2) - \gamma(t_1)| \leq c_f\kappa(\|u\|_2, t)(1 + |x|)\|u\|_2|t_2 - t_1|^{\frac{1}{2}}, \quad 0 \leq t_1 \leq t_2 \leq t. \tag{5.4}$$

Proof. We begin by proving (5.3). For any $s \in [0,t]$ we have that

$$\begin{aligned}
|\gamma(s)| &\leq |x| + \int_0^t |F(\gamma(s))||u(s)|\ ds \\
&\leq |x| + \int_0^t c_f(1 + |\gamma(s)|)|u(s)|\ ds \\
&\leq |x| + c_f\left(\int_0^t (1 + |\gamma(s)|)^2\ ds\right)^{\frac{1}{2}} \|u\|_2.
\end{aligned}$$

Thus, we get

$$|\gamma(s)|^2 \leq C\left(|x|^2 + c_f^2 t\|u\|_2^2 + c_f^2\|u\|_2^2 \int_0^t |\gamma(s)|^2\ ds\right)$$

which implies the (5.3) by Gronwall's inequality.

We now proceed to show (5.4). Fix t_1, t_2 such that $0 \leq t_1 \leq t_2 \leq t$. Then, we have that

$$\begin{aligned}
|\gamma(t_2) - \gamma(t_1)| &\leq \int_{t_1}^{t_2} |F(\gamma(s))||u(s)|\ ds \\
&\leq c_f \int_{t_1}^{t_2} (1 + |\gamma(s)|)|u(s)|\ ds \\
&\leq c_f\kappa(\|u\|_2, t)(1 + |x|) \int_{t_1}^{t_2} |u(s)|\ ds
\end{aligned}$$

where the last inequality holds by (5.3). Hence, by Hölder's inequality we obtain

$$|\gamma(t_2) - \gamma(t_1)| \leq c_f\kappa(\|u\|_2, t)(1 + |x|)\|u\|_2|t_2 - t_1|^{\frac{1}{2}}.$$

This completes the proof of the lemma. □

Let the Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ be such that

(L0) $L \in C^2(\mathbb{R}^d \times \mathbb{R}^m)$ and $L(x,u) \geq L(x,0)$ for any $(x,u) \in \mathbb{R}^d \times \mathbb{R}^m$.

(L1) There exist a non-decreasing function $\beta : [0,\infty) \to \mathbb{R}$ and a constant $\ell_1 \geq 0$ such that

$$\begin{aligned}
L(x,u) \leq&\ \beta(|x|)(1 + |u|^2), \quad \forall\ (x,u) \in \mathbb{R}^d \times \mathbb{R}^m \\
|D_xL(x,u)| \leq&\ \ell_1(1 + |u|^2), \quad \forall\ (x,u) \in \mathbb{R}^d \times \mathbb{R}^m \\
D_u^2L(x,u) \geq&\ \frac{1}{\ell_1}, \quad \forall\ (x,u) \in \mathbb{R}^d \times \mathbb{R}^m.
\end{aligned}$$

(**L2**) There exists a compact set $\mathcal{K}_L \subset \mathbb{R}^d$ and a constant $\delta_L > 0$ such that

$$\inf_{x\in\mathbb{R}^d\setminus\mathcal{K}_L} L(x,0) \geq \delta_L + \min_{x\in\mathcal{K}_L} L(x,0). \tag{5.5}$$

Observe that a special class of functions L which satisfy **(L0)** is the class of Lagrangians $L \in C^2(\mathbb{R}^d \times \mathbb{R}^m)$ which are convex w.r.t. $u \in \mathbb{R}^d$ and reversible, that is, $L(x,u) = L(x,-u)$ for any $(x,u) \in \mathbb{R}^d \times \mathbb{R}^m$. Moreover, note that by **(L0)**, **(L1)** and **(L2)** we obtain

$$L(x,u) \geq \frac{1}{2\ell_1}|u|^2 + L(x^*,0), \quad \forall\ (x,u) \in \mathbb{R}^d \times \mathbb{R}^m \tag{5.6}$$

where $x^* \in \mathcal{K}_L$ is such that

$$L(x^*,0) = \min_{x\in\mathcal{K}_L} L(x,0).$$

Furthermore, set

$$\delta^*(x) = d_{\mathrm{SR}}(x,x^*), \quad \forall\ x \in \mathbb{R}^d$$

and observe that, by Corollary 1.7, there exists a nondecreasing function $D : \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\delta^*(x) \leq D(|x|), \quad \forall\ x \in \mathbb{R}^d. \tag{5.7}$$

5.2 Boundedness of optimal trajectories

We consider the following minimisation problem: for any $T > 0$ and $x \in \mathbb{R}^d$

$$\text{to minimize } \int_0^T L(\gamma(s),u(s))\,ds \text{ over all } (\gamma,u) \in \Gamma^{x\to}_{0,T} \tag{5.8}$$

and we set

$$V_T(x) = \inf_{(\gamma,u)\in\Gamma^{x\to}_{0,T}} \int_0^T L(\gamma(s),u(s))\,ds, \quad \forall\ x \in \mathbb{R}^d. \tag{5.9}$$

For any $x \in \mathbb{R}^d$ we say that a trajectory-control pair $(\gamma,u) \in \Gamma^{x\to}_{0,T}$ is optimal if it solves (5.8).

Remark 5.3. We observe that by using classical technics from optimal control theory one can easily obtain the existence of optimal trajectory-control pairs for (5.8) (see, for instance, [26, Theorem 7.4.4]).

In this section, we prove the uniform boundedness of optimal trajectories for (5.8) starting from a given compact set. We begin by deriving a uniform bound for the Lebsegue measure of all times at which an optimal trajectory may lie outside the compact set $\mathcal{K}_L$ of assumption **(L3)**.

Proposition 5.4. *Assume* **(F0)**, *and* **(L0)** – **(L2)**. *For any $R \geq 0$ there exists a constant $M_R \geq 0$ such that for any $x \in \overline{B}_R$, any $T \geq \delta^*(x)$, and any optimal pair $(\gamma_x,u_x) \in \Gamma^{x\to}_{0,T}$ for (5.8) we have that*

$$\mathcal{L}^1\left(\{t \in [0,T] : \gamma_x(t) \notin \mathcal{K}_L\}\right) \leq M_R. \tag{5.10}$$

Proof. Fix $R \geq 0$ and let $x \in \overline{B}_R$. Let $(\bar{\gamma}_x, \bar{u}_x) \in \Gamma^{x \to x^*}_{0,\delta^*(x)}$ be a solution of (1.5) and recall that, as observed in (5.7), $\delta^*(x) \leq D(R)$. Define the control

$$\widehat{u}_x(t) = \begin{cases} \bar{u}_x(t), & t \in [0, \delta^*(x)] \\ 0, & t \in (\delta^*(x), T]. \end{cases}$$

Then, $(\widehat{\gamma}_x, \widehat{u}_x) \in \Gamma^{x \to x^*}_{0,T}$ and we obtain

$$\begin{aligned} V_T(x) &\leq \int_0^{\delta^*(x)} L(\bar{\gamma}_x(t), \bar{u}_x(t))\, dt + (T - \delta^*(x)) L(x^*, 0) \\ &=: c_1(x, \delta^*(x)) + (T - \delta^*(x)) L(x^*, 0). \end{aligned} \tag{5.11}$$

Invoking Corollary 1.7 once again, we have that $c_1(x, \delta^*(x)) \leq C_1(R)$ for some positive constant $C_1(R)$. Now, let $(\gamma_x, u_x) \in \Gamma^{x \to}_{0,T}$ be optimal for (5.8). Then, we have that

$$\begin{aligned} V_T(x) = &\int_0^T L(\gamma_x(t), u_x(t))\, dt \geq \int_0^T L(\gamma_x(t), 0)\, dt \\ \geq &\, c_2(x, \delta^*(x)) + \int_{\delta^*(x)}^T L(\gamma_x(t), 0) \mathbf{1}_{\mathcal{K}_L}(\gamma_x(t))\, dt + \int_{\delta^*(x)}^T L(\gamma_x(t), 0) \mathbf{1}_{\mathcal{K}_L^c}(\gamma_x(t))\, dt \\ \geq &\, c_2(x, \delta^*(x)) + L(x^*, 0) \mathcal{L}^1\big(\{t \in [\delta^*(x), T] :\ \gamma_x(t) \in \mathcal{K}_L\}\big) \\ + &\left(\inf_{x \in \mathbb{R}^d \setminus \mathcal{K}_L} L(x, 0) \right) \mathcal{L}^1\big(\{t \in [\delta^*(x), T] :\ \gamma_x(t) \notin \mathcal{K}_L\}\big) \\ = &\, c_2(x, \delta^*(x)) + L(x^*, 0) \mathcal{L}^1\big(\{t \in [\delta^*(x), T] :\ \gamma_x(t) \in \mathcal{K}_L\}\big) \\ + &\left(\inf_{x \in \mathbb{R}^d \setminus \mathcal{K}_L} L(x, 0) \right) \big(T - \delta^*(x) - \mathcal{L}^1\big(\{t \in [\delta^*(x), T] :\ \gamma_x(t) \in \mathcal{K}_L\}\big)\big) \end{aligned} \tag{5.12}$$

where

$$c_2(x, \delta^*(x)) := \int_0^{\delta^*(x)} L(\gamma_x(t), 0)\, dt.$$

From (5.12) it also follows that

$$V_T(x) \geq c_2(x, \delta^*(x)) + L(x^*, 0)(T - \delta^*(x)),$$

which, together with (5.11), yields

$$c_2(x, \delta^*(x)) \leq c_1(x, \delta^*(x)) \leq C_1(R). \tag{5.13}$$

Hence, we have that

$$c_3(\delta^*(x)) := c_1(\delta^*(x)) - c_2(\delta^*(x)) \geq 0, \quad c_3(\delta^*(x)) \leq C_1(R).$$

So, combining (5.11) and (5.12) and recalling **(L2)**, we deduce that

$$\begin{aligned} c_3(\delta^*(x)) &\geq \left(\inf_{x \in \mathbb{R}^d \setminus \mathcal{K}_L} L(x, 0) - L(x^*, 0) \right) \big(T - \delta^*(x) - \mathcal{L}^1\big(\{t \in [\delta^*(x), T] :\ \gamma_u(t) \in \mathcal{K}_L\}\big)\big) \\ &\geq \delta_L \mathcal{L}^1\big(\{t \in [\delta^*(x), T] :\ \gamma_x(t) \notin \mathcal{K}_L\}\big). \end{aligned}$$

Therefore,

$$\mathcal{L}^1\big(\{t \in [0, T] :\ \gamma_x(t) \notin \mathcal{K}_L\}\big) \leq \frac{c_3(\delta^*(x))}{\delta_L} + \delta^*(x).$$

Recalling that $\delta^*(x) \leq D(R)$ and setting

$$M_R := \frac{C_1(R)}{\delta_L} + D(R)$$

we obtain the conclusion. □

Theorem 5.5. *Assume* **(F0)**, *and* **(L0)** – **(L2)**. *For any $R \geq 0$ there exist two constants $P_R, Q_R \geq 0$ such that for any $x \in \overline{B}_R$, any $T \geq \delta^*(x)$, and any optimal pair $(\gamma_x, u_x) \in \Gamma_{0,T}^{x\rightarrow}$ for (5.8) we have that*

$$\int_0^T |u_x(t)|^2 \, dt \leq P_R \tag{5.14}$$

and

$$|\gamma_x(t)| \leq Q_R, \quad \forall \, t \in [0,T]. \tag{5.15}$$

Proof. We begin with the proof of (5.14). Since $(\gamma_x, u_x) \in \Gamma_{0,T}^{x\rightarrow}$ is optimal for (5.8) we have that

$$V_T(x) = \int_0^T L(\gamma_x(t), u_x(t)) \, dt \geq \frac{1}{2\ell_1} \int_0^T |u_x(t)|^2 \, dt + TL(x^*, 0). \tag{5.16}$$

On the other hand, let $(\bar{\gamma}_x, \bar{u}_x) \in \Gamma_{0,\delta^*(x)}^{x\rightarrow x^*}$ be a solution of (1.5) and define the control

$$\widehat{u}_x(t) = \begin{cases} \bar{u}_x(t), & t \in [0, \delta^*(x)] \\ 0, & t \in (\delta^*(x), T], \end{cases}$$

that is, $(\widehat{\gamma}_x, \widehat{u}_x) \in \Gamma_{0,T}^{x\rightarrow x^*}$. By the definition of V_T we deduce that

$$V_T(x) \leq \int_0^{\delta^*(x)} L(\bar{\gamma}_x(t), \bar{u}_x(t)) \, dt + (T - \delta^*(x))L(x^*, 0). \tag{5.17}$$

Combining (5.16) and (5.17) we obtain

$$\frac{1}{2\ell_1} \int_0^T |u_x(t)|^2 \, dt \leq \int_0^{\delta^*(x)} L(\bar{\gamma}_x(t), \bar{u}_x(t)) \, dt - \delta^*(x)L(x^*, 0).$$

In order to prove (5.14), we need an upper bound for the term

$$\int_0^{\delta^*(x)} L(\bar{\gamma}_x(t), \bar{u}_x(t)) \, dt.$$

Observe that, since $\|\bar{u}_x\|_{\infty,[0,\delta^*(x)]} \leq 1$ and $\delta^*(x) \leq D(R)$, Lemma 5.1 ensures that

$$|\bar{\gamma}_x(t)| \leq (R + c_f D(R))e^{c_f D(R)} =: \Lambda(R), \quad \forall \, t \in [0, \delta^*(x)]. \tag{5.18}$$

Therefore, by assumption **(L1)** we deduce that

$$\int_0^{\delta^*(x)} L(\bar{\gamma}_x(t), \bar{u}_x(t)) \, dt \leq 2\beta(\Lambda(R)).$$

Hence, (5.14) follows taking

$$P_R = 2\ell_1(2\beta(\Lambda(R)) + D(R)L(x^*, 0)).$$

We now proceed to prove (5.15). Let $(\gamma_x, u_x) \in \Gamma^{x \to}_{0,T}$ be a solution of (5.8). Clearly, we just need to estimate $|\gamma_x(t)|$ for all times at which the optimal trajectory lies outside the compact set $\mathcal{K}_L$. Let $\bar{t} \in [0,T]$ be such that $\gamma_x(\,\bar{t}\,) \notin \mathcal{K}_L$ and set

$$t_0 = \begin{cases} \sup\{t \in [0, \bar{t}\,] : \gamma_x(t) \in \mathcal{K}_L\}, & \text{if } \{t \in [0, \bar{t}\,] : \gamma_x(t) \in \mathcal{K}_L\} \neq \emptyset \\ 0, & \text{if } \{t \in [0, \bar{t}\,] : \gamma_x(t) \in \mathcal{K}_L\} = \emptyset \end{cases}$$

We only consider the case of $t_0 \neq 0$ since the reasoning is similar when $t_0 = 0$. Since γ_x is a solution of (5.1) we deduce that for any $t \in [t_0, \bar{t}]$

$$\begin{aligned} |\gamma_x(t)| \leq & |\gamma_x(t_0)| + c_f \int_{t_0}^{t} (1 + |\gamma_x(s)|)|u_x(s)| \, ds \\ \leq & |\gamma_x(t_0)| + c_f \|u_x\|_{2,[t_0,\bar{t}]} \left(\int_{t_0}^{t} \left(1 + |\gamma_x(s)|^2\right) \, ds \right)^{\frac{1}{2}}. \end{aligned}$$

Hence, appealing to (5.10) and (5.14) we deduce that

$$\begin{aligned} |\gamma_x(t)|^2 \leq & \bar{C} \left(|\gamma_x(t_0)|^2 + \|u_x\|^2_{2,[t_0,\bar{t}]} M_R + \|u_x\|^2_{2,[t_0,\bar{t}]} \int_{t_0}^{t} |\gamma_x(s)|^2 \, ds \right) \\ \leq & \bar{C} \left(|\gamma_x(t_0)|^2 + P_R M_R + P_R \int_{t_0}^{t} |\gamma_x(s)|^2 \, ds \right) \end{aligned}$$

for some constant $\bar{C} \geq 0$. Thus, recalling that $|t - t_0| \leq M_R$ by Proposition 5.4, the Gronwall inequality yields

$$|\gamma_x(t)| \leq \bar{C} |\gamma_x(t_0)|^2 P_R M_R e^{P_R M_R}, \quad \forall\, t \in [t_0, \bar{t}]$$

and we set $Q_R := \bar{C}|\gamma_x(t_0)|^2 P_R M_R e^{P_R M_R}$. Since $|\gamma_x(t_0)| \leq \max\{|y| : y \in \mathcal{K}_L\}$ and $|\gamma_x(t)| \leq \max\{|y| : y \in \mathcal{K}_L\}$ for all times t at which $\gamma_x(t) \in \mathcal{K}_L$, we get the conclusion. □

5.3 Long-time average and ergodic constant

In this section, we investigate the existence of the limit

$$\lim_{T \to \infty} \frac{1}{T} V_T(x) \quad (x \in \mathbb{R}^d),$$

where $V_T(x)$ is defined in (5.9), as well as the related problem of the existence of solutions to the ergodic Hamilton-Jacobi equation

$$c + H(x, D\chi(x)) = 0 \quad (x \in \mathbb{R}^d) \tag{5.19}$$

for some $c \in \mathbb{R}$, where $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the Hamiltonian associated with L, that is,

$$H(x,p) = \sup_{u \in \mathbb{R}^m} \left\{ \sum_{i=1}^{m} u_i \langle p, f_i(x) \rangle - L(x,u) \right\}, \quad \forall\, (x,p) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{5.20}$$

5.3.1 Long-time average

In order to prove the main result of this section, that is, Theorem 5.8, we need to show, first, that the value function V_T is locally equicontinuous uniformly in T.

Lemma 5.6. *Assume* **(F0)**, *and* **(L0)** – **(L2)**. *For any $R \geq 0$ there exist two constants $T_R \geq 0$ and $K(R) \geq 0$ such that*

$$|V_T(x) - V_T(y)| \leq K(R) d_{\mathrm{SR}}(x, y), \quad \forall\, T \geq T_R \quad \forall\, x, y \in \overline{B}_R.$$

Proof. Let x, $y \in \overline{B}_R$, set $\delta = d_{\mathrm{SR}}(x, y)$ and let $T \geq d_{\mathrm{SR}}(x, y) =: T_R$. Let $(\bar{\gamma}_y, \bar{u}_y) \in \Gamma_{0,\delta}^{y \to x}$ be a solution of (1.5), let $(\gamma_x, u_x) \in \Gamma_{0,T}^{x \to}$ be a solution of (5.8), and define the control

$$\widehat{u}_y(t) = \begin{cases} \bar{u}_y(t), & t \in [0, \delta] \\ u_x(t - \delta), & t \in (\delta, T]. \end{cases}$$

Then, we have that

$$\begin{aligned} V_T(y) - V_T(x) \leq & \int_0^T L(\widehat{\gamma}_y(t), \widehat{u}_y(t))\, dt - \int_0^T L(\gamma_x(t), u_x(t))\, dt \\ \leq & \int_0^\delta L(\bar{\gamma}_y(t), \bar{u}_y(t))\, dt + \int_\delta^T L(\gamma_x(t - \delta), u_x(t - \delta))\, dt - \int_0^T L(\gamma_x(t), u_x(t))\, dt \\ = & \int_0^\delta L(\bar{\gamma}_y(t), \bar{u}_y(t))\, dt + \int_0^{T-\delta} L(\gamma_x(s), u_x(s))\, ds - \int_0^T L(\gamma_x(s), u_x(s))\, ds \\ = & \int_0^\delta L(\bar{\gamma}_y(t), \bar{u}_y(t))\, dt - \int_{T-\delta}^T L(\gamma_x(s), u_x(s))\, ds. \end{aligned} \tag{5.21}$$

First, by (5.6) we get

$$\int_{T-\delta}^T L(\gamma_x(s), u_x(s))\, ds \geq \delta L(x^*, 0).$$

Then, since $\|\bar{u}_y\|_\infty \leq 1$ and $\delta \leq c(R)$, by Lemma 5.1 we have that

$$|\bar{\gamma}_y(t)| \leq (R + c_f c(R)) e^{c_f c(R)} =: \Lambda(R), \quad \forall\, t \in [0, \delta]. \tag{5.22}$$

Thus, by **(L1)** we obtain

$$\int_0^\delta L(\bar{\gamma}_y(s), \bar{u}_y(s))\, ds \leq 2\delta\beta(\Lambda(R)).$$

Hence, going back to (5.21), from Corollary 1.7 it follows that

$$\begin{aligned} & V_T(y) - V_T(x) \\ \leq & \int_0^\delta L(\bar{\gamma}_y(s), \bar{u}_y(s))\, ds - \int_{T-\delta}^T L(\gamma_x(s), u_x(s))\, ds \\ \leq & \delta(2\beta(\Lambda(R)) - L(x^*, 0)) = K(R) d_{\mathrm{SR}}(x, y) \end{aligned}$$

where $K(R) := \tilde{c}_2(2\beta(\Lambda(R)) - L(x^*, 0))$. Switching x and y in the above reasoning completes the proof. □

Lemma 5.7. *Assume* **(F0)**, *and* **(L0)** – **(L2)**. *For any $R \geq 0$ there exists a constant $C_R \geq 0$ such that for any $x \in \overline{B}_R$, any $T > 0$, and any optimal pair $(\gamma_x, u_x) \in \Gamma_{0,T}^{x \to}$ of* (5.8) *there exists a pair $(\gamma_T, u_T) \in \Gamma_{0,T}^{x \to x}$ such that*

$$\int_0^T L(\gamma_T(t), u_T(t))\, dt \leq \int_0^T L(\gamma_x(t), u_x(t))\, dt + C_R.$$

Proof. Fix $R \geq 0$, $x \in \overline{B}_R$, and take an optimal pair $(\gamma_x, u_x) \in \Gamma_{0,T}^{x\to}$. If $\gamma_x(T) = x$ then it is enough to take $C_R = 0$ and $(\gamma_T, u_T) = (\gamma_x, u_x)$. If this is not the case, let $\delta_0 \in (0, T)$ be such that

$$\delta_0 = d_{\mathrm{SR}}(\gamma_x(T - \delta_0), x).$$

Note that such a number δ_0 exists since $g(\delta) := d_{\mathrm{SR}}(\gamma_x(T - \delta), x) - \delta$, for $\delta \in [0, T]$, is a continuous function satisfying

$$g(T) = -T < 0$$
$$g(0) = d_{\mathrm{SR}}(\gamma_x(T), x) > 0.$$

For simplicity of notation set $y = \gamma_x(T-\delta_0)$ and observe that $|y| \leq Q_R$ by Theorem 5.5. Let $(\bar{\gamma}_y, \bar{u}_y) \in \Gamma_{0,\delta_0}^{y\to x}$ be a solution of (1.5) and define the control

$$u_T(t) = \begin{cases} u_x(t), & t \in [0, T - \delta_0] \\ \bar{u}_y(t + \delta_0 - T), & t \in (T - \delta_0, T]. \end{cases}$$

Then

$$\int_0^T L(\gamma_T(t), u_T(t))\, dt = \int_0^{T-\delta_0} L(\gamma_x(t), u_x(t))\, dt + \int_0^{\delta_0} L(\bar{\gamma}_y(t), \bar{u}_y(t))\, dt$$
$$= \int_0^T L(\gamma_x(t), u_x(t))\, dt - \int_{T-\delta_0}^T L(\gamma_x(t), u_x(t))\, dt + \int_0^{\delta_0} L(\bar{\gamma}_y(t), \bar{u}_y(t))\, dt.$$

By (5.6) we obtain

$$\int_{T-\delta_0}^T L(\gamma_x(t), u_x(t))\, dt \geq \delta_0 L(x^*, 0).$$

Let $R' = \max\{R, Q_R\}$. Then, since $\|\bar{u}_y\|_{\infty,[0,\delta_0]} \leq 1$ and $|y| \leq Q_R$, by Lemma 5.1 we also have that

$$|\bar{\gamma}_y(t)| \leq (Q_R + c_f R')e^{c_f R'} =: \Lambda(R)$$

for any $t \in [0, \delta_0]$. So, we obtain

$$-\int_{T-\delta_0}^T L(\gamma_x(t), u_x(t))\, dt + \int_0^{\delta_0} L(\bar{\gamma}_y(t), \bar{u}_y(t))\, dt$$
$$\leq -\delta_0 L(x^*, 0) + \int_0^{\delta_0} L(\bar{\gamma}_y(t), \bar{u}_y(t))\, dt \leq \delta_0\big(2\beta(\Lambda(R)) - L(x^*, 0)\big).$$

The conclusion follows taking

$$C_R = R'\big(2\beta(\Lambda(R)) - L(x^*, 0)\big).$$ □

Theorem 5.8 **(Existence of the critical constant).** *Assume* **(F0)**, *and* **(L0)** – **(L2)**. *There exists a constant $\alpha(L) \in \mathbb{R}$, called the critical constant (or Mañé's critical value), such that*

$$\lim_{T\to\infty} \sup_{x \in \overline{B}_R} \left| \frac{1}{T} V_T(x) - \alpha(L) \right| = 0, \quad \forall\, R > 0. \tag{5.23}$$

Proof. Let $R \geq 0$. By Lemma 5.6, for all $x \in \overline{B}_R$ we deduce that

$$|V_T(x) - V_T(0)| \leq K'(R). \tag{5.24}$$

Hence, to obtain the conclusion it suffices to prove the existence of the limit

$$\lim_{T\to\infty} \frac{1}{T} V_T(0) =: \alpha(L). \tag{5.25}$$

For this purpose let $\{T_n\}_{n\in\mathbb{N}}$ and $\{(\gamma_n, u_n)\}_{n\in\mathbb{N}} \subset \Gamma^{0\to}_{0,T_n}$ be such that

$$\begin{aligned} \liminf_{T\to\infty} \frac{1}{T} V_T(0) &= \lim_{n\to\infty} \frac{1}{T_n} \inf_{(\gamma,u)\in\Gamma^{0\to}_{0,T_n}} \int_0^{T_n} L(\gamma(t), u(t))\, dt \\ &= \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} L(\gamma_n(t), u_n(t))\, dt. \end{aligned} \tag{5.26}$$

By Lemma 5.7 there exists a sequence $(\gamma_n^0, u_n^0) \in \Gamma^{0\to 0}_{0,T_n}$ and a constant $C_0 \geq 0$ such that

$$\int_0^{T_n} L(\gamma_n^0(t), u_n^0(t))\, dt \leq \int_0^{T_n} L(\gamma_n(t), u_n(t))\, dt + C_0. \tag{5.27}$$

Next, for any $n \in \mathbb{N}$ let $\widehat{u}_n^0$ be the periodic extension of u_n^0, i.e., $\widehat{u}_n^0$ is T_n-periodic and $\widehat{u}_n^0(t) = u_n^0(t)$ for any $t \in [0, T_n]$. Then, we have that

$$\limsup_{T\to\infty} \frac{1}{T} V_T(0) \leq \limsup_{T\to\infty} \frac{1}{T} \int_0^T L(\widehat{\gamma}_n^0(t), \widehat{u}_n^0(t))\, dt, \quad \forall\, n \in \mathbb{N} \tag{5.28}$$

by using $\widehat{u}_n^0$ as a competitor in (5.9). Then, by periodicity and (5.27) we obtain

$$\begin{aligned} &\limsup_{T\to\infty} \frac{1}{T} \int_0^T L(\widehat{\gamma}_n^0(t), \widehat{u}_n^0(t))\, dt \\ &\qquad = \frac{1}{T_n} \int_0^{T_n} L(\gamma_n^0(t), u_n^0(t))\, dt \leq \frac{1}{T_n} \int_0^{T_n} L(\gamma_n(t), u_n(t))\, dt + \frac{C_0}{T_n}. \end{aligned}$$

Therefore, recalling (5.28) and (5.26) we conclude that

$$\begin{aligned} \limsup_{T\to\infty} \frac{1}{T} V_T(0) &\leq \lim_{n\to\infty} \left(\frac{1}{T_n} \int_0^{T_n} L(\gamma_n(t), u_n(t))\, dt + \frac{C_0}{T_n} \right) \\ &= \liminf_{T\to\infty} \frac{1}{T} V_T(0). \end{aligned}$$

This yields (5.25), thus completing the proof. ☐

Corollary 5.9. *Assume* **(F0)**, *and* **(L0)** – **(L2)**. *Then, we have that*

$$\alpha(L) = L(x^*, 0).$$

Proof. First, we recall that

$$\alpha(L) = \lim_{T\to\infty} \frac{1}{T} V_T(0) = \lim_{T\to\infty} \frac{1}{T} \inf_{(\gamma,u)\in\Gamma^{0\to}_{0,T}} \int_0^T L(\gamma(s), u(s))\, ds.$$

So, taking $(\gamma_x, u_x) \in \Gamma_{0,T}^{0\to}$ optimal for $V_T(0)$ we obtain

$$\alpha(L) = \lim_{T\to\infty} \frac{1}{T} \int_0^T L(\gamma_x(s), u_x(s))\, ds \geq \lim_{T\to\infty} \int_0^T L(x^*, 0)\, ds = L(x^*, 0)$$

since, by assumption **(L0)** and **(L2)**, we have that $L(x, u) \geq L(x^*, 0)$ for any $(x, u) \in \mathbb{R}^d \times \mathbb{R}^m$.

On the other hand, we observe that, owing to Theorem 5.8, the value of $\alpha(L)$ could be computed replacing 0 in (5.25) with any other point of $\mathbb{R}^d$. So,

$$\alpha(L) = \lim_{T\to\infty} \frac{1}{T} V_T(x^*).$$

This implies that

$$\alpha(L) = \lim_{T\to\infty} \frac{1}{T} \inf_{(\gamma,u)\in\Gamma_{0,T}^{x^*\to}} \int_0^T L(\gamma(s), u(s))\, ds \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T L(x^*, 0)\, ds = L(x^*, 0)$$

which yields the conclusion. □

Remark 5.10. Note that in view of Theorem 5.8 we have that

$$\lim_{T\to\infty} \sup_{x\in\overline{B}_R} \left| \frac{1}{T} \inf_{(\gamma,u)\in\Gamma_{0,T}^{x\to}} \int_0^T \big(L(\gamma(s), u(s)) - \alpha(L)\big)\, ds \right| = 0, \quad \forall\, R \geq 0.$$

Moreover, from Corollary 5.9 we deduce that

$$\min_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^m} L(x, u) - \alpha(L) = 0.$$

Therefore, by replacing L with $\widehat{L}(x, u) := L(x, u) - \alpha(L)$ one can reduce the analysis to the case of $\alpha(\widehat{L}) = \min_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^m} \widehat{L}(x, u) = 0$.

5.3.2 Application to Abel means

Now, we move to the analysis of the ergodic equation

$$c + H(x, D\chi) = 0, \quad x \in \mathbb{R}^d$$

showing the existence of viscosity solutions to such an equation by studying the limit behavior of solutions to the discounted problem

$$\lambda v_\lambda(x) + H(x, Dv_\lambda(x)) = 0, \quad x \in \mathbb{R}^d \tag{5.29}$$

as $\lambda \downarrow 0$. To do so, define the function

$$v_\lambda(x) = \inf_{(\gamma,u)\in\Gamma_{0,\infty}^{x\to}(e^{-\lambda t}dt)} \left\{ \int_0^{+\infty} e^{-\lambda t} L(\gamma(t), u(t))\, dt \right\}, \tag{5.30}$$

where

$$\Gamma_{0,\infty}^{x\to}(e^{-\lambda t}dt) := \Big\{(\gamma, u) \in L^\infty_{\text{loc}}(0, \infty; \mathbb{R}^d) \times L^2_{\text{loc}}(0, \infty; \mathbb{R}^m) : \\ (\gamma, u) \in \Gamma_{0,T}^{x\to} \;\; \forall\, T > 0, \text{ and } \int_0^\infty e^{-\lambda t} |u(t)|^2\, dt < \infty\Big\}.$$

Hereafter, we assume the following.

(L2') There exists a compact set $\mathcal{K} \subset \mathbb{R}^d$ such that

$$\min_{x \in \mathcal{K}} L(x,0) = 0, \quad \text{and} \quad \inf_{x \in \mathbb{R}^d \setminus \mathcal{K}} L(x,0) > 0.$$

We recall that in view of Remark 5.10 assumption **(L2')** is not restrictive and, moreover, by Corollary 5.9 we have that $\alpha(L) = 0$. Furthermore, **(L2')** stands for the corresponding of **(L2)** given so far.

Note that $v_\lambda(x) \geq 0$ for any $x \in \mathbb{R}^d$. Then v_λ is the continuous viscosity solution of (5.29).

Proposition 5.11. *Assume* **(F0)**, *and* **(L0)** – **(L2')**. *Then, for any* $R \geq 0$ *we have that:*

(i) $\{\lambda v_\lambda\}_{\lambda > 0}$ *is equibounded on* $\overline{B}_R$*;*

(ii) *there exists a constant* $C_R \geq 0$ *such that*

$$|v_\lambda(x) - v_\lambda(y)| \leq C_R d_{\text{SR}}(x,y), \quad \forall\, x, y \in \overline{B}_R. \tag{5.31}$$

Remark 5.12. *Recalling that* r_R *is the uniform degree of nonholonomy of the distribution* $\{f_i\}_{i=1,\dots,m}$ *associated with the compact* $\overline{B}_R$*, Corollary 1.7 and* (5.31) *yield*

$$|v_\lambda(x) - v_\lambda(y)| \leq C_R \widetilde{c}_2 |x-y|^{\frac{1}{r_R}} \quad \forall\, x, y \in \overline{B}_R.$$

Proof of Proposition 5.11: Let $R \geq 0$ and let $x \in \overline{B}_R$. Taking $(\bar{\gamma}, \bar{u}) \in \Gamma_{0,\infty}^{x \to}(e^{-\lambda t} dt)$ such that $(\bar{\gamma}(t), \bar{u}(t)) \equiv (x, 0)$, by **(L1)** we get

$$\begin{aligned} \lambda v_\lambda(x) &\leq \lambda \int_0^{+\infty} e^{-\lambda t} L(x,0)\, dt \\ &\leq \beta(R) \int_0^{+\infty} \lambda e^{-\lambda t}\, dt = \beta(R). \end{aligned}$$

On the other hand, by **(L2')** we have that

$$\lambda v_\lambda(x) \geq 0.$$

Thus, for any $\lambda > 0$ we conclude that

$$\lambda |v_\lambda(x)| \leq \beta(R), \quad \forall\, x \in \overline{B}_R.$$

In order to prove (ii), for any fixed x, $y \in \overline{B}_R$ set $\delta = d_{\text{SR}}(x,y)$. Let $(\bar{\gamma}_y, \bar{u}_y) \in \Gamma_{0,\delta}^{y \to x}$ be a solution of (1.5). Let $(\gamma_x, u_x) \in \Gamma_{0,+\infty}^{x \to}(e^{-\lambda t} dt)$ be such that

$$\int_0^\infty e^{-\lambda t} L(\gamma_x(t), u_x(t))\, dt \leq v_\lambda(x) + \lambda.$$

Define a new control

$$\widehat{u}_y(t) = \begin{cases} \bar{u}_y(t), & t \in [0,\delta] \\ u_x(t-\delta), & t \in (\delta, +\infty), \end{cases}$$

and so $(\widehat{\gamma}_y, \widehat{u}_y) \in \Gamma_{0,\infty}^{y\to}(e^{-\lambda t}dt)$. Then, we have that

$$\begin{aligned}
&v_\lambda(y) - v_\lambda(x) \\
&\leq \int_0^\delta e^{-\lambda t} L(\bar{\gamma}_y(t), \bar{u}_y(t))\, dt + \int_\delta^{+\infty} e^{-\lambda t} L(\gamma_x(t-\delta), u_x(t-\delta))\, dt \\
&- \int_0^{+\infty} e^{-\lambda t} L(\gamma_x(t), u_x(t))\, dt + \lambda \\
&= \int_0^\delta e^{-\lambda t} L(\bar{\gamma}_y(t), \bar{u}_y(t))\, dt + (e^{-\lambda\delta} - 1) \int_0^{+\infty} e^{-\lambda s} L(\gamma_x(s), u_x(s))\, ds + \lambda \\
&= \int_0^\delta e^{-\lambda t} L(\bar{\gamma}_y(t), \bar{u}_y(t))\, dt + \big(-\delta\lambda + \mathrm{o}(\delta\lambda)\big)(v_\lambda(x) + \lambda) + \lambda
\end{aligned}$$

where

$$\lim_{q \to 0} \frac{\mathrm{o}(q)}{q} = 0.$$

By point (i) we have that $\delta\lambda v_\lambda(x) \leq \delta\beta(R)$ and for $\lambda \leq 1$ we obtain $o(\delta\lambda) \leq \delta$. Moreover, by Lemma 5.1 we know that

$$|\bar{\gamma}_y(t)| \leq (|y| + c_f\delta)e^{c_f\delta} =: \Lambda(R), \quad \forall\, t \in [0, \delta]$$

since $\|\bar{u}_y\|_{\infty,[0,\delta]} \leq 1$. Thus, by **(L1)** we deduce that

$$\int_0^\delta e^{-\lambda t} L(\bar{\gamma}_y(t), \bar{u}_y(t))\, dt \leq \int_0^\delta \beta(|\bar{\gamma}_y(t)|)(1 + |\bar{u}_y(t)|^2)\, dt \leq 2\delta\beta(\Lambda(R)).$$

Therefore, setting $C_R = 2\beta(\Lambda(R))$ we obtain (5.31) recalling that $\delta = d_{\mathrm{SR}}(x, y)$. □

Note that, the above proof fails for general control systems, i.e., of the form (5.1), under the assumption **(LUGC)** since, a priori, T_R might not be of the order of $|x - y|$.

Theorem 5.13 (**Existence of correctors**). *Assume* **(F0)**, *and* **(L0)** – **(L2')**. *Then there exists a continuous function* $\chi : \mathbb{R}^d \to \mathbb{R}$ *and a sequence* $\lambda_n \downarrow 0$ *such that, for any* $R \geq 0$,

$$\lim_{n\to\infty} v_{\lambda_n}(x) = \chi(x), \quad \textit{uniformly on } \overline{B}_R.$$

Moreover, we have that:

(i) $\chi(x) \geq 0$, $\chi(x^*) = 0$ *and* χ *is locally Lipschitz continuous w.r.t.* d_{SR}, *that is, for any* $R \geq 0$ *there exists a constant* $\ell_R \geq 0$ *such that*

$$|\chi(x) - \chi(y)| \leq \ell_R d_{\mathrm{SR}}(x, y), \quad \forall\, x, y \in \overline{B}_R. \tag{5.32}$$

(ii) χ *is a viscosity solution of the ergodic Hamilton-Jacobi equation*

$$H(x, D\chi(x)) = 0 \quad (x \in \mathbb{R}^d). \tag{5.33}$$

Proof. First, we observe that, by an adaptation of [8, Theorem 5] (see Theorem 5.21 in Section 5.5.1), we have that

$$0 = \lim_{T\to+\infty} \frac{1}{T} V^T(x) = \lim_{\lambda\to 0} \lambda v_\lambda(x) \tag{5.34}$$

locally uniformly in space. We recall that $v_\lambda(x)$ is a continuous viscosity solution of

$$\lambda v_\lambda(x) + H(x, Dv_\lambda(x)) = 0 \quad (x \in \mathbb{R}^d)$$

Since $v_\lambda(x^*) = 0$, by Proposition 5.11 we deduce that $\{v_\lambda\}_{\lambda>0}$ is equibounded and equicontinuous. So, applying the Ascoli-Arzelá Theorem and a diagonal argument we deduce that there exists a sequence $\lambda_n \downarrow 0$ such that $\{v_{\lambda_n}(x)\}_{n\in\mathbb{N}}$ is locally uniformly convergent, i.e., for any $R \geq 0$

$$\lim_{n\to\infty} v_{\lambda_n}(x) =: \chi(x) \quad \text{uniformly on} \quad \overline{B}_R.$$

Hence, from **(L2')** we immediately deduce that $\chi(x) \geq 0$ and, again, since $v_\lambda(x^*) = 0$ we get $\chi(x^*) = 0$. Furthermore, from (5.31) we get (5.32). Finally, the stability of viscosity solutions ensures that χ is a solution of (5.33), which proves (ii). □

Definition 5.14 (Critical equation and critical solutions). *The equation*

$$H(x, D\chi(x)) = 0 \quad (x \in \mathbb{R}^d) \tag{5.35}$$

is called the critical *(or,* ergodic*) Hamilton-Jacobi equation. A continuous function χ is called a* critical subsolution *(resp.* supersolution*) if it is a viscosity subsolution (resp. supersolution) of* (5.35) *and a* critical solution *if it is both a subsolution and a supersolution.*

5.4 Representation formula

In this last section, we construct a critical solution that can be represented as the value function of a sub-Riemannian optimal control problem. Such a solution, which is useful to develop the Aubry-Mather theory in the sub-Riemannian case, will be obtained as the asymptotic limit as $t \to \infty$ of the Lax-Oleinik semigroup, applied to χ given by Theorem 5.13.

We begin by giving the definition of dominated functions.

Definition 5.15 (Dominated functions). *Let a, $b \in \mathbb{R}$ such that $a < b$ and let x, $y \in \mathbb{R}^d$. Let ϕ be a continuous function on $\mathbb{R}^d$. We say that ϕ is dominated by $L - c$, and we denote this by $\phi \prec L - c$, if for any trajectory-control pair $(\gamma, u) \in \Gamma_{a,b}^{x\to y}$ we have that*

$$\phi(y) - \phi(x) \leq \int_a^b L(\gamma(s), u(s))\, ds - c\,(b - a).$$

Let us introduce, now, the following class of functions

$$\mathcal{S} = \left\{ \varphi \in C(\mathbb{R}^d) : \varphi(x) \geq 0 \ \forall\, x \in \mathbb{R}^d,\ \varphi \prec L \right\}$$

endowed with the topology induced by the uniform convergence on compact sets. Then, for any $x \in \mathbb{R}^d$, any $t \geq 0$ and any $\varphi \in \mathcal{S}$ define the functional

$$\mathcal{F}_\varphi : \Gamma_{0,t}^{\to x} \to \mathbb{R}$$

as

$$\mathcal{F}_\varphi(\gamma, u) = \varphi(\gamma(0)) + \int_0^t L(\gamma(s), u(s))\, ds$$

and

$$T_t\varphi(x) = \inf_{(\gamma,u)\in\Gamma_{0,t}^{\to x}} \left\{ \varphi(\gamma(0)) + \int_0^t L(\gamma(s), u(s))\, ds \right\}. \tag{5.36}$$

Before proceeding to derive several properties of $T_t\varphi$, including the fact that $T_t\varphi(x) \geq 0$, we first show that the class $\mathcal{S}$ is non-empty.

Lemma 5.16. *Assume* **(F0)**, *and* **(L0)** – **(L2')**. *Then, the function* χ *constructed in Theorem 5.13 belongs to* $\mathcal{S}$.

Proof. Let χ be the critical solution given in Theorem 5.13, i.e.,

$$\chi(x) = \lim_{n\to\infty} v_{\lambda_n}(x)$$

where the limit is uniform on compact subsets of $\mathbb{R}^d$. Recall that

$$v_\lambda(x) = \inf_{(\gamma,u)\in\Gamma^{x\to}_{0,\infty}(e^{\lambda t}\ dt)} \int_0^\infty e^{-\lambda t} L(\gamma(t), u(t))\ dt.$$

Next, we show that $\chi \in \mathcal{S}$. From Theorem 5.13 we know that $\chi(x) \geq 0$ for any $x \in \mathbb{R}^d$. Hence, we only need to prove that $\chi \prec L$. To do so, let $R \geq 0$ and let $x, y \in \overline{B}_R$. Fix $a, b \in \mathbb{R}$ and let $(\gamma, u) \in \Gamma^{x\to y}_{a,b}$. Let $(\gamma_y, u_y) \in \Gamma^{y\to}_{0,\infty}(e^{-\lambda t}\ dt)$ be λ-optimal for $v_\lambda(y)$, that is,

$$\int_0^\infty e^{-\lambda t} L(\gamma_y(t), u_y(t))\ dt \leq v_\lambda(y) + \lambda.$$

and define the control

$$\widetilde{u}(t) = \begin{cases} u(t+a), & t \in [0, b-a] \\ u_y(t-a+b), & t \in (b-a, \infty). \end{cases}$$

Then, $(\widetilde{\gamma}, \widetilde{u}) \in \Gamma^{x\to}_{0,\infty}$

$$\begin{aligned}
v_\lambda(x) - v_\lambda(y) \leq & \int_0^\infty e^{-\lambda t} L(\widetilde{\gamma}(t), \widetilde{u}(t))\ dt - \int_0^\infty e^{-\lambda t} L(\gamma_y(t), u_y(t))\ dt + \lambda \\
= & \int_0^{b-a} e^{-\lambda t} L(\gamma(t+a), u(t+a))\ dt \\
+ & \int_{b-a}^\infty e^{-\lambda t} L(\gamma_y(t), u_y(t))\ dt - \int_0^\infty e^{-\lambda t} L(\gamma_y(t), u_y(t))\ dt + \lambda \\
\leq & \int_a^b L(\gamma(t), u(t))\ dt + \left(e^{-\lambda(b-a)} - 1\right) \int_0^\infty e^{-\lambda t} L(\gamma_y(t), u_y(t))\ dt + \lambda \\
= & \int_a^b L(\gamma(t), u(t))\ dt - (b-a)\lambda v_\lambda(y)(1 + o(1)) + \lambda.
\end{aligned}$$

Therefore, since $\lambda = \lambda_n$ as $n \to \infty$ from the previous estimate we get

$$\chi(x) - \chi(y) \leq \int_a^b L(\gamma(t), u(t))\ dt$$

which completes the proof. □

Theorem 5.17 (Lax-Oleinik semigroup). *Assume* **(F0)**, *and* **(L0)** – **(L2')**. *The following holds.*

1. *For any* $\varphi \in \mathcal{S}$ *there exists a function* $N_\varphi : \mathbb{R}^d \to \mathbb{R}$, *which is bounded on compact sets, such that for any* $(t, x) \in [0, \infty) \times \mathbb{R}^d$ *there exists a trajectory-control pair* $(\gamma_x, u_x) \in \Gamma^{\to x}_{0,t}$ *such that*
$$\mathcal{F}_\varphi(\gamma_x, u_x) \leq N_\varphi(x). \tag{5.37}$$

2. For any $\varphi \in \mathcal{S}$ the following holds. For any $R \geq 0$ there exists a nondecreasing function $C_\varphi : [0,\infty) \to [0,\infty)$ such that for any $(t,x) \in [0,\infty) \times \overline{B}_R$ and any $(\gamma, u) \in \Gamma_{0,t}^{\to x}$ satisfying (5.37) we have that

$$d_{\mathrm{SR}}(x, \gamma(0)) \leq C_\varphi(R) := \beta(R)D(R) + \max_{x \in \overline{B}_R} \varphi(x). \tag{5.38}$$

3. For any $x \in \mathbb{R}^d$, any $t \geq 0$ and any $\varphi \in \mathcal{S}$ we have that $T_t\varphi(x) \geq 0$. Moreover, for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$ the infimum in (5.36) is attained.

4. For any $\varphi \in \mathcal{S}$ and any $c \in \mathbb{R}$ we have that $T_t(\varphi + c) = T_t\varphi + c$ for all $t \geq 0$.

5. T_t is a semigroup on $\mathcal{S}$, i.e., $T_t : \mathcal{S} \to \mathcal{S}$ and for any s, $t \geq 0$ and $\varphi \in \mathcal{S}$

$$T_0\varphi = \varphi, \quad T_s(T_t\varphi) = T_{s+t}\varphi.$$

6. T_t is continuous on $\mathcal{S}$ w.r.t. the topology induced by the uniform convergence on compact subsets.

Remark 5.18. We recall that, according to [2, Theorem 3.31], a set $\mathcal{K}$ is compact in $(\mathbb{R}^d, d_{\mathrm{SR}})$ if and only if $\mathcal{K}$ is compact in $\mathbb{R}^d$ w.r.t. the Euclidean distance.

Proof. We begin by proving (1). To do so, we consider two cases: first, we take $(t,x) \in [D(|x|), \infty) \times \mathbb{R}^d$ and, then, $(t,x) \in [0, D(|x|)) \times \mathbb{R}^d$. Recall that $D(\cdot)$ is defined in (5.7) and satisfies $\delta^*(x) \leq D(|x|)$.

Define the function $N_\varphi : \mathbb{R}^d \to \mathbb{R}$ as

$$N_\varphi(x) = \begin{cases} \varphi(x^*) + D(|x|)\beta(|x|), & (t,x) \in [D(|x|), \infty) \times \mathbb{R}^d \\ \varphi(x) + D(|x|)\beta(|x|), & (t,x) \in [0, D(|x|)) \times \mathbb{R}^d. \end{cases}$$

Note that, since $\varphi \in \mathcal{S}$ we deduce that N_φ is bounded on any compact subset of $\mathbb{R}^d$.

We now proceed with the first part of the proof, i.e., we show that for any $(t,x) \in [D(|x|), \infty) \times \mathbb{R}^d$ there exists $(\gamma_x, u_x) \in \Gamma_{0,t}^{\to x}$ such that

$$\mathcal{F}_\varphi(\gamma_x, u_x) \leq N_\varphi(x).$$

Let $(\gamma_0, u_0) \in \Gamma_{0,\delta^*(x)}^{x^* \to x}$ be optimal for (1.5) and define the control

$$u_x(s) = \begin{cases} 0, & s \in [0, t - \delta^*(x)) \\ u_0(s - t + \delta^*(x)), & s \in [t - \delta^*(x), t]. \end{cases}$$

so that $(\gamma_x, u_x) \in \Gamma_{0,t}^{x^* \to x}$. Then

$$\begin{aligned} \mathcal{F}_\varphi(\gamma_x, u_x) &= \varphi(x^*) + \int_{t-\delta^*(x)}^{t} L(\gamma_0(s - t + \delta^*(x)), u_0(s - t + \delta^*(x)))\, ds \\ &= \varphi(x^*) + \int_0^{\delta^*(x)} L(\gamma_0(s), u_0(s))\, ds \leq \varphi(x^*) + \int_0^{\delta^*(x)} L(\gamma_0(s), u_0(s))\, ds \end{aligned}$$

Let us estimate the rightmost term above. Recalling that $|u_0(s)| \leq 1$ for any $s \in [0, \delta^*(x)]$ we have that

$$|\gamma_0(t)| \leq (|x^*| + c_f\delta^*(x))e^{c_f\delta^*(x)} =: \Lambda(|x|), \quad \forall\, t \in [0, \delta^*].$$

Thus, we get

$$\int_0^{\delta^*(x)} L(\gamma_0(s), u_0(s))\, ds \le \delta^*(x)\beta(\Lambda(|x|)) \le D(|x|)\beta(\Lambda(|x|)).$$

Hence, we obtain

$$\mathcal{F}_\varphi(\gamma, u) \le \varphi(x^*) + D(|x|)\beta(\Lambda(|x|)) \tag{5.39}$$

which completes the proof of (1) for $(t, x) \in [D(|x|), \infty) \times \mathbb{R}^d$.

We now consider the case $(t, x) \in [0, D(|x|)) \times \mathbb{R}^d$. Let $(\gamma_x, u_x) \in \Gamma_{0,t}^{\to x}$ be defined as

$$u_x(s) = 0, \quad \gamma_x(s) \equiv x, \quad s \in [0, D(|x|)).$$

Then

$$\mathcal{F}_\varphi(x, 0) \le \ \varphi(x) + tL(x, 0) \le \varphi(x) + D(|x|)\beta(|x|). \tag{5.40}$$

This completes the proof of (1).

We proceed now with the proof of (2). In order to prove (5.38) we estimate from below $\mathcal{F}_\varphi(\gamma, u)$, for any $(\gamma, u) \in \Gamma_{0,t}^{\to x}$ satisfying (5.37), and then we combine such estimate with the definition of $N_\varphi(\cdot)$. In view of (1) we also analyze two cases: first, we show that the conclusion holds for any $(t, x) \in [D(|x|), \infty) \times \mathbb{R}^d$ and then we do the same for $(t, x) \in [0, D(|x|)) \times \mathbb{R}^d$.

Let $(t, x) \in [D(|x|), \infty) \times \mathbb{R}^d$ and let $(\gamma, u) \in \Gamma_{0,t}^{\to x}$ satisfy (5.37). Then, by (5.6) we have that

$$\mathcal{F}_\varphi(\gamma, u) \ge \varphi(\gamma(0)) + \frac{1}{2\ell_1}\int_0^t |u(s)|^2\, ds \ge \ \frac{1}{2\ell_1} d_{\mathrm{SR}}(x, \gamma(0))^2. \tag{5.41}$$

Therefore, combining (5.41) with (5.39) we have that

$$\frac{1}{2\ell_1} d_{\mathrm{SR}}(x, \gamma(0))^2 \le D(|x|)\beta(|x|) + \varphi(x^*) \tag{5.42}$$

which implies (5.38) for $(t, x) \in [D(|x|), \infty) \times \mathbb{R}^d$ by the continuity of φ.

Now, let $(t, x) \in [0, D(|x|)) \times \mathbb{R}^d$ and observe that inequality (5.41) still holds true. So, we combine such estimate with (5.40) to obtain

$$\frac{1}{2\ell_1} d_{\mathrm{SR}}(x, \gamma(0))^2 \le D(|x|)\beta(|x|) + \varphi(x) \tag{5.43}$$

which implies (5.38) for any $(t, x) \in [0, D(|x|)) \times \mathbb{R}^d$, again, by the continuity of φ. Hence, from (5.42) and (5.43) we get, for any $R \ge 0$, any $(t, x) \in [0, \infty) \times \overline{B}_R$ and any $(\gamma, u) \in \Gamma_{0,t}^{\to x}$ satisfying (5.37),

$$d_{\mathrm{SR}}(x, \gamma(0)) \le C_\varphi(R) := \beta(R)D(R) + \max_{x \in \overline{B}_R} \varphi(x).$$

Now, given $(t, x) \in [0, \infty) \times \mathbb{R}^d$, let $\varphi \in \mathcal{S}$ and let $(\gamma, u) \in \Gamma_{0,t}^{\to x}$ be optimal for $T_t\varphi(x)$. Then, by definition we have that

$$T_t\varphi(x) \ge \varphi(\gamma(0)) + \frac{1}{2\ell_1}\int_0^t |u(s)|^2\, ds \ge \ 0.$$

The existence of minimizing pairs $(\gamma, u) \in \Gamma_{0,t}^{\to x}$ follows by classical results in optimal control theory (see, for instance, [26, Theorem 7.4.4]). This completes the proof of (3). Then, (4) is a direct consequence of the definition of $T_t\varphi$.

In order to prove (5), from the previous point we know that for any $(t, x) \in [0, \infty) \times \mathbb{R}^d$ and any $\varphi \in \mathcal{S}$ we have that $T_t\varphi(x) \geq 0$. Moreover, the proof of the fact that $T_t\varphi(x) \prec L$ and of the semigroup property, the latter being based on the dynamic programming principle, is similar to the proof of [42, (1) of Proposition 3.3] and [41, Proposition 4.6.2.] and will be omitted here.

We finally show (6). Let $R \geq 0$, let $x \in \overline{B}_R$ and let $t \geq 0$. Let $\{\varphi_n\}_{n\in\mathbb{N}} \in \mathcal{S}$ and let $\varphi \in \mathcal{S}$ be such that $\varphi_n \to \varphi$ locally uniformly. Then, on the one hand, taking $(\gamma_x^\varphi, u_x^\varphi) \in \Gamma_{0,t}^{\to x}$ optimal for $T_t\varphi(x)$ we obtain

$$T_t\varphi_n(x) - T_t\varphi(x) \leq \varphi_n(\gamma_x^\varphi(0)) - \varphi(\gamma_x^\varphi(0)).$$

Hence, from (2) we deduce that there exists a constant $C_\varphi(R) \geq 0$ such that

$$T_t\varphi_n(x) - T_t\varphi(x) \leq \|\varphi_n(\cdot) - \varphi(\cdot)\|_{\infty, \overline{B}_{C_\varphi(R)}}. \tag{5.44}$$

Similarly, on the other hand, let $(\gamma_x^n, u_x^n) \in \Gamma_{0,t}^{\to x}$ optimal for $T_t\varphi_n(x)$. Then, from (2) there exists a constant $C_{\varphi_n}(R) \geq 0$ such that

$$T_t\varphi(x) - T_t\varphi_n(x) \leq \varphi(\gamma_x^n(0)) - \varphi_n(\gamma_x^n(0)) \leq \|\varphi_n(\cdot) - \varphi(\cdot)\|_{\infty, \overline{B}_{C_{\varphi_n}(R)}}. \tag{5.45}$$

Again in view of (2) by the locally uniform convergence of the sequence φ_n the constant $C_{\varphi_n}(R)$ can be chosen uniform w.r.t. $n \in \mathbb{N}$. Therefore, combining (5.44) and (5.45) the proof of (6) is complete. □

We call T_t the Lax-Oleinik semigroup, adapted to the sub-Riemannian systems. Now, recall that we are interested in finding a critical solution $\overline{\chi}$ such that

$$\overline{\chi}(x) = T_t\overline{\chi}(x), \quad \forall\, t \geq 0,\ \forall\, x \in \mathbb{R}^d.$$

Hereafter, we take χ a critical solution in $\mathcal{S}$ (the existence of which is guaranteed by Theorem 5.13). We will show that $T_t\chi(x)$ converges as the $t \to \infty$ to the function $\overline{\chi}$ we are looking for.

Proposition 5.19. *Assume* **(F0)**, *and* **(L0)** – **(L2')**. *Then, for any $R \geq 0$ we have that*

(i) $\{T_t\chi\}_{t\geq 0}$ *is equibounded on* $\overline{B}_R$*;*

(ii) $\{T_t\chi\}_{t\geq 1}$ *is equicontinuous on* $\overline{B}_R$.

Proof. In order to prove(i) we argue as in Theorem 5.17. Let $R \geq 0$, let $t \geq 0$ let $x \in \overline{B}_R$. Let $(\gamma_x, u_x) \in \Gamma_{0,t}^{\to x}$ be optimal for $T_t\chi(x)$. Since χ is Lipschitz continuous w.r.t. d_{SR} the following holds

$$\chi(x) \leq \ell_R d_{\mathrm{SR}}(x, 0), \quad \forall\, x \in \overline{B}_R. \tag{5.46}$$

Then, since $\chi \geq 0$ from (5.6) we obtain

$$T_t\chi(x) \geq \ \chi(\gamma(0)) + \frac{1}{2\ell_1}\int_0^t |u_x(s)|^2\, ds = \ \frac{1}{2\ell_1} d_{\mathrm{SR}}(x, \gamma_x(0))^2 \tag{5.47}$$

which is bounded by (2) in Theorem 5.17. Then, on the one hand, if $t \in [D(R), \infty)$ by (5.39) and (5.46) we obtain

$$T_t\chi(x) \leq \chi(x^*) + D(R)\beta(R) \leq \ \ell_{R_{\mathcal{K}}} d_{\mathrm{SR}}(x^*, 0) + D(R)\beta(R) \tag{5.48}$$

where $R_{\mathcal{K}}$ stands for the diameter of $\mathcal{K}$. On the other hand, if $t \in [0, D(R))$ by (5.40) and (5.46) we get

$$T_t\chi(x) \leq \chi(x) + D(R)\beta(R) \leq \ \ell_R d_{\mathrm{SR}}(x, 0) + D(R)\beta(R). \tag{5.49}$$

Hence, combining (5.47) with (5.48) and, also, (5.47) with (5.49) the proof of (i) is complete.

We proceed to show (ii), that is, the equicontinuity of $T_t\chi(x)$ for $t \geq 1$. Let $R \geq 0$, let x, $y \in \overline{B}_R$ and let $t \geq 1$.

To begin with, assume that $d_{\mathrm{SR}}(x, y) > 1$. Then, we have that

$$|T_t\chi(x) - T_t\chi(y)| \leq 2\|T_t\chi\|_{\infty, \overline{B}_R} \leq 2\|T_t\chi\|_{\infty, \overline{B}_R} d_{\mathrm{SR}}(x, y).$$

We now consider the other case, i.e., $d_{\mathrm{SR}}(x, y) \leq 1$. Let $(\gamma_0, u_0) \in \Gamma^{y \to x}_{0, d_{\mathrm{SR}}(x,y)}$ be optimal for (1.5) and let $(\gamma_y, u_y) \in \Gamma^{\to y}_{0,t}$ be optimal for $T_t\chi(y)$. Then, define the control

$$\widetilde{u}(s) = \begin{cases} u_y(s + d_{\mathrm{SR}}(x, y)), & s \in [0, t - d_{\mathrm{SR}}(x, y)] \\ u_0(s - t + d_{\mathrm{SR}}(x, y)), & s \in (t - d_{\mathrm{SR}}(x, y), t] \end{cases}$$

and call $\widetilde{\gamma}$ the corresponding trajectory, that is, $(\widetilde{\gamma}, \widetilde{u}) \in \Gamma^{\to x}_{0,t}$. Note that $\widetilde{u}$ can be used to estimate $T_t\chi(x)$ from above. We have that

$$\begin{aligned}
&T_t\chi(x) - T_t\chi(y) \\
&\leq \chi(\widetilde{\gamma}(0)) - \chi(\gamma_y(0)) + \int_0^t L(\widetilde{\gamma}(s), \widetilde{u}(s))\, ds - \int_0^t L(\gamma_y(s), u_y(s))\, ds \\
&= \chi(\gamma_y(d_{\mathrm{SR}}(x, y))) - \chi(\gamma_y(0)) + \int_0^t L(\gamma_y(s), u_y(s))\, ds \\
&- \int_0^{d_{\mathrm{SR}}(x,y)} L(\gamma_y(s), u_y(s))\, ds + \int_0^{d_{\mathrm{SR}}(x,y)} L(\gamma_0(s), u_0(s))\, ds - \int_0^t L(\gamma_y(s), u_y(s))\, ds \\
&= \chi(\gamma_y(d_{\mathrm{SR}}(x, y))) - \chi(\gamma_y(0)) - \int_0^{d_{\mathrm{SR}}(x,y)} L(\gamma_y(s), u_y(s))\, ds + \int_0^{d_{\mathrm{SR}}(x,y)} L(\gamma_0(s), u_0(s))\, ds.
\end{aligned} \tag{5.50}$$

We estimate first the integral terms. By (5.6) we immediately obtain

$$\int_0^{d_{\mathrm{SR}}(x,y)} L(\gamma_y(s), u_y(s))\, ds \geq 0.$$

Moreover, since $\|u_0\|_{\infty, [0, d_{\mathrm{SR}}(x,y)]} \leq 1$ we have that

$$|\gamma_0(t)| \leq (|x| + c_f d_{\mathrm{SR}}(x, y)) e^{c_f d_{\mathrm{SR}}(x,y)}, \quad \forall\, t \in [0, d_{\mathrm{SR}}(x, y)].$$

So, we get

$$\int_0^{d_{\mathrm{SR}}(x,y)} L(\gamma_0(s), u_0(s))\, ds \leq d_{\mathrm{SR}}(x, y)\big(|x| + c_f d_{\mathrm{SR}}(x, y)\big) e^{c_f d_{\mathrm{SR}}(x,y)}.$$

Combining both inequalities we have that

$$\begin{aligned} &-\int_0^{d_{\mathrm{SR}}(x,y)} L(\gamma_y(s),u_y(s))\ ds+\int_0^{d_{\mathrm{SR}}(x,y)} L(\gamma_0(s),u_0(s))\ ds \\ &\leq d_{\mathrm{SR}}(x,y)\big(|x|+c_f d_{\mathrm{SR}}(x,y)\big)e^{c_f d_{\mathrm{SR}}(x,y)}. \end{aligned} \tag{5.51}$$

Therefore, in order to obtain the results we need to estimate

$$\chi(\gamma_y(d_{\mathrm{SR}}(x,y)))-\chi(\gamma_y(0)). \tag{5.52}$$

First, we claim that $|\gamma_y(0)|$ and $|\gamma_y(d_{\mathrm{SR}}(x,y))|$ are bounded. Indeed, observe that from (2) in Theorem 5.17 and the equivalence of the sub-Riemannian topology with the Euclidean one we deduce that

$$|\gamma_y(0)|\leq 2\max\{R,C_\chi(R)\}.$$

Moreover, by Lemma 5.2 we know that

$$|\gamma_y(s)|\leq \kappa(\|u_y\|_2,1)(1+|\gamma_y(0)|),\quad \forall\ s\in[0,d_{\mathrm{SR}}(x,y)].$$

So, in particular,

$$|\gamma_y(d_{\mathrm{SR}}(x,y))|\leq\ \kappa(\|u_y\|_2,1)(1+|\gamma_y(0)|).$$

We claim that $\|u_y\|_{2,[0,d_{\mathrm{SR}}(x,y)]}$ is bounded by a constant that only depends on R. Indeed, from (i) we know that $T_t\chi(y)$ is locally uniformly bounded and by (5.6) we know that

$$T_t\chi(y)\geq\ \chi(\gamma_y(0))+\frac{1}{2\ell_1}\int_0^t |u_y(s)|^2\ ds\geq\ \frac{1}{2\ell_1}\int_0^t |u_y(s)|^2\ ds.$$

Thus, we obtain

$$\frac{1}{2\ell_1}\int_0^t |u_y(s)|^2\ ds\leq \|T_t\chi(y)\|_{\infty,\overline{B}_R}$$

and this completes the proof of the claim since $T_t\chi$ is locally equibounded by (i). For simplicity of notation, let $R_y\geq 0$ be such that

$$|\gamma_y(d_{\mathrm{SR}}(x,y))|\leq R_y,\quad |\gamma_y(0)|\leq R_y.$$

Moreover, we denote by $r_y\geq 1$ the degree of nonholonomy associated with the compact set $\overline{B}_{R_y}$.

Hence, going back to by the Lipschitz continuity of χ w.r.t. d_{SR} we get

$$\chi(\gamma_y(d_{\mathrm{SR}}(x,y)))-\chi(\gamma_y(0))\leq \ell_{R_y} d_{\mathrm{SR}}(\gamma_y(d_{\mathrm{SR}}(x,y)),\gamma_y(0)).$$

Then, by Corollary 1.7 we have that

$$\chi(\gamma_y(d_{\mathrm{SR}}(x,y)))-\chi(\gamma_y(0))\leq \widetilde{c}_2|\gamma_y(d_{\mathrm{SR}}(x,y))-\gamma_y(0)|^{\frac{1}{r_y}} \tag{5.53}$$

where $\widetilde{c}_2$ depends only on R_y. Next, from Lemma 5.2 we have that

$$|\gamma_y(d_{\mathrm{SR}}(x,y))-\gamma_y(0)|\leq \kappa\big(\|u_y\|_{2,[0,d_{\mathrm{SR}}(x,y)]},d_{\mathrm{SR}}(x,y)\big)(1+|\gamma_y(0)|)d_{\mathrm{SR}}(x,y)^{\frac{1}{2}}.$$

Hence, we get that there exists a constant $C'_R\geq 0$ such that

$$\chi(\gamma_y(d_{\mathrm{SR}}(x,y))-\chi(\gamma_y(0))\leq C'_R d_{\mathrm{SR}}(x,y)^{\frac{1}{2r_y}}. \tag{5.54}$$

Therefore, combining (5.50), (5.51) and (5.54) we obtain

$$T_t\chi(x)-T_t\chi(y)\leq\ d_{\mathrm{SR}}(x,y)\big(|x|+c_f d_{\mathrm{SR}}(x,y)\big)e^{c_f d_{\mathrm{SR}}(x,y)}+C'_R d_{\mathrm{SR}}(x,y)^{\frac{1}{2r_y}}.$$

Finally, exchanging the role of x and y the proof of the equicontinuity is complete. □

Theorem 5.20. *Assume* **(F0)**, *and* **(L0)** – **(L2')**. *Then, there exists a continuous function* $\overline{\chi}$ *such that*

$$\lim_{t\to\infty} T_t\chi(x) = \overline{\chi}(x) \tag{5.55}$$

uniformly on $\overline{B}_R$ *for any* $R \geq 0$. *Moreover, we have that*

$$\overline{\chi}(x) = T_t\overline{\chi}(x), \quad t \geq 0, \ \ x \in \mathbb{R}^d$$

and $\overline{\chi}$ *satisfies*

$$H(x, D\overline{\chi}(x)) = 0, \quad (x \in \mathbb{R}^d) \tag{5.56}$$

in the viscosity sense.

Proof. In order to prove the existence of the limit in (5.55), we first show that the map

$$t \mapsto T_t\chi(x)$$

is nondecreasing for any $x \in \mathbb{R}^d$. Indeed, we have that

$$T_t\chi(x) \leq T_t\left(T_s\chi(x)\right) = T_{t+s}\chi(x)$$

where the inequality holds since $\chi \prec L$. This implies that

$$T_t\chi(x) \leq T_{t+s}\chi(x)$$

and so we have that

$$T_t\chi(x) \leq T_{t'}\chi(x), \quad \forall\, t \leq t'.$$

Therefore, since by Proposition 5.19 we have that $T_t\chi$ is locally equibounded it follows that the pointwise limit

$$\lim_{t\to\infty} T_t\chi(x)$$

exists for all $x \in \mathbb{R}^d$. Moreover, again by Proposition 5.19 we know that the family $T_t\chi$ is locally equicontinuous. Thus the above limit is locally uniform.

Let us set

$$\overline{\chi}(x) = \lim_{t\to\infty} T_t\chi(x), \quad \forall\, x \in \mathbb{R}^d$$

Next, we show that $\overline{\chi}(x) = T_t\overline{\chi}(x)$ for any $x \in \mathbb{R}^d$ and any $t \geq 0$. Indeed, let $s \geq 0$. Then

$$T_s\overline{\chi}(x) = \lim_{t\to\infty} T_s\big(T_t\chi(x)\big) = \lim_{t\to\infty} T_{s+t}\chi(x)$$

where we have used the continuity of the semigroup T_t and property (4) in Theorem 5.17. Hence, we get

$$T_s\overline{\chi}(x) = \lim_{t\to\infty} T_{s+t}\chi(x) = \overline{\chi}(x).$$

So, we have that

$$\overline{\chi}(x) = \ T_t\overline{\chi}(x) = \inf_{(\gamma,u)\in\Gamma_{0,t}^{\to x}} \left\{\overline{\chi}(x) + \int_0^t L(\gamma(s), u(s))\, ds\right\} \tag{5.57}$$

The proof of the fact that from (5.57) the function $\overline{\chi}$ solves (5.56) in the viscosity sense is similar to the proof of [42, Proposition 5.1, Proposition 5.2]. □

5.5 Appendix

5.5.1 Abelian-Tauberian Theorem

In this appendix, we give a new formulation of the Abelian-Tauberian Theorem, stated in [8, Theorem 5], tailored for the proof of Theorem 5.13.

Theorem 5.21. *Let $\psi(t,x)$ be the solution of*

$$\begin{cases} \partial_t \psi(t,x) + H(x, D\psi(t,x)) = 0, & (t,x) \in [0,T] \times \mathbb{R}^d \\ \psi(T,x) = 0, & x \in \mathbb{R}^d. \end{cases}$$

For any $\lambda > 0$, let $\psi_\lambda(x)$ be the solution of

$$\lambda \psi(x) + H(x, D\psi(x)) = 0, \quad x \in \mathbb{R}^d.$$

Then:

(i) if $\{\lambda \psi_\lambda(\cdot\ )\}_{\lambda>0}$ locally uniformly converges to a constant $\bar{d} \in \mathbb{R}$ as $\lambda \downarrow 0$, then $\{\frac{1}{T}\psi(0,\cdot\ )\}_{T>0}$ locally uniformly converges to $\bar{d}$ as $T \to \infty$;

(ii) if $\{\frac{1}{T}\psi(0,\cdot\ )\}_{T>0}$ locally uniformly converges to a constant $\bar{d} \in \mathbb{R}$ as $T \to \infty$, then $\{\lambda \psi_\lambda(\cdot)\}_{\lambda>0}$ locally uniformly converges to $\bar{d}$ as $\lambda \downarrow 0$.

This result can be proved arguing as in [8, Theorem 5] keeping in mind the following differences:

1. the uniform convergence on the full space $\overline{\Omega}$ is replaced by the locally uniform convergence on $\mathbb{R}^d$;
2. whenever the boundedness assumption on L is used in [8] one here has to invoke the boundedness of optimal pairs (γ, u) in $L^\infty(0,T;\mathbb{R}^d) \times L^2(0,T;\mathbb{R}^m)$.

Chapter 6

Aubry-Mather Theory for sub-Riemannian control systems

6.1 Settings and assumptions

For $m \in \mathbb{N}$ and $i = 1, \dots, m$, let

$$f_i : \mathbb{R}^d \to \mathbb{R}^d$$

and

$$u_i : [0, \infty) \to \mathbb{R}$$

be smooth vector fields and measurable controls, respectively, and consider the following controlled dynamics of sub-Riemannian type

$$\dot{\gamma}(t) = \sum_{i=1}^{m} f_i(\gamma(t))u_i(t) = F(\gamma(t))U(t), \quad t \in [0, +\infty) \tag{6.1}$$

where $F(x) = [f_1(x)| \dots |f_m(x)]$ is an $d \times m$ real matrix and $U(t) = (u_1(t), \dots, u_m(t))^{\star}$[1].

For any s_0, $s_1 \in \mathbb{R}$ such that $s_0 < s_1$ and x, $y \in \mathbb{R}^d$ we set

$$\begin{aligned}
\Gamma_{s_0,s_1}^{x\to} &= \{(\gamma, u) \in \mathrm{AC}([s_0, s_1]; \mathbb{R}^d) \times L^2(s_0, s_1; \mathbb{R}^m) : \dot{\gamma}(t) = F(\gamma(t))u(t),\ \gamma(s_0) = x\}, \\
\Gamma_{s_0,s_1}^{\to y} &= \{(\gamma, u) \in \mathrm{AC}([s_0, s_1]; \mathbb{R}^d) \times L^2(s_0, s_1; \mathbb{R}^m) : \dot{\gamma}(t) = F(\gamma(t))u(t),\ \gamma(s_1) = y\}, \\
\Gamma_{s_0,s_1}^{x\to y} &= \Gamma_{s_0,s_1}^{x\to} \cap \Gamma_{s_0,s_1}^{\to y}.
\end{aligned}$$

Throughout the paper we assume the vector fields f_i to satisfy the following.

(F0) There exists an integer $r_0 \geq 1$ such that $f_i \in C^{r_0 - 1}(\mathbb{R}^d)$, for any $i = 1, \dots, m$, and

$$\Delta^{r_0}(x) = \mathbb{R}^d, \quad \forall x \in \mathbb{R}^d;$$

(F1) There exists a constant $c_f \geq 1$ such that for any $i = 1, \dots, m$

$$|f_i(x)| \leq c_f(1 + |x|), \quad \forall\, x \in \mathbb{R}^d; \tag{6.2}$$

(F2) $m \leq d$, $F \in C^{1,1}_{loc}(\mathbb{R}^d)$ and for any $x \in \mathbb{R}^d$ the matrix $F(x)$ has full rank m.

Observe that, from **(F2)** we have that the vector fields f_i are linearly independent.

By **(F1)** and Gronwall inequality we get the following estimate on solutions of (6.1).

[1] $(u_1, \dots, u_m)^{\star}$ denotes the transpose of $(u_1, \dots, u_m)$

Lemma 6.1. *Let $x \in \mathbb{R}^d$, $t \geq 0$ and $(\gamma, u) \in \Gamma_{0,t}^{x \rightarrow}$. If $u \in L^\infty(0,t;\mathbb{R}^m)$ then we have that*

$$|\gamma(s)| \leq (|x| + c_f \|u\|_\infty t) e^{c_f \|u\|_\infty t}, \quad \forall\ s \in [0,t].$$

We now state the assumptions on the Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$.

(**L0**) $L \in C^2(\mathbb{R}^d \times \mathbb{R}^m)$ is reversible, that is $L(x,u) = L(x,-u)$ for any $(x,u) \in \mathbb{R}^d \times \mathbb{R}^m$;

(L1) There exists a positive constant ℓ_1, C_1 such that

$$D_u^2 L(x,u) \geq \frac{1}{\ell_1}, \quad (x,u) \in \mathbb{R}^d \times \mathbb{R}^m$$
$$|D_x L(x,u)| \leq C_1(1 + |u|^2), \quad (x,u) \in \mathbb{R}^d \times \mathbb{R}^m;$$

and L is locally semiconcave in space uniformly w.r.t. $u \in \mathbb{R}^m$.

(**L2**) There exists $\ell_2 \geq 0$ such that

$$L(x,u) \leq \ell_1 |u|^2 + \ell_2, \quad (x,u) \in \mathbb{R}^d \times \mathbb{R}^m$$

(**L3**) There exists a compact set $\mathcal{K}_L \subset \mathbb{R}^d$ and a constant $\delta_L > 0$ such that

$$\inf_{x \in \mathbb{R}^d \setminus \mathcal{K}_L} L(x,0) \geq \delta_L + \min_{x \in \mathcal{K}_L} L(x,0); \tag{6.3}$$

Note that by **(L0)**, **(L1)** and **(L3)** we obtain

$$L(x,u) \geq \frac{1}{2\ell_1}|u|^2 + L(x^*,0), \quad \forall\ (x,u) \in \mathbb{R}^d \times \mathbb{R}^m \tag{6.4}$$

where $x^* \in \mathcal{K}_L$ is such that

$$L(x^*,0) = \min_{x \in \mathcal{K}_L} L(x,0).$$

Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be the Hamiltonian associated with L, that is,

$$H(x,p) = \sup_{u \in \mathbb{R}^m} \left\{ \sum_{i=1}^m u_i \langle p, f_i(x) \rangle - L(x,u) \right\}, \quad \forall\ (x,p) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{6.5}$$

Since the subject of this Chapter is deeply connected with the problem studied in Chapter 5, we first recall the main results of such Chapter. Consider the following minimization problem: for any $T > 0$ and any $x \in \mathbb{R}^d$

$$\text{to minimize } \int_0^T L(\gamma(s), u(s))\, ds \text{ over all } (\gamma,u) \in \Gamma_{0,T}^{x \rightarrow} \tag{6.6}$$

and define the function $V_T : \mathbb{R}^d \to \mathbb{R}$ by

$$V_T(x) = \inf_{(\gamma,u) \in \Gamma_{0,T}^{x \rightarrow}} \int_0^T L(\gamma(s), u(s))\, ds, \quad \forall\ x \in \mathbb{R}^d. \tag{6.7}$$

For any $x \in \mathbb{R}^d$ we say that a trajectory-control pair $(\gamma,u) \in \Gamma_{0,T}^{x \rightarrow}$ is optimal if it solves (6.6). Note that, the existence of optimal trajectory-control pairs for (6.6) is a well-known result (see, e.g., [26, Theorem 7.4.4]).

Then, we know that for any $R \geq 0$ there exist two constants P_R, $Q_R \geq 0$ such that for any $x \in \overline{B}_R$, any $T \geq d_{\mathrm{SR}}(x, x^*)$, and any optimal pair $(\gamma_x, u_x) \in \Gamma^{x \to}_{0,T}$ for (6.6) the following holds:

$$\int_0^T |u_x(t)|^2 \, dt \leq P_R \tag{6.8}$$

and

$$|\gamma_x(t)| \leq Q_R, \quad \forall\, t \in [0, T]. \tag{6.9}$$

Moreover, there exists $\alpha(L) \in \mathbb{R}$ such that for any $R \geq 0$

$$\lim_{T \to +\infty} \frac{1}{T} V_T(x) = \alpha(L), \quad \text{uniformly on } \overline{B}_R \tag{6.10}$$

and a continuous viscosity solution $\chi : \mathbb{R}^d \to \mathbb{R}$ of the ergodic Hamilton-Jacobi equation

$$\alpha(L) + H(x, D\chi(x)) = 0, \quad x \in \mathbb{R}^d. \tag{6.11}$$

Furthermore, such a solution che ne represented as

$$\chi(x) = \inf_{(\gamma,u) \in \Gamma^{\to x}_{0,t}} \left\{ \chi(\gamma(0)) + \int_0^t L(\gamma(s), u(s)) \, ds \right\} - \alpha(L)\, t \tag{6.12}$$

for any $(t, x) \in [0, T] \times \mathbb{R}^d$.

We say that $\alpha(L)$ is the critical constant *for our problem and χ is any* critical solution *to the ergodic equation* (6.11)*. Hereafter, we will always work with critical solutions that satisfy* (6.12).

Differently from Chapter 5 here we also need to assume the following on the sub-Riemannian system:

(S) there are no singular minimizing controls of problem (6.6).

The above extra assumption is needed for the critical solution χ to be more regular. Indeed, under **(S)** any critical solution is locally semiconcave and consequently locally Lipschitz continuous, see [25, Theorem 1].

Remark 6.2. (*i*) Observe that the sub-Riemannian systems in Example 1.5 fit assumption **(S)**, see for instance [25, Theorem 5.1].

(*ii*) In view of the assumptions on L, we deduce that for any $R \geq 0$ there exists a constant $\widetilde{C}_R \geq 0$ such that

$$|H(x, p) - H(y, p)| \leq \widetilde{C}_R (1 + |p|)|x - y|, \quad \forall\, x, y \in \overline{B}_R. \tag{6.13}$$

6.2 Characterization of the ergodic constant

We begin by introducing a class of probability measures that adapts the notion of closed measures to sub-Riemannian control systems. Set

$$\mathscr{P}^2_c(\mathbb{R}^d \times \mathbb{R}^m) = \left\{ \mu \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^m) : \int_{\mathbb{R}^d \times \mathbb{R}^m} |u|^2 \, \mu(dx, du) < +\infty,\ \mathrm{spt}(\pi_1 \sharp \mu) \text{ compact} \right\}$$

where $\pi_1 : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ denotes the projection onto the first factor, i.e. $\pi_1(x, u) = x$.

Recall that $F(x) = [f_1(x)| \ldots |f_m(x)]$ is the real $d \times m$ matrix in (6.1).

Definition 6.3 (F**-closed measure).** *We say that* $\mu \in \mathscr{P}_c^2(\mathbb{R}^d \times \mathbb{R}^m)$ *is an* F*-closed measure if*

$$\int_{\mathbb{R}^d\times\mathbb{R}^m} \langle F^\star(x)D\varphi(x), u\rangle \; \mu(dx,du) = 0, \quad \forall\, \varphi \in C^1(\mathbb{R}^d).$$

We denote by $\mathcal{C}_F$ *the set of all* F*-closed measures.*

Closed measures were first introduced in [43] in order to overcome the lack of regularity of the Lagrangian L. Indeed, if L is merely continuous, then there is no Euler flow and, consequently, it makes no sense to introduce invariant measure as in [41]. Similarly, in our setting such a flow does not exists and for this reason the use of closed measures turns out to be necessary. Moreover, as we will show in the next result, such measures collect the behavior of minimizing trajectories for (6.7) as the time horizon T goes to infinity. We now proceed to construct one closed measure that will be particularly useful to study the Aubry set.

Fix $x_0 \in \mathbb{R}^d$ and for any $T > 0$ let the pair $(\gamma_{x_0}, u_{x_0}) \in \Gamma_{0,T}^{x_0 \to}$ be optimal for (6.6). Define the probability measure $\mu_{x_0}^T$ by

$$\int_{\mathbb{R}^d\times\mathbb{R}^m} \varphi(x,u)\; \mu_{x_0}^T(dx,du) = \frac{1}{T}\int_0^T \varphi(\gamma_{x_0}(t), u_{x_0}(t))\; dt, \quad \forall\, \varphi \in C_b(\mathbb{R}^d\times\mathbb{R}^m). \tag{6.14}$$

Then, we have the following.

Proposition 6.4. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Then,* $\{\mu_{x_0}^T\}_{T>0}$ *is tight and there exists a sequence* $T_n \to \infty$ *such that* $\mu_{x_0}^{T_n}$ *weakly-* converges to an* F*-closed measure* $\mu_{x_0}^\infty$.

Proof. First, from (6.9) it follows that $\{\pi_1\sharp\mu_{x_0}^T\}_{T>0}$ has compact support, uniformly in T. Thus, such a family of measures is tight. Let us prove that $\{\pi_2\sharp\mu^T\}_{T>0}$ is also tight.

On the one hand, taking the null control we have that

$$\frac{1}{T}v^T(x_0) \leq \frac{1}{T}\int_0^T L(x_0, 0)\; ds \leq \ell_2.$$

On the other hand, since (γ_{x_0}, u_{x_0}) is a minimizing pair for $V_T(x_0)$, from (6.4) we get

$$\begin{aligned}\frac{1}{T}v^T(x_0) &= \frac{1}{T}\int_0^T L(\gamma_{x_0}(t), u_{x_0}(t))\; dt\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\; \mu_{x_0}^T(dx,du) \geq \int_{\mathbb{R}^d\times\mathbb{R}^m} \left(\frac{1}{2\ell_1}|u|^2 + L(x^*,0)\right)\; \mu_{x_0}^T(dx,du)\end{aligned}$$

which implies that

$$\frac{1}{2\ell_1}\int_{\mathbb{R}^d\times\mathbb{R}^m} |u|^2\; \mu_{x_0}^T(dx,du) \leq \ell_2 - L(x^*,0).$$

Consequently, the family of probability measures $\{\pi_2\sharp\mu_{x_0}^T\}_{T>0}$ has bounded second order moment (w.r.t. T). So, $\{\pi_2\sharp\mu_{x_0}^T\}_{T>0}$ is tight.

Since $\{\pi_1\sharp\mu_{x_0}^T\}_{T>0}$ and $\{\pi_2\sharp\mu_{x_0}^T\}_{T>0}$ are tight, so is $\{\mu_{x_0}^T\}_{T>0}$ by [7, Theorem 5.2.2]. Therefore, by Prokhorov's Theorem there exists $\{T_n\}_{n\in\mathbb{N}}$, with $T_n \to \infty$, and $\mu_{x_0}^\infty \in \mathscr{P}_c^2(\mathbb{R}^d\times\mathbb{R}^m)$ such that $\mu_{x_0}^{T_n} \rightharpoonup^* \mu_{x_0}^\infty$.

We now show that $\mu_{x_0}^{\infty}$ is an F-closed measure, that is

$$\int_{\mathbb{R}^d\times\mathbb{R}^m} \langle F^{\star}(x)D\psi(x),u\rangle\ \mu_{x_0}^{\infty}(dx,du)=0, \quad \forall\ \psi\in C^1(\mathbb{R}^d).$$

By definition we have that

$$\int_{\mathbb{R}^d\times\mathbb{R}^m} \langle F^{\star}(x)D\psi(x),u\rangle \mu_{x_0}^{T_n}(dx,du)=\frac{1}{T_n}\int_0^{T_n}\langle F^{\star}(\gamma_{x_0}(t)D\psi(\gamma_{x_0}(t)),u_{x_0}(t)\rangle\ dt$$
$$=\frac{1}{T_n}\int_0^{T_n}\langle D\psi(\gamma_{x_0}(t),\dot\gamma_{x_0}(t)\rangle\ dt=\frac{\psi(\gamma_{x_0}(T_n))-\psi(x_0)}{T_n}.$$

Then, from (6.9) we know that $\gamma_{x_0}(T_n)\in \overline{B}_{Q_{|x_0|}}$. So, we get

$$\lim_{n\to\infty}\frac{\psi(\gamma_{x_0}(T_n))-\psi(x_0)}{T_n}=0$$

and, consequently,

$$\int_{\mathbb{R}^d\times\mathbb{R}^m} \langle F^{\star}(x)D\psi(x),u\rangle\ \mu_{x_0}^{\infty}(dx,du)=0. \qquad \square$$

Set

$$\mathscr{P}_R^2(\mathbb{R}^d\times\mathbb{R}^m)=\left\{\mu\in\mathscr{P}_c^2(\mathbb{R}^d\times\mathbb{R}^m):\mathrm{spt}(\pi_1\sharp\mu)\subset\overline{B}_R\right\}.$$

The following property, which is interesting in its own right, will be crucial for the characterization of the critical constant derived in Theorem 6.8 below.

Proposition 6.5. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Then, for any* $R\geq Q_0$, *where* Q_0 *is given in* (6.9), *we have that*

$$\inf_{\mu\in\mathcal{C}_{\mathrm{F}}\cap\mathscr{P}_R^2(\mathbb{R}^d\times\mathbb{R}^m)}\int_{\mathbb{R}^d\times\mathbb{R}^m}L(x,u)\ \mu(dx,du)=-\inf_{\psi\in C^1(\mathbb{R}^d)}\sup_{x\in\overline{B}_R}H(x,D\psi(x)). \tag{6.15}$$

Lemma 6.6. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Then, for any* $R\geq Q_0$, *where* Q_0 *is given in* (6.9), *we have that*

$$\begin{aligned}&\inf_{\mu\in\mathcal{C}_{\mathrm{F}}\cap\mathscr{P}_R^2(\mathbb{R}^d\times\mathbb{R}^m)}\int_{\mathbb{R}^d\times\mathbb{R}^m}L(x,u)\ \mu(dx,du)\\ &=\inf_{\mu\in\mathscr{P}_R^2(\mathbb{R}^d\times\mathbb{R}^m)}\sup_{\psi\in C^1(\mathbb{R}^d)}\int_{\mathbb{R}^d\times\mathbb{R}^m}\Big(L(x,u)-\langle F^{\star}(x)D\psi(x),u\rangle\Big)\ \mu(dx,du).\end{aligned} \tag{6.16}$$

The proof of the above lemma is based on an argument which is quite common in optimal transport theory see, for instance, [67, Theorem 1.3]. We give the reasoning for the reader's convenience.

Proof. Since L is bounded below we have that

$$\begin{aligned}&\inf_{\mu\in\mathcal{C}_{\mathrm{F}}\cap\mathscr{P}_R^2(\mathbb{R}^d\times\mathbb{R}^m)}\int_{\mathbb{R}^d\times\mathbb{R}^m}L(x,u)\ \mu(dx,du)\\ &=\inf_{\mu\in\mathscr{P}_R^2(\mathbb{R}^d\times\mathbb{R}^m)}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^m}L(x,u)\ \mu(dx,du)+\omega(\mu)\right\}\end{aligned}$$

where

$$\omega(\mu) = \begin{cases} 0, & \mu \in \mathcal{C}_{\mathrm{F}} \\ +\infty, & \mu \notin \mathcal{C}_{\mathrm{F}}. \end{cases}$$

So, observing that

$$\omega(\mu) = \sup_{\psi \in C^1(\mathbb{R}^d)} - \int_{\mathbb{R}^d \times \mathbb{R}^m} \langle F^\star(x) D\psi(x), u \rangle \; \mu(dx, du)$$

we obtain (6.16). □

Lemma 6.7. *Let $\phi \in C(\mathbb{R}^d \times \mathbb{R}^m)$ be such that*

$$\phi_0 \leq \phi(x,u) \leq C_\phi(1 + |u|^2), \quad \forall \; (x,u) \in \mathbb{R}^d \times \mathbb{R}^m$$

for some constants $\phi_0 \in \mathbb{R}$ and $C_\phi \geq 0$. Let $\{\mu_j\}_{j \in \mathbb{N}} \in \mathscr{P}^2(\mathbb{R}^d \times \mathbb{R}^m)$ and let $\mu \in \mathscr{P}^2(\mathbb{R}^d \times \mathbb{R}^m)$ be such that $\mu_j \rightharpoonup^* \mu$ as $j \to \infty$. Then, we have that

$$\liminf_{j \to \infty} \int_{\mathbb{R}^d \times \mathbb{R}^m} \phi(x,u) \; \mu_j(dx,du) \geq \int_{\mathbb{R}^d \times \mathbb{R}^m} \phi(x,u) \; \mu(dx,du). \tag{6.17}$$

Proof. We first prove (6.17) assuming that $\phi_0 = 0$ and then we remove such a constraint.

For any $\varepsilon > 0$ we have that

$$\int_{\mathbb{R}^d \times \mathbb{R}^m} \phi(x,u) \; \mu_{x0}^{T_j}(dx,du) = \int_{\mathbb{R}^d \times \mathbb{R}^m} \frac{\phi(x,u)}{1 + \varepsilon \, |u|^2}(1 + \varepsilon \, |u|^2) \; \mu_{x0}^{T_j}(dx,du)$$
$$\geq \int_{\mathbb{R}^d \times \mathbb{R}^m} \frac{\phi(x,u)}{1 + \varepsilon \, |u|^2} \; \mu_{x0}^{T_j}(dx,du).$$

From the growth assumption on ϕ we deduce that the function $\frac{\phi(x,u)}{1+\varepsilon\,|u|^2}$ is bounded and so by weak-$*$ convergence we get

$$\liminf_{j \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^m} \phi(x,u) \; \mu_{x0}^{T_j}(dx,du) \geq \int_{\mathbb{R}^d \times \mathbb{R}^m} \frac{\phi(x,u)}{1 + \varepsilon \, |u|^2} \; \mu_{x0}^{\infty}(dx,du).$$

Therefore, as $\varepsilon \downarrow 0$ we obtain (6.17).

For $\phi_0 \neq 0$, we have that

$$\int_{\mathbb{R}^d \times \mathbb{R}^m} [(\phi(x,u) - \phi_0) + \phi_0] \; \mu_{x0}^{T_j}(dx,du)$$
$$= \int_{\mathbb{R}^d \times \mathbb{R}^m} \frac{\phi(x,u) - \phi_0}{1 + \varepsilon \, |u|^2}(1 + \varepsilon \, |u|^2) \; \mu_{x0}^{T_j}(dx,du) + \phi_0$$
$$\geq \int_{\mathbb{R}^d \times \mathbb{R}^m} \frac{\phi(x,u) - \phi_0}{1 + \varepsilon \, |u|^2} \; \mu_{x0}^{T_j}(dx,du) + \phi_0.$$

Thus, we obtain

$$\liminf_{j \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^m} \phi(x,u) \; \mu_{x0}^{T_j}(dx,du) \geq \int_{\mathbb{R}^d \times \mathbb{R}^m} \frac{\phi(x,u) - \phi_0}{1 + \varepsilon \, |u|^2} \; \mu_{x0}^{\infty}(dx,du) + \phi_0$$

which in turn yields the result as $\varepsilon \downarrow 0$. □

Proof of Proposition 6.5. We divide the proof into two steps.

(1): Define $\mathcal{F} : C^1(\mathbb{R}^d) \times \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m) \to \mathbb{R}$ by

$$\mathcal{F}(\psi, \mu) = \int_{\mathbb{R}^d \times \mathbb{R}^m} \Big(L(x,u) - \langle F^\star(x) D\psi(x), u\rangle \Big) \; \mu(dx, du).$$

We will apply the Minimax Theorem ([64, Theorem A.1]) to prove that

$$\begin{aligned} &\inf_{\mu \in \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \sup_{\psi \in C^1(\mathbb{R}^d)} \int_{\mathbb{R}^d \times \mathbb{R}^m} \Big(L(x,u) - \langle F^\star(x) D\psi(x), u\rangle \Big) \; \mu(dx, du) \\ = &\sup_{\psi \in C^1(\mathbb{R}^d)} \inf_{\mu \in \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \int_{\mathbb{R}^d \times \mathbb{R}^m} \Big(L(x,u) - \langle F^\star(x) D\psi(x), u\rangle \Big) \; \mu(dx, du). \end{aligned}$$

In order to check that the hypothesis of such a theorem are satisfied, let us define

$$c^* = 1 + L(x^*, 0).$$

We claim that the level set

$$\mathcal{E} := \big\{ \mu \in \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m) : \; \mathcal{F}(0, \mu) \leq c^* \big\}$$

is compact in $\big(\mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m), d_1\big)$. Indeed, for any given $\mu \in \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)$ we know that $\pi_1 \sharp \mu$ has compact support contained in $\overline{B}_R$. Moreover, the coercivity of L implies that for any given $\mu \in \mathcal{E}$ we have that $\pi_2 \sharp \mu$ has bounded second moment which in turn yields the tightness of the family $\pi_2 \sharp \mu$ for any $\mu \in \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)$. Thus, the level set is compact by Prokhorov's Theorem and [7, Theorem 5.2.2]. Moreover, from Lemma 6.7 we have that $\mathcal{F}(\psi, \mu)$ is lower-semicontinuous w.r.t. μ in $\mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)$. Therefore, applying the Minimax Theorem ([64, Theorem A.1]) we obtain

$$\begin{aligned} &\inf_{\mu \in \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \sup_{\psi \in C^1(\mathbb{R}^d)} \int_{\mathbb{R}^d \times \mathbb{R}^m} \Big(L(x,u) - \langle F^\star(x) D\psi(x), u\rangle \Big) \; \mu(dx, du) \\ = &\sup_{\psi \in C^1(\mathbb{R}^d)} \inf_{\mu \in \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \int_{\mathbb{R}^d \times \mathbb{R}^m} \Big(L(x,u) - \langle F^\star(x) D\psi(x), u\rangle \Big) \; \mu(dx, du). \end{aligned}$$

(2): Proof of (6.15). By (6.16) we get

$$\begin{aligned} &\inf_{\mu \in \mathcal{C}_{\mathrm{F}} \cap \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \int_{\mathbb{R}^d \times \mathbb{R}^m} L(x,u) \; \mu(dx, du) \\ = &\inf_{\mu \in \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \sup_{\psi \in C^1(\mathbb{R}^d)} \int_{\mathbb{R}^d \times \mathbb{R}^m} \Big(L(x,u) - \langle F^\star(x) D\psi(x), u\rangle \Big) \; \mu(dx, du) \\ = &\sup_{\psi \in C^1(\mathbb{R}^d)} \inf_{\mu \in \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \int_{\mathbb{R}^d \times \mathbb{R}^m} \Big(L(x,u) - \langle F^\star(x) D\psi(x), u\rangle \Big) \; \mu(dx, du). \end{aligned}$$

Now, the coercivity of L ensures the existence of the

$$\min_{(x,u) \in \overline{B}_R \times \mathbb{R}^m} \Big\{ L(x,u) - \langle F^\star(x) D\psi(x), u\rangle \Big\}.$$

Therefore, by taking a Dirac mass centered at any minimizer of the above function, one deduce that

$$
\begin{aligned}
&\sup_{\psi\in C^1(\mathbb{R}^d)} \inf_{\mu\in\mathscr{P}^2_R(\mathbb{R}^d\times\mathbb{R}^m)} \int_{\mathbb{R}^d\times\mathbb{R}^m} \Big(L(x,u)-\langle F^\star(x)D\psi(x),u\rangle\Big)\ \mu(dx,du)\\
&\sup_{\psi\in C^1(\mathbb{R}^d)} \min_{(x,u)\in\overline{B}_R\times\mathbb{R}^m} \Big\{L(x,u)-\langle F^\star(x)D\psi(x),u\rangle\Big\}.\\
=&\sup_{\psi\in C^1(\mathbb{R}^d)}\left(-\max_{(x,u)\in\overline{B}_R\times\mathbb{R}^m}\Big\{L(x,u)-\langle F^\star(x)D\psi(x),u\rangle\Big\}\right)\\
=&-\inf_{\psi\in C^1(\mathbb{R}^d)} \max_{(x,u)\in\overline{B}_R\times\mathbb{R}^m}\Big\{L(x,u)-\langle F^\star(x)D\psi(x),u\rangle\Big\}\\
=&-\inf_{\psi\in C^1(\mathbb{R}^d)}\max_{x\in\overline{B}_R} H(x,D\psi(x))
\end{aligned}
$$

where the last equality holds true observing that

$$
\begin{aligned}
&\max_{(x,u)\in\overline{B}_R\times\mathbb{R}^m}\Big\{\langle F^\star(x)D\psi(x),u\rangle-L(x,u)\Big\}\\
=&\max_{x\in\overline{B}_R}\sup_{u\in\mathbb{R}^m}\Big\{\langle F^\star(x)D\psi(x),u\rangle-L(x,u)\Big\}=\sup_{x\in\overline{B}_R} H(x,D\psi(x)).
\end{aligned}
$$

This completes the proof. □

The following characterization of the critical value is essential for the analysis in Section 6.3.

Theorem 6.8. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Then, for any* $R\geq Q_0$, *where* Q_0 *is given in* (6.9), *we have that*

$$
\alpha(L)=\inf_{\mu\in\mathcal{C}_F\cap\mathscr{P}^2_R(\mathbb{R}^d\times\mathbb{R}^m)}\int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu(dx,du)=\inf_{\mu\in\mathcal{C}_F}\int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu(dx,du). \tag{6.18}
$$

Lemma 6.9. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Let* χ *be a critical solution. Then, for any* $R\geq 0$ *there exists a constant* $\kappa_R\geq 0$ *such that for any* $\varepsilon>0$

$$
\alpha(L)+H(x,D\chi_\varepsilon(x))\leq\kappa_R\,\varepsilon,\quad \forall\ x\in\overline{B}_R \tag{6.19}
$$

where $\chi_\varepsilon(x)=\chi\star\xi_\varepsilon(x)$ *and* ξ_ε *is a smooth mollifier.*

Proof. From **(S)** we have that χ belongs to $W^{1,\infty}_{\text{loc}}(\mathbb{R}^d)$. So,

$$
\alpha(L)+H(x,D\chi(x))=0,\quad \text{a.e. } x\in\mathbb{R}^d. \tag{6.20}
$$

Let $R\geq 0$ and let $x_0\in\overline{B}_R$. Then, by Jensen's inequality we get

$$
\begin{aligned}
\alpha(L)+H(x_0,D\chi_\varepsilon(x_0))&=\alpha(L)+H\left(x_0,\int_{\mathbb{R}^d} D\chi(x_0-y)\xi_\varepsilon(y)\ dy\right)\\
&\leq\int_{\mathbb{R}^d}\big[\alpha(L)+H(x_0,D\chi(x_0-y))\big]\xi_\varepsilon(y)\ dy.
\end{aligned}
$$

Moreover, writing

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \big[\,\alpha(L) + H(x_0, D\chi(x_0-y))\big]\xi_\varepsilon(y)\ dy \\
&= \underbrace{\int_{\mathbb{R}^d} \big[\,\alpha(L) + H(x_0 - y, D\chi(x_0-y))\big]\xi_\varepsilon(y)\ dy}_{\mathbf{I}} \\
&+ \underbrace{\int_{\mathbb{R}^d} \big[H(x_0, D\chi(x_0-y)) - H(x_0-y, D\chi(x_0-y))\big]\xi_\varepsilon(y)\ dy}_{\mathbf{II}},
\end{aligned}
$$

by (6.20) we deduce that $\mathbf{I} = 0$ and by (6.13) we get $\mathbf{II} \leq \kappa_R\,\varepsilon$. □

Proof of Theorem 6.8. We divide the proof into two steps.

Step 1: We first show that for any $R \geq Q_0$, where Q_0 is given in (6.9),

$$
\alpha(L) = \inf_{\mu \in \mathcal{C}_F \cap \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \int_{\mathbb{R}^d \times \mathbb{R}^m} L(x,u)\ \mu(dx,du).
$$

Indeed, by (6.10) we know that

$$
\alpha(L) = \lim_{T \to +\infty} \frac{1}{T} v^T(0).
$$

Hence, appealing to Lemma 6.7 and recalling that $L(x,u) \geq L(x^*,0)$ we obtain

$$
\alpha(L) = \lim_{T \to \infty} \int_{\mathbb{R}^d \times \mathbb{R}^m} L(x,u)\ \mu_0^T(dx,du) \geq \int_{\mathbb{R}^d \times \mathbb{R}^m} L(x,u)\ \mu_0^\infty(dx,du). \tag{6.21}
$$

Recalling that $\mu_0^\infty \in \mathcal{C}_{\mathrm{F}} \cap \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)$ for any $R \geq Q_0$, we deduce that

$$
\alpha(L) \geq \inf_{\mu \in \mathcal{C}_{\mathrm{F}} \cap \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \int_{\mathbb{R}^d \times \mathbb{R}^m} L(x,u)\ \mu(dx,du).
$$

Next, by Proposition 6.5 we have that for any $\psi \in C^1_c(\mathbb{R}^d)$

$$
\inf_{\mu \in \mathcal{C}_{\mathrm{F}} \cap \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \int_{\mathbb{R}^d \times \mathbb{R}^m} L(x,u)\ \mu(dx,du) = -\inf_{\psi \in C^1(\mathbb{R}^d)} \sup_{x \in \overline{B}_R} H(x, D\psi(x)). \tag{6.22}
$$

Let χ be a critical solution. For $\varepsilon \geq 0$ let $\chi_\varepsilon(x) = \chi \star \xi^\varepsilon(x)$, where ξ^ε is a smooth mollifier. From Lemma 6.9 we know that for any $R \geq 0$

$$
\alpha(L) + H(x, D\chi_\varepsilon(x)) \leq \kappa_R\,\varepsilon, \quad x \in \overline{B}_R.
$$

Then, using χ_ε in (6.22) we obtain

$$
\begin{aligned}
&\inf_{\mu \in \mathcal{C}_{\mathrm{F}} \cap \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \int_{\mathbb{R}^d \times \mathbb{R}^m} L(x,u)\ \mu(dx,du) \\
&\geq -\sup_{x \in \overline{B}_R} H(x, D\chi_\varepsilon(x)) \geq \alpha(L) - \kappa_R\,\varepsilon\,.
\end{aligned}
$$

Hence, as $\varepsilon \downarrow 0$ we get

$$
\inf_{\mu \in \mathcal{C}_{\mathrm{F}} \cap \mathscr{P}^2_R(\mathbb{R}^d \times \mathbb{R}^m)} \int_{\mathbb{R}^d \times \mathbb{R}^m} L(x,u)\ \mu(dx,du) \geq \alpha(L)
$$

and this completes the first step.

Step 2: Now we prove that

$$\alpha(L) = \inf_{\mu\in\mathcal{C}_F} \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu(dx,du),$$

that is, we remove the constraint $\mu \in \mathscr{P}^2_R(\mathbb{R}^d\times\mathbb{R}^m)$.

Let $\{\mu_j\}_{j\in\mathbb{N}} \subset \mathcal{C}_F$ be such that

$$\lim_{j\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu_j(dx,du) = \inf_{\mu\in\mathcal{C}_F} \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu(dx,du). \tag{6.23}$$

Since $\mu_j \in \mathcal{C}_F \subset \mathscr{P}^2_c(\mathbb{R}^d\times\mathbb{R}^m)$ we deduce that there exists $\{R_j\}_{j\in\mathbb{N}}$ such that

$$\mathrm{spt}(\mu_j) \subset \overline{B}_{R_j}.$$

Moreover, without loss of generality, we can assume that for any $j\in\mathbb{N}$

$$\inf_{\mu\in\mathcal{C}_F\cap\mathscr{P}^2_{R_j}(\mathbb{R}^d\times\mathbb{R}^m)} \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu(dx,du) = \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu_j(dx,du).$$

Since, for j sufficiently large, we have proved that

$$\alpha(L) = \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu_j(dx,du)$$

the conclusion follows from (6.23). □

Corollary 6.10. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Then the following holds.*

(i) $\alpha(L) = L(x^*,0) = \min_{x\in\mathcal{K}_L} L(x,0)$.

(ii) For any $x_0\in\mathbb{R}^d$ *we have that*

$$\alpha(L) = \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu^\infty_{x_0}(dx,du)$$

where $\mu^\infty_{x_0}$ *is given in Proposition 6.4.*

Remark 6.11. Note that point (i) of the conclusion has beed already proved in [23, Corollary 5.4]. Here we propose a different approach which relies on (6.18).

Proof. (i) On the one hand, by Theorem 6.8, we have that

$$\alpha(L) = \inf_{\mu\in\mathcal{C}_F} \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu(dx,du) \geq L(x^*,0)$$

where the inequality holds true by (6.4).

On the other hand we observe that the Dirac measure $\delta_{(x^*,0)}$ is F-closed. So,

$$\alpha(L) = \inf_{\mu\in\mathcal{C}_F} \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu(dx,du) \leq \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \delta_{(x^*,0)}(dx,du) = L(x^*,0).$$

(ii) Recalling Lemma 6.7 we obtain

$$\alpha(L) = \lim_{T\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu^T_0(dx,du) \geq \int_{\mathbb{R}^d\times\mathbb{R}^m} L(x,u)\ \mu^\infty_0(dx,du).$$

Thus, the conclusion follows from Theorem 6.8 recalling that $\mu^\infty_{x_0}$ is F-closed by Proposition 6.4. □

6.3 Aubry set

We denote by L^* the Legendre Transform of L, that is,

$$L^*(x,p) = \sup_{u\in\mathbb{R}^m} \big\{\langle p, v\rangle - L(x,u)\big\},$$

and we observe that

$$H(x,p) = L^*(x, F^*(x)p), \quad (x,p) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{6.24}$$

Moreover, since L satisfies **(L0)** – **(L2)** we know that L^* is coercive and strictly convex in p.

Definition 6.12 (Dominated functions and calibrated curves). *Let $c \in \mathbb{R}$ and let φ be a continuous function on $\mathbb{R}^d$.*

1. *We say that φ is dominated by $L - c$ and we denote this by $\varphi \prec L - c$, if for any a, $b \in \mathbb{R}$, with $a < b$, and any trajectory-control pair $(\gamma, u) \in \Gamma_{a,b}^{\gamma(a)\to\gamma(b)}$ we have that*
$$\varphi(\gamma(b)) - \varphi(\gamma(a)) \leq \int_a^b L(\gamma(s), u(s))\, ds - c\,(b-a).$$

2. *We say that the first component $\gamma : [a,b] \to \mathbb{R}^d$, with a, $b \in \mathbb{R}$ and $a < b$, of a trajectory-control pair $(\gamma, u) \in \Gamma_{a,b}^{\gamma(a)\to\gamma(b)}$ is a calibrated curve for φ if*
$$\varphi(\gamma(b)) - \varphi(\gamma(a)) = \int_a^b L(\gamma(s), u(s))\, ds - c\,(b-a).$$

   We denote by $\mathrm{Cal}(\varphi)$ *the set of all calibrated curves for φ.*

For any $t \geq 0$ and for any x, $y \in \mathbb{R}^d$ we denote by $A_t(x,y)$ the action functional, also called fundamental solution of the critical equation, i.e.,

$$A_t(x,y) = \inf_{(\gamma,u)\in\Gamma_{0,t}^{x\to y}} \left\{ \int_0^t L(\gamma(s), u(s))\, ds \right\}.$$

We note that $\varphi \prec L - \alpha(L)$ if and only if for any x, y in $\mathbb{R}^d$ and for any $t \geq 0$ we have that

$$\varphi(y) - \varphi(x) \leq A_t(x,y) - \alpha(L)\, t. \tag{6.25}$$

Then, Peierls's barrier is defined as

$$\mathrm{A}_\infty(x,y) = \liminf_{t\to\infty} \big[A_t(x,y) - \alpha(L)\, t\big], \quad x, y \in \mathbb{R}^d. \tag{6.26}$$

Lemma 6.13. *The following properties hold.*

(i) *For any x, $y \in \mathbb{R}^d$ we have that $0 \leq \mathrm{A}_\infty(x,y) < \infty$.*

(ii) *For any x, y, $z \in \mathbb{R}^d$ we have that*

$$\mathrm{A}_\infty(x,z) \leq \mathrm{A}_\infty(x,y) + \mathrm{A}_\infty(y,z) \tag{6.27}$$

and, for any $t \geq 0$ we have that

$$\mathrm{A}_\infty(x,z) \leq \mathrm{A}_\infty(x,y) + A_t(y,z) - \alpha(L)\, t. \tag{6.28}$$

Proof. Point (i) follow by **(L2)** and the reversibility of L, respectively. Point (ii) follows by similar arguments as in [43]. □

Definition 6.14 (Projected Aubry set). *The projected Aubry set $\mathcal{A}$ is defined by*

$$\mathcal{A} = \{x \in \mathbb{R}^d : \mathrm{A}_\infty(x, x) = 0\}.$$

Lemma 6.15. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Let $(x, y) \in \mathbb{R}^{2d}$ be such that*

$$h := \mathrm{A}_\infty(x, y) \in \mathbb{R}.$$

Let $\{t_n\}_{n\in\mathbb{N}} \in \mathbb{R}$ and $(\gamma_n, u_n) \in \Gamma^{x\to y}_{0,t_n}$ be such that

$$t_n \to +\infty \quad and \quad \lim_{n\to+\infty} \int_0^{t_n} L(\gamma_{u_n}(s), u_n(s))\, ds - \alpha(L)\, t_n = h. \tag{6.29}$$

Then, there exists a subsequence, still denoted by (γ_n, u_n), and a trajectory-control pair $(\bar{\gamma}, \bar{u}) \in \Gamma^{x\to y}_{0,\infty}$ such that

(i) $\{u_n\}_{n\in\mathbb{N}}$ weakly converges to $\bar{u}$ in L^2 on any compact subset of $[0, \infty)$;

(ii) $\{\gamma_n\}_{n\in\mathbb{N}}$ uniformly converges to $\bar{\gamma}$ on every compact subset of $[0, \infty)$.

Proof. From (6.26) it follows that there exists $\bar{n} \in \mathbb{N}$ such that for any $n \geq \bar{n}$ we have that

$$\int_0^{t_n} L(\gamma_n(s), u_n(s))\, ds - \alpha(L)\, t_n \leq h + 1.$$

On the other hand, by **(L2)** we obtain

$$\int_0^{t_n} L(\gamma_n(s), u_n(s))\, ds - \alpha(L)\, t_n \geq \frac{1}{2\ell_1} \int_0^{t_n} |u_n(s)|^2\, ds - (L(x^*, 0) + \alpha(L))t_n.$$

Appealing to (i) in Corollary 6.10 we have that $L(x^*, 0) + \alpha(L) = 0$. So,

$$\int_0^{t_n} |u_n(s)|^2\, ds \leq 2\ell_1(h + 1), \quad \forall n \geq \bar{n}.$$

Therefore, there exists a subsequence, still denoted by $\{u_n\}$, that weakly converges to an admissible control $\bar{u}$ in L^2 on any compact subset of $[0, +\infty)$. Moreover, let $R \geq 0$ be such that $|x| \leq R$. Then, by (6.9) for any $t > 0$ we have that

$$|\gamma_n(s)|^2 \leq Q_R, \quad \forall\ s \in [0, t],\ \forall\ n \geq \bar{n}$$

and

$$\int_0^{t_n} |\dot{\gamma}_n(s)|^2\, ds \leq \int_0^{t_n} c_f^2 \big(1 + |\gamma_n(s)|\big)^2 |u_n(s)|^2\, ds$$
$$\leq c_f^2(1 + Q_R) 2\ell_1(h + 1), \quad \forall\ s \in [0, t],\ \forall\ n \geq \bar{n}.$$

Hence, $\{\gamma_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(0, t; \mathbb{R}^d)$ for any $t > 0$. Then, by the Ascoli-Arzela Theorem, up to extracting a further subsequence, $\{\gamma_n\}_{n\in\mathbb{N}}$ uniformly converges to a curve $\bar{\gamma}$ on every compact subset of $[0, +\infty)$.

Now, we claim that $(\bar{\gamma}, \bar{u})$ satisfies (6.1). Indeed, for any $t \geq 0$ we have that

$$\gamma_n(t) = x + \sum_{i=1}^{m} \int_0^t u_i^n(s) f_i(\gamma_n(s))\, ds.$$

Thus, by the locally uniform convergence of γ_n it follows that $f_i(\gamma_n(t)) \to f_i(\bar\gamma(t))$, locally uniformly, for any $t \geq 0$, as $n \to +\infty$ for any $i = 1, \dots, m$. Therefore, taking $v \in \mathbb{R}^d$ we deduce that

$$\langle v, \gamma_n(t)\rangle = \langle v, x\rangle + \sum_{i=1}^{m} \int_0^t u_i^n(s)\langle f_i(\gamma_n(s)), v\rangle \, ds, \quad \forall\, t \geq 0.$$

As $n \to +\infty$ we get

$$\langle v, \bar\gamma(t)\rangle = \langle v, x\rangle + \sum_{i=1}^{m} \int_0^t \bar u_i(s)\langle f_i(\bar\gamma(s)), v\rangle \, ds, \quad \forall\, t \geq 0.$$

Since $v \in \mathbb{R}^d$ is arbitrary the conclusion follows. □

Remark 6.16. Arguing as in the proof of Lemma 6.15, one can prove the following. Given $h \in \mathbb{R}$, $\{t_n\}_{n\in\mathbb{N}}$ and $(\gamma_n, u_n) \in \Gamma^{x\to y}_{-t_n,0}$ such that

$$t_n \to +\infty \quad \text{and} \quad \lim_{n\to+\infty} \int_{-t_n}^{0} L(\gamma_{u_n}(s), u_n(s)) \, ds - \alpha(L)\, t_n = h.$$

Then, there exists a subsequence, still denoted by (γ_n, u_n), and a trajectory-control pair $(\bar\gamma, \bar u)$ such that

(*i*) $\{u_n\}_{n\in\mathbb{N}}$ weakly converges to $\bar u$ in L^2 on any compact subset of $(-\infty, 0]$;

(*ii*) $\{\gamma_n\}_{n\in\mathbb{N}}$ uniformly converges to $\bar\gamma$ on every compact subset of $(-\infty, 0]$.

Proposition 6.17. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *For each x, $y \in \mathbb{R}^d$ there exists $(\bar\gamma, \bar u) \in \Gamma^{\to y}_{-\infty,0}$ such that*

$$\mathrm{A}_\infty(x, y) - \mathrm{A}_\infty(x, \bar\gamma(-t)) = \int_{-t}^{0} L(\bar\gamma(s), \bar u(s)) \, ds - \alpha(L)\, t, \quad \forall\, t \geq 0. \tag{6.30}$$

Moreover, for each $x \in \mathbb{R}^d$ the map $y \mapsto \mathrm{A}_\infty(x, y)$ is a critical solution on $\mathbb{R}^d$.

Proof. Fix x, $y \in \mathbb{R}^d$ and let $\{t_n\}_{n\in\mathbb{N}}$, $(\gamma_n, u_n) \in \Gamma^{x\to y}_{-t_n,0}$ be such that

$$t_n \to \infty, \quad \text{and} \quad \lim_{n\to\infty} \int_{-t_n}^{0} L(\gamma_n(s), u_n(s)) \, ds - \alpha(L)\, t_n = \mathrm{A}_\infty(x, y).$$

Then, from Remark 6.16 there exists $(\bar\gamma, \bar u)$ such that u_n weakly converges to $\bar u$ and γ_n uniformly converges to $\bar\gamma$, on every compact subset of $(-\infty, 0]$.

Let $R \geq 0$ be such that $|x| \leq R$. Fix $t \in [0, \infty)$, take $n \in \mathbb{N}$ such that $d_n = d_{\mathrm{SR}}(\bar\gamma(-t), \gamma_n(-t)) \leq 1$ and $t_n > t+1$. Let $(\gamma_0, u_0) \in \Gamma^{\gamma_n(-t)\to\bar\gamma(-t)}_{-t,-t+d_n}$ be optimal for (1.5) and let $\widetilde u_n \in L^2(-t_n, -t + d_n)$ be given by

$$\widetilde u_n(s) = \begin{cases} u_n(s), & s \in [-t_n, -t] \\ u_0(s), & s \in (-t, -t + d_n]. \end{cases}$$

We denote by $\widetilde\gamma_n$ the associated trajectory, that is, $(\widetilde\gamma_n, \widetilde u_n) \in \Gamma^{x\to\bar\gamma(-t)}_{-t_n,-t+d_n}$. Then, defining the control $\widehat u_n(s) = \widetilde u_n(s - t_n)$, denoting by $\widehat\gamma_n$ the associated trajectory by **(L2)** and the fact that $\|u_0\|_\infty \leq 1$ we get

$$
\begin{aligned}
& A_{t_n-t+d_n}(x,\bar\gamma(-t)) - \alpha(L)(t_n-t+d_n) \\
\leq & \int_0^{t_n-t+d_n} L(\widehat\gamma_n(s),\widehat u_n(s))\,ds - \alpha(L)(t_n-t+d_n) \\
\leq & \int_{-t_n}^{-t} L(\gamma_n(s),u_n(s))\,ds + \int_{-t}^{-t+d_n} L(\gamma_0(s),u_0(s))\,ds - \alpha(L)(t_n-t+d_n) \\
\leq & \int_{-t_n}^{-t} L(\gamma_n(s),u_n(s))\,ds + (\ell_2+\ell_1-\alpha(L))d_n - \alpha(L)(t_n-t).
\end{aligned}
$$

Hence, from the lower-semicontinuity of the action we obtain

$$
\begin{aligned}
& \mathrm{A}_\infty(x,\bar\gamma(-t)) + \int_{-t}^0 L(\bar\gamma(s),\bar u(s))\,ds - \alpha(L)\,t \\
\leq & \liminf_{n\to+\infty}\left\{A_{t_n-t+d_n}(x,\bar\gamma(-t)) - \alpha(L)(t_n-t-d_n)\right\} + \int_{-t}^0 L(\bar\gamma(s),\bar u(s))\,ds - \alpha(L)\,t \\
\leq & \liminf_{n\to+\infty}\left\{(\ell_2+\ell_1-\alpha(L))d_n + \int_{-t_n}^{-t} L(\gamma_n(s),u_n(s))\,ds - \alpha(L)(t_n-t)\right\} \\
+ & \liminf_{n\to+\infty}\left\{\int_{-t}^0 L(\gamma_n(s),u_n(s))\,ds - \alpha(L)\,t\right\}.
\end{aligned}
$$

By combining together the terms inside the brackets we get

$$
\begin{aligned}
& \mathrm{A}_\infty(x,\bar\gamma(-t)) + \int_{-t}^0 L(\bar\gamma(s),\bar u(s))\,ds - \alpha(L)\,t \\
\leq & \liminf_{n\to+\infty}\left\{(\ell_2+\ell_1-\alpha(L))d_n + \int_{-t_n}^0 L(\gamma_n(s),u_n(s))\,ds - \alpha(L)\,t_n\right\} = \mathrm{A}_\infty(x,y).
\end{aligned}
$$

Therefore, we obtain

$$
\mathrm{A}_\infty(x,y) - \mathrm{A}_\infty(x,\bar\gamma(-t)) \geq \int_{-t}^0 L(\bar\gamma(s),\bar u(s))\,ds - \alpha(L)\,t. \tag{6.31}
$$

Next, we claim that

$$
\mathrm{A}_\infty(x,y) - \mathrm{A}_\infty(x,\bar\gamma(-t)) \leq \int_{-t}^0 L(\bar\gamma(s),\bar u(s))\,ds - \alpha(L)\,t. \tag{6.32}
$$

Indeed, by (6.28) we have that

$$
\mathrm{A}_\infty(x,y) - \mathrm{A}_\infty(x,\bar\gamma(-t)) \leq A_t(\bar\gamma(-t),y) - \alpha(L)\,t.
$$

Hence, defining the control

$$
\widehat u(s) = \bar u(s-t), \quad s \geq 0
$$

and denoting by $\widehat\gamma$ the associated trajectory, we deduce that

$$
\begin{aligned}
& \mathrm{A}_\infty(x,y) - \mathrm{A}_\infty(x,\bar\gamma(-t)) \leq A_t(\bar\gamma(-t),y) - \alpha(L)\,t \\
& \qquad \leq \int_0^t L(\widehat\gamma(s),\widehat u(s))\,ds - \alpha(L)\,t = \int_{-t}^0 L(\bar\gamma(s),\bar u(s))\,ds - \alpha(L)\,t.
\end{aligned}
$$

By combining (6.31) and (6.32) we obtain (6.30). The fact that $y \mapsto \mathrm{A}_\infty(x,y)$ is a critical solution for any $x \in \mathbb{R}^d$ can be proved by a standard argument which uses the dynamic programming principle. □

6.3.1 Compactness of the Aubry set

In this section, we prove that the projected Aubry set $\mathcal{A}$ is a compact subset of $\mathbb{R}^d$. We begin with some preliminaries.

Proposition 6.18. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *For any $x \in \mathbb{R}^d$ there exists $T_x \geq 0$ such that, for any $t \geq T_x$, any optimal pair $(\gamma_x, u_x) \in \Gamma_{0,t}^{x\to}$ for (6.6) satisfies*

$$\mathcal{L}^1 \left(\{s \in [0,t] : \gamma_x(s) \in \mathcal{K}_L\}\right) > 0.$$

Proof. We proceed by contradiction,. Suppose that there exist $x_0 \in \mathbb{R}^d$, $\{t_k\}_{k\in\mathbb{N}}$ with $t_k \to \infty$, and a sequence of optimal pairs $(\gamma_k, u_k) \in \Gamma_{0,t_k}^{x_0\to}$ of (6.6) such that

$$\mathcal{L}^1 \left(\{s \in [0,t_k] : \gamma_k(s) \in \mathcal{K}_L\}\right) = 0.$$

On the one hand, we have that

$$\int_0^{t_k} L(\gamma_k(s), u_k(s))\, ds > t_k \inf_{y\in\mathcal{K}_L^c} L(y,0). \tag{6.33}$$

On the other hand, having fixed any optimal pair $(\gamma_0, u_0) \in \Gamma_{0,\delta(x_0)}^{x_0\to x^*}$ for (1.5) and for any $k \in \mathbb{N}$ such that $t_k > \delta(x_0)$ define the control

$$\widetilde{u}_k(s) = \begin{cases} u_0(s), & s \in [0, \delta(x_0)] \\ 0, & s \in (\delta(x_0), t_k]. \end{cases}$$

Then, since $\|u_0\|_\infty \leq 1$ it follows that

$$\begin{aligned} \int_0^{t_k} L(\gamma_k(s), u_k(s))\, ds \leq & \int_0^{t_k} L(\widetilde{\gamma}_k(s), \widetilde{u}_k(s))\, ds \\ & \leq \delta(x_0)(\ell_1 + \ell_2) + (t_k - \delta(x_0))L(x^*, 0). \end{aligned} \tag{6.34}$$

Thus, combining (6.33) and (6.34) and dividing by t_k we get

$$\inf_{y\in\mathcal{K}_L^c} L(y,0) < \frac{1}{t_k}\delta(x_0)(\ell_1+\ell_2) + \left(1 - \frac{\delta(x_0)}{t_k}\right) L(x^*,0)$$

Moreover, by **(L3)** we deduce that

$$L(x^*,0) + \delta_L < \frac{1}{t_k}\delta(x_0)(\ell_1+\ell_2) + \left(1 - \frac{\delta(x_0)}{t_k}\right) L(x^*,0)$$

With $\delta_L > 0$. Taking the limit as $k \to \infty$ in the above inequalities yields $\delta_L \leq 0$ which is a contradiction. □

In view of the reversibility of L, the above Lemma implies the following.

Corollary 6.19. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *For any $x \in \mathbb{R}^d$ there exists $T_x \geq 0$ such that for any $t \geq T_x$, any optimal pair $(\gamma_x, u_x) \in \Gamma_{-t,0}^{\to x}$ for problem (6.6) we have that*

$$\mathcal{L}^1 \left(\{s \in [-t,0] : \gamma_x(s) \in \mathcal{K}_L\}\right) > 0.$$

We observe that since calibrated curves are, in particular, minimizing trajectories for (6.6) then Corollary 6.19 can be applied to such curves. This is a key point to deduce that the projected Aubry set is bounded, as we show below.

Proposition 6.20. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)**, *and* **(S)**. *Then $\mathcal{A}$ is bounded.*

Proof. Let $x_0 \in \mathbb{R}^d$ be such that $\mathrm{A}_\infty(x_0, x_0) = 0$. By Proposition 6.17 there exists $(\bar{\gamma}, \bar{u}) \in \Gamma^{\to x_0}_{-\infty,0}$ such that $\bar{\gamma}$ is a calibrated curve for $\mathrm{A}_\infty(x_0, \cdot\,)$ and, by Corollary 6.19, we know that there exists $t_0 \in (-\infty, 0]$ such that

$$\bar{\gamma}(t_0) \in \mathcal{K}_L.$$

Thus, the the trajectory $\widetilde{\gamma}$ associated with the control

$$\widetilde{u}(s) = \bar{u}(s + t_0),$$

for $s \in [0, -t_0]$, with $\widetilde{\gamma}(-t_0) = x_0$, is a calibrated curve for $\mathrm{A}_\infty(x_0, \cdot\,)$ such that $\widetilde{\gamma}(0) \in \mathcal{K}_L$. Then, from (6.9) this implies that there exists $R_L \geq 0$ such that $x_0 \in \overline{B}_{R_L}$. □

Next, we show that the projected Aubry set is closed.

Proposition 6.21. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *$\mathcal{A}$ is a closed subset of $\mathbb{R}^d$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ such that $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}^d$. Then, we have to show that $x \in \mathcal{A}$.

By definition we have that there exist sequence $\{t_{k_n}\}_{n\in\mathbb{N}}$ and $\{(\gamma_{k_n}, u_{k_n})\}_{n\in\mathbb{N}} \in \Gamma^{x_n \to x_n}_{0,t_{k_n}}$ such that

$$\int_0^{t_{k_n}} L(\gamma_{k_n}(s), u_{k_n}(s))\, ds - \alpha(L)\, t_{k_n} \leq \frac{1}{n}.$$

Then, by Lemma 6.15 there exists $(\bar{\gamma}, \bar{u})$ such that u_{k_n} weakly converges to $\bar{u}$ and γ_{k_n} uniformly converges to $\bar{\gamma}$, on every compact subset of $[0, \infty)$, respectively. Let us define $d_n = d_{\mathrm{SR}}(x_n, x)$ and the control

$$\widetilde{u}_n(s) \begin{cases} u_1^n(s), & s \in [-d_n, 0] \\ u_{k_n}(s), & s \in (0, t_{k_n}] \\ u_2^n(s), & s \in (t_{k_n}, t_{k_n} + d_{k_n}] \end{cases}$$

where $(\gamma_1^n, u_1^n) \in \Gamma^{x \to x_n}_{-d_n,0}$ and $(\gamma_2^n, u_2^n) \in \Gamma^{x_n \to x}_{t_{k_n}, t_{k_n}+d_{k_n}}$ are optimal for (1.5), on their respective intervals. Hence, we have that $(\widetilde{\gamma}_n, \widetilde{u}_n) \in \Gamma^{x\to x}_{-d_k, t_{k_n}+d_{k_n}}$ and, thus, we get

$$\begin{aligned} \mathrm{A}_\infty(x,x) &\leq \liminf_{n\to\infty} \left[A_{t_{k_n}+2d_n}(x,x) - \alpha(L)(t_{k_n} + 2d_n)\right] \\ &\leq \liminf_{n\to\infty} \Big(\int_{-d_n}^0 L(\gamma_1^n(s), u_1^n(s))\, ds + \int_0^{t_{k_n}} L(\gamma_{k_n}(s), u_{k_n}(s))\, ds - \alpha(L)\, t_{k_n} \\ &+ \int_{t_{k_n}}^{t_{k_n}+d_n} L(\gamma_2^n(s), u_2^n(s))\, ds - 2\,\alpha(L)\, d_n \Big) \leq \lim_{n\to\infty} \left(d_n(\ell_1 + \ell_2) + \frac{1}{n} \right) = 0. \end{aligned}$$

The proof is thus complete since, by definition, $\mathrm{A}_\infty(x,x) \geq 0$ for any $x \in \mathbb{R}^d$. □

Theorem 6.22 (**Compactness of the Aubry set**). *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Then, $\mathcal{A}$ is a nonempty compact set.*

Proof. The fact that $\mathcal{A}$ is compact follows from Proposition 6.20 and Proposition 6.21. Moreover, **(L3)** and Corollary 6.10 ensure that $\mathcal{A}$ is nonempty since $x^* \in \mathcal{A}$. □

6.4 Horizontal regularity of critical solutions

In this section we show that any critical solution is differentiable along the range of F, see the definition below, at any point lying on the projected Aubry set.

Definition 6.23 (**Horizontal differentiability).** *We say that a continuous function ψ on $\mathbb{R}^d$ is differentiable at $x \in \mathbb{R}^d$ along the range of $F(x)$ (or, horizontally differentiable at x) if there exists $q_x \in \mathbb{R}^m$ such that*

$$\lim_{v \to 0} \frac{\psi(x + F(x)v) - \psi(x) - \langle q_x, v \rangle}{|v|} = 0. \tag{6.35}$$

Clearly, if ψ is Frechét differentiable at x, than ψ is differentiable along the range of $F(x)$ and $q_x = F^\star(x)D\psi(x)$.

For any $\psi \in C(\mathbb{R}^d)$ we set $D^+_F\psi(x) = F^*(x)D^+\psi(x)$.

Lemma 6.24. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Let $\psi \in C(\mathbb{R}^d)$ be locally semiconcave. Then, ψ is differentiable at $x \in \mathbb{R}^d$ along the range of $F(x)$ if and only if $D^+_F\psi(x) = \{q_x\}$.*

Proof. We first prove that if $D^+_F\psi(x)$ is a singleton then ψ is differentiable at x along the range of $F(x)$. Let $\{q_x\} = D^+_F\psi(x)$ and take $p_x \in D^+\psi(x)$. Then

$$\psi(x + F(x)v) - \psi(x) - \langle p_x, F(x)v \rangle \le o(|F(x)v|) \le o(|v|).$$

Therefore, we deduce that

$$\limsup_{v \to 0} \frac{\psi(x + F(x)v) - \psi(x) - \langle q_x, v \rangle}{|v|} \le 0.$$

In order to prove the reverse inequality for the lim inf, let $\{v_k\}_{k \in \mathbb{N}}$ be any sequence such that $v_k \neq 0$, $v_k \to 0$ as $k \to +\infty$ and let

$$p_k \in D^+\psi(x + F(x)v_k).$$

Then

$$\begin{aligned}
&\frac{1}{|v_k|}\big(\psi(x + F(x)v_k) - \psi(x) - \langle p_x, F(x)v_k \rangle\big) \\
=\ &\frac{1}{|v_k|}\big(\psi(x + F(x)v_k) - \psi(x) - \langle p_k, F(x)v_k \rangle + \langle p_k - p_x, F(x)v_k \rangle\big) \\
\ge\ &\frac{1}{|v_k|} o(|F(x)v_k|) - |F^*(x)p_k - q_x||v_k|.
\end{aligned}$$

By the upper-semicontinuity of $D^+\psi$ we have that $|F^*(x)p_k - q_x| \to 0$ as $k \uparrow \infty$. Since since this is true for any sequence $v_k \to 0$, we conclude that

$$\liminf_{v \to 0} \frac{\psi(x + F(x)v) - \psi(x) - \langle q_x, v \rangle}{|v|} \ge 0.$$

We now prove that, if ψ is differentiable along the range of $F(x)$, then $D^+_F\psi(x)$ is a singleton. To do so, let $p \in D^+\psi(x)$ and let $q_x \in \mathbb{R}^m$ be as in (6.35). Then, we know that

$$\lim_{h \downarrow 0} \frac{\psi(x + hF(x)\theta) - \psi(x)}{h} \ge \langle q_x, \theta \rangle.$$

Moreover, by definition we have that for any $\theta \in \mathbb{R}^d$

$$\lim_{h\downarrow 0} \frac{\psi(x + hF(x)\theta) - \psi(x)}{h} \leq \langle F^* p, \theta\rangle.$$

Therefore,

$$\langle q_x, \theta\rangle \leq \langle F^* p, \theta\rangle, \quad \forall\, \theta \in \mathbb{R}^d.$$

Thus $F^*(x)p = q_x$. $\square$

Hereafter, the vector q_x given in Definition 6.23 will be called the horizontal differential of ψ at $x \in \mathbb{R}^d$ and will be denoted by $D_F\psi(x)$.

The next two propositions ensure that any critical solution χ is differentiable along the range of F at any point lying on a calibrated curve γ. The proof consists of showing that $D_F^+\chi$ is a singleton on γ. We recall that

$$L^*(x,p) = \sup_{v\in\mathbb{R}^d} \left\{\langle p, v\rangle - L(x,v)\right\}$$

is the Legendre Transform of L. We will rather write the critical equation using L^*, instead of the Hamiltonian H, to underline the role of horizontal differentiability.

Proposition 6.25. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Let χ be a critical subsolution and let (γ, u) be such that $\gamma : [0,\infty) \to \mathbb{R}^d$ is calibrated for χ. Then we have that*

$$\alpha(L) + L^*(\gamma(\tau), p) = 0, \quad \forall\, p \in D_F^+\chi(\gamma(\tau))$$

for all $\tau > 0$.

Proof. On the one hand, since χ is a subsolution of (6.11) we have that

$$\alpha(L) + H(\gamma(\tau), p) \leq 0, \quad \forall\, p \in D^+\chi(\gamma(\tau)).$$

So, from (6.24) and recalling that $D_F^+\chi(x) = F^*(x)D^+\chi(x)$ for any $x \in \mathbb{R}^d$ we get

$$\alpha(L) + L^*(\gamma(\tau), p) \leq 0, \quad \forall\, p \in D_F^+\chi(\gamma(\tau)).$$

Thus, it is enough to prove the reverse inequality.

Let $h \geq 0$, then since γ is a calibrated curve for χ we have that

$$\chi(\gamma(\tau)) - \chi(\gamma(\tau - h)) = \int_{\tau-h}^{\tau} L(\gamma(s), u(s))\, ds - \alpha(L)\, h.$$

Then, by the definition of super-differential we get

$$\begin{aligned}\chi(\gamma(\tau)) - \chi(\gamma(\tau - h)) &\leq \langle p, \gamma(\tau) - \gamma(\tau - h)\rangle + o(h)\\ = \langle p, \int_{\tau-h}^{\tau} \dot\gamma(s)\, ds\rangle + o(h) &= \int_{\tau-h}^{\tau} \langle F^*(\gamma(s))p, u(s)\rangle\, ds + o(h)\end{aligned}$$

Therefore, we conclude that

$$\int_{\tau-h}^{\tau} L(\gamma(s), u(s))\, ds - \alpha(L)\, h \leq \int_{\tau-h}^{\tau} \langle F^*(\gamma(s))p, u(s)\rangle\, ds + o(h)$$

or

$$\begin{aligned}-\alpha(L) &\leq \frac{1}{h}\int_{\tau-h}^{\tau} \Big(\langle F^*(\gamma(s))p, u(s)\rangle - L(\gamma(s), u(s))\Big)\, ds + o(1)\\ &\leq \frac{1}{h}\int_{\tau-h}^{\tau} L^*(\gamma(s), F^*(\gamma(s))p)\, ds + o(1).\end{aligned}$$

Thus, for $h \to 0$ we obtain the conclusion. $\square$

Proposition 6.26. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Let* χ *be a critical solution and let* (γ, u) *be such that* $\gamma : [0, \infty) \to \mathbb{R}^d$ *is calibrated for* χ. *Then, for any* $\tau > 0$ *we have that* χ *is differentiable at* $\gamma(\tau)$ *along the range of* $F(\gamma(\tau))$.

Proof. We recall that from [25, Theorem 1] we have that χ is semiconcave. By Proposition 6.25 we know that

$$\alpha(L) + L^*(\gamma(\tau), p) = 0$$

for any $p \in D_F^+\chi(\gamma(\tau))$. Moreover, we have that $L^*(x, \cdot)$ is strictly convex and the set $D_F^+\chi(x)$ is convex. Therefore, the above equality implies that $D_F^+\chi(\gamma(\tau))$ is a singleton. Consequently, Lemma 6.24 ensure that χ is differentiable at $\gamma(\tau)$ along the range of $F(\gamma(\tau))$. □

We are now ready to prove the differentiability of any critical solution on the Aubry set.

Theorem 6.27 (**Horizontal differentiability on the Aubry set**). *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Let* χ *be a critical solution. Then, the following holds.*

(I) *For any* $x \in \mathcal{A}$ *there exists a trajectory-control pair* $(\gamma_x, u_x) \in \Gamma^{\to x}_{-\infty,0} \cap \Gamma^{x\to}_{0,\infty}$ *such that*

$$A_\infty(\gamma_x(t), x) = -\int_0^t L(\gamma_x(s), u_x(s))\, ds + \alpha(L)\, t \tag{6.36}$$

and

$$A_\infty(x, \gamma_x(-t)) = -\int_{-t}^0 L(\gamma_x(s), u_x(s))\, ds + \alpha(L)\, t \tag{6.37}$$

(ii) $\gamma_x : \mathbb{R} \to \mathbb{R}^d$ *is calibrated for* χ.

(iii) χ *is horizontally differentiable at* $x \in \mathcal{A}$.

Proof. We start by proving (6.36). Since $x \in \mathcal{A}$. We have that $A_\infty(x, x) = 0$. So there exist $\{t_n\}_{n\in\mathbb{N}}$ and $(\gamma_n^+, u_n^+) \in \Gamma^{x\to x}_{0,t_n}$ such that

$$t_n \to +\infty \quad \text{and} \quad \lim_{n\to+\infty} \int_0^{t_n} L(\gamma_n^+(s), u_n^+(s))\, ds - \alpha(L)\, t_n = 0. \tag{6.38}$$

Then, by Lemma 6.15 there exists (γ_x^+, u_x^+) such that u_n^+ weakly converges to u_x^+ and γ_n^+ uniformly converges to γ_x^+, on every compact subset of $[0, \infty)$, respectively. Fix $t \in [0, +\infty)$, fix n large enough such that $d_n := d_{\mathrm{SR}}(\gamma_x^+(t), \gamma_n^+(t)) \le 1$ and $t + 1 < t_n$. Let $(\gamma_0, u_0) \in \Gamma^{\gamma_x^+(t)\to\ \gamma_n^+(t)}_{t,t+d_n}$ be a solution of (1.5) and let $\widetilde{u}_n \in L^2(t, t_n + d_n)$ be such that

$$\widetilde{u}_n(s) = \begin{cases} u_0(s), & s \in [t, t + d_n] \\ u_n^+(s), & s \in (t + d_n, t_n]. \end{cases}$$

Then, recalling that $\|u_0\|_\infty \le 1$ we obtain

$$\begin{aligned} \int_t^{t_n} L(\widetilde{\gamma}_n(s), \widetilde{u}_n(s))\, ds &= \int_t^{t+d_n} L(\gamma_0(s), u_0(s))\, ds + \int_{t+d_n}^{t_n} L(\gamma_n(s), u_n(s))\, ds \\ &\le (\ell_1 + \ell_2)d_n + \int_{t+d_n}^{t_n} L(\gamma_n(s), u_n(s))\, ds. \end{aligned}$$

Now, defining $\widehat{u}_n(s) = \widetilde{u}_n(s-t)$ and denoting by $\widehat{\gamma}_n$ the associated trajectory we get

$$
\begin{aligned}
&\mathrm{A}_\infty(\gamma_x^+(t), x) \leq \liminf_{n\to+\infty} \left[A_{t_n-t} - \alpha(L)(t_n - t)\right] \\
&\leq \liminf_{n\to+\infty} \left[\int_0^{t_n-t} L(\widehat{\gamma}_n(s), \widehat{u}_n(s))\, ds - \alpha(L)(t_n-t)\right] \\
&\leq \liminf_{n\to+\infty} \left[\int_t^{t+d_n} L(\gamma_0(s), u_0(s))\, ds + \int_{t+d_n}^{t_n} L(\gamma_n(s), u_n(s))\, ds - \alpha(L)(t_n-t)\right] \\
&\leq \liminf_{n\to+\infty} \left[(\ell_1+\ell_2)d_n + \int_{t+d_n}^{t_n} L(\gamma_n^+(s), u_n^+(s))\, ds - \alpha(L)(t_n-t)\right] \\
&= \liminf_{n\to+\infty} \left[(\ell_1+\ell_2)d_n + \int_0^{t_n} L(\gamma_n^+(s), u_n^+(s))\, ds - \alpha(L)\, t_n - \int_0^{t+d_n} L(\gamma_n^+(s), u_n^+(s))\, ds + \alpha(L)\, t\right]
\end{aligned}
$$

Then, by (6.39), the uniform convergence of γ_n and the fact that $d_n \downarrow 0$ we deduce that

$$\int_0^t L(\gamma^+(s), u_x^+(s))\, ds - \alpha(L)\, t + \mathrm{A}_\infty(\gamma_x^+(t), x) \leq 0.$$

Moreover, we also have that

$$\mathrm{A}_\infty(x, \gamma_x^+(t)) = \mathrm{A}_\infty(x, \gamma_x^+(t)) - \mathrm{A}_\infty(x,x) \leq \int_0^t L(\gamma_n^+(s), u_n^+(s))\, ds - \alpha(L)\, t$$

and $\mathrm{A}_\infty(x, \gamma_x^+(t)) + \mathrm{A}_\infty(\gamma_x^+(t), x) \geq 0$. Therefore, we obtain

$$\int_0^t L(\gamma_x^+(s), u_x^+(s))\, ds - \alpha(L)\, t + \mathrm{A}_\infty(\gamma_x^+(t), x) = 0.$$

Similar arguments show that there exists $(\gamma_x^-, u_x^-) \in \Gamma_{-\infty,0}^{\to x}$ such that (6.37) holds. Indeed, it is enough to consider $\{t_n\}_{n\in\mathbb{N}}$ and $(\gamma_n^-, u_n^-) \in \Gamma_{-t_n,0}^{x\to x}$ such that

$$t_n \to +\infty \quad \text{and} \quad \lim_{n\to+\infty} \int_{-t_n}^0 L(\gamma_n^-(s), u_n^-(s))\, ds - \alpha(L)\, t_n = 0. \tag{6.39}$$

Then, by Remark 6.16 there exists (γ_x^-, u_x^-) such that u_n^- weakly converges to u_x^- and γ_n^- uniformly converges to γ_x^-, on every compact subset of $[0, \infty)$, respectively. Therefore, defining the control

$$u_x(s) = \begin{cases} u_x^+(s), & s \in [0,\infty) \\ u_x^-(s), & s \in (-\infty, 0] \end{cases}$$

and denoting by γ_x the associated trajectory the proof of (i) is complete.

Next, we prove (ii), that is, γ_x is a calibrated curve for χ. From (6.12) we know that $\chi \prec L - \alpha(L)$ and thus for any $t \geq 0$ the following hold

$$\chi(\gamma_x(t)) - \chi(x) \leq \int_0^t L(\gamma_x(s), u_x(s))\, ds - \alpha(L)\, t.$$

Moreover, again from (6.12) we deduce that

$$\chi(x) - \chi(\gamma_x(t)) \leq A_s(\gamma_x(t), x) - \alpha(L)\, s$$

for any $s \geq 0$. Thus, we get

$$\chi(x) - \chi(\gamma_x(t)) \leq \mathrm{A}_\infty(\gamma_x(t), x) = -\int_0^t L(\gamma_x(s), u_x(s))\, ds + \alpha(L)\, t.$$

This proves that γ_x is a calibrated curve for χ on $[0, \infty)$. Similarly, one can prove that the same holds on $(-\infty, 0]$. Moreover, if we consider $-s < 0 < t$ we can write

$$\begin{aligned}\chi(\gamma_x(t)) - \chi(\gamma_x(-s)) &= \chi(\gamma_x(t)) - \chi(x) + \chi(x) - \chi(\gamma_x(-s)) \\ &= \int_0^t L(\gamma_x(\tau), u_x(\tau))\, d\tau - \alpha(L)\, t + \int_{-s}^0 L(\gamma_x(\tau), u_x(\tau))\, d\tau - \alpha(L)\, s \\ &= \int_{-s}^t L(\gamma_x(\tau), u_x(\tau))\, d\tau - \alpha(L)(t+s),\end{aligned}$$

and this completes the proof of (ii).

Finally, by (ii) and Proposition 6.26 we deduce that χ is differentiable at $x \in \mathcal{A}$ along the range of $F(x)$. □

Proposition 6.28. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Let χ be a critical solution. Let $x \in \mathcal{A}$ and let (γ_x, u_x) be such that γ_x is a calibrated curve for χ on $\mathbb{R}$ with $\gamma_x(0) = x$. Then, we have that*

$$\gamma_x(t) \in \mathcal{A}, \quad t \geq 0. \tag{6.40}$$

Proof. In order to prove (6.40) it is enough to show that

$$\mathrm{A}_\infty(\gamma_x(t), \gamma_x(t)) \leq 0, \quad t \geq 0 \tag{6.41}$$

since it is always true that $\mathrm{A}_\infty(\gamma_x(t), \gamma_x(t)) \geq 0$ for any $t \geq 0$. From (6.27) the following holds

$$\mathrm{A}_\infty(\gamma_x(t), \gamma_x(t)) \leq \mathrm{A}_\infty(\gamma_x(t), x) + \mathrm{A}_\infty(x, \gamma_x(t)), \quad t \geq 0. \tag{6.42}$$

Since γ_x is calibrated for χ we deduce that

$$\mathrm{A}_\infty(x, \gamma_x(t)) = \int_0^t L(\gamma_x(s), u_x(s))\, ds - \alpha(L)\, t. \tag{6.43}$$

and

$$\mathrm{A}_\infty(\gamma_x(t), x) = -\int_0^t L(\gamma_x(s), u_x(s))\, ds + \alpha(L)\, t. \tag{6.44}$$

Hence, combining (6.43) and (6.44) with (6.42) we get (6.41) which we recall that it implies (6.40). □

Corollary 6.29. *Assume* **(F0)** – **(F2)**, **(L0)** – **(L3)** *and* **(S)**. *Let χ be a critical solution, let $x \in \mathcal{A}$ and let γ_x be calibrated for χ. Then, γ_x satisfies the state equation with control*

$$u_x(t) = D_p L^*(\gamma_x(t), D_F\chi(\gamma_x(t))), \quad t \geq 0.$$

Moreover,

$$D_F\chi(\gamma_x(t)) = D_u L(\gamma_x(t), u_x(t)), \quad t \geq 0.$$

Proof. Let χ be a critical solution, let $x \in \mathcal{A}$ and let γ_x be a calibrated curve for χ. Let u_x be the control associated with γ_x. Then, from the Maximum Principle and the inclusion of the dual arc into the superdifferential of the corresponding value function, e.g. [26, Theorem 7.4.17], we have that

$$\langle D_F\chi(\gamma_x(t)), u_x(t)\rangle = L(\gamma_x(t), u_x(t)) + L^*(\gamma_x(t), D_F\chi(\gamma_x(t)))$$

for any $t \geq 0$. Note that, $D_F\chi(\gamma_x(t))$ is well-defined by Proposition 6.26. Hence, by the properties of the Legendre Transform we obtain

$$u_x(t) = D_p L^*(\gamma_x(t), D_F\chi(\gamma_x(t))), \quad t \geq 0$$

and

$$D_F\chi(\gamma_x(t)) = D_u L(\gamma_x(t), u_x(t)), \quad t \geq 0. \qquad \square$$

Remark 6.30. Following the classical Aubry-Mather theory for Tonelli Hamiltonian systems, one can define the Aubry set $\widetilde{\mathcal{A}} \subset \mathbb{R}^d \times \mathbb{R}^m$ as

$$\widetilde{\mathcal{A}} = \bigcap \{(x, u) \in \mathcal{A} \times \mathbb{R}^m : D_F\chi(x) = D_u L(x, u)\}$$

where the intersection is taken over all the critical solutions χ. Note that such a set is nonempty since $(x^*, 0) \in \widetilde{\mathcal{A}}$.

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