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Bose-Einstein condensation for two dimensional interacting bosons: mean field and Gross-Pitaevskii scalings

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Abstract

This thesis is concerned with static properties of large bosonic systems in two dimensions. These systems at very low-temperatures are expected to exhibit *Bose-Einstein condensation*. From a mathematical and physical, point of view it is interesting to provide conditions for the occurrence of Bose-Einstein condensation. Obviously, studying a system of N particles, where N is large, is very challenging. However, to overcome this problem we can rely on effective theories, which describe the collective behaviour of the particles.

The aim of the manuscript is to present new results regarding the occurrence of Bose-Einstein condensation in two-dimensional bosonic systems in suitable scaling limits.

Our first result consists of the rigorous derivation of complete Bose-Einstein condensation of low-energy states in a regime where the interaction potential scales as $N^{2\beta}V(N^{\beta}\cdot)$, for $\beta > 0$ such that $\log_{N\to\infty}(\log N^{\beta})/N = 0$, where N is the number of particles. We show that the system exhibits complete Bose-Einstein condensation with a uniform bound on the number of the excitations and we prove upper and lower bounds on the ground state energy of the system up to $\mathcal{O}(1)$.

Our second result regards the *Gross-Pitaevskii regime*. In this scenario the range of the two-body potential is exponentially small with respect to the number of particles, *i.e.* the potential scales as $e^{2N}V(e^{N}\cdot)$. The strong singularity of the interaction implies that the correlations among particles play a crucial role. In this limit, we improve existing results on the emergence of Bose-Einstein condensation, providing an almost optimal bound on the rate of condensation and more precise bounds on the ground state energy.

In the conclusion, we briefly analyze our progress concerning the project of verifying the predictions of Bogoliubov theory for 2d bosons in the Gross-Pitaevskii regime. The goal consists here in deriving an asymptotic expansion of the ground state energy up to the second order and the low-energy spectrum of the Hamiltonian related to the system.

Throughout the manuscript we highlight the main differences between the two scaling regimes that we are considering. On the contrary, the last chapter, (Chapter 5), is focused on the Gross-Pitaevskii regime.

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Contents

1 Introduction **1**

1.1 Bose-Einstein condensation for 2d bosons interacting through singular potentials . . . 6

1.2 Bose-Einstein condensation in the Gross-Pitaevskii regime . . . 8

2 Bose-Einstein condensation: outline of the proof **12**

2.1 The Fock space setting: focusing on excited particles . . . 12

2.2 Renormalization of the excitation Hamiltonian $\mathcal{L}_N^\beta$ . . . 18

2.2.1 Quadratic renormalization . . . 22

2.2.2 Cubic Renormalization . . . 28

2.2.3 Proof of Theorem 1.1 . . . 29

2.3 Renormalization of the excitation Hamiltonian $\mathcal{L}_N$ . . . 33

2.3.1 Quadratic renormalization . . . 37

2.3.2 Cubic Renormalization . . . 38

2.3.3 Proof of Theorem 1.3 . . . 43

3 Analysis of the Renormalized Hamiltonian for bosons interacting through singular potentials **48**

3.1 Analysis of the quadratically renormalized excitation Hamiltonian $\mathcal{G}_{N,\ell}^\beta$ . . . 48

3.1.1 Analysis of $\mathcal{G}_{N,\ell}^{\beta,(0)} = e^{-B_H}\mathcal{L}_N^{\beta,(0)}e^{B_H}$ . . . 49

3.1.2 Analysis of $\mathcal{G}_{N,\ell}^{\beta,(2)} = e^{-B_H}\mathcal{L}_N^{\beta,(2)}e^{B_H}$ . . . 51

3.1.3 Analysis of $\mathcal{G}_{N,\ell}^{\beta,(3)} = e^{-B_H}\mathcal{L}_N^{\beta,(3)}e^{B_H}$ . . . 64

3.1.4 Analysis of $\mathcal{G}_{N,\ell}^{\beta,(4)} = e^{-B_H}\mathcal{L}_N^{\beta,(4)}e^{B_H}$ . . . 70

3.1.5 Proof of Propositions 2.4 . . . 78

3.2 Analysis of the cubically renormalized excitation Hamiltonian $\mathcal{R}_{N,\ell}^\beta$ 82

3.2.1 A priori bounds on the energy . . . 83

3.2.2 Analysis of $e^{-A_H}\mathcal{O}_N e^{A_H}$ . . . 88

3.2.3 Contributions from $e^{-A_H}\mathcal{K}e^{A_H}$ . . . 90

3.2.4 Analysis of $e^{-A_H}\mathcal{Q}_N e^{A_H}$ . . . 93

3.2.5 Contributions from $e^{-A_H}\mathcal{C}_N e^{A_H}$ . . . 95

3.2.6 Proof of Proposition 2.8 . . . 98

4 Analysis of the Renormalized Gross - Pitaevskii Hamiltonian 103
4.1 Analysis of the quadratically renormalized excitation Hamiltonian $\mathcal{G}_{N,\alpha}$ 103
4.1.1 Analysis of $\mathcal{G}^{(0)}_{N,\alpha} = e^{-B}\mathcal{L}^{(0)}_N e^B$ 104
4.1.2 Analysis of $\mathcal{G}^{(2)}_{N,\alpha} = e^{-B}\mathcal{L}^{(2)}_N e^B$ 104
4.1.3 Analysis of $\mathcal{G}^{(3)}_{N,\alpha} = e^{-B}\mathcal{L}^{(3)}_N e^B$ 113
4.1.4 Analysis of $\mathcal{G}^{(4)}_{N,\alpha} = e^{-B}\mathcal{L}^{(4)}_N e^B$ 116
4.1.5 Proof of Proposition 2.11 123
4.2 Analysis of the cubically renormalized excitation Hamiltonian $\mathcal{R}_N$ 126
4.2.1 A priori bounds on the energy 127
4.2.2 Analysis of $e^{-A}\mathcal{O}_N e^A$ 131
4.2.3 Contributions from $e^{-A}\mathcal{K}e^A$ 132
4.2.4 Analysis of $e^{-A}\mathcal{Z}_N e^A$ 137
4.2.5 Contributions from $e^{-A}\mathcal{C}_N e^A$ 139
4.2.6 Proof of Proposition 2.14 142

5 Conclusion and Perspectives 146

A Properties of the Scattering Function in the GP scaling 150

B Properties of the Scattering Function through singular potentials 160

C Ground state for many-bosons system through singular potentials 166
C.1 Introduction 166
C.2 Generalized Dyson Lemma 168
C.3 Second moment estimate 174
C.4 Three body-estimate 180
C.5 Energy lower bound and convergence of states 183
C.5.1 Convergence of density matrices 186

Chapter 1

Introduction

A *Bose-Einstein condensate* is an exciting state of matter which occurs in dilute gases of bosonic atoms (and even in more general bosonic systems) at very low-temperatures. It roughly consists of a macroscopic fraction of the particles behaving as they occupy the same one-particle state. The first theoretical prediction of this phenomenon was given by Einstein in 1925 for cold atomic gases [24, 25], based on previous ideas of the physicist Bose in 1924 [12].

After seventy years, the groups of Cornell, Ketterle and Wieman got some fruitful results at an experimental level, verifying the phenomenon in laboratories. Their discovery led them to win the Nobel prize in 2001. To observe BEC experimentally, physicists cool down atomic gases to very low-temperatures in extremely dilute regimes (this is to prevent the gases from solidifying); see [13, Chapter 1] for a review on experiments.

Although Einstein predicted theoretically the realization of BEC for non-interacting bosons, one is clearly interested in considering the more realistic case of interacting systems. From a mathematical point of view, the problem of showing the occurrence of BEC for interacting particles, has been addressed later. The first trace can be found in the work of Bogoliubov in 1947, [11]. Bogoliubov theory, under the assumption of BEC, showed that dilute interacting bosons exhibit superfluidity, by predicting a linear spectrum of excitations for small momenta. Although Bogoliubov approach is an illuminating treatment, it is not mathematically rigorous. We will spend some more precise words on Bogoliubov theory in Chapter 5.

The rigorous justification of the occurrence of Bose-Einstein condensation and of the validity of Bogoliubov predictions in interacting Bose gas is a challenging problem. So far, realization of BEC for interacting particles has been established for an homogeneous system only in the special case of hard core bosons on a lattice at half-filling in dimension greater or equal three [23]. The only available results for a general dilute gas in the *thermodynamic limit*, i.e. where N interacting bosons are confined in a periodic box with volume L^3, $L > 0$, with density $\rho = N/L^3$ kept fixed and the volume of the confining box increases to infinity (limit as $L \to \infty$), concern the expression for the ground state energy (see [73, 30] for latest results).

Even though to affirm that a *phase transition* really occurs one needs to study the thermodynamic limit, it is interesting to investigate the equilibrium properties of interacting bosons in simpler, but still physically relevant, dilute regimes, where the effective range of interaction, described by the *scattering length* (later denoted by $\mathfrak{a}$) of the interacting potential, is let to depend on the number of particles in such a way that $\rho\mathfrak{a}^2 \to 0$ as $N \to \infty$. In these settings, the N- dependent potential might be understood as an effective description for interactions occurring in large many particles systems. Examples of these effective theories are, for instance, the Gross-Pitaevskii theory for strongly interacting systems, and the Hartree theory for weak interactions. For both scaling, there are many results in three dimensions. BEC for three-dimensional bosons has been established in the *mean-field* regime (see reference in [15, page 3] for a complete list of result in this regime) in the Gross-Pitaevskii regime, and intermediate regimes interpolating the two (a complete list of these results appears throughout [67, Chap. 4, Chap.5] respectively). In the same regimes the predictions of Bogoliubov theory have been verified [70, 35, 44, 9, 8].

While three dimensional settings have been studied intensively, the problem in lower dimensions, both from an experimental and theoretical point of view, has got attention later. Experiments for bosons confined in a optical traps have been first realized in two-dimensions in 2001 by Görlitz et al. [34] (again see [13] for a review on other related experiments).

From a theoretical point of view, the two-dimensional case is critical. In fact, while a very general theorem due to Mermin-Wagner-Hohenberg [57, 36] rules out the occurrence of BEC in two-dimensional systems at any positive temperature, a phase transition is expected at zero temperature. This criticality makes the study of Bose-Einstein condensation in two-dimensions even more challenging and interesting.

Turning to the simpler problem of characterizing the equilibrium properties of 2d Bose gases, the first prediction for the ground state energy in the thermodynamic limit was obtained by Schick in the '70s [68], later confirmed by Lieb-Yngvason in [55]. The expected expression for the second order correction has been proved for 2d bosons restricted to quasi-free states by Fournais-Napiorkowski-Reuvers-Solovej in [29]. Furthermore, systems of two-dimensional bosons in the Gross-Pitaevskii regime confined by external trapping potentials have been studied in [51, 46, 47]. If one consider intermediate scaling limits, between Gross-Pitaevski and mean-field, results have been obtained in [41, 42, 59] (see also next subsections).

The aim of this thesis is to present new results concerning the occurrence of Bose-Einstein condensation in two-dimensional bosons in suitable scaling limits.

Our setting

Let us briefly sketch the mathematical setting and the scaling limits we will be interested in. What we are going to say is standard and can be found for instance in [66, Chapter II] and in [49].

In this thesis we are interested in studying static properties of large two-dimensional bosonic systems.

A single particle in quantum mechanics is identified by a -normalized- vector

$\psi \in \mathfrak{h}$, $\|\psi\|_{\mathfrak{h}} = 1$, where $\mathfrak{h}$ is a complex, separable Hilbert space. A system of $N \in \mathbb{N}$ particles is described by a vector $\psi_N \in \mathfrak{h}^N = \mathfrak{h}^{\otimes N}$. We are considering particles which obey Bose-Einstein statistics, they are called bosons. We deal with a system of N bosons, which is described by a vector $\psi_N \in \mathfrak{h}^N$, invariant under permutations, namely:

$$\psi_N(x_1, \dots, x_j, \dots, x_k, \dots, x_N) = \psi_N(x_1, \dots, x_k, \dots, x_j, \dots, x_N),$$

for any $j \neq k = 1, \dots, N$.

Throughout this thesis, our Hilbert space for one particle is $\mathfrak{h} := L^2(\Lambda)$, where $\Lambda = [-1/2; 1/2]^2$ a unit box in two dimensions with periodic boundary conditions, identified with the two-dimensional unit torus. Hence, the N-particle bosonic system is described by $\mathfrak{h}_s^N := L_s^2(\Lambda^N)$.

The energy of N interacting bosons can be generally described by a Hamilton operator H_N which acts as a self-adjoint operator in $\mathfrak{h}_s^N$ and has the form

$$H_N = \sum_{i=1}^{N} (-\Delta_{x_i} + V_{ext}(x_i)) + \sum_{1 \le i < j \le N} v(x_i - x_j). \tag{1.1}$$

In (1.1) the one-body operator $-\Delta_{x_i}$, $i = 1, \dots, N$ is the Laplacian with respect to the variable $x_i \in \Lambda$, it measures the kinetic energy of the i-th particle. The operator V_{ext} denotes an external potential. In our proof we will consider bosons trapped in a box, with periodic boundary conditions without external potential, but we expect our results to hold in this more general setting. Finally, v is a real-valued, measurable function which models the interactions among particles and acts as a multiplication operator. We assume v radially symmetric and non-negative. Notice that, here, we are considering a two-body interaction. In fact, since we deal with very dilute regimes we can neglect all interactions involving three or more particles.

With the conditions on the interaction potential that we are going to impose, we can ensure that $H_N : D(H_N) \to L_s^2(\Lambda^N)$, is a densely-defined, self-adjoint operator, bounded from below. Thus, we define the *ground state energy* E_N as

$$E_N = \inf_{\substack{\psi_N \in D(H_N), \\ \|\psi_N\| = 1}} \langle \psi_N, H_N \, \psi_N \rangle. \tag{1.2}$$

The ground state energy corresponds to the lowest possible energy of the system. Since we deal with large bosonic systems, an exact computation of E_N is not feasible. However, one can aim to get a good approximation in the limit for large N, i.e. $N \to \infty$. Moreover, one can also investigate the energy levels above the ground state energy, called *excitation spectrum*, and the conditions on the trapping potential also ensure that the spectrum is purely discrete. We will give a hint about how the techniques introduced in the thesis can be used to investigate the excitation spectrum in Chapter 5.

As mentioned above, we aim to prove Bose-Einstein condensation for low-energy states at zero temperature in two dimensions. A mathematical formalization of this phenomenon for general interacting system was given by Onsager and Penrose [61]. Their definition makes use of the so-called *one-particle reduced density matrix* $\gamma_N^{(1)} \in \mathcal{L}(L^2(\Lambda))$, of a many-body wave function ψ_N, where with $\mathcal{L}(L^2(\Lambda))$ we denote the space of trace-class operators. The one-particle reduced density matrix is defined through its integral kernel

$$\gamma_N^{(1)}(x;y) = \int_{\Lambda^{N-1}} \psi_N(x, x_2, \dots, x_N)\overline{\psi}_N(y, x_2, \dots, x_N)dx_2 \dots dx_N \tag{1.3}$$

for $x, y \in \Lambda$. Equivalently, it is defined as

$$\gamma_N^{(1)} := \mathrm{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|,$$

normalized such that $\mathrm{tr}\gamma_N^{(1)} = 1$. Then a sequence of many-body wave functions $(\psi_N)_{N\in\mathbb{N}} \in L^2_s(\Lambda^N)$ with associated sequence of one-particle reduced density matrices $(\gamma_N^{(1)})_{N\in\mathbb{N}}$ exhibits complete Bose-Einstein condensation in the one-particle wave function $\varphi \in L^2(\Lambda)$ with associated orthogonal rank-one projection $|\varphi\rangle\langle\varphi| \in \mathcal{L}(L^2(\Lambda))$ if

$$\lim_{\substack{N,\Lambda\to\infty, \\ N/|\Lambda|=\mathrm{const}}} \mathrm{tr}\big|\gamma_N^{(1)} - |\varphi\rangle\langle\varphi|\big| = 0. \tag{1.4}$$

One can also consider different scaling regimes, keeping $|\Lambda| = 1$ and setting the interaction potential depending on the number of particles N. In these settings, where condensation is said to occur if the convergence (1.4) is established in the limit $N \to \infty$ (without requiring the density to be constant), there are many results both for the investigation of the ground state and the excitation spectrum. For a discussion of the results on the dynamical and static properties of 3d and 2d bosons, including very recent results on BEC for positive temperature we refer the reader to [67, 21, 5, 33] and reference therein.

From now on we will focus only on the two-dimensional case. In the following we will briefly recall the 2d scalings that will be investigated in the manuscript.

From the mean-field to the Gross-Pitaevskii regime

First, we consider H_N being of the form

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + \frac{1}{N} \sum_{1\leq i<j\leq N} V(x_i - x_j), \tag{1.5}$$

and acting on $L^2([-1/2; 1/2]^{2N})$; this is the so-called *mean-field* regime. Notice that the kinetic and the potential parts are of the same order with respect to the number of particles N, hence they give the same contribution to the energy. In this limit each particle interacts essentially with all the others, since the range of the interaction is comparable to the extension of the whole system. Thus, we can say that interactions are frequent, but weak.

We can go one step further and investigate even low density regimes where the interaction is more singular and converges to a Dirac mass. Namely, we consider interaction potentials that scales as $N^{2\beta-1}V(N^{\beta}\cdot)$. Actually, for $\beta < 1/2$ the behavior is similar to the mean-field regime $\beta = 0$, since the range of the interaction is much smaller than the full system size, but it stays much larger than the typical inter-particle distance $N^{-1/2}$. On the other hand, when $\beta > 1/2$ the range of the interactions is much smaller than the typical inter-particle distance (see [67] for a detailed explanation), hence interaction becomes stronger.

In two dimensions, a critical threshold is determined by the limit $\ell = \lim_{N\to\infty}(\log \mathfrak{a}_N)/N$, with $\mathfrak{a}_N$ is the scattering length of the interaction, (i.e. the effective range of interaction between particles), when $\ell \neq 0$ the system is in the so-called *Gross-Pitaevskii regime*. In general, this setting describes a situation where N particles are confined in a two-dimensional box of side-length L and interact through a potential whose scattering length is of order e^{-N} (remember that we consider unit boxes). Hence, we deal with an extreme dilute regime.

There are a few results in the literature concerning the investigation of the ground state energy for regimes where $\mathfrak{a}_N \sim N^{-\beta}$ and $\ell = 0$. Lewin, Nam and Rougerie [41, 42, 59], obtained convergence of the ground state energy and of the one-particle density matrices for $\beta < 1$. Remarkably, for $\beta < 1$ they can also consider non-positive potentials (see also [67] and reference therein for a review). We consider this regime in Section 1.1 explaining which results we obtained.

Let us now focus on the more challenging case, the Gross-Pitaevskii limit. In this scaling the range of the interaction is exponentially small with respect to the number of particles, *i.e.* the potential scales as $e^{2N}V(e^{N}\cdot)$. This regime has been first studied by Lieb-Seiringer-Yngvason [47, 51, 50, 46]. In these papers they proved the exhibition of condensation and the expression for the ground state energy. To be precise, in [47] they considered the harder case with a magnetic potential (see also [49] for details and a review on these results).

Similar results have been obtained starting from a three dimensional Bose gas, trapped by a potential which is strongly confining in one direction, so that the system becomes effectively two-dimensional [69]. Finally, it is worth to mention [37, 14], where rigorous results on the time-evolution in the two-dimensional Gross-Pitaevskii regime have been established (in [14], the focus is on the dynamics of a three-dimensional gas, with strong confinement in one direction).

In [20] we improve the result in [55] and prove optimal condensation up to logarithmic corrections. This work is part of this thesis, we explain our result in Section 1.2, Chapter 2 and 4.

In the next sections we explain our main results in two-dimensions and we give a general idea of the proof, which stems from ideas introduced in [7, 9, 10] for the analysis of 3d bosons.

1.1 Bose-Einstein condensation for 2d bosons interacting through singular potentials

We consider N bosons in a box in $\mathbb{R}^2$ of side-length one, i.e. $\Lambda = [-1/2; 1/2]^2$, described by the Hamilton operator

$$H_N^\beta = \sum_{j=1}^N -\Delta_{x_j} + \sum_{i<j}^N N^{2\beta-1} V(N^\beta(x_i - x_j)), \tag{1.6}$$

where $\beta > 0$ such that $\lim_{N\to\infty} (\log N^\beta)/N = 0$.

In this regime we establish an upper and lower bound for the ground state energy of the system up to $\mathcal{O}(1)$ corrections and obtain a proof of condensation with optimal rate for all low energy states, as described by the following theorem.

Theorem 1.1. *Let $V \in L^2(\mathbb{R}^2)$ have compact support and be pointwise non-negative. Let $\beta > 0$. Then there exists a constant $C > 0$ such that the ground state energy E_N^β of (1.10) satisfies*

$$\left| E_N^\beta - \frac{\widehat{V}(0)}{2} N + \frac{\widehat{V}(0)^2}{8\pi} \log N^\beta \right| \le C\,. \tag{1.7}$$

Furthermore, consider a sequence $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and such that

$$\langle \psi_N, H_N^\beta \psi_N \rangle \le \frac{\widehat{V}(0)}{2} N - \frac{\widehat{V}(0)^2}{8\pi} \log N^\beta + K$$

for a $K > 0$. Then the reduced density matrix $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N} |\psi_N\rangle\langle\psi_N|$ associated with ψ_N satisfies

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \le \frac{C(K+1)}{N} \tag{1.8}$$

for all $N \in \mathbb{N}$ large enough.

Remark.

1. The condition $V \in L^2(\mathbb{R}^2)$ comes from the proof of properties of the scattering function associated to the potential $N^{2\beta-1}V(N^\beta x)$, that we need in our analysis, see Chapter B for more details.

2. Notice that Theorem 1.1 eventually holds for any $\beta > 0$ such that $\lim_{N\to\infty}(\log N^\beta)/N = 0$. Indeed it might depend on N.

The strategy of the proof of Theorem 1.1 follows [10]. We will work in a second quantization setting to describe the fact that the number of excitations (i.e. particles outside the condensate) vary. Using a unitary map introduced by Lewin-Nam-Serfaty-Solovej [44] (see Section 2.1 for the precise definition) we factor out particles in the condensate from the Hamiltonian H_N^β. Hence, we are able to rewrite H_N^β as a new excited Hamiltonian - that we will call $\mathcal{L}_N^\beta$- where the particles in the condensate do not appear anymore.

However, this excitation Hamiltonian is not enough to get the correct ground state energy or to show condensation. In fact, in $\mathcal{L}_N^\beta$ there are still important constant contributions hidden in the cubic and quartic terms. This is because, the action of the map U_N, which factors out the condensate, does not take into account the correlation between particles. Hence, to overcome this problem, we construct a unitary map which is obtained by taking the exponential of an anti-symmetric operator B_H quadratic in the excitations. This map was first introduced in [4], in the Fock space, then implemented in [17] and [7], in the 3d setting, and it is constructed in such a way that the coefficients take into account the correlation structure. In particular, these coefficients come from the solution to the scattering equation. The action of the unitary map e^{B_H} allows us to obtain a new Hamiltonian, called $\mathcal{G}^{\beta}_{N,\ell}$.

Now, if we consider a sufficiently small factor in front of the interaction potential, then the result for the ground state energy immediately appears. However, without restriction on the size of the potential, the renormalized Hamiltonian $\mathcal{G}^{\beta}_{N,\ell}$ is not enough. To solve this issue, we use another unitary operator e^{A_H}, defined through the operator A_H, cubic in annihilation and creation operators over excited particles. With the action of e^{A_H}, we get a renormalized Hamiltonian $\mathcal{R}^{\beta}_{N,\ell}$.

At this point it is worth to stress that, in order to show BEC, one would ideally obtain a quadratic Hamiltonian, which can be diagonalized, thus obtaining an upper bound for the number of excited particles in terms of the energy. However, this is not the case. The renormalized Hamiltonian will not be really quadratic, but there are still terms that need to be controlled through localization techniques on the number of particles developed by Lewin-Nam-Serfaty-Solovej [44] (and before by Lieb-Solovej [53]). This requires an a-priori knowledge on the occurrence of condensation in the regimes described by (1.6), as stated in the following theorem.

Theorem 1.2. *Let H^{β}_N be defined in (1.6) with V non negative, radially symmetric and compactly supported. Let $\beta > 0$. Then*

$$\lim_{N\to\infty} \inf_{\|\psi\|=1} \frac{\langle \psi, H^{\beta}_N \psi\rangle}{N} = \frac{\widehat{V}(0)}{2}.$$

Moreover, if ψ_N is an approximate ground state for H_N, namely

$$\lim_{N\to\infty} \frac{\langle \psi_N, H^{\beta}_N \psi_N\rangle}{N} = \frac{\widehat{V}(0)}{2},$$

and $\gamma^{(k)}_N = \mathrm{Tr}_{k+1\to N}|\psi_N\rangle\langle\psi_N|$ is the k- particle reduced density matrix of ψ_N, then there is complete Bose-Einstein condensation

$$\lim_{N\to\infty} \mathrm{Tr}\left|\gamma^{(k)}_N - |\varphi_0^{\otimes k}\rangle\langle\varphi_0^{\otimes k}|\right| = 0, \quad \forall k \in \mathbb{N}. \tag{1.9}$$

with $\varphi_0(x) \in L^2(\mathbb{R}^2)$ the zero momentum mode defined by $\varphi_0(x) = 1$ for all $x \in \Lambda$.

Remark. As in Theorem 1.1, also Theorem 1.2 holds for any $\beta > 0$ such that $\lim_{N\to\infty}(\log N^{\beta})/N = 0$.

The proof of Theorem 1.2 for $\beta < 1$ can be found in [41, 42, 59]. For larger β the same statement can be shown following the strategy applied in [60] for the three-dimensional case, see Appendix C.

1.2 Bose-Einstein condensation in the Gross-Pitaevskii regime

We analyze, now, the limit where the interaction between particles is exponential in N, namely the Gross-Pitaevskii regime. The achieved result is part of a joint paper with Serena Cenatiempo and Benjamin Schlein, submitted for publication to a peer review journal [20]. The GP regime is more interesting and intricate than the other regimes we mentioned in previous sections. Indeed, the integral of the potential is no longer of order $\mathcal{O}(N^{-1})$, but it is of order $\mathcal{O}(1)$, moreover, the interaction is more singular, as it is clear from (1.10).

The Hamilton operator for this setting is of the form

$$H_N^{\mathrm{GP}} = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} e^{2N} V(e^N(x_i - x_j)), \tag{1.10}$$

again acting on a dense subspace of $L^2(\mathbb{R}^2)$. Here we assume $V \in L^3(\mathbb{R}^2)$, which comes from the correlation structure defined by the zero-energy scattering equation. We denote by $\mathfrak{a}$ the scattering length of the unscaled potential V. We recall that in two dimensions and for a potential V with finite range R_0, the scattering length is defined by

$$\frac{2\pi}{\log(R/\mathfrak{a})} = \inf_{\phi} \int_{B_R} \left[|\nabla\phi|^2 + \frac{1}{2} V |\phi|^2 \right] dx, \tag{1.11}$$

where $R > R_0$, B_R is the disk of radius R centered at the origin and the infimum is taken over functions $\phi \in H^1(B_R)$ with $\phi(x) = 1$ for all x with $|x| = R$. The unique minimizer of the variational problem on the r.h.s. of (1.11) is non-negative, radially symmetric and satisfies the scattering equation

$$-\Delta\phi^{(R)} + \frac{1}{2} V \phi^{(R)} = 0,$$

in the sense of distributions. For $R_0 < |x| \le R$, we have

$$\phi^{(R)}(x) = \frac{\log(|x|/\mathfrak{a})}{\log(R/\mathfrak{a})}\,.$$

By scaling, $\phi_N(x) := \phi^{(e^N R)}(e^N x)$ is such that

$$-\Delta\phi_N + \frac{1}{2} e^{2N} V(e^N x)\phi_N = 0.$$

We have

$$\phi_N(x) = \frac{\log(|x|/\mathfrak{a}_N)}{\log(R/\mathfrak{a}_N)} \qquad \forall x \in \mathbb{R}^2 : e^{-N} R_0 < |x| \le R\,,$$

for all $x \in \mathbb{R}^2$ with $e^{-N}R_0 < |x| \leq R$. Here $\mathfrak{a}_N = e^{-N}\mathfrak{a}$. See [49, Appendix C] for a more detailed explanation.

As we said in the introduction, the properties of trapped two dimensional bosons in the Gross-Pitaevskii regime (in the more general case where the bosons are confined by external trapping potentials) have been first studied in [51, 46, 47]. These results can be translated to the Hamilton operator (1.10), defined on the torus, with no external potential. They imply that the ground state energy E_N of (1.10) is such that

$$E_N = 2\pi N\big(1 + O(N^{-1/5})\big)\,. \tag{1.12}$$

Moreover, they imply Bose-Einstein condensation in the zero-momentum mode $\varphi_0(x) = 1$ for all $x \in \Lambda$, for any approximate ground state of (1.10). More precisely, it follows from [46] that, for any sequence $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and

$$\lim_{N\to\infty} \frac{1}{N}\langle \psi_N, H_N^{\text{GP}} \psi_N\rangle = 2\pi, \tag{1.13}$$

the one-particle reduced density matrix $\gamma_N^{(1)} = \text{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|$ is such that

$$1 - \langle \varphi_0, \gamma_N^{(1)}\varphi_0\rangle \leq CN^{-\bar{\delta}} \tag{1.14}$$

for a sufficiently small $\bar{\delta} > 0$. The estimate (1.14) states that, in many-body states satisfying (1.13) (approximate ground states), almost all particles are described by the one-particle orbital φ_0, with at most $N^{1-\delta} \ll N$ orthogonal excitations.

In the theorem we are going to state, under the assumption $V \in L^3(\mathbb{R}^2)$, we improve the results (1.12) and (1.14) by providing more precise bounds on the ground state energy and on the number of excitations. In particular, we prove the following result.

Theorem 1.3. *Let $V \in L^3(\mathbb{R}^2)$ have compact support, be spherically symmetric and pointwise non-negative. Then there exists a constant $C > 0$ such that the ground state energy E_N of (1.10) satisfies*

$$2\pi N - C \leq E_N \leq 2\pi N + C\log N. \tag{1.15}$$

Furthermore, consider a sequence $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and such that

$$\langle \psi_N, H_N^{GP}\psi_N\rangle \leq 2\pi N + K \tag{1.16}$$

for a $K > 0$. Then the reduced density matrix $\gamma_N^{(1)} = \text{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|$ associated with ψ_N is such that

$$1 - \langle \varphi_0, \gamma_N^{(1)}\varphi_0\rangle \leq \frac{C(1+K)}{N} \tag{1.17}$$

for all $N \in \mathbb{N}$ large enough.

It is interesting to compare the Gross-Pitaevskii regime with the thermodynamic limit, where a Bose gas of N particles interacting through a fixed potential with scattering length $\mathfrak{a}$ is confined in a box with area L^2, so that $N, L \to \infty$ with the density $\rho = N/L^2$ kept fixed. Let $b = |\log(\rho\mathfrak{a}^2)|^{-1}$. Then, in the dilute limit $\rho\mathfrak{a}^2 \ll 1$, the ground state energy per particle in the thermodynamic limit is expected to satisfy

$$e_0(\rho) = 4\pi\rho^2 b\Big(1 + b\log b + \big(1/2 + 2\gamma + \log\pi\big)b + o(b)\Big)\,, \tag{1.18}$$

with γ the Euler's constant. The leading order term on the r.h.s. of (1.18) has been first derived in [68] and then rigorously established in [55], with an error rate $b^{-1/5}$. The corrections up to order b have been predicted in [1, 58, 62]. To date, there is no rigorous proof of (1.18). Some partial result, based on the restriction to quasi-free states, has been recently obtained in [29, Theorem 1].

Notice that, for a fixed $\mathfrak{a}$ Eq. (1.18) leads to

$$e_0(\rho) = \frac{4\pi\rho^2}{|\log\rho|}\left(1 - \frac{\log|\log\rho|}{|\log\rho|} + \big(1/2 + 2\gamma + \log(\pi/\mathfrak{a}^2)\big)\frac{1}{|\log\rho|} + o\Big(\frac{\log|\log\rho|}{(\log\rho)^2}\Big)\right)\,.$$

Extrapolating from (1.18), in the Gross-Pitaevskii regime we expect $|E_N - 2\pi N| \leq C$. While our estimate (1.15) captures the correct lower bound, the upper bound is off by a logarithmic correction. Eq. (1.17), on the other hand, is expected to be optimal (but of course, by (1.15), we need to choose $K = C\log N$ to be sure that (1.16) can be satisfied).

The proof of Theorem 1.3 follows the strategy cited in the previous section 1.1, that has been recently introduced in [10]. But let us stress that there are additional obstacles in the two-dimensional case, requiring new ideas. To appreciate the difference between the case of singular interacting potentials in two dimensions, as well as the Gross-Pitaevskii regime in two- and three-dimensions, we can compute the energy of the trivial wave function $\psi_N \equiv 1$. The expectation of (1.10) in this state is of order N^2. It is only through correlations that the energy can approach (1.15). Also in three dimensions, uncorrelated many-body wave functions have large energy, but in that case the difference with respect to the ground state energy is only of order N ($N\widehat{V}(0)/2$ rather than $4\pi\mathfrak{a}_0 N$). This observation is a sign that correlations in two-dimensions are stronger and play a more important role than in three dimensions (this creates problems in handling error terms that, in the other setting considered, were simply estimated in terms of the integral of the potential).

Summary

The thesis is organized as follows. In Chapter 2 we introduce our setting, based on a description of orthogonal excitations of the condensate on a truncated Fock space. Moreover, we describe separately for the two different regimes considered in this thesis the main steps to prove Theorems 1.1 and 1.3. Namely, in Section 2.2 we show how to renormalize the excitation Hamiltonian H_N^β, to regularise the singular interaction, through the action of unitary operators. Section 2.3 is dedicated, analogously, to the renormalization of the Hamiltonian $H_N^{\rm GP}$.

The technical bounds establishing the properties of the renormalized Hamiltonians described in Chapter 2 are the content of Chapter 3 (where we show Prop. 2.4, Prop. 2.8 as well as Theorem 1.1) and Chapter 4 (where we prove Prop. 2.11, Prop. 2.14 and Theorem 1.3).

Finally in Chapter 5 we have a look at the future perspective. Namely, the result of Theorem 1.3, are the starting point to investigate the validity of Bogoliubov theory for the Gross-Pitaevskii regime. We explain how one can obtain the next-to-leading order term in the expansion of the ground state energy as well as information on the low energy excitation spectrum.

We defer to Appendices A and B respectively the proof of the two crucial Lemmas 2.10 and 2.1 establishing properties of the solution of the Neumann problem associated with the two-body potential V in both regimes. Finally, in Appendix C we give a sketch of the proof of Theorem 1.2, following the strategy in [60].

Chapter 2

Bose-Einstein condensation: outline of the proof

2.1 The Fock space setting: focusing on excited particles

The mathematical framework we use throughout the thesis is the Fock space, useful to describe excitations around a Bose-Einstein condensate. It is the aim of this section is to describe this setting, i.e. we introduce the Fock space, first over a generic Hilbert space $\mathfrak{h}$, which in our model is $\mathfrak{h} = L^2(\Lambda)$. What follows is well-known, we recall just some useful definitions. The reader can find proofs and details, for instance, in [5].

We define the bosonic Fock space as the Hilbert space

$$\mathcal{F} = \bigoplus_{n\geq 0} \mathfrak{h}^n = \bigoplus_{n\geq 0} \mathfrak{h}^{\otimes_s n}$$

where $\mathfrak{h}^n_s$ is the dense subspace of $\mathfrak{h}^n$ consisting of vectors that are symmetric with respect to permutations. This space is provided with an inner product, for $\Psi, \Phi \in \mathcal{F}$

$$\langle \Psi,\ \Phi \rangle = \sum_{n\geq 0} \langle \psi^{(n)},\ \phi^{(n)} \rangle$$

and the corresponding norm for $\Psi \in \mathcal{F}$ is given by

$$\|\Psi\|^2 = \sum_{n\geq 0} \|\psi^{(n)}\|_2^2.$$

On $\mathcal{F}$ we denote by $\Omega = \{1, 0, \dots\} \in \mathcal{F}$ the vacuum vector, which describes a state where no particles are present at all.

We can now define, for a function $f \in \mathfrak{h}$, the creation operator $a^*(f)$ and the annihilation operator $a(f)$ by

$$(a^*(f)\Psi)^{(n)}(x_1, \dots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \Psi^{(n-1)}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$$

$$(a(f)\Psi)^{(n)}(x_1, \dots, x_n) = \sqrt{n+1} \int_\Lambda \bar{f}(x) \Psi^{(n+1)}(x, x_1, \dots, x_n)\, dx.$$

Indeed, $a^*(f)$ creates a new particle with a wave function f, on the contrary $a(f)$ annihilates such a particle. Notice that $a^*(f)$ is the adjoint of $a(f)$ and that they satisfy the canonical commutation relations

$$[a(f), a^*(g)] = \langle f, g\rangle, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0$$

for all $f, g \in \mathfrak{h}$ (here we are indicating with $\langle g, h\rangle$ the inner product on $L^2(\mathfrak{h})$). In our setting, as we already said in Chapter 1, we are considering the Hilbert space $\mathfrak{h} = L^2(\Lambda)$, with Λ a unit box with periodic boundary conditions. For most of the analysis, it will be convenient for us to work in momentum space. The plane waves $\varphi_p(x) = e^{-ip\cdot x}$ form a basis for $L^2(\Lambda)$. Then, we define the operators

$$a_p^* = a^*(\varphi_p), \quad \text{and} \quad a_p = a(\varphi_p)$$

creating and, respectively, annihilating a particle with momentum p.

However, to exploit the non-negativity of the interaction potential V, sometimes it will be useful to switch to position space. For this purpose, we introduce operator valued distributions $\check{a}_x, \check{a}_x^*$ such that

$$a(f) = \int \bar{f}(x)\, \check{a}_x\, dx, \quad a^*(f) = \int f(x)\, \check{a}_x^*\, dx,$$

which in turn satisfy

$$[\check{a}_x, \check{a}_y^*] = \delta(x-y), \qquad [\check{a}_x, \check{a}_y] = [\check{a}_x^*, \check{a}_y^*] = 0.$$

The number of particles operator, defined on a dense subspace of $\mathcal{F}$ by $(\mathcal{N}\Psi)^{(n)} = n\psi^{(n)} \in \mathcal{F}$, for any $\Psi = \{\psi^{(n)}_{n\geq 0}\}$ can be expressed both in momentum and position space as

$$\mathcal{N} = \sum_{p\in\Lambda^*} a_p^* a_p = \int \check{a}_x^* \check{a}_x\, dx\,.$$

It is then easy to check that creation and annihilation operators are bounded with respect to the square root of $\mathcal{N}$, i.e.

$$\|a(f)\Psi\| \leq \|f\| \|\mathcal{N}^{1/2}\Psi\|, \quad \|a^*(f)\Psi\| \leq \|f\| \|(\mathcal{N}+1)^{1/2}\Psi\|$$

for all $f \in L^2(\Lambda)$.

As we did for $\mathcal{N}$, we can express the second quantization of any one-particle operator in terms of the operator-valued distribution $\check{a}_x, \check{a}_x^*$. Consider $J^{(1)}$ be a one-particle operator on the space $\mathfrak{h}$. The second quantized operator $d\Gamma(J^{(1)})$ on the Fock space $\mathcal{F}$ is defined by

$$(d\Gamma(J^{(1)})\Psi)^{(n)} = \sum_{i=1}^{n} J^{(1)}\psi^{(n)},$$

where $J^{(1)}$ denotes the operator acting on $\mathfrak{h}^n$ as $J^{(1)}$ on the i-th particle and as the identity on the other $(n-1)$ particles. If $J^{(1)}$ has integral kernel $J^{(1)}(x;y)$, it is easy to show that

$$d\Gamma(J^{(1)}) = \int dxdy J^{(1)}(x;y) a_x^* a_y.$$

In particular, we can use this representation to define the one-particle density operator $\gamma_\Psi^{(1)} : \mathfrak{h} \to \mathfrak{h}$ associated with a vector Ψ on $\mathcal{F}$ through its integral kernel

$$\gamma_\Psi^{(1)}(x;y) = \frac{1}{\langle \Psi, \mathcal{N}\Psi\rangle}\langle \Psi, \check{a}_y^* \check{a}_x \Psi\rangle,$$

which for N-particles states coincide with the definition of $\gamma_N^{(1)}$ in (1.3).

With the tools introduced above, we are able to rewrite Hamilton operators of the form (1.1) (with no external potential) as follows.

$$H_N = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{1}{2}\sum_{p,q,r\in\Lambda^*} \widehat{v}(r) a_{p+r}^* a_q^* a_p a_{q+r} \tag{2.1}$$

where

$$\widehat{v}(k) = \int_{\mathbb{R}^2} v(x) e^{-ik\cdot x} dx$$

is the Fourier transform of v, defined for all $k \in \mathbb{R}^2$. For (1.6) and (1.10), v is of the form $v(x) = N^{2\beta-1}V(N^\beta x)$ and $v(x) = e^{2N}V(e^N x)$ respectively. In particular, we will have

$$H_N^\beta = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{1}{2N}\sum_{p,q,r\in\Lambda^*} \widehat{V}(r/N^\beta) a_{p+r}^* a_q^* a_p a_{q+r} \tag{2.2}$$

and

$$H_N^{\mathrm{GP}} = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{1}{2}\sum_{p,q,r\in\Lambda^*} \widehat{V}(r/e^N) a_{p+r}^* a_q^* a_p a_{q+r}. \tag{2.3}$$

Notice that, (1.10), (1.6) are the restriction of (2.1) to the N-particle sector of the Fock space $\mathcal{F}$. Moreover, there is a slight abuse of notation for the Hamiltonians: the same notation is used both for L^2- space and Fock space.

Next, we want to construct another Fock space. We denote by $L^2_\perp(\Lambda)$ the orthogonal complement in $L^2(\Lambda)$ of the one dimensional space spanned by φ_0, which, we recall, is the zero-momentum mode in $L^2(\Lambda)$, normalized for all $x \in \Lambda$. We construct the Fock space over $L^2_\perp(\Lambda)$, generated by the annihilation and creation operators defined above a_p^* with $p \in \Lambda_+^* := 2\pi\mathbb{Z}^2 \setminus \{0\}$. This will be denoted by

$$\mathcal{F}_+ = \bigoplus_{n\geq 0} L^2_\perp(\Lambda)^{\otimes_s n}.$$

Moreover, we indicate the number of particles operator on $\mathcal{F}_+$ as

$$\mathcal{N}_+ = \sum_{p\in\Lambda_+^*} a_p^* a_p.$$

Our aim is to factor out particles in the Bose-Einstein condensate from low-energy states of H_N. To this end, first we need to introduce for $N \in \mathbb{N}$ the truncated Fock space

$$\mathcal{F}_+^{\leq N} = \bigoplus_{n=0}^{N} L^2_\perp(\Lambda)^{\otimes_s n}.$$

Following [44, 43] for the zero-momentum mode $\varphi_0 \in L^2(\Lambda)$, *i.e.* $\varphi_0(x) \equiv 1$ for all $x \in \Lambda$, every $\psi_N \in L^2_s(\Lambda^N)$ can be uniquely represented as

$$\psi_N = \sum_{n=0}^{N} \alpha_n \otimes_s \varphi_0^{\otimes(N-n)} = \alpha_0 \otimes_s \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes(N-1)} + \cdots + \alpha_N$$

for a sequence $\alpha_j \in L^2_\perp(\Lambda)^{\otimes_s j}$, for all $j = 0 \ldots N$. Here, $L^2_\perp(\Lambda)^{\otimes_s j}$ indicates the symmetric tensor product of j copies of the orthogonal complement $L^2_\perp(\Lambda)$ of φ_0.

We can therefore introduce the unitary map $U_N : L^2_s(\Lambda^N) \to \mathcal{F}_+^{\leq N}$ defining $U_N(\varphi_0)\psi_N = \{\alpha_0, \alpha_1, \ldots, \alpha_N\} \in \mathcal{F}_+^{\leq N}$. This map removes the condensate described by the one-particle wave function φ_0 and allows us to focus on its orthogonal excitations. We can also define U_N identifying ψ_N with the Fock space vector $\{0, 0, \ldots, \psi_N, 0, \ldots\}$ and using creation and annihilation operators; we find

$$U_N\, \psi_N = \bigoplus_{n=0}^{N} (1 - |\varphi_0\rangle\langle\varphi_0|)^{\otimes n} \frac{a(\varphi_0)^{N-n}}{\sqrt{(N-n)!}}\, \psi_N$$

for all $\psi_N \in L^2_s(\Lambda^N)$. One can easily check that the $U_N^* : \mathcal{F}_+^{\leq N} \to L^2_s(\Lambda^N)$ is given by

$$U_N^*\, \{\alpha^{(0)}, \ldots, \alpha^{(N)}\} = \sum_{n=0}^{N} \frac{a^*(\varphi_0)^{N-n}}{\sqrt{(N-n)!}}\, \alpha^{(n)}$$

and that $U_N^* U_N = 1$, ie. U_N is unitary.

Using U_N, we can define the excitation Hamiltonian $\mathcal{L} := U_N H_N U_N^*$, acting on a dense subspace of $\mathcal{F}_+^{\leq N}$. From (2.1) we can compute the excitation Hamiltonian $\mathcal{L}$ using the following rules, whose proof can be found in [44], describing the action of the unitary operator U_N on products of a creation and an annihilation operator (products of the form $a_p^* a_q$ can be thought of as operators mapping $L^2_s(\Lambda^N)$ to itself). For any $p, q \in \Lambda_+^* = 2\pi\mathbb{Z}^2 \backslash \{0\}$,

$$\begin{aligned} U_N\, a_0^* a_0\, U_N^* &= N - \mathcal{N}_+ \\ U_N\, a_p^* a_0\, U_N^* &= a_p^* \sqrt{N - \mathcal{N}_+} \\ U_N\, a_0^* a_p\, U_N^* &= \sqrt{N - \mathcal{N}_+}\, a_p \\ U_N\, a_p^* a_q\, U_N^* &= a_p^* a_q. \end{aligned} \tag{2.4}$$

It is useful to introduce generalized creation and annihilation operators

$$b_p^* = a_p^* \sqrt{\frac{N - \mathcal{N}_+}{N}}, \qquad \text{and} \qquad b_p = \sqrt{\frac{N - \mathcal{N}_+}{N}}\, a_p$$

for all $p \in \Lambda_+^*$. Their definition is a natural consequence of the action of the map U_N. In fact, we get

$$U_N^* b_p^* U_N = a_p^* \frac{a_0}{\sqrt{N}}, \qquad U_N^* b_p U_N = \frac{a_0^*}{\sqrt{N}} a_p,$$

this means that b_p^* creates a particle with momentum $p \in \Lambda_+^*$ but, at the same time, it annihilates a particle from the condensate. Moreover, differently from the standard creation and annihilation operators, b_p^* and b_p leave the total number of particles in the system invariant. On states exhibiting complete Bose-Einstein condensation in the zero-momentum mode φ_0, we have $a_0, a_0^* \simeq \sqrt{N}$ and we can therefore expect that $b_p^* \simeq a_p^*$ and that $b_p \simeq a_p$. These operators satisfy the commutation relations

$$\begin{aligned} [b_p, b_q^*] &= \left(1 - \frac{\mathcal{N}_+}{N}\right) \delta_{p,q} - \frac{1}{N} a_q^* a_p \\ [b_p, b_q] &= [b_p^*, b_q^*] = 0. \end{aligned} \tag{2.5}$$

Furthermore, we find

$$[b_p, a_q^* a_r] = \delta_{pq} b_r, \qquad [b_p^*, a_q^* a_r] = -\delta_{pr} b_q^* \tag{2.6}$$

for all $p, q, r \in \Lambda_+^*$; this implies in particular that $[b_p, \mathcal{N}_+] = b_p$, $[b_p^*, \mathcal{N}_+] = -b_p^*$. It is also useful to notice that the operators b_p^*, b_p, like a_p^*, a_p, can be bounded by the square root of the number of particles operators; we find

$$\|b_p \xi\| \leq \|\mathcal{N}_+^{1/2} \xi\|, \qquad \|b_p^* \xi\| \leq \|(\mathcal{N}_+ + 1)^{1/2} \xi\|$$

for all $\xi \in \mathcal{F}_+^{\leq N}$. Since $\mathcal{N}_+ \leq N$ on $\mathcal{F}_+^{\leq N}$, the operators b_p^*, b_p are bounded, with $\|b_p\|, \|b_p^*\| \leq (N+1)^{1/2}$.

We can also define modified operator valued distributions

$$\check{b}_x = \sqrt{\frac{N - \mathcal{N}_+}{N}}\, \check{a}_x, \qquad \text{and} \qquad \check{b}_x^* = \check{a}_x^* \sqrt{\frac{N - \mathcal{N}_+}{N}}$$

in position space, for $x \in \Lambda$. The commutation relations (2.5) take the form

$$\begin{aligned} [\check{b}_x, \check{b}_y^*] &= \left(1 - \frac{\mathcal{N}_+}{N}\right) \delta(x-y) - \frac{1}{N} \check{a}_y^* \check{a}_x \\ [\check{b}_x, \check{b}_y] &= [\check{b}_x^*, \check{b}_y^*] = 0 \end{aligned}$$

Moreover, (2.6) translates to

$$[\check{b}_x, \check{a}_y^* \check{a}_z] = \delta(x-y) \check{b}_z, \qquad [\check{b}_x^*, \check{a}_y^* \check{a}_z] = -\delta(x-z) \check{b}_y^*$$

which also implies that $[\check{b}_x, \mathcal{N}_+] = \check{b}_x$, $[\check{b}_x^*, \mathcal{N}_+] = -\check{b}_x^*$.

Going back to the two settings that we are considering, we obtain on one hand for the Hamiltonian H_N^β,

$$\mathcal{L}_N^\beta := U_N H_N^\beta U_N^* = \mathcal{L}_N^{\beta,(0)} + \mathcal{L}_N^{\beta,(2)} + \mathcal{L}_N^{\beta,(3)} + \mathcal{L}_N^{\beta,(4)}$$

with

$$\begin{aligned}
\mathcal{L}_N^{\beta,(0)} &= \frac{\widehat{V}(0)}{2N}(N-1)(N-\mathcal{N}_+) + \frac{\widehat{V}(0)}{2N}\mathcal{N}_+(N-\mathcal{N}_+) \\
\mathcal{L}_N^{\beta,(2)} &= \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p + \sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)\left[b_p^* b_p - \frac{1}{N}a_p^* a_p\right] \\
&\quad + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)\left[b_p^* b_{-p}^* + b_p b_{-p}\right] \\
\mathcal{L}_N^{\beta,(3)} &= \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N^\beta)\left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q}\right] \\
\mathcal{L}_N^{\beta,(4)} &= \frac{1}{2N} \sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*: \\ r\neq -p,-q}} \widehat{V}(r/N^\beta) a_{p+r}^* a_q^* a_p a_{q+r}.
\end{aligned} \tag{2.7}$$

For the Gross-Pitaeviskii Hamiltonian Eq.(1.10) we have

$$\mathcal{L}_N = U_N H_N^{\mathrm{GP}} U_N^* = \mathcal{L}_N^{(0)} + \mathcal{L}_N^{(2)} + \mathcal{L}_N^{(3)} + \mathcal{L}_N^{(4)} \tag{2.8}$$

with

$$\begin{aligned}
\mathcal{L}_N^{(0)} &= \frac{1}{2}\widehat{V}(0)(N-1)(N-\mathcal{N}_+) + \frac{1}{2}\widehat{V}(0)\mathcal{N}_+(N-\mathcal{N}_+) \\
\mathcal{L}_N^{(2)} &= \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p + N\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\left[b_p^* b_p - \frac{1}{N}a_p^* a_p\right] \\
&\quad + \frac{N}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\left[b_p^* b_{-p}^* + b_p b_{-p}\right] \\
\mathcal{L}_N^{(3)} &= \sqrt{N} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/e^N)\left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q}\right] \\
\mathcal{L}_N^{(4)} &= \frac{1}{2} \sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*: \\ r\neq -p,-q}} \widehat{V}(r/e^N) a_{p+r}^* a_q^* a_p a_{q+r}\,.
\end{aligned} \tag{2.9}$$

The expressions (2.7) and (2.9) show clearly why dealing with the Gross-Pitaevskii regime is harder. Indeed, if we take the expectation on the vacuum state Ω, while for $\mathcal{L}_N^\beta$ it is of order N, for $\mathcal{L}_N$ it is of order N^2. Moreover, while $\mathcal{L}_N^{\beta,(3)}$ and $\mathcal{L}_N^{\beta,(4)}$ have some small factors $1/\sqrt{N}$ and $1/N$ in front, the cubic and quartic terms in (2.9) are much larger. Indeed, the analysis of the excitation Hamiltonian $\mathcal{L}_N^\beta$ is done closely following [10], while the one for the Gross-Pitaevskii regime requires additional ideas.

In the following section we describe the analysis of $\mathcal{L}_N^\beta$ and $\mathcal{L}_N$ separately, showing how to obtain a proof of Theorems 1.1 and 1.3 respectively.

From (2.7) and (2.9) we see that conjugation with U_N extracts, from the original quartic interaction in (2.2) and (2.3), some large constant and quadratic contributions, collected in $\mathcal{L}_N^{\beta,(0)}$, $\mathcal{L}_N^{(0)}$, and $\mathcal{L}_N^{\beta,(2)}$, $\mathcal{L}_N^{(2)}$ respectively. However, in the two regimes we are considering, this is not enough; in fact, there are still large contributions to the energy hidden among cubic and quartic terms in $\mathcal{L}_N^{\beta,(3)}$ and $\mathcal{L}_N^{\beta,(4)}$ as well as in $\mathcal{L}_N^{(3)}$ and $\mathcal{L}_N^{(4)}$ for the Gross-Pitaevskii case (as we already mentioned the expectation of $\mathcal{L}_N$ on the vacuum state Ω is of order N^2, which is a clear indication of the fact that there are other large contributions to the energy).

Since U_N only removes products of the zero-energy mode φ_0, correlations among particles remain in the excitation vector $U_N\psi_N$. This means correlations play a crucial role in this regime. In the following section we explain how to take into account the correlation structure. This will lead to a renormalization of the excitation Hamiltonians $\mathcal{L}_N^\beta$ and $\mathcal{L}_N$ in Equations (2.7), (2.9) which allows us to show condensation. As we will see, the analysis of the two regimes, although sharing a similar strategy, requires different ideas.

2.2 Renormalization of the excitation Hamiltonian $\mathcal{L}_N^\beta$

To take into account the short scale correlation structure on top of the condensate, we consider the ground state f_ℓ of the Neumann problem

$$\Big(-\Delta + \frac{1}{2N}V(x)\Big) f_\ell(x) = \lambda_\ell\, f_\ell(x) \tag{2.10}$$

on the ball $|x| \leq N^\beta \ell$, normalized so that $f_\ell(x) = 1$ for $|x| = N^\beta \ell$. Notice that also f_ℓ and λ_ℓ depend on N, but for convenience we omit its dependence. By scaling, we observe that $f_\ell(N^\beta \cdot)$ satisfies

$$\Big(-\Delta + \frac{N^{2\beta}}{2N}V(N^\beta x)\Big) f_\ell(N^\beta x) = N^{2\beta}\lambda_\ell\, f_\ell(N^\beta x)$$

on the ball $|x| \leq \ell$. We choose $0 < \ell < 1/2$, so that the ball of radius ℓ is contained in the box $\Lambda = [-1/2; 1/2]^2$. We extend then $f_\ell(N^\beta .)$ to Λ, by setting $f_{N,\ell}(x) = f_\ell(N^\beta x)$, if $|x| \leq \ell$ and $f_{N,\ell}(x) = 1$ for $x \in \Lambda$, with $|x| > \ell$. Then

$$\Big(-\Delta + \frac{N^{2\beta}}{2N}V(N^\beta(x))\Big) f_{N,\ell}(x) = N^{2\beta}\lambda_\ell\, f_{N,\ell}(x)\chi_\ell(x) \tag{2.11}$$

where χ_ℓ is the characteristic function of the ball of radius ℓ. The Fourier coefficients of the function $f_{N,\ell}$ are given by

$$\widehat{f}_{N,\ell}(p) := \int_\Lambda f_\ell(N^\beta x) e^{-ip\cdot x} dx$$

for all $p \in \Lambda^*$. We also introduce the function

$$w_\ell(x) = 1 - f_\ell(x)$$

for $|x| \leq N^\beta \ell$ and we extend it by setting $w_\ell(x) = 0$ for $|x| > N^\beta \ell$. Its re-scaled version is defined by the function $w_{N,\ell} : \Lambda \to \mathbb{R}$, such that $w_{N,\ell}(x) = w_\ell(N^\beta x)$ if

$|x| \le \ell$ and $w_{N,\ell} = 0$ if $x \in \Lambda$ with $|x| > \ell$.

The Fourier coefficients of the re-scaled function $w_{N,\ell}$ are given by

$$\widehat{w}_{N,\ell}(p) = \int_\Lambda w_\ell(N^\beta x) e^{-ip\cdot x} dx = N^{-2\beta} \widehat{w}_\ell(p/N^\beta). \tag{2.12}$$

We find $\widehat{f}_{N,\ell}(p) = \delta_{p,0} - N^{-2\beta}\widehat{w}_\ell(p/N^\beta)$. From the Neumann problem (2.11) we obtain

$$-p^2 N^{-2\beta}\widehat{w}_\ell(p/N^\beta) + \frac{1}{2N}\sum_{q\in\Lambda^*}\widehat{V}((p-q)/N^\beta)\widehat{f}_{N,\ell}(q) = N^{2\beta}\lambda_\ell \sum_{q\in\Lambda^*}\widehat{\chi}(p-q)\widehat{f}_{N,\ell}(q), \tag{2.13}$$

where we used the notation $\widehat{\chi}_\ell$ for the Fourier coefficients of the characteristic function on the ball of radius ℓ. Note that $\widehat{\chi}_\ell(p) = \ell^2\,\widehat{\chi}(\ell p)$, with $\widehat{\chi}(p)$ the Fourier coefficients of the characteristic function on the ball of radius one.
In the next lemma, we collect some important properties of the solution of (2.11).

Lemma 2.1. *Let $V \in L^2(\mathbb{R}^2)$ be non-negative, compactly supported (with range R_0) and spherically symmetric, and denote its scattering length by $\mathfrak{a}$. Fix $0 < \ell < 1/2$, $N^\beta\ell > 0$ sufficiently large and let f_ℓ denote the solution of* (2.10)*. Then*

i)

$$0 \le f_\ell(x) \le 1 \qquad \forall x \in \mathbb{R}^2 : |x| \le N^\beta \ell$$

ii) We have

$$\left|\lambda_\ell - \frac{1}{(N^\beta\ell)^2}\frac{\widehat{V}(0)}{2\pi N}\left(1 - \frac{\widehat{V}(0)}{4\pi N}\log(N^\beta)\right)\right| \le \frac{C}{(N^\beta\ell)^2 N^2}$$

iii) We have

$$\left|\frac{1}{N}\int dx\, V(x) f_\ell(x) - \frac{\widehat{V}(0)}{N}\left(1 - \frac{\widehat{V}(0)}{4\pi N}\log(N^\beta)\right)\right| \le \frac{C}{N^2} \tag{2.14}$$

iv) There exists a constant $C > 0$ such that

$$\begin{aligned} |w_\ell(x)| &\le C \qquad \text{if } |x| \le R_0 \\ \left|w_\ell(x) - \frac{\widehat{V}(0)}{4\pi N}\log\left(N^\beta\ell/|x|\right)\right| &\le \frac{C}{N} \quad \text{if } R_0 \le |x| \le N^\beta \ell. \end{aligned} \tag{2.15}$$

v) There exists a constant $C > 0$ such that

$$|\nabla w_\ell(x)| \le \frac{C}{N}\frac{1}{|x|+1}.$$

vi) Let $w_{N,\ell} = 1 - f_{N,\ell}$ with $f_{N,\ell} = f_\ell(N^\beta x)$. Then for the Fourier coefficients of the function $w_{N,\ell}$ defined in (2.12) *the following holds*

$$|\widehat{w}_{N,\ell}(p)| \leq \frac{c}{p^2 N}. \tag{2.16}$$

Proof. The proof of points i)-v) is deferred to Appendix B. To prove point vi) we use the scattering equation (2.13):

$$\widehat{w}_\ell(p/N^\beta) = \frac{N^{2\beta}}{2p^2} \sum_{q\in\Lambda^*} \widehat{V}((p-q)/N^\beta)\widehat{f}_{N,\ell}(q) - \frac{N^{4\beta}}{p^2}\lambda_\ell \sum_{q\in\Lambda^*} \widehat{\chi}_\ell(p-q)\widehat{f}_{N,\ell}(q).$$

Using the fact that $N^{2\beta}\lambda_\ell \leq C\ell^{-2}N^{-1}$, from point ii), and that $f_\ell \leq 1$, we end up with

$$\begin{aligned}
|\widehat{w}_\ell(p/N^\beta)| &\leq \frac{N^{2\beta}}{2p^2}\left[\frac{1}{N}\left|(\widehat{V}(\cdot/N^\beta) * \widehat{f}_{N,\ell})(p)\right| + 2N^{2\beta}\lambda_\ell \left|(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(p)\right|\right] \\
&\leq \frac{N^{2\beta}}{2p^2}\left[\frac{1}{N}(\widehat{V}(\cdot/N^\beta) * \widehat{f}_{N,\ell})(0) + \frac{C}{\ell^2 N}\,(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(0)\right] \\
&\leq \frac{N^{2\beta}}{2p^2}\left[\frac{1}{N}\int V(x)f_\ell(x)dx + \frac{C}{\ell^2 N}\int \chi_\ell(x)f_\ell(N^\beta x)dx\right] \\
&\leq C\frac{N^{2\beta}}{p^2 N}.
\end{aligned}$$

□

We now define the function $\check{\eta} : \Lambda \to \mathbb{R}$ through its Fourier coefficients $\eta : \Lambda^* \to \mathbb{R}$

$$\eta_p = -N\widehat{w}_{N,\ell}(p) = -N^{1-2\beta}\widehat{w}_\ell(p/N^\beta). \tag{2.17}$$

Using Lemma 2.10, we can bound

$$|\eta_p| \leq \frac{C}{|p|^2} \tag{2.18}$$

for all $p \in \Lambda^*_+ = 2\pi\mathbb{Z}^2\backslash\{0\}$, and for some constant $C > 0$ independent of N and $\ell \in (0; \frac{1}{2})$, if N is large enough. We can also rewrite the scattering equation (2.13) in terms of η_p, we find

$$p^2\eta_p + \frac{1}{2}(\widehat{V}(\cdot/N^\beta) * \widehat{f}_{N,\ell})(p) = N^{1+2\beta}\lambda_\ell(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(p) \tag{2.19}$$

or equivalently, expressing the other terms through the coefficients η_p,

$$\begin{aligned}
p^2\eta_p + \frac{1}{2}\widehat{V}(p/N^\beta) + \frac{1}{2N}\sum_{q\in\Lambda^*}\widehat{V}((p-q)/N^\beta)\eta_q& \\
= N^{1+2\beta}\lambda_\ell\widehat{\chi}_\ell(p) + N^{2\beta}\lambda_\ell \sum_{q\in\Lambda^*} \widehat{\chi}_\ell(p-q)\eta_q&.
\end{aligned} \tag{2.20}$$

Moreover, with the bounds in (2.15), we can estimate the L^2-norm as

$$\|\eta\|^2 = \|\check{\eta}\|^2 = C\int \log^2(\ell/|x|)\chi(|x| \le \ell)d^2x \le C\ell^2. \tag{2.21}$$

For $\alpha > 0$, we now want to define the momentum set

$$P_H = \{p \in \Lambda_+^* : |p| \ge \ell^{-\alpha}\},$$

with $\ell \in (0; 1/2)$. We set

$$\eta_H(p) = \eta_p\, \chi(p \in P_H) = \eta_p \chi(|p| \ge \ell^{-\alpha})\,. \tag{2.22}$$

Eq. (2.18) implies that

$$\|\eta_H\| \le C\ell^\alpha. \tag{2.23}$$

Notice that for $\alpha > 1$, the last bound improves (2.21). For our analysis, this improvement, due to the cutoff on high momenta, will be crucial. We will mostly use the coefficients η_p with $p \neq 0$. Sometimes, however, it will be useful to have an estimate on η_0 (because Eq. (2.20) involves η_0). From (2.17) and Lemma 2.1, part iii) we find

$$|\eta_0| \le N\int_{|x|\le\ell} w_\ell(N^\beta x)d^2x \le C\int_{|x|\le\ell} |\log(\ell/|x|)|d^2x \le C\ell^2\,. \tag{2.24}$$

We can also consider some bounds for the function $\check{\eta}$. Writing

$$\eta_H(p) = \eta_p - \eta_p\chi(|p| \le \ell^{-\alpha}),$$

we obtain

$$\check{\eta}_H(x) = \check{\eta}(x) - \sum_{\substack{p\in\Lambda^*:\\ |p|\le\ell^{-\alpha}}} \eta_p e^{ip\cdot x} = -Nw_\ell(N^\beta x) - \sum_{\substack{p\in\Lambda^*:\\ |p|\le\ell^{-\alpha}}} \eta_p e^{ip\cdot x}.$$

We thus find

$$|\check{\eta}_H(x)| \le C\log N^\beta + \sum_{\substack{p\in\Lambda^*:\\ |p|\le\ell^{-\alpha}}} |p|^{-2} \le C\big(\log N^\beta + \alpha\log\ell\big) \le C\log N^\beta \tag{2.25}$$

for all $x \in \Lambda$, α independent on N and $N \in \mathbb{N}$ large enough. Moreover, the H^1-norms of η diverge, as $N \to \infty$. From (2.17) and Lemma 2.1, part iv) we find

$$\begin{aligned}\|\check{\eta}_H\|_{H_1}^2 \le \|\check{\eta}\|_{H_1}^2 &= \int_{|x|\le\ell} N^2|\nabla w_\ell(N^\beta x)|^2 d^2x \\ &= \int_{|x|\le N^\beta\ell} N^2|\nabla w_\ell(x)|^2 d^2x \\ &\le C\int_{|x|\le N^\beta\ell} \frac{1}{(|x|+1)^2}\, d^2x \le C\log N\end{aligned}$$

for all $\ell \in (0; 1/2)$ and $N \in \mathbb{N}$ large enough.

2.2.1 Quadratic renormalization

To factor out correlation, one could think to conjugate $\mathcal{L}_N^\beta$ with a Bogoliubov transformation of the form

$$e^{\widetilde{B}} = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^*}\left(\eta_p a_p^* a_{-p}^* - \bar{\eta}_p a_p a_{-p}\right)\right] \tag{2.26}$$

defined through the standard creation and annihilation operators, where the coefficient η_p is defined as in 2.17. This idea was used first in [27] and later exploited in [4] to study the effective dynamics of large system of bosons in the Gross-Pitaevskii regime. Although the action of standard Bogoliubov transformation on creation and annihilation operators can be explicitly written as (see [4], [3, Chapter 2.2])

$$e^{-\widetilde{B}} a_p e^{\widetilde{B}} = \cosh(\eta_p) a_p + \sinh(\eta_p) a_{-p}^*$$

this unitary operator does not leave the truncated Fock space $\mathcal{F}_+^{\leq N}$ invariant. This is why we need to define *generalized Bogoliubov transformation* through the anti-symmetric operator

$$B = \frac{1}{2}\sum_{p\in\Lambda_+^*}\left(\eta_p b_p^* b_{-p}^* - \bar{\eta}_p b_p b_{-p}\right) \tag{2.27}$$

with $\eta_{-p} = \eta_p$ for all $p \in \Lambda_+^*$, and we consider the unitary operator

$$e^{B} = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^*}\left(\eta_p b_p^* b_{-p}^* - \bar{\eta}_p b_p b_{-p}\right)\right]. \tag{2.28}$$

Their action ensures that the truncated Fock space $\mathcal{F}_+^{\leq N}$ remains invariant. They have been first introduced in [17] (in position space) and then translated to the momentum space in [7]. Their definition and their main properties will be discussed in this subsection.

Conjugation with (2.28) leaves the number of particles essentially invariant, as confirmed by the following lemma.

Lemma 2.2. *Assume B is defined as in (2.27), with $\eta \in \ell^2(\Lambda^*)$ and $\eta_p = \eta_{-p}$ for all $p \in \Lambda_+^*$. Then, for every $n \in \mathbb{N}$ there exists a constant $C > 0$ such that, on $\mathcal{F}_+^{\leq N}$,*

$$e^{-B}(\mathcal{N}_+ + 1)^n e^{B} \leq Ce^{C\|\eta\|}(\mathcal{N}_+ + 1)^n\,. \tag{2.29}$$

as an operator inequality on $\mathcal{F}_+^{\leq N}$.

The proof of (2.29) can be found in [17, Lemma 3.1] (a similar result has been previously established in [70]).

We collect now important properties about the action of unitary operators of the form e^B, as defined in (2.28). As shown in [7, Lemma 2.5 and 2.6] (or see

[10, Lemma 3.2]), we have, if $\|\eta\|$ is sufficiently small,

$$\begin{aligned} e^{-B} b_p e^{B} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{ad}_B^{(n)}(b_p) \\ e^{-B} b_p^* e^{B} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{ad}_B^{(n)}(b_p^*) \end{aligned} \tag{2.30}$$

where the series converge absolutely and ad_B is defined recursively as

$$\mathrm{ad}_B^{(0)}(A) = A \quad \text{and} \quad \mathrm{ad}_B^{(n)}(A) = [B, \mathrm{ad}_B^{(n-1)}(A)].$$

To confirm the expectation that generalized Bogoliubov transformation act similarly to standard Bogoliubov transformations, on states with few excitations, we define from (2.30) (for $\|\eta\|$ small enough) the remainder operators d_p, d_p^*

$$e^{-B} b_q e^{B} = \gamma_q b_q + \sigma_q b_{-q}^* + d_q, \qquad e^{-B} b_q^* e^{B} = \gamma_q b_q^* + \sigma_q b_{-q} + d_q^* \tag{2.31}$$

where we introduced the notation $\gamma_q = \cosh(\eta_q)$ and $\sigma_q = \sinh(\eta_q)$. An explicit definition of the operators d_p, d_p^* can be found in [10, Eq. 3.17]. It will also be useful to introduce remainder operators in position space. For $x \in \Lambda$, we define the operator valued distributions $\check{d}_x, \check{d}_x^*$ through

$$e^{-B} \check{b}_x e^{B} = b(\check{\gamma}_x) + b^*(\check{\sigma}_x) + \check{d}_x, \qquad e^{-B} \check{b}_x^* e^{B} = b^*(\check{\gamma}_x) + b(\check{\sigma}_x) + \check{d}_x^* \tag{2.32}$$

where $\check{\gamma}_x(y) = \sum_{q \in \Lambda^*} \cosh(\eta_q) e^{iq\cdot(x-y)}$ and $\check{\sigma}_x(y) = \sum_{q \in \Lambda^*} \sinh(\eta_q) e^{iq\cdot(x-y)}$.

Throughout our analysis we are going to use pointwise bounds for the quantities defined above, namely, γ_q, σ_q as well as $\check{\gamma}_x(y) = \delta(x) + \check{r}(x), \check{\sigma}_x(y)$. In momentum space, using their definitions and expanding the hyperbolic functions we have that for all $q \in \Lambda_+^*$

$$\begin{aligned} &|\sigma_q| \le |\eta_q| \le \frac{C}{|q|^2}, \quad |\sigma_q - \eta_q| \le |\eta_q|^3 \le \frac{C}{|q|^6}, \quad |\gamma_q| \le C, \\ &|\gamma_q - 1| \le |\eta_q|^2 \le \frac{C}{|q|^4}, \quad |\gamma_q \sigma_q - \eta_q| \le |(1+\eta_q)(\eta_q + \eta_q^3) - \eta_q| \le |\eta_q|^3 \le \frac{C}{|q|^6}\,. \end{aligned} \tag{2.33}$$

In position space, we obtain from (2.25) the estimates

$$\|\check{\sigma}\|_2 \le C, \quad \|\check{\sigma}\|_\infty \le C \log N, \quad \|\check{\sigma} * \check{\gamma}\|_\infty \le C \log N\,.$$

The definition of operators d_p, d_p^* will be crucial in the analysis shown in Chapter 3.

The next lemma is from [10, Lemma 3.4] and gives us estimates necessary in our analysis

Lemma 2.3. *Let $\eta \in \ell^2(\Lambda_+^*)$, $n \in \mathbb{Z}$. For $p \in \Lambda_+^*$, let d_p be defined as in (2.31). If $\|\eta\|$ is small enough, there exists $C > 0$ such that*

$$\begin{aligned} \|(\mathcal{N}_+ + 1)^{n/2} d_p \xi\| &\le \frac{C}{N} \left[|\eta_p| \|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| + \|\eta\| \|b_p (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| \right], \\ \|(\mathcal{N}_+ + 1)^{n/2} d_p^* \xi\| &\le \frac{C}{N} \|\eta\| \, \|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| \end{aligned} \tag{2.34}$$

for all $p \in \Lambda_+^*, \xi \in \mathcal{F}_+^{\leq N}$. *With* $\bar{\bar{d}}_p = d_p + N^{-1} \sum_{q \in \Lambda_+^*} \eta_q b_q^* a_{-q}^* a_p$, *we also have, for* $p \notin$ *supp* η, *the improved bound*

$$\|(\mathcal{N}_+ + 1)^{n/2} \bar{\bar{d}}_p \xi\| \leq \frac{C}{N} \|\eta\|^2 \|a_p (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\|. \tag{2.35}$$

In position space, with $\check{d}_x$ *defined as in (2.32), we find*

$$\|(\mathcal{N}_+ + 1)^{n/2} \check{d}_x \xi\| \leq \frac{C}{N} \|\eta\| \Big[\|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| + \|b_x (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| \Big]. \tag{2.36}$$

Furthermore, letting $\check{\bar{d}}_x = \check{d}_x + (\mathcal{N}_+/N) b^*(\check{\eta}_x)$, *we find*

$$\begin{aligned} &\|(\mathcal{N}_+ + 1)^{n/2} \check{a}_y \check{\bar{d}}_x \xi\| \\ &\quad \leq \frac{C}{N} \Big[\|\eta\|^2 \|(\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| + \|\eta\| |\check{\eta}(x-y)| \|(\mathcal{N} + 1)^{(n+2)/2} \xi\| \\ &\qquad + \|\eta\| \|\check{a}_x (\mathcal{N}_+ + 1)^{(n+1)/2} \xi\| + \|\eta\|^2 \|\check{a}_y (\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| \\ &\qquad + \|\eta\| \|\check{a}_x \check{a}_y (\mathcal{N} + 1)^{(n+2)/2} \xi\| \Big] \end{aligned} \tag{2.37}$$

and, finally,

$$\begin{aligned} &\|(\mathcal{N}_+ + 1)^{n/2} \check{d}_x \check{d}_y \xi\| \\ &\quad \leq \frac{C}{N^2} \Big[\|\eta\|^2 \|(\mathcal{N}_+ + 1)^{(n+6)/2} \xi\| + \|\eta\| |\check{\eta}(x-y)| \|(\mathcal{N}_+ + 1)^{(n+4)/2} \xi\| \\ &\qquad + \|\eta\|^2 \|a_x (\mathcal{N}_+ + 1)^{(n+5)/2} \xi\| + \|\eta\|^2 \|a_y (\mathcal{N}_+ + 1)^{(n+5)/2} \xi\| \\ &\qquad + \|\eta\|^2 \, \|a_x a_y (\mathcal{N}_+ + 1)^{(n+4)/2} \xi\| \Big] \end{aligned} \tag{2.38}$$

for all $\xi \in \mathcal{F}_+^{\leq n}$.

From [10, Corollary 3.3] and Lemma 2.3 next corollary follows. This controls the double commutator of the remainder operators d_p, d_p^* with smooth functions $f(\mathcal{N}_+/M)$ of the number of particles operator, varying on the scale M. This will be necessary to localize the number of particles operator in Prop. 2.5.

Corollary 2.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be smooth and bounded. For* $M \in \mathbb{N} \backslash \{0\}$, *let* $f_M = f(\mathcal{N}_+/M)$. *The bounds in (2.34), (2.36) and (2.37) remain true if we replace, on the left hand side,* d_p *by* $[f_M, [f_M, d_p]]$, $\bar{\bar{d}}_p$ *by* $[f_M, [f_M, \bar{\bar{d}}_p]]$, $\check{d}_x$ *by* $[f_M, [f_M, \check{d}_x]]$, $\check{a}_y \check{\bar{d}}_x$ *by* $[f_M, [f_M, \check{a}_y \check{\bar{d}}_x]]$ *and* $\check{d}_x \check{d}_y$ *by* $[f_M, [f_M, \check{d}_x \check{d}_y]]$ *and, on the right hand side, the constant* C *by* $CM^{-2} \|f'\|_\infty^2$. *For example, the first bound in (2.34) becomes*

$$\begin{aligned} &\big\|(\mathcal{N}_+ + 1)^{n/2} [f_M, [f_M, d_p]] \xi \big\| \\ &\qquad\qquad \leq \frac{C \|f'\|_\infty^2}{N M^2} \left[|\eta_p| \|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| + \|\eta\| \|b_p (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| \right]. \end{aligned}$$

We can now construct the generalized Bogoliubov transformation as in (2.28), with the coefficients introduced in (2.22), $e^{B_H} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$. With e^{B_H}, we define a new, renormalized, excitation Hamiltonian $\mathcal{G}_{N,\ell}^{\beta} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ by setting

$$\mathcal{G}_{N,\ell}^{\beta} = e^{-B_H} \mathcal{L}_N^{\beta} e^{B_H} = e^{-B_H} U_N H_N^{\beta} U_N^* e^{B_H}. \tag{2.39}$$

Notice that $\mathcal{G}_{N,\ell}^{\beta}$ depends also on α, which appears in the definition of the unitary operator e^{B_H}. For convenience we do not keep track of its dependence in the notation of $\mathcal{G}_{N,\ell}^{\beta}$.

In the next proposition, we collect some important properties of the renormalized excitation Hamiltonian $\mathcal{G}_{N,\ell}^{\beta}$. In the following, we will use the notation

$$\mathcal{K} = \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p \qquad \text{and} \qquad \mathcal{V}_N^{\beta} = \frac{1}{2N} \sum_{\substack{p,q \in \Lambda_+^*, r \in \Lambda^*: \\ r \neq -p,-q}} \widehat{V}(r/N^{\beta}) a_{p+r}^* a_q^* a_{q+r} a_p \tag{2.40}$$

for the kinetic and potential energy operators, restricted on $\mathcal{F}_+^{\leq N}$. We will also write

$$\mathcal{H}_N^{\beta} = \mathcal{K} + \mathcal{V}_N^{\beta}\,.$$

Proposition 2.4. *Let $V \in L^2(\mathbb{R}^2)$ be compactly supported, pointwise non-negative and spherically symmetric. Let $\mathcal{G}_{N,\ell}^{\beta}$ be defined as in* (2.39) *and let*

$$\begin{aligned}
\mathcal{G}_{N,\ell}^{\beta,\text{eff}} :=& \left[\frac{\widehat{V}(0)}{2} - \frac{\widehat{V}(0)^2}{8\pi N} \log N^{\beta} \right] (N - \mathcal{N}_+) \\
&+ \left[\frac{\widehat{V}(0)}{2} + \frac{\widehat{V}(0)^2}{8\pi N} \log N^{\beta} \right] \mathcal{N}_+ \left(\frac{N - \mathcal{N}_+}{N} \right) \\
&+ \widehat{V}(0) \sum_{p \in P_H^c} a_p^* a_p \left(1 - \frac{\mathcal{N}_+}{N} \right) + \frac{\widehat{V}(0)}{2} \sum_{p \in P_H^c} \left(b_p b_{-p} + b_{-p}^* b_p^* \right) \\
&+ \frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N^{\beta}) \left[b_{p+q}^* a_{-p}^* a_q + \text{h.c.} \right] + \mathcal{H}_N^{\beta}\,,
\end{aligned} \tag{2.41}$$

where $P_H^c = \Lambda_+^* \setminus P_H$. Then for all $\alpha > 1$, $\ell \in (0; 1/2)$ there exists a constant $C > 0$ such that $\mathcal{E}_{N,\ell}^{\beta} = \mathcal{G}_{N,\ell}^{\beta} - \mathcal{G}_{N,\ell}^{\beta,\text{eff}}$ is bounded by

$$\pm \mathcal{E}_{N,\ell}^{\beta} \leq C \ell^{\alpha-1} \mathcal{H}_N^{\beta} + C |\log \ell|. \tag{2.42}$$

for all N large enough. Moreover, there exists a constant $C > 0$ such that

$$\pm \left[f(\mathcal{N}_+/M), \left[f(\mathcal{N}_+/M), \mathcal{E}_{N,\ell}^{\beta} \right] \right] \leq C \ell^{\alpha-1} M^{-2} \|f'\|_\infty^2 \left(\mathcal{H}_N^{\beta} + 1 \right) \tag{2.43}$$

for all $\alpha > 1$, $\ell \in (0; 1/2)$ small enough, $f : \mathbb{R} \to \mathbb{R}$ smooth and bounded, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ large enough.

The analysis of $\mathcal{G}^\beta_{N,\ell}$ as well as the proof Prop. 2.4 will be discussed in details in Chapter 3. In the next proposition we give more detailed information on $\mathcal{G}^\beta_{N,\ell}$ as well as a localization estimate for the renormalized Hamiltonian.

Proposition 2.5. *Let $V \in L^2(\mathbb{R}^2)$ be compactly supported, pointwise non-negative and spherically symmetric. Then the lower bound*

$$\mathcal{G}^\beta_{N,\ell} \geq \frac{\widehat{V}(0)}{2}N - \frac{\widehat{V}(0)^2}{8\pi}\log N^\beta + c\mathcal{H}^\beta_N - C\mathcal{N}_+ - C|\log \ell|, \tag{2.44}$$

holds true for all $\alpha > 1$, $\ell \in (0;1/2)$ small enough, $N \in \mathbb{N}$ large enough. Under the same conditions we can also write

$$\mathcal{G}^\beta_{N,\ell} = \frac{\widehat{V}(0)}{2}N - \frac{\widehat{V}(0)^2}{8\pi}\log N^\beta + \mathcal{H}^\beta_N + \theta^\beta_{N,\ell} \tag{2.45}$$

where for every $\delta > 0$ there exists a constant $C > 0$ such that

$$\pm\, \theta^\beta_{N,\ell} \leq \delta\mathcal{H}^\beta_N + C\,|\log \ell|(\mathcal{N}_+ + 1), \tag{2.46}$$

and there exists a constant $C > 0$ such that

$$\pm\Big[f(\mathcal{N}_+/M), \Big[f(\mathcal{N}_+/M), \theta^\beta_{N,\ell}\Big]\Big] \leq C\,|\log \ell|^{1/2}M^{-2}\|f'\|^2_\infty \left(\mathcal{H}^\beta_N + 1\right) \tag{2.47}$$

for all $\alpha > 1$, $\ell \in (0;1/2)$ small enough, $f : \mathbb{R} \to \mathbb{R}$ smooth and bounded, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ large enough.

Moreover, Let $f, g : \mathbb{R} \to [0;1]$ be smooth, with $f^2(x)+g^2(x) = 1$ for all $x \in \mathbb{R}$. For $M \in \mathbb{N}$, let $f_M := f(\mathcal{N}_+/M)$ and $g_M := g(\mathcal{N}_+/M)$. There exists $C > 0$ such that

$$\mathcal{G}^\beta_{N,\ell} = f_M\, \mathcal{G}^\beta_{N,\ell}\, f_M + g_M\, \mathcal{G}^\beta_{N,\ell}\, g_M + \mathcal{E}^\beta_M \tag{2.48}$$

with

$$\pm\mathcal{E}^\beta_M \leq \frac{C\,|\log \ell|^{1/2}}{M^2}\big(\|f'\|^2_\infty + \|g'\|^2_\infty\big)\big(\mathcal{H}^\beta_N + 1\big)$$

for all $\alpha > 1$, $\ell \in (0;1/2)$ small enough, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ large enough.

Proof. First we prove (2.44),(2.45) and the bound in (2.47). We have to control the off-diagonal quadratic term $\frac{\widehat{V}(0)}{2}\sum_{p\in P^c_H}\left(b_pb_{-p} + b^*_{-p}b^*_p\right)$ and the cubic term $\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda^*_+:p+q\neq 0}\widehat{V}(p/N^\beta)\left[b^*_{p+q}a^*_{-p}a_q + \text{h.c.}\right]$ appearing in $\mathcal{G}^{\beta,\text{eff}}_{N,\ell}$, defined in (2.41). We observe, first of all, that

$$\begin{aligned}\Big|\frac{\widehat{V}(0)}{2}\sum_{p\in P^c_H}\langle\xi, (b_pb_{-p} + b^*_{-p}b^*_p)\xi\rangle\Big| &\leq \frac{\widehat{V}(0)}{2}\sum_{p\in P^c_H}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|b_p\xi\| \\ &\leq C|\log \ell|^{1/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\,.\end{aligned} \tag{2.49}$$

Moreover, we have

$$\begin{aligned}&[f_M, [f_M, b_pb_{-p}]] \\ &\quad= f_M(f_M\, b_pb_{-p} - b_pb_{-p}f_M) - (f_M\, b_pb_{-p} - b_pb_{-p}f_M)f_M \\ &\quad= f(\mathcal{N}_+/M)^2b_pb_{-p} - 2f(\mathcal{N}_+/M)f(\mathcal{N}_+ + 2/M)b_pb_{-p} + f(\mathcal{N}_+ + 2/M)^2b_pb_{-p} \\ &\quad= (f(\mathcal{N}_+/M) - f((\mathcal{N}_+ + 2)/M))^2b_pb_{-p},\end{aligned}$$

where we used the definition of f_M, and the equality $f(\mathcal{N}_+ + 1)a_p = a_p f(\mathcal{N}_+)$. Using this identity and a similar one for $[f_M, [f_M, b_p^* b_{-p}^*]]$, we also obtain

$$\Big|\frac{\widehat{V}(0)}{2}\sum_{p\in P_H^c}\langle \xi, \big[f_M, \big[f_M, \big(b_p b_{-p} + b_p^* b_{-p}^*\big)\big]\big]\xi\rangle\Big| \\ \leq CM^{-2}|\log \ell|^{1/2}\|f'\|_\infty^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|. \tag{2.50}$$

On the other hand, it is possible to show an improved lower bound for the operator on the l.h.s. of (2.49), by noticing that, for an arbitrary $\delta > 0$,

$$\begin{aligned} 0 &\leq \sum_{p\in P_H^c}\left(\sqrt{\delta}|p|b_p^* + \frac{\widehat{V}(0)}{2\sqrt{\delta}|p|}b_{-p}\right)\left(\sqrt{\delta}|p|b_p + \frac{\widehat{V}(0)}{2\sqrt{\delta}|p|}b_{-p}^*\right) \\ &= \delta\sum_{p\in P_H^c} p^2 b_p^* b_p + \frac{\widehat{V}(0)^2}{4\delta}\sum_{p\in P_H^c}\frac{1}{p^2}b_{-p}b_{-p}^* + \frac{\widehat{V}(0)}{2}\sum_{p\in P_H^c}(b_{-p}b_p + b_p^* b_{-p}^*). \end{aligned}$$

From commutation relations in (2.5), we have

$$b_{-p}b_{-p}^* = b_{-p}^* b_{-p} + (1 - \mathcal{N}_+/N) - N^{-1}a_{-p}^* a_{-p}\,.$$

Observing that

$$b_p^* b_p = a_p^* \frac{N - \mathcal{N}_+}{N} a_p \leq a_p^* a_p$$

and that $\sum_{p\in P_H^c}|p|^{-2} \leq C|\log \ell|$, we conclude that there exists a constant $C > 0$, independent of $\ell \in (0; 1/2)$ and of N, such that

$$\frac{\widehat{V}(0)}{2}\sum_{p\in P_H^c}(b_{-p}b_p + b_p^* b_{-p}^*) \geq -\delta\mathcal{K} - C\delta^{-1}\mathcal{N}_+ - C\delta^{-1}|\log \ell| \tag{2.51}$$

for any $\delta > 0$. As for the cubic term on the r.h.s. of (2.91), we have, switching to position space,

$$\Big|\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*: p+q\neq 0}\widehat{V}(p/N^\beta)\langle\xi, \big(b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\big)\xi\rangle\Big| \\ \leq N^{1/2}\int_{\Lambda^2} dxdy\, N^{2\beta-1}V(N^\beta(x-y))\|\check{a}_x\xi\|\|\check{a}_x\check{a}_y\xi\| \leq C\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{V}_N^\beta)^{1/2}\,\xi\| \tag{2.52}$$

and analogously

$$\Big|\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*: p+q\neq 0}\widehat{V}(p/N^\beta)\langle\xi, \big[f_M, \big[f_M, \big(b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\big)\big]\big]\xi\rangle\Big| \\ \leq CM^{-2}\|f'\|_\infty^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{V}_N^\beta)^{1/2}\,\xi\|. \tag{2.53}$$

Combining (2.42) with (2.49) and (2.52), we obtain (2.46). From (2.42), (2.51) and (2.52), we infer (2.44). Combining instead the bound in (2.43), with (2.50) and (2.53) we find (2.47), since all other contributions to $\mathcal{G}_{N,\ell}^{\beta,\text{eff}}$ commute with $\mathcal{N}_+$. Next, we prove (2.48). One can easily check that $\mathcal{G}_{N,\ell}^{\beta}$ can be rewritten as (see also [53, 44])

$$\mathcal{G}_{N,\ell}^{\beta} = f_M \mathcal{G}_{N,\ell}^{\beta} f_M + g_M \mathcal{G}_{N,\ell}^{\beta} g_M + \frac{1}{2}\Big([f_M,[f_M,\mathcal{G}_{N,\ell}^{\beta}]] + [g_M,[g_M,\mathcal{G}_{N,\ell}^{\beta}]]\Big).$$

Writing as in (2.45), $\mathcal{G}_{N,\ell}^{\beta} = D_N + \mathcal{H}_N^{\beta} + \theta_{N,\ell}^{\beta}$, with $D_N = \widehat{V}(0)N/2 - \widehat{V}(0)^2(\log N^{\beta})/8\pi$, and noticing that D_N and $\mathcal{H}_N^{\beta}$ commute with f_M, g_M, and using the bound in (2.43), we conclude that

$$\pm\Big([f_M,[f_M,\mathcal{G}_{N,\ell}^{\beta}]] + [g_M,[g_M,\mathcal{G}_{N,\ell}^{\beta}]]\Big) \leq \frac{C\,|\log \ell|^{1/2}}{M^2}\big(\|f_M'\|_\infty^2 + \|g_M'\|_\infty^2\big)\big(\mathcal{H}_N^{\beta}+1\big).$$

□

2.2.2 Cubic Renormalization

Conjugation through the generalized Bogoliubov transformation (2.28) is not enough to prove Theorem 1.1. In order to estimate the number of excitations $\mathcal{N}_+$ through the energy and show Bose-Einstein condensation, we still need to renormalize the cubic term on the last line (2.41).

To obtain this, we conjugate the main part of $\mathcal{G}_{N,\ell}^{\beta}$, namely $\mathcal{G}_{N,\ell}^{\beta,\text{eff}}$, with an additional unitary operator, given by the exponential of the anti-symmetric operator $A_H : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$

$$A_H := \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_H^c} \eta_r \big[b_{r+v}^* a_{-r}^* a_v - \text{h.c.}\big], \tag{2.54}$$

with coefficients $\eta_H(p)$ defined as in (2.22) and

$$P_H^c = \Lambda_+^* \setminus P_H = \{p \in \Lambda_+^* : |p| \leq \ell^{-\alpha}\},$$

for $\ell \in (0;1/2)$, $\alpha > 1$ introduced to make the norm of η_p small.

An important observation is that while conjugation with e^{A_H} allows to renormalize the large contribution in $\mathcal{G}_{N,\ell}^{\beta}$, it does not substantially change the number of excitations. The following proposition can be proved as in [10, Proposition 5.1].

Proposition 2.6. *Suppose that A_H is defined as in (2.54). For any $k \in \mathbb{N}$ there exists a constant $C > 0$ such that the operator inequality*

$$e^{-A_H}(\mathcal{N}_+ + 1)^k e^{A_H} \leq C(\mathcal{N}_+ + 1)^k$$

holds true on $\mathcal{F}_+^{\leq N}$, for all $\alpha > 0$, and N large enough.

We will need to control also the growth of the expectation of the total energy operator $\mathcal{H}_N^{\beta} = \mathcal{K} + \mathcal{V}_N^{\beta}$ with respect to the cubic conjugation, as stated in the following lemma.

Proposition 2.7. *There exists a constant $C > 0$ such that*

$$e^{-sA_H}\mathcal{H}_N^\beta\, e^{sA_H} \leq C\,\mathcal{H}_N^\beta + C\,|\log \ell|\,(\mathcal{N}_+ + 1) \tag{2.55}$$

for all $\alpha > 0$, $s \in [0;1]$ and $N \in \mathbb{N}$ large enough.

The proof of Prop. 2.7 can be found in Chapter 3, Section 3.2. We now use the cubic phase e^{A_H} to introduce a new excitation Hamiltonian, defining

$$\mathcal{R}_{N,\ell}^\beta := e^{-A_H}\,\mathcal{G}_{N,\ell}^{\beta,\text{eff}}\,e^{A_H} \tag{2.56}$$

on a dense subset of $\mathcal{F}_+^{\leq N}$. The operator $\mathcal{G}_{N,\ell}^{\beta,\text{eff}}$ is defined as in (2.41).This allows us to show the following proposition.

Proposition 2.8. *Let $V \in L^2(\mathbb{R}^2)$ be compactly supported, pointwise non-negative and spherically symmetric. Then, for all $\alpha > 1$, there exists a constant $C > 0$ such that*

$$\mathcal{R}_{N,\ell}^\beta \geq \frac{\widehat{V}(0)}{2}N - \frac{\widehat{V}(0)^2}{8\pi}\log N^\beta + \big(1 - C\,\ell^\alpha \log \ell\big)\mathcal{H}_N^\beta - C\ell^{-2\alpha}\mathcal{N}_+^2/N - C\ell^{-2\alpha} \tag{2.57}$$

for all $\ell \in (0;1/2)$ small enough and N large enough.

As for $\mathcal{G}_{N,\ell}^\beta$, the detailed analysis of $\mathcal{R}_{N,\ell}^\beta$ and the proof of Proposition 2.8 will be given in Chapter 3.

2.2.3 Proof of Theorem 1.1

To show Theorem 1.1 we will use Theorem 1.2, as stated in Section 1.1, which shows Bose-Einstein condensation for approximate minimizers of the Hamiltonian (1.6). The next proposition combines the results of Prop. 2.4, Prop. 2.5 and Prop. 2.8 with Theorem 1.2. We make use of localization of the number of particles techniques, a technique borrowed from Lewin-Nam-Serfaty-Solovej [44] (inspired by previous work of Lieb-Solovej [53]).

Proposition 2.9. *Let $V \in L^2(\mathbb{R}^2)$ be compactly supported, pointwise non-negative and spherically symmetric. Let $\mathcal{G}_{N,\ell}^\beta$ be the renormalized excitation Hamiltonian defined as in (2.86). Then, for every $\alpha > 1$, $\ell \in (0;1/2)$ small enough, there exist constants $C, c > 0$ such that*

$$\mathcal{G}_{N,\ell}^\beta - \frac{\widehat{V}(0)}{2}N + \frac{\widehat{V}(0)^2}{8\pi}\log N^\beta \geq c\,N_+ - C \tag{2.58}$$

for all $N \in \mathbb{N}$ sufficiently large.

Proof. As in Proposition 2.5, let $f, g : \mathbb{R} \to [0;1]$ be smooth, with $f^2(x)+g^2(x) = 1$ for all $x \in \mathbb{R}$. Moreover, assume that $f(x) = 0$ for $x > 1$ and $f(x) = 1$ for

$x < 1/2$, we fix $M = \ell^{3\alpha} N$, and we set $f_M = f(\mathcal{N}_+/M), g_M = g(\mathcal{N}_+/M)$. We also define

$$D_N = \frac{\widehat{V}(0)}{2} N - \frac{\widehat{V}(0)^2}{8\pi} \log N^\beta \, . \tag{2.59}$$

It follows from Proposition 2.5, Eq. (2.48) that

$$\begin{aligned} \mathcal{G}^\beta_{N,\ell} - D_N \geq f_M \big(\mathcal{G}^\beta_{N,\ell} - D_N\big) f_M + g_M \big(\mathcal{G}^\beta_{N,\ell} - D_N\big) g_M \\ - C |\log \ell|^{1/2}\, \ell^{-6\alpha} N^{-2} (\mathcal{H}^\beta_N + 1) \, . \end{aligned} \tag{2.60}$$

Let us consider the first term on the r.h.s. of (2.60). From Prop. 2.4, there exists $C > 0$ such that

$$\mathcal{G}^\beta_{N,\ell} - D_N \geq \mathcal{G}^{\beta,\text{eff}}_{N,\ell} - D_N - C\ell^{\alpha-1} \mathcal{H}^\beta_N - C\, |\log \ell|$$

and also, from (2.45),

$$\mathcal{G}^\beta_{N,\ell} - D_N \geq \frac{1}{2} \mathcal{H}^\beta_N - C\mathcal{N}_+ - C\, |\log \ell| \tag{2.61}$$

for all $\alpha > 1$, $\ell \in (0; 1/2)$ small enough and N large enough. The last two bounds combined together imply that

$$\mathcal{G}^\beta_{N,\ell} - D_N \geq (1 - C\ell^{\alpha-1}) \big(\mathcal{G}^{\beta,\text{eff}}_{N,\ell} - D_N\big) - C\ell^{\alpha-1} \mathcal{N}_+ - C\, |\log \ell|,$$

and in turn, for $\ell > 0$ small enough,

$$\mathcal{G}^\beta_{N,\ell} - D_N \geq \frac{1}{2} \big(\mathcal{G}^{\beta,\text{eff}}_{N,\ell} - D_N\big) - C\ell^{\alpha-1} \mathcal{N}_+ - C\, |\log \ell|.$$

Now, using Prop. 2.8, choosing $\alpha > 1$ we find

$$\begin{aligned} & f_M (\mathcal{G}^\beta_{N,\ell} - D_N) f_M \\ & \geq \frac{1}{2} f_M \Big(\mathcal{G}^{\beta,\text{eff}}_{N,\ell} - D_N\Big) f_M - C\ell^{\alpha-1} f_M^2 \mathcal{N}_+ - C |\log \ell| f_M^2 \\ & \geq \frac{1}{2} f_M e^{A_H} \left[(1 - C\ell^\alpha |\log \ell|\,) \mathcal{H}^\beta_N - C\ell^{-2\alpha} \frac{\mathcal{N}_+^2}{N} - C\ell^{-2\alpha} \right] e^{-A_H} f_M \\ & \qquad\qquad - C\ell^{\alpha-1} f_M^2 \mathcal{N}_+ - C\, |\log \ell| f_M^2 \\ & \geq \frac{1}{2} f_M e^{A_H} \Big[(1 - C\ell^\alpha |\log \ell|\,) \mathcal{H}^\beta_N - C\ell^\alpha \mathcal{N}_+ \Big] e^{-A_H} f_M - C\ell^{\alpha-1} f_M^2 \mathcal{N}_+ - C\ell^{-2\alpha} f_M^2, \end{aligned}$$

where in the last inequality, we used Prop. 2.6 to estimate

$$\begin{aligned} f_M e^{-A_H} \mathcal{N}_+^2 e^{A_H} f_M & \leq C f_M (\mathcal{N}_+ + 1)^2 f_M \\ & \leq C N \ell^{3\alpha} f_M (\mathcal{N}_+ + 1) f_M \leq C N \ell^{3\alpha} f_M e^{-A_H} (\mathcal{N}_+ + 1) e^{A_H} f_M \end{aligned}$$

due to the choice of $M = \ell^{3\alpha} N$. Since now $\mathcal{N}_+ \leq C\mathcal{K} \leq C\mathcal{H}^\beta_N$, we obtain that, for $\ell \in (0; 1/2)$ small enough,

$$f_M \big(\mathcal{G}^\beta_{N,\ell} - D_N\big) f_M \geq C f_M e^{A_H} \mathcal{N}_+ e^{-A_H} f_M - C\ell^{\alpha-1} f_M^2 \mathcal{N}_+ - C\ell^{-2\alpha} f_M^2 \, .$$

With Prop. 2.6, we conclude that, again for $\ell > 0$ small enough,

$$f_M\Big(\mathcal{G}_{N,\ell}^{\beta} - D_N\Big)f_M \geq C f_M^2 \mathcal{N}_+ - C\ell^{-2\alpha} f_M^2. \tag{2.62}$$

We now focus on the second term on the r.h.s. of (2.60). We want that Eq. (2.62) holds true, so we keep $\ell > 0$ fixed, and we will only worry about the dependence on N. We claim that there exists a constant $C > 0$ such that

$$g_M\big(\mathcal{G}_{N,\ell}^{\beta} - D_N\big)g_M \geq g_M\big(\mathcal{G}_{N,\ell}^{\beta} - \widehat{V}(0)N/2\big)g_M \geq CN g_M^2 \tag{2.63}$$

for all N sufficiently large. To prove (2.63) we observe that, since $g(x) = 0$ for all $x \leq 1/2$,

$$g_M\Big(\mathcal{G}_{N,\ell}^{\beta} - \widehat{V}(0)N/2\Big)g_M \geq \left[\inf_{\xi\in\mathcal{F}_{\geq M/2}^{\leq N}:\|\xi\|=1} \frac{1}{N}\langle\xi, \mathcal{G}_{N,\ell}^{\beta}\xi\rangle - \frac{\widehat{V}(0)}{2}\right] N g_M^2$$

where $\mathcal{F}_{\geq M/2}^{\leq N} = \{\xi \in \mathcal{F}_+^{\leq N} : \xi = \chi(\mathcal{N}_+ \geq M/2)\xi\}$ is the subspace of $\mathcal{F}_+^{\leq N}$ where states with at least $M/2$ excitations are described (recall that $M = \ell^{3\alpha}N$). To prove (2.63) it is enough to show that there exists $C > 0$ with

$$\inf_{\xi\in\mathcal{F}_{\geq M/2}^{\leq N}:\|\xi\|=1} \frac{1}{N}\langle\xi, \mathcal{G}_{N,\ell}^{\beta}\,\xi\rangle - \frac{\widehat{V}(0)}{2} \geq C \tag{2.64}$$

for all N large enough. From Theorem 1.2 we know that

$$\begin{aligned}\inf_{\xi\in\mathcal{F}_{\geq M/2}^{\leq N}:\|\xi\|=1} \frac{1}{N}\langle\xi, \mathcal{G}_{N,\ell}^{\beta}\,\xi\rangle - \frac{\widehat{V}(0)}{2} &\geq \inf_{\xi\in\mathcal{F}_{+}^{\leq N}:\|\xi\|=1} \frac{1}{N}\langle\xi, \mathcal{G}_{N,\ell}^{\beta}\xi\rangle - \frac{\widehat{V}(0)}{2} \\ &= \frac{E_N^{\beta}}{N} - \frac{\widehat{V}(0)}{2} \to 0\end{aligned}$$

as $N \to \infty$. Hence, if we assume by contradiction that (2.64) does not hold true, then we can find a subsequence $N_j \to \infty$ with

$$\inf_{\xi\in\mathcal{F}_{\geq M_j/2}^{\leq N_j}:\|\xi\|=1} \frac{1}{N_j}\langle\xi, \mathcal{G}_{N_j,\ell}^{\beta}\,\xi\rangle - \frac{\widehat{V}(0)}{2} \to 0$$

as $j \to \infty$ (here we used the notation $M_j = \ell^{3\alpha}N_j$). This implies that there exists a sequence $\xi_{N_j} \in \mathcal{F}_{\geq M_j/2}^{\leq N_j}$ with $\|\xi_{N_j}\| = 1$ for all $j \in \mathbb{N}$ such that

$$\lim_{j\to\infty} \frac{1}{N_j}\langle\xi_{N_j}, \mathcal{G}_{N_j,\ell}^{\beta}\,\xi_{N_j}\rangle = \frac{\widehat{V}(0)}{2}\,.$$

Let now $S := \{N_j : j \in \mathbb{N}\} \subset \mathbb{N}$ and denote by ξ_N a normalized minimizer of $\mathcal{G}_{N,\ell}^{\beta}$ for all $N \in \mathbb{N}\setminus S$. Setting $\psi_N = U_N^* e^{B_H}\xi_N$, for all $N \in \mathbb{N}$, we obtain that $\|\psi_N\| = 1$ and that

$$\lim_{N\to\infty} \frac{1}{N}\langle\psi_N, H_N^{\beta}\psi_N\rangle = \lim_{N\to\infty} \frac{1}{N}\langle\xi_N, \mathcal{G}_{N,\ell}^{\beta}\,\xi_N\rangle = \frac{\widehat{V}(0)}{2}.$$

In other words, the sequence ψ_N is an approximate ground state of H_N. From (1.9), we conclude that ψ_N exhibits complete Bose-Einstein condensation in the zero-momentum mode φ_0, meaning that

$$\lim_{N\to\infty}\big(1-\langle\varphi_0,\gamma_N\varphi_0\rangle\big)=0\,.$$

Using Lemma 2.2 and the rules (2.4), we observe that

$$\begin{aligned}\frac{1}{N}\langle\xi_N,\mathcal{N}_+\xi_N\rangle&=\frac{1}{N}\langle e^{-B_H}U_N\psi_N,\mathcal{N}_+e^{-B_H}U_N\psi_N\rangle\\&\le\frac{C}{N}\langle\psi_N,U_N^*(\mathcal{N}_++1)U_N\psi_N\rangle\\&=\frac{C}{N}+C\left[1-\frac{1}{N}\langle\psi_N,a^*(\varphi_0)a(\varphi_0)\psi_N\rangle\right]\\&=\frac{C}{N}+C\,[1-\langle\varphi_0,\gamma_N\varphi_0\rangle]\to 0\end{aligned}\tag{2.65}$$

as $N\to\infty$. On the other hand, for $N\in S=\{N_j:j\in\mathbb{N}\}$, we have $\xi_N=\chi(\mathcal{N}_+\ge M/2)\xi_N$ and therefore

$$\frac{1}{N}\langle\xi_N,\mathcal{N}_+\xi_N\rangle\ge\frac{M}{2N}=\frac{\ell^{3\alpha}}{2}$$

in contradiction with (2.65). This proves (2.64), (2.63) and therefore also

$$g_M\big(\mathcal{G}^\beta_{N,\ell}-D_N\big)g_M\ge C\mathcal{N}_+g_M^2.\tag{2.66}$$

Inserting (2.62) and (2.66) on the r.h.s. of (2.60), we obtain that

$$\mathcal{G}^\beta_{N,\ell}-D_N\ge C\mathcal{N}_+-CN^{-2}\mathcal{H}^\beta_N-C\tag{2.67}$$

for N large enough (the constants C are now allowed to depend on ℓ, since ℓ has been fixed once and for all after (2.62)). From Eq. (2.67) together with (2.61), we obtain

$$\begin{aligned}\mathcal{G}^\beta_{N,\ell}-D_N&\ge C\mathcal{N}_+-CN^{-2}\mathcal{H}^\beta_N-C\\&\ge C\mathcal{N}_+-CN^{-2}(\mathcal{G}^\beta_{N,\ell}-D_N)-CN^{-2}\mathcal{N}_+-CN^{-2}|\log\ell|,\end{aligned}$$

which clearly leads to (2.58). □

We are now ready to show our main theorem.

Proof of Theorem 1.1. First of all, (2.45) and (2.46) in Prop. 2.4 imply that

$$\mathcal{G}^\beta_{N,\ell}-D_N\le 2\mathcal{H}^\beta_N+C\mathcal{N}_++C$$

with D_N defined in (2.59). With the vacuum Ω as trial state, we obtain the upper bound $E^\beta_N\le D_N+C$ for the ground state energy E^β_N of $\mathcal{G}^\beta_{N,\ell}$ (which coincides with the ground state energy of H^β_N). With Eq. (2.58), we also find the lower bound $E^\beta_N\ge D_N-C$. This proves (1.7).

Let now be $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and

$$\langle \psi_N, H_N^\beta \psi_N \rangle \le D_N + K\,.$$

We define the excitation vector $\xi_N = e^{-B_H} U_N \psi_N$. Then $\|\xi_N\| = 1$ and, recalling that $\mathcal{G}^\beta_{N,\ell} = e^{-B_H} U_N H_N^\beta U_N^* e^{B_H}$, we have

$$\langle \xi_N, \mathcal{N}_+ \xi_N \rangle \le C \big\langle \xi_N, \Big(\mathcal{G}^\beta_{N,\ell} - D_N \Big) \xi_N \big\rangle + C \le C(K+1)\,.$$

If γ_N denotes the one-particle reduced density matrix associated with ψ_N, we obtain

$$\begin{aligned} 1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle &= 1 - \frac{1}{N} \langle \psi_N, a^*(\varphi_0) a(\varphi_0) \psi_N \rangle \\ &= 1 - \frac{1}{N} \langle U_N^* e^{B_H} \xi_N, a^*(\varphi_0) a(\varphi_0) U_N^* e^{B_H} \xi_N \rangle \\ &= \frac{1}{N} \langle e^{B_H} \xi_N, \mathcal{N}_+ e^{B_H} \xi_N \rangle \le \frac{C}{N} \langle \xi_N, \mathcal{N}_+ \xi_N \rangle \le \frac{C(K+1)}{N} \end{aligned}$$

which concludes the proof of (1.8). □

2.3 Renormalization of the excitation Hamiltonian $\mathcal{L}_N$

In this section, we proceed as in Section 2.2. Namely, we will suitably renormalize the Gross-Pitaevskii excitation Hamiltonian $\mathcal{L}_N$ defined in Eq. (2.9). This section follows [20, Section 3, 4, 5], a joint work with Serena Cenatiempo and Benjamin Schlein.

As before we consider the scattering problem associated to the Gross-Pitaevskii interaction to take into account the short scale correlation structure on top of the condensate. In particular, we consider the solution f_ℓ[1] of the equation

$$\Big(-\Delta + \frac{1}{2} V(x) \Big) f_\ell(x) = \lambda_\ell \, f_\ell(x) \tag{2.68}$$

associated with the smallest possible eigenvalue λ_ℓ, on the ball $|x| \le e^N \ell$, with Neumann boundary conditions and normalized so that $f_\ell(x) = 1$ for $|x| = e^N \ell$. Here and in the following we omit the N-dependence in the notation for f_ℓ and for λ_ℓ. By scaling, we observe that $f_\ell(e^N \cdot)$ satisfies

$$\Big(-\Delta + \frac{e^{2N}}{2} V(e^N x) \Big) f_\ell(e^N x) = e^{2N} \lambda_\ell \, f_\ell(e^N x)$$

on the ball $|x| \le \ell$. We choose $\ell < 1/2$, so that the ball of radius ℓ is contained in the box $\Lambda = [-1/2; 1/2]^2$. We extend then $f_\ell(e^N .)$ to Λ, by setting $f_{N,\ell}(x) = f_\ell(e^N x)$, if $|x| \le \ell$ and $f_{N,\ell}(x) = 1$ for $x \in \Lambda$, with $|x| > \ell$. Then, assuming also

[1] The reader can notice that we are using the same notation as in Section 2.2. This is just for our convenience, the two f_ℓ solve different Neumann problems.

that $R_0 e^{-N} < \ell$ (later we will choose $\ell = N^{-\alpha}$, so this condition is satisfied, for all N large enough),

$$\Big(-\Delta + \frac{e^{2N}}{2} V(e^N x) \Big) f_{N,\ell}(x) = e^{2N} \lambda_\ell \, f_{N,\ell}(x) \chi_\ell(x) \,, \tag{2.69}$$

where χ_ℓ is the characteristic function of the ball of radius ℓ. The Fourier coefficients of the function $f_{N,\ell}$ are given by

$$\widehat{f}_{N,\ell}(p) := \int_\Lambda f_\ell(e^N x) e^{-ip\cdot x} dx$$

for all $p \in \Lambda^*$. We also introduce the function $w_\ell(x) = 1 - f_\ell(x)$ for $|x| \le e^N \ell$ and extend it by setting $w_\ell(x) = 0$ for $|x| > e^N \ell$. Its re-scaled version is defined by $w_{N,\ell} : \Lambda \to \mathbb{R}$ $w_{N,\ell}(x) = w_\ell(e^N x)$ if $|x| \le \ell$ and $w_{N,\ell} = 0$ if $x \in \Lambda$ with $|x| > \ell$.

The Fourier coefficients of the re-scaled function $w_{N,\ell}$ are given by

$$\widehat{w}_{N,\ell}(p) = \int_\Lambda w_\ell(e^N x) e^{-ip\cdot x} dx = e^{-2N} \widehat{w}_\ell \left(e^{-N} p \right) . \tag{2.70}$$

We find $\widehat{f}_{N,\ell}(p) = \delta_{p,0} - e^{-2N} \widehat{w}_\ell(e^{-N}p)$. From the Neumann problem (2.69) we obtain

$$-p^2 e^{-2N} \widehat{w}_\ell(e^{-N}p) + \frac{1}{2} \sum_{q\in\Lambda^*} \widehat{V}(e^{-N}(p-q)) \widehat{f}_{N,\ell}(q) = e^{2N} \lambda_\ell \sum_{q\in\Lambda^*} \widehat{\chi}_\ell(p-q) \widehat{f}_{N,\ell}(q). \tag{2.71}$$

where we used the notation $\widehat{\chi}_\ell$ for the Fourier coefficients of the characteristic function on the ball of radius ℓ. Note that $\widehat{\chi}_\ell(p) = \ell^2 \, \widehat{\chi}(\ell p)$ with $\widehat{\chi}(p)$ the Fourier coefficients of the characteristic function on the ball of radius one.

In the next lemma, we collect some important properties of the solution of (2.68).

Lemma 2.10. *Let $V \in L^3(\mathbb{R}^2)$ be non-negative, compactly supported (with range R_0) and spherically symmetric, and denote its scattering length by $\mathfrak{a}$, as in Eq. (1.11). Fix $0 < \ell < 1/2$, N sufficiently large and let f_ℓ denote the solution of (2.69). Then*

i)

$$0 \le f_\ell(x) \le 1 \qquad \forall \, |x| \le e^N \ell \, .$$

ii) We have

$$\left| \lambda_\ell - \frac{2}{(e^N\ell)^2 \log(e^N\ell/\mathfrak{a})} \right| \le \frac{C}{(e^N\ell)^2 \log^2(e^N\ell/\mathfrak{a})} \tag{2.72}$$

iii) There exist a constant $C > 0$ such that

$$\left| \int dx \, V(x) f_\ell(x) - \frac{4\pi}{\log(e^N\ell/\mathfrak{a})} \right| \le \frac{C}{\log^2(e^N\ell/\mathfrak{a})} \tag{2.73}$$

iv) There exists a constant $C > 0$ such that

$$|w_\ell(x)| \le \begin{cases} C & \text{if } |x| \le R_0 \\ C\,\frac{\log(e^N\ell/|x|)}{\log(e^N\ell/\mathfrak{a})} & \text{if } R_0 \le |x| \le e^N\ell \end{cases}$$
$$|\nabla w_\ell(x)| \le \frac{C}{\log(e^N\ell/\mathfrak{a})}\frac{1}{|x|+1} \qquad \text{for all } |x| \le e^N\ell \tag{2.74}$$

v) Let $w_{N,\ell} = 1 - f_{N,\ell}$ with $f_{\ell,N} = f_\ell(e^N x)$. Then the Fourier coefficients of the function $w_{N,\ell}$ defined in (2.70) are such that

$$|\widehat{w}_{N,\ell}(p)| \le \frac{C}{p^2\log(e^N\ell/\mathfrak{a})}. \tag{2.75}$$

Proof. The proof of points i)-iv) is deferred in Appendix A. To prove point v) we use the scattering equation (2.71):

$$\widehat{w}_\ell(e^{-N}p) = \frac{e^{2N}}{2p^2}\sum_{q\in\Lambda^*}\widehat{V}(e^{-N}(p-q))\widehat{f}_{N,\ell}(q) - \frac{e^{4N}}{p^2}\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\widehat{f}_{N,\ell}(q).$$

From point ii) $e^{2N}\lambda_\ell \le C\ell^{-2}|\ln(e^N\ell/\mathfrak{a})|^{-1}$ and using that $0 \le f_\ell \le 1$, we end up with

$$\begin{aligned} |\widehat{w}_\ell(e^{-N}p)| &\le \frac{e^{2N}}{2p^2}\Big[\big|(\widehat{V}(e^{-N}\cdot) * \widehat{f}_{N,\ell})(p)\big| + 2e^{2N}\lambda_\ell\,\big|(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(p)\big|\Big] \\ &\le \frac{e^{2N}}{2p^2}\left[\int V(x)f_\ell(x)dx + C\ell^{-2}|\log(e^N\ell/\mathfrak{a})|^{-1}\int \chi_\ell(x)f_\ell(e^N x)dx\right] \\ &\le \frac{Ce^{2N}}{p^2\log(e^N\ell/\mathfrak{a})}. \end{aligned}$$

□

We now define $\check{\eta} : \Lambda \to \mathbb{R}$ through

$$\check{\eta}(x) = -Nw_{N,\ell}(x) = -Nw_\ell(e^N x)\,. \tag{2.76}$$

With (2.74) we find

$$|\check{\eta}(x)| \le \begin{cases} CN & \text{if } |x| \le e^{-N}R_0 \\ C\log(\ell/|x|) & \text{if } e^{-N}R_0 \le |x| \le \ell \end{cases} \tag{2.77}$$

and in particular, recalling that $e^{-N}R_0 < \ell \le 1/2$,

$$|\check{\eta}(x)| \le C\max(N, \log(\ell/|x|)) \le CN \tag{2.78}$$

for all $x \in \Lambda$. Using (2.77) we find

$$\|\eta\|^2 = \|\check{\eta}\|^2 \le C\int_{|x|\le\ell}|\log(\ell/|x|)|^2 d^2x \le C\ell^2\int_0^1(\log r)^2 r\,dr \le C\ell^2\,.$$

In the following we choose $\ell = N^{-\alpha}$, for some $\alpha > 0$ to be fixed later, so that

$$\|\eta\| \leq CN^{-\alpha}\,. \tag{2.79}$$

This choice of ℓ will be crucial for our analysis, as commented below. Notice, on the other hand, that the H^1-norms of η diverge, as $N \to \infty$. From (2.76) and Lemma 2.10, part iv) we find

$$\begin{aligned}\|\check{\eta}\|_{H_1}^2 = \int_{|x|\leq \ell} e^{2N} N^2 |(\nabla w_\ell)(e^N x)|^2 d^2x &= \int_{|x|\leq e^N \ell} N^2 |\nabla w_\ell(x)|^2 d^2x \\ &\leq C \int_{|x|\leq e^N\ell} \frac{1}{(|x|+1)^2}\, d^2x \leq CN\end{aligned}$$

for $N \in \mathbb{N}$ large enough. We denote with $\eta : \Lambda^* \to \mathbb{R}$ the Fourier transform of $\check{\eta}$, or equivalently

$$\eta_p = -N\widehat{w}_{N,\ell}(p) = -Ne^{-2N}\widehat{w}_\ell(p/e^N)\,. \tag{2.80}$$

With (2.75) we can bound (since $\ell = N^{-\alpha}$)

$$|\eta_p| \leq \frac{C}{|p|^2} \tag{2.81}$$

for all $p \in \Lambda_+^* = 2\pi\mathbb{Z}^2\backslash\{0\}$, and for some constant $C > 0$ independent of N, if N is large enough. From (2.79) we also have

$$\|\eta\|_\infty \leq CN^{-\alpha}\,. \tag{2.82}$$

Remark. Notice that in this scaling we need to choose the smallness of the L^2-norm of η in terms of some power of N. On the contrary, in Section 2.2 was sufficient to choose $\|\eta_H\| \leq C\ell^\alpha$, with $\ell \in (0; 1/2)$ of order $\mathcal{O}(1)$. Indeed, a constant of order $\mathcal{O}(1)$. In fact, in this setting, a small constant would not be sufficient to control the error terms.

Moreover, (2.71) implies the relation

$$p^2\eta_p + \frac{N}{2}(\widehat{V}(./e^N) * \widehat{f}_{N,\ell})(p) = Ne^{2N}\lambda_\ell(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(p) \tag{2.83}$$

or equivalently, expressing also the other terms through the coefficients η_p,

$$\begin{aligned}p^2\eta_p + \frac{N}{2}\widehat{V}(p/e^N) + \frac{1}{2}\sum_{q\in\Lambda^*}\widehat{V}((p-q)/e^N)\eta_q \\ = Ne^{2N}\lambda_\ell\widehat{\chi}_\ell(p) + e^{2N}\lambda_\ell \sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q\,.\end{aligned} \tag{2.84}$$

We will mostly use the coefficients η_p with $p \neq 0$. Sometimes, however, it will be useful to have an estimate on η_0 (because Eq. (2.84) involves η_0). From (2.80) and Lemma 2.10, part iv) we find

$$|\eta_0| \leq N\int_{|x|\leq\ell} w_\ell(e^N x)d^2x \leq C\int_{|x|\leq\ell}\log(\ell/|x|)d^2x + CNe^{-N} \leq C\ell^2\,. \tag{2.85}$$

2.3.1 Quadratic renormalization

We introduce generalized Bogoliubov transformation, as we did in Section 2.2, this allows us implement the correlation structure keeping invariant the truncated Fock space $\mathcal{F}_+^{\leq N}$. Again we use the coefficients defined in (2.80) to construct an anti-symmetric operator as (2.27) and in turn the unitary operator e^B as in (2.28). Here, as opposed to the coefficients in (2.22), we do not need to introduce the cut-off, and so, the unitary operator e^B will act over all the momenta in Λ_+^*. Moreover, Lemma 2.2 is still valid, indeed the proof only requires that $\|\eta\| \leq C$.

Now, using the generalized Bogoliubov transformation $e^B : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$, we define a new, renormalized, excitation Hamiltonian $\mathcal{G}_{N,\alpha} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ by setting

$$\mathcal{G}_{N,\alpha} = e^{-B}\mathcal{L}_N e^B = e^{-B} U_N H_N U_N^* e^B \,. \tag{2.86}$$

In the next proposition, we collect important properties $\mathcal{G}_{N,\alpha}$. We will use the notation

$$\mathcal{K} = \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p \qquad \text{and} \qquad \mathcal{V}_N = \frac{1}{2}\sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*:\\ r\neq -p,-q}} \widehat{V}(r/e^N) a_{p+r}^* a_q^* a_{q+r} a_p \tag{2.87}$$

for the kinetic and potential energy operators, restricted on $\mathcal{F}_+^{\leq N}$, and $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$. We also introduce a renormalized interaction potential $\omega_N \in L^\infty(\Lambda)$, which is defined as the function with Fourier coefficients $\widehat{\omega}_N$

$$\widehat{\omega}_N(p) := g_N\, \widehat{\chi}(p/N^\alpha) \,, \qquad g_N = 2N^{1-2\alpha} e^{2N} \lambda_\ell \tag{2.88}$$

for any $p \in \Lambda_+^*$, and

$$\widehat{\omega}_N(0) = g_N \widehat{\chi}(0) = \pi g_N \,. \tag{2.89}$$

with $\widehat{\chi}(p)$ the Fourier coefficients of the characteristic function of the ball of radius one. From (2.72) and $\ell = N^{-\alpha}$ one has $|g_N| \leq C$. Note in particular that the potential $\widehat{\omega}_N(p)$ decays on momenta of order N^α, which are much smaller than e^N. From Lemma 2.10 parts i) and iii) we find

$$\big|\widehat{\omega}_N(0) - N\|Vf_\ell\|_1\big| \leq \frac{C}{N} \,, \qquad \Big|\widehat{\omega}_N(0) - 4\pi\left(1 + \alpha\,\tfrac{\log N}{N}\right)\Big| \leq \frac{C}{N} \,. \tag{2.90}$$

Proposition 2.11. *Let $V \in L^3(\mathbb{R}^2)$ be compactly supported, pointwise non-negative and spherically symmetric. Let $\mathcal{G}_{N,\alpha}$ be defined as in* (2.86) *and define*

$$\begin{aligned}\mathcal{G}_{N,\alpha}^{\mathit{eff}} := {} & \frac{1}{2}\widehat{\omega}_N(0)(N-1)\left(1 - \frac{\mathcal{N}_+}{N}\right) + \left[2N\widehat{V}(0) - \frac{1}{2}\widehat{\omega}_N(0)\right]\mathcal{N}_+\left(1 - \frac{\mathcal{N}_+}{N}\right)\\ & + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)(b_p b_{-p} + \text{h.c.}) + \sqrt{N}\sum_{\substack{p,q\in\Lambda_+^*:\\ p+q\neq 0}} \widehat{V}(p/e^N)\left[b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\right]\\ & + \mathcal{H}_N \,.\end{aligned} \tag{2.91}$$

Then there exists a constant $C > 0$ such that $\mathcal{E}_\mathcal{G} = \mathcal{G}_{N,\alpha} - \mathcal{G}^{\text{eff}}_{N,\alpha}$ is bounded by

$$|\langle \xi, \mathcal{E}_\mathcal{G}\, \xi\rangle| \leq C\big(N^{1/2-\alpha} + N^{-1}(\log N)^{1/2}\big)\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\|\xi\|^2 \tag{2.92}$$

for all $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$ and $N \in \mathbb{N}$ large enough.

The reader could notice that in Eq. (2.91) the original potential $\widehat{V}(p/e^N)$ is replaced in the constant and in the off-diagonal quadratic terms, namely $\frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{\omega}_N(p)(b_p b_{-p} + \text{h.c.})$, by the faster decaying potential $\widehat{\omega}_N(p)$. Whereas, in Prop. 2.4, Eq. (2.41), the original potential $\widehat{V}(p/N^\beta)$ was replaced in the same terms by $\widehat{V}(0)$.

The proof of Prop. 2.11 is very similar to the proof of [8, Prop. 4.2], as well as the one in Chapter 3. For completeness, we write it Chapter 4, Section 4.1.

2.3.2 Cubic Renormalization

Conjugation with the generalized Bogoliubov transformation (2.26) renormalizes constant and off-diagonal quadratic terms on the r.h.s. of (2.91). In order to estimate the number of excitations $\mathcal{N}_+$ through the energy and show Bose-Einstein condensation, we still need to renormalize the diagonal quadratic term (the part proportional to $N\widehat{V}(0)\mathcal{N}_+$, on the first line of (2.91)) and the cubic term on the second line of (2.91). To this end, we conjugate $\mathcal{G}^{\text{eff}}_{N,\alpha}$ with an additional unitary operator, given by the exponential of the anti-symmetric operator

$$A := \frac{1}{\sqrt{N}} \sum_{r,v\in\Lambda_+^*} \eta_r \big[b^*_{r+v} a^*_{-r} a_v - \text{h.c.}\big] \tag{2.93}$$

with η_p defined in (2.80). Notice that, differently from (2.54), here it is not necessary to introduce a cut-off, even on low momenta. Again the smallness, that allows us to control error terms, is gained by choosing $\ell = N^{-\alpha}$, see Eq. (2.79).

As for Prop.2.7, also in this setting we observe that while conjugation with e^A allows to renormalize the large terms in $\mathcal{G}_{N,\alpha}$, it does not substantially change the number of excitations. Similarly to Prop. 2.6 one can show that the growth of the number of particles operator under the action of e^A is almost invariant.

Proposition 2.12. *Suppose that A is defined as in (2.93). Then, for any $k \in \mathbb{N}$ there exists a constant $C > 0$ such that the operator inequality*

$$e^{-A}(\mathcal{N}_+ + 1)^k e^A \leq C(\mathcal{N}_+ + 1)^k$$

holds true on $\mathcal{F}_+^{\leq N}$, for any $\alpha > 0$ (recall the choice $\ell = N^{-\alpha}$ in the definition (2.80) of the coefficients η_r), and N large enough.

We will also need to control the growth of the expectation of the energy $\mathcal{H}_N$ with respect to the cubic conjugation. This is the content of the following proposition, which is proved in Subsection 4.2.1.

Proposition 2.13. *Let A be defined as in (2.93). Then there exists a constant $C > 0$ such that*

$$e^{-sA}\mathcal{H}_N e^{sA} \leq C\mathcal{H}_N + CN(\mathcal{N}_+ + 1) \tag{2.94}$$

for all $\alpha \geq 1$, $s \in [0;1]$ and $N \in \mathbb{N}$ large enough.

Remark. It is interesting to compare Prop. 2.13 with Prop. 2.7. While in 2.7, under the action of the cubic renormalization we loose a factor $|\log \ell|$, of order $\mathcal{O}(1)$, in terms of the number of particles, in the GP regime the action of e^A on $\mathcal{H}_N$ leads to the appearance of large terms $N\mathcal{N}_+$, as in Eq.(2.94). This makes the analysis of the cubic operator substantially more difficult.

We use now the cubic phase e^A to introduce a new excitation Hamiltonian, obtained by conjugating the main part $\mathcal{G}^{\text{eff}}_{N,\alpha}$ of $\mathcal{G}_{N,\alpha}$. We define

$$\mathcal{R}_{N,\alpha} := e^{-A}\,\mathcal{G}^{\text{eff}}_{N,\alpha}\,e^{A} \tag{2.95}$$

on a dense subset of $\mathcal{F}_+^{\leq N}$. Conjugation with e^A renormalizes both the contribution proportional to $\mathcal{N}_+$ (in the first line on the r.h.s. of (2.91)) and the cubic term on the r.h.s. of (2.91), effectively replacing the singular potential $\widehat{V}(p/e^N)$ by the renormalized potential $\widehat{\omega}_N(p)$ defined in (2.88). This follows from the following proposition.

Proposition 2.14. *Let $V \in L^3(\mathbb{R}^2)$ be compactly supported, pointwise non-negative and spherically symmetric. Let $\mathcal{R}_{N,\alpha}$ be defined in* (2.95) *and define*

$$\begin{aligned}\mathcal{R}^{\textit{eff}}_{N,\alpha} = {} & \frac{1}{2}(N-1)\,\widehat{\omega}_N(0)(1-\mathcal{N}_+/N) + \frac{1}{2}\widehat{\omega}_N(0)\,\mathcal{N}_+\,(1-\mathcal{N}_+/N) \\ & + \widehat{\omega}_N(0)\sum_{p\in\Lambda^*_+} a^*_p a_p\Big(1-\frac{\mathcal{N}_+}{N}\Big) + \frac{1}{2}\sum_{p\in\Lambda^*_+}\widehat{\omega}_N(p)\big[b^*_p b^*_{-p} + b_p b_{-p}\big] \\ & + \frac{1}{\sqrt{N}}\sum_{\substack{r,v\in\Lambda^*_+:\\ r\neq -v}}\widehat{\omega}_N(r)\big[b^*_{r+v}a^*_{-r}a_v + h.c.\big] + \mathcal{H}_N\,.\end{aligned} \tag{2.96}$$

Then for $\ell = N^{-\alpha}$ and $\alpha > 2$ there exists a constant $C > 0$ such that $\mathcal{E}_{\mathcal{R}} = \mathcal{R}_{N,\alpha} - \mathcal{R}^{\textit{eff}}_{N,\alpha}$ is bounded by

$$\pm\,\mathcal{E}_{\mathcal{R}} \leq C[N^{2-\alpha} + N^{-1/2}(\log N)^{1/2}](\mathcal{H}_N + 1)\,, \tag{2.97}$$

for $N \in \mathbb{N}$ sufficiently large.

The proof of Proposition 2.14 will be given in Chapter 4, Section 4.2. We will also need more detailed information on $\mathcal{R}^{\text{eff}}_{N,\alpha}$, as contained in the following proposition.

Proposition 2.15. *Let $\mathcal{R}^{\textit{eff}}_{N,\alpha}$ be defined in* (2.96)*. Then, for every $c > 0$ there is a constant $C > 0$ (large enough) such that*

$$\mathcal{R}^{\textit{eff}}_{N,\alpha} \geq 2\pi N + \frac{\widehat{\omega}_N(0)}{2}\,\mathcal{N}_+ + \frac{c}{\log N}\,\mathcal{H}_N - C(\log N)^2\,\frac{\mathcal{N}_+^2}{N} - C \tag{2.98}$$

for all $\alpha > 2$ and $N \in \mathbb{N}$ large enough.

Moreover, let $f, g : \mathbb{R} \to [0;1]$ be smooth, with $f^2(x) + g^2(x) = 1$ for all $x \in \mathbb{R}$. For $M \in \mathbb{N}$, let $f_M := f(\mathcal{N}_+/M)$ and $g_M := g(\mathcal{N}_+/M)$. Then there exists $C > 0$ such that

$$\mathcal{R}^{eff}_{N,\alpha} = f_M \, \mathcal{R}^{eff}_{N,\alpha} \, f_M + g_M \, \mathcal{R}^{eff}_{N,\alpha} \, g_M + \Theta_M \tag{2.99}$$

with

$$\pm \Theta_M \leq \frac{C \log N}{M^2} \big(\|f'\|^2_\infty + \|g'\|^2_\infty \big) \big(\mathcal{H}_N + 1 \big)$$

for all $\alpha > 2$, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ large enough.

Proof. From (2.96), using that $|\widehat{\omega}_N(0)| \leq C$ we have

$$\begin{aligned} \mathcal{R}^{\text{eff}}_{N,\alpha} \geq \; & \frac{N}{2} \, \widehat{\omega}_N(0) + \widehat{\omega}_N(0) \, \mathcal{N}_+ + \frac{1}{2} \sum_{p \in \Lambda^*_+} \widehat{\omega}_N(p) \big[b^*_p b^*_{-p} + b_p b_{-p} \big] \\ & + \frac{1}{\sqrt{N}} \sum_{\substack{r,v \in \Lambda^*_+ : \\ r \neq -v}} \widehat{\omega}_N(r) \big[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \big] + \mathcal{H}_N - C \, \frac{\mathcal{N}^2_+}{N} - C \, . \end{aligned} \tag{2.100}$$

For the cubic term on the r.h.s. of (2.100), with

$$\sum_{p \in \Lambda^*_+} \frac{|\widehat{\omega}_N(p)|^2}{p^2} \leq C \log N \tag{2.101}$$

we can bound

$$\begin{aligned} & \Big| \frac{1}{\sqrt{N}} \sum_{\substack{r,v \in \Lambda^*_+ \\ r \neq -v}} \widehat{\omega}_N(r) \langle \xi, b^*_{r+v} a^*_{-r} a_v \xi \rangle \Big| \\ & \leq \frac{1}{\sqrt{N}} \sum_{\substack{r,v \in \Lambda^*_+ \\ r \neq -v}} |\widehat{\omega}_N(r)| \| (\mathcal{N}_+ + 1)^{-1/2} b_{r+v} a_{-r} \xi \| \| (\mathcal{N}_+ + 1)^{1/2} a_v \xi \| \\ & \leq \frac{1}{\sqrt{N}} \Big[\sum_{\substack{r,v \in \Lambda^*_+ \\ r \neq -v}} |r|^2 \| (\mathcal{N}_+ + 1)^{-1/2} b_{r+v} a_{-r} \xi \|^2 \Big]^{1/2} \\ & \qquad \times \Big[\sum_{\substack{r,v \in \Lambda^*_+ \\ r \neq -v}} \frac{|\widehat{\omega}_N(r)|^2}{|r|^2} \| (\mathcal{N}_+ + 1)^{1/2} a_v \xi \|^2 \Big]^{1/2} \\ & \leq \frac{C (\log N)^{1/2}}{\sqrt{N}} \, \| \mathcal{K}^{1/2} \xi \| \| (\mathcal{N}_+ + 1) \xi \| \, . \end{aligned} \tag{2.102}$$

As for the off-diagonal quadratic term on the r.h.s of (2.100), we combine it with part of the kinetic energy to estimate. For any $0 < \mu < 1$, we have

$$\begin{aligned}\frac{1}{2}\sum_{p\in\Lambda_+^*}&\widehat{\omega}_N(p)\left[b_p^*b_{-p}^*+b_{-p}b_p\right]+(1-\mu)\sum_{p\in\Lambda_+^*}p^2a_p^*a_p\\ &=(1-\mu)\sum_{p\in\Lambda_+^*}p^2\left[b_p^*+\frac{\widehat{\omega}_N(p)}{2(1-\mu)p^2}b_{-p}\right]\left[b_p+\frac{\widehat{\omega}_N(p)}{2(1-\mu)p^2}b_{-p}^*\right]\\ &\quad-\frac{1}{4(1-\mu)}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2}b_pb_p^*+(1-\mu)\sum_{p\in\Lambda_+^*}p^2a_p^*\frac{\mathcal{N}_+}{N}a_p\end{aligned}$$

since $a_p^*a_p - b_p^*b_p = a_p^*(\mathcal{N}_+/N)a_p$. With (2.5), we conclude that

$$\begin{aligned}\frac{1}{2}\sum_{p\in\Lambda_+^*}&\widehat{\omega}_N(p)\left[b_p^*b_{-p}^*+b_{-p}b_p\right]+(1-\mu)\sum_{p\in\Lambda_+^*}p^2a_p^*a_p\\ &\geq-\frac{1}{4(1-\mu)}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2}a_p^*a_p-\frac{1}{4(1-\mu)}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2}.\end{aligned}$$

With the choice $\mu = C/\log N$ and with (2.101), we obtain

$$\begin{aligned}\frac{1}{2}\sum_{p\in\Lambda_+^*}&\widehat{\omega}_N(p)\left[b_p^*b_{-p}^*+b_{-p}b_p\right]+(1-\mu)\sum_{p\in\Lambda_+^*}p^2a_p^*a_p\\ &\geq-\frac{1}{4(1-\mu)}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2}a_p^*a_p-\frac{1}{4}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2}-C.\end{aligned}\tag{2.103}$$

To bound the first contribution on the r.h.s. of the last equation, we use the term $\widehat{\omega}_N(0)\mathcal{N}_+$, in (2.100). To this end, we observe that, with (2.90),

$$\frac{|\widehat{\omega}_N(p)|^2}{4(1-\mu)p^2}\leq\frac{|\widehat{\omega}_N(0)|^2}{4(1-\mu)p^2}\leq\frac{\widehat{\omega}_N(0)}{4(1-\mu)\pi}\left(1+C\frac{\log N}{N}\right)\leq\frac{\widehat{\omega}_N(0)}{2}$$

for every $p \in \Lambda_+^*$ (notice that $|p| \geq 2\pi$, for every $p \in \Lambda_+^*$) and for N large enough (recall the choice $\mu = C/\log N$). Inserting (2.102) and (2.103) in (2.100) and using the kinetic energy $\mu\mathcal{K} = C(\log N)^{-1}\mathcal{K}$ (remaining after subtracting the term $(1-\mu)\mathcal{K}$ needed on the l.h.s. of (2.103)) to bound the r.h.s. of (2.102), we find

$$\begin{aligned}\mathcal{R}_{N,\alpha}^{\text{eff}}\geq\frac{N}{2}\widehat{\omega}_N(0)&-\frac{1}{4}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2}+\frac{\widehat{\omega}_N(0)}{2}\mathcal{N}_+\\ &+\frac{c}{\log N}\mathcal{H}_N-C\frac{(\log N)^2}{N}\mathcal{N}_+^2-C.\end{aligned}\tag{2.104}$$

Let us now consider the second term on the r.h.s more carefully. Using that, from (2.88), $\widehat{\omega}_N(p) = g_N\widehat{\chi}(p/N^\alpha)$, we can bound, for any fixed $K > 0$,

$$\frac{1}{4}\sum_{p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2}\leq C+\frac{1}{4}\sum_{\substack{p\in\Lambda_+^*:\\ K<|p|\leq N^\alpha}}\frac{|\widehat{\omega}_N(p)|^2}{p^2}.$$

With $|\widehat{\omega}_N(p) - \widehat{\omega}_N(0)| \leq C|p|/N^\alpha$, we obtain

$$\frac{1}{4}\sum_{p\in\Lambda_+^*} \frac{|\widehat{\omega}_N(p)|^2}{p^2} \leq C + \frac{|\widehat{\omega}_N(0)|^2}{4} \sum_{\substack{p\in\Lambda_+^*:\\ K<|p|\leq N^\alpha}} \frac{1}{p^2} \leq C + 4\pi^2 \sum_{\substack{p\in\Lambda_+^*:\\ K<|p|\leq N^\alpha}} \frac{1}{p^2}\,. \tag{2.105}$$

For $q \in \mathbb{R}^2$, let us define $h(q) = 1/p^2$, if q is contained in the square of side length 2π centered at $p \in \Lambda_+^*$ (with an arbitrary choice on the boundary of the squares). We can then estimate, for K large enough,

$$4\pi^2 \sum_{\substack{p\in\Lambda_+^*:\\ K<|p|\leq N^\alpha}} \frac{1}{p^2} \leq \int_{K/2<|q|\leq N^\alpha+K} h(q)dq\,.$$

For q in the square centered at $p \in \Lambda_+^*$, we bound

$$\left| h(q) - \frac{1}{q^2}\right| = \frac{|p^2-q^2|}{p^2\, q^2} \leq \frac{C}{|q|^3}\,.$$

Hence

$$4\pi^2 \sum_{\substack{p\in\Lambda_+^*:\\ K<|p|\leq N^\alpha}} \frac{1}{p^2} \leq \int_{K/2<|q|<N^\alpha+K} \frac{1}{q^2} dq + C \leq 2\pi\alpha \log N + C\,.$$

Inserting in (2.105), we conclude that

$$\frac{1}{4}\sum_{p\in\Lambda_+^*} \frac{|\widehat{\omega}_N(p)|^2}{p^2} \leq 2\pi\alpha\log N + C\,.$$

Combining the last bound with (2.90) (and noticing that the contribution proportional to $\log N$ cancels exactly), from (2.104) we obtain

$$\mathcal{R}^{\text{eff}}_{N,\alpha} \geq\; 2\pi N + \frac{\widehat{\omega}_N(0)}{2}\mathcal{N}_+ + \frac{c}{\log N}\mathcal{H}_N - C\frac{(\log N)^2}{N}\mathcal{N}_+^2 - C$$

which proves (2.98).

Next we prove (2.99). From (2.96), with $|\widehat{\omega}_N(0)| \leq C$, the bound (2.102) and since, by (2.101),

$$\begin{aligned}\left|\sum_{p\in\Lambda_+^*} \widehat{\omega}_N(p)\langle \xi, b_p^* b_{-p}^*\xi\rangle\right| &\leq \sum_{p\in\Lambda_+^*} |\widehat{\omega}_N(p)| \|b_p\xi\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\leq \left[\sum_{p\in\Lambda_+^*} \frac{|\widehat{\omega}_N(p)|^2}{p^2}\right]^{1/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\| \\ &\leq C(\log N)^{1/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\|\end{aligned}$$

it follows that

$$\mathcal{R}^{\text{eff}}_{N,\alpha} = 2\pi N + \mathcal{H}_N + \theta_{N,\alpha} \tag{2.106}$$

where for arbitrary $\delta > 0$, there exists a constant $C > 0$ such that

$$\pm\, \theta_{N,\alpha} \leq \delta \mathcal{H}_N + C(\log N)\,(\mathcal{N}_+ + 1)\,. \tag{2.107}$$

We now note that for $f : \mathbb{R} \to \mathbb{R}$ smooth and bounded and $\theta_{N,\alpha}$ defined above, there exists a constant $C > 0$ such that

$$\pm\, [f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), \theta_{N,\alpha}]] \leq C\,\frac{\log N}{M^2}\|f'\|_\infty^2(\mathcal{H}_N + 1) \tag{2.108}$$

for all $\alpha > 2$ and $N \in \mathbb{N}$ large enough. The proof of (2.108) follows analogously to the one for (2.107), since the bounds leading to (2.107) remain true if we replace the operators $b_p^\#$, $\# = \{\cdot, *\}$, and $a_p^* a_q$ with $[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), b_p^\#]]$ or $[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), a_p^* a_q]]$ respectively, provided we multiply the r.h.s. by an additional factor $M^{-2}\|f'\|_\infty^2$, since, for example

$$[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), b_p]] = \big(f(\mathcal{N}_+/M) - f((\mathcal{N}_+ + 1)/M)\big)^2 b_p$$

and $\|f(\mathcal{N}_+/M) - f((\mathcal{N}_+ + 1)/M)\| \leq CM^{-1}\|f'\|_\infty$. With an explicit computation we obtain

$$\mathcal{R}^{\text{eff}}_{N,\alpha} = f_M \mathcal{R}^{\text{eff}}_{N,\alpha} f_M + g_M \mathcal{R}^{\text{eff}}_{N,\alpha} g_M + \frac{1}{2}\Big([f_M, [f_M, \mathcal{R}^{\text{eff}}_{N,\alpha}]] + [g_M, [g_M, \mathcal{R}^{\text{eff}}_{N,\alpha}]]\Big)\,.$$

Writing $\mathcal{R}^{\text{eff}}_{N,\alpha}$ as in (2.106) and using (2.108) we get

$$\pm\Big([f_M, [f_M, \mathcal{R}^{\text{eff}}_{N,\alpha}]] + [g_M, [g_M, \mathcal{R}^{\text{eff}}_{N,\alpha}]]\Big) \leq \frac{C\log N}{M^2}\big(\|f'\|_\infty^2 + \|g'\|_\infty^2\big)\big(\mathcal{H}_N + 1\big)\,.$$

□

2.3.3 Proof of Theorem 1.3

The next proposition combines the results of Prop. 2.11, Prop. 2.14 and Prop. 2.15. Its proof makes use of localization in the number of particle and is an adaptation of the proof of [10, Proposition 6.1]. The main difference w.r.t. [10] is that here we need to localize on sectors of $\mathcal{F}^{\leq N}$ where the number of particles is $o(N)$, in the limit $N \to \infty$.

Proposition 2.16. *Let $V \in L^3(\mathbb{R}^2)$ be compactly supported, pointwise non-negative and spherically symmetric. Let $\mathcal{G}_{N,\alpha}$ be the renormalized excitation Hamiltonian defined as in (2.86). Then, for every $\alpha \geq 5/2$, there exist constants $C, c > 0$ such that*

$$\mathcal{G}_{N,\alpha} - 2\pi N \geq c\mathcal{N}_+ - C \tag{2.109}$$

for all $N \in \mathbb{N}$ sufficiently large.

Remark. Eq. (2.109) actually holds also for $\alpha > 2$. We pick $\alpha \geq 5/2$ in order to have a uniform bound in Eq. (2.116).

Proof. Let $f, g : \mathbb{R} \to [0;1]$ be smooth, with $f^2(x) + g^2(x) = 1$ for all $x \in \mathbb{R}$. Moreover, assume that $f(x) = 0$ for $x > 1$ and $f(x) = 1$ for $x < 1/2$. For a small $\varepsilon > 0$, we fix $M = N^{1-\varepsilon}$ and we set $f_M = f(\mathcal{N}_+/M), g_M = g(\mathcal{N}_+/M)$. It follows from Prop. 2.15 that

$$\begin{aligned}\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \geq f_M\big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\big)f_M + g_M\big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\big)g_M \\ - CN^{2\varepsilon-2}(\log N)(\mathcal{H}_N + 1)\end{aligned} \tag{2.110}$$

Let us consider the first term on the r.h.s. of (2.110). From Prop. 2.15, for all $\alpha > 2$ there exist $c, C > 0$ such that

$$\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \geq c\mathcal{N}_+ - \frac{C}{N}(\log N)^2\mathcal{N}_+^2 - C\,. \tag{2.111}$$

On the other hand, with (2.106) and (2.107) we also find

$$\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \geq c\mathcal{H}_N - C(\log N)\,(\mathcal{N}_+ + 1) \tag{2.112}$$

for all $\alpha > 2$ and N large enough. Moreover, due to the choice $M = N^{1-\varepsilon}$, we have

$$\frac{(\log N)^2}{N} f_M\mathcal{N}_+^2 f_M \leq \frac{(\log N)^2}{N^\varepsilon} f_M^2\mathcal{N}_+\,.$$

With the last bound, Eq. (2.111) implies that

$$f_M\Big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\Big)f_M \geq cf_M^2\mathcal{N}_+ - C \tag{2.113}$$

for N large enough.

Let us next consider the second term on the r.h.s. of (2.110). We claim that there exists a constant $c > 0$ such that

$$g_M\Big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\Big)g_M \geq cNg_M^2 \tag{2.114}$$

for all N sufficiently large. To prove (2.114) we observe that, since $g(x) = 0$ for all $x \leq 1/2$,

$$g_M\Big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N\Big)g_M \geq \left[\inf_{\xi\in\mathcal{F}^{\leq N}_{\geq M/2}:\|\xi\|=1} \frac{1}{N}\langle\xi, \mathcal{R}^{\text{eff}}_{N,\alpha}\xi\rangle - 2\pi\right] Ng_M^2$$

where $\mathcal{F}^{\leq N}_{\geq M/2} = \{\xi \in \mathcal{F}^{\leq N}_+ : \xi = \chi(\mathcal{N}_+ \geq M/2)\xi\}$ is the subspace of $\mathcal{F}^{\leq N}_+$ where states with at least $M/2$ excitations are described (recall that $M = N^{1-\varepsilon}$). To prove (2.114) it is enough to show that there exists $C > 0$ with

$$\inf_{\xi\in\mathcal{F}^{\leq N}_{\geq M/2}:\|\xi\|=1} \frac{1}{N}\langle\xi, \mathcal{R}^{\text{eff}}_{N,\alpha}\xi\rangle - 2\pi \geq C \tag{2.115}$$

for all N large enough. On the other hand, using the definitions of $\mathcal{G}_{N,\alpha}$ in (2.91), $\mathcal{R}_{N,\alpha}$ and $\mathcal{R}^{\text{eff}}_{N,\alpha}$ in (2.96), we obtain that the ground state energy E_N of the system is given by

$$E_N = \inf_{\xi \in \mathcal{F}_+^{\leq N} : \|\xi\|=1} \big\langle \xi, e^{-A} \mathcal{G}_{N,\alpha} e^{A} \xi \big\rangle = \inf_{\xi \in \mathcal{F}_+^{\leq N} : \|\xi\|=1} \big\langle \xi, \big(\mathcal{R}^{\text{eff}}_{N,\alpha} + \mathcal{E}_L\big)\xi \big\rangle$$

with $\mathcal{E}_L = \mathcal{E}_{\mathcal{R}} + e^{-A}\mathcal{E}_{\mathcal{G}} e^{A}$. The bounds (2.92) and (2.97), together with Prop. 2.12 and Prop. 2.13, imply that for any $\alpha \geq 5/2$ there exists $C > 0$ such that

$$\begin{aligned} \pm \mathcal{E}_L &\leq C N^{-1/2} (\log N)^{1/2} \big[(\mathcal{H}_N + 1) + e^{-A} \big(N^{-1}(\mathcal{H}_N + 1) + (\mathcal{N}_+ + 1) \big) e^{A} \big] + C \\ &\leq C N^{-1/2} (\log N)^{1/2} (\mathcal{H}_N + 1) + C \, . \end{aligned} \tag{2.116}$$

With (2.112) we obtain

$$\pm \, \mathcal{E}_L \leq C N^{-1/2} (\log N)^{1/2} \big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \big) + C N^{-1/2} (\log N)^{3/2} \mathcal{N}_+ + C \, , \tag{2.117}$$

and therefore, with $\mathcal{N}_+ \leq N$

$$E_N - 2\pi N \leq C \inf_{\xi \in \mathcal{F}_+^{\leq N} : \|\xi\|=1} \big\langle \xi, \big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \big) \xi \big\rangle + C N^{1/2} (\log N)^{3/2} + C \, .$$

From the result (1.12) of [51, 46, 47]

$$\begin{aligned} \inf_{\xi \in \mathcal{F}^{\leq N}_{\geq M/2} : \|\xi\|=1} \frac{1}{N} \langle \xi, \mathcal{R}^{\text{eff}}_{N,\alpha} \xi \rangle - 2\pi &\geq \inf_{\xi \in \mathcal{F}_+^{\leq N} : \|\xi\|=1} \frac{1}{N} \langle \xi, \big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N \big) \xi \rangle \\ &\geq c \left(\frac{E_N}{N} - 2\pi \right) - \frac{C}{\sqrt{N}} (\log N)^{3/2} - C N^{-1} \to 0 \end{aligned}$$

as $N \to \infty$. If we assume for contradiction that (2.115) does not hold true, then we can find a subsequence $N_j \to \infty$ with

$$\inf_{\xi \in \mathcal{F}^{\leq N_j}_{\geq M_j/2} : \|\xi\|=1} \frac{1}{N_j} \langle \xi, \mathcal{R}^{\text{eff}}_{N_j,\alpha} \xi \rangle - 2\pi \to 0$$

as $j \to \infty$ (here we used the notation $M_j = N_j^{1-\varepsilon}$). This implies that there exists a sequence $\tilde{\xi}_{N_j} \in \mathcal{F}^{\leq N_j}_{\geq M_j/2}$ with $\|\tilde{\xi}_{N_j}\| = 1$ for all $j \in \mathbb{N}$ such that

$$\lim_{j \to \infty} \frac{1}{N_j} \langle \tilde{\xi}_{N_j}, \mathcal{R}^{\text{eff}}_{N_j,\alpha} \tilde{\xi}_{N_j} \rangle = 2\pi \, .$$

On the other hand, using the relation $\mathcal{R}^{\text{eff}}_{N_j,\alpha} = e^{-A} \mathcal{G}_{N_j,\alpha} e^{A} - \mathcal{E}_{L,j}$ with $\mathcal{E}_{L,j}$ satisfying the bound (2.117) (with $\mathcal{N}_+ \leq N_j$), we obtain that there exist constants $c_1, c_2, C > 0$ such that

$$\begin{aligned} c_1 \langle \tilde{\xi}_{N_j}, & \big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N_j \big) \tilde{\xi}_{N_j} \rangle - C N_j^{1/2} (\log N_j)^{3/2} \\ &\leq \langle e^{A} \tilde{\xi}_{N_j}, \big(\mathcal{G}_{N_j,\alpha} - \, 2\pi N_j \big) e^{A} \tilde{\xi}_{N_j} \rangle \\ &\qquad \leq c_2 \langle \tilde{\xi}_{N_j}, \big(\mathcal{R}^{\text{eff}}_{N,\alpha} - 2\pi N_j \big) \tilde{\xi}_{N_j} \rangle + C N_j^{1/2} (\log N_j)^{3/2} \end{aligned}$$

Hence for $\xi_{N_j} = e^A \tilde{\xi}_{N_j}$ we have

$$\lim_{N_j \to \infty} \frac{1}{N_j} \langle \xi_{N_j}, \mathcal{G}_{N_j,\alpha} \xi_{N_j} \rangle = 2\pi \,.$$

Let now $S := \{N_j : j \in \mathbb{N}\} \subset \mathbb{N}$ and denote by ξ_N a normalized minimizer of $\mathcal{G}_{N,\alpha}$ for all $N \in \mathbb{N} \setminus S$. Setting $\psi_N = U_N^* e^B \xi_N$, for all $N \in \mathbb{N}$, we obtain that $\|\psi_N\| = 1$ and that

$$\lim_{N\to\infty} \frac{1}{N} \langle \psi_N, H_N^{\mathrm{GP}} \psi_N \rangle = \lim_{N\to\infty} \frac{1}{N} \langle \xi_N, \mathcal{G}_{N,\alpha} \xi_N \rangle = 2\pi \tag{2.118}$$

Eq. (2.118) shows that the sequence ψ_N is an approximate ground state of H_N^{GP}. From (1.14), we conclude that ψ_N exhibits complete Bose-Einstein condensation in the zero-momentum mode φ_0, and in particular that there exists $\bar{\delta} > 0$ such that

$$|1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle| \leq C N^{-\bar{\delta}} \,.$$

Using Lemma 2.2, Prop. 2.12 and the rules (2.4), we observe that

$$\begin{aligned} \frac{1}{N} \langle \xi_N, \mathcal{N}_+ \xi_N \rangle &= \frac{1}{N} \langle e^{-B} U_N \psi_N, \mathcal{N}_+ e^{-B} U_N \psi_N \rangle \\ &\leq \frac{C}{N} \langle \psi_N, U_N^* (\mathcal{N}_+ + 1) U_N \psi_N \rangle \\ &= \frac{C}{N} + C \left[1 - \frac{1}{N} \langle \psi_N, a^*(\varphi_0) a(\varphi_0) \psi_N \rangle \right] \\ &= \frac{C}{N} + C \left[1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \right] \leq C N^{-\bar{\delta}} \end{aligned} \tag{2.119}$$

as $N \to \infty$.

On the other hand, for $N \in S = \{N_j : j \in \mathbb{N}\}$, we have $\xi_N = \chi(\mathcal{N}_+ \geq M/2) \xi_N$ and therefore

$$\frac{1}{N} \langle \xi_N, \mathcal{N}_+ \xi_N \rangle \geq \frac{M}{2N} = \frac{N^{-\varepsilon}}{2} \,.$$

Choosing $\varepsilon < \bar{\delta}$ and N large enough we get a contradiction with (2.119). This proves (2.115), (2.114) and therefore also

$$g_M \Big(\mathcal{R}_{N,\alpha}^{\mathrm{eff}} - 2\pi N \Big) g_M \geq c \mathcal{N}_+ g_M^2 \,. \tag{2.120}$$

Inserting (2.113) and (2.120) on the r.h.s. of (2.110), we obtain that

$$\mathcal{R}_{N,\alpha}^{\mathrm{eff}} - 2\pi N \geq c \mathcal{N}_+ - C (\log N) N^{2\varepsilon - 2} (\mathcal{H}_N + 1) - C \tag{2.121}$$

for N large enough. With (2.112), (2.121) implies

$$\mathcal{R}_{N,\alpha}^{\mathrm{eff}} - 2\pi N \geq c \mathcal{N}_+ - C.$$

To conclude, we use the relation $e^{-A} \mathcal{G}_{N,\alpha} e^A = \mathcal{R}_{N,\alpha}^{\mathrm{eff}} + \mathcal{E}_L$ and the bound (2.117). We have that for $\alpha \geq 5/2$ there exist $c, C > 0$ such that

$$\begin{aligned} \mathcal{G}_{N,\alpha} - 2\pi N &\geq c e^A \big(\mathcal{R}_{N,\alpha}^{\mathrm{eff}} - 2\pi N \big) e^{-A} - C N^{-1/2} (\log N)^{3/2} e^A \mathcal{N}_+ e^A - C \\ &\geq c\, e^A \mathcal{N}_+ e^{-A} - C \geq c \mathcal{N}_+ - C \end{aligned}$$

where we used (2.121) and Prop. 2.12. □

We are now ready to show our main theorem.

Proof of Theorem 1.3. Let E_N be the ground state energy of H_N^{GP}. Evaluating (2.91) and (2.92) on the vacuum $\Omega \in \mathcal{F}_+^{\leq N}$ and using (2.89), we obtain the upper bound

$$E_N \leq 2\pi N + C \log N\,.$$

With Eq. (2.109) we also find the lower bound $E_N \geq 2\pi N - C$. This proves (1.15).

Let now $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and

$$\langle \psi_N, H_N^{\text{GP}} \psi_N \rangle \leq 2\pi N + K\,. \tag{2.122}$$

We define the excitation vector $\xi_N = e^{-B} U_N \psi_N$. Then $\|\xi_N\| = 1$ and, recalling that $\mathcal{G}_{N,\alpha} = e^{-B} U_N H_N^{\text{GP}} U_N^* e^B$ we have, with (2.109),

$$\big\langle \psi_N, (H_N^{\text{GP}} - 2\pi N)\psi_N \big\rangle = \big\langle \xi_N, (\mathcal{G}_{N,\alpha} - 2\pi N)\xi_N \big\rangle \geq c\big\langle \xi_N, \mathcal{N}_+ \xi_N \big\rangle - C\,. \tag{2.123}$$

From Eqs. (2.122) and (2.123) we conclude that

$$\langle \xi_N, \mathcal{N}_+ \xi_N \rangle \leq C(1+K)\,. \tag{2.124}$$

If γ_N denotes the one-particle reduced density matrix associated with ψ_N, using Lemma 2.2 we obtain

$$\begin{aligned} 1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle &= 1 - \frac{1}{N} \langle \psi_N, a^*(\varphi_0) a(\varphi_0) \psi_N \rangle \\ &= 1 - \frac{1}{N} \langle U_N^* e^B \xi_N, a^*(\varphi_0) a(\varphi_0) U_N^* e^B \xi_N \rangle \\ &= \frac{1}{N} \langle e^B \xi_N, \mathcal{N}_+ e^B \xi_N \rangle \leq \frac{C}{N} \langle \xi_N, \mathcal{N}_+ \xi_N \rangle \leq \frac{C(1+K)}{N} \end{aligned}$$

which concludes the proof of (1.17). □

Chapter 3

Analysis of the Renormalized Hamiltonian for bosons interacting through singular potentials

In this chapter we show Prop. 2.4 and Prop. 2.8 establishing properties of the renormalized excitation Hamiltonians $\mathcal{G}^{\beta}_{N,\ell}$ and $\mathcal{R}^{\beta}_{N,\ell}$ defined in Eq. (2.39) and Eq. (2.56)

While this analysis follows closely the one in [10, Section 7, 8] appropriate adjustments (due to the different scaling and dimension) are needed. We write all the details for the reader convenience.

3.1 Analysis of the quadratically renormalized excitation Hamiltonian $\mathcal{G}^{\beta}_{N,\ell}$

From (2.8) and (2.86), we can decompose

$$\mathcal{G}^{\beta}_{N,\ell} = e^{-B_H}\mathcal{L}^{\beta}_{N}e^{B_H} = \mathcal{G}^{\beta,(0)}_{N,\ell} + \mathcal{G}^{\beta,(2)}_{N,\ell} + \mathcal{G}^{\beta,(3)}_{N,\ell} + \mathcal{G}^{\beta,(4)}_{N,\ell}$$

with

$$\mathcal{G}^{\beta,(j)}_{N,\ell} = e^{-B_H}\mathcal{L}^{\beta,(j)}_{N}e^{B_H}$$

In the next subsection, we prove separate bounds for the operators $\mathcal{G}^{\beta,(j)}_{N,\ell}$, $j = 0, 2, 3, 4$. As stated in Chapter 2, we will assume the potential $V \in L^2(\mathbb{R}^2)$ to be compactly supported, pointwise non-negative and spherically symmetric.

As already said in Section 2.2, Lemma 2.3 will be crucial throughout our analysis. A first simple application of it is the following bound on the growth of the expectation of $\mathcal{N}_+$.

Lemma 3.1. *Assume B is defined as in* (2.27)*, with $\eta \in \ell^2(\Lambda^*)$ and $\eta_p = \eta_{-p}$ for all $p \in \Lambda^*_+$. Then, there exists a constant $C > 0$ such that*

$$\left|\langle \xi, \left[e^{-B}\mathcal{N}_+ e^{B} - \mathcal{N}_+\right]\xi\rangle\right| \leq \|\eta\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2$$

for all $\xi \in \mathcal{F}^{\leq N}_+$.

Proof. With (2.31) we write

$$\begin{aligned}
&e^{-B}\mathcal{N}_+e^{B}-\mathcal{N}_+\\
&=\int_0^1 e^{-sB}[\mathcal{N}_+,B]e^{sB}ds\\
&=\int_0^1\sum_{p\in\Lambda_+^*}\eta_p\,e^{-sB}(b_pb_{-p}+b_p^*b_{-p}^*)e^{sB}\,ds\\
&=\int_0^1\sum_{p\in\Lambda_+^*}\eta_p\,\Big[(\gamma_p^{(s)}b_p+\sigma_p^{(s)}b_{-p}^*+d_p^{(s)})(\gamma_p^{(s)}b_{-p}+\sigma_p^{(s)}b_{-p}^*+d_{-p}^{(s)})+\text{h.c.}\Big]\,ds
\end{aligned}$$

with $\gamma_p^{(s)}=\cosh(s\eta_p)$, $\sigma_p^{(s)}=\sinh(s\eta_p)$. Using $|\gamma_p^{(s)}|\le C$ and $|\sigma_p^{(s)}|\le C|\eta_p|$, (2.34) in Lemma 2.3 we arrive at

$$\begin{aligned}
&\Big|\langle\xi,\big[e^{-B}\mathcal{N}_+e^{B}-\mathcal{N}_+\big]\xi\rangle\Big|\\
&\qquad\le C\|(\mathcal{N}_++1)^{1/2}\xi\|\sum_{p\in\Lambda_+^*}|\eta_p|\,\big[|\eta_p|\|(\mathcal{N}_++1)^{1/2}\xi\|+\|b_p\xi\|\big]\\
&\qquad\le C\|\eta\|\|(\mathcal{N}_++1)^{1/2}\xi\|^2.
\end{aligned}$$

□

3.1.1 Analysis of $\mathcal{G}_{N,\ell}^{\beta,(0)}=e^{-B_H}\mathcal{L}_N^{\beta,(0)}e^{B_H}$

From (2.9), recall that

$$\mathcal{L}_N^{\beta,(0)}=\frac{\widehat{V}(0)}{2N}(N-1)(N-\mathcal{N}_+)+\frac{\widehat{V}(0)}{2N}\mathcal{N}_+(N-\mathcal{N}_+).\tag{3.1}$$

Hence, we define the error operator $\mathcal{E}_N^{(0)}$ through the identity

$$\mathcal{G}_{N,\ell}^{\beta,(0)}=e^{-B_H}\mathcal{L}_N^{\beta,(0)}e^{B_H}=\frac{\widehat{V}(0)}{2N}(N-1)(N-\mathcal{N}_+)+\frac{\widehat{V}(0)}{2N}\mathcal{N}_+(N-\mathcal{N}_+)+\mathcal{E}_{N,\ell}^{\beta,(0)}\tag{3.2}$$

In the next proposition, we estimate $\mathcal{E}_{N,\ell}^{\beta,(0)}$ with a smooth and bounded function of $\mathcal{N}_+$.

Proposition 3.2. *There exists a constant $C>0$ such that*

$$\pm\,\mathcal{E}_{N,\ell}^{\beta,(0)}\le C\ell^{\alpha}(\mathcal{N}_++1)\tag{3.3}$$

and

$$\pm\,[f(\mathcal{N}_+/M),[f(\mathcal{N}_+/M),\mathcal{E}_{N,\ell}^{\beta,(0)}]]\le C\ell^{\alpha}M^{-2}\|f'\|_\infty^2(\mathcal{N}_++1)\tag{3.4}$$

for all $\alpha>0$, $\ell\in(0;1/2)$, f smooth and bounded, $M\in\mathbb{N}$ and $N\in\mathbb{N}$ large enough.

Proof. From (3.1) we have

$$\mathcal{L}_N^{\beta,(0)} = \frac{(N-1)}{2}\widehat{V}(0) + \frac{\widehat{V}(0)}{2N}\mathcal{N}_+ - \frac{\widehat{V}(0)}{2N}\mathcal{N}_+^2. \tag{3.5}$$

We use the following identity to rewrite the last term on the right hand side

$$-\frac{\mathcal{N}_+^2}{N} = \mathcal{N}_+\frac{N-\mathcal{N}_+}{N} - \mathcal{N}_+ = \sum_{q\in\Lambda_+^*} b_q^* b_q - \frac{\mathcal{N}_+}{N} - \mathcal{N}_+,$$

we insert it in (3.5), and we obtain

$$\mathcal{L}_N^{\beta,(0)} = \frac{(N-1)}{2}\widehat{V}(0) + \frac{1}{2}\widehat{V}(0)\left[\sum_{q\in\Lambda_+^*} b_q^* b_q - \mathcal{N}_+\right].$$

From (3.2), it follows that

$$\mathcal{E}_{N,\ell}^{\beta,(0)} = \frac{1}{2}\widehat{V}(0)\sum_{q\in\Lambda_+^*}\left[e^{-B_H}b_q^*b_q e^{B_H} - b_q^*b_q\right] - \frac{1}{2}\widehat{V}(0)\left[e^{-B_H}\mathcal{N}_+e^{B_H} - \mathcal{N}_+\right]. \tag{3.6}$$

With (2.31), we can express

$$\sum_{q\in\Lambda_+^*} e^{-B_H}b_q^*b_q e^{B_H} = \sum_{q\in\Lambda_+^*}\left[\gamma_q b_q^* + \sigma_q b_{-q} + d_q^*\right]\left[\gamma_q b_q + \sigma_q b_{-q}^* + d_q\right]$$

where we set $\gamma_q = \cosh\eta_H(q)$, $\sigma_q = \sinh\eta_H(q)$ and where d_q, d_q^* are defined as in (2.31), with η replaced by $\eta_H(q) = \eta_q\chi(q\in P_H)$. From (2.33) we have $|\gamma_q^2-1| = |\gamma_q-1||\gamma_q+1| \le C\eta_H(q)^2$, $|\sigma_q| \le C|\eta_H(q)|$, the first bound in (2.34), Cauchy-Schwarz and the estimate $\|\eta_H\| \le C\ell^\alpha$ from (2.23), we conclude that first term on the r.h.s. of (3.6) can be bounded by

$$\Big|\sum_{q\in\Lambda_+^*}\langle\xi, \left[e^{-B_H}b_q^*b_q e^{B_H} - b_q^*b_q\right]\xi\rangle\Big| \le C\ell^\alpha\|(\mathcal{N}_++1)^{1/2}\xi\|^2.$$

As for the second term on the r.h.s. of (3.6), we use Lemma 3.1, with η replaced by η_H. This concludes the proof of (3.3).

Now consider the bound (3.4). It follows similarly, because, as observed in Corollary 2.1, the estimates (2.34) in Lemma 2.3 remain true if we replace d_p and d_p^* by $[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), d_p]]$ and, respectively, $[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), d_p^*]]$, provided we multiply the r.h.s. by an additional factor $M^{-2}\|f'\|_\infty^2$. The same observation holds true for bounds involving the operators b_p, b_p^*, since, for example,

$$[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), b_p]] = (f(\mathcal{N}_+/M) - f((\mathcal{N}_++1)/M))^2 b_p \tag{3.7}$$

and $\|f(\mathcal{N}_+/M) - f((\mathcal{N}_++1)/M)\| \le CM^{-1}\|f'\|_\infty$. □

3.1.2 Analysis of $\mathcal{G}_{N,\ell}^{\beta,(2)} = e^{-B_H}\mathcal{L}_N^{\beta,(2)}e^{B_H}$

From (2.9), we can decompose in two parts $\mathcal{L}_N^{\beta,(2)} = \mathcal{K} + \mathcal{L}_N^{\beta,(2,V)}$, where $\mathcal{K} = \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p$ is the kinetic energy operator and

$$\mathcal{L}_N^{\beta,(2,V)} = \sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)\left[b_p^*b_p - \frac{1}{N}a_p^*a_p\right] + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)\left[b_p^*b_{-p}^* + b_pb_{-p}\right]. \tag{3.8}$$

Hence, we have

$$\mathcal{G}_{N,\ell}^{\beta,(2)} = e^{-B_H}\mathcal{K}e^{B_H} + e^{-B_H}\mathcal{L}_N^{\beta,(2,V)}e^{B_H}. \tag{3.9}$$

In the next two propositions, we analyze the two terms on the r.h.s. of the last equation. We start with the analysis of the action of e^{B_H} on the kinetic energy operator.

Proposition 3.3. *There exists $C > 0$ such that*

$$\begin{aligned} e^{-B_H}\mathcal{K}e^{B_H} = \mathcal{K} &+ \sum_{p\in P_H} p^2\eta_p(b_pb_{-p} + b_p^*b_{-p}^*) \\ &+ \sum_{p\in P_H} p^2\eta_p^2\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big) + \mathcal{E}_{N,\ell}^{(K)} \end{aligned} \tag{3.10}$$

where

$$\pm\mathcal{E}_{N,\ell}^{(K)} \le C\ell^{\alpha-1}(\mathcal{H}_N^\beta + 1) \tag{3.11}$$

and

$$\pm\Big[f(\mathcal{N}_+/M),\Big[f(\mathcal{N}_+/M),\mathcal{E}_{N,\ell}^{(K)}\Big]\Big] \le CM^{-2}\|f'\|_\infty^2\,\ell^{\alpha-1}\big(\mathcal{H}_N^\beta + 1\big) \tag{3.12}$$

for all $\alpha > 1$, $\ell \in (0;1/2)$ small enough, f smooth and bounded, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ large enough.

Proof. To show (3.11), we write

$$\begin{aligned} e^{-B_H}\mathcal{K}e^{B_H} - \mathcal{K} &= \int_0^1 e^{-sB_H}[\mathcal{K}, B_H]e^{sB_H}ds \\ &= \int_0^1 \sum_{p\in P_H} p^2\eta_p\left[e^{-sB_H}b_pb_{-p}e^{sB_H} + e^{-sB_H}b_p^*b_{-p}^*e^{sB_H}\right]ds. \end{aligned}$$

Using the relations (2.31), we can write

$$\begin{aligned}
&e^{-B_H}\mathcal{K}e^{B_H}-\mathcal{K}\\
&=\int_0^1 ds\sum_{p\in P_H}p^2\eta_p\Big[\big(\gamma_p^{(s)}b_p+\sigma_p^{(s)}b^*_{-p}\big)\big(\gamma_p^{(s)}b_{-p}+\sigma_p^{(s)}b^*_p\big)\;+\text{h.c.}\Big]\\
&\quad+\int_0^1 ds\sum_{p\in P_H}p^2\eta_p\big[\big(\gamma_p^{(s)}b_p+\sigma_p^{(s)}b^*_{-p}\big)d^{(s)}_{-p}+d^{(s)}_p\big(\gamma_p^{(s)}b_{-p}+\sigma_p^{(s)}b^*_p\big)+\text{h.c.}\big]\\
&\quad+\int_0^1 ds\sum_{p\in P_H}p^2\eta_p\big[d^{(s)}_pd^{(s)}_{-p}+\text{h.c.}\big]\\
&=:\mathrm{G}_1+\mathrm{G}_2+\mathrm{G}_3
\end{aligned}\tag{3.13}$$

with the notation $\gamma_p^{(s)}=\cosh(s\eta_H(p))$, $\sigma_p^{(s)}=\sinh(s\eta_H(p))$ and where $d_p^{(s)}$ is defined as in (2.31), with η_p replaced by $s\eta_H(p)$ (recall that $\eta_H(p)=\eta_p\chi(p\in P_H)$). We start by analysing G_1, and we obtain

$$\begin{aligned}
\mathrm{G}_1=&\int_0^1 ds\sum_{p\in P_H}p^2\eta_p\Big[\big((\gamma_p^{(s)})^2+(\sigma_p^{(s)})^2\big)\big(b_pb_{-p}+b^*_{-p}b^*_p\big)\\
&\qquad\qquad+\gamma_p^{(s)}\sigma_p^{(s)}(4b^*_pb_p-2N^{-1}a^*_pa_p)\big)\Big]\\
&+2\int_0^1 ds\sum_{p\in P_H}p^2\eta_p\gamma_p^{(s)}\sigma_p^{(s)}\left(1-\frac{\mathcal{N}_+}{N}\right)\\
=&\sum_{p\in P_H}p^2\eta_p\big(b_pb_{-p}+b^*_{-p}b^*_p\big)+\sum_{p\in P_H}p^2\eta_p^2\left(1-\frac{\mathcal{N}_+}{N}\right)+\mathcal{E}_1^K
\end{aligned}\tag{3.14}$$

with

$$\begin{aligned}
\mathcal{E}_1^K=&\int_0^1 ds\sum_{p\in P_H}p^2\eta_p\big[\big((\gamma_p^{(s)})^2-1\big)+(\sigma_p^{(s)})^2\big]\big(b_pb_{-p}+b^*_{-p}b^*_p\big)\\
&+\int_0^1 ds\sum_{p\in P_H}p^2\eta_p\gamma_p^{(s)}\sigma_p^{(s)}(4b^*_pb_p-2N^{-1}a^*_pa_p))\\
&+2\int_0^1 ds\sum_{p\in P_H}p^2\eta_p\left[(\gamma_p^{(s)}-1)\sigma_p^{(s)}+(\sigma_p^{(s)}-s\eta_p)\right]\left(1-\frac{\mathcal{N}_+}{N}\right).
\end{aligned}$$

For an arbitrary $\xi\in\mathcal{F}_+^{\leq N}$, we bound

$$\begin{aligned}
|\langle\xi,\,\mathcal{E}_1^K\xi\rangle|\leq\;& C\sum_{p\in P_H}p^2|\eta_p|^3\|b_p\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|+C(1+N^{-1})\sum_{p\in P_H}p^2\eta_p^2\|a_p\xi\|^2\\
&+C\sum_{p\in P_H}p^2\eta_p^4\\
\leq\;& C\ell^\alpha\|(\mathcal{N}_++1)^{1/2}\xi\|^2,
\end{aligned}\tag{3.15}$$

since $|\big((\gamma_p^{(s)})^2-1\big)| \leq C\eta_p^2$, $(\sigma_p^{(s)})^2 \leq C\eta_p^2$ and $p^2\eta_p \leq C$, for all $p \in P_H$.

We consider now G_2 in (3.13). We split it as $\mathrm{G}_2 = \mathrm{G}_{21}+\mathrm{G}_{22}+\mathrm{G}_{23}+\mathrm{G}_{24}$, with

$$\begin{aligned}
\mathrm{G}_{21} &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p \left(\gamma_p^{(s)} b_p d_{-p}^{(s)} + \text{h.c.}\right), \\
\mathrm{G}_{22} &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p \left(\sigma_p^{(s)} b_{-p}^* d_{-p}^{(s)} + \text{h.c.}\right), \\
\mathrm{G}_{23} &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p \left(\gamma_p^{(s)} d_p^{(s)} b_{-p} + \text{h.c.}\right), \\
\mathrm{G}_{24} &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p \left(\sigma_p^{(s)} d_p^{(s)} b_p^* + \text{h.c.}\right).
\end{aligned} \tag{3.16}$$

We start from G_{21}. We write

$$\begin{aligned}
\mathrm{G}_{21} = & \int_0^1 ds \sum_{p\in P_H} p^2\eta_p(\gamma_p^{(s)}-1) b_p d_{-p}^{(s)} + \int_0^1 ds \sum_{p\in \Lambda_+^*} p^2\eta_p b_p d_{-p}^{(s)} \\
& - \int_0^1 ds \sum_{p\in P_H^c} p^2\eta_p b_p \left[d_{-p}^{(s)} + \frac{1}{N}\sum_{q\in P_H} s\eta_q b_q^* a_{-q}^* a_{-p}\right] \\
& + \int_0^1 ds\, \frac{s}{N} \sum_{p\in P_H^c, q\in P_H} p^2\eta_p\eta_q b_p b_q^* a_{-q}^* a_{-p} + \text{h.c.}
\end{aligned}$$

where with P_H^c we mean $P_H^c = \Lambda_+^* \setminus P_H$. Using (2.5) we rewrite

$$b_p \mathcal{N}_+ b_p^* = (\mathcal{N}_+ +1) b_p b_p^* = (\mathcal{N}_+ +1)(1-\mathcal{N}_+/N) + (\mathcal{N}_+ +1)(b_p^* b_p - N^{-1} a_p^* a_p), \tag{3.17}$$

to manipulate a bit the second term and we arrive at

$$\mathrm{G}_{21} = -\sum_{p\in P_H} p^2\eta_p \frac{\mathcal{N}_+ +1}{N}\frac{N-\mathcal{N}_+}{N} + \left[\mathcal{E}_2^K + \text{h.c.}\right] \tag{3.18}$$

where $\mathcal{E}_2^K = \sum_{j=1}^5 \mathcal{E}_{2j}^K$, with

$$\begin{aligned}
\mathcal{E}_{21}^K &= \frac{1}{2N}\sum_{p\in P_H} p^2\eta_p^2(\mathcal{N}_+ +1)\big(b_p^* b_p - \frac{1}{N} a_p^* a_p\big), \quad \mathcal{E}_{24}^K = -\int_0^1 ds \sum_{p\in P_H^c} p^2\eta_p b_p \bar{\bar{d}}_{-p}^{(s)} \\
\mathcal{E}_{22}^K &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p(\gamma_p^{(s)}-1) b_p d_{-p}^{(s)}, \quad \mathcal{E}_{25}^K = \frac{1}{2N}\sum_{p\in P_H^c, q\in P_H} p^2\eta_p\eta_q b_p b_q^* a_{-q}^* a_{-p} \\
\mathcal{E}_{23}^K &= \int_0^1 ds \sum_{p\in \Lambda_+^*} p^2\eta_p b_p \bar{d}_{-p}^{(s)},
\end{aligned} \tag{3.19}$$

where

$$\bar{d}^{(s)}_{-p} = d^{(s)}_{-p} + s\eta_H(p)\frac{\mathcal{N}_+}{N}b^*_p, \qquad \text{and} \qquad \bar{\bar{d}}^{(s)}_{-p} = d^{(s)}_{-p} + \frac{1}{N}\sum_{q\in P_H} s\eta_q b^*_q a^*_{-q} a_{-p}. \tag{3.20}$$

Let us consider the first term in (3.19), this can be easily bound

$$|\langle \xi, \mathcal{E}^K_{21}\xi\rangle| \le C\sum_{p\in P_H} p^2\eta_p^2\|a_p\xi\|^2 \le C\ell^\alpha\|\mathcal{N}_+^{1/2}\xi\|^2 \tag{3.21}$$

and, using $|\gamma_p^{(s)} - 1| \le C\eta_p^2$ and (2.34) in Lemma 2.3,

$$\begin{aligned}
|\langle \xi, \mathcal{E}^K_{22}\xi\rangle| &\le \sum_{p\in P_H} p^2|\eta_p|^3\|\mathcal{N}_+^{1/2}\xi\|\|d^{(s)}_{-p}\xi\| \\
&\le \sum_{p\in P_H} p^2|\eta_p|^3\|\mathcal{N}_+^{1/2}\xi\|\left[|\eta_p|\|\mathcal{N}_+^{1/2}\xi\| + \|\eta_H\|\|a_p\xi\|\right] \\
&\le C(\ell^{2\alpha} + \ell^{3\alpha})\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.
\end{aligned} \tag{3.22}$$

With (2.35) in Lemma 2.3, we can also estimate

$$\begin{aligned}
|\langle \xi, \mathcal{E}^K_{24}\xi\rangle| &\le \int_0^1 ds \sum_{p\in P_H^c} p^2|\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\bar{\bar{d}}^{(s)}_{-p}\xi\| \\
&\le C\|\eta_H\|^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \sum_{p\in P_H^c}\|a_p\xi\| \\
&\le C\|\eta_H\|^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\Big[\sum_{|p|\in P_H^c}\|a_p\|^2\Big]^{1/2}\Big[\sum_{|p|\le\ell^{-\alpha}} 1\Big]^{1/2} \\
&\le C\ell^\alpha\|(\mathcal{N}_+ + 1)^{1/2}\xi\|.
\end{aligned} \tag{3.23}$$

To bound the last term in (3.19), we commute b_p to the right (note that $p \ne q$). We find

$$\begin{aligned}
|\langle \xi, \mathcal{E}^K_{25}\xi\rangle| \le CN^{-1} &\sum_{p\in P_H^c, q\in P_H} p^2|\eta_p||\eta_q|\|a_q a_{-q}\xi\|\|a_p a_{-p}\xi\| \\
&\le C\sum_{p\in P_H^c, q\in P_H} p^2|\eta_p||\eta_q|\|a_q\xi\|\|a_p\xi\| \\
&\le C\Big[\sum_{p\in P_H^c, q\in P_H} p^2\eta_q^2\|a_p\xi\|^2\Big]^{1/2}\Big[\sum_{p\in P_H^c, q\in P_H} 1\cdot|q|^2\eta_q^2\|a_q\xi\|^2\Big]^{1/2}\sup_{q\in P_H}\frac{1}{|q|} \\
&\le C\|\eta_H\|\ell^\alpha\ell^{-\alpha}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\
&\le C\ell^\alpha\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|.
\end{aligned} \tag{3.24}$$

To control the third term in (3.19), we first use the scattering equation (2.19) to write

$$\mathcal{E}_{23}^K = \int_0^1 ds \frac{1}{2} \sum_{p\in\Lambda_+^*} \Big(\widehat{V}(./N^\beta) * \widehat{f}_{N,\ell}\Big)(p) b_p \bar{d}_{-p}^{(s)} + \int_0^1 ds\, N^{1+2\beta} \lambda_\ell \sum_{p\in\Lambda_+^*} \Big(\widehat{\chi}_\ell * \widehat{f}_{N,\ell}\Big)(p) b_p \bar{d}_{-p}^{(s)}.$$

Switching to position space, we obtain

$$\begin{aligned}\mathcal{E}_{23}^K = &-\int_0^1 ds \int_{\Lambda^2} dxdy N^{2\beta} V(N^\beta(x-y)) f_{N,\ell}(x-y) \check{b}_x \check{\bar{d}}_y^{(s)} \\ &+ \int_0^1 ds N^{1+2\beta} \lambda_\ell \int_{\Lambda^2} dxdy \chi_\ell(x-y) f_{N,\ell}(x-y) \check{b}_x \check{\bar{d}}_y^{(s)}.\end{aligned}$$

With Lemma 2.1, we find

$$|\langle \xi, \mathcal{E}_{23}^K \xi\rangle| \leq C \int_0^1 ds \int_{\Lambda^2} dxdy \left[N^{2\beta} V(N^\beta(x-y)) + \ell^{-2}\chi_\ell(x-y)\right] \times \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|(\mathcal{N}_+ +1)^{-1/2} \check{a}_x \check{\bar{d}}_y^{(s)} \xi\|.$$

Hence, with Eq. (2.37) in Lemma 2.3,

$$\begin{aligned}|\langle \xi, \mathcal{E}_{23}^K \xi\rangle| &\leq CN^{-1}\|\eta_H\| \int_0^1 ds \int_{\Lambda^2} dxdy \left[N^{2\beta} V(N^\beta(x-y)) + \ell^{-2}\chi_\ell(x-y)\right] \\ &\quad \times \|(\mathcal{N}_+ +1)^{1/2}\xi\| \Big[\log N \|(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\eta_H\| \|\check{a}_x(\mathcal{N}_+ +1)\xi\| \\ &\quad + \|\check{a}_y \xi\| + \|\check{a}_x \check{a}_y (\mathcal{N}_+ 1)^{1/2}\xi\|\Big] \\ &\leq C\ell^{\alpha-1} \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + C\ell^\alpha \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|(\mathcal{V}_N^\beta)^{1/2}\,\xi\|,\end{aligned}$$

for N large enough. Combining the last bound with (3.21), (3.22), (3.23), (3.24), we conclude that

$$\pm \left[\mathcal{E}_2^K + \text{h.c.}\right] \leq C\ell^{\alpha-1}(\mathcal{H}_N^\beta + 1). \tag{3.25}$$

Next, we consider the term G_{22} in (3.16). With (2.34) in Lemma 2.3, we find

$$\begin{aligned}|\langle \xi, \mathrm{G}_{22}\xi\rangle| &\leq C \sum_{p\in P_H} p^2 \eta_p^2 \|b_{-p}\xi\| \|d_{-p}\xi\| \\ &\leq C \sum_{p\in P_H} p^2 \eta_p^2 \|b_{-p}\xi\| \left[|\eta_p| \|(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\eta_H\| \|b_p\xi\|\right] \\ &\leq C\ell^{2\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2.\end{aligned} \tag{3.26}$$

As for the term G_{23}, defined in (3.16), we split it as $\mathrm{G}_{23} = \sum_{j=1}^4 \mathcal{E}_{3j}^K + \text{h.c.}$, with

$$\mathcal{E}_{31}^K = \int_0^1 ds \sum_{p\in P_H} p^2 \eta_p \big(\gamma_p^{(s)} - 1\big) d_p^{(s)} b_{-p}\,, \qquad \mathcal{E}_{32}^K = \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2 \eta_p d_p^{(s)} b_{-p}$$

$$\mathcal{E}_{33}^K = \frac{1}{2N} \sum_{p\in P_H^c, q\in P_H} p^2 \eta_p \eta_q b_q^* a_{-q}^* a_p b_{-p}\,, \qquad \mathcal{E}_{34}^K = -\int_0^1 ds \sum_{p\in P_H^c} p^2 \eta_p \bar{\bar{d}}_p^{(s)} b_{-p}$$

with the notation for $\bar{\bar{d}}_p^{(s)}$ introduced in (3.20). With (2.34) in Lemma 2.3, we find

$$\begin{aligned}|\langle \xi, \mathcal{E}^K_{31}\xi\rangle| &\leq C\int_0^1 \sum_{p\in P_H} p^2|\eta_p|^3 \|(\mathcal{N}_+ +1)^{1/2}\xi\|\|d_p^{(s)} b_{-p}\xi\| \\ &\leq C\|\eta_H\|\|(\mathcal{N}_+ +1)^{1/2}\xi\| \sum_{p\in P_H} p^2|\eta_p|^3\|b_p\xi\| \leq C\ell^{3\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2\end{aligned}$$

and also, proceeding as in (3.23),

$$\begin{aligned}|\langle \xi, \mathcal{E}^K_{34}\xi\rangle| &\leq C\int_0^1 ds \sum_{p\in P_H^c} p^2|\eta_p| \|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{N}_+ +1)^{-1/2}\bar{\bar{d}}_p^{(s)} b_{-p}\xi\| \\ &\leq C\|\eta_H\|^2\|(\mathcal{N}_+ +1)^{1/2}\xi\| \sum_{p\in P_H^c} p^2|\eta_p|\|b_{-p}\xi\| \\ &\leq C\ell^{\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2.\end{aligned} \tag{3.27}$$

The term $\mathcal{E}^K_{33}$ coincides with the contribution $\mathcal{E}^K_{25}$ in (3.19); from (3.24) we obtain $\pm\mathcal{E}^K_{33} \leq C\ell^{2\alpha}(\mathcal{H}_N^\beta+1)$. As for $\mathcal{E}^K_{32}$, we use (2.19) and we switch to position space. Proceeding as we did above to control the term $\mathcal{E}^K_{23}$, we arrive at

$$\begin{aligned}|\langle \xi, \mathcal{E}^K_{32}\xi\rangle| \leq C\int_0^1 ds \int_{\Lambda^2} dxdy \, &\big[N^{2\beta}V(N^\beta(x-y)) + \ell^{-2}\chi_\ell(x-y)\big] \\ &\times \|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{N}_+ +1)^{-1/2}\check{d}_x^{(s)}\check{a}_y\xi\|\end{aligned}$$

With (2.36) in Lemma 2.3, we find

$$\begin{aligned}|\langle \xi, \mathcal{E}^K_{32}\xi\rangle| \leq C\|\eta_H\| &\int_{\Lambda^2} dxdy \, \big[N^{2\beta}V(N^\beta(x-y)) + \ell^{-2}\chi_\ell(x-y)\big] \\ &\times \|\xi\| \big[\|\check{a}_y(\mathcal{N}_+ +1)\xi\| + N^{-1/2}\|\check{a}_x\check{a}_y\xi\|\big] \\ \leq C\ell^{\alpha-1}&\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + C\ell^{\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{V}_N^\beta)^{1/2}\xi\|.\end{aligned}$$

Combining the last bounds, we conclude that

$$\pm\mathrm{G}_{23} \leq C\ell^{\alpha-1}(\mathcal{H}_N^\beta+1). \tag{3.28}$$

To estimate the term G_{24} in (3.16), we use (2.34) in Lemma 2.3; with (2.2), we find

$$
\begin{aligned}
&|\langle \xi, \mathrm{G}_{24}\xi\rangle| \\
&\quad \leq C\int_0^1 ds \sum_{p\in P_H} p^2\eta_p^2 \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|(\mathcal{N}_+ +1)^{-1/2} d_p^{(s)} b_p^*\xi\| \\
&\quad \leq C\|(\mathcal{N}_+ +1)^{1/2}\xi\| \sum_{p\in P_H} p^2\eta_p^2 \left[|\eta_p| \|(\mathcal{N}_+ +1)^{3/2}\xi\| \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. + N^{-1}\|\eta_H\| \|b_p b_p^* (\mathcal{N}_+ +1)^{1/2}\xi\| \right] \\
&\quad \leq C\sum_{p\in P_H} |\eta_p|^2 \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 \\
&\quad\quad + CN^{-1}\|\eta_H\| \sum_{p\in P_H} |\eta_p| \left[\|b_p^* b_p (\mathcal{N}_+ +1)^{1/2}\xi\| + \|(\mathcal{N}_+ +1)^{1/2}\xi\| \right] \|(\mathcal{N}_+ +1)^{1/2}\xi\| \\
&\quad \leq C(\ell^{2\alpha} + \ell^{\alpha}) \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2.
\end{aligned}
$$

Together with (3.18), (3.25), (3.26), (3.28), this implies that

$$
\mathrm{G}_2 = -\sum_{p\in P_H} p^2 \eta_p \frac{\mathcal{N}_+ + 1}{N} \frac{N - \mathcal{N}_+}{N} + \mathcal{E}_4^K
$$

where

$$
\pm \mathcal{E}_4^K \leq C\ell^{\alpha-1}(\mathcal{H}_N^\beta + 1). \tag{3.29}
$$

We still have to analyze G_3, defined in (3.13). We split it as

$$
\mathrm{G}_3 = \mathcal{E}_{51}^K + \mathcal{E}_{52}^K + \mathrm{h.c.}
$$

with

$$
\mathcal{E}_{51}^K = \int_0^1 ds \sum_{p\in \Lambda_+^*} p^2 \eta_p d_p^{(s)} d_{-p}^{(s)}, \qquad \mathcal{E}_{52}^K = -\int_0^1 ds \sum_{p\in P_H^c} p^2 \eta_p d_p^{(s)} d_{-p}^{(s)}.
$$

With (2.34) in Lemma 2.3 (using $\eta_H(p) = 0$ for $p \in P_H^c$) and proceeding as in (3.27), we obtain

$$
\begin{aligned}
|\langle \xi, \mathcal{E}_{52}^K \xi\rangle| &\leq C\|\eta_H\| \sum_{p\in P_H^c} p^2 |\eta_p| \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|d_{-p}\xi\| \\
&\leq C\|\eta_H\|^2 \|(\mathcal{N}_+ +1)^{1/2}\xi\| \sum_{p\in P_H^c} p^2|\eta_p| \|b_{-p}\xi\| \leq C\ell^{\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2.
\end{aligned}
$$

To estimate $\mathcal{E}_{51}^K$, we use (2.19) and we switch to position space. Again, we proceed as in the analysis of the terms $\mathcal{E}_{23}^K$ and $\mathcal{E}_{32}^K$ above, we obtain

$$
\begin{aligned}
|\langle \xi, \mathcal{E}_{51}^K \xi\rangle| \leq C\|(\mathcal{N}_+ +1)^{1/2}\xi\| \int_0^1 ds \int_{\Lambda^2} dxdy \left[N^{2\beta} V(N^\beta(x-y)) + \ell^{-2}\chi_\ell(x-y) \right] \\
\times \|(\mathcal{N}_+ +1)^{-1/2} \check{d}_x^{(s)} \check{d}_y^{(s)} \xi\|.
\end{aligned}
$$

With (2.38) in Lemma 2.3, we arrive at

$$\begin{aligned}
&|\langle \xi, \mathcal{E}^K_{51}\xi\rangle| \\
&\leq C\|(\mathcal{N}_+ +1)^{1/2}\xi\| \int_0^1 ds \int_{\Lambda^2} dxdy\, \big[N^{2\beta}V(N^\beta(x-y)) + \ell^{-2}\chi_\ell(x-y)\big] \\
&\quad \times \Big[(\|\eta_H\|^2 + \|\eta_H\| N^{-1}\log N)\|(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\eta_H\|^2\|\check{a}_x\xi\| \\
&\qquad\qquad + \|\eta_H\|^2\|\check{a}_y\xi\| + \|\eta_H\|^2 N^{-1/2}\|\check{a}_x\check{a}_y\xi\|\Big] \\
&\leq C\ell^{2\alpha-1}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + C\ell^{2\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{V}_N^\beta)^{1/2}\xi\|.
\end{aligned}$$

Hence, $\pm \mathrm{G}_3 \leq C\ell^{2\alpha-1}(\mathcal{H}_N^\beta + 1)$. With (3.14), (3.15), (3.29), we obtain (3.10) and (3.11), as desired.

As explained in Corollary 2.1, the bounds in Lemma 2.3 continue to hold, with an additional factor $M^{-2}\|f'\|^2_\infty$ on the r.h.s., if we replace the operators d_p, d^*_p, $\bar{\bar{d}}_p$, $\check{a}_y\check{\bar{d}}_x$, $\check{d}_x\check{d}_y$ by their double commutators with $f(\mathcal{N}_+/M)$. From (3.7) we conclude that also bounds involving b_p and b^*_p or, analogously $\check{b}_x$ and $\check{b}^*_x$ remain true if we replace them by their double commutator with $f(\mathcal{N}_+/M)$. As a consequence, (3.12) follows through the same arguments that led us to (3.11). □

In the next proposition, we pass to the study of the second term on the r.h.s. of (3.9).

Proposition 3.4. *There is a constant $C > 0$ such that*

$$\begin{aligned}
&e^{-B_H}\mathcal{L}_N^{\beta,(2,V)}e^{B_H} \\
&= \sum_{p\in P_H} \widehat{V}(p/N^\beta)\eta_p\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big) \\
&\quad + \sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)a^*_pa_p\left(1-\frac{\mathcal{N}_+}{N}\right) + \frac{1}{2}\sum_{p\in\Lambda^*_+}\widehat{V}(p/N^\beta)\big(b_pb_{-p} + b^*_{-p}b^*_p\big) + \mathcal{E}^{\beta,(V)}_{N,\ell}
\end{aligned} \tag{3.30}$$

where

$$\pm\mathcal{E}^{\beta,(V)}_{N,\ell} \leq C\,\ell^\alpha(\mathcal{H}^\beta_N + 1) \tag{3.31}$$

and

$$\pm\Big[f(\mathcal{N}_+/M), \Big[f(\mathcal{N}_+/M), \mathcal{E}^{\beta,(V)}_{N,\ell}\Big]\Big] \leq C\ell^\alpha M^{-2}\|f'\|^2_\infty\big(\mathcal{H}^\beta_N+1\big) \tag{3.32}$$

for all $\alpha > 0$, $\ell \in (0;1/2)$ small enough, f smooth and bounded, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ large enough.

Proof. To show (3.31), we start from (3.8) and we decompose

$$\begin{aligned} e^{-B_H}\mathcal{L}^{(2,V)}_{N,\beta}e^{B_H} = &\sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)e^{-B_H}b_p^*b_pe^{B_H} \\ &-\frac{1}{N}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)e^{B_H}a_p^*a_pe^{-B_H} \\ &+\frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)e^{-B_H}\big[b_pb_{-p}+b_p^*b_{-p}^*\big]e^{B_H} \\ =:&\ \mathrm{F}_1+\mathrm{F}_2+\mathrm{F}_3. \end{aligned} \tag{3.33}$$

With equations (2.31), we split F_1 as

$$\begin{aligned} \mathrm{F}_1 = &\sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)\big[\gamma_pb_p^*+\sigma_pb_{-p}\big]\big[\gamma_pb_p+\sigma_pb_{-p}^*\big] \\ &+\sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)\big[(\gamma_pb_p^*+\sigma_pb_{-p})d_p+d_p^*(\gamma_pb_p+\sigma_pb_{-p}^*)+d_p^*d_p\big] \\ =:&\ \mathrm{F}_{11}+\mathrm{F}_{12} \end{aligned}$$

with the notation $\gamma_p = \cosh\eta_H(p)$, $\sigma_p = \sinh\eta_H(p)$ and the operators d_p, as defined in (2.31), with η replaced by η_H. We decompose

$$\mathrm{F}_{11} = \sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)b_p^*b_p+\mathcal{E}_1^V$$

with

$$\begin{aligned} \mathcal{E}_{11}^V = \sum_{p\in P_H}\widehat{V}(p/N^\beta)\Big[&(\gamma_p^2-1)b_p^*b_p+\gamma_p\sigma_p(b_{-p}b_p+b_p^*b_{-p}^*) \\ &+\sigma_p^2(b_p^*b_p-N^{-1}a_p^*a_p)+\sigma_p^2\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big] \end{aligned}$$

where we used $\gamma_p = 1$ and $\sigma_p = 0$ for $p\in P_H^c$ to restrict the second sum. With $|\gamma_p^2-1|\le C\eta_p^2$, $|\sigma_p|\le C|\eta_p|$ for all $p\in P_H$ and since $\|\eta_H\|\le\ell^\alpha$, we find

$$\pm\mathcal{E}_{11}^V\le C\ell^\alpha(\mathcal{N}_++1)$$

if N is large enough. With Lemma 2.3 (with η replaced by η_H), we can also bound $\pm\mathrm{F}_{12}\le C\ell^\alpha(\mathcal{N}_++1)$. We conclude that

$$\mathrm{F}_1 = \sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)b_p^*b_p+\mathcal{E}_1^V \tag{3.34}$$

with $\pm\mathcal{E}_1^V\le C\ell^\alpha(\mathcal{N}_++1)$.

Let us now focus on the second contribution on the r.h.s. of (3.33). We have $-\mathrm{F}_2\ge 0$, and, by Lemma 2.2,

$$\begin{aligned} -\mathrm{F}_2 &= \frac{1}{N}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)e^{-B_H}a_p^*a_pe^{B_H} \\ &\le\frac{\|\widehat{V}\|_\infty}{N}e^{-B_H}\mathcal{N}_+e^{B_H}\le\ \frac{C}{N}\mathcal{N}_+\le\ell^\alpha(\mathcal{N}_++1) \end{aligned} \tag{3.35}$$

if $N \in \mathbb{N}$ is large enough. In fact, the smallness in terms of N guarantees the bound $-F_2 \leq C\ell^\alpha(\mathcal{N}_+ + 1)$.

Finally, we turn our attention to the last term on the r.h.s. of (3.33). With (2.31), we decompose F_3 as

$$\begin{aligned}
\mathrm{F}_3 &= \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)\left[\gamma_p b_p + \sigma_p b_{-p}^*\right]\left[\gamma_p b_{-p} + \sigma_p b_p^*\right] + \text{h.c.} \\
&\quad + \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)\ \left[\left(\gamma_p b_p + \sigma_p b_{-p}^*\right) d_{-p} + d_p\left(\gamma_p b_{-p} + \sigma_p b_p^*\right)\right] + \text{h.c.} \\
&\quad + \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta) d_p d_{-p} + \text{h.c.} \\
&=: \mathrm{F}_{31} + \mathrm{F}_{32} + \mathrm{F}_{33} + \text{h.c.}
\end{aligned} \tag{3.36}$$

We decompose in turn the first term as

$$\mathrm{F}_{31} = \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)\big(b_p b_{-p} + b_{-p}^* b_p^*\big) + \sum_{p\in P_H} \widehat{V}(p/N^\beta)\eta_p \frac{N-\mathcal{N}_+}{N} + \mathcal{E}_3^V \tag{3.37}$$

with (recall that $\gamma_p = 1$ and $\sigma_p = 0$ for $p \in P_H^c$)

$$\begin{aligned}
\mathcal{E}_3^V = \sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)\Bigg[&\frac{1}{2}(\gamma_p^2 - 1 + \sigma_p^2)\big(b_p b_{-p} + b_{-p}^* b_p^*\big) + 2\sigma_p\gamma_p b_p^* b_p \\
&- N^{-1}\gamma_p\sigma_p a_p^* a_p + (\gamma_p\sigma_p - \eta_p)\frac{N-\mathcal{N}_+}{N}\Bigg].
\end{aligned}$$

Using again the estimates $|\gamma_p^2 - 1| \leq C\eta_p^2$ and $|\sigma_p| \leq C|\eta_p|$ for all $p \in \Lambda_+^*$, we find

$$\pm\mathcal{E}_3^V \leq C\ell^\alpha(\mathcal{N}_+ + 1). \tag{3.38}$$

Consider now F_{32} in (3.36), which we split into four parts

$$\begin{aligned}
\mathrm{F}_{32} &= \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)\ \left[\left(\gamma_p b_p + \sigma_p b_{-p}^*\right) d_{-p} + d_p\left(\gamma_p b_{-p} + \sigma_p b_p^*\right)\right] + \text{h.c.} \\
&=: \mathrm{F}_{321} + \mathrm{F}_{322} + \mathrm{F}_{323} + \mathrm{F}_{324}.
\end{aligned} \tag{3.39}$$

Starting with F_{321}, we decompose it again as

$$\begin{aligned}
\mathrm{F}_{321} &= \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)(\gamma_p - 1) b_p d_{-p} + \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta) b_p\left[d_{-p} + \eta_H(p)\frac{\mathcal{N}_+}{N} b_p^*\right] \\
&\quad - \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)\eta_H(p) b_p \frac{\mathcal{N}_+}{N} b_p^* + \text{h.c.}
\end{aligned}$$

Using (3.17), as we did in the proof of Prop. 3.3, we arrive at

$$\mathrm{F}_{321} = -\sum_{p\in P_H} \widehat{V}(p/N^\beta)\eta_p\left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{\mathcal{N}_+ + 1}{N}\right) + \mathcal{E}_4^V$$

where $\mathcal{E}_4^V = \mathcal{E}_{41}^V + \mathcal{E}_{42}^V + \mathcal{E}_{43}^V + \text{h.c.}$, with

$$\mathcal{E}_{41}^V = \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)\,(\gamma_p - 1) b_p d_{-p}\,, \qquad \mathcal{E}_{42}^V = \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta) b_p \bar{d}_{-p}$$

$$\mathcal{E}_{43}^V = -\frac{1}{2}\sum_{p\in P_H} \widehat{V}(p/N^\beta)\eta_p \frac{\mathcal{N}_+ + 1}{N}(b_p^* b_p - N^{-1} a_p^* a_p)$$

and with the notation $\bar{d}_{-p} = d_{-p} + N^{-1}\eta_H(p)\mathcal{N}_+ b_p^*$. Since $|\gamma_p - 1| \le C\eta_p^2$, we find easily with (2.34) in Lemma 2.3 that

$$\begin{aligned}
|\langle \xi, \mathcal{E}_{41}^V \xi\rangle| &\le C \sum_{p\in P_H} \eta_p^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \left[|\eta_p| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta_H\| \|a_p \xi\|\right] \\
&\le C\ell^{2\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.
\end{aligned}$$

Moreover,

$$|\langle \xi, \mathcal{E}_{43}^V \xi\rangle| \le C \sum_{p\in P_H} \eta_p \|a_p\xi\|^2 \le C\ell^\alpha \|\mathcal{N}_+^{1/2}\xi\|^2.$$

While, to control $\mathcal{E}_{42}^V$ we switch to position space. With (2.37) in Lemma 2.3, we find

$$\begin{aligned}
|\langle \xi, \mathcal{E}_{42}^V \xi\rangle| &\le C \int_{\Lambda^2} dx dy\, N^{2\beta} V(N^\beta(x-y)) \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{-1/2} \check{a}_x \check{\bar{d}}_y \xi\| \\
&\le C\|\eta_H\| \int_{\Lambda^2} dx dy\, N^{2\beta} V(N^\beta(x-y)) \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\
&\qquad \times \Big[(N^{-1}\|\eta_H\| + \log N/N) \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta_H\| \|\check{a}_x \xi\| \\
&\qquad\quad + N^{-1} \|\check{a}_y \xi\| + N^{-1/2} \|\check{a}_x \check{a}_y \xi\|\Big] \\
&\le C\ell^\alpha \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\ell^\alpha \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|(\mathcal{V}_N^\beta)^{1/2}\,\xi\|.
\end{aligned}$$

We conclude that

$$\mathcal{E}_4^V \le C\ell^\alpha (\mathcal{H}_N^\beta + 1).$$

To estimate the term F_{322} in (3.39), we use (2.34) in Lemma 2.3 and $|\sigma_p| \le C|\eta_p|$; we obtain

$$\begin{aligned}
|\langle \xi, \mathrm{F}_{322} \xi\rangle| &\le C \sum_{p\in P_H} |\eta_p| \|b_{-p}\xi\| \|d_{-p}\xi\| \\
&\le C \sum_{p\in P_H} |\eta_p| \|b_{-p}\xi\| \left[|\eta_p| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta_H\| \|b_{-p}\xi\|\right] \\
&\le C\ell^{2\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.
\end{aligned}$$

Let us now consider the term F_{323} on the r.h.s. of (3.39). Here, we proceed as we did above to estimate F_{321}. We write $\mathrm{F}_{323} = \mathcal{E}_{51}^V + \mathcal{E}_{52}^V + \text{h.c.}$, with

$$\mathcal{E}_{51}^V = \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)\,(\gamma_p - 1)\, d_p b_{-p}\,, \qquad \mathcal{E}_{52}^V = \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)\, d_p b_{-p}.$$

With $|\gamma_p - 1| \leq C\eta_p^2$, we obtain

$$|\langle \xi, \mathcal{E}_{51}^V \xi\rangle| \leq C \sum_{p\in P_H} \eta_p^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|a_p\xi\|\|\eta_H\| \leq C\ell^{3\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.$$

While for $\mathcal{E}_{52}^V$ we switch to position space, and we find, by (2.36),

$$\begin{aligned} |\langle \xi, \mathcal{E}_{52}^V \xi\rangle| &\leq C \int_{\Lambda^2} dxdy\, N^{2\beta} V(N^\beta(x-y)) \|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{-1/2}\check{d}_x \check{a}_y \xi\| \\ &\leq C\|\eta\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \int_{\Lambda^2} dxdy\, N^{2\beta} V(N^\beta(x-y)) \\ &\qquad\qquad \times \left[\|\check{a}_y\xi\| + N^{-1/2}\|\check{a}_x\check{a}_y\xi\|\right] \\ &\leq C\ell^\alpha \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\ell^\alpha\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{V}_N^\beta)^{1/2}\xi\|. \end{aligned}$$

Hence, $\pm \mathrm{F}_{323} \leq C\ell^{2\alpha}(\mathcal{H}_N^\beta + 1)$.

To estimate the term F_{324} in (3.39), we use (2.34) in Lemma 2.3 and the estimate

$$\sum_{p\in\Lambda_+^*} |\widehat{V}(p/N^\beta)||\eta_p| \leq C \sum_{p\in\Lambda_+^*:|p|\leq N^\beta} \frac{1}{|p|^2} + C \sum_{p\in\Lambda_+^*:|p|\geq N^\beta} \frac{|\widehat{V}(p/N^\beta)|}{|p|^2} \leq C\log N^\beta + C, \tag{3.40}$$

where we used $|\eta_p| \leq C|p|^{-2}$; we find

$$\begin{aligned} |\langle \xi, \mathrm{F}_{324}\xi\rangle| \leq\; & C\sum_{p\in P_H} \left|\widehat{V}(p/N^\beta)\right||\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{-1/2} d_p\, b_p^*\xi\| \\ \leq\; & C\sum_{p\in P_H} \left|\widehat{V}(p/N^\beta)\right||\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ & \times \left[|\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + N^{-1}\|\eta_H\|\|b_p b_p^*(\mathcal{N}_+ + 1)^{1/2}\xi\|\right] \\ \leq\; & C\sum_{p\in P_H} \left|\widehat{V}(p/N^\beta)\right||\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ & \times \left[|\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + N^{-1}\|\eta_H\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta_H\|\|a_p\xi\|\right] \\ \leq\; & C(\ell^{2\alpha} + \ell^\alpha N^{-1}\log N)\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2. \end{aligned}$$

Combining the last bounds, we conclude that

$$\mathrm{F}_{32} = \sum_{p\in P_H} \widehat{V}(p/N^\beta)\eta_p \left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{-\mathcal{N}_+ - 1}{N}\right) + \mathcal{E}_6^V$$

with

$$\pm\mathcal{E}_6^V \leq C\ell^\alpha(\mathcal{H}_N^\beta + 1). \tag{3.41}$$

To bound the last term F_{33} in (3.36), we switch to position space. With

Lemma 2.3, specifically (2.38), and (2.25), we obtain

$$\begin{aligned}|\langle\xi, \mathrm{F}_{33}\xi\rangle| &\leq C\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \int_{\Lambda^2} dxdy\, N^{2\beta}V(N^\beta(x-y))\|(\mathcal{N}_+ + 1)^{-1/2}\check{d}_x\check{d}_y\xi\| \\ &\leq C\|\eta_H\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \int_{\Lambda^2} dxdy\, N^{2\beta}V(N^\beta(x-y)) \\ &\qquad\qquad \times \big[(\|\eta_H\| + 1)\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta_H\|\|\check{a}_x\xi\| \\ &\qquad\qquad\qquad + \|\eta_H\|\|\check{a}_y\xi\| + N^{-1/2}\|\check{a}_x\check{a}_y\xi\|\big] \\ &\leq C\ell^\alpha\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\ell^{2\alpha}\|\mathcal{N}_+^{1/2}\xi\|\|(\mathcal{V}_N^\beta)^{1/2}\xi\|.\end{aligned}$$

The last equation, combined with (3.36), (3.37), (3.38) and (3.41), implies that

$$\begin{aligned}\mathrm{F}_3 &= \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N^\beta)(b_pb_{-p} + b_{-p}^*b_p^*) \\ &\quad + \sum_{p\in P_H} \widehat{V}(p/N^\beta)\eta_p \left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{N-\mathcal{N}_+ - 1}{N}\right) + \mathcal{E}_7^V\end{aligned}$$

with

$$\pm\mathcal{E}_7^V \leq C\ell^\alpha(\mathcal{H}_N^\beta + 1).$$

Together with (3.34) and with (3.35), we obtain (3.30) with (3.31). Finally, to prove Eq. (3.32) we argue similarly as we did at the end of the proof of Prop. 3.3 to show (3.12). □

In conclusion of this subsection, we combine the results of Prop. 3.3 and Prop. 3.4.

Proposition 3.5. *There exists a constant $C > 0$ such that*

$$\begin{aligned}\mathcal{G}_{N,\ell}^{\beta,(2)} &= \mathcal{K} + \sum_{p\in P_H}\Big[p^2\eta_p^2 + \widehat{V}(p/N^\beta)\eta_p\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+ - 1}{N}\Big) \\ &\quad + \sum_{p\in P_H} p^2\eta_p\big(b_p^*b_{-p}^* + b_pb_{-p}\big) + \sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)a_p^*a_p\frac{N-\mathcal{N}_+}{N} \\ &\quad + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)\big(b_pb_{-p} + b_{-p}^*b_p^*\big) + \mathcal{E}_{N,\ell}^{\beta,(2)}\end{aligned}$$

where

$$\pm\mathcal{E}_{N,\ell}^{\beta,(2)} \leq C\ell^{\alpha-1}(\mathcal{H}_N^\beta + 1)$$

and

$$\pm\Big[f(\mathcal{N}_+/M), \Big[f(\mathcal{N}_+/M), \mathcal{E}_{N,\ell}^{(2)}\Big]\Big] \leq C\ell^{\alpha-1}M^{-2}\|f'\|_\infty^2\left(\mathcal{H}_N^\beta + 1\right)$$

for all $\alpha > 1$, $\ell \in (0;1/2)$ small enough, f smooth and bounded, $M \in \mathbb{N}$, $N \in \mathbb{N}$ large enough.

3.1.3 Analysis of $\mathcal{G}_{N,\ell}^{\beta,(3)} = e^{-B_H}\mathcal{L}_N^{\beta,(3)}e^{B_H}$

From (2.9), we have

$$\mathcal{G}_{N,\ell}^{\beta,(3)} = \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N^\beta) e^{-B_H} b_{p+q}^* a_{-p}^* a_q e^{B_H} + \text{h.c.} \tag{3.42}$$

Proposition 3.6. *There exists a constant $C > 0$ such that*

$$\mathcal{G}_{N,\ell}^{\beta,(3)} = \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N^\beta) \left[b_{p+q}^* a_{-p}^* a_q + \text{h.c.} \right] + \mathcal{E}_{N,\ell}^{\beta,(3)} \tag{3.43}$$

where

$$\pm\mathcal{E}_{N,\ell}^{\beta,(3)} \leq C\ell^\alpha(\mathcal{H}_N^\beta + 1) \tag{3.44}$$

and

$$\pm\,[f(\mathcal{N}_+/M),[f(\mathcal{N}_+/M),\mathcal{E}_{N,\ell}^{\beta,(3)}]] \leq CM^{-2}\|f'\|_\infty^2 \ell^\alpha \big(\mathcal{H}_N^\beta + 1\big) \tag{3.45}$$

for all $\alpha > 0$, $\ell \in (0;1/2)$ small enough, f smooth and bounded, $M \in \mathbb{N}$, $N \in \mathbb{N}$ large enough.

Proof of Proposition 3.6. We start by writing

$$\begin{aligned} e^{-B_H} a_{-p}^* a_q e^{B_H} &= a_{-p}^* a_q + \int_0^1 ds\, e^{-sB_H}[a_{-p}^* a_q, B_H] e^{sB_H} \\ &= a_{-p}^* a_q + \int_0^1 ds e^{-sB_H}(\eta_p b_q b_p + \eta_q b_{-p}^* b_{-q}^*) e^{sB_H}. \end{aligned}$$

From (3.42), we find

$$\begin{aligned} \mathcal{G}_{N,\ell}^{\beta,(3)} &= \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*:\\ p+q\neq 0}} \widehat{V}(p/N^\beta) e^{-B_H} b_{p+q}^* e^{B_H}\, a_{-p}^* a_q \\ &\quad + \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*,\\ p+q\neq 0}} \widehat{V}(p/N^\beta)\eta_H(p)\, e^{-B_H} b_{p+q}^* e^{B_H} \int_0^1 ds\, e^{-sB_H} b_p b_q e^{sB_H} \\ &\quad + \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*,\\ p+q\neq 0}} \widehat{V}(p/N^\beta)\eta_H(q)\, e^{-B_H} b_{p+q}^* e^{B_H} \int_0^1 ds\, e^{-sB_H} b_{-p}^* b_{-q}^* e^{sB_H} \\ &\quad + \text{h.c.} \end{aligned}$$

Using (2.31) we arrive at (3.43), with

$$\begin{aligned}\mathcal{E}_{N,\ell}^{\beta,(3)} &= \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*:\\ p+q\neq 0}} \widehat{V}(p/N^\beta)\big((\gamma_{p+q}-1)b^*_{p+q} + \sigma_{p+q}b_{-p-q} + d^*_{p+q}\big)\, a^*_{-p}a_q \\ &\quad + \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*,\\ p+q\neq 0}} \widehat{V}(p/N^\beta)\eta_H(p)\, e^{-B_H} b^*_{p+q} e^{B_H} \int_0^1 ds\, e^{-sB_H} b_p b_q e^{sB_H} \\ &\quad + \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*,\\ p+q\neq 0}} \widehat{V}(p/N^\beta)\eta_H(q)\, e^{-B_H} b^*_{p+q} e^{B_H} \int_0^1 ds\, e^{-sB_H} b^*_{-p} b^*_{-q} e^{sB_H} \\ &\quad + \text{h.c.} \\ &=: \; \mathcal{E}_1^{(3)} + \mathcal{E}_2^{(3)} + \mathcal{E}_3^{(3)} + \text{h.c.}\end{aligned} \tag{3.46}$$

where we defined $\gamma_p = \cosh \eta_p$, $\sigma_p = \sinh \eta_p$ and where the operator d_p is defined as in (2.31),with η replaced by η_H. To conclude Prop. 3.6, we have to show that the three error terms $\mathcal{E}_1^{(3)}, \mathcal{E}_2^{(3)}, \mathcal{E}_3^{(3)}$ all satisfy the bounds (3.44), (3.45). We consider $\mathcal{E}_1^{(3)}$ first, and we proceed decomposing it as

$$\begin{aligned}\mathcal{E}_1^{(3)} &= \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N^\beta)\big((\gamma_{p+q}-1)b^*_{p+q} + \sigma_{p+q}b_{-p-q} + d^*_{p+q}\big)\, a^*_{-p}a_q \\ &=: \mathcal{E}_{11}^{(3)} + \mathcal{E}_{12}^{(3)} + \mathcal{E}_{13}^{(3)}.\end{aligned}$$

Since $|\gamma_{p+q}-1| \leq |\eta_H(p+q)|^2$ and $\|\eta_H\| \leq C\ell^\alpha$, we have

$$\begin{aligned}|\langle \xi, \mathcal{E}_{11}^{(3)}\xi\rangle| &\leq C\frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} |\widehat{V}(p/N^\beta)||\eta_H(p+q)|^2\, \|b_{p+q}a_{-p}\xi\|\|a_q\xi\| \\ &\leq C\frac{1}{\sqrt{N}}\Big[\sum_{p,q\in\Lambda_+^*:p+q\neq 0} |\eta_H(p+q)|^2\, \|a_{-p}(\mathcal{N}_++1)^{1/2}\xi\|^2\Big]^{1/2} \\ &\qquad\qquad \times \Big[\sum_{p,q\in\Lambda_+^*:p+q\neq 0} |\eta_H(p+q)|^2 \|a_q\xi\|^2\Big]^{1/2} \\ &\leq C\|\eta_H\|^2\|(\mathcal{N}_++1)^{1/2}\xi\|^2 \leq C\ell^{2\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2.\end{aligned} \tag{3.47}$$

To bound $\mathcal{E}_{12}^{(3)}$ we move a^*_{-p} to the left of b_{-p-q} (using $[a_{-p-q}, a^*_{-p}] = 0$, since $q \neq 0$). With $|\sigma_{p+q}| \leq C|\eta_H(p+q)|$, we obtain

$$\begin{aligned}|\langle \xi, \mathcal{E}_{12}^{(3)}\xi\rangle| &\leq C\frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} |\widehat{V}(p/N^\beta)||\eta_H(p+q)|\, \|a_{-p}\xi\|\|a_q b_{-p-q}\xi\| \\ &\leq C\frac{1}{\sqrt{N}}\Big[\sum_{p,q\in\Lambda_+^*:p+q\neq 0} |\eta_H(p+q)|^2\, \|a_{-p}\xi\|^2\Big]^{1/2} \\ &\qquad\qquad \times \Big[\sum_{p,q\in\Lambda_+^*:p+q\neq 0} \|a_q b_{-p-q}\xi\|^2\Big]^{1/2} \\ &\leq C\ell^\alpha\|(\mathcal{N}_++1)^{1/2}\xi\|^2.\end{aligned} \tag{3.48}$$

In $\mathcal{E}_{13}^{(3)}$, on the other hand, we write $d_{p+q}^* = \bar{d}_{p+q}^* - \frac{(\mathcal{N}_++1)}{N}\eta_H(p+q)b_{-p-q}$. We split $\mathcal{E}_{13}^{(3)} = \mathcal{E}_{131}^{(3)} + \mathcal{E}_{132}^{(3)}$, with

$$\begin{aligned}\mathcal{E}_{131}^{(3)} &= \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N^\beta)\, \bar{d}_{p+q}^* a_{-p}^* a_q \\ \mathcal{E}_{132}^{(3)} &= -\frac{(\mathcal{N}_++1)}{N^{3/2}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N^\beta)\eta_H(p+q)\, b_{-p-q}a_{-p}^* a_q.\end{aligned}$$

Similarly as we did for $\mathcal{E}_{12}^{(3)}$, to bound the term $\mathcal{E}_{132}^{(3)}$, we commute a_{-p}^* to the left of b_{-p-q} and we find $\pm\mathcal{E}_{132}^{(3)} \leq CN^{-1/2}\ell^\alpha(\mathcal{N}_++1)$. As for the term $\mathcal{E}_{131}^{(3)}$, we switch to position space:

$$\begin{aligned}\mathcal{E}_{131}^{(3)} &= \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*:p+q\neq 0} \widehat{V}(p/N^\beta)\, \bar{d}_{p+q}^* a_{-p}^* a_q \\ &= \frac{1}{\sqrt{N}} \int_{\Lambda^2} dxdy N^{2\beta} V(N^\beta(x-y)) \check{\bar{d}}_x^* \check{a}_y^* \check{a}_x.\end{aligned}$$

With (2.37), we bound

$$\begin{aligned}|\langle \xi, \mathcal{E}_{131}^{(3)}\xi\rangle| &\leq \frac{1}{\sqrt{N}} \int_{\Lambda^2} dxdy N^{2\beta} V(N^\beta(x-y)) \|\check{a}_x\xi\| \|\check{a}_y \check{\bar{d}}_x \xi\| \\ &\leq \frac{C}{\sqrt{N}}\|\eta_H\| \int_{\Lambda^2} dxdy N^{2\beta} V(N^\beta(x-y)) \|\check{a}_x\xi\| \\ &\quad \times \big[(N^{-1/2}\|\eta_H\| + \log N/N^{1/2}) \|(\mathcal{N}_++1)^{1/2}\xi\| + N^{-1/2}\|\check{a}_x\xi\| \\ &\qquad\qquad + \|\eta_H\| \|\check{a}_y \mathcal{N}_+^{1/2}\xi\| + \|\check{a}_x\check{a}_y\xi\|\big] \\ &\leq C(\ell^{2\alpha}N^{-1} + \log N/N\ell^\alpha + N^{-1}\ell^\alpha + \ell^{2\alpha}) \|(\mathcal{N}_++1)^{1/2}\xi\|^2 \\ &\qquad\qquad + C\ell^\alpha \|(\mathcal{N}_++1)^{1/2}\xi\| \|(\mathcal{V}_N^\beta)^{1/2}\,\xi\|.\end{aligned}$$

Combining the last bound with (3.47) and (3.48) we conclude that

$$\pm\mathcal{E}_1^{(3)} \leq C\ell^\alpha(\mathcal{H}_N^\beta + 1). \tag{3.49}$$

Next, we consider the term $\mathcal{E}_2^{(3)}$, defined in (3.46). Using Eq. (2.31) we rewrite

$$\begin{aligned}\mathcal{E}_2^{(3)} &= \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*,p+q\neq 0} \widehat{V}(p/N^\beta)\eta_H(p)\, e^{-B_H} b^*_{p+q} e^{B_H} \\ &\qquad \times \int_0^1 ds\, \big(\gamma_p^{(s)}\gamma_q^{(s)} b_p b_q + \sigma_p^{(s)}\sigma_q^{(s)} b^*_{-p} b^*_{-q} + \gamma_p^{(s)}\sigma_q^{(s)} b^*_{-q} b_p + \sigma_p^{(s)}\gamma_q^{(s)} b^*_{-p} b_q\big) \\ &\quad + \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*,p+q\neq 0} \widehat{V}(p/N^\beta)\eta_H(p)\, e^{-B_H} b^*_{p+q} e^{B_H} \int_0^1 ds\, \gamma_p^{(s)}\sigma_q^{(s)} [b_p, b^*_{-q}] \\ &\quad + \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*,p+q\neq 0} \widehat{V}(p/N^\beta)\eta_H(p)\, e^{-B_H} b^*_{p+q} e^{B_H} \\ &\qquad \times \int_0^1 ds\, \Big[d_p^{(s)}\big(\gamma_q^{(s)} b_q + \sigma_q^{(s)} b^*_{-q}\big) + \big(\gamma_p^{(s)} b_p + \sigma_p^{(s)} b^*_{-p}\big) d_q^{(s)} + d_p^{(s)} d_q^{(s)}\Big] \\ &=: \mathcal{E}_{21}^{(3)} + \mathcal{E}_{22}^{(3)} + \mathcal{E}_{23}^{(3)}\end{aligned} \tag{3.50}$$

where, for any $s \in [0;1]$ and $p \in \Lambda_+^*$, $\gamma_p^{(s)} = \cosh(s\eta_H(p))$, $\sigma_p^{(s)} = \sinh(s\eta_H(p))$ and $d_p^{(s)}$ is the operator defined as in (2.31), with η replaced by η_H. First we bound

$$\begin{aligned}|\langle \xi, \mathcal{E}_{21}^{(3)} \xi\rangle| &\leq C\frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p\neq -q} |\eta_H(p)| \|b_{p+q} e^{B_H}\xi\| \Big[\|b_p b_q \xi\| \\ &\qquad + |\eta_H(p)| \|b_q(\mathcal{N}_+ + 1)^{1/2}\xi\| + |\eta_H(q)| \|b_p(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\qquad + |\eta_H(p)||\eta_H(q)| \|(\mathcal{N}_+ + 1)\xi\|\Big] \\ &\leq C\frac{1}{\sqrt{N}} \big(\|\eta_H\| + \|\eta_H\|^2 + \|\eta_H\|^3\big) \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \\ &\leq C N^{-1/2} (\ell^\alpha + \ell^{2\alpha} + \ell^{3\alpha}) \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.\end{aligned} \tag{3.51}$$

Since $[b_p, b^*_{-q}] = -a^*_{-q} a_p / N$ for all $p \neq -q$, we find

$$\begin{aligned}|\langle \xi, \mathcal{E}_{22}^{(3)} \xi\rangle| &\leq C N^{-3/2} \sum_{p,q\in\Lambda_+^*,p+q\neq 0} |\eta_H(p)||\eta_H(q)| \|b_{p+q} e^{B_H}\xi\| \|a_p(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\leq C N^{-1} \|\eta\|^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \leq C N^{-1} \ell^{2\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.\end{aligned} \tag{3.52}$$

To bound the third term on the r.h.s. of (3.50), we switch to position space. We obtain

$$\begin{aligned}\mathcal{E}_{23}^{(3)} &= \frac{1}{\sqrt{N}} \int_{\Lambda^3} dx dy dz N^{2\beta} V(N^\beta(x-z)) \check{\eta}(z-y)\, e^{-B_H} \check{b}^*_x e^{B_H} \\ &\qquad \times \int_0^1 ds\, \Big[\check{d}_y^{(s)}\big(b(\check{\gamma}_x^{(s)}) + b^*(\check{\sigma}_x^{(s)})\big) + \big(b(\check{\gamma}_y^{(s)}) + b^*(\check{\sigma}_y^{(s)})\big)\check{d}_x^{(s)} + \check{d}_y^{(s)}\check{d}_x^{(s)}\Big].\end{aligned}$$

Using the bounds (2.36), (2.37), (2.38) and Lemma 2.2 we arrive at

$$\begin{aligned}|\langle \xi, \mathcal{E}_{23}^{(3)} \xi\rangle| &\leq CN^{-1/2}\|\eta_H\| \int_{\Lambda^3} dxdydz\, N^{2\beta}V(N^\beta(x-z))|\check{\eta}(y-z)|\, \|\check{b}_x e^{B_H}\xi\| \\ &\quad\times \Big[\|\check{b}_x\check{b}_y\xi\| + \|(\mathcal{N}_+ +1)\xi\| + \|\check{b}_x(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\check{b}_y(\mathcal{N}_+ +1)^{1/2}\xi\|\Big] \\ &\leq \frac{C\|\eta_H\|^2}{\sqrt{N}} \|\mathcal{N}_+^{1/2} e^{B_H}\xi\| \|(\mathcal{N}_+ +1)\xi\| \\ &\leq C\ell^{2\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2.\end{aligned}$$

Combined with (3.51) and (3.52), the last bound implies that

$$\pm \mathcal{E}_2^{(3)} \leq C\ell^\alpha(\mathcal{N}_+ +1). \tag{3.53}$$

Finally, we consider the last term on the r.h.s. of (3.46). In this case, we estimate the expectation of its adjoint -in absolute value- because it is more convenient. We split it as follows

$$\begin{aligned}\mathcal{E}_3^{(3)*} &= \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*,\\ p+q\neq 0}} \widehat{V}(p/N^\beta)\eta_H(q) \int_0^1 ds\, e^{-sB_H} b_{-q} e^{sB_H} \\ &\qquad\times \big(\gamma_p^{(s)} b_{-p} + \sigma_p^{(s)} b_p^* + d_{-p}^{(s)}\big)\big(\gamma_{p+q}b_{p+q} + \sigma_{p+q}b^*_{-p-q} + d_{p+q}\big) \\ &= \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*,\\ p+q\neq 0}} \widehat{V}(p/N^\beta)\eta_H(q) \int_0^1 ds\, e^{-sB_H} b_{-q} e^{sB_H} \\ &\quad\times \Big[\gamma_p^{(s)}\gamma_{p+q}b_{-p}b_{p+q} + \sigma_p^{(s)}\sigma_{p+q}b_p^*b^*_{-p-q} + \gamma_p^{(s)}\sigma_{p+q}b^*_{-p-q}b_{-p} + \gamma_{p+q}\sigma_p^{(s)}b_p^*b_{p+q} \\ &\qquad + d_{-p}^{(s)}\big(\gamma_{p+q}b_{p+q} + \sigma_{p+q}b^*_{-p-q}\big) + \big(\gamma_p^{(s)}b_{-p} + \sigma_p^{(s)}b_p^*\big)d_{p+q} + d_{-p}^{(s)}d_{p+q}\Big] \\ &\quad+ \frac{1}{\sqrt{N}} \sum_{\substack{p,q\in\Lambda_+^*,\\ p+q\neq 0}} \widehat{V}(p/N^\beta)\eta_H(q) \int_0^1 ds\, e^{-sB_H} b_{-q} e^{sB_H}\gamma_p^{(s)}\sigma_{p+q}[b_{-p}, b^*_{-p-q}] \\ &=: \mathcal{E}_{31}^{(3)} + \mathcal{E}_{32}^{(3)}.\end{aligned}$$

Using that $q \neq 0$ and thus that $[b_{-p}, b^*_{-p-q}] = -a^*_{-p-q}a_{-p}/N$, we can estimate the second term by

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{32}^{(3)} \xi\rangle| \\
&\quad \leq C \frac{1}{N^{3/2}} \int_0^1 ds \sum_{\substack{p,q\in\Lambda_+^*,\\ p+q\neq 0}} |\eta_H(q)||\eta_H(p+q)| \, \|a_{-p-q}\, e^{-sB_H} b^*_{-q} e^{sB_H}\xi\| \|a_{-p}\xi\| \\
&\quad \leq C \frac{1}{N^{3/2}} \int_0^1 ds \Big[\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}} |\eta_H(q)|^2 \, \|a_{-p-q}\, e^{-sB_H} b^*_{-q} e^{sB_H}\xi\|^2 \Big]^{1/2} \\
&\qquad\qquad\qquad\qquad \times \Big[\sum_{\substack{p,q\in\Lambda_+^*\\ p+q\neq 0}} |\eta_H(p+q)|^2 \|a_{-p}\xi\|^2 \Big]^{1/2} \\
&\quad \leq C N^{-1} \|\eta\|^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \leq C\ell^{2\alpha} N^{-1} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.
\end{aligned} \tag{3.54}$$

To bound the expectation of $\mathcal{E}_{31}^{(3)}$, it is convenient to switch to position space. We find

$$\begin{aligned}
\mathcal{E}_{31}^{(3)} = \frac{1}{\sqrt{N}} \int_0^1 ds \int_{\Lambda^2} dxdy \, N^{2\beta} V(N^\beta(x-y))\, e^{-sB_H} b(\check{\eta}_{H,x}) e^{sB_H} \\
\times \Big[b(\check{\gamma}_x^{(s)}) b(\check{\gamma}_y) + b^*(\check{\sigma}_x^{(s)}) b^*(\check{\sigma}_y) + b^*(\check{\sigma}_y) b(\check{\gamma}_x^{(s)}) + b^*(\check{\sigma}_x^{(s)}) b(\check{\gamma}_y) \\
+ \check{d}_x^{(s)} \big(b(\check{\gamma}_y) + b^*(\check{\sigma}_y)\big) + \big(b(\check{\gamma}_x^{(s)}) + b^*(\check{\sigma}_x^{(s)})\big) \check{d}_y + \check{d}_x^{(s)} \check{d}_y \Big]
\end{aligned}$$

where we used the notation $\check{\eta}$, $\check{\gamma}^{(s)}$ and $\check{\sigma}^{(s)}$ to indicate the functions on Λ with Fourier coefficients $\eta_H(p)$, $\cosh(s\eta_H(p))$ and, respectively, $\sinh(s\eta_H(p))$, and where $\check{\eta}_{H,x}$, $\check{\gamma}_x$ and $\check{\sigma}_x$ denote the functions defined by $\check{\eta}_{H,x}(z) = \check{\eta}_H(z-x)$, $\check{\gamma}_x(z) = \check{\gamma}(z-x)$ and $\check{\sigma}_x(z) = \check{\sigma}(z-x)$. Using (2.36), (2.37), (2.38) and the bound (2.25), we find, for N large enough,

$$\begin{aligned}
|\langle \xi, \mathcal{E}_{31}^{(3)} \xi\rangle| \leq \frac{C}{\sqrt{N}} \int_0^1 ds \int_{\Lambda^2} dxdy \, N^{2\beta} V(N^\beta(x-y)) \|b^*(\check{\eta}_{H,x}) e^{sB_H}\xi\| \\
\times \Big[\|\check{b}_x \check{b}_y \xi\| + \|\check{b}_x (\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{b}_y (\mathcal{N}_+ + 1)^{1/2}\xi\| + \|(\mathcal{N}_+ + 1)\xi\| \Big].
\end{aligned}$$

With Lemma 2.2, we estimate

$$\|b^*(\check{\eta}_{H,x}) e^{sB_H}\xi\| \leq C\|\eta_H\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\|$$

and so, we conclude that

$$|\langle \xi, \mathcal{E}_{31}^{(3)} \xi\rangle| \leq C\ell^\alpha \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\ell^\alpha \|(\mathcal{V}_N^\beta)^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\|.$$

From (3.54), we find

$$\pm \mathcal{E}_3^{(3)} \leq C\ell^\alpha (\mathcal{H}_N^\beta + 1)$$

and thus, combining this bound with (3.46), (3.49) and (3.53), we arrive at

$$\pm\mathcal{E}^{(3)}_{N,\ell} \leq C\ell^\alpha(\mathcal{H}^\beta_N + 1)$$

This proves (3.44). The bound (3.45) follows similarly, arguing as we did at the end of the proof of Prop. 3.3 to show (3.12). □

3.1.4 Analysis of $\mathcal{G}^{\beta,(4)}_{N,\ell} = e^{-B_H}\mathcal{L}^{\beta,(4)}_N e^{B_H}$

With $\mathcal{L}^{\beta,(4)}_N$ as defined in (2.7), we write

$$\begin{aligned}\mathcal{G}^{\beta,(4)}_{N,\ell} &= e^{-B_H}\mathcal{L}^{\beta,(4)}_N e^{B_H}\\ &= \mathcal{V}^\beta_N + \frac{1}{2N}\sum_{\substack{q\in\Lambda^*_+, r\in\Lambda^*\\ q,\, q+r\in P_H}} \widehat{V}(r/N^\beta)\eta_{q+r}\eta_q\left(1-\frac{\mathcal{N}_+}{N}\right)\left(1-\frac{\mathcal{N}_+ +1}{N}\right)\\ &\quad + \frac{1}{2N}\sum_{\substack{q\in\Lambda^*_+, r\in\Lambda^*:\\ q+r\in P_H}} \widehat{V}(r/N^\beta)\,\eta_{q+r}\left(b_q b_{-q} + b^*_q b^*_{-q}\right) + \mathcal{E}^{\beta,(4)}_{N,\ell}.\end{aligned}$$

Proposition 3.7. *There exists a constant $C > 0$ such that*

$$\pm\mathcal{E}^{\beta,(4)}_{N,\ell} \leq C\ell^\alpha(\mathcal{H}^\beta_N + 1) \tag{3.55}$$

and

$$\pm\,[f(\mathcal{N}_+/M),[f(\mathcal{N}_+/M),\mathcal{E}^{\beta,(4)}_{N,\ell}]] \leq CM^{-2}\|f'\|^2_\infty\ell^\alpha\big(\mathcal{H}^\beta_N + 1\big) \tag{3.56}$$

for all $\alpha > 0$, $\ell \in (0;1/2)$ small enough, f smooth and bounded, $M \in \mathbb{N}$, $N \in \mathbb{N}$ large enough.

In the proof of Prop. 3.7, we are going to use the following lemma, which is an adaptation to this scaling of [10, Lemma 7.7], due to the different L^∞ norm of $\check{\eta}(x)$, and it simplify computations for the proof of Prop. 3.7.

Lemma 3.8. *Let $\eta \in \ell^2(\Lambda^*)$, as defined in (2.22). Then there exists a constant $C > 0$ such that*

$$\begin{aligned}&\|(\mathcal{N}_+ +1)^{n/2}e^{-B}\check{b}_x\check{b}_y e^B\xi\|\\ &\leq C\Big[\ \|(\mathcal{N}_+ +1)^{(n+2)/2}\xi\| + \log N\|(\mathcal{N}_+ +1)^{n/2}\xi\|\\ &\qquad + \|\check{a}_y(\mathcal{N}_+ +1)^{(n+1)/2}\xi\| + \|\check{a}_x(\mathcal{N}_+ +1)^{(n+1)/2}\xi\| + \|\check{a}_x\check{a}_y(\mathcal{N}_+ +1)^{n/2}\xi\|\Big]\end{aligned} \tag{3.57}$$

for all $\xi \in \mathcal{F}^{\leq N}_+$, $n \in \mathbb{Z}$.

Proof. We consider $n = 0$, the general case follows similarly. With the notation $\gamma_p = \cosh\eta_p$, $r_p = 1 - \gamma_p$, $\sigma_p = \sinh\eta_p$ and denoting by $\check{\sigma}$, $\check{r}$ the functions in $L^2(\Lambda)$ with Fourier coefficients σ_p and r_p, we use (2.31) to write

$$\begin{aligned}\|e^{-B}\check{b}_x\check{b}_y e^B\xi\| &= \|\big(\check{b}_x + b(\check{r}_x) + b^*(\check{\sigma}_x) + \check{d}_x\big)\big(\check{b}_y + b(\check{r}_y) + b^*(\check{\sigma}_y) + \check{d}_y\big)\xi\| \\ &\leq \|\check{b}_x\check{b}_y\xi\| + C(\|\check{b}_x\mathcal{N}_+^{1/2}\xi\| + \|\check{b}_y\mathcal{N}_+^{1/2}\xi\|) + C|\check{\sigma}(x-y)|\|\xi\| \\ &\quad + \|\check{b}_x\check{d}_y\xi\| + \|\check{d}_x\big(\check{b}_y + b(\check{r}_y) + b^*(\check{\sigma}_y) + \check{d}_y\big)\xi\|\end{aligned}$$

because $\|r\|, \|\sigma\| \leq C\|\eta\| \leq C$. Using Eq. (2.38) and (after writing $\check{b}_x\check{d}_y = \check{b}_x\check{\bar{d}}_y - \check{b}_x(\mathcal{N}_+/N)b^*(\check{\eta}_y)$) Eq. (2.37), and with the bound (2.25) (which also implies $|\check{\sigma}(x)| \leq C\log N$), we obtain (3.57). $\square$

Proof of Prop. 3.7. We start by writing

$$\begin{aligned}&e^{-B_H}\mathcal{L}_N^{\beta,(4)}e^{B_H} \\ &= \frac{1}{2N}\sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*:\\ r\neq -p,q}} \widehat{V}(r/N^\beta)e^{-B_H}a_p^*a_q^*a_{q-r}a_{p+r}e^{B_H} \\ &= \mathcal{V}_N^\beta + \frac{1}{2N}\sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*:\\ r\neq -p,q}} \widehat{V}(r/N^\beta)\int_0^1 ds\, e^{-sB_H}\left[a_p^*a_q^*a_{q-r}a_{p+r}, B_H\right]e^{sB_H} \\ &= \mathcal{V}_N^\beta + \frac{1}{2N}\sum_{\substack{q\in\Lambda_+^*, r\in\Lambda^*:\\ r\neq -q}} \widehat{V}(r/N^\beta)\eta_H(q+r)\int_0^1 ds\, \left(e^{-sB_H}b_q^*b_{-q}^*e^{sB_H} + \text{h.c.}\right) \\ &+ \frac{1}{N}\sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*:\\ r\neq p,-q}} \widehat{V}(r/N^\beta)\eta_H(q+r)\int_0^1 ds\, \left(e^{-sB_H}b_{p+r}^*b_q^*a_{-q-r}^*a_pe^{sB_H} + \text{h.c.}\right).\end{aligned} \tag{3.58}$$

Notice that

$$\begin{aligned}&e^{-sB_H}a_{-q-r}^*a_pe^{sB_H} \\ &\qquad = a_{-q-r}^*a_p + \int_0^s d\tau\, e^{-\tau B_H}\left[a_{-q-r}^*a_p, B_H\right]e^{-\tau B_H} \\ &\qquad = a_{-q-r}^*a_p + \int_0^s d\tau\, e^{-\tau B_H}\left(\eta_H(p)b_{-p}^*b_{-q-r}^* + \eta_H(q+r)b_pb_{q+r}\right)e^{-\tau B_H}.\end{aligned}$$

Inserting in (3.58), we can rewrite

$$\mathcal{G}_{N,\ell}^{\beta,(4)} - \mathcal{V}_N^\beta = \mathrm{W}_1 + \mathrm{W}_2 + \mathrm{W}_3 + \mathrm{W}_4$$

with

$$\begin{aligned}
\mathrm{W}_1 &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds \big(e^{-sB_H} b_q b_{-q}\, e^{sB_H} + \text{h.c.}\big) \\
\mathrm{W}_2 &= \frac{1}{N} \sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*:\\ r\neq p,-q}} \widehat{V}(r/N^\beta)\, \eta_H(q+r) \int_0^1 ds \big(e^{-sB_H} b_q^* b_{-q}^* e^{sB_H} a_{-q-r}^* a_p + \text{h.c.}\big) \\
\mathrm{W}_3 &= \frac{1}{N} \sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq -p-q} \widehat{V}(r/N^\beta)\eta_H(q+r)\eta_H(p) \\
&\qquad\qquad \times \int_0^1 ds \int_0^s d\tau \,\big(e^{-sB_H} b_{p+r}^* b_q^* e^{sB_H} e^{-\tau B_H} b_{-p}^* b_{-q-r}^* e^{\tau B_H} + \text{h.c.}\,\big) \\
\mathrm{W}_4 &= \frac{1}{N} \sum_{p,q\in\Lambda_+^*, r\in\Lambda^*: r\neq -p-q} \widehat{V}(r/N^\beta)\, \eta_H^2(q+r) \\
&\qquad\qquad \times \int_0^1 ds \int_0^s d\tau \,\big(e^{-sB_H} b_{p+r}^* b_q^* e^{sB_H} e^{-\tau B_H} b_p b_{q+r} e^{\tau B_H} + \text{h.c.}\,\big).
\end{aligned} \tag{3.59}$$

Let us first consider the term W_1. With (2.31), we find

$$\begin{aligned}
\mathrm{W}_1 &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \\
&\qquad \times \int_0^1 ds \big(\gamma_q^{(s)} b_q + \sigma_q^{(s)} b_{-q}^* + d_q^{(s)}\big)\big(\gamma_q^{(s)} b_{-q} + \sigma_q^{(s)} b_q^* + d_{-q}^{(s)}\big) + \text{h.c.}
\end{aligned}$$

where we defined $\gamma_q^{(s)} = \cosh(s\eta_H(q))$, $\sigma_q^{(s)} = \sinh(s\eta_H(q))$ and where $d_q^{(s)}$ is defined as in (2.31), with η replaced by $s\eta_H$. We split

$$\begin{aligned}
\mathrm{W}_1 &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds (\gamma_q^{(s)})^2 (b_q b_{-q} + \text{h.c.}\,) \\
&\quad + \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds\, \gamma_q^{(s)}\sigma_q^{(s)} \big([b_q, b_q^*] + \text{h.c.}\,\big) \\
&\quad + \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds\, \gamma_q^{(s)} \big(b_q d_{-q}^{(s)} + \text{h.c.}\,\big) + \mathcal{E}_{10}^{(4)} \\
&=: \mathrm{W}_{11} + \mathrm{W}_{12} + \mathrm{W}_{13} + \mathcal{E}_{10}^{(4)}
\end{aligned} \tag{3.60}$$

where

$$\mathcal{E}_{10}^{(4)} = \mathcal{E}_{101}^{(4)} + \mathcal{E}_{102}^{(4)} + \mathcal{E}_{103}^{(4)} + \mathcal{E}_{104}^{(4)} + \mathcal{E}_{105}^{(4)} \tag{3.61}$$

with the errors

$$\begin{aligned}
\mathcal{E}_{101}^{(4)} &= \frac{1}{2N} \sum_{\substack{q\in\Lambda_+^*, r\in\Lambda^*:\\ r\neq -q}} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds \Big[2\gamma_q^{(s)}\sigma_q^{(s)} b_q^* b_q + (\sigma_q^{(s)})^2 b_{-q}^* b_q^* + \text{h.c.} \Big] \\
\mathcal{E}_{102}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds\, \sigma_q^{(s)} \big(b_{-q}^* d_{-q}^{(s)} + \text{h.c.} \big) \\
\mathcal{E}_{103}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds\, \sigma_q^{(s)} \big(d_q^{(s)} b_q^* + \text{h.c.} \big) \\
\mathcal{E}_{104}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds\, \gamma_q^{(s)} \big(d_q^{(s)} b_{-q} + \text{h.c.} \big) \\
\mathcal{E}_{105}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds \big(d_q^{(s)} d_{-q}^{(s)} + \text{h.c.} \big).
\end{aligned} \tag{3.62}$$

From the bound (3.40), for all $\beta > 0$ considered in Theorem 1.3 we have

$$\frac{1}{N} \sup_{q\in\Lambda_+^*} \sum_{r\in\Lambda_+^*} |\widehat{V}(r/N^\beta)||\eta_{q+r}| \le C \tag{3.63}$$

uniformly in $N \in \mathbb{N}$. We can bound the first term in (3.62) by

$$\begin{aligned}
|\langle \xi, \mathcal{E}_{101}^{(4)} \xi\rangle| &\le C \sum_{q\in\Lambda_+^*} \big[|\eta_q| \|b_q\xi\|^2 + \eta_q^2 \|b_q\xi\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \big] \\
&\le C\ell^\alpha \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.
\end{aligned}$$

To estimate the second term in (3.62), we use (3.63) and Lemma 2.3; we find

$$\begin{aligned}
|\langle \xi, \mathcal{E}_{102}^{(4)} \xi\rangle| &\le C \sum_{q\in\Lambda_+^*} |\eta_H(q)| \|b_{-q}\xi\| \big[|\eta_H(q)| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta_H\| \|b_{-q}\xi\| \big] \\
&\le C\ell^{2\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.
\end{aligned}$$

For the third term in (3.62) we switch to position space and use (2.36) as well as (2.25):

$$\begin{aligned}
|\langle \xi, \mathcal{E}_{103}^{(4)} \xi\rangle| &\le \frac{C}{N} \int dx dy N^{2\beta} V(N^\beta(x-y)) |\check{\eta}_H(x-y)| \\
&\quad \times \int_0^1 ds\, \|(\mathcal{N}+1)^{-1/2} \check{d}_y b^*(\check{\sigma}_x^{(s)})\xi\| \|(\mathcal{N}+1)^{1/2}\xi\| \\
&\le \frac{C}{N} \|\check{\eta}\|_\infty \|\eta\| \int dx dy N^{2\beta} V(N^\beta(x-y)) \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \int_0^1 ds \\
&\quad \times \Big[\|b^*(\check{\sigma}_x^{(s)})\xi\| + \frac{1}{N} |\check{\eta}^{(s)}(x-y)| \|(\mathcal{N}+1)^{1/2}\xi\| + \frac{1}{\sqrt{N}} \|b^*(\check{\sigma}_x^{(s)})\check{b}_y\xi\| \Big] \\
&\le C \frac{(\log N)^2}{N^2} \ell^\alpha \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 .
\end{aligned}$$

Consider now the fourth term in (3.62). We write $\mathcal{E}_{104}^{(4)} = \mathcal{E}_{1041}^{(4)} + \mathcal{E}_{1042}^{(4)}$, with

$$\begin{aligned}\mathcal{E}_{1041}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds\, (\gamma_q^{(s)}-1)d_q b_{-q} \\ \mathcal{E}_{1042}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds\, d_q^{(s)} b_{-q}.\end{aligned}$$

With $|\gamma_q^{(s)}-1| \leq C|\eta_H(q)|^2$, (3.63) and Lemma 2.3, we easily find

$$|\langle \xi, \mathcal{E}_{1041}^{(4)}\xi\rangle| \leq C\ell^{2\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2.$$

While to study the term $\mathcal{E}_{1042}^{(4)}$, we switch to position space. Using (2.25) and (2.36) in Lemma 2.3, we obtain

$$\begin{aligned}|\langle \xi, \mathcal{E}_{1042}^{(4)}\xi\rangle| &= \Big|\frac{1}{2N}\int_0^1 ds \int_{\Lambda^2} dxdy N^{2\beta}V(N^\beta(x-y))\check{\eta}_H(x-y)\langle \xi, \check{d}_x^{(s)}\check{b}_y\xi\rangle\Big| \\ &\leq C\frac{\log N}{N}\int_0^1\int_{\Lambda^2} dxdy N^{2\beta}V(N^\beta(x-y))\|(\mathcal{N}_+ +1)^{1/2}\xi\|\times \\ &\qquad\qquad\qquad\qquad \times \|(\mathcal{N}_+ +1)^{-1/2}\check{d}_x^{(s)}\check{b}_y\xi\| \\ &\leq C\frac{\log N}{N}\|\eta_H\|\int_0^1\int_{\Lambda^2} dxdy N^{2\beta}V(N^\beta(x-y))\|(\mathcal{N}_+ +1)^{1/2}\xi\| \\ &\qquad\qquad\qquad \times N^{-1}\Big[\|\check{a}_y\mathcal{N}_+\xi\| + \|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|\Big] \\ &\leq C\ell^\alpha\frac{\log N}{N}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + C\ell^\alpha\frac{\log N}{N}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{V}_N^\beta)^{1/2}\xi\|.\end{aligned}$$

Let us now consider the last term in (3.62). Switching to position space and using (2.38) in Lemma 2.3 and again (2.25), we arrive at

$$\begin{aligned}|\langle \xi, \mathcal{E}_{105}^{(4)}\xi\rangle| &\leq C\frac{\log N}{N}\int_{\Lambda^2} dxdy\, N^{2\beta}V(N^\beta(x-y))\|(\mathcal{N}_+ +1)^{1/2}\xi\| \\ &\qquad\qquad\qquad\qquad \times \|(\mathcal{N}_+ +1)^{-1/2}\check{d}_x\check{d}_y\xi\| \\ &\leq C\frac{\log N}{N}\|\eta_H\|\|(\mathcal{N}_+ +1)^{1/2}\xi\|\int_{\Lambda^2} dxdy\, N^{2\beta}V(N^\beta(x-y)) \\ &\quad \times \big[(\|\eta_H\|+1)\|(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\eta_H\|\|\check{a}_x\xi\| + \|\check{a}_y\xi\| + N^{-1/2}\|\check{a}_x\check{a}_y\xi\|\big] \\ &\leq C\frac{\log N}{N}\ell^\alpha\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 \\ &\qquad\qquad\qquad + C\frac{\log N}{N}\ell^\alpha\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{V}_N^\beta)^{1/2}\xi\|.\end{aligned}$$

We conclude that the error term (3.61) can be estimated by

$$\pm\mathcal{E}_{10}^{(4)} \leq C\ell^\alpha(\mathcal{H}_N^\beta + 1).$$

Next, we come back to the terms $\mathrm{W}_{11}, \mathrm{W}_{12}, \mathrm{W}_{13}$ defined in (3.60). Using (3.63) and $|\gamma_q^{(s)} - 1| \leq C\eta_H(q)^2$, we can write

$$\mathrm{W}_{11} = \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r)(b_q b_{-q} + \text{h.c.}) + \mathcal{E}_{11}^{(4)} \tag{3.64}$$

where $\mathcal{E}_{11}^{(4)}$ satisfies the estimate

$$\begin{aligned} |\langle \xi, \mathcal{E}_{11}^{(4)} \xi\rangle| &\leq \frac{C}{N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} |\widehat{V}(r/N^\beta)||\eta_H(q+r)||\eta_H(q)|^2 \|b_q\xi\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\leq C\ell^{2\alpha} \|(\mathcal{N}_+ + 1)\xi\|^2. \end{aligned}$$

The second term in (3.60) can be decomposed as

$$\mathrm{W}_{12} = \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r)\eta_H(q)\left(1 - \frac{\mathcal{N}_+}{N}\right) + \mathcal{E}_{12}^{(4)} \tag{3.65}$$

where the error

$$\begin{aligned} \mathcal{E}_{12}^{(4)} = &\frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds(\gamma_q^{(s)}\sigma_q^{(s)} - s\eta_H(q)) \left(1 - \frac{\mathcal{N}_+}{N}\right) \\ &- \frac{1}{2N^2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds \gamma_q^{(s)}\sigma_q^{(s)} a_q^* a_q \end{aligned}$$

can be bounded, using (3.63) and $|\gamma_q^{(s)}\sigma_q^{(s)} - s\eta_H(q))|| \leq C|\eta_q|^3$, by

$$\pm \mathcal{E}_{12}^{(4)} \leq C\ell^\alpha(\mathcal{N}_+ + 1).$$

As for the third term on the r.h.s. of (3.60), we write

$$\mathrm{W}_{13} = -\frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r)\eta_H(q)\left(1 - \frac{\mathcal{N}_+}{N}\right)\frac{\mathcal{N}_+ + 1}{N} + \mathcal{E}_{13}^{(4)} \tag{3.66}$$

where $\mathcal{E}_{13}^{(4)} = \mathcal{E}_{131}^{(4)} + \mathcal{E}_{132}^{(4)} + \mathcal{E}_{133}^{(4)} + \mathcal{E}_{134}^{(4)}$, with

$$\begin{aligned} \mathcal{E}_{131}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds(\gamma_q^{(s)} - 1) b_q d_{-q}^{(s)} + \text{h.c.} \\ \mathcal{E}_{132}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r) \int_0^1 ds\, b_q \left[d_{-q}^{(s)} + s\eta_H(q)\frac{\mathcal{N}_+}{N} b_q^*\right] + \text{h.c.} \\ \mathcal{E}_{133}^{(4)} &= -\frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r)\eta_H(q) b_q^* b_q \frac{\mathcal{N}_+ + 1}{N} \\ \mathcal{E}_{134}^{(4)} &= \frac{1}{2N^2} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N^\beta)\eta_H(q+r)\eta_H(q) a_q^* a_q \frac{\mathcal{N}_+ + 1}{N}. \end{aligned}$$

It is easy to estimate the last two terms: with (3.63), we have

$$\pm\mathcal{E}^{(4)}_{133} \leq C(\log N)/N\ell^\alpha(\mathcal{N}_+ + 1), \qquad \pm\mathcal{E}^{(4)}_{134} \leq C(\log N)/N\ell^\alpha(\mathcal{N}_+ + 1).$$

With $|\gamma^{(s)}_q - 1| \leq C\eta_H(q)^2$, Lemma 2.3 and, again, (3.63), we also find

$$\begin{aligned}
|\langle \xi, \mathcal{E}^{(4)}_{131}\xi\rangle| &\leq \frac{C}{2N}\sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} |\widehat{V}(r/N^\beta)||\eta_H(q+r)||\eta_H(q)|^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\
&\qquad\qquad \times \big[|\eta_q|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta\|\|b_q\xi\|\big] \\
&\leq C\frac{\log N}{N}\ell^{2\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.
\end{aligned}$$

Let us now focus on $\mathcal{E}^{(4)}_{132}$. Switching to position space making use of the notation $\check{\bar{d}}^{(s)}_y = d^{(s)}_y + s(\mathcal{N}_+/N)b^*(\check{\eta}_{H,y})$ and using Lemma 2.3, specifically (2.37), we obtain

$$\begin{aligned}
|\langle \xi, \mathcal{E}^{(4)}_{132}\xi\rangle| &= \Big|\frac{C}{2N}\int_0^1 ds \int_{\Lambda^2} dxdy N^{2\beta}V(N^\beta(x-y))\check{\eta_H}(x-y)\langle \xi, \check{b}_x\check{d}_y\xi\rangle\Big| \\
&\leq C\frac{\log N}{2N}\|\eta_H\|\int_{\Lambda^2} dxdy N^{2\beta}V(N^\beta(x-y))\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\
&\qquad \times \big[(N^{-1} + N^{-1}\log N)\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + N^{-1}\|\check{a}_x\xi\| \\
&\qquad + \|\eta_H\|\|\check{a}_y\xi\| + N^{-1}\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|\big] \\
&\leq C\frac{\log N}{N}\ell^\alpha\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \\
&\qquad + C\frac{\log N}{N}\ell^\alpha\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{V}^\beta_N)^{1/2}\,\xi\|.
\end{aligned}$$

We conclude that $\pm\mathcal{E}^{(4)}_{13} \leq C\ell^\alpha(\mathcal{H}^\beta_N + 1)$. Combining this with (3.64), (3.65), (3.66), we obtain

$$\begin{aligned}
\mathrm{W}_1 &= \frac{1}{2N}\sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q}\widehat{V}(r/N^\beta)\eta_H(q+r)\big(b_q b_{-q} + \text{h.c.}\big) \\
&\quad + \frac{1}{2N}\sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q}\widehat{V}(r/N^\beta)\eta_H(q+r)\eta_H(q)\left(1 - \frac{\mathcal{N}_+}{N}\right)\left(1 - \frac{\mathcal{N}_+ + 1}{N}\right) \\
&\quad + \mathcal{E}^{(4)}_1
\end{aligned} \tag{3.67}$$

with

$$\pm\mathcal{E}^{(4)}_1 \leq C\ell^\alpha(\mathcal{H}^\beta_N + 1).$$

We focus now on the remaining terms in Eq. (3.59). First, consider W_2, we switch to position space and we find

$$\mathrm{W}_2 = \frac{1}{N}\int_{\Lambda^2} dxdy N^{2\beta}V(N^\beta(x-y))\int_0^1 ds\big(e^{-sB_H}\check{b}^*_x\check{b}^*_y e^{sB_H}a^*(\check{\eta}_{H,x})\check{a}_y + \text{h.c.}\big)$$

with the notation $\check{\eta}_{H,x}(z) = \check{\eta}_H(x-z)$. By Cauchy-Schwarz, we have

$$\begin{aligned}|\langle \xi, \mathrm{W}_2 \xi\rangle| \leq \frac{C}{N} \int_{\Lambda^2} dxdy\, N^{2\beta} V(N^\beta(x-y)) \int_0^1 ds \\ \times \|(\mathcal{N}_+ +1)^{1/2} e^{-sB_H} \check{b}_x \check{b}_y e^{sB_H} \xi\| \|(\mathcal{N}_+ +1)^{-1/2} a^*(\check{\eta}_{H,x}) \check{a}_y \xi\|.\end{aligned}$$

With

$$\|(\mathcal{N}_+ +1)^{-1/2} a^*(\check{\eta}_{H,x}) \check{a}_y \xi\| \leq C \|\eta_H\| \|\check{a}_y \xi\| \leq C \ell^\alpha \|\check{a}_y \xi\|$$

and using Lemma 3.8, we obtain

$$\begin{aligned}|\langle \xi, \mathrm{W}_2 \xi\rangle| &\leq \frac{C\ell^\alpha}{N} \int_{\Lambda^2} dxdy\, N^{2\beta} V(N^\beta(x-y)) \|\check{a}_y \xi\| \\ &\quad \times \Big\{ (\log N + N) \|(\mathcal{N}_+ +1)^{1/2} \xi\| + N \|\check{a}_x \xi\| + N \|\check{a}_y \xi\| + N^{1/2} \|\check{a}_x \check{a}_y \xi\| \Big\} \\ &\leq C\ell^\alpha \, \|(\mathcal{N}_+ +1)^{1/2} \xi\|^2 + C\ell^\alpha \|(\mathcal{N}_+ +1)^{1/2} \xi\| \|(\mathcal{V}_N^\beta)^{1/2} \, \xi\|. \end{aligned} \tag{3.68}$$

Also for the term W_3 in (3.59), we switch to position space. We find

$$\begin{aligned}\mathrm{W}_3 = \frac{C}{N} \int_{\Lambda^2} dxdy\, N^{2\beta} V(N^\beta(x-y)) \\ \times \int_0^1 ds \int_0^s d\tau \, \big(e^{-sB_H} \check{b}_x^* \check{b}_y^* e^{sB_H} \, e^{-\tau B_H} b^*(\check{\eta}_{H,x}) b^*(\check{\eta}_{H,y}) e^{\tau B_H} + \text{h.c.} \big)\end{aligned}$$

and so

$$\begin{aligned}|\langle \xi, \mathrm{W}_3 \xi\rangle| \leq \frac{C}{N} \int_{\Lambda^2} dxdy\, N^{2\beta} V(N^\beta(x-y)) \\ \times \int_0^1 ds \int_0^s d\tau \, \|(\mathcal{N}_+ +1)^{1/2} e^{-sB_H} \check{b}_x \check{b}_y e^{sB_H} \xi\| \\ \times \|(\mathcal{N}_+ +1)^{-1/2} e^{-\tau B_H} b^*(\check{\eta}_{H,x}) b^*(\check{\eta}_{H,y}) e^{\tau B_H} \xi\| \, .\end{aligned}$$

With Lemma 2.2, we find

$$\|(\mathcal{N}_+ +1)^{-1/2} e^{-\tau B_H} b^*(\check{\eta}_{H,x})) b^*(\check{\eta}_{H,y}) e^{\tau B_H} \xi\| \leq C \|\eta_H\|^2 \|(\mathcal{N}_+ +1)^{1/2} \xi\|.$$

Using Lemma 3.8, we conclude that

$$\begin{aligned}|\langle \xi, \mathrm{W}_3 \xi\rangle| &\leq \frac{C}{N} \|\eta_H\|^2 \int_{\Lambda^2} dxdy\, N^{2\beta} V(N^\beta(x-y)) \|(\mathcal{N}_+ +1)^{1/2} \xi\| \\ &\quad \times \Big\{ (\log N + N) \|(\mathcal{N}_+ +1)^{1/2} \xi\| + N \|\check{a}_x \xi\| + N \|\check{a}_y \xi\| + N^{1/2} \|\check{a}_x \check{a}_y \xi\| \Big\} \\ &\leq C\ell^{2\alpha} \, \|(\mathcal{N}_+ +1)^{1/2} \xi\|^2 + C\ell^{2\alpha} \|(\mathcal{V}_N^\beta)^{1/2} \, \xi\| \|(\mathcal{N}_+ +1)^{1/2} \xi\|. \end{aligned} \tag{3.69}$$

The term W_4 in (3.59) can be bounded similarly. Switching to position space, we find

$$\begin{aligned}\mathrm{W}_4 = \frac{1}{N} \int dxdy\, N^{2\beta} V(N^\beta(x-y)) \\ \times \int_0^1 ds \int_0^s d\tau \, \big(e^{-sB_H} \check{b}_x^* \check{b}_y^* \, e^{sB_H} \, e^{-\tau B_H} b(\check{\eta}_{H,x}^2) \check{b}_y e^{\tau B_H} + \text{h.c.} \big)\end{aligned}$$

where $\check{\eta_H}^2$ denotes the function with Fourier coefficients $\eta_H^2(q)$, for $q \in \Lambda^*$, and where $\check{\eta}^2_{H,x}(y) := \check{\eta_H}^2(x-y)$. We conclude that $\|\check{\eta}^2_x\| = \|\eta^2_H\| \leq C\ell^{3\alpha}$. With Cauchy-Schwarz, we arrive at

$$\begin{aligned}|\langle \xi, \mathrm{W}_4 \xi\rangle| \leq \frac{C\ell^{3\alpha}}{N} \int_0^1 ds \int_0^s d\tau \int dxdy N^{2\beta} V(N^\beta(x-y)) \\ \times \|(\mathcal{N}_+ + 1)^{1/2} e^{-sB_H} \check{b}_y \check{b}_x e^{sB_H}\xi\| \|\check{b}_y e^{\tau B_H}\xi\|.\end{aligned}$$

Applying Lemma 3.8 and then Lemma 2.2, we obtain

$$\begin{aligned}|\langle \xi, \mathrm{W}_4 \xi\rangle| \leq & \frac{C\ell^{3\alpha}}{N} \int_0^1 ds \int_0^s d\tau \int dxdy N^{2\beta} V(N^\beta(x-y)) \|\check{b}_y e^{\tau B_H}\xi\| \\ & \times \big\{(\log N + N)\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + N\|\check{a}_x \xi\| \\ & \qquad + N\|\check{a}_y\xi\| + N^{1/2}\|\check{a}_x\check{a}_y\xi\|\big\} \\ \leq & \, C\ell^{3\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\ell^{3\alpha}\|(\mathcal{V}_N^\beta)^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|.\end{aligned}$$

Finally, combining (3.67), (3.68), (3.69) together with the last bound, we end up with

$$\begin{aligned}\mathcal{G}_{N,\ell}^{\beta,(4)} = & \, \mathcal{V}_N^\beta + \frac{1}{2N} \sum_{\substack{q\in\Lambda_+^*, r\in\Lambda^*: \\ r\neq -q}} \widehat{V}(r/N^\beta)\eta_H(q+r)\big(b_q b_{-q} + \text{h.c.}\big) \\ & + \frac{1}{2N} \sum_{\substack{q\in\Lambda_+^*, r\in\Lambda^*: \\ r\neq -q}} \widehat{V}(r/N^\beta)\eta_H(q+r)\eta_H(q)\left(1 - \frac{\mathcal{N}_+}{N}\right)\left(1 - \frac{\mathcal{N}_+ + 1}{N}\right) \\ & + \mathcal{E}_{N,\ell}^{\beta,(4)}\end{aligned}$$

where $\mathcal{E}_{N,\ell}^{\beta,(4)}$ satisfies (3.55). As for the bound (3.56), again, we argue similarly as we did at the end of the proof of Prop. 3.3 to show (3.12).

□

3.1.5 Proof of Propositions 2.4

Aim of this subsection is to prove Proposition 2.4. To this end we combine the results of subsections 3.1.1 - 3.1.4. Indeed, from Propositions 3.2, 3.5, 3.6, 3.7, we conclude that the excitation Hamiltonian $\mathcal{G}_{N,\ell}^\beta$ defined in (2.39) is such that

$$\begin{aligned}
\mathcal{G}_{N,\ell}^{\beta} = \; & \frac{\widehat{V}(0)}{2}\,(N+\mathcal{N}_+-1)\Big(1-\frac{\mathcal{N}_+}{N}\Big) \\
& + \sum_{p\in P_H}\eta_p\Big[p^2\eta_p+\widehat{V}(p/N^\beta)+\frac{1}{2N}\sum_{\substack{r\in\Lambda^*\\ p+r\in P_H}}\widehat{V}(r/N^\beta)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big) \\
& + \mathcal{K} + \sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)a_p^*a_p\frac{N-\mathcal{N}_+}{N} \\
& + \sum_{p\in P_H}\Big[\,p^2\eta_p+\frac{1}{2}\widehat{V}(p/N^\beta)+\frac{1}{2N}\sum_{r\in\Lambda^*:\,p+r\in P_H}\widehat{V}(r/N^\beta)\eta_{p+r}\,\Big]\big(b_p^*b_{-p}^*+b_pb_{-p}\big) \\
& + \frac{1}{2}\sum_{p\in P_H^c}\Big[\widehat{V}(p/N^\beta)+\frac{1}{2N}\sum_{r\in\Lambda^*:\,p+r\in P_H}\widehat{V}(r/N^\beta)\eta_{p+r}\Big]\big(b_pb_{-p}+b_{-p}^*b_p^*\big) \\
& + \frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/N^\beta)\,\big[b_{p+q}^*a_{-p}^*a_q+\mathrm{h.c.}\,\big]+\mathcal{V}_N^\beta+\mathcal{E}_1^\beta
\end{aligned} \tag{3.70}$$

where

$$\pm\mathcal{E}_1^\beta \le C\ell^{\alpha-1}(\mathcal{H}_N^\beta+1)$$

and, with the notation $f_M = f(\mathcal{N}_+/M)$,

$$\pm[f_M,[f_M,\mathcal{E}_1^\beta]] \le C\ell^{\alpha-1}M^{-2}\|f'\|_\infty^2\big(\mathcal{H}_N^\beta+1\big)$$

for every f bounded and smooth and $M\in\mathbb{N}$.

First, we want to show (2.42). Using the scattering equation (2.20), we have

$$\begin{aligned}
&\sum_{p\in P_H}\eta_p\Big[p^2\eta_p+\widehat{V}(p/N^\beta)+\frac{1}{2N}\sum_{r\in\Lambda^*:\,p+r\in P_H}\widehat{V}(r/N^\beta)\eta_{p+r}\Big] \\
&\qquad = \sum_{p\in P_H}\eta_p\Big[\,\frac{1}{2}\widehat{V}(p/N^\beta)+N^{1+2\beta}\lambda_\ell\widehat{\chi}_\ell(p)+N^{2\beta}\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q\,\Big] \\
&\qquad\quad - \frac{1}{2N}\sum_{\substack{p,q\in\Lambda^*:\\ p\in P_H,\,q\in P_H^c}}\widehat{V}((p-q)/N^\beta)\eta_p\eta_q.
\end{aligned}$$

With Lemma 2.1 and estimating

$$\|\widehat{\chi}_\ell\| = \|\chi_\ell\| \le C\ell, \qquad \|\eta_H\|\le\ell^\alpha, \qquad \|\widehat{\chi}_\ell*\eta_H\| = \|\chi_\ell\check{\eta}_H\| \le \|\check{\eta}_H\|\le\ell^\alpha,$$

we conclude that

$$\begin{aligned}
&\sum_{p\in P_H}\eta_p\Big[p^2\eta_p+\widehat{V}(p/N^\beta)+\frac{1}{2N}\sum_{\substack{r\in\Lambda^*\\ p+r\in P_H}}\widehat{V}(r/N^\beta)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big) \\
&\qquad = \frac{1}{2}\sum_{p\in P_H}\widehat{V}(p/N^\beta)\eta_p\left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{N-\mathcal{N}_+-1}{N}\right)+\mathcal{E}_2^\beta
\end{aligned}$$

with $\pm\mathcal{E}_2^\beta \le C(|\log \ell^\alpha| N^{-1}\log N + \ell^{\alpha-1} + \ell^{2\alpha-2}N^{-1})$. Since $\sum_{p\in P_H^c} |\widehat{V}(p/N^\beta)||\eta_p| \le C|\log \ell^\alpha|$, and from (2.14), we further obtain

$$\begin{aligned}\sum_{p\in P_H} \eta_p \Big[p^2\eta_p + \widehat{V}(p/N^\beta) + \frac{1}{2N}\sum_{\substack{r\in\Lambda^*\\ p+r\in P_H}} \widehat{V}(r/N^\beta)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big)\\ = \left[-\frac{\widehat{V}^2(0)}{8\pi N}\log N^\beta\right](N-\mathcal{N}_+-1)\left(\frac{N-\mathcal{N}_+}{N}\right) + \mathcal{E}_3^\beta\end{aligned} \tag{3.71}$$

where $\pm\mathcal{E}_3^\beta \le C(|\log \ell| N^{-1}\log N + \ell^{\alpha-1} + \ell^{2\alpha-2}N^{-1})$.

Using once more (2.20), we can also handle the fourth line of (3.70); we find

$$\begin{aligned}\sum_{p\in P_H}\Big[p^2\eta_p + \frac{1}{2}\widehat{V}(p/N^\beta) + \frac{1}{2N}\sum_{r\in\Lambda^*:\, p+r\in P_H}\widehat{V}(r/N^\beta)\eta_{p+r}\Big](b_p^*b_{-p}^* + b_pb_{-p})\\ = \sum_{p\in P_H}\Big[N^{1+2\beta}\lambda_\ell\widehat{\chi}_\ell(p) + N^{2\beta}\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q\Big](b_p^*b_{-p}^* + b_pb_{-p})\\ - \frac{1}{2N}\sum_{\substack{p,q\in\Lambda^*:\\ p\in P_H,\, q\in P_H^c}}\widehat{V}((p-q)/N^\beta)\eta_q(b_p^*b_{-p}^* + b_pb_{-p}).\end{aligned} \tag{3.72}$$

Notice that

$$\begin{aligned}\Big|\langle\xi, N^{1+2\beta}\lambda_\ell\sum_{p\in P_H}\widehat{\chi}_\ell(p)b_pb_{-p}\xi\rangle\Big| &\le C\ell^{-2}\|(\mathcal{N}_++1)^{1/2}\xi\|\sum_{p\in P_H}|\widehat{\chi}_\ell(p)|\|b_p\xi\|\\ &\le C\ell^{-2+1/2+3/2\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\\ &\le C\ell^{3/2(\alpha-1)}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned}$$

Using $\widehat{\chi}_\ell * \eta = \eta$ (because $\chi_\ell(x)w_\ell(x) = w_\ell(x)$ in position space), we can also bound

$$\Big|\langle\xi, N^{2\beta}\lambda_\ell\sum_{p\in P_H, q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q(b_p^*b_{-p}^* + b_pb_{-p})\xi\rangle\Big| \le CN^{-1}\ell^{-2+\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2.$$

Furthermore, we have

$$\begin{aligned}&\Big|\langle\xi, \frac{1}{2N}\sum_{\substack{p,q\in\Lambda^*:\\ p\in P_H,\, q\in P_H^c}}\widehat{V}((p-q)/N^\beta)\eta_q b_pb_{-p}\xi\rangle\Big|\\ &\le \frac{C}{N}\Big[\sum_{\substack{p,q\in\Lambda^*:\\ p\in P_H, q\in P_H^c}}\frac{1}{|q|^2}\frac{|\widehat{V}((p-q)/N^\beta)|^2}{|p^2|}\Big]^{1/2}\\ &\qquad\times\Big[\sum_{\substack{p,q\in\Lambda^*:\\ p\in P_H,\, q\in P_H^c}}\frac{1}{|q|^2}|p|^2\|b_p\xi\|^2\Big]^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\\ &\le C|\log\ell|(\log N)^{1/2}N^{-1}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|.\end{aligned} \tag{3.73}$$

We conclude that Eq. (3.72) can be included in the error term as

$$\begin{aligned}\pm \sum_{p\in P_H}\Big[p^2\eta_p + \frac{1}{2}\widehat{V}(p/N^\beta) + \frac{1}{2N}\sum_{\substack{r\in\Lambda^*:\\ p+r\in P_H}} \widehat{V}(r/N^\beta)\eta_{p+r}\Big]\big(b_p^* b_{-p}^* + b_p b_{-p}\big)\\ \le C(\ell^{3/2(\alpha-1)} + \ell^{\alpha-2}N^{-1} + |\log \ell|(\log N)^{1/2}N^{-1})(\mathcal{K}+1)\\ \le C\ell^{3/2(\alpha-1)}(\mathcal{K}+1)\end{aligned} \tag{3.74}$$

for N large enough. Proceeding in the same way for the fifth line on the r.h.s. of (3.70), we can write it as

$$\begin{aligned}\frac{1}{2}\sum_{p\in P_H^c}\Big[\widehat{V}(p/N^\beta) + \frac{1}{N}\sum_{r\in\Lambda^*:\, p+r\in P_H}\widehat{V}(r/N^\beta)\eta_{p+r}\Big]\big(b_p b_{-p} + b_{-p}^* b_p^*\big)\\ = \frac{1}{2}\sum_{p\in P_H^c}(\widehat{V}(\cdot/N^\beta) * \widehat{f}_{N,\ell})(p)\big(b_p b_{-p} + b_{-p}^* b_p^*\big) + \mathcal{E}_4^\beta\end{aligned} \tag{3.75}$$

where the error operator

$$\mathcal{E}_4^\beta = \frac{1}{2N}\sum_{\substack{p,q\in\Lambda^*:\\ p,\,q\in P_H^c}} \widehat{V}((p-q)/N^\beta)\eta_q\big(b_p b_{-p} + b_{-p}^* b_p^*\big)$$

can be bounded by $\pm\mathcal{E}_4^\beta \le C|\log\ell|(\log N)^{1/2}N^{-1}(\mathcal{K}+1)$, similarly as in (3.73).
Combining (3.70) with (3.71), (3.74) and (3.75), we conclude that

$$\begin{aligned}\mathcal{G}_{N,\ell}^\beta = &\Big[\frac{\widehat{V}(0)}{2} - \frac{\widehat{V}(0)^2}{8\pi N}\log N^\beta\Big](N-1)\left(\frac{N-\mathcal{N}_+}{N}\right)\\ &+\left[\frac{\widehat{V}(0)}{2} + \frac{\widehat{V}(0)^2}{8\pi N}\log N^\beta\right]\mathcal{N}_+\left(\frac{N-\mathcal{N}_+}{N}\right)\\ &+\mathcal{K} + \sum_{p\in\Lambda_+^*}\widehat{V}(p/N^\beta)a_p^* a_p\frac{N-\mathcal{N}_+}{N} + \frac{1}{2}\sum_{p\in P_H^c}(\widehat{V}(\cdot/N^\beta) * \widehat{f}_{N,\ell})_p\big(b_p b_{-p} + b_{-p}^* b_p^*\big)\\ &+\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*: p+q\neq 0}\widehat{V}(p/N^\beta)\left[b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\right] + \mathcal{V}_N^\beta + \mathcal{E}_5^\beta\end{aligned}$$

with

$$\pm\mathcal{E}_5^\beta \le C\ell^{\alpha-1}(\mathcal{H}_N^\beta + 1) + C|\log\ell|.$$

Observing that

$$\pm\sum_{p\in P_H}\widehat{V}(p/N^\beta)a_p^* a_p \le C\ell^{2\alpha}(\mathcal{K}+1)\,,$$

that $|\widehat{V}(p/N^\beta) - \widehat{V}(0)| \le C|p|N^{-\beta}$, and that, by (2.14),

$$\begin{aligned}&|(\widehat{V}(\cdot/N^\beta) * \widehat{f}_{N,\ell})_p - \widehat{V}(0)|\\ &\le \int dx\, N^{2\beta}V(N^\beta x)f_\ell(N^\beta x)\big|e^{ip\cdot x} - 1\big| + \left|\int N^{2\beta}V(N^\beta x)f_\ell(N^\beta x) - \widehat{V}(0)\right|\\ &\le C(|p|N^{-\beta} + (\log N^\beta)/N)\end{aligned} \tag{3.76}$$

we arrive, with $\mathcal{G}_{N,\ell}^{\beta,\text{eff}}$ defined as in (2.41), at $\mathcal{G}_{N,\ell}^{\beta} = \mathcal{G}_{N,\ell}^{\beta,\text{eff}} + \mathcal{E}_{N,\ell}^{\beta}$, with an error $\mathcal{E}_{N,\ell}^{\beta}$ that satisfies

$$\pm\mathcal{E}_{N,\ell}^{\beta} \leq C\ell^{\alpha-1}\mathcal{H}_N^{\beta} + C|\log \ell|$$

for all N large enough. This completes the proof of (2.42). The second bound in (2.43) follows similarly, arguing as we did at the end of Prop. 3.3.

3.2 Analysis of the cubically renormalized excitation Hamiltonian $\mathcal{R}_{N,\ell}^{\beta}$

In this section we want to prove Proposition 2.8, which gives a lower bound on the excitation Hamiltonian $\mathcal{R}_{N,\ell}^{\beta} = e^{-A_H}\mathcal{G}_{N,\ell}^{\beta,\text{eff}}e^{A_H}$, with $\mathcal{G}_{N,\ell}^{\beta,\text{eff}}$ as in (2.41) and the cubic phase

$$A_H = \frac{1}{\sqrt{N}} \sum_{r\in P_H, v\in P_H^c} \eta_r \left[b_{r+v}^* a_{-r}^* a_v - \text{h.c.}\right] \tag{3.77}$$

introduced in (2.54). Here we indicate the high momentum set $P_H = \{p \in \Lambda_+^* : |p| \geq \ell^{-\alpha}\}$ and the complementary set $P_H^c = \{p \in \Lambda_+^* : |p| \leq \ell^{-\alpha}\}$ for $\alpha > 0$. To this aim, we decompose for convenience

$$\mathcal{G}_{N,\ell}^{\beta,\text{eff}} = \mathcal{O}_N + \mathcal{K} + \mathcal{Q}_N + \mathcal{C}_N + \mathcal{V}_N^{\beta}$$

with $\mathcal{K}$ and $\mathcal{V}_N^{\beta}$ being the kinetic and the potential energy operators, as in (2.40), and

$$\begin{aligned}
\mathcal{O}_N &= \left[\frac{\widehat{V}(0)}{2} + \frac{\widehat{V}(0)^2}{8\pi N}\log N^{\beta}\right]\mathcal{N}_+\left(\frac{N-\mathcal{N}_+}{N}\right) \\
&\quad + \left[\frac{\widehat{V}(0)}{2} - \frac{\widehat{V}(0)^2}{8\pi N}\log N^{\beta}\right](N-1)\left(\frac{N-\mathcal{N}_+}{N}\right) \\
\mathcal{Q}_N &= \widehat{V}(0)\sum_{p\in P_H^c} a_p^* a_p\left(1-\frac{\mathcal{N}_+}{N}\right) + \frac{\widehat{V}(0)}{2}\sum_{p\in P_H^c}\left[b_p^* b_{-p}^* + b_p b_{-p}\right] \\
\mathcal{C}_N &= \frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/N^{\beta})\left[b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\right].
\end{aligned} \tag{3.78}$$

with $P_H^c = \Lambda_+^* \backslash P_H$. To study the contributions of these operators to $\mathcal{R}_{N,\ell}^{\beta}$ and so to prove Proposition 2.8 we will need a-priori bounds controlling the growth of the expectation of the energy $\mathcal{H}_N^{\beta} = \mathcal{K} + \mathcal{V}_N^{\beta}$ through cubic conjugation, in the next subsection we obtain these bounds. Throughout this section, as in section 3.1, we will always assume that $V \in L^2(\mathbb{R}^2)$ is compactly supported, pointwise non-negative and spherically symmetric.

As we already stressed at the beginning of this chapter, this analysis follows from the one in [10], with some slight modifications. In particular, our analysis is simplified by the fact that in two-dimensions we do not need to introduce a cut-off on low momenta, restricting the momentum v appearing in Eq. (3.77) to a subset of P_H^c. On the contrary, in [10], the authors need to restrict v to the set of low-momenta $P_L = \{p \in \Lambda_+^* : \ |p| \leq \ell^{-\beta}\}$, with $\beta < \alpha$, rather than considering v in $P_H^c = \{p \in \Lambda_+^* : \ |p| \leq \ell^{-\alpha}\}$. Furthermore, the growth of the total energy operator $\mathcal{H}_N^\beta$ as stated in Prop. 2.7, is the best that we can achieve (due to the logarithmic growth, and properties of the logarithm). On the other hand in [10, Section 8.1], the authors need to work harder to control the growth of the total energy (a-priori bounds on the growth of the kinetic energy operator on low-momenta $\mathcal{K}_L = \sum_{|p|\leq \ell^{-\nu}} p^2 a_p^* a_p$, allow them to treat better the growth of the total energy operator).

3.2.1 A priori bounds on the energy

To get a-priori estimates on the growth of the expectation of the energy operator, we proceed first in controlling the commutator of the cubic phase (3.77) with the potential energy operator $\mathcal{V}_N^\beta$.

Proposition 3.9. *There exists a constant $C > 0$ such that*

$$[\mathcal{V}_N^\beta, A_H] = \frac{1}{N^{3/2}} \sum_{\substack{r \in \Lambda_+^*, v \in P_H^c \\ r \neq -v}} (\widehat{V}(\cdot/N^\beta) * \eta)(r) \big[b_{r+v}^* a_{-r}^* a_v + \text{h.c.} \big] + \delta_{\mathcal{V}_N^\beta} \tag{3.79}$$

where

$$|\langle \xi, \delta_{\mathcal{V}_N^\beta} \xi \rangle| \leq C(|\log \ell|^{1/2} \ell^\alpha + \ell^\alpha) \|(\mathcal{H}_N^\beta + 1)^{1/2} \xi\|^2 \tag{3.80}$$

for all $\alpha > 0$, $\ell \in (0; 1/2)$ and N large enough.

Proof. With

$$\begin{aligned}
&[a_{p+u}^* a_q^* a_p a_{q+u}, b_{r+v}^* a_{-r}^* a_v] \\
&\quad = [a_{p+u}^* a_q^* a_p a_{q+u}, a_{r+v}^*] \sqrt{1 - (\mathcal{N}_+/N)} a_{-r}^* a_v + b_{r+v}^* [a_{p+u}^* a_q^* a_p a_{q+u}, a_{-r}^* a_v] \\
&\quad = b_{p+u}^* a_q^* a_{q+u} a_{-r}^* a_v \delta_{p,r+v} + b_{p+u}^* a_q^* a_p a_{-r}^* a_v \delta_{q+u,r+v} \\
&\qquad + b_{r+v}^* a_{p+u}^* a_q^* a_p a_v \delta_{-r,q+u} + b_{r+v}^* a_{p+u}^* a_q^* a_{q+u} a_v \delta_{-r,p} \\
&\qquad - b_{r+v}^* a_{-r}^* a_{p+u}^* a_p a_{q+u} \delta_{q,v} - b_{r+v}^* a_{-r}^* a_q^* a_p a_{q+u} \delta_{v,p+u}
\end{aligned}$$

and normal ordering the first two terms, we obtain

$$[\mathcal{V}_N^\beta, A_H] = \frac{1}{N^{3/2}} \sum_{u \in \Lambda^*, r \in P_H, v \in P_H^c}^{*} \widehat{V}((u-r)/N^\beta) \eta_r b_{u+v}^* a_{-u}^* a_v + \Theta_2 + \Theta_3 + \Theta_4 + \text{h.c.}$$

with

$$\begin{aligned}
\Theta_2 &:= \frac{1}{N^{3/2}} \sum^*_{\substack{u\in\Lambda^*, p\in\Lambda^*_+,\\ r\in P_H, v\in P_H^c}} \widehat{V}(u/N^\beta)\eta_r b^*_{p+u} a^*_{r+v-u} a^*_{-r} a_p a_v \\
\Theta_3 &:= \frac{1}{N^{3/2}} \sum^*_{\substack{u\in\Lambda^*, p\in\Lambda^*_+,\\ r\in P_H, v\in P_H^c}} \widehat{V}(u/N^\beta)\eta_r b^*_{r+v} a^*_{p+u} a^*_{-r-u} a_p a_v \\
\Theta_4 &:= -\frac{1}{N^{3/2}} \sum^*_{\substack{u\in\Lambda^*, p\in\Lambda^*_+,\\ r\in P_H, v\in P_H^c}} \widehat{V}(u/N^\beta)\eta_r b^*_{r+v} a^*_{-r} a^*_{p+u} a_p a_{v+u}.
\end{aligned} \tag{3.81}$$

Where with $\sum^*$ we indicate that we exclude choices of momenta for which the argument of a creation or annihilation operator vanishes. Writing

$$\begin{aligned}
&\frac{1}{N^{3/2}} \sum^*_{\substack{u\in\Lambda^*\\ r\in P_H, v\in P_H^c}} \widehat{V}((u-r)/N^\beta)\eta_r b^*_{u+v} a^*_{-u} a_v \\
&\qquad = \frac{1}{N^{3/2}} \sum^*_{\substack{u,r\in\Lambda^*,\\ v\in P_H^c}} \widehat{V}((u-r)/N^\beta)\eta_r b^*_{u+v} a^*_{-u} a_v \\
&\qquad\quad - \frac{1}{N^{3/2}} \sum^*_{\substack{u\in\Lambda^*, v\in P_H^c,\\ r\in P_H^c\cup\{0\}}} \widehat{V}((u-r)/N^\beta)\eta_r b^*_{u+v} a^*_{-u} a_v
\end{aligned}$$

and comparing with (3.79), we conclude that $\delta_{\mathcal{V}_N^\beta} = \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \text{h.c.}$, with

$$\Theta_1 = -\frac{1}{N^{3/2}} \sum^*_{\substack{u\in\Lambda^*, v\in P_H^c,\\ r\in P_H^c\cup\{0\}}} \widehat{V}((u-r)/N^\beta)\eta_r b^*_{u+v} a^*_{-u} a_v$$

and with $\Theta_2, \Theta_3, \Theta_4$ as defined in (3.81).

To conclude the proof of the lemma, we show next that each error term Θ_j, with $j = 1, 2, 3, 4$, satisfies (3.80). We start with Θ_1. For any $\xi \in \mathcal{F}_+^{\leq N}$, switching (partly) to position space and applying Cauchy-Schwarz, we find

$$\begin{aligned}
|\langle \xi, \Theta_1 \xi\rangle| &\leq \frac{1}{\sqrt{N}} \Big[\int_{\Lambda^2} dxdy\; N^{2\beta-1} V(N^\beta(x-y)) \sum_{\substack{r\in\{0\}\cup P_H^c,\\ v\in P_H^c}} |\eta_r| |v|^{-2} \|\check{b}_x \check{a}_y \xi\|^2 \Big]^{1/2} \\
&\qquad \times \Big[\int_{\Lambda^2} dxdy\; N^{2\beta-1} V(N^\beta(x-y)) \sum_{\substack{r\in\{0\}\cup P_H^c,\\ v\in P_H^c}} |\eta_r| |v|^2 \|a_v \xi\|^2 \Big]^{1/2} \\
&\leq \frac{C}{N} |\log \ell|^{3/2} \|(\mathcal{V}_N^\beta)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\|.
\end{aligned} \tag{3.82}$$

Denoting by $\check{\eta}_H \in L^2(\Lambda)$ the function with Fourier coefficients $\eta_H(p) = \eta_p \chi(p \in P_H)$ and using (2.23), we can bound the term Θ_2 on the r.h.s. of (3.81) by

$$\begin{aligned}|\langle \xi, \Theta_2 \xi \rangle| &= \left| \frac{1}{N^{1/2}} \int_{\Lambda^2} dxdy\ N^{2\beta-1} V(N^\beta(x-y)) \sum_{v \in P_H^c} e^{ivy} \langle \xi, \check{b}_x^* \check{a}_y^* a^*(\check{\eta}_{H,y}) \check{a}_x a_v \xi \rangle \right| \\ &\leq \frac{\|\check{\eta}_H\|}{N^{1/2}} \left[\int_{\Lambda^2} dxdy\ N^{2\beta-1} V(N^\beta(x-y)) \sum_{v \in P_H^c} |v|^{-2} \|\mathcal{N}_+^{1/2} \check{b}_x \check{a}_y \xi\|^2 \right]^{1/2} \\ &\qquad \times \left[\int_{\Lambda^2} dxdy\ N^{2\beta-1} V(N^\beta(x-y)) \sum_{v \in P_H^c} |v|^2 \|\check{a}_x a_v \xi\|^2 \right]^{1/2} \\ &\leq C\ell^\alpha |\log \ell|^{1/2} \|(\mathcal{V}_N^\beta)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\|.\end{aligned}$$

The remaining contributions Θ_3 and Θ_4 can be controlled similarly. We find

$$\begin{aligned}|\langle \xi, \Theta_3 \xi \rangle| &= \left| \frac{1}{N^{1/2}} \int_{\Lambda^2} dxdy\ N^{2\beta-1} V(N^\beta(x-y)) \sum_{\substack{r \in P_H, \\ v \in P_H^c}} e^{-iry} \eta_r \langle \xi, b_{r+v}^* \check{a}_x^* \check{a}_y^* \check{a}_x a_v \xi \rangle \right| \\ &\leq \frac{1}{N^{1/2}} \left[\int_{\Lambda^2} dxdy\ N^{2\beta-1} V(N^\beta(x-y)) \sum_{\substack{r \in P_H, \\ v \in P_H^c}} |v|^{-2} \|b_{r+v} \check{a}_x \check{a}_y \xi\|^2 \right]^{1/2} \\ &\qquad \times \left[\int_{\Lambda^2} dxdy\ N^{2\beta-1} V(N^\beta(x-y)) \sum_{\substack{r \in P_H, \\ v \in P_H^c}} \eta_r^2 |v|^2 \|\check{a}_x a_v \xi\|^2 \right]^{1/2} \\ &\leq \frac{C|\log \ell|^{1/2} \|\eta_H\|}{N^{1/2}} \|\mathcal{N}_+^{1/2} (\mathcal{V}_N^\beta)^{1/2} \xi\| \|\mathcal{N}_+^{1/2} \mathcal{K}^{1/2} \xi\| \\ &\leq C\ell^\alpha |\log \ell|^{1/2} \|(\mathcal{V}_N^\beta)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\|\end{aligned}$$

as well as

$$\begin{aligned}|\langle \xi, \Theta_4 \xi \rangle| &= \left| \frac{1}{N^{1/2}} \int_{\Lambda^2} dxdy\ N^{2\beta-1} V(N^\beta(x-y)) \sum_{\substack{r \in P_H, \\ v \in P_H^c}} \eta_r e^{-ivy} \langle \xi, b_{r+v}^* a_{-r}^* \check{a}_x^* \check{a}_x \check{a}_y \xi \rangle \right| \\ &\leq \frac{1}{N^{1/2}} \left[\int_{\Lambda^2} dxdy\ N^{2\beta-1} V(N^\beta(x-y)) \sum_{\substack{r \in P_H, \\ v \in P_H^c}} |r|^{-2} |\eta_r|^2 \|\check{a}_x \check{a}_y \xi\|^2 \right]^{1/2} \\ &\qquad \times \left[\int_{\Lambda^2} dxdy\ N^{2\beta-1} V(N^\beta(x-y)) \sum_{\substack{r \in P_H, \\ v \in P_H^c}} |r|^2 \|b_{r+v} a_{-r} \check{a}_x \xi\|^2 \right]^{1/2} \\ &\leq C\ell^\alpha \|(\mathcal{V}_N^\beta)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\|.\end{aligned}$$

Choosing $N > |\log \ell|^{3/2}$ (to control the r.h.s. of (3.82)), we obtain (3.80). □

With the help of Prop. 3.9, we are now ready to prove Prop. 2.7.

Proof of Prop. 2.7. We apply Gronwall's lemma. For a fixed $\xi \in \mathcal{F}_+^{\leq N}$ and $s \in [0;1]$, we define

$$f_\xi(s) := \langle \xi, e^{-sA_H} \mathcal{H}_N^\beta \, e^{sA_H} \xi \rangle.$$

Then

$$f'_\xi(s) = \langle \xi, e^{-sA_H}[\mathcal{K}, A_H] e^{sA_H} \xi \rangle + \langle \xi, e^{-sA_H}[\mathcal{V}_N^\beta, A_H] e^{sA_H} \xi \rangle. \tag{3.83}$$

Let us first consider the second term. From Prop. 3.9, we find

$$[\mathcal{V}_N^\beta, A_H] = \frac{1}{N^{3/2}} \sum_{\substack{r \in \Lambda_+^* \\ v \in P_H^c, r \neq -v}} (\widehat{V}(\cdot/N^\beta) * \eta)(r) \left[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \right] + \delta_{\mathcal{V}_N^\beta}$$

where the operator $\delta_{\mathcal{V}_N^\beta}$ satisfies (3.80). Switching to position space and applying Cauchy-Schwarz, we find

$$\begin{aligned}
&\left| \frac{1}{N^{3/2}} \sum_{r \in \Lambda_+^*, v \in P_H^c, r \neq -v} (\widehat{V}(\cdot/N^\beta) * \eta)(r) \langle \xi, e^{-sA_H} b^*_{r+v} a^*_{-r} a_v e^{sA_H} \xi \rangle \right| \\
&= \left| \frac{1}{N^{1/2}} \int_{\Lambda^2} dxdy \; N^{2\beta-1} V(N^\beta(x-y)) \check{\eta}(x-y) \sum_{v \in P_H^c} e^{ivx} \langle \xi, e^{-sA_H} \check{a}^*_x \check{a}^*_y a_v e^{sA_H} \xi \rangle \right| \\
&\leq \frac{C \|\check{\eta}\|_\infty}{N^{1/2}} \|(\mathcal{V}_N^\beta)^{1/2} e^{sA_H} \xi\| \left[\int_{\Lambda^2} dxdy \; N^{2\beta-1} V(N^\beta(x-y)) \Big\| \sum_{v \in P_H^c} e^{ivx} a_v e^{sA_H} \xi \Big\|^2 \right]^{1/2} \\
&\leq C \frac{\log N}{N^{1/2}} \|(\mathcal{V}_N^\beta)^{1/2} e^{sA_H} \xi\| \|\mathcal{N}_+^{1/2} e^{sA_H} \xi\|
\end{aligned} \tag{3.84}$$

because, by (2.25), $\|\check{\eta}\|_\infty \leq C \log N$ and

$$\int_\Lambda dx \Big\| \sum_{v \in P_H^c} e^{ivx} a_v e^{sA_H} \xi \Big\|^2 = \sum_{v \in P_L} \langle e^{sA_H} \xi, a_v^* a_v e^{sA_H} \xi \rangle \leq \langle e^{sA_H} \xi, \mathcal{N}_+ e^{sA_H} \xi \rangle.$$

Together with (3.80), for $\alpha > 0$, we conclude that

$$\left| \langle \xi, e^{-sA_H} [\mathcal{V}_N^\beta, A_H] e^{sA_H} \xi \rangle \right| \leq C \langle \xi, e^{-sA_H} \mathcal{H}_N^\beta \, e^{sA_H} \xi \rangle \tag{3.85}$$

if N is large enough. Let us consider the first term on the r.h.s. of (3.83). We compute

$$\begin{aligned}
[\mathcal{K}, A_H] &= \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_H^c} 2r^2 \eta_r \left[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \right] \\
&\qquad + \frac{2}{\sqrt{N}} \sum_{r \in P_H, v \in P_H^c} r \cdot v \; \eta_r \left[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \right] \\
&=: \mathrm{T}_1 + \mathrm{T}_2.
\end{aligned} \tag{3.86}$$

We use (2.20) to rewrite the first term on the r.h.s. of (3.86) as

$$\begin{aligned}\mathrm{T}_1 = &-\frac{1}{\sqrt{N}} \sum_{\substack{r\in\Lambda_+^*, v\in P_H^c,\\ r\neq -v}} (\widehat{V}(\cdot/N^\beta) * \widehat{f}_{N,\ell})(r)\big[b_{r+v}^* a_{-r}^* a_v + \text{h.c.}\big] \\ &+ \frac{1}{\sqrt{N}} \sum_{\substack{r,v\in P_H^c,\\ r\neq -v}} (\widehat{V}(\cdot/N^\beta) * \widehat{f}_{N,\ell})(r)\big[b_{r+v}^* a_{-r}^* a_v + \text{h.c.}\big] \\ &+ \frac{1}{\sqrt{N}} \sum_{r\in P_H, v\in P_H^c} N^{1+2\beta}\lambda_\ell(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(r)\big[b_{r+v}^* a_{-r}^* a_v + \text{h.c.}\big] \\ =:&\ \mathrm{T}_{11} + \mathrm{T}_{12} + \mathrm{T}_{13}\end{aligned} \tag{3.87}$$

Since $\|f_\ell\|_\infty \leq 1$ and using Lemma 2.1, part iii), the contribution of T_{11} can be estimated similarly as in (3.84); we obtain

$$\big|\langle \xi, e^{-sA_H}\mathrm{T}_{11}\, e^{sA_H}\xi\rangle\big| \leq C\|(\mathcal{V}_N^\beta)^{1/2}\, e^{sA_H}\xi\|\|\mathcal{N}_+^{1/2} e^{sA_H}\xi\|.$$

The second term in (3.87) can be controlled by

$$\begin{aligned}\big|\langle \xi, e^{-sA_H}\mathrm{T}_{12}\, e^{sA_H}\xi\rangle\big| &\leq \frac{1}{N^{1/2}}\Bigg[\sum_{r,v\in P_H^c, r\neq -v} |r|^2\|b_{r+v}a_{-r}e^{sA_H}\xi\|^2\Bigg]^{1/2} \\ &\qquad\qquad \times \Bigg[\sum_{r,v\in P_H^c, r\neq -v} |r|^{-2}\|a_v e^{sA_H}\xi\|^2\Bigg]^{1/2} \\ &\leq C|\log\ell|^{1/2}\|\mathcal{K}^{1/2}e^{sA_H}\xi\|\|\mathcal{N}_+^{1/2}e^{sA_H}\xi\|.\end{aligned}$$

Finally, since $(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(p) = \widehat{\chi}_\ell(p) + N^{-1}\eta_r$, the explicit expression in spherical coordinates

$$\widehat{\chi}_\ell(p) = 2\pi\int_0^\ell\int_0^\pi e^{-i|p|\cdot r\cos(\theta)} r dr d\theta = 2\pi\ell\frac{J_1(\ell|p|)}{|p|},$$

where J_1 is the Bessel function of the first kind, and for high momenta can be bounded by ([56])

$$\widehat{\chi}_\ell(p) \leq \frac{C\ell^{1/2}}{|p|^{3/2}}$$

and the bound (2.18) imply that $|(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(p)| \leq C\ell^{1/2}|p|^{-3/2}$, for N large enough. With Lemma 2.1, the third term on the r.h.s. of (3.87) can thus be estimated by

$$\begin{aligned}&\big|\langle \xi, e^{-sA_H}\mathrm{T}_{13}e^{sA_H}\xi\rangle\big| \\ &\quad\leq C\ell^{-2+1/2}N^{-1/2}\Bigg[\sum_{r\in P_H, v\in P_H^c} |r|^2\|\|b_{r+v}a_{-r}e^{sA_H}\xi\|^2\Bigg]^{1/2}\Bigg[\sum_{r\in P_H, v\in P_H^c} |r|^{-5}\|a_v e^{sA_H}\xi\|^2\Bigg]^{1/2} \\ &\quad\leq C\ell^{3/2(\alpha-1)}\|\mathcal{K}^{1/2}e^{sA_H}\xi\|\|\mathcal{N}_+^{1/2}e^{sA_H}\xi\| \leq C\|\mathcal{K}^{1/2}e^{sA_H}\xi\|\|\mathcal{N}_+^{1/2}e^{sA_H}\xi\|.\end{aligned} \tag{3.88}$$

So far, we proved that

$$|\langle \xi, \mathrm{T}_1 \xi \rangle| \le C|\log \ell|^{1/2} \|(\mathcal{H}_N^\beta)^{1/2} e^{sA_H} \xi\| \|\mathcal{N}_+^{1/2} e^{sA_H} \xi\| \tag{3.89}$$

for all $\xi \in \mathcal{F}_+^{\le N}$. Let us now consider the second term on the r.h.s. of (3.86). We find

$$\begin{aligned} &\big|\langle \xi, e^{-sA_H} \mathrm{T}_2 e^{sA_H} \xi \rangle\big| \\ &\le \frac{C}{\sqrt{N}} \Big[\sum_{r \in P_H, v \in P_H^c} |r|^2 \| b_{r+v} a_{-r} e^{sA_H} \xi\|^2 \Big]^{1/2} \Big[\sum_{r \in P_H, v \in P_H^c} |v|^2 \eta_r^2 \| a_v e^{sA_H} \xi\|^2 \Big]^{1/2} \\ &\le C\ell^\alpha \|\mathcal{K}^{1/2} e^{sA_H} \xi\|^2 . \end{aligned} \tag{3.90}$$

Together with (3.89), we conclude that

$$|\langle \xi, e^{-sA_H} [\mathcal{K}, A_H] e^{sA_H} \xi \rangle| \le C \langle \xi, e^{-sA_H} \mathcal{H}_N^\beta e^{sA_H} \xi \rangle + C|\log \ell| \langle \xi, e^{-sA_H} \mathcal{N}_+ e^{-sA_H} \xi \rangle. \tag{3.91}$$

With Eq. (3.85), Eq. (3.91), and Prop. 2.6, we obtain the differential inequality

$$|f_\xi'(s)| \le C f_\xi(s) + C|\log \ell| \langle \xi, (\mathcal{N}_+ + 1)\xi \rangle.$$

By Gronwall's Lemma, we find (2.55). □

3.2.2 Analysis of $e^{-A_H} \mathcal{O}_N e^{A_H}$

In this section we study the contribution to $\mathcal{R}_{N,\ell}^\beta$ arising from the operator $\mathcal{O}_N$, defined in (3.78). To this end, it is convenient to use the following lemma.

Lemma 3.10. *There exists a constant $C > 0$ such that*

$$\begin{aligned} &\Big| \sum_{p \in \Lambda_+^*} F_p \langle \xi_1, (e^{-A_H} a_p^* a_p e^{A_H} - a_p^* a_p) \xi_2 \rangle \Big| \\ &\qquad\qquad \le C\ell^\alpha \|F\|_\infty \|(\mathcal{N}_+ + 1)^{1/2} \xi_1\| \|(\mathcal{N}_+ + 1)^{1/2} \xi_2\| \end{aligned} \tag{3.92}$$

for all $\alpha > 0$, $\xi_1, \xi_2 \in \mathcal{F}_+^{\le N}$, $F \in \ell^\infty(\Lambda_+^*)$, $\ell \in (0;1/2)$ and $N \in \mathbb{N}$ large enough.

The proof follows directly from [10, Lemma 8.6], we report it for convenience of the reader.

Proof. It is a simple consequence of Proposition 2.6. We write

$$\sum_{p \in \Lambda_+^*} F_p (e^{-A_H} a_p^* a_p e^{A_H} - a_p^* a_p) = \int_0^1 ds \sum_{p \in \Lambda_+^*} F_p e^{-sA_H} [a_p^* a_p, A_H] e^{sA_H}$$

and compute

$$\sum_{p \in \Lambda_+^*} F_p [a_p^* a_p, A_H] = \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_H^c} (F_{r+v} + F_{-r} - F_v) \eta_r b_{r+v}^* a_{-r}^* a_v + \text{h.c.}$$

By Cauchy-Schwarz, we find with the help of Proposition 2.6 that

$$\begin{aligned}\Big|\frac{1}{\sqrt{N}}&\sum_{r\in P_H, v\in P_H^c}(F_{r+v}+F_{-r}-F_v)\eta_r\langle e^{sA_H}\xi_1, b^*_{r+v}a^*_{-r}a_v e^{sA_H}\xi_2\rangle\Big|\\ &\leq \frac{C\|F\|_\infty}{\sqrt{N}}\sum_{r\in P_H, v\in P_H^c}|\eta_r|\|a_v e^{sA_H}\xi_2\|\|a_{-r}b_{r+v}e^{sA_H}\xi_1\|\\ &\leq C\ell^\alpha\|F\|_\infty\|(\mathcal{N}_++1)^{1/2}\xi_1\|\|(\mathcal{N}_++1)^{1/2}\xi_2\|.\end{aligned}$$

Since the bound is uniform in the integration variable $s\in[0;1]$, we obtain (3.92). □

With the following notation

$$D'_N=\frac{\widehat{V}(0)}{2}+\frac{\widehat{V}(0)^2}{8\pi N}\log N^\beta,\qquad D_N=\frac{\widehat{V}(0)}{2}-\frac{\widehat{V}(0)^2}{8\pi N}\log N^\beta\,,$$

the following statement holds.

Proposition 3.11. *There exists a constant $C>0$ such that*

$$e^{-A_H}\mathcal{O}_N e^{A_H}=D'_N\mathcal{N}_+\Big(\frac{N-\mathcal{N}_+}{N}\Big)+D_N(N-\mathcal{N}_+)+\delta_{\mathcal{O}_N}$$

where

$$|\langle\xi,\delta_{\mathcal{O}_N}\xi\rangle|\leq C\ell^\alpha\langle\xi,(\mathcal{N}_++1)\xi\rangle$$

for all $\alpha>0$, $\xi\in\mathcal{F}_+^{\leq N}$ and $N\in\mathbb{N}$ large enough.

Proof. Recall from (3.78) that

$$\mathcal{O}_N=D'_N\mathcal{N}_+\Big(\frac{N-\mathcal{N}_+}{N}\Big)+D_N\left(N-\mathcal{N}_+\right)$$

Lemma 3.10 implies that

$$\begin{aligned}\pm\left\{e^{-A_H}\left[D'_N\mathcal{N}_++D_N\left(N-\mathcal{N}_+\right)\right]e^{A_H}-\left[D'_N\mathcal{N}_++D_N\left(N-\mathcal{N}_+\right)\right]\right\}\\ \leq C\ell^\alpha(\mathcal{N}_++1).\end{aligned}$$

As for the contribution quadratic in $\mathcal{N}_+$, we can write

$$\begin{aligned}&N^{-1}\left\langle\xi,\left[e^{-A_H}\mathcal{N}_+^2e^{A_H}-\mathcal{N}_+^2\right]\xi\right\rangle\\ &\quad=N^{-1}\left\langle\xi_1,\left[e^{-A_H}\mathcal{N}_+e^{A_H}-\mathcal{N}_+\right]\xi\right\rangle+N^{-1}\left\langle\xi,\left[e^{-A_H}\mathcal{N}_+e^{A_H}-\mathcal{N}_+\right]\xi_2\right\rangle\end{aligned}$$

with $\xi_1=e^{-A_H}\mathcal{N}_+e^{A_H}\xi$ and $\xi_2=\mathcal{N}_+\xi$. Applying again Lemma 3.10, we obtain

$$\begin{aligned}&\left|N^{-1}\left\langle\xi,\left[e^{-A_H}\mathcal{N}_+^2e^{A_H}-\mathcal{N}_+^2\right]\xi\right\rangle\right|\\ &\qquad\leq CN^{-1}\ell^\alpha\|(\mathcal{N}_++1)^{1/2}\xi\|\left[\|(\mathcal{N}_++1)^{1/2}\xi_1\|+\|(\mathcal{N}_++1)^{1/2}\xi_2\|\right].\end{aligned}$$

Using (twice) Prop. 2.6, we find

$$\|(\mathcal{N}_++1)^{1/2}\xi_1\|=\|(\mathcal{N}_++1)^{1/2}e^{-A_H}\mathcal{N}_+e^{A_H}\xi\|\leq C\|(\mathcal{N}_++1)^{3/2}\xi\|.$$

Hence,we conclude that

$$\begin{aligned}&\left|\left\langle\xi,\left[e^{-A_H}\mathcal{N}_+^2e^{A_H}-\mathcal{N}_+^2\right]\xi\right\rangle\right|\\ &\qquad\leq CN^{-1}\ell^\alpha\|(\mathcal{N}_++1)^{1/2}\xi\|\|(\mathcal{N}_++1)^{3/2}\xi\|\leq C\ell^\alpha\|(\mathcal{N}_++1)^{1/2}\xi\|^2.\end{aligned}$$

□

3.2.3 Contributions from $e^{-A_H}\mathcal{K}e^{A_H}$

In this subsection, we consider contributions to $\mathcal{R}^{\beta}_{N,\ell}$ arising from conjugation of the kinetic energy operator $\mathcal{K} = \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p$. In particular, in the next proposition, we establish properties of the commutator $[\mathcal{K}, A_H]$.

Proposition 3.12. *There exists a constant $C > 0$ such that*

$$\begin{aligned}[\mathcal{K}, A_H] = &-\frac{1}{\sqrt{N}} \sum_{p\in\Lambda_+^*, q\in P_H^c, p\neq -q} (\widehat{V}(\cdot/N^\beta) * \widehat{f}_{N,\ell})(p)(b^*_{p+q}a^*_{-p}a_q + \text{h.c.}) \\ &+ \frac{\widehat{V}(0)}{\sqrt{N}} \sum_{p\in P_H^c, q\in P_H^c, p\neq -q} \big[b^*_{p+q}a^*_{-p}a_q + \text{h.c.}\big] + \delta_{\mathcal{K}}\end{aligned}$$

where

$$|\langle \xi, \delta_{\mathcal{K}}\xi\rangle| \leq C\ell^{3/2(\alpha-1)}\|(\mathcal{H}_N^\beta + 1)^{1/2}\xi\|^2 \tag{3.93}$$

for all $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$, $N \in \mathbb{N}$ large enough. Moreover, we have

$$\begin{aligned}\Big|\frac{\widehat{V}(0)}{\sqrt{N}} \sum_{p,q\in P_H^c, p\neq -q} \langle \xi, \big[b^*_{p+q}a^*_{-p}a_q, A_H\big]\xi\rangle\Big| \\ \leq C(\ell^\alpha|\log\ell|^{1/2} + \ell^\alpha)\|(\mathcal{H}_N^\beta + 1)^{1/2}\xi\|^2\end{aligned} \tag{3.94}$$

for all $\alpha > 0$, $\xi \in \mathcal{F}_+^{\leq N}$ and $N \in \mathbb{N}$ large enough.

Proof. The bound (3.93) is a consequence of Eqs. (3.86), (3.87), (3.88), (3.90) in the proof of Prop. 2.7, and of the observation that, from the estimate (3.76),

$$\begin{aligned}&\Big|\frac{1}{\sqrt{N}} \sum_{p,q\in P_H^c, p\neq -q} \big[(\widehat{V}(\cdot/N) * \widehat{f}_{N,\ell})(p) - \widehat{V}(0)\big]\langle \xi, b^*_{p+q}a^*_{-p}a_q\xi\rangle\Big| \\ &\leq CN^{-1/2-\beta} \sum_{p,q\in P_H^c, p\neq -q} |p|\|b_{p+q}a_{-p}\xi\|\|a_q\xi\| \\ &\quad + C(\log N^\beta\ell)N^{-3/2} \sum_{p,q\in P_H^c, p\neq -q} \|b_{p+q}a_{-p}\xi\|\|a_q\xi\| \\ &\leq C(N^{-\beta}\ell^{-\alpha} + (\log N^\beta\ell)N^{-1})\|\mathcal{K}^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|.\end{aligned}$$

Let us now focus on (3.94). We have

$$\begin{aligned}&\frac{\widehat{V}(0)}{\sqrt{N}} \sum_{p,q\in P_H^c, p\neq -q} \big[b^*_{p+q}a^*_{-p}a_q, A_H\big] + \text{h.c.} \\ &= \frac{\widehat{V}(0)}{N} \sum_{\substack{r\in P_H, p,q,v\in P_H^c, \\ p\neq -q, r\neq -v}} \eta_r\big[b^*_{p+q}a^*_{-p}a_q, b^*_{r+v}a^*_{-r}a_v - a^*_v a_{-r}b_{r+v}\big] + \text{h.c.}\end{aligned} \tag{3.95}$$

We split the commutator into the four summands

$$[b^*_{p+q}a^*_{-p}a_q, b^*_{r+v}a^*_{-r}a_v - a^*_v a_{-r}b_{r+v}] = \big([b^*_{p+q}, b^*_{r+v}a^*_{-r}a_v] + [a^*_v a_{-r}b_{r+v}, b^*_{p+q}]\big)a^*_{-p}a_q \\ + b^*_{p+q}\big([a^*_{-p}a_q, b^*_{r+v}a^*_{-r}a_v] + [a^*_v a_{-r}b_{r+v}, a^*_{-p}a_q]\big). \tag{3.96}$$

We compute

$$[b^*_{p+q}, b^*_{r+v}a^*_{-r}a_v]a^*_{-p}a_q = -b^*_{r+v}b^*_{-r}a^*_{-p}a_q\delta_{p+q,v} = -b^*_{r+v}b^*_{-r}a^*_{q-v}a_q\delta_{p+q,v} \tag{3.97}$$

as well as

$$\begin{aligned}&[a^*_v a_{-r}b_{r+v}, b^*_{p+q}]a^*_{-p}a_q \\ &= (1-\mathcal{N}_+/N)a^*_v a^*_{r+q}a_q a_{r+v}\delta_{p+q,-r} + (1-\mathcal{N}_+/N)a^*_v a_v\delta_{p+q,-r}\delta_{r+v,-p} \\ &\quad + (1-\mathcal{N}_+/N)a^*_v a^*_{q-r-v}a_{-r}a_q\delta_{p+q,r+v} + (1-\mathcal{N}_+/N)a^*_v a_v\delta_{p+q,r+v}\delta_{r,p} \\ &\quad - N^{-1}a^*_v a^*_{p+q}a^*_{-p}a_{-r}a_{r+v}a_q - N^{-1}a^*_v a^*_{q-r-v}a_{-r}a_q\delta_{r+v,-p} - N^{-1}a^*_v a^*_{q+r}a_{r+v}a_q\delta_{p,r}.\end{aligned} \tag{3.98}$$

Similarly, we find

$$\begin{aligned}b^*_{p+q}[a^*_{-p}a_q, b^*_{r+v}a^*_{-r}a_v] &= b^*_{p+r+v}b^*_{-p}a^*_{-r}a_v\delta_{q,r+v} + b^*_{p-r}b^*_{r+v}a^*_{-p}a_v\delta_{q,-r} \\ &\quad - b^*_{q-v}b^*_{r+v}a^*_{-r}a_q\delta_{-p,v}\end{aligned} \tag{3.99}$$

and

$$\begin{aligned}b^*_{p+q}[a^*_v a_{-r}b_{r+v}, a^*_{-p}a_q] &= b^*_{q+r}a^*_v a_q b_{r+v}\delta_{r,p} - b^*_{p+v}a^*_{-p}a_{-r}b_{r+v}\delta_{q,v} \\ &\quad + b^*_{q-r-v}a^*_v a_{-r}b_q\delta_{r+v,-p}.\end{aligned} \tag{3.100}$$

Taking into account that $\delta_{r,p} = \delta_{q,-r} = \delta_{r+v,q} = 0$ for $r \in P_H, p, q, v \in P_H^c$ we obtain, inserting these formulas into (3.95),

$$\frac{\widehat{V}(0)}{\sqrt{N}} \sum_{p,q\in P_H^c, p\neq -q} \big[b^*_{p+q}a^*_{-p}a_q, A_H\big] + \text{h.c.} = \sum_{j=1}^{7} \Upsilon_j + \text{h.c.}$$

where

$$
\begin{aligned}
\Upsilon_1 &:= -\frac{2\widehat{V}(0)}{N} \sum_{\substack{r\in P_H;q,v\in P_H^c,\\ q\neq v, r\neq -v}} \eta_r b^*_{r+v} b^*_{-r} a^*_{q-v} a_q, \\
\Upsilon_2 &:= \frac{\widehat{V}(0)}{N} \sum_{\substack{r\in P_H;q,v\in P_H^c,\\ q+rP_H^c, r\neq -q, r\neq -v}} \eta_r (1-\mathcal{N}_+/N) a^*_v a^*_{r+q} a_q a_{r+v}, \\
\Upsilon_3 &:= \frac{\widehat{V}(0)}{N} \sum_{\substack{r\in P_H, v\in P_H^c,\\ r+v\in P_H^c}} \eta_r (1-\mathcal{N}_+/N) a^*_v a_v, \\
\Upsilon_4 &:= \frac{\widehat{V}(0)}{N} \sum_{\substack{r\in P_H;q,v\in P_H^c,\\ q-r-v\in P_H^c}} \eta_r (1-\mathcal{N}_+/N) a^*_v a^*_{q-r-v} a_{-r} a_q, \\
\Upsilon_5 &:= -\frac{\widehat{V}(0)}{N^2} \sum_{\substack{r\in P_H, p,q,v\in P_H^c,\\ p\neq -q, r\neq -v}} \eta_r a^*_v a^*_{p+q} a^*_{-p} a_{-r} a_{r+v} a_q, \\
\Upsilon_6 &:= -\frac{\widehat{V}(0)}{N^2} \sum_{\substack{r\in P_H;q,v\in P_H^c,\\ r+v\in P_H^c, q\neq r+v}} \eta_r a^*_v a^*_{q-r-v} a_{-r} a_q, \\
\Upsilon_7 &:= -\frac{\widehat{V}(0)}{N} \sum_{\substack{r\in P_H, p,v\in P_H^c,\\ p,r\neq -v}} \eta_r b^*_{p+v} a^*_{-p} a_{-r} b_{r+v}, \\
\Upsilon_8 &:= \frac{\widehat{V}(0)}{N} \sum_{\substack{r\in P_H;q,v\in P_H^c,\\ r+v\in P_H^c, q\neq r+v}} \eta_r b^*_{q-r-v} a^*_v a_{-r} b_q.
\end{aligned}
\tag{3.101}
$$

In fact, Υ_1 collects the contribution from (3.97) and the non-vanishing contribution from (3.99), $\Upsilon_2 - \Upsilon_6$ corresponds to the five non-vanishing terms on the r.h.s. of (3.98), Υ_7 and Υ_8 reflect the two non-vanishing terms on the r.h.s. of (3.100).

To conclude the proof of Prop. 3.12, it remains to show that all operators in (3.101) satisfy (3.94). By Cauchy-Schwarz, we observe that

$$
\begin{aligned}
\big|\langle \xi, \Upsilon_1 \xi\rangle\big| &\leq \frac{C\ell^\alpha}{N} \sum_{\substack{r\in P_H;q,v\in P_H^c,\\ q\neq v, r\neq -v}} |\eta_r| \|a_q (\mathcal{N}_+ +1)^{1/2}\xi\| |r| \|a_{-r} a_{q-v} a_{r+v} (\mathcal{N}_+ +1)^{-1/2}\xi\| \\
&\leq C\ell^\alpha \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\|.
\end{aligned}
$$

While the expectation of Υ_2 is bounded by

$$
\begin{aligned}
\big|\langle \xi, \Upsilon_2 \xi\rangle\big| &\leq \frac{C}{N} \sum_{\substack{r\in P_H;q,v\in P_H^c,\\ q+r\in P_H^c, r\neq -q, r\neq -v}} |\eta_r| |q| \|a_q a_{r+v}\xi\| |q|^{-1} \|a_v a_{r+q}\xi\| \\
&\leq C\ell^\alpha |\log \ell|^{1/2} \|\mathcal{K}^{1/2}\xi\| \|\mathcal{N}_+^{1/2}\xi\|.
\end{aligned}
$$

On the other hand one can easily see that $\pm\Upsilon_3 \leq CN^{-1}|\log\ell|\mathcal{N}_+ \leq C\ell^\alpha(\mathcal{N}_+ + 1)$, since we already said $N > |\log\ell|$, and the expectations of the terms Υ_4, Υ_6 and Υ_8 can all be estimated by the expectation

$$\begin{aligned}\big|\langle\xi,(\Upsilon_4+\Upsilon_6+\Upsilon_8)\xi\rangle\big| &\leq \frac{C}{N}\sum_{\substack{r\in P_H;q,v\in P_H^c,\\ q-r-v\neq 0}}|\eta_r||v|\|a_v a_{q-r-v}\xi\||v|^{-1}\|a_{-r}a_q\xi\| \\ &\leq C\ell^\alpha|\log\ell|^{1/2}\|\mathcal{K}^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|.\end{aligned}$$

Finally, the expectations of Υ_5 and Υ_7 can be bounded by

$$\begin{aligned}\big|\langle\xi,\Upsilon_5\xi\rangle\big| &\leq \frac{C\ell^\alpha}{N^2}\sum_{\substack{r\in P_H,p,q,v\in P_H^c,\\ p\neq -q,r\neq -v}}|\eta_r||p|\|a_{-p}a_v a_{p+q}\xi\||p|^{-1}|r|\|a_{-r}a_{r+v}a_q\xi\| \\ &\leq C\ell^{2\alpha}|\log\ell|^{1/2}\|\mathcal{K}^{1/2}\xi\|^2\end{aligned}$$

and by

$$\begin{aligned}\big|\langle\xi,\Upsilon_7\xi\rangle\big| &\leq \frac{C\ell^\alpha}{N}\sum_{\substack{r\in P_H,p,v\in P_H^c,\\ p,r\neq -v}}|\eta_r||p|\|a_{-p}a_{p+v}\xi\||p|^{-1}|r|\|a_{-r}a_{r+v}\xi\| \\ &\leq C\ell^{2\alpha}|\log\ell|^{1/2}\|\mathcal{K}^{1/2}\xi\|^2.\end{aligned}$$

□

3.2.4 Analysis of $e^{-A_H}\mathcal{Q}_N e^{A_H}$

In this subsection, we consider contributions to $\mathcal{R}^\beta_{N,\ell}$ arising from conjugation of $\mathcal{Q}_N$, as defined in (3.78).

Proposition 3.13. *There exists a constant $C > 0$ such that*

$$e^{A_H}\mathcal{Q}_N e^{-A_H} = \widehat{V}(0)\sum_{p\in P_H^c}a_p^*a_p\Big(1-\frac{\mathcal{N}_+}{N}\Big) + \frac{\widehat{V}(0)}{2}\sum_{p\in P_H^c}\big[b_p^*b_{-p}^* + b_p b_{-p}\big] + \delta_{\mathcal{Q}_N}$$

where

$$\pm\,\delta_{\mathcal{Q}_N} \leq C\ell^\alpha|\log\ell|^{3/2}(\mathcal{H}_N^\beta + 1) \tag{3.102}$$

for all $\alpha > 0$, and $N \in \mathbb{N}$ large enough.

Proof. Proceeding as in the proof of Proposition 3.11, it follows from Lemma 3.10 that

$$\begin{aligned}\pm\Big[\widehat{V}(0)\sum_{p\in P_H^c}e^{-A_H}a_p^*a_p(1-N/\mathcal{N}_+)e^{A_H} - \widehat{V}(0)\sum_{p\in P_H^c}a_p^*a_p(1-N/\mathcal{N}_+)\Big]& \\ \leq C\ell^\alpha(\mathcal{N}_+ + 1).&\end{aligned} \tag{3.103}$$

Let us thus focus on the remaining part of $\mathcal{Q}_N$. We expand

$$\begin{aligned}\frac{\widehat{V}(0)}{2}\sum_{p\in P_H^c}\Big(e^{-A_H}\big[b_p^*b_{-p}^*+b_pb_{-p}\big]e^{A_H}-\big[b_p^*b_{-p}^*+b_pb_{-p}\big]\Big)\\ =\frac{\widehat{V}(0)}{2}\int_0^1 ds\sum_{p\in P_H^c}e^{-sA_H}\big[b_p^*b_{-p}^*,A_H\big]e^{sA_H}+\text{h.c.}\end{aligned}\tag{3.104}$$

We compute

$$\big[b_p^*b_{-p}^*,b_{r+v}^*a_{-r}^*a_v-a_v^*a_{-r}b_{r+v}\big]=b_{r+v}^*\big[b_p^*b_{-p}^*,a_{-r}^*a_v\big]+\big[a_v^*a_{-r}b_{r+v},b_p^*b_{-p}^*\big]$$

where

$$b_{r+v}^*\big[b_p^*b_{-p}^*,a_{-r}^*a_v\big]=-b_{r+v}^*b_{-v}^*b_{-r}^*(\delta_{-p,v}+\delta_{p,v})$$

and

$$\begin{aligned}\big[a_v^*a_{-r}b_{r+v},b_p^*b_{-p}^*\big]&=b_v^*b_r^*b_{r+v}(\delta_{-r,p}+\delta_{r,p})\\&\quad+(1-\mathcal{N}_+/N)b_{-r-v}^*a_v^*a_{-r}(\delta_{r+v,-p}+\delta_{r+v,-p})\\&\quad-2N^{-1}b_v^*a_r^*a_{r+v}(\delta_{p,-r}+\delta_{r,p})-2N^{-1}b_p^*a_{-p}^*a_v^*a_{-r}a_{r+v}.\end{aligned}$$

Using the fact that $\delta_{p,-r}=\delta_{p,r}=0$ for $r\in P_H$ and $p\in P_H^c$, we find that $\sum_{p\in P_H^c}\big[b_p^*b_{-p}^*,A_H\big]+\text{h.c.}=\sum_{i=1}^3(\Phi_i+\text{h.c.})$, where

$$\begin{aligned}\Phi_1&:=-\frac{2}{\sqrt{N}}\sum_{r\in P_H,v\in P_H^c}\eta_r b_{r+v}^*b_{-r}^*b_{-v}^*,\\ \Phi_2&:=\frac{2}{\sqrt{N}}\sum_{r\in P_H,v\in P_H^c:r+v\in P_H^c}\eta_r(1-\mathcal{N}_+/N)b_{-r-v}^*a_v^*a_{-r},\\ \Phi_3&:=-\frac{2}{N^{3/2}}\sum_{r\in P_H,v,p\in P_H^c}\eta_r b_p^*a_{-p}^*a_v^*a_{-r}a_{r+v}.\end{aligned}$$

Let us now bound the expectation of the operators $\Phi_i, i=1,2,3,$. By Cauchy-Schwarz, we find that

$$\begin{aligned}|\langle\xi,\Phi_1\xi\rangle|&\le\left|\frac{2}{\sqrt{N}}\sum_{r\in P_H,v\in P_H^c}\eta_r\langle\xi,b_{r+v}^*b_{-r}^*b_{-v}^*\xi\rangle\right|\\&\le\frac{C}{\sqrt{N}}\sum_{r\in P_H,v\in P_H^c}|\eta_r||v|^{-1}\|(\mathcal{N}_++1)^{1/2}\xi\|\,|v|\|b_{-v}b_{r+v}b_{-r}(\mathcal{N}_++1)^{-1/2}\xi\|\\&\le C\ell^\alpha|\log\ell|^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\end{aligned}$$

as well as

$$\begin{aligned}|\langle\xi,\Phi_2\xi\rangle|&\le\left|\frac{2}{\sqrt{N}}\sum_{r\in P_H,v\in P_H^c:r+v\in P_H^c}\eta_r\langle\xi,(1-\mathcal{N}_+/N)b_{-r-v}^*a_v^*a_{-r}\xi\rangle\right|\\&\le\frac{C}{\sqrt{N}}\sum_{r\in P_H,v\in P_H^c}|\eta_r||v|^{-1}\|(\mathcal{N}_++1)^{1/2}\xi\||v|\|a_{-v}b_{r+v}\xi\|\\&\le C\ell^\alpha|\log\ell|^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned}$$

To bound Φ_3 we notice that

$$\begin{aligned}\big|\langle \xi, \Phi_3 \xi\rangle\big| &\leq \frac{C\ell^\alpha}{N^{3/2}} \sum_{r\in P_H, v,p\in P_H^c} |\eta_r||p|\|a_p a_v (\mathcal{N}_+ +1)^{1/2}\xi\| |p|^{-1}|r|\|a_{-r}a_{r+v}\xi\| \\ &\leq C\ell^{2\alpha}|\log \ell|^{1/2}\|\mathcal{K}^{1/2}\xi\|^2.\end{aligned}$$

With (3.104), we conclude that

$$\begin{aligned}\pm \Big[\frac{\widehat{V}(0)}{2}\sum_{p\in P_H^c}\Big(e^{-A_H}\big[b_p^* b_{-p}^* + b_p b_{-p}\big]e^{A_H} - \big[b_p^* b_{-p}^* + b_p b_{-p}\big]\Big)\Big] \\ \leq C\int_0^1 ds\; e^{-sA}\big[\ell^\alpha |\log \ell|^{1/2}(\mathcal{K}+\mathcal{N}_+ +1) + \ell^{2\alpha}|\log\ell|^{1/2}\mathcal{K}\big]e^{sA} \\ \leq C\int_0^1 ds\; e^{-sA_H}\big[\ell^\alpha|\log \ell|^{1/2}(\mathcal{H}_N^\beta +1)\big]e^{sA_H},\end{aligned}$$

Finally, we apply Prop. 2.7 to conclude that

$$\begin{aligned}\pm \Big[\frac{\widehat{V}(0)}{2}\sum_{p\in P_H^c}\Big(e^{-A_H}\big[b_p^* b_{-p}^* + b_p b_{-p}\big]e^{A_H} - \big[b_p^* b_{-p}^* + b_p b_{-p}\big]\Big)\Big] \\ \leq C\ell^\alpha|\log\ell|^{1/2}(\mathcal{H}_N^\beta+1) + C\ell^\alpha|\log \ell|^{3/2}(\mathcal{N}_+ +1).\end{aligned}$$

Together with the estimate (3.103), we arrive at (3.102). □

3.2.5 Contributions from $e^{-A_H}\mathcal{C}_N e^{A_H}$

Finally, in this subsection, we consider contributions to $\mathcal{R}^\beta_{N,\ell}$ arising from conjugation of the cubic operator $\mathcal{C}_N$ defined in (3.78). In particular, in the next proposition, we establish properties of the commutator $[\mathcal{C}_N, A_H]$.

Proposition 3.14. *There exists a constant $C>0$ such that*

$$\big[\mathcal{C}_N, A_H\big] = \frac{2}{N}\sum_{r\in P_H, v\in P_H^c}\big[\widehat{V}(r/N^\beta)\eta_r + \widehat{V}((r+v)/N^\beta)\eta_r\big]a_v^* a_v \frac{(N-\mathcal{N}_+)}{N} + \delta_{\mathcal{C}_N}$$

where

$$|\langle \xi, \delta_{\mathcal{C}_N}\xi\rangle| \leq C(\ell^\alpha + \ell^\alpha|\log \ell|^{1/2})\|(\mathcal{H}_N^\beta+1)^{1/2}\xi\|^2 \tag{3.105}$$

for all $\alpha>0$ and $N\in\mathbb{N}$ large enough.

Proof. We have

$$\big[\mathcal{C}_N, A_H\big] = \frac{1}{N}\sum_{\substack{p,q\in\Lambda_+^*: p+q\neq 0\\ r\in P_H, v\in P_H^c}} \widehat{V}(p/N^\beta)\eta_r\big[b_{p+q}^* a_{-p}^* a_q, b_{r+v}^* a_{-r}^* a_v - a_v^* a_{-r} b_{r+v}\big] + \text{h.c.}$$

From (3.96), (3.97), (3.98), (3.99) and (3.100) we arrive at

$$\begin{aligned}\big[\mathcal{C}_N, A_H\big] = \frac{2}{N} \sum_{r\in P_H, v\in P_H^c} \big[\widehat{V}(r/N^\beta)\eta_r + \widehat{V}((r+v)/N^\beta)\eta_r\big] a_v^* a_v \Big(1 - \frac{\mathcal{N}_+}{N}\Big) \\ + \sum_{j=1}^{12} (\Xi_j + \text{h.c.})\end{aligned}$$

where

$$\begin{aligned}
\Xi_1 &:= -\frac{1}{N} \sum_{\substack{r\in P_H, v\in P_H^c,\\ p\in\Lambda_+^*: p\neq v}} \widehat{V}(p/N^\beta)\eta_r b_{r+v}^* b_{-r}^* a_{-p}^* a_{v-p},\\
\Xi_2 &:= \frac{1}{N} \sum_{\substack{r\in P_H, v\in P_H^c,\\ p\in\Lambda_+^*: r\neq -p}} \widehat{V}(p/N^\beta)\eta_r (1-\mathcal{N}_+/N) a_v^* a_{-p}^* a_{-r-p} a_{r+v},\\
\Xi_3 &:= \frac{1}{N} \sum_{\substack{r\in P_H, v\in P_H^c,\\ p\in\Lambda_+^*: r+v\neq p}} \widehat{V}(p/N^\beta)\eta_r (1-\mathcal{N}_+/N) a_v^* a_{-p}^* a_{-r} a_{r+v-p},\\
\Xi_4 &:= -\frac{1}{N^2} \sum_{\substack{r\in P_H, v\in P_H^c,\\ p,q\in\Lambda_+^*: p+q\neq 0}} \widehat{V}(p/N^\beta)\eta_r a_v^* a_{p+q}^* a_{-p}^* a_{-r} a_{r+v} a_q,\\
\Xi_5 &:= -\frac{1}{N^2} \sum_{\substack{r\in P_H, v\in P_H^c,\\ q\in\Lambda_+^*: r+v\neq q}} \widehat{V}((r+v)/N^\beta)\eta_r a_v^* a_{q-r-v}^* a_{-r} a_q,\\
\Xi_6 &:= -\frac{1}{N^2} \sum_{\substack{r\in P_H, v\in P_H^c,\\ q\in\Lambda_+^*: r\neq -q}} \widehat{V}(r/N^\beta)\eta_r a_v^* a_{q+r}^* a_{r+v} a_q,\\
\Xi_7 &:= \frac{1}{N} \sum_{\substack{r\in P_H, v\in P_H^c,\\ p\in\Lambda_+^*: r+v\neq -p}} \widehat{V}(p/N^\beta)\eta_r b_{p+r+v}^* b_{-p}^* a_{-r}^* a_v,\\
\Xi_8 &:= \frac{1}{N} \sum_{\substack{r\in P_H, v\in P_H^c,\\ p\in\Lambda_+^*: r\neq -p}} \widehat{V}(p/N^\beta)\eta_r b_{p-r}^* b_{r+v}^* a_{-p}^* a_v,\\
\Xi_9 &:= -\frac{1}{N} \sum_{\substack{r\in P_H, v\in P_H^c,\\ q\in\Lambda_+^*: q\neq v}} \widehat{V}(v/N^\beta)\eta_r b_{q-v}^* b_{r+v}^* a_{-r}^* a_q,\\
\Xi_{10} &:= \frac{1}{N} \sum_{\substack{r\in P_H, v\in P_H^c,\\ q\in\Lambda_+^*: r\neq -q}} \widehat{V}(r/N^\beta)\eta_r b_{q+r}^* a_v^* a_q b_{r+v},
\end{aligned}$$

as well as

$$\Xi_{11} := -\frac{1}{N}\sum_{\substack{r\in P_H, v\in P_H^c,\\ p\in\Lambda_+^*: p\neq -v}} \widehat{V}(p/N^\beta)\eta_r b_{p+v}^* a_{-p}^* a_{-r} b_{r+v},$$
$$\Xi_{12} := \frac{1}{N}\sum_{\substack{r\in P_H, v\in P_H^c,\\ q\in\Lambda_+^*: q\neq r+v}} \widehat{V}((r+v)/N^\beta)\eta_r b_{q-r-v}^* a_v^* a_{-r} b_q.$$

To conclude the proof of the proposition, we have to show that all terms Ξ_j, $j=1,\dots,12$, satisfy the bound (3.105). The expectation of Ξ_1 can be controlled with Cauchy-Schwarz by

$$\begin{aligned}\big|\langle\xi,\Xi_1\xi\rangle\big| &\leq \frac{C\ell^\alpha}{N}\sum_{\substack{r\in P_H, v\in P_H^c,\\ p\in\Lambda_+^*: p\neq v}} |\eta_r|\|(\mathcal{N}_+ +1)^{1/2}a_{v-p}\xi\||r|\|a_{-r}a_{r+v}a_{-p}(\mathcal{N}_+ +1)^{-1/2}\xi\| \\ &\leq C\ell^\alpha\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned}$$

The same bound holds (after relabeling) for Ξ_9; we find

$$\big|\langle\xi,\Xi_9\xi\rangle\big| \leq C\ell^\alpha\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.$$

Also the expectations of the terms Ξ_2, Ξ_3 and (again after relabeling) of the terms Ξ_5, Ξ_6, Ξ_{10}, Ξ_{12} can be bounded similarly. We find

$$\begin{aligned}&|\langle\xi,\Xi_2\xi\rangle| + |\langle\xi,\Xi_3\xi\rangle| + |\langle\xi,\Xi_5\xi\rangle| + |\langle\xi,\Xi_6\xi\rangle| + |\langle\xi,\Xi_{10}\xi\rangle| + |\langle\xi,\Xi_{12}\xi\rangle| \\ &\leq \frac{C\ell^\alpha}{N}\sum_{\substack{r\in P_H,\\ v\in P_H^c,\\ p\in\Lambda_+^*}}\Big(|\eta_r|\|a_v a_{-p}\xi\||r+v|\|a_{r+v}a_{-r-p}\xi\| + |\eta_r|\|a_{-p}a_v\xi\||r|\|a_{-r}a_{r+v-p}\xi\| \\ &\qquad + |\eta_r|\|a_v a_{p-r-v}\xi\||r|\|a_{-r}a_p\xi\| + |\eta_r|\|a_v a_{p+r}\xi\||r+v|\|a_{r+v}a_p\xi\| \\ &\qquad + |\eta_r|\|a_{p+r}a_v\xi\||r+v|\|a_{r+v}a_p\xi\| + |\eta_r|\|a_{p-r-v}a_v\xi\||r|\|a_{-r}a_p\xi\|\Big) \\ &\leq C\ell^\alpha\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned}$$

To control the remaining terms, we switch to position space and use the potential energy operator $\mathcal{V}_N^\beta$. We start with Ξ_4. Applying Cauchy-Schwarz, we find

$$\begin{aligned}|\langle\xi,\Xi_4\xi\rangle| &= \left|\frac{1}{N^2}\int_{\Lambda^2}dxdy\; N^{2\beta}V(N^\beta(x-y))\sum_{\substack{r\in P_H,\\ v\in P_H^c}}\eta_r\langle\xi,\check{a}_x^*\check{a}_y^*a_v^*a_{-r}a_{r+v}\check{a}_x\xi\rangle\right| \\ &\leq \frac{1}{N}\int_{\Lambda^2}dxdy\; N^{2\beta-1}V(N^\beta(x-y))\sum_{\substack{r\in P_H,\\ v\in P_H^c}}|\eta_r|\|a_v\check{a}_x\check{a}_y\xi\|\|a_{-r}a_{r+v}\check{a}_x\xi\| \\ &\leq C\ell^\alpha\|(\mathcal{V}_N^\beta)^{1/2}\,\xi\|\|\mathcal{N}_+^{1/2}\xi\|.\end{aligned}$$

Next, we rewrite Ξ_7, Ξ_8 and Ξ_{11}- partially- in position spaceas

$$\Xi_7 = \int_{\Lambda^2} dxdy\; N^{2\beta-1}V(N^\beta(x-y)) \sum_{r\in P_H, v\in P_H^c} e^{i(r+v)x}\eta_r \check{b}_x^* \check{b}_y^* a_{-r}^* a_v,$$

$$\Xi_8 = \int_{\Lambda^2} dxdy\; N^{2\beta-1}V(N^\beta(x-y)) \sum_{r\in P_H, v\in P_H^c} e^{-irx}\eta_r \check{b}_x^* \check{b}_y^* a_{r+v}^* a_v,$$

$$\Xi_{11} = -\int_{\Lambda^2} dxdy\; N^{2\beta-1}V(N^\beta(x-y)) \sum_{r\in P_H, v\in P_H^c} e^{ivx}\eta_r \check{b}_x^* \check{b}_y^* a_{-r} b_{r+v}.$$

Thus, we obtain

$$\begin{aligned} |\langle \xi, \Xi_7 \xi\rangle| &\leq \int_{\Lambda^2} dxdy\; N^{2\beta-1}V(N^\beta(x-y)) \sum_{r\in P_H} \|\check{a}_x \check{a}_y a_{-r}\xi\| |\eta_r| \Big\| \sum_{v\in P_H^c} e^{ivx} a_v \xi\Big\| \\ &\leq C\ell^\alpha \|(\mathcal{V}_N^\beta)^{1/2}\xi\| \Big[\int_\Lambda dx \sum_{v,v'\in P_H^c} e^{i(v-v')x}\langle \xi, a_{v'}^* a_v \xi\rangle \Big]^{1/2} \\ &\leq C\ell^\alpha \|(\mathcal{V}_N^\beta)^{1/2}\,\xi\| \|\mathcal{N}_+^{1/2}\xi\| \end{aligned}$$

as well as

$$\begin{aligned} &|\langle \xi, \Xi_8 \xi\rangle| + |\langle \xi, \Xi_{11}\xi\rangle| \\ &\leq C \int_{\Lambda^2} dxdy\; N^{2\beta-1}V(N^\beta(x-y)) \\ &\quad \times \sum_{r\in P_H, v\in P_H^c} \Big(|v|^{-1}\|\check{a}_x\check{a}_y a_{r+v}\xi\| |\eta_r| |v| \|a_v\xi\| + C\ell^\alpha |\eta_r| \|\check{a}_x\check{a}_y\xi\| |r| \|a_{-r}b_{r+v}\xi\|\Big) \\ &\leq C(\ell^\alpha |\log \ell|^{1/2} + \ell^\alpha) \|(\mathcal{V}_N^\beta)^{1/2}\,\xi\| \|\mathcal{K}^{1/2}\xi\|\,. \end{aligned}$$

Collecting all the bounds above, we arrive at (3.105). □

3.2.6 Proof of Proposition 2.8

In this last subsection we recombine the results of Sections 3.2.1-3.2.5 to prove Proposition 2.8. We are assuming $\alpha \geq 3$.

From Prop. 3.11 and Prop. 3.13 we obtain that

$$\begin{aligned} \mathcal{R}_{N,\ell}^\beta \geq\; & \frac{\widehat{V}(0)}{2}\mathcal{N}_+\Big(1-\frac{\mathcal{N}_+}{N}\Big) + \frac{\widehat{V}(0)}{2}(N-\mathcal{N}_+) - \frac{\widehat{V}(0)^2}{8\pi}\log N^\beta \\ & + \widehat{V}(0)\sum_{p\in P_H^c} a_p^* a_p (1-\mathcal{N}_+/N) + \frac{\widehat{V}(0)}{2}\sum_{p\in P_H^c}\big[b_p^* b_{-p}^* + b_p b_{-p}\big] \\ & + \mathcal{K} + \mathcal{C}_N + \mathcal{V}_N^\beta + \int_0^1 ds\; e^{-sA_H}\big[\mathcal{K}+\mathcal{C}_N+\mathcal{V}_N^\beta, A_H\big]e^{sA_H} \\ & - C\ell^\alpha |\log\ell|^{3/2}(\mathcal{H}_N^\beta + 1) \end{aligned}$$

with $\mathcal{C}_N$ defined as in (3.78). From Prop. 3.9, Prop. 3.12 and Prop. 3.14, we can write, for N large enough,

$$\begin{aligned}
&[\mathcal{K}+\mathcal{C}_N+\mathcal{V}_N^\beta, A_H] \\
&\geq -\frac{1}{\sqrt{N}}\sum_{\substack{p\in\Lambda_+^*,\\ q\in P_H^c,\\ p\neq -q}}\widehat{V}(p/N^\beta)\big[b^*_{p+q}a^*_{-p}a_q+\text{h.c.}\big]+\frac{\widehat{V}(0)}{\sqrt{N}}\sum_{\substack{p\in P_H^c,\\ q\in P_H^c,\\ p\neq -q}}\big[b^*_{p+q}a^*_{-p}a_q+\text{h.c.}\big] \\
&\quad+\frac{2}{N}\sum_{r\in P_H, v\in P_H^c}\big[\widehat{V}(r/N^\beta)\eta_r+\widehat{V}((r+v)/N^\beta)\eta_r\big]a^*_v a_v\Big(1-\frac{\mathcal{N}_+}{N}\Big) \\
&\quad-C\ell^\alpha|\log\ell|(\mathcal{H}_N^\beta+1).
\end{aligned}$$

From Prop. 2.6, Prop. 2.7, and recalling the definition (3.78) of the operator $\mathcal{C}_N$, we deduce that

$$\begin{aligned}
&\int_0^1 ds\; e^{-sA_H}[\mathcal{K}+\mathcal{C}_N+\mathcal{V}_N^\beta, A_H]e^{sA_H} \\
&\geq \int_0^1 ds\; e^{-sA_H}\Big[-\mathcal{C}_N+\frac{\widehat{V}(0)}{\sqrt{N}}\sum_{\substack{p\in P_H^c, q\in P_H^c,\\ p\neq -q}}\big[b^*_{p+q}a^*_{-p}a_q+\text{h.c.}\big] \\
&\qquad+\frac{2}{N}\sum_{r\in P_H, v\in P_H^c}\big[\widehat{V}(r/N^\beta)\eta_r+\widehat{V}((r+v)/N^\beta)\eta_r\big]a^*_v a_v\Big(1-\frac{\mathcal{N}_+}{N}\Big)\Big]e^{sA_H} \\
&\quad+\frac{1}{\sqrt{N}}\int_0^1 ds\sum_{\substack{p\in\Lambda_+^*, q\in P_H,\\ p\neq -q}}\widehat{V}(p/N^\beta)e^{-sA_H}\big[b^*_{p+q}a^*_{-p}a_q+\text{h.c.}\big]e^{sA_H} \\
&\quad-C\ell^\alpha|\log\ell|^2(\mathcal{H}_N^\beta+1)\,.
\end{aligned} \tag{3.106}$$

We consider the expectation of the operator on the fourth line, this can be estimated after switching to position space as

$$\begin{aligned}
&\Big|\frac{1}{N^{1/2}}\int_0^1 ds\sum_{\substack{p\in\Lambda_+^*, q\in P_H,\\ p\neq -q}}\widehat{V}(p/N^\beta)\langle\xi, e^{-sA_H}b^*_{p+q}a^*_{-p}a_q e^{sA_H}\xi\rangle\Big| \\
&\leq N^{1/2}\int_0^1 ds\int_{\Lambda^2}dxdy\; N^{2\beta-1}V(N^\beta(x-y))\|\check{a}_x\check{a}_y e^{sA_H}\xi\|\Big\|\sum_{q\in P_H}e^{iqx}a_q e^{sA_H}\xi\Big\| \\
&\leq C\int_0^1 ds\;\|(\mathcal{V}_N^\beta)^{1/2}\,e^{sA_H}\xi\|\Big[\int_\Lambda dx\sum_{q,q'\in P_H}e^{i(q-q')x}\langle e^{sA_H}\xi, a^*_{q'}a_q e^{sA_H}\xi\rangle\Big]^{1/2} \\
&\leq C\ell^\alpha\int_0^1 ds\;\|(\mathcal{V}_N^\beta)^{1/2}\,e^{sA_H}\xi\|\|\mathcal{K}^{1/2}e^{sA_H}\xi\| \\
&\leq C\ell^\alpha\|(\mathcal{H}_N^\beta+1)^{1/2}\xi\|^2+C\ell^\alpha|\log\ell|\|(\mathcal{N}_++1)^{1/2}\xi\|^2.
\end{aligned} \tag{3.107}$$

Next, to bound the term on the third line of (3.106) we use the fact that $\Big|\sum_{r\in\Lambda_+^*}\widehat{V}(r/N^\beta)\eta_r\Big| \leq C\log N$. Hence,

$$\pm\frac{2}{N}\int_0^1 ds\, e^{-sA_H}\Big[\sum_{r\in P_H, v\in P_H^c}\big[\widehat{V}(r/N^\beta)\eta_r + \widehat{V}((r+v)/N^\beta)\eta_r\big]a_v^*a_v\Big(1-\frac{\mathcal{N}_+}{N}\Big)\Big]e^{sA_H} \leq C\frac{\log N}{N}.$$

To handle the second term on the second line of (3.106), we apply Prop. 3.12 and then Prop. 2.6, Prop 2.7

$$\begin{aligned}
&\pm\Bigg(\frac{\widehat{V}(0)}{\sqrt{N}}\int_0^1 ds \sum_{\substack{p,q\in P_H^c,\\ p\neq -q}}\Big[e^{-sA_H}b_{p+q}^*a_{-p}^*a_q e^{sA_H} - b_{p+q}^*a_{-p}^*a_q\Big] + \text{h.c.}\Bigg)\\
&=\pm\Bigg(\frac{\widehat{V}(0)}{\sqrt{N}}\int_0^1 ds\int_0^s dt \sum_{\substack{p,q\in P_H^c,\\ p\neq -q}} e^{-tA_H}\Big[b_{p+q}^*a_{-p}^*a_q, A_H\Big]e^{tA_H}\Bigg)\\
&\leq C\ell^\alpha|\log\ell|^{3/2}(\mathcal{H}_N^\beta+1)\,.
\end{aligned}$$

As for the first term on the second line of (3.106), we use again Prop. 3.14. Proceeding then as in (3.2.6), we have

$$\begin{aligned}
\int_0^1 ds\, e^{-sA_H}\mathcal{C}_N e^{sA_H} &= \mathcal{C}_N + \int_0^1 ds\int_0^s dt\, e^{-tA_H}[\mathcal{C}_N, A_H]e^{tA_H}\\
&\leq \mathcal{C}_N + C\ell^\alpha\,(\mathcal{H}_N^\beta+1) + C\ell^\alpha|\log\ell|(\mathcal{N}_++1).
\end{aligned}\tag{3.108}$$

Inserting the bounds (3.107)-(3.108) into (3.106) and using additionally the simple bounds

$$0\leq\sum_{p\in P_H}a_p^*a_p \leq \ell^{2\alpha}\mathcal{K}$$

and

$$\begin{aligned}
\Big|\frac{\widehat{V}(0)}{\sqrt{N}}\sum_{\substack{p\in P_H^c, q\in P_H,\\ p\neq -q}}\langle\xi, b_{p+q}^*a_{-p}^*a_q\xi\rangle\Big| &\leq \frac{C\ell^\alpha}{\sqrt{N}}\sum_{\substack{p\in P_H^c, q\in P_H,\\ p\neq -q}}|p|\|a_{-p}a_{p+q}\xi\||p|^{-1}|q|\|a_q\xi\|\\
&\leq \frac{C}{\sqrt{N}}\ell^\alpha|\log\ell|^{1/2}\,\|\mathcal{K}^{1/2}\mathcal{N}_+^{1/2}\xi\|\Big[\sum_{q\in P_H}|q|^2\|a_q\xi\|^2\Big]^{1/2}\\
&\leq C\ell^\alpha|\log\ell|^{1/2}\|\mathcal{K}^{1/2}\xi\|^2
\end{aligned}$$

we end up with

$$\begin{aligned}\mathcal{R}^{\beta}_{N,\ell} \geq\ & \frac{\widehat{V}(0)}{2}\mathcal{N}_+\Big(1-\frac{\mathcal{N}_+}{N}\Big)+\frac{\widehat{V}(0)}{2}(N-\mathcal{N}_+)-\frac{\widehat{V}(0)^2}{8\pi}\log N^{\beta}\\ &+\widehat{V}(0)\sum_{p\in P_H^c}a_p^*a_p\Big(1-\frac{\mathcal{N}_+}{N}\Big)+\frac{\widehat{V}(0)}{2}\sum_{p\in P_H^c}\big[b_p^*b_{-p}^*+b_pb_{-p}\big]\\ &+\frac{\widehat{V}(0)}{\sqrt{N}}\sum_{p\in P_H^c,q\in\Lambda_+^*:p\neq -q}\big[b^*_{p+q}a^*_{-p}a_q+\text{h.c.}\big]+\big(1-C\ell^{\alpha}|\log\ell|^2\big)(\mathcal{H}_N^{\beta}+1)\end{aligned} \tag{3.109}$$

under the assumptions $\alpha > 3$.

We define now the function $v_\beta \in L^\infty(\Lambda)$ by setting

$$v_\beta(x) := \widehat{V}(0)\sum_{p\in\{0\}\cup P_H^c} e^{ip\cdot x} = \widehat{V}(0)\sum_{p\in\Lambda^*:|p|\leq \ell^{-\alpha}} e^{ip\cdot x}.$$

In other words, v_β is defined so that $\widehat{v}_\beta(p) = \widehat{V}(0)$ for all $p \in \Lambda^*$ with $|p| \leq \ell^{-\alpha}$ and $\widehat{v}_\beta(p) = 0$ otherwise. Observe, in particular, that $\widehat{v}_\beta(p) \geq 0$ for all $p \in \Lambda^*$. Proceeding as in (2.9), but now with $\widehat{V}(p/N^\beta)$ replaced by $\widehat{v}_\beta(p)$, we find that

$$\begin{aligned}U_N\left[\frac{1}{N}\sum_{i<j}^{N}v_\beta(x_i-x_j)\right]U_N^* =\ & \frac{\widehat{V}(0)}{2N}(N-1)(N-\mathcal{N}_+)+\frac{\widehat{V}(0)}{2N}\mathcal{N}_+(N-\mathcal{N}_+)\\ &+\widehat{V}(0)\sum_{p\in P_H^c}a_p^*a_p\Big(1-\frac{\mathcal{N}_+}{N}\Big)\\ &+\frac{\widehat{V}(0)}{2}\sum_{p\in P_H^c}(b_p^*b_{-p}^*+b_pb_{-p})\\ &+\frac{\widehat{V}(0)}{\sqrt{N}}\sum_{p\in P_H^c,q\in\Lambda_+^*,p\neq -q}\big[b^*_{p+q}a^*_{-p}a_q+a_q^*a_{-p}b_{p+q}\big]\\ &+\frac{\widehat{V}(0)}{2N}\sum_{p,q\in\Lambda_+^*,r\in P_H^c:r\neq -p,-q}a^*_{p+r}a^*_qa_pa_{q+r}.\end{aligned}$$

Comparing with (3.109) and noticing that

$$\begin{aligned}\frac{\widehat{V}(0)}{N}\sum_{\substack{p,q\in\Lambda_+^*,r\in P_H^c:\\ r\neq -p,-q}}\langle\xi,a^*_{p+r}a^*_qa_pa_{q+r}\xi\rangle &\leq \frac{C}{N}\sum_{\substack{p,q\in\Lambda_+^*,r\in P_H^c:\\ r\neq -p,-q}}\|a_{p+r}a_q\xi\|\|a_pa_{q+r}\xi\|\\ &\leq \frac{C\ell^{-2\alpha}}{N}\|\mathcal{N}_+\xi\|^2\end{aligned}$$

we conclude that

$$\mathcal{R}^{\beta}_{N,\ell} \geq \; U_N \left[\frac{1}{N} \sum_{i<j}^{N} v_\beta(x_i - x_j) \right] U_N^* - \frac{\widehat{V}(0)^2}{8\pi} \log N^\beta - C\ell^{-2\alpha} \mathcal{N}_+^2/N \\ + \big(1 - C\ell^\alpha |\log \ell|^2\big) \mathcal{H}_N^\beta - C\ell^\alpha |\log \ell|^2. \tag{3.110}$$

By standard arguments, see for instance [70, Lemma 1], we observe now that, since $\widehat{v}_\beta(p) \geq 0$ for all $p \in \Lambda^*$,

$$\begin{aligned} 0 &\leq \int_{\Lambda^2} dxdy \, v_\beta(x-y) \left[\sum_{j=1}^{N} \delta(x - x_j) - N \right] \left[\sum_{i=1}^{N} \delta(y - x_i) + N \right] \\ &= \sum_{i,j=1}^{N} v_\beta(x_i - x_j) - N^2 \widehat{v}_\beta(0) = 2 \sum_{i<j}^{N} v_\beta(x_i - x_j) + N v_\beta(0) - N^2 \widehat{v}_\beta(0). \end{aligned}$$

This implies that

$$\frac{1}{N} \sum_{i<j}^{N} v_\beta(x_i - x_j) \geq \frac{N}{2} \widehat{v}_\beta(0) - v_\beta(0) \geq \frac{\widehat{V}(0)}{2} N - C\ell^{-2\alpha}.$$

From (3.110), we finally obtain

$$\mathcal{R}^{\beta}_{N,\ell} \geq \frac{\widehat{V}(0)}{2} N - \frac{\widehat{V}(0)^2}{8\pi} \log N^\beta + (1 - C\ell^\alpha |\log \ell|^2) \mathcal{H}_N^\beta - C\ell^{-2\alpha} \mathcal{N}_+^2/N - C\ell^{-2\alpha}.$$

This completes the proof of Proposition 2.8.

Chapter 4

Analysis of the Renormalized Gross - Pitaevskii Hamiltonian

In this chapter, we proceed similarly as we did in Chapter 3, namely, we write explicitly all the bounds needed to prove properties of $\mathcal{G}_{N,\alpha}$ and $\mathcal{R}_{N,\alpha}$, defined as in Eq. (2.86) and Eq. (2.95) respectively, established in Prop. 2.11 and 2.14. These propositions are the key ingredient to prove Theorem 1.3. The analysis in Section 4.1 follows closely that of [10, Section 7] with some slight modifications due to the different scaling of the interaction potential and the fact that the kernel η_p of e^B is different from zero for all $p \in \Lambda_+^*$ (in [10] η_p is different from zero only for momenta larger than a sufficiently large cutoff of order one). Moreover, while in three dimensions, as well as in the dilute regime showed in Chapter 3, it was sufficient to choose the function η_p appearing in the generalized Bogoliubov transformation with $\|\eta\|$ sufficiently small but of order one, we need here $\|\eta\|$ to be of order $N^{-\alpha}$ for some $\alpha > 0$ large enough. As discussed in Chapter 2 this is achieved by considering the Neumann problem for the scattering equation in (2.69) on a ball of radius $\ell = N^{-\alpha}$; as a consequence some terms depending on ℓ will be large, compared to the analogous terms in [10].

On the other hand, in Section 4.2 we describe in details the analysis of $\mathcal{R}_{N,\alpha}$ and we end up in proving Proposition 2.8.

The calculations in the following sections are reported as in [20, Section 6 and Appendix A], with obvious modification to avoid overlapping with the analysis in Chapter 3.

4.1 Analysis of the quadratically renormalized excitation Hamiltonian $\mathcal{G}_{N,\alpha}$

The aim of this section is to show Prop. 2.11. From (2.8) and (2.86), we can decompose

$$\mathcal{G}_{N,\alpha} = e^{-B}\mathcal{L}_N e^B = \mathcal{G}_{N,\alpha}^{(0)} + \mathcal{G}_{N,\alpha}^{(2)} + \mathcal{G}_{N,\alpha}^{(3)} + \mathcal{G}_{N,\alpha}^{(4)}$$

with

$$\mathcal{G}_{N,\alpha}^{(j)} = e^{-B}\mathcal{L}_N^{(j)} e^B \, .$$

To analyse $\mathcal{G}_{N,\alpha}$ we need precise informations on the action of the generalized Bogoliubov transformation e^B, with B the antisymmetric operator defined in (2.27), as explained in Chapter 2, Section 2.2. Then, in the subsections 4.1.1–4.1.4 we prove separate bounds for the operators $\mathcal{G}_{N,\alpha}^{(j)}$, $j = 0, 2, 3, 4$, which we combine in Subsection 4.1.5 to prove Prop. 2.11. In the analysis we will make use, again, of Eq. (2.33), Lemma 2.3 and Lemma 3.1 with the appropriate smallness of the norm of η.

4.1.1 Analysis of $\mathcal{G}^{(0)}_{N,\alpha} = e^{-B}\mathcal{L}^{(0)}_N e^B$

We define $\mathcal{E}^{(0)}_N$ so that

$$\mathcal{G}^{(0)}_{N,\alpha} = e^{-B}\mathcal{L}^{(0)}_N e^B = \frac{1}{2}\widehat{V}(0)(N+\mathcal{N}_+-1)(N-\mathcal{N}_+) + \mathcal{E}^{(0)}_{N,\alpha}\,.$$

where we recall from (2.9) that

$$\mathcal{L}^{(0)}_N = \frac{1}{2}\widehat{V}(0)(N-1+\mathcal{N}_+)(N-\mathcal{N}_+)\,.$$

Proposition 4.1. *Under the assumptions of Prop. 2.11, there exists a constant $C>0$ such that*

$$\pm\mathcal{E}^{(0)}_{N,\alpha} \le CN^{1-\alpha}(\mathcal{N}_+ + 1)$$

for all $\alpha>0$ and $N\in\mathbb{N}$ large enough.

Proof. The proof follows [10, Prop. 7.1].

We write

$$\mathcal{L}^{(0)}_N = \frac{N(N-1)}{2}\widehat{V}(0) + \frac{N}{2}\widehat{V}(0)\Big[\sum_{q\in\Lambda^*_+} b^*_q b_q - \mathcal{N}_+\Big]\,.$$

Hence,

$$\mathcal{E}^{(0)}_N = \frac{N}{2}\widehat{V}(0)\sum_{q\in\Lambda^*_+}\left[e^{-B}b^*_q b_q e^B - b^*_q b_q\right] - \frac{N}{2}\widehat{V}(0)\left[e^{-B}\mathcal{N}_+ e^B - \mathcal{N}_+\right]\,.$$

To bound the first term we use (2.31), $|\gamma_q^2 - 1| \le C\eta_q^2$, $|\sigma_q| \le C|\eta_q|$, the first bound in (2.34), Cauchy-Schwarz and the estimate $\|\eta\| \le CN^{-\alpha}$. To bound the second term, we use Lemma 3.1. We conclude that

$$|\langle\xi, \mathcal{E}^{(0)}_N\xi\rangle| \le CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,.$$

□

4.1.2 Analysis of $\mathcal{G}^{(2)}_{N,\alpha} = e^{-B}\mathcal{L}^{(2)}_N e^B$

We consider first conjugation of the kinetic energy operator.

Proposition 4.2. *Under the assumptions of Prop. 2.11, there exists $C>0$ such that*

$$\begin{aligned} e^{-B}\mathcal{K}e^B = \mathcal{K} &+ \sum_{p\in\Lambda^*_+} p^2\eta_p(b_p b_{-p} + b^*_p b^*_{-p}) \\ &+ \sum_{p\in\Lambda^*_+} p^2\eta_p^2\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big) + \mathcal{E}^{(K)}_{N,\alpha} \end{aligned} \tag{4.1}$$

where

$$|\langle \xi, \mathcal{E}^{(K)}_{N,\alpha}\xi\rangle| \leq CN^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ +1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 \quad (4.2)$$

for any $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$ and $N \in \mathbb{N}$ large enough.

Proof. We proceed as in the proof of [10, Prop. 7.2]. We write

$$\begin{aligned}
&e^{-B}\mathcal{K}e^{B} - \mathcal{K} \\
&= \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p\Big[\big(\gamma_p^{(s)}b_p + \sigma_p^{(s)}b^*_{-p}\big)\big(\gamma_p^{(s)}b_{-p} + \sigma_p^{(s)}b^*_p\big) + \text{h.c.}\Big] \\
&\quad + \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p\big[\big(\gamma_p^{(s)}b_p + \sigma_p^{(s)}b^*_{-p}\big)d^{(s)}_{-p} + d^{(s)}_p\big(\gamma_p^{(s)}b_{-p} + \sigma_p^{(s)}b^*_p\big) + \text{h.c.}\big] \quad (4.3)\\
&\quad + \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p\big[d^{(s)}_p d^{(s)}_{-p} + \text{h.c.}\big] \\
&=: \mathrm{G}_1 + \mathrm{G}_2 + \mathrm{G}_3
\end{aligned}$$

with $\gamma_p^{(s)} = \cosh(s\eta_p)$, $\sigma_p^{(s)} = \sinh(s\eta_p)$ and where $d_p^{(s)}$ is defined as in (2.31), with η_p replaced by $s\eta_p$. We find

$$\mathrm{G}_1 = \sum_{p\in\Lambda_+^*} p^2\eta_p\big(b_pb_{-p} + b^*_{-p}b^*_p\big) + \sum_{p\in\Lambda_+^*} p^2\eta_p^2\left(1 - \frac{\mathcal{N}_+}{N}\right) + \mathcal{E}_1^K$$

with

$$\begin{aligned}
\mathcal{E}_1^K =& \,2\int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p(\sigma_p^{(s)})^2\big(b_pb_{-p} + b^*_{-p}b^*_p\big) \\
&+ \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p\gamma_p^{(s)}\sigma_p^{(s)}(4b^*_pb_p - 2N^{-1}a^*_pa_p) \\
&+ 2\int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p\left[(\gamma_p^{(s)}-1)\sigma_p^{(s)} + (\sigma_p^{(s)} - s\eta_p)\right]\left(1 - \frac{\mathcal{N}_+}{N}\right).
\end{aligned}$$

Since $|\big((\gamma_p^{(s)})^2 - 1\big)| \leq C\eta_p^2$, $(\sigma_p^{(s)})^2 \leq C\eta_p^2$, $p^2|\eta_p| \leq C$, $\|\eta\|_\infty \leq N^{-\alpha}$, we can estimate

$$\begin{aligned}
&|\langle\xi, \mathcal{E}_1^K\xi\rangle| \\
&\leq C\sum_{p\in\Lambda_+^*} p^2|\eta_p|^3\|b_p\xi\|\|(\mathcal{N}_+ +1)^{1/2}\xi\| + C\sum_{p\in\Lambda_+^*} p^2\eta_p^2\|a_p\xi\|^2 + C\sum_{p\in\Lambda_+^*} p^2\eta_p^4\|\xi\|^2 \\
&\leq C\|\eta\|\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 \leq CN^{-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2,
\end{aligned} \quad (4.4)$$

for any $\xi \in \mathcal{F}_+^{\leq N}$. To bound the term G_3 in (4.3), we switch to position space:

$$\begin{aligned}
|\langle\xi, \mathrm{G}_3\xi\rangle| \leq CN\int_0^1 ds\int_{\Lambda^2} dxdy\,&\left[e^{2N}V(e^N(x-y)) + N^{2\alpha-1}\chi(|x-y| \leq N^{-\alpha})\right] \\
&\times \|(\mathcal{N}_+ +1)^{-1/2}\check{d}_x^{(s)}\check{d}_y^{(s)}\xi\|\|(\mathcal{N}_+ +1)^{1/2}\xi\|
\end{aligned}$$

With (2.38), we obtain

$$\begin{aligned}
&|\langle \xi, \mathrm{G}_3 \xi\rangle| \\
&\leq CN^{1-\alpha} \int_{\Lambda^2} dxdy \left[e^{2N} V(e^N(x-y)) + N^{2\alpha-1}\chi(|x-y| \leq N^{-\alpha})\right] \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \\
&+ CN^{-2\alpha} \int_{\Lambda^2} dxdy \left[e^{2N} V(e^N(x-y)) + N^{2\alpha-1}\chi(|x-y| \leq N^{-\alpha})\right] \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\
&\qquad\qquad \times \Big[\|\check{a}_x (\mathcal{N}_+ + 1)\xi\| + \|\check{a}_y (\mathcal{N}_+ + 1)\xi\| + \|\check{a}_x \check{a}_y (\mathcal{N}_+ + 1)^{1/2}\xi\| \Big] \\
&\leq CN^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\| \,.
\end{aligned} \tag{4.5}$$

Finally, we consider G_2 in (4.3). We split it as $\mathrm{G}_2 = \mathrm{G}_{21} + \mathrm{G}_{22} + \mathrm{G}_{23} + \mathrm{G}_{24}$, with

$$\begin{aligned}
\mathrm{G}_{21} &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p \left(\gamma_p^{(s)} b_p d_{-p}^{(s)} + \text{h.c.} \right), \\
\mathrm{G}_{22} &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p \left(\sigma_p^{(s)} b_{-p}^* d_{-p}^{(s)} + \text{h.c.} \right) \\
\mathrm{G}_{23} &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p \left(\gamma_p^{(s)} d_p^{(s)} b_{-p} + \text{h.c.} \right), \\
\mathrm{G}_{24} &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p \left(\sigma_p^{(s)} d_p^{(s)} b_p^* + \text{h.c.} \right) \,.
\end{aligned} \tag{4.6}$$

We consider G_{21} first. We write

$$\mathrm{G}_{21} = -\sum_{p \in \Lambda_+^*} p^2 \eta_p^2 \frac{\mathcal{N}_+ + 1}{N} \frac{N - \mathcal{N}_+}{N} + \left[\mathcal{E}_2^K + \text{h.c.} \right]$$

where $\mathcal{E}_2^K = \sum_{j=1}^3 \mathcal{E}_{2j}^K$, with

$$\begin{aligned}
\mathcal{E}_{21}^K &= \frac{1}{2N} \sum_{p \in \Lambda_+^*} p^2 \eta_p^2 (\mathcal{N}_+ + 1) \big(b_p^* b_p - \frac{1}{N} a_p^* a_p \big), \\
\mathcal{E}_{22}^K &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p (\gamma_p^{(s)} - 1) b_p d_{-p}^{(s)}, \\
\mathcal{E}_{23}^K &= \int_0^1 ds \sum_{p \in \Lambda_+^*} p^2 \eta_p b_p \overline{d}_{-p}^{(s)}.
\end{aligned} \tag{4.7}$$

and where we introduced the notation $\overline{d}_{-p}^{(s)} = d_{-p}^{(s)} + s\eta_p (\mathcal{N}_+/N) b_p^*$. With (2.82), we find

$$|\langle \xi, \mathcal{E}_{21}^K \xi\rangle| \leq C \sum_{p \in \Lambda_+^*} \eta_p \|a_p \xi\|^2 \leq CN^{-\alpha} \|\mathcal{N}_+^{1/2}\xi\|^2 \tag{4.8}$$

Using $|\gamma_p^{(s)}-1| \leq C\eta_p^2$ and (2.34), we obtain

$$|\langle \xi, \mathcal{E}_{22}^K \xi\rangle| \leq \sum_{p\in\Lambda_+^*} p^2|\eta_p|^3 \|\mathcal{N}_+^{1/2}\xi\|\|d_{-p}^{(s)}\xi\| \leq CN^{-3\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2. \tag{4.9}$$

To control the third term in (4.7), we use (2.83) and we switch to position space. We find

$$\begin{aligned}\mathcal{E}_{23}^K = & -N\int_0^1 ds\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))f_{N,\ell}(x-y)\check{b}_x\check{\bar{d}}_y^{(s)}\\ &+N\int_0^1 dse^{2N}\lambda_\ell\int_{\Lambda^2} dxdy\,\chi_\ell(x-y)f_{N,\ell}(x-y)\check{b}_x\check{\bar{d}}_y^{(s)}\\ =&\ \mathcal{E}_{231}^K+\mathcal{E}_{232}^K\,.\end{aligned} \tag{4.10}$$

With (2.37) and $|\check{\eta}(x-y)| \leq CN$, we obtain

$$\begin{aligned}|\langle \xi, \mathcal{E}_{231}^K\xi\rangle| \leq N\int_0^1 ds\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))&\\ \times \|(\mathcal{N}_++1)^{1/2}\xi\|\|(\mathcal{N}_++1)^{-1/2}\check{a}_x\check{\bar{d}}_y^{(s)}\xi\|&\\ \leq CN^{1-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2+CN^{1/2-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|.&\end{aligned} \tag{4.11}$$

As for $\mathcal{E}_{232}^K$, with (2.37) and Lemma 2.10 (recalling $\ell = N^{-\alpha}$), we find

$$\begin{aligned}|\langle \xi, \mathcal{E}_{232}^K\xi\rangle| \leq\ & CN^{-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2\\ &+\int_{\Lambda^2} dxdy\,\chi(|x-y|\leq N^{-\alpha})\|(\mathcal{N}_++1)^{1/2}\xi\|\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|\end{aligned} \tag{4.12}$$

To bound the last term on the r.h.s. of (4.12) we use Hölder's and Sobolev inequality $\|u\|_q \leq Cq^{1/2}\|u\|_{H^1}$, valid for any $2 \leq q < \infty$. We find

$$\begin{aligned}&\int_{\Lambda^2} dxdy\,\chi(|x-y|\leq N^{-\alpha})\|(\mathcal{N}_++1)^{1/2}\xi\|\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|\\ &\leq C\|(\mathcal{N}_++1)^{1/2}\xi\|\int_\Lambda dx\left(\int_\Lambda dy\,\chi(|x-y|\leq N^{-\alpha})\right)^{1-1/q}\left(\int_\Lambda dy\,\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|^q\right)^{1/q}\\ &\leq CN^{2\alpha/q-2\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\int_\Lambda dx\left(\int_\Lambda dy\,\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|^q\right)^{1/q}\\ &\leq Cq^{1/2}N^{2\alpha/q-2\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\\ &\qquad\times\left[\int_{\Lambda^2} dxdy\,\|\check{a}_x\nabla_y\check{a}_y\mathcal{N}_+^{1/2}\xi\|^2+\int_{\Lambda^2} dxdy\,\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|^2\right]^{1/2}\\ &\leq Cq^{1/2}N^{2\alpha/q-2\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\left[\ \|\mathcal{K}^{1/2}\mathcal{N}_+\xi\|+\|\mathcal{N}_+^{3/2}\xi\|\right]\,.\end{aligned}$$

Choosing $q = \log N$, we get

$$\begin{aligned}&\int_{\Lambda^2} dxdy\,\chi(|x-y|\leq N^{-\alpha})\|(\mathcal{N}_++1)^{1/2}\xi\|\|\check{a}_x\check{a}_y(\mathcal{N}_++1)^{1/2}\xi\|\\ &\quad\leq CN^{1-2\alpha}(\log N)^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned} \tag{4.13}$$

Therefore, for any $\xi \in \mathcal{F}_+^{\leq N}$,

$$|\langle \xi, \mathcal{E}_{232}^K \xi\rangle| \leq \; N^{1-2\alpha}(\log N)^{1/2}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + N^{-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \,.$$

Combining the last bound with (4.8), (4.9) and (4.11), we conclude that

$$|\langle \xi, \mathcal{E}_2^K \xi\rangle| \leq CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \,. \tag{4.14}$$

for any $\alpha > 1$, $N \in \mathbb{N}$ large enough, $\xi \in \mathcal{F}_+^{\leq N}$.

The term G_{22} in (4.6) can be bounded using (2.34). We find

$$|\langle \xi, \mathrm{G}_{22}\xi\rangle| \leq CN^{-2\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \,. \tag{4.15}$$

We split $\mathrm{G}_{23} = \mathcal{E}_{31}^K + \mathcal{E}_{32}^K + \text{h.c.}$, with

$$\mathcal{E}_{31}^K = \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p\big(\gamma_p^{(s)} - 1\big)d_p^{(s)}b_{-p} \,, \qquad \mathcal{E}_{32}^K = \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p d_p^{(s)}b_{-p}$$

With (2.34), we find

$$|\langle \xi, \mathcal{E}_{31}^K \xi\rangle| \leq C\int_0^1 ds \sum_{p\in\Lambda_+^*} p^2|\eta_p|^3\|(d_p^{(s)})^*\xi\|\|b_{-p}\xi\|ds \leq CN^{-3\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2$$

To estimate $\mathcal{E}_{32}^K$, we use (2.83) and we switch to position space. Proceeding as we did in (4.10), (4.11), (4.12), we obtain

$$\begin{aligned}|\langle \xi, \mathcal{E}_{32}^K \xi\rangle| \leq CN\int_0^1 ds\int_{\Lambda^2} dxdy\, &\big[e^{2N}V(e^N(x-y)) + N^{2\alpha-1}\chi(|x-y|\leq N^{-\alpha})\big] \\ &\times \|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{-1/2}\check{d}_x^{(s)}\check{b}_y\xi\| \,.\end{aligned}$$

With (2.36) and (4.13) we find

$$\begin{aligned}|\langle \xi, \mathcal{E}_{32}^K \xi\rangle| &\leq CN^{-\alpha}\int_{\Lambda^2} dxdy\, \big[e^{2N}V(e^N(x-y)) + N^{2\alpha-1}\chi(|x-y|\leq N^{-\alpha})\big] \\ &\qquad\qquad \times \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \big[\|\check{a}_y(\mathcal{N}_+ + 1)\xi\| + \|\check{a}_x\check{a}_y(\mathcal{N}_+ + 1)^{1/2}\xi\|\big] \\ &\leq CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\| \\ &\qquad + CN^{1-2\alpha}(\log N)^{1/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\| \,.\end{aligned}$$

Combining the bounds for $\mathcal{E}_{31}^K$ and $\mathcal{E}_{32}^K$, we conclude that, if $\alpha > 1$,

$$|\langle \xi, \mathrm{G}_{23}\xi\rangle| \leq CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{H}_N^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \tag{4.16}$$

To bound G_{24} in (4.6), we use (2.34), the bounds (2.81) and $\|\eta\|_{H_1}^2 \leq CN$, and the commutator (2.5):

$$\begin{aligned}&|\langle \xi, \mathrm{G}_{24}\xi\rangle| \\ &\quad\leq C\int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{-1/2}d_p^{(s)}b_p^*\xi\| \\ &\quad\leq C\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\sum_{p\in\Lambda_+^*} p^2\eta_p^2\big[|\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + N^{-1}\|\eta\|\|b_pb_p^*(\mathcal{N}_+ + 1)^{1/2}\xi\|\big] \\ &\quad\leq CN^{-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \,.\end{aligned}$$

Together with (4.6), (4.14), (4.15) and (4.16), this implies that

$$\mathrm{G}_2 = -\sum_{p\in\Lambda_+^*} p^2\eta_p^2 \frac{\mathcal{N}_+ + 1}{N}\frac{N-\mathcal{N}_+}{N} + \mathcal{E}_4^K$$

with

$$|\langle\xi, \mathcal{E}_4^K\xi\rangle| \leq CN^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \,. \quad (4.17)$$

Combining (4.4), (4.5) and (4.17), we obtain (4.1) and (4.2). □

In the next proposition, we consider the conjugation of the operator

$$\mathcal{L}_N^{(2,V)} = N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\left[b_p^*b_p - \frac{1}{N}a_p^*a_p\right] + \frac{N}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\left[b_p^*b_{-p}^* + b_pb_{-p}\right]$$

Proposition 4.3. *Under the assumptions of Prop. 2.11, there is a constant $C > 0$ such that*

$$\begin{aligned} e^{-B}\mathcal{L}_N^{(2,V)}e^B = {} & N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\eta_p\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big) \\ & + N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)a_p^*a_p\left(1-\frac{\mathcal{N}_+}{N}\right) \\ & + \frac{N}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\big(b_pb_{-p} + b_{-p}^*b_p^*\big) + \mathcal{E}_N^{(V)} \end{aligned} \quad (4.18)$$

where

$$|\langle\xi, \mathcal{E}_N^{(V)}\xi\rangle| \leq CN^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \,. \quad (4.19)$$

for any $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$ and $N \in \mathbb{N}$ large enough.

Proof. We write

$$\begin{aligned} e^{-B}\mathcal{L}_N^{(2,V)}e^B = {} & N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)e^{-B}b_p^*b_pe^B - \sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)e^{-B}a_p^*a_pe^B \\ & + \frac{N}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)e^{-B}\big[b_pb_{-p} + b_p^*b_{-p}^*\big]e^B \\ =: {} & \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3 \,. \end{aligned} \quad (4.20)$$

With (2.31), we find

$$\begin{aligned} \mathrm{F}_1 = {} & N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\big[\gamma_pb_p^* + \sigma_pb_{-p}\big]\big[\gamma_pb_p + \sigma_pb_{-p}^*\big] \\ & + N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\big[(\gamma_pb_p^* + \sigma_pb_{-p})d_p + d_p^*(\gamma_pb_p + \sigma_pb_{-p}^*) + d_p^*d_p\big] \end{aligned}$$

where $\gamma_p = \cosh \eta_p$, $\sigma_p = \sinh \eta_p$ and the operators d_p are defined in (2.31). Using $|1-\gamma_p| \leq \eta_p^2$, $|\sigma_p| \leq C|\eta_p|$ and using Lemma 2.3 for the terms on the second line, we find

$$\mathrm{F}_1 = N \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N) b_p^* b_p + \mathcal{E}_1^V \tag{4.21}$$

with $\pm\mathcal{E}_1^V \leq CN^{1-\alpha}(\mathcal{N}_+ + 1)$.

Let us now consider the second contribution on the r.h.s. of (4.20). We find

$$-\mathrm{F}_2 = \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N) a_p^* a_p + \mathcal{E}_2^V \tag{4.22}$$

with

$$\mathcal{E}_2^V = \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N) \int_0^1 e^{-sB}(\eta_p b_{-p} b_p + \text{h.c.}) e^{sB} ds.$$

With Lemma 2.2, we easily find $\pm\mathcal{E}_2^V \leq CN^{-\alpha}(\mathcal{N}_+ + 1)$.

Finally, we consider the last term on the r.h.s. of (4.20). With (2.31), we obtain

$$\begin{aligned}
\mathrm{F}_3 = &\frac{N}{2} \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N) \left[\gamma_p b_p + \sigma_p b_{-p}^*\right] \left[\gamma_p b_{-p} + \sigma_p b_p^*\right] + \text{h.c.} \\
&+ \frac{N}{2} \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N) \left[(\gamma_p b_p + \sigma_p b_{-p}^*)\, d_{-p} + d_p\,(\gamma_p b_{-p} + \sigma_p b_p^*)\right] + \text{h.c.} \\
&+ \frac{N}{2} \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N) d_p d_{-p} + \text{h.c.} \\
=: &\, \mathrm{F}_{31} + \mathrm{F}_{32} + \mathrm{F}_{33}\,.
\end{aligned} \tag{4.23}$$

Using $|1-\gamma_p| \leq C\eta_p^2$, $|\sigma_p| \leq C|\eta_p|$, we obtain

$$\mathrm{F}_{31} = \frac{N}{2} \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N) \big(b_p b_{-p} + b_{-p}^* b_p^*\big) + N \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N) \eta_p \frac{N-\mathcal{N}_+}{N} + \mathcal{E}_3^V \tag{4.24}$$

with $\pm\mathcal{E}_3^V \leq CN^{1-\alpha}(\mathcal{N}_+ + 1)$. As for F_{32} in (4.23), we divide it into four parts

$$\begin{aligned}
\mathrm{F}_{32} = &\frac{N}{2} \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N) \left[(\gamma_p b_p + \sigma_p b_{-p}^*)\, d_{-p} + d_p\,(\gamma_p b_{-p} + \sigma_p b_p^*)\right] + \text{h.c.} \\
=: &\, \mathrm{F}_{321} + \mathrm{F}_{322} + \mathrm{F}_{323} + \mathrm{F}_{324}\,.
\end{aligned} \tag{4.25}$$

We start with F_{321}, which we write as

$$\mathrm{F}_{321} = -N \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N) \eta_p \left(\frac{N-\mathcal{N}_+}{N}\right) \left(\frac{\mathcal{N}_+ + 1}{N}\right) + \mathcal{E}_4^V$$

where $\mathcal{E}_4^V = \mathcal{E}_{41}^V + \mathcal{E}_{42}^V + \mathcal{E}_{43}^V + \text{h.c.}$, with

$$\begin{aligned}\mathcal{E}_{41}^V &= \frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\,(\gamma_p - 1)b_p d_{-p}\,, \qquad \mathcal{E}_{42}^V = \frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N) b_p \overline{d}_{-p} \\ \mathcal{E}_{43}^V &= -\frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_p \frac{\mathcal{N}_+ + 1}{N}(b_p^* b_p - N^{-1}a_p^* a_p)\end{aligned}$$

and with the notation $\overline{d}_{-p} = d_{-p} + N^{-1}\eta_p \mathcal{N}_+ b_p^*$. Since $|\gamma_p - 1| \le C\eta_p^2$, $\|\eta\|_\infty \le CN^{-\alpha}$, we find easily with (2.34) that

$$|\langle \xi, \mathcal{E}_{41}^V \xi\rangle| \le CN^{1-3\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 .$$

Moreover

$$|\langle \xi, \mathcal{E}_{43}^V \xi\rangle| \le CN \sum_{p\in\Lambda_+^*} \eta_p \|a_p\xi\|^2 \le CN^{1-\alpha}\|\mathcal{N}_+^{1/2}\xi\|^2 .$$

As for $\mathcal{E}_{42}^V$, we switch to position space and we use (2.37). We obtain

$$\begin{aligned}|\langle \xi, \mathcal{E}_{42}^V \xi\rangle| &\le CN \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{-1/2} \check{a}_x \check{\overline{d}}_y \xi\| \\ &\le CN^{1-\alpha} \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\qquad \times \Big[\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_x\xi\| + \|\check{a}_y\xi\| + N^{-1/2}\|\check{a}_x\check{a}_y\xi\| \Big] \\ &\le CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\| .\end{aligned}$$

We conclude that

$$|\langle \xi, \mathcal{E}_4^V \xi\rangle| \le CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2$$

To bound the term F_{322} in (4.25), we use (2.34) and $|\sigma_p| \le C|\eta_p|$; we obtain

$$\begin{aligned}|\langle \xi, \mathrm{F}_{322}\xi\rangle| &\le CN \sum_{p\in\Lambda_+^*} |\eta_p| \|b_{-p}\xi\| \left[|\eta_p| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta\| \|b_{-p}\xi\| \right] \\ &\le CN^{1-2\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 .\end{aligned}$$

Let us now consider the term F_{323} on the r.h.s. of (4.25). We write $\mathrm{F}_{323} = \mathcal{E}_{51}^V + \mathcal{E}_{52}^V + \text{h.c.}$, with

$$\mathcal{E}_{51}^V = \frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\,(\gamma_p - 1)\, d_p b_{-p}\,, \qquad \mathcal{E}_{52}^V = \frac{N}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\, d_p b_{-p}\,.$$

With $|\gamma_p - 1| \le C\eta_p^2$ and (2.34) we obtain

$$|\langle \xi, \mathcal{E}_{51}^V \xi\rangle| \le CN \sum_{p\in\Lambda_+^*} \eta_p^2\, \|d_p^*\xi\| \|a_p\xi\| \le CN^{1-3\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 .$$

We find, switching to position space and using (2.36),

$$\begin{aligned}
&|\langle \xi, \mathcal{E}^V_{52}\xi\rangle| \\
&\leq CN \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|(\mathcal{N}_+ +1)^{-1/2} \check{d}_x \check{a}_y \xi\| \\
&\leq CN^{1-\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\| \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \left[\|\check{a}_y\xi\| + N^{-1/2} \|\check{a}_x \check{a}_y \xi\| \right] \\
&\leq CN^{1-\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + CN^{1/2-\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\| \,.
\end{aligned}$$

Hence,

$$|\langle \xi, \mathrm{F}_{323}\xi\rangle| \leq CN^{1-\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + CN^{1/2-\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\|$$

To estimate the term F_{324} in (4.25) we use (2.34) and the bound

$$\begin{aligned}
\sum_{p\in\Lambda_+^*} |\widehat{V}(p/e^N)||\eta_p| &\leq C \sum_{p\in\Lambda_+^*,\, |p|\leq e^N} \frac{1}{p^2} + C \sum_{p\in\Lambda_+^*,\, |p|>e^N} \frac{|\widehat{V}(p/e^N)|}{p^2} \\
&\leq CN + C \Bigg(\sum_{p\in\Lambda_+^*} |\widehat{V}(p/e^N)|^2 \Bigg)^{1/2} \Bigg(\sum_{p\in\Lambda_+^*,\, |p|>e^N} \frac{1}{p^4} \Bigg)^{1/2} \\
&\leq CN
\end{aligned}$$

We find

$$\begin{aligned}
|\langle \xi, \mathrm{F}_{324}\xi\rangle| \leq\; & CN \sum_{p\in\Lambda_+^*} \big|\widehat{V}(p/e^N)\big| |\eta_p| \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|(\mathcal{N}_+ +1)^{-1/2} d_p\, b_p^* \xi\| \\
\leq\; & CN \sum_{p\in\Lambda_+^*} \big|\widehat{V}(p/e^N)\big| |\eta_p| \|(\mathcal{N}_+ +1)^{1/2}\xi\| \\
& \quad \times \left[|\eta_p| \|(\mathcal{N}_+ +1)^{1/2}\xi\| + N^{-1} \|\eta\| \|b_p b_p^* (\mathcal{N}_+ +1)^{1/2}\xi\| \right] \\
\leq\; & CN \sum_{p\in\Lambda_+^*} \big|\widehat{V}(p/e^N)\big| |\eta_p| \|(\mathcal{N}_+ +1)^{1/2}\xi\| \\
& \quad \times \left[|\eta_p| \|(\mathcal{N}_+ +1)^{1/2}\xi\| + N^{-1} \|\eta\| \|(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\eta\| \|a_p\xi\| \right] \\
\leq\; & CN^{1-\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 \,.
\end{aligned}$$

Combining the last bounds, we arrive at

$$\mathrm{F}_{32} = N \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_p \left(\frac{N-\mathcal{N}_+}{N} \right) \left(\frac{-\mathcal{N}_+ -1}{N} \right) + \mathcal{E}^V_6$$

with

$$|\langle \xi, \mathcal{E}^V_6\xi\rangle| \leq CN^{1-\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + CN^{1/2-\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\| \,. \quad (4.26)$$

To control the last contribution F_{33} in (4.23), we switch to position space. With (2.38) and (2.78) we obtain

$$\begin{aligned}
|\langle \xi, \mathrm{F}_{33}\xi\rangle| &\leq CN \|(\mathcal{N}_+ +1)^{1/2}\xi\| \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ +1)^{-1/2} \check{d}_x \check{d}_y \xi\| \\
&\leq CN^{1-\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + CN^{1/2-2\alpha} \|(\mathcal{N}_+ +1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\| \,.
\end{aligned}$$

The last equation, combined with (4.23), (4.24) and (4.26), implies that

$$\begin{aligned}\mathrm{F}_3 = &\frac{N}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)(b_pb_{-p}+b^*_{-p}b^*_p)\\ &+N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\eta_p\left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{N-\mathcal{N}_+-1}{N}\right)+\mathcal{E}_7^V\end{aligned}$$

with

$$|\langle\xi,\mathcal{E}_7^V\xi\rangle|\le CN^{1-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2+CN^{1/2-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\,.$$

Together with (4.21) and with (4.22), and recalling that $b_p^*b_p-N^{-1}a_p^*a_p=a_p^*a_p(1-\mathcal{N}_+/N)$, we obtain (4.18) with (4.19). □

4.1.3 Analysis of $\mathcal{G}_{N,\alpha}^{(3)}=e^{-B}\mathcal{L}_N^{(3)}e^B$

We consider here the conjugation of the cubic term $\mathcal{L}_N^{(3)}$, defined in (2.9).

Proposition 4.4. *Under the assumptions of Prop. 2.11, there exists a constant $C>0$ such that*

$$\mathcal{G}_{N,\alpha}^{(3)}=e^{-B}\mathcal{L}_N^{(3)}e^B=\sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq0}\widehat{V}(p/e^N)\left[b^*_{p+q}a^*_{-p}a_q+\text{h.c.}\right]+\mathcal{E}_N^{(3)}$$

where

$$|\langle\xi,\mathcal{E}_N^{(3)}\xi\rangle|\le CN^{1/2-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|+CN^{1-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2 \quad (4.27)$$

for any $\alpha>1$ and $N\in\mathbb{N}$ large enough.

Proof. This proof is similar to the proof of [10, Prop. 7.5]. Expanding $e^{-B}a^*_{-p}a_qe^B$, we arrive at

$$\begin{aligned}\mathcal{E}_N^{(3)}=&\sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq0}\widehat{V}(p/e^N)\big((\gamma_{p+q}-1)b^*_{p+q}+\sigma_{p+q}b_{-p-q}+d^*_{p+q}\big)\,a^*_{-p}a_q\\ &+\sqrt{N}\sum_{p,q\in\Lambda_+^*,p+q\neq0}\widehat{V}(p/e^N)\eta_p\,e^{-B}b^*_{p+q}e^B\int_0^1ds\,e^{-sB}b_pb_qe^{sB}\\ &+\sqrt{N}\sum_{p,q\in\Lambda_+^*,p+q\neq0}\widehat{V}(p/e^N)\eta_q\,e^{-B}b^*_{p+q}e^B\int_0^1ds\,e^{-sB}b^*_{-p}b^*_{-q}e^{sB}\\ &+\text{h.c.}\\ =:&\;\mathcal{E}_1^{(3)}+\mathcal{E}_2^{(3)}+\mathcal{E}_3^{(3)}+\text{h.c.}\end{aligned} \quad (4.28)$$

where, as usual, $\gamma_p=\cosh\eta(p)$, $\sigma_p=\sinh\eta(p)$ and d_p is as in (2.31). We consider $\mathcal{E}_1^{(3)}$. To this end, we write

$$\begin{aligned}\mathcal{E}_1^{(3)}=&\sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq0}\widehat{V}(p/e^N)\big((\gamma_{p+q}-1)b^*_{p+q}+\sigma_{p+q}b_{-p-q}+d^*_{p+q}\big)\,a^*_{-p}a_q\\ =:&\,\mathcal{E}_{11}^{(3)}+\mathcal{E}_{12}^{(3)}+\mathcal{E}_{13}^{(3)}\,.\end{aligned}$$

Since $|\gamma_{p+q}-1| \leq |\eta_{p+q}|^2$ and $\|\eta\| \leq CN^{-\alpha}$, we find

$$|\langle \xi, \mathcal{E}_{11}^{(3)} \xi\rangle| \leq CN\|\eta\|^2 \|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 \leq CN^{1-2\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 . \tag{4.29}$$

As for $\mathcal{E}_{12}^{(3)}$, we commute a^*_{-p} through b_{-p-q} (recall $q\neq 0$). With $|\sigma_{p+q}| \leq C|\eta_{p+q}|$, we obtain

$$|\langle \xi, \mathcal{E}_{12}^{(3)} \xi\rangle| \leq CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 . \tag{4.30}$$

We decompose now $\mathcal{E}_{13}^{(3)} = \mathcal{E}_{131}^{(3)} + \mathcal{E}_{132}^{(3)}$, with

$$\begin{aligned}
\mathcal{E}_{131}^{(3)} &= \sqrt{N} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/e^N)\, \bar{d}^*_{p+q} a^*_{-p} a_q \\
\mathcal{E}_{132}^{(3)} &= -\frac{(\mathcal{N}_+ +1)}{N}\sqrt{N} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/e^N)\eta_{p+q}\, b_{-p-q} a^*_{-p} a_q \, .
\end{aligned}$$

where we defined $d^*_{p+q} = \overline{d}^*_{p+q} - \frac{(\mathcal{N}_+ +1)}{N}\eta_{p+q} b_{-p-q}$. The term $\mathcal{E}_{132}^{(3)}$ is estimated similarly to $\mathcal{E}_{12}^{(3)}$, moving a^*_{-p} to the left of b_{-p-q}; we find $\pm\mathcal{E}_{132}^{(3)} \leq CN^{1-\alpha}(\mathcal{N}_+ +1)$. We bound $\mathcal{E}_{131}^{(3)}$ in position space. We find

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{131}^{(3)} \xi\rangle| \\
&\leq N^{1/2}\int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|\check{a}_x \xi\| \|\check{a}_y \check{\bar{d}}_x \xi\| \\
&\leq CN^{1/2-\alpha}\int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|\check{a}_x \xi\| \\
&\quad \times \big[\, \|(\mathcal{N}_+ +1)\xi\| + N^{-1}\|\check{a}_x(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\eta\|\|\check{a}_y(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\check{a}_x\check{a}_y\xi\|\big] \\
&\leq CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\| .
\end{aligned}$$

With (4.29) and (4.30) we obtain

$$|\langle \xi, \mathcal{E}_{1}^{(3)} \xi\rangle| \leq CN^{1/2-\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|(\mathcal{N}_+ +1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 . \tag{4.31}$$

Next, we focus on $\mathcal{E}_{2}^{(3)}$, defined in (4.28). With Eq. (2.31), we find

$$\begin{aligned}
\mathcal{E}_{2}^{(3)} = \sqrt{N} &\sum_{p,q\in\Lambda_+^*, p+q\neq 0} \widehat{V}(p/e^N)\eta_p\, e^{-B} b^*_{p+q} e^B \\
&\times \int_0^1 ds\, \big(\gamma_p^{(s)}\gamma_q^{(s)} b_p b_q + \sigma_p^{(s)}\sigma_q^{(s)} b^*_{-p} b^*_{-q} + \gamma_p^{(s)}\sigma_q^{(s)} b^*_{-q} b_p + \sigma_p^{(s)}\gamma_q^{(s)} b^*_{-p} b_q\big) \\
&+ \sqrt{N} \sum_{p,q\in\Lambda_+^*, p+q\neq 0} \widehat{V}(p/e^N)\eta_p\, e^{-B} b^*_{p+q} e^B \int_0^1 ds\, \gamma_p^{(s)}\sigma_q^{(s)} [b_p, b^*_{-q}] \\
&+ \sqrt{N} \sum_{p,q\in\Lambda_+^*, p+q\neq 0} \widehat{V}(p/e^N)\eta_p\, e^{-B} b^*_{p+q} e^B \\
&\qquad \times \int_0^1 ds \Big[d_p^{(s)}\big(\gamma_q^{(s)} b_q + \sigma_q^{(s)} b^*_{-q}\big) + \big(\gamma_p^{(s)} b_p + \sigma_p^{(s)} b^*_{-p}\big) d_q^{(s)} + d_p^{(s)} d_q^{(s)}\Big] \\
&=: \mathcal{E}_{21}^{(3)} + \mathcal{E}_{22}^{(3)} + \mathcal{E}_{23}^{(3)}
\end{aligned} \tag{4.32}$$

with $\gamma_p^{(s)} = \cosh(s\eta_p)$, $\sigma_p^{(s)} = \sinh(s\eta_p)$ and $d_p^{(s)}$ defined as in (2.31), with η replaced by $s\eta$. With Lemma 2.2, we get

$$|\langle \xi, \mathcal{E}_{21}^{(3)} \xi \rangle| \leq C N^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 . \tag{4.33}$$

Since $[b_p, b^*_{-q}] = -a^*_{-q} a_p / N$ for $p \neq -q$, we find

$$|\langle \xi, \mathcal{E}_{22}^{(3)} \xi \rangle| \leq C N^{-2\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 . \tag{4.34}$$

As for the third term on the r.h.s. of (4.32), we switch to position space. We find

$$\begin{aligned}\mathcal{E}_{23}^{(3)} = \sqrt{N} \int_{\Lambda^3} & dx dy dz \, e^{2N} V(e^N(x-z)) \check{\eta}(y-z) \, e^{-B} \check{b}^*_x e^B \\ & \times \int_0^1 ds \left[\check{d}_y^{(s)} \big(b(\check{\gamma}_x^{(s)}) + b^*(\check{\sigma}_x^{(s)}) \big) + \big(b(\check{\gamma}_y^{(s)}) + b^*(\check{\sigma}_y^{(s)}) \big) \check{d}_x^{(s)} + \check{d}_y^{(s)} \check{d}_x^{(s)} \right] .\end{aligned}$$

Using the bounds (2.36), (2.37), (2.38) and Lemma 2.2 we arrive at

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{23}^{(3)} \xi \rangle| \\
&\leq C\sqrt{N} \int_{\Lambda^3} dx dy dz \, e^{2N} V(e^N(x-z)) |\check{\eta}(y-z)| \|\check{b}_x e^B \xi\| \int_0^1 ds \\
&\times \Big[\|\check{d}_y^{(s)} \big(\check{b}_x + b(\check{r}_x^{(s)}) + b^*(\check{\sigma}_x^{(s)}) \big) \xi\| + \| \big(\check{b}_y + b(\check{r}_y^{(s)}) + b^*(\check{\sigma}_y^{(s)}) \big) \check{d}_x^{(s)} \xi\| + \|\check{d}_x^{(s)} \check{d}_y^{(s)} \xi\| \Big] \\
&\leq C\sqrt{N} \int_{\Lambda^3} dx dy dz \, e^{2N} V(e^N(x-z)) |\check{\eta}(y-z)| \|\check{b}_x e^B \xi\| \Big[N^{-1} |\check{\eta}(x-y)| \|(\mathcal{N}_+ + 1)\xi\| \\
&\quad + \|\eta\| \|\check{b}_x \check{b}_y \xi\| + \|\eta\| \|(\mathcal{N}_+ + 1)\xi\| + \|\eta\| \|\check{b}_x (\mathcal{N}_+ + 1)^{1/2} \xi\| + \|\eta\| \|\check{b}_y (\mathcal{N}_+ + 1)^{1/2} \xi\| \Big] \\
&\leq C N^{1-\alpha} \|\mathcal{N}_+^{1/2} e^B \xi\| \|(\mathcal{N}_+ + 1)\xi\| \\
&\leq C N^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2
\end{aligned}$$

where $\check{r}$ indicates the function in $L^2(\Lambda)$ with Fourier coefficients $r_p = 1 - \gamma_p$, and the fact that $\|\check{\eta}\|, \|\check{r}\|, \|\check{\sigma}\| \leq C N^{-\alpha}$. Combined with (4.33) and (4.34), the last bound implies that

$$\pm \mathcal{E}_2^{(3)} \leq C N^{1-\alpha} (\mathcal{N}_+ + 1) . \tag{4.35}$$

To bound the last contribution on the r.h.s. of (4.28), it is convenient to bound (in absolute value) the expectation of its adjoint

$$\begin{aligned}\mathcal{E}_3^{(3)*} = \sqrt{N} \sum_{p,q \in \Lambda_+^*, p+q \neq 0} & \widehat{V}(p/e^N) \eta_q \int_0^1 ds \, e^{-sB} b_{-q} e^{sB} \\ & \times \big(\gamma_p^{(s)} b_{-p} + \sigma_p^{(s)} b^*_p + d_{-p}^{(s)} \big) \big(\gamma_{p+q} b_{p+q} + \sigma_{p+q} b^*_{-p-q} + d_{p+q} \big)\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_3^{(3)*} &= \sqrt{N} \sum_{p,q\in\Lambda_+^*, p+q\neq 0} \widehat{V}(p/e^N)\eta_q \int_0^1 ds\, e^{-sB} b_{-q} e^{sB} \\
&\quad \times \Big[\gamma_p^{(s)}\gamma_{p+q} b_{-p} b_{p+q} + \sigma_p^{(s)}\sigma_{p+q} b_p^* b_{-p-q}^* + \gamma_p^{(s)}\sigma_{p+q} b_{-p-q}^* b_{-p} + \gamma_{p+q}\sigma_p^{(s)} b_p^* b_{p+q} \\
&\qquad + d_{-p}^{(s)}\big(\gamma_{p+q} b_{p+q} + \sigma_{p+q} b_{-p-q}^*\big) + \big(\gamma_p^{(s)} b_{-p} + \sigma_p^{(s)} b_p^*\big) d_{p+q} + d_{-p}^{(s)} d_{p+q}\Big] \\
&\quad + \sqrt{N} \sum_{p,q\in\Lambda_+^*, p+q\neq 0} \widehat{V}(p/e^N)\eta_q \int_0^1 ds\, e^{-sB} b_{-q} e^{sB} \gamma_p^{(s)} \sigma_{p+q} [b_{-p}, b_{-p-q}^*] \\
&=: \mathcal{E}_{31}^{(3)} + \mathcal{E}_{32}^{(3)} \,.
\end{aligned}$$

Since $q \neq 0$, $[b_{-p}, b_{-p-q}^*] = -a_{-p-q}^* a_{-p}/N$. Thus, we can estimate

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{32}^{(3)} \xi\rangle| \\
&\quad \leq C N^{-1/2} \int_0^1 ds \sum_{p,q\in\Lambda_+^*, p+q\neq 0} |\eta_q||\eta_{p+q}| \, \|a_{-p-q}\, e^{-sB} b_{-q}^* e^{sB}\xi\| \|a_{-p}\xi\| \\
&\quad \leq C\|\eta\|^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \leq C N^{-2\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \,.
\end{aligned} \tag{4.36}$$

To bound the expectation of $\mathcal{E}_{31}^{(3)}$, we switch to position space. We find

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{31}^{(3)}\xi\rangle| \\
&\leq N^{1/2} \int_0^1 ds \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|b^*(\check{\eta}_x) e^{sB}\xi\| \Big[\|\check{b}_x \check{b}_y \xi\| \\
&+ \|\eta\| \|\check{b}_x (\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta\|\|\check{b}_y(\mathcal{N}_+ + 1)^{1/2}\xi\| + N^{-1}|\check{\eta}(x-y)| \|(\mathcal{N}_+ + 1)\xi\|\Big] \,.
\end{aligned}$$

With Lemma 2.2, we conclude that

$$|\langle \xi, \mathcal{E}_{31}^{(3)}\xi\rangle| \leq C N^{1/2-\alpha} \|\mathcal{V}_N^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + C N^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \,. \tag{4.37}$$

From (4.36) and (4.37) we obtain

$$|\langle \xi, \mathcal{E}_{3}^{(3)}\xi\rangle| \leq C N^{1/2-\alpha} \|\mathcal{V}_N^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + C N^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \,.$$

Together with (4.28), (4.31) and (4.35), we arrive at (4.27). □

4.1.4 Analysis of $\mathcal{G}_{N,\alpha}^{(4)} = e^{-B}\mathcal{L}_N^{(4)} e^B$

Finally, we consider the conjugation of the quartic term $\mathcal{L}_N^{(4)}$. We define the error operator $\mathcal{E}_N^{(4)}$ through

$$\begin{aligned}
\mathcal{G}_{N,\alpha}^{(4)} = e^{-B}\mathcal{L}_N^{(4)} e^B &= \mathcal{V}_N + \frac{1}{2} \sum_{\substack{q\in\Lambda_+^*, r\in\Lambda^* \\ r\neq -q}} \widehat{V}(r/e^N)\eta_{q+r}\eta_q \left(1 - \frac{\mathcal{N}_+}{N}\right)\left(1 - \frac{\mathcal{N}_+ + 1}{N}\right) \\
&\quad + \frac{1}{2} \sum_{\substack{q\in\Lambda_+^*, r\in\Lambda^*: \\ r\neq -q}} \widehat{V}(r/e^N)\, \eta_{q+r} \left(b_q b_{-q} + b_q^* b_{-q}^*\right) + \mathcal{E}_N^{(4)}
\end{aligned}$$

Proposition 4.5. *Under the assumptions of Prop.2.11 there exists a constant $C > 0$ such that*

$$|\langle \xi, \mathcal{E}_N^{(4)} \xi \rangle| \leq C N^{1/2-\alpha} \|\mathcal{V}_N^{1/2} \xi\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\| + C N^{1-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 \quad (4.38)$$

for any $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$ and $N \in \mathbb{N}$ large enough.

To show Prop. 4.5, we use the following lemma, whose proof can be obtained as in [10, Lemma 7.7].

Lemma 4.6. *Let $\eta \in \ell^2(\Lambda^*)$ as defined in (2.80). Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\|(\mathcal{N}_+ + 1)^{n/2} e^{-B} \check{b}_x \check{b}_y e^B \xi\| \\ &\qquad \leq C \Big[N \|(\mathcal{N}_+ + 1)^{n/2} \xi\| + \|\check{a}_y (\mathcal{N}_+ + 1)^{(n+1)/2} \xi\| \\ &\qquad\quad + \|\check{a}_x (\mathcal{N}_+ + 1)^{(n+1)/2} \xi\| + \|\check{a}_x \check{a}_y (\mathcal{N}_+ + 1)^{n/2} \xi\| \Big] \end{aligned}$$

for all $\xi \in \mathcal{F}_+^{\leq N}$, $n \in \mathbb{Z}$.

Proof of Prop. 4.5. We follow the proof of [10, Prop. 7.6]. We write

$$\mathcal{G}_{N,\alpha}^{(4)} = \mathcal{V}_N + W_1 + W_2 + W_3 + W_4$$

with

$$\begin{aligned} \mathrm{W}_1 &= \frac{1}{2} \sum_{q \in \Lambda_+^*, r \in \Lambda^* : r \neq -q} \widehat{V}(r/e^N) \eta_{q+r} \int_0^1 ds \, \big(e^{-sB} b_q b_{-q} \, e^{sB} + \text{h.c.} \big) \\ \mathrm{W}_2 &= \sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq p, -q} \widehat{V}(r/e^N) \, \eta_{q+r} \int_0^1 ds \, \big(e^{-sB} b_{p+r}^* b_q^* e^{sB} a_{-q-r}^* a_p + \text{h.c.} \big) \end{aligned} \quad (4.39)$$

and

$$\begin{aligned} \mathrm{W}_3 = &\sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq -p-q} \widehat{V}(r/e^N) \eta_{q+r} \eta_p \\ &\quad \times \int_0^1 ds \int_0^s d\tau \, \big(e^{-sB} b_{p+r}^* b_q^* e^{sB} e^{-\tau B} b_{-p}^* b_{-q-r}^* e^{\tau B} + \text{h.c.} \big) \\ \mathrm{W}_4 = &\sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq -p-q} \widehat{V}(r/e^N) \, \eta_{q+r}^2 \\ &\quad \times \int_0^1 ds \int_0^s d\tau \, \big(e^{-sB} b_{p+r}^* b_q^* e^{sB} e^{-\tau B} b_p b_{q+r} e^{\tau B} + \text{h.c.} \big) . \end{aligned} \quad (4.40)$$

Let us first consider the term W_1. With (2.31), we find

$$\begin{aligned}\mathrm{W}_1 &= \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds(\gamma_q^{(s)})^2(b_q b_{-q} + \text{h.c.})\\ &\quad + \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds\,\gamma_q^{(s)}\sigma_q^{(s)}\big([b_q, b_q^*] + \text{h.c.}\big)\\ &\quad + \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds\,\gamma_q^{(s)}\big(b_q d_{-q}^{(s)} + \text{h.c.}\big) + \mathcal{E}_{10}^{(4)}\\ &=: \mathrm{W}_{11} + \mathrm{W}_{12} + \mathrm{W}_{13} + \mathcal{E}_{10}^{(4)}\end{aligned} \tag{4.41}$$

where

$$\mathcal{E}_{10}^{(4)} = \mathcal{E}_{101}^{(4)} + \mathcal{E}_{102}^{(4)} + \mathcal{E}_{103}^{(4)} + \mathcal{E}_{104}^{(4)} + \mathcal{E}_{105}^{(4)} \tag{4.42}$$

with

$$\begin{aligned}\mathcal{E}_{101}^{(4)} &= \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds\Big[2\gamma_q^{(s)}\sigma_q^{(s)}b_q^*b_q + (\sigma_q^{(s)})^2 b_{-q}^* b_q^* + \text{h.c.}\Big]\\ \mathcal{E}_{102}^{(4)} &= \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds\,\sigma_q^{(s)}\big(b_{-q}^* d_{-q}^{(s)} + \text{h.c.}\big)\\ \mathcal{E}_{103}^{(4)} &= \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds\,\sigma_q^{(s)}\big(d_q^{(s)} b_q^* + \text{h.c.}\big)\\ \mathcal{E}_{104}^{(4)} &= \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds\,\gamma_q^{(s)}\big(d_q^{(s)} b_{-q} + \text{h.c.}\big)\\ \mathcal{E}_{105}^{(4)} &= \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds\big(d_q^{(s)} d_{-q}^{(s)} + \text{h.c.}\big)\,.\end{aligned} \tag{4.43}$$

With

$$\frac{1}{N}\sup_{q\in\Lambda_+^*}\sum_{r\in\Lambda_+^*} |\widehat{V}(r/e^N)||\eta_{q+r}| \leq C < \infty \tag{4.44}$$

uniformly in $N \in \mathbb{N}$, we can estimate the first term in (4.43) by

$$|\langle \xi, \mathcal{E}_{101}^{(4)}\xi\rangle| \leq CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,.$$

Using (4.44) and (2.34) we also find

$$|\langle \xi, \mathcal{E}_{102}^{(4)}\xi\rangle| \leq CN^{1-2\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,.$$

For the third term in (4.43) we switch to position space and use (2.36):

$$\begin{aligned}|\langle \xi, \mathcal{E}_{103}^{(4)}\xi\rangle| &\leq \frac{1}{2}\int dxdy e^{2N}V(e^N(x-y))|\check{\eta}(x-y)| \\ &\quad \times \int_0^1 ds\, \|(\mathcal{N}+1)^{-1/2}\check{d}_y b^*(\check{\sigma}_x^{(s)})\xi\| \|(\mathcal{N}+1)^{1/2}\xi\| \\ &\leq C\|\check{\eta}\|_\infty \|\eta\| \int dxdy e^{2N}V(e^N(x-y))\|(\mathcal{N}_+ +1)^{1/2}\xi\| \int_0^1 ds \\ &\quad \times \Big[\|b^*(\check{\sigma}_x^{(s)})\xi\| + \frac{1}{N}|\check{\eta}^{(s)}(x-y)| \|(\mathcal{N}+1)^{1/2}\xi\| + \frac{1}{\sqrt{N}}\|b^*(\check{\sigma}_x^{(s)})\check{b}_y\xi\|\Big] \\ &\leq CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 .\end{aligned}$$

Consider now the fourth term in (4.43). We write $\mathcal{E}_{104}^{(4)} = \mathcal{E}_{1041}^{(4)} + \mathcal{E}_{1042}^{(4)}$, with

$$\begin{aligned}\mathcal{E}_{1041}^{(4)} &= \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds\, (\gamma_q^{(s)}-1)d_q^{(s)}b_{-q} \\ \mathcal{E}_{1042}^{(4)} &= \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds\, d_q^{(s)}b_{-q}\end{aligned}$$

With $|\gamma_q^{(s)}-1| \leq C|\eta_q|^2$, (4.44) and $\|d_q^*\xi\| \leq C\|\eta\|\|(\mathcal{N}_+ +1)^{1/2}\xi\|$, we find

$$|\langle \xi, \mathcal{E}_{1041}^{(4)}\xi\rangle| \leq CN^{1-3\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2$$

As for $\mathcal{E}_{1042}^{(4)}$, we switch to position space. Using (2.78) and (2.36), we obtain

$$\begin{aligned}&|\langle \xi, \mathcal{E}_{1042}^{(4)}\xi\rangle| \\ &\quad = \Big|\frac{1}{2}\int_0^1 ds \int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\check{\eta}(x-y)\langle \xi, \check{d}_x^{(s)}\check{b}_y\xi\rangle\Big| \\ &\quad \leq CN\int_0^1 ds\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{N}_+ +1)^{-1/2}\check{d}_x^{(s)}\check{b}_y\xi\| \\ &\quad \leq CN\|\eta\|\int_0^1 ds\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\|(\mathcal{N}_+ +1)^{1/2}\xi\| \\ &\qquad\qquad \times N^{-1}\Big[\|\check{a}_y\mathcal{N}_+\xi\| + \|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|\Big] \\ &\quad \leq CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\end{aligned}$$

Let us consider the last term in (4.43). Switching to position space and using (2.38) in Lemma 2.3 and again (2.78), we arrive at

$$\begin{aligned}&|\langle \xi, \mathcal{E}_{105}^{(4)}\xi\rangle| \\ &\leq CN\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\|(\mathcal{N}_+ +1)^{1/2}\xi\|\int_0^1 ds\|(\mathcal{N}_+ +1)^{-1/2}\check{d}_x^{(s)}\check{d}_y^{(s)}\xi\| \\ &\leq CN\|\eta\|\|(\mathcal{N}_+ +1)^{1/2}\xi\|\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y)) \\ &\qquad \times \big[\|(\mathcal{N}_+ +1)^{1/2}\xi\| + \|\eta\|\|\check{a}_x\xi\| + \|\eta\|\|\check{a}_y\xi\| + N^{-1/2}\|\eta\|\|\check{a}_x\check{a}_y\xi\|\big] \\ &\leq CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2 + CN^{1/2-2\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\| .\end{aligned}$$

Summarizing, we have shown that (4.42) can be bounded by

$$|\langle \xi, \mathcal{E}_{10}^{(4)}\xi\rangle| \leq CN^{1/2-\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2 \quad (4.45)$$

for any $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$. Next, we come back to the terms $\mathrm{W}_{11}, \mathrm{W}_{12}, \mathrm{W}_{13}$ introduced in (4.41). Using (4.44) and $|\gamma_q^{(s)} - 1| \leq C\eta_q^2$, we can write

$$\mathrm{W}_{11} = \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}(b_q b_{-q} + \text{h.c.}) + \mathcal{E}_{11}^{(4)}\,, \quad (4.46)$$

where $\mathcal{E}_{11}^{(4)}$ is such that

$$|\langle \xi, \mathcal{E}_{11}^{(4)}\xi\rangle| \leq CN^{1-2\alpha}\|(\mathcal{N}_++1)\xi\|^2\,.$$

Next, we can decompose the second term in (4.41) as

$$\mathrm{W}_{12} = \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\eta_q\left(1 - \frac{\mathcal{N}_+}{N}\right) + \mathcal{E}_{12}^{(4)} \quad (4.47)$$

where $\pm\mathcal{E}_{12}^{(4)} \leq CN^{-\alpha}\mathcal{N}_+ + N^{1-3\alpha}$.

The third term on the r.h.s. of (4.41) can be written as

$$\mathrm{W}_{13} = -\frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\eta_q\left(1 - \frac{\mathcal{N}_+}{N}\right)\frac{\mathcal{N}_++1}{N} + \mathcal{E}_{13}^{(4)} \quad (4.48)$$

where $\mathcal{E}_{13}^{(4)} = \mathcal{E}_{131}^{(4)} + \mathcal{E}_{132}^{(4)} + \mathcal{E}_{133}^{(4)} + \mathcal{E}_{134}^{(4)}$, with

$$\begin{aligned}
\mathcal{E}_{131}^{(4)} &= \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds\,(\gamma_q^{(s)} - 1)b_q d_{-q}^{(s)} + \text{h.c.}\\
\mathcal{E}_{132}^{(4)} &= \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\int_0^1 ds\, b_q\left[d_{-q}^{(s)} + s\eta_q\frac{\mathcal{N}_+}{N}b_q^*\right] + \text{h.c.}\\
\mathcal{E}_{133}^{(4)} &= -\frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\eta_q b_q^* b_q\frac{\mathcal{N}_++1}{N}\\
\mathcal{E}_{134}^{(4)} &= \frac{1}{2N}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\eta_q a_q^* a_q\frac{\mathcal{N}_++1}{N}\,.
\end{aligned}$$

With (4.44), we immediately find

$$\pm\mathcal{E}_{133}^{(4)} \leq CN^{1-\alpha}(\mathcal{N}_++1), \qquad \pm\mathcal{E}_{134}^{(4)} \leq CN^{-\alpha}(\mathcal{N}_++1)\,.$$

With $|\gamma_q^{(s)} - 1| \leq C\eta_q^2$, Lemma 2.3 and, again, (4.44), we also obtain

$$|\langle \xi, \mathcal{E}_{131}^{(4)}\xi\rangle| \leq CN^{1-3\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2\,.$$

Let us now consider $\mathcal{E}_{132}^{(4)}$. In position space, with $\check{\bar{d}}_y^{(s)} = d_y^{(s)} + (\mathcal{N}_+/N)b^*(\check{\eta}_y)$ and using (2.37), we obtain

$$\begin{aligned}|\langle\xi, \mathcal{E}_{132}^{(4)}\xi\rangle| &= \Big|\frac{1}{2}\int_0^1 ds \int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\check{\eta}(x-y)\langle\xi, \check{b}_x \check{\bar{d}}_y^{(s)}\xi\rangle\Big| \\ &\le CN^{1-\alpha}\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\qquad \times \Big[\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_y\xi\| + \|\check{a}_x\xi\| + N^{-1}\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|\Big] \\ &\le CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\,.\end{aligned}$$

It follows that

$$|\langle\xi, \mathcal{E}_{13}^{(4)}\rangle| \le CN^{1/2-\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.$$

With (4.45), (4.46), (4.47), (4.48), we obtain

$$\begin{aligned}\mathrm{W}_1 = &\frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\big(b_q b_{-q} + \text{h.c.}\big) \\ &+ \frac{1}{2}\sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\eta_q\left(1 - \frac{\mathcal{N}_+}{N}\right)\left(1 - \frac{\mathcal{N}_+ + 1}{N}\right) + \mathcal{E}_1^{(4)}\end{aligned} \tag{4.49}$$

where

$$|\langle\xi, \mathcal{E}_1^{(4)}\xi\rangle| \le CN^{1/2-\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,,$$

Next, we control the term W_2, from (4.39). In position space, we find

$$\mathrm{W}_2 = \int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\int_0^1 ds\big(e^{-sB}\check{b}_x^*\check{b}_y^* e^{sB}a^*(\check{\eta}_x)\check{a}_y + \text{h.c.}\big)$$

with $\check{\eta}_x(z) = \check{\eta}(x-z)$. By Cauchy-Schwarz, we have

$$\begin{aligned}|\langle\xi, \mathrm{W}_2\xi\rangle| \le &\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\int_0^1 ds \\ &\times \|(\mathcal{N}_+ + 1)^{1/2}e^{-sB}\check{b}_x\check{b}_y e^{sB}\xi\|\|(\mathcal{N}_+ + 1)^{-1/2}a^*(\check{\eta}_x)\check{a}_y\xi\|\,.\end{aligned}$$

With

$$\|(\mathcal{N}_+ + 1)^{-1/2}a^*(\check{\eta}_x)\check{a}_y\xi\| \le C\|\eta\|\|\check{a}_y\xi\| \le CN^{-\alpha}\|\check{a}_y\xi\|$$

and using Lemma 4.6, we obtain

$$\begin{aligned}|\langle\xi, \mathrm{W}_2\xi\rangle| \le CN^{-\alpha}&\int_{\Lambda^2} dxdy\, e^{2N}V(e^N(x-y))\|\check{a}_y\xi\| \\ &\times \Big\{N\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + N\|\check{a}_x\xi\| + N\|\check{a}_y\xi\| + N^{1/2}\|\check{a}_x\check{a}_y\xi\|\Big\} \\ \le CN^{1-\alpha}&\,\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + CN^{1/2-\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\,.\end{aligned} \tag{4.50}$$

Also for the term W_3 in (4.40), we switch to position space. We find

$$\begin{aligned}\mathrm{W}_3 = &\int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \\ &\times \int_0^1 ds \int_0^s d\tau \left(e^{-sB} \check{b}_x^* \check{b}_y^* e^{sB}\, e^{-\tau B} b^*(\check{\eta}_x) b^*(\check{\eta}_y) e^{\tau B} + \text{h.c.} \right).\end{aligned}$$

and thus

$$\begin{aligned}|\langle \xi, \mathrm{W}_3 \xi\rangle| \leq \int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \int_0^1 ds \int_0^s d\tau\, &\|(\mathcal{N}_+ + 1)^{1/2} e^{-sB} \check{b}_x \check{b}_y e^{sB} \xi\| \\ &\times \|(\mathcal{N}_+ + 1)^{-1/2} e^{-\tau B} b^*(\check{\eta}_x)\, b^*(\check{\eta}_y) e^{\tau B} \xi\|.\end{aligned}$$

With Lemma 2.2, we find

$$\|(\mathcal{N}_+ + 1)^{-1/2} e^{-\tau B} b^*(\check{\eta}_x)\, b^*(\check{\eta}_y) e^{\tau B} \xi\| \leq C\|\eta\|^2 \|(\mathcal{N}_+ + 1)^{1/2} \xi\|.$$

Using Lemma 4.6, we conclude that

$$\begin{aligned}|\langle \xi, \mathrm{W}_3 \xi\rangle| \leq C\|\eta\|^2 &\int_{\Lambda^2} dxdy\, e^{2N} V(e^N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \\ &\times \Big\{ N\|(\mathcal{N}_+ + 1)^{1/2} \xi\| + N\|\check{a}_x \xi\| + N\|\check{a}_y \xi\| + N^{1/2} \|\check{a}_x \check{a}_y \xi\| \Big\} \\ \leq CN^{1-2\alpha} &\|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 + CN^{1/2-2\alpha} \|\mathcal{V}_N^{1/2} \xi\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\|.\end{aligned} \tag{4.51}$$

The term W_4 in (4.40) can be bounded similarly. In position space, we find

$$\begin{aligned}\mathrm{W}_4 = &\int dxdy\, e^{2N} V(e^N(x-y)) \\ &\times \int_0^1 ds \int_0^s d\tau \left(e^{-sB} \check{b}_x^* \check{b}_y^*\, e^{sB}\, e^{-\tau B} b(\check{\eta_x^2}) \check{b}_y e^{\tau B} + \text{h.c.} \right)\end{aligned}$$

with $\check{\eta^2}$ the function with Fourier coefficients η_q^2, for $q \in \Lambda^*$, and where $\check{\eta_x^2}(y) := \check{\eta^2}(x-y)$. Clearly $\|\check{\eta_x^2}\| \leq C\|\check{\eta}\|^2 \leq CN^{-2\alpha}$. With Cauchy-Schwarz and Lemma 2.2, we obtain

$$\begin{aligned}|\langle \xi, \mathrm{W}_4 \xi\rangle| \leq CN^{-2\alpha} \int_0^1 ds \int_0^s d\tau \int dxdy\, &e^{2N} V(e^N(x-y)) \\ &\times \|(\mathcal{N}_+ + 1)^{1/2} \check{b}_y \check{b}_x e^{sB} \xi\| \|\check{b}_y e^{\tau B} \xi\|.\end{aligned}$$

Applying Lemma 4.6 and then Lemma 2.2, we obtain

$$\begin{aligned}|\langle \xi, \mathrm{W}_4 \xi\rangle| \leq CN^{-2\alpha} &\int_0^1 ds \int_0^s d\tau \int dxdy\, e^{2N} V(e^N(x-y)) \|\check{b}_y e^{\tau B} \xi\| \\ &\times \{ N\|(\mathcal{N}_+ + 1)^{1/2} \xi\| + N\|\check{a}_x \xi\| + N\|\check{a}_y \xi\| + N^{1/2} \|\check{a}_x \check{a}_y \xi\| \} \\ \leq CN^{1-2\alpha} &\|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2 + CN^{1/2-2\alpha} \|\mathcal{V}_N^{1/2} \xi\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\|.\end{aligned}$$

From (4.49), (4.50), (4.51) and the last bound, we conclude that

$$\mathcal{G}^{(4)}_{N,\alpha} = \mathcal{V}_N + \frac{1}{2}\sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\big(b_q b_{-q} + \text{h.c.}\big)$$
$$+ \frac{1}{2}\sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} \widehat{V}(r/e^N)\eta_{q+r}\eta_q \left(1-\frac{\mathcal{N}_+}{N}\right)\left(1-\frac{\mathcal{N}_+ +1}{N}\right) + \mathcal{E}^{(4)}_{N,\alpha}$$

where $\mathcal{E}^{(4)}_{N,\alpha}$ satisfies (4.38).

□

4.1.5 Proof of Proposition 2.11

With the results established in Subsections 4.1.1 - 4.1.4, we cam now show Prop. 2.11. Propositions 4.1, 4.2, 4.3, 4.4, 4.5, imply that

$$\begin{aligned}
\mathcal{G}_{N,\alpha} &= \frac{\widehat{V}(0)}{2}\left(N+\mathcal{N}_+ -1\right)\left(N-\mathcal{N}_+\right) \\
&+ \sum_{p\in\Lambda^*_+} \eta_p\Big[p^2\eta_p + N\widehat{V}(p/e^N) + \frac{1}{2}\sum_{\substack{r\in\Lambda^*\\ p+r\neq 0}} \widehat{V}(r/e^N)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+ -1}{N}\Big) \\
&+ \mathcal{K} + N\sum_{p\in\Lambda^*_+}\widehat{V}(p/e^N)a^*_p a_p\Big(1-\frac{\mathcal{N}_+}{N}\Big) \\
&+ \sum_{p\in\Lambda^*_+}\Big[\, p^2\eta_p + \frac{N}{2}\widehat{V}(p/e^N) + \frac{1}{2}\sum_{r\in\Lambda^*:\, p+r\neq 0}\widehat{V}(r/e^N)\eta_{p+r}\,\Big]\big(b^*_p b^*_{-p} + b_p b_{-p}\big) \\
&+ \sqrt{N}\sum_{p,q\in\Lambda^*_+:\, p+q\neq 0}\widehat{V}(p/e^N)\left[b^*_{p+q}a^*_{-p}a_q + \text{h.c.}\,\right] + \mathcal{V}_N + \mathcal{E}_1
\end{aligned} \tag{4.52}$$

where

$$|\langle \xi, \mathcal{E}_1\xi\rangle| \leq CN^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_+ +1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2$$

for any $\alpha > 1$ and $\xi\in\mathcal{F}^{\leq N}_+$. With (2.84), we find

$$\begin{aligned}
&\sum_{p\in\Lambda^*_+}\eta_p\Big[p^2\eta_p + N\widehat{V}(p/e^N) + \frac{1}{2}\sum_{r\in\Lambda^*:\, p+r\neq 0}\widehat{V}(r/e^N)\eta_{p+r}\Big] \\
&= \sum_{p\in\Lambda^*_+}\eta_p\Big[\,\frac{N}{2}\widehat{V}(p/e^N) + Ne^{2N}\lambda_\ell\widehat{\chi}_\ell(p) + e^{2N}\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q - \frac{1}{2}\widehat{V}(p/e^N)\eta_0\,\Big]
\end{aligned}$$

From Lemma 2.10 and estimating $\|\widehat{\chi}_\ell\| = \|\chi_\ell\| \leq CN^{-\alpha}$, $\|\eta\|\leq CN^{-\alpha}$ and $\|\widehat{\chi}_\ell * \eta\| = \|\chi_\ell\check{\eta}\| \leq \|\check{\eta}\| \leq CN^{-\alpha}$, we have

$$\Big|Ne^{2N}\lambda_\ell\sum_{p\in\Lambda^*_+}\eta_p\widehat{\chi}_\ell(p)\Big| \leq CN^{2\alpha}\|\widehat{\chi}_\ell\|\|\eta\| \leq C,$$

and

$$\Big|e^{2N}\lambda_\ell \sum_{p\in\Lambda_+^*,\, q\in\Lambda^*} \widehat{\chi}_\ell(p-q)\eta_q\eta_p\Big| \leq CN^{2\alpha-1}\|\hat{\chi}_\ell * \eta\|\|\eta\| \leq CN^{-1}.$$

Moreover, using (4.44) and the bound (2.85) we find

$$\Big|\frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_p\eta_0\Big| \leq CN^{1-2\alpha}\,.$$

We obtain

$$\begin{aligned}
&\sum_{p\in\Lambda_+^*}\eta_p\Big[p^2\eta_p + N\widehat{V}(p/e^N) + \frac{1}{2}\sum_{\substack{r\in\Lambda^*\\ p+r\in\Lambda_+^*}}\widehat{V}(r/e^N)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big)\\
&\qquad = \frac{N}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\eta_p\left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{N-\mathcal{N}_+-1}{N}\right) + \mathcal{E}_2
\end{aligned}$$

with $\pm\mathcal{E}_2 \leq C$ for all $\alpha \geq 1/2$. On the other hand, using (2.85) we have

$$\begin{aligned}
\frac{N}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\eta_p &= \frac{N}{2}\big(\widehat{V}(\cdot/e^N)*\eta\big)(0) - \frac{N}{2}\widehat{V}(0)\eta_0\\
&= \frac{N^2}{2}\Big(\int dx V(x)f_\ell(x) - \widehat{V}(0)\Big) + \tilde{\mathcal{E}}_2
\end{aligned}$$

with $\pm\tilde{\mathcal{E}}_2 \leq CN^{1-2\alpha}$. With the first bound in (2.90) we conclude that

$$\begin{aligned}
&\sum_{p\in\Lambda_+^*}\eta_p\Big[p^2\eta_p + N\widehat{V}(p/e^N) + \frac{1}{2}\sum_{\substack{r\in\Lambda^*\\ p+r\in\Lambda_+^*}}\widehat{V}(r/e^N)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big)\\
&= \frac{1}{2N}\left[\widehat{\omega}_N(0) - N\widehat{V}(0)\right](N-\mathcal{N}_+-1)(N-\mathcal{N}_+) + \mathcal{E}_3
\end{aligned} \tag{4.53}$$

where $\pm\mathcal{E}_3 \leq C$, if $\alpha \geq 1/2$. Using (2.84), we can also handle the fourth line of (4.52); we find

$$\begin{aligned}
&\sum_{p\in\Lambda_+^*}\left[p^2\eta_p + \frac{N}{2}\widehat{V}(p/e^N) + \frac{1}{2}\sum_{r\in\Lambda^*:\, p+r\in\Lambda_+^*}\widehat{V}(r/e^N)\eta_{p+r}\right]\big(b_p^*b_{-p}^* + b_pb_{-p}\big)\\
&\quad = \sum_{p\in\Lambda_+^*}\Big[Ne^{2N}\lambda_\ell\widehat{\chi}_\ell(p) + e^{2N}\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q - \frac{1}{2}\widehat{V}(p/e^N)\eta_0\Big]\big(b_p^*b_{-p}^* + b_pb_{-p}\big)\,.
\end{aligned} \tag{4.54}$$

The last two terms on the right hand side of (4.54) are error terms. With (2.85) and (4.44) we have

$$\begin{aligned}
\Big| \sum_{p\in\Lambda_+^*} \widehat{V}(p/e^N)\eta_0\big(b_p^*b_{-p}^* + b_pb_{-p}\big)\Big| \\
&\leq CN^{-2\alpha}\bigg[\sum_{p\in\Lambda_+^*}\frac{|\widehat{V}(p/e^N)|^2}{p^2}\bigg]^{1/2}\bigg[\sum_{p\in\Lambda_+^*}p^2\|a_p\xi\|^2\bigg]^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\| \\
&\leq CN^{1/2-2\alpha}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|\,.
\end{aligned}$$

The second term on the right hand side of (4.54) can be bounded in position space:

$$\begin{aligned}
&\Big|\langle\xi,\ e^{2N}\lambda_\ell\sum_{p\in\Lambda_+^*}(\widehat{\chi}_\ell*\eta)(p)(b_p^*b_{-p}^*+b_pb_{-p})\xi\rangle\Big| \\
&\leq CN^{2\alpha-1}\|(\mathcal{N}_++1)^{1/2}\xi\|\int_{\Lambda^2}dxdy\,\chi_\ell(x-y)|\check{\eta}(x-y)|\|(\mathcal{N}_++1)^{-1/2}\check{b}_x\check{b}_y\xi\| \\
&\leq CN^{\alpha-1}\|(\mathcal{N}_++1)^{1/2}\xi\|\left[\int_{\Lambda^2}dxdy\,\chi_\ell(x-y)\|(\mathcal{N}_++1)^{-1/2}\check{a}_x\check{a}_y\xi\|^2\right]^{1/2}.
\end{aligned}$$

The term in parenthesis can be bounded similarly as in (4.62). Namely,

$$\int_{\Lambda^2}dxdy\,\chi_\ell(x-y)\|(\mathcal{N}_++1)^{-1/2}\check{a}_x\check{a}_y\xi\|^2\leq CqN^{-2\alpha/q'}\|\mathcal{K}^{1/2}\xi\|^2$$

for any $q>2$ and $1<q'<2$ with $1/q+1/q'=1$. Choosing $q=\log N$, we get

$$\begin{aligned}
&\Big|\langle\xi,e^{2N}\lambda_\ell\sum_{p\in\Lambda_+^*}(\widehat{\chi}_\ell*\eta)(p)(b_p^*b_{-p}^*+b_pb_{-p})\xi\rangle\Big| \\
&\qquad\leq CN^{-1}(\log N)^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\,,
\end{aligned}$$

and, from (4.54), we conclude that

$$\begin{aligned}
&\sum_{p\in\Lambda_+^*}\Big[p^2\eta_p+\frac{N}{2}\widehat{V}(p/e^N)+\frac{1}{2}\sum_{\substack{r\in\Lambda^*:\\ p+r\in\Lambda_+^*}}\widehat{V}(r/e^N)\eta_{p+r}\Big]\big(b_p^*b_{-p}^*+b_pb_{-p}\big) \\
&\quad=\sum_{p\in\Lambda_+^*}Ne^{2N}\lambda_\ell\widehat{\chi}_\ell(p)\big(b_p^*b_{-p}^*+b_pb_{-p}\big)+\mathcal{E}_4\,,
\end{aligned}\tag{4.55}$$

with

$$|\langle\xi,\mathcal{E}_4\xi\rangle|\leq CN^{-1}(\log N)^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|$$

if $\alpha>1$. Combining (4.52) with (4.53) and (4.55), and using the definition (2.88) we conclude that

$$\begin{aligned}\mathcal{G}_{N,\alpha} &= \frac{1}{2}\,\widehat{\omega}_N(0)(N-1)\Big(1-\frac{\mathcal{N}_+}{N}\Big) + \Big[N\widehat{V}(0) - \frac{1}{2}\,\widehat{\omega}_N(0)\Big]\,\mathcal{N}_+\Big(1-\frac{\mathcal{N}_+}{N}\Big) \\ &\quad + N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)a_p^*a_p\left(1-\frac{\mathcal{N}_+}{N}\right) + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)(b_pb_{-p}+\text{h.c.}\,) \\ &\quad + \sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq0}\widehat{V}(p/e^N)\left[b_{p+q}^*a_{-p}^*a_q+\text{h.c.}\right] \\ &\quad + \mathcal{K}+\mathcal{V}_N+\mathcal{E}_5\,,\end{aligned} \tag{4.56}$$

with

$$\begin{aligned}|\langle\xi,\mathcal{E}_5\xi\rangle| &\leq CN^{1/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\| + CN^{1-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2 \\ &\quad + CN^{-1}(\log N)^{1/2}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\| + C\|\xi\|^2\,,\end{aligned}$$

for any $\alpha>1$. Observing that $|\widehat{V}(p/e^N)-\widehat{V}(0)|\leq C|p|e^{-N}$ in the second line on the r.h.s. of (4.56), we arrive at $\mathcal{G}_{N,\alpha}=\mathcal{G}_{N,\alpha}^{\text{eff}}+\mathcal{E}_{\mathcal{G}}$, with $\mathcal{G}_{N,\alpha}^{\text{eff}}$ defined as in (2.91) and with $\mathcal{E}_{\mathcal{G}}$ that satisfies (2.92).

4.2 Analysis of the cubically renormalized excitation Hamiltonian $\mathcal{R}_N$

In this section, we show Prop. 2.14, where we establish a lower bound for the operator $\mathcal{R}_{N,\alpha}=e^{-A}\mathcal{G}_{N,\alpha}^{\text{eff}}e^A$, with $\mathcal{G}_{N,\alpha}^{\text{eff}}$ as defined in (2.91) and with

$$A=\frac{1}{\sqrt{N}}\sum_{r,v\in\Lambda_+^*}\eta_r\left[b_{r+v}^*a_{-r}^*a_v-\text{h.c.}\right]. \tag{4.57}$$

We decompose

$$\mathcal{G}_{N,\alpha}^{\text{eff}}=\mathcal{O}_N+\mathcal{K}+\mathcal{Z}_N+\mathcal{C}_N+\mathcal{V}_N \tag{4.58}$$

with $\mathcal{K}$ and $\mathcal{V}_N$ as in (2.87), and with

$$\begin{aligned}\mathcal{O}_N &= \frac{1}{2}\widehat{\omega}_N(0)(N-1)\Big(1-\frac{\mathcal{N}_+}{N}\Big) + \big[2N\widehat{V}(0)-\frac{1}{2}\widehat{\omega}_N(0)\big]\mathcal{N}_+\Big(1-\frac{\mathcal{N}_+}{N}\Big), \\ \mathcal{Z}_N &= \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)(b_pb_{-p}+\text{h.c.}\,) \\ \mathcal{C}_N &= \sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq0}\widehat{V}(p/e^N)\left[b_{p+q}^*a_{-p}^*a_q+\text{h.c.}\right]\,.\end{aligned} \tag{4.59}$$

We will analyze the conjugation of all terms on the r.h.s. of (4.58) in Subsections 4.2.2–4.2.6. The estimates emerging from these subsections will then be combined in Subsection 4.2.6 to conclude the proof of Prop. 2.14. Throughout the section, we will need Prop. 2.13 to control the growth of the expectation of the energy $\mathcal{H}_N=\mathcal{K}+\mathcal{V}_N$ under the action of (4.57); the proof of Prop. 2.13 is contained in Subsection 4.2.1.

In this section, we will always assume that $V\in L^3(\mathbb{R}^2)$ is compactly supported, pointwise non-negative and spherically symmetric.

4.2.1 A priori bounds on the energy

In this section, we show Prop. 2.13. To this end, we will need the following proposition.

Proposition 4.7. *Let $\mathcal{V}_N$ and A be defined in* (2.87) *and* (2.93) *respectively. Then, there exists a constant $C > 0$ such that*

$$[\mathcal{V}_N, A] = \frac{1}{N^{1/2}} \sum_{\substack{u,r,v \in \Lambda_+^* \\ u \neq -v}} \widehat{V}((u-r)/e^N)\eta_r \big[b^*_{u+v} a^*_{-u} a_v + \text{h.c.}\big] + \delta_{\mathcal{V}_N}$$

where

$$|\langle \xi, \delta_{\mathcal{V}_N} \xi \rangle| \leq C(\log N)^{1/2} N^{1/2-\alpha} \|\mathcal{H}_N^{1/2}\xi\|^2 \tag{4.60}$$

for any $\alpha > 0$, for all $\xi \in \mathcal{F}_+^{\leq N}$, and $N \in \mathbb{N}$ large enough.

Proof. We proceed as in [10, Prop. 8.1], computing $[a^*_{p+u} a^*_q a_p a_{q+u}, b^*_{r+v} a^*_{-r} a_v]$. We obtain

$$[\mathcal{V}_N, A] = \frac{1}{N^{1/2}} \sum^*_{u \in \Lambda^*, r,v \in \Lambda_+^*} \widehat{V}((u-r)/e^N)\eta_r b^*_{u+v} a^*_{-u} a_v + \Theta_1 + \Theta_2 + \Theta_3 + \text{h.c.}$$

with

$$\begin{aligned}
\Theta_1 :=& \ \frac{1}{\sqrt{N}} \sum^*_{\substack{u \in \Lambda^* \\ r,p,v \in \Lambda_+^*}} \widehat{V}(u/e^N)\eta_r b^*_{p+u} a^*_{r+v-u} a^*_{-r} a_p a_v \, , \\
\Theta_2 :=& \ \frac{1}{\sqrt{N}} \sum^*_{\substack{u \in \Lambda^* \\ p,r,v \in \Lambda_+^*}} \widehat{V}(u/e^N)\eta_r b^*_{r+v} a^*_{p+u} a^*_{-r-u} a_p a_v \, , \\
\Theta_3 :=& \ -\frac{1}{\sqrt{N}} \sum^*_{u \in \Lambda^*, p,r,v \in \Lambda_+^*} \widehat{V}(u/e^N)\eta_r b^*_{r+v} a^*_{-r} a^*_{p+u} a_p a_{v+u} \, .
\end{aligned} \tag{4.61}$$

and with $\sum^*$ running over all momenta, except choices for which the argument of a creation or annihilation operator vanishes. We conclude that $\delta_{\mathcal{V}_N} = \Theta_1 + \Theta_2 + \Theta_3 + \text{h.c.}$ Next, we show that each error term Θ_j, with $j = 1, 2, 3$, satisfies (4.60). To bound Θ_1 we switch to position space and apply Cauchy-Schwarz. We find

$$\begin{aligned}
|\langle \xi, \Theta_1 \xi \rangle| \leq& \frac{1}{\sqrt{N}} \int_{\Lambda^2} dxdy \; e^{2N} V(e^N(x-y)) \|\check{a}(\check{\eta}_y)\check{a}_y \check{a}_x \xi\| \|\check{a}_y \check{a}_x \xi\| \\
\leq& \; C\|\eta\| \int_{\Lambda^2} dxdy \, e^{2N} V(e^N(x-y)) \|\check{a}_y \check{a}_x \xi\|^2 \\
\leq& C N^{-\alpha} \|\mathcal{V}_N^{1/2}\xi\|^2 \, ,
\end{aligned}$$

for any $\xi \in \mathcal{F}_+^{\leq N}$ The term Θ_3 can be controlled similarly. We find

$$\begin{aligned} |\langle \xi, \Theta_3 \xi\rangle| &= \left| \frac{1}{\sqrt{N}} \int_{\Lambda^2} dxdy \; e^{2N} V(e^N(x-y)) \langle \xi, \check{b}^*_x \check{a}^*(\check{\eta}_x) \check{a}^*_y \check{a}_x \check{a}_y \xi \rangle \right| \\ &\leq C N^{-\alpha} \|\mathcal{V}_N^{1/2} \xi\|^2 . \end{aligned}$$

It remains to bound the term Θ_2 on the r.h.s. of (4.61). Passing to position space we obtain, by Cauchy-Schwarz,

$$\begin{aligned} |\langle \xi, \Theta_2 \xi\rangle| =& \left| \frac{1}{\sqrt{N}} \int_{\Lambda^3} dxdydz \; e^{2N} V(e^N(y-z)) \check{\eta}(x-z) \langle \xi, \check{b}^*_x \check{a}^*_y \check{a}^*_z \check{a}_x \check{a}_y \xi\rangle \right| \\ &\leq C N^{-1/2} \int_{\Lambda^3} dxdydz \; e^{2N} V(e^N(y-z)) |\check{\eta}(x-z)| \|\check{a}_x \check{a}_y \check{a}_z \xi\| \|\check{a}_x \check{a}_y \xi\| \\ &\leq C N^{-1/2} \|\mathcal{V}_N^{1/2} \mathcal{N}_+^{1/2} \xi\| \left[\int_{\Lambda^3} dxdydz \, e^{2N} V(e^N(y-z)) |\check{\eta}(x-z)|^2 \|\check{a}_x \check{a}_y \xi\|^2 \right]^{1/2} , \end{aligned}$$

To bound the term in the square bracket, we write it in first quantized form and, for any $2 < q < \infty$, we apply Hölder inequality and the Sobolev inequality $\|u\|_q \leq C\sqrt{q}\, \|u\|_{H^1}$ -derived from [45, Theorem 8.5.ii)]- to estimate (denoting by $1 < q' < 2$ the dual index to q),

$$\begin{aligned} \sum_{n=2}^N \sum_{i<j}^n & \int \left[e^{2N} V(e^N \cdot) * |\check{\eta}|^2 \right] (x_i - x_j) \, |\xi^{(n)}(x_1, \dots, x_n)|^2 dx_1 \dots dx_n \\ &\leq Cq \|e^{2N} V(e^N \cdot) * |\check{\eta}|^2\|_{q'} \\ &\qquad \times \sum_{n=2}^N n \sum_{i=1}^n \int \left[|\nabla_{x_i} \xi^{(n)}(x_1, \dots, x_n)|^2 + |\xi^{(n)}(x_1, \dots, x_n)|^2 \right] dx_1 \dots dx_n \\ &\leq Cq \|\check{\eta}\|_{2q'}^2 \|(\mathcal{K} + \mathcal{N}_+)^{1/2} \mathcal{N}_+^{1/2} \xi\|^2. \end{aligned} \tag{4.62}$$

With the bounds (2.78), (2.79),

$$\|\check{\eta}\|_{2q'}^2 \leq \|\check{\eta}\|_2^{2/q'} \|\check{\eta}\|_\infty^{2(q'-1)/q'} \leq N^{-2\alpha/q'} N^{2(q'-1)/q'}$$

we conclude that

$$\begin{aligned} |\langle \xi, \Theta_2 \xi\rangle| &\leq C q^{1/2} N^{-1/2} N^{-\alpha/q'} N^{1/q} \|\mathcal{V}_N^{1/2} \mathcal{N}_+^{1/2} \xi\| \|(\mathcal{K} + \mathcal{N}_+)^{1/2} \mathcal{N}_+^{1/2} \xi\| \\ &\leq C q^{1/2} N^{1/2} N^{-\alpha/q'} N^{1/q} \|\mathcal{V}_N^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\| \end{aligned}$$

for any $2 < q < \infty$, if $1/q + 1/q' = 1$. Choosing $q = \log N$, we obtain that

$$|\langle \xi, \Theta_2 \xi\rangle| \leq C (\log N)^{1/2} N^{1/2-\alpha} \|\mathcal{H}_N^{1/2} \xi\|^2 .$$

□

Using Prop. 4.7, we can now show Proposition 2.13.

Proof of Prop. 2.13. The proof follows a strategy similar to [10, Lemma 8.2]. For fixed $\xi \in \mathcal{F}_+^{\leq N}$ and $s \in [0;1]$, we define

$$f_\xi(s) := \langle \xi, e^{-sA} \mathcal{H}_N e^{sA} \xi \rangle \,.$$

We compute

$$f'_\xi(s) = \langle \xi, e^{-sA} [\mathcal{K}, A] e^{sA} \xi \rangle + \langle \xi, e^{-sA} [\mathcal{V}_N, A] e^{sA} \xi \rangle \,. \tag{4.63}$$

With Prop. 4.7, we have

$$[\mathcal{V}_N, A] = \frac{1}{\sqrt{N}} \sum_{u,v \in \Lambda_+^*, u \neq -v} (\widehat{V}(\cdot/e^N) * \eta)(u) \left[b^*_{u+v} a^*_{-u} a_v + \text{h.c.} \right] + \delta_{\mathcal{V}_N}$$

with $\delta_{\mathcal{V}_N}$ satisfying (4.60). Switching to position space and using Prop. 2.12 we find , using (2.78) to bound $\|\check{\eta}\|_\infty \leq CN$,

$$\begin{aligned} &\left| \frac{1}{\sqrt{N}} \sum_{u,v \in \Lambda_+^*} (\widehat{V}(\cdot/e^N) * \eta)(u) \langle \xi, e^{-sA} b^*_{u+v} a^*_{-u} a_v e^{sA} \xi \rangle \right| \\ &= \left| \frac{1}{\sqrt{N}} \int_{\Lambda^2} dxdy \; e^{2N} V(e^N(x-y)) \check{\eta}(x-y) \langle \xi, e^{-sA} \check{a}^*_x \check{a}^*_y \check{a}_y e^{sA} \xi \rangle \right| \\ &\leq N^{1/2} \left[\int_{\Lambda^2} dxdy \; e^{2N} V(e^N(x-y)) \| \check{a}_x \check{a}_y e^{sA} \xi \|^2 \right]^{1/2} \\ &\qquad \times \left[\int_{\Lambda^2} dxdy \; e^{2N} V(e^N(x-y)) \| \check{a}_y e^{sA} \xi \|^2 \right]^{1/2} \\ &\leq C N^{1/2} \| \mathcal{V}_N^{1/2} e^{sA} \xi \| \| \mathcal{N}_+^{1/2} e^{sA} \xi \| \end{aligned} \tag{4.64}$$

Together with (4.60) we conclude that for any $\alpha > 1/2$

$$\left| \langle \xi, e^{-sA} [\mathcal{V}_N, A] e^{sA} \xi \rangle \right| \leq C \langle \xi, e^{-sA} \mathcal{H}_N e^{sA} \xi \rangle + CN \langle \xi, e^{-sA} (\mathcal{N}_+ + 1) e^{sA} \xi \rangle \tag{4.65}$$

if N is large enough. Next, we analyze the first term on the r.h.s. of (4.63). We compute

$$\begin{aligned} [\mathcal{K}, A] = &\frac{1}{\sqrt{N}} \sum_{r,v \in \Lambda_+^*} 2r^2 \eta_r \left[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \right] \\ &\qquad + \frac{2}{\sqrt{N}} \sum_{r,v \in \Lambda_+^*} r \cdot v \; \eta_r \left[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \right] \\ =: &\, \mathrm{T}_1 + \mathrm{T}_2 \,. \end{aligned} \tag{4.66}$$

With (2.84), we write

$$\begin{aligned} \mathrm{T}_1 = &- \sqrt{N} \sum_{\substack{r,v \in \Lambda_+^* \\ r \neq -v}} (\widehat{V}(\cdot/e^N) * \widehat{f}_{N,\ell})(r) \left[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \right] \\ &+ 2\sqrt{N} \sum_{r,v \in \Lambda_+^*} e^{2N} \lambda_\ell (\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(r) \left[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \right] \\ =: &\, \mathrm{T}_{11} + \mathrm{T}_{12} \,. \end{aligned} \tag{4.67}$$

The contribution of T_{11} can be estimated similarly as in (4.64); switching to position space and using (2.73), we obtain

$$\begin{aligned}\big|\langle\xi_1,\mathrm{T}_{11}\,\xi_2\rangle\big| &\leq C\sqrt{N}\int dxdy e^{2N}V(e^N(x-y))f_\ell(e^N(x-y))\|\check{a}_x\check{a}_y\xi\|\|a_y\xi\| \\ &\leq C\sqrt{N}\Big[\int dxdy e^{2N}V(e^N(x-y))\|\check{a}_x\check{a}_y\xi\|^2\Big]^{1/2} \\ &\qquad\times\Big[\int dxdy e^{2N}V(e^N(x-y))f_\ell(e^N(x-y))\|a_y\xi\|^2\Big]^{1/2} \\ &\leq C\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|\,.\end{aligned} \tag{4.68}$$

for any $\xi\in\mathcal{F}_+^{\leq N}$. The second term in (4.67) can be controlled using that for any $\xi\in\mathcal{F}_+^{\leq N}$ and $2\leq q<\infty$ we have

$$\begin{aligned}&N^{2\alpha}\int_{\Lambda^2}dxdy\,\chi(|x-y|\leq N^{-\alpha})\|\check{a}_x\check{a}_y\xi\|\|\check{a}_x\xi\| \\ &\leq N^{2\alpha}\int_{\Lambda^2}dx\|\check{a}_x\xi\|\Big(\int dy\,\chi(|x-y|\leq N^{-\alpha})\Big)^{1-1/q}\Big(\int dy\|\check{a}_x\check{a}_y\xi\|^q\Big)^{1/q} \\ &\leq CN^{2\alpha/q}q^{1/2}\Big[\int dx\|\check{a}_x\xi\|^2\Big]^{1/2}\Big[\int dxdy\|\check{a}_x\nabla_y\check{a}_y\xi\|^2+\int dxdy\|\check{a}_x\check{a}_y\xi\|^2\Big]^{1/2} \\ &\leq CN^{2\alpha/q}q^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\left[\|\mathcal{K}^{1/2}(\mathcal{N}_++1)^{1/2}\xi\|+\|(\mathcal{N}_++1)\xi\|\right].\end{aligned} \tag{4.69}$$

Hence, choosing $q=\log N$,

$$\begin{aligned}&\big|\langle\xi,\mathrm{T}_{12}\xi\rangle\big| \\ &\quad=\Big|\sqrt{N}e^{2N}\lambda_\ell\int_{\Lambda^2}dxdy\,\chi(|x-y|\leq N^{-\alpha})f_{N,\ell}(x-y)\langle\xi,\check{b}_x^*\check{a}_y^*\check{a}_x\xi\rangle\Big| \\ &\quad\leq CN^{2\alpha-1/2}\int_{\Lambda^2}dxdy\,\chi(|x-y|\leq N^{-\alpha})\|\check{a}_x\check{a}_y\xi\|\|\check{a}_x\xi\| \\ &\quad\leq C(\log N)^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\left[\|\mathcal{K}^{1/2}\xi\|+\|(\mathcal{N}_++1)^{1/2}\xi\|\right]\,,\end{aligned} \tag{4.70}$$

With (4.68) and (4.70) we conclude that

$$|\langle\xi,e^{-A}\mathrm{T}_1e^{A}\xi\rangle|\leq C(\log N)^{1/2}\|(\mathcal{H}_N+1)^{1/2}e^{sA}\xi\|\|(\mathcal{N}_++1)^{1/2}e^{sA}\xi\|\,. \tag{4.71}$$

for all $\xi\in\mathcal{F}_+^{\leq N}$. As for the second term on the r.h.s. of (4.66) we have

$$\begin{aligned}&\big|\langle\xi,\mathrm{T}_2\xi\rangle\big| \\ &\quad\leq\frac{C}{\sqrt{N}}\Big[\sum_{r\in\Lambda_+^*}|r|^2\|\mathcal{N}_+^{1/2}a_{-r}\xi\|^2\Big]^{1/2}\Big[\sum_{r,v\in\Lambda_+^*}|v|^2\eta_r^2\|a_v\xi\|^2\Big]^{1/2} \\ &\quad\leq CN^{-\alpha}\|\mathcal{K}^{1/2}\xi\|^2.\end{aligned} \tag{4.72}$$

for any $\xi\in\mathcal{F}_+^{\leq N}$. With (4.71) and Prop. 2.12, we conclude that

$$|\langle\xi,e^{-sA}[\mathcal{K},A]e^{sA}\xi\rangle|\leq C\langle\xi,e^{-sA}\mathcal{H}_Ne^{sA}\xi\rangle+C\log N\langle\xi,e^{-sA}\mathcal{N}_+e^{sA}\xi\rangle\,.$$

Combining with Eq. (4.65) we obtain

$$|\langle \xi, e^{-sA}[\mathcal{H}_N, A]e^{sA}\xi\rangle| \leq C\langle \xi, e^{-sA}\mathcal{H}_N e^{sA}\xi\rangle + CN\langle \xi, e^{-sA}\mathcal{N}_+ e^{sA}\xi\rangle \,.$$

With Prop. 2.12 we obtain the differential inequality

$$|f'_\xi(s)| \leq C f_\xi(s) + CN\langle \xi, (\mathcal{N}_+ + 1)\xi\rangle \,.$$

By Gronwall's Lemma, we find (2.94). □

4.2.2 Analysis of $e^{-A}\mathcal{O}_N e^A$

In this section we study the contribution to $\mathcal{R}_{N,\alpha}$ arising from the operator $\mathcal{O}_N$, defined in (4.59). To this end, it is convenient to use the following lemma.

Lemma 4.8. *Let A be defined in* (2.93)*. Then, there exists a constant $C > 0$ such that*

$$\sum_{p\in\Lambda_+^*} F_p\, e^{-A}a_p^* a_p e^A = \sum_{p\in\Lambda_+^*} F_p\, a_p^* a_p + \mathcal{E}_F$$

where

$$|\langle \xi_1, \mathcal{E}_F \xi_2\rangle| \leq CN^{-\alpha}\|F\|_\infty \|(\mathcal{N}_+ + 1)^{1/2}\xi_1\| \|(\mathcal{N}_+ + 1)^{1/2}\xi_2\|$$

for all $\alpha > 0$, $\xi_1, \xi_2 \in \mathcal{F}_+^{\leq N}$, $F \in \ell^\infty(\Lambda_+^*)$, and $N \in \mathbb{N}$ large enough.

Proof. The lemma is analogous to [10, Lemma 8.6]. We estimate

$$\begin{aligned}
&\Big| \sum_{p\in\Lambda_+^*} F_p(\langle \xi_1, e^{-A}a_p^* a_p e^A \xi_2\rangle - \langle \xi_1, a_p^* a_p \xi_2\rangle)\Big| \\
&\qquad = \Big| \int_0^1 ds \sum_{p\in\Lambda_+^*} F_p \langle \xi_1, e^{-sA}[a_p^* a_p, A]e^{sA}\xi_2\rangle \Big| \\
&\qquad \leq \frac{1}{\sqrt{N}} \int_0^1 ds \sum_{r,v\in\Lambda_+^*} |F_{r+v} + F_{-r} - F_v||\eta_r||\langle e^{sA}\xi_1, b_{r+v}^* a_{-r}^* a_v e^{sA}\xi_2\rangle| \\
&\qquad \leq C\|\eta\|\|F\|_\infty \|(\mathcal{N}_+ + 1)^{1/2}\xi_1\|\|(\mathcal{N}_+ + 1)^{1/2}\xi_2\| \,.
\end{aligned}$$

where we used Prop. 2.12. □

We consider now the action of e^A on the operator $\mathcal{O}_N$, as defined in (4.59).

Proposition 4.9. *Let A be defined in* (2.93)*. Then there exists a constant $C > 0$ such that*

$$e^{-A}\mathcal{O}_N e^A = \frac{1}{2}\widehat{\omega}_N(0)(N-1)\left(1 - \frac{\mathcal{N}_+}{N}\right) + \big[2N\widehat{V}(0) - \frac{1}{2}\widehat{\omega}_N(0)\big]\mathcal{N}_+(1-\mathcal{N}_+/N) + \delta_{\mathcal{O}_N}$$

where

$$\pm\delta_{\mathcal{O}_N} \leq CN^{1-\alpha}(\mathcal{N}_+ + 1)$$

for all $\alpha > 0$, and $N \in \mathbb{N}$ large enough.

Proof. The proof is very similar to [10, Prop. 8.7]. First of all, with Lemma 4.8 we can bound

$$\begin{aligned}\pm\Big\{ e^{-A}\Big[\frac{1}{2}\widehat{\omega}_N(0)(N-1)\Big(1-\frac{\mathcal{N}_+}{N}\Big)+\big[2N\widehat{V}(0)-\frac{1}{2}\widehat{\omega}_N(0)\big]\mathcal{N}_+\Big]e^{A} \\ -\Big[\frac{1}{2}\widehat{\omega}_N(0)(N-1)\Big(1-\frac{\mathcal{N}_+}{N}\Big)+\big[2N\widehat{V}(0)-\frac{1}{2}\widehat{\omega}_N(0)\big]\mathcal{N}_+\Big]\Big\} \\ \leq CN^{1-\alpha}(\mathcal{N}_++1)\,.\end{aligned}$$

Moreover, for the contribution quadratic in $\mathcal{N}_+$, we can decompose

$$\begin{aligned}&\big\langle \xi, \big[e^{-A}\mathcal{N}_+^2e^{A}-\mathcal{N}_+^2\big]\,\xi\big\rangle \\ &\qquad= \big\langle \xi_1, \big[e^{-A}\mathcal{N}_+e^{A}-\mathcal{N}_+\big]\,\xi\big\rangle + \big\langle \xi, \big[e^{-A}\mathcal{N}_+e^{A}-\mathcal{N}_+\big]\,\xi_2\big\rangle\end{aligned}$$

with $\xi_1 = e^{-A}\mathcal{N}_+e^{A}\xi$ and $\xi_2 = \mathcal{N}_+\xi$, and estimate, again with Lemma 4.8,

$$\begin{aligned}&\big|\big\langle \xi, \big[e^{-A}\mathcal{N}_+^2e^{A}-\mathcal{N}_+^2\big]\,\xi\big\rangle\big| \\ &\qquad\leq CN^{-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\,\big[\|(\mathcal{N}_++1)^{1/2}\xi_1\|+\|(\mathcal{N}_++1)^{1/2}\xi_2\|\big]\ .\end{aligned}$$

With Prop. 2.12, we have $\|(\mathcal{N}_++1)^{1/2}\xi_1\| \leq C\|(\mathcal{N}_++1)^{3/2}\xi\|$. □

4.2.3 Contributions from $e^{-A}\mathcal{K}e^{A}$

In Section 4.2.6 we will analyse the contributions to $\mathcal{R}_{N,\alpha}$ arising from conjugation of the kinetic energy operator $\mathcal{K} = \sum_{p\in\Lambda_+^*} p^2a_p^*a_p$. To this aim we will exploit properties of the commutator $[\mathcal{K}, A]$, collected in the following proposition.

Proposition 4.10. *Let A be defined as in* (2.93) *and $\widehat{\omega}_N(r)$ be defined in* (2.88)*. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}[\mathcal{K}, A] =\; &-\sqrt{N}\sum_{p,q\in\Lambda_+^*,p\neq -q}(\widehat{V}(\cdot/e^N)*\widehat{f}_{N,\ell})(p)(b_{p+q}^*a_{-p}^*a_q+\text{h.c.}) \\ &+\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*,p\neq -q}\widehat{\omega}_N(p)\big[b_{p+q}^*a_{-p}^*a_q+\text{h.c.}\big]+\delta_{\mathcal{K}}\end{aligned}$$

where

$$\big|\langle\xi,\delta_{\mathcal{K}}\xi\rangle\big| \leq CN^{-1}(\log N)^{1/2}\|\mathcal{K}^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\| + CN^{-\alpha}\|\mathcal{K}^{1/2}\xi\|^2 \tag{4.73}$$

for all $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$, and $N \in \mathbb{N}$ large enough. Moreover, the operator

$$\Delta_{\mathcal{K}} = \frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*,p\neq -q}\widehat{\omega}_N(p)\big[b_{p+q}^*a_{-p}^*a_q, A\big]$$

satisfies

$$\big|\langle\xi,\Delta_{\mathcal{K}}\xi\rangle\big| \leq CN^{-\alpha}(\log N)^{1/2}\|\mathcal{K}^{1/2}\xi\|^2 + CN^{-1}\|(\mathcal{N}_++1)^{1/2}\xi\|^2 \tag{4.74}$$

for all $\alpha > 1$, $\xi \in \mathcal{F}_+^{\leq N}$, and $N \in \mathbb{N}$ large enough.

Proof. To show (4.73) we recall from Eqs. (4.66), (4.67) that

$$\begin{aligned}[\mathcal{K}, A] = & -\sqrt{N} \sum_{\substack{r,v\in\Lambda_+^* \\ r\neq -v}} (\widehat{V}(\cdot/e^N) * \widehat{f}_{N,\ell})(r)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] \\ & + 2\sqrt{N} \sum_{r,v\in\Lambda_+^*} e^{2N}\lambda_\ell(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(r)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] \\ & \qquad + \frac{2}{\sqrt{N}} \sum_{r,v\in\Lambda_+^*} r\cdot v\; \eta_r \big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] \\ = & \; \text{T}_{11} + \text{T}_{12} + \text{T}_2 \,. \end{aligned}$$

with T_2 satisfying (4.72). Using the definition $\widehat{\omega}_N(p) = 2Ne^{2N}\lambda_\ell\widehat{\chi}_\ell(p)$ we write

$$\begin{aligned}\text{T}_{12} = & \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p\neq -q} \widehat{\omega}_N(p)\big[b^*_{p+q}a^*_{-p}a_q + \text{h.c.}\,\big] \\ & + \frac{2}{\sqrt{N}}\, e^{2N}\lambda_\ell \sum_{p,q\in\Lambda_+^*, p\neq -q} (\widehat{\chi}_\ell * \eta)(p)\big[b^*_{p+q}a^*_{-p}a_q + \text{h.c.}\,\big] \\ = & \; \text{T}_{121} + \text{T}_{122}. \end{aligned}$$

Hence, $\delta_K = T_2 + T_{122}$. To bound T_{122} we switch to position space:

$$\begin{aligned} & |\langle \xi, \text{T}_{122}\xi\rangle| \\ & \leq CN^{2\alpha-3/2} \int_{\Lambda^2} \chi_\ell(x-y)\check{\eta}(x-y)\|\check{a}_x\check{a}_y\xi\|\|\check{a}_x\xi\| \\ & \leq CN^{2\alpha-3/2}\left[\int_{\Lambda^2} \chi_\ell(x-y)\|\check{a}_x\check{a}_y\xi\|^2 dxdy\right]^{1/2} \left[\int_{\Lambda^2} |\check{\eta}(x-y)|^2\|\check{a}_x\xi\|^2 dxdy\right]^{1/2} \\ & \leq CN^{\alpha-3/2}\|\mathcal{N}_+^{1/2}\xi\| \left[\int_{\Lambda^2} \chi_\ell(x-y)\|\check{a}_x\check{a}_y\xi\|^2 dxdy\right]^{1/2}. \end{aligned}$$

To bound the term in the parenthesis, we proceed similarly as in (4.62). We find

$$\int_{\Lambda^2} \chi_\ell(x-y)\|\check{a}_x\check{a}_y\xi\|^2 dxdy \leq Cq\|\chi_\ell\|_{q'}\|\mathcal{K}^{1/2}\mathcal{N}_+^{1/2}\xi\|^2 \leq CqN^{1-2\alpha/q'}\|\mathcal{K}^{1/2}\xi\|^2$$

for any $q > 2$ and $1 < q' < 2$ with $1/q + 1/q' = 1$. Choosing $q = \log N$, we obtain

$$|\langle \xi, \text{T}_{122}\xi\rangle| \leq CN^{-1}(\log N)^{1/2}\|\mathcal{N}_+^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|$$

With (4.72), this implies (4.73).

Let us now focus on (4.74). We have

$$\begin{aligned} & \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p\neq -q} \widehat{\omega}_N(p)\big[b^*_{p+q}a^*_{-p}a_q, A\big] \\ & \quad = \frac{1}{N} \sum_{\substack{r,p,q,v\in\Lambda_+^*, \\ p\neq -q, r\neq -v}} \widehat{\omega}_N(p)\eta_r\big[b^*_{p+q}a^*_{-p}a_q, b^*_{r+v}a^*_{-r}a_v - a^*_v a_{-r}b_{r+v}\big]\,. \end{aligned}$$

With the commutators from the proof of Prop. 8.8 in [10], we arrive at

$$\frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p\neq -q} \widehat{\omega}_N(p)\big[b^*_{p+q}a^*_{-p}a_q, A\big] + \text{h.c.} = \sum_{j=1}^{12} \Upsilon_j + \text{h.c.}$$

where

$$\begin{aligned}
\Upsilon_1 &:= -\frac{1}{N} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ q\neq v, r\neq -v}} \big(\widehat{\omega}_N(v-q)+\widehat{\omega}_N(v)\big)\eta_r b^*_{r+v}b^*_{-r}a^*_{q-v}a_q\,,\\
\Upsilon_2 &:= \frac{1}{N} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ r\neq -v, r\neq -q}} \widehat{\omega}_N(r+q)\eta_r(1-\mathcal{N}_+/N)a^*_v a^*_{r+q}a_q a_{r+v}\,,\\
\Upsilon_3 &:= \frac{1}{N} \sum_{\substack{r,v\in\Lambda_+^*,\\ r\neq -v}} \big(\widehat{\omega}_N(r+v)+\widehat{\omega}_N(r)\big)\eta_r(1-\mathcal{N}_+/N)a^*_v a_v,\\
\Upsilon_4 &:= \frac{1}{N} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ q\neq v, r\neq -v}} \widehat{\omega}_N(r+v-q)\eta_r(1-\mathcal{N}_+/N)a^*_v a^*_{q-r-v}a_{-r}a_q,\\
\Upsilon_5 &:= -\frac{1}{N^2} \sum_{\substack{p,q,r,v\in\Lambda_+^*,\\ p\neq -q, r\neq -v}} \widehat{\omega}_N(p)\eta_r a^*_v a^*_{p+q}a^*_{-p}a_{-r}a_{r+v}a_q\,,\\
\Upsilon_6 &:= -\frac{1}{N^2} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ q\neq r+v}} \widehat{\omega}_N(r+v)\eta_r a^*_v a^*_{q-r-v}a_{-r}a_q,\\
\Upsilon_7 &:= -\frac{1}{N^2} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ q\neq -r, r\neq -v}} \widehat{\omega}_N(r)\eta_r a^*_v a^*_{q+r}a_{r+v}a_q\,,\\
\Upsilon_8 &:= \frac{1}{N} \sum_{\substack{r,v,p\in\Lambda_+^*,\\ p\neq -r-v}} \widehat{\omega}_N(p)\eta_r b^*_{p+r+v}b^*_{-p}a^*_{-r}a_v\,,\\
\Upsilon_9 &:= \frac{1}{N} \sum_{\substack{p,r,v\in\Lambda_+^*,\\ p\neq r, r\neq -v}} \widehat{\omega}_N(p)\eta_r b^*_{p-r}b^*_{r+v}a^*_{-p}a_v\,,\\
\Upsilon_{10} &:= \frac{1}{N} \sum_{\substack{q,r,v\in\Lambda_+^*,\\ q\neq -r, r\neq -v}} \widehat{\omega}_N(r)\eta_r b^*_{q+r}a^*_v a_q b_{r+v}\,,\\
\Upsilon_{11} &:= -\frac{1}{N} \sum_{\substack{p,r,v\in\Lambda_+^*,\\ p\neq -v, r\neq -v}} \widehat{\omega}_N(p)\eta_r b^*_{p+v}a^*_{-p}a_{-r}b_{r+v}\,,\\
\Upsilon_{12} &:= \frac{1}{N} \sum_{\substack{q,r,v\in\Lambda_+^*\\ r\neq q-v, -v}} \widehat{\omega}_N(r+v)\eta_r b^*_{q-r-v}a^*_v a_{-r}b_q\,.
\end{aligned} \tag{4.75}$$

To conclude the proof of Prop. 4.10, we show that all operators in (4.75) satisfy (4.74). To study all these terms it is convenient to switch to position space. We recall that $\widehat{\omega}_N(p) = g_N \widehat{\chi}(\ell p)$ with $|g_N| \leq C$ and $\ell = N^{-\alpha}$. Using (4.69) we find:

$$\begin{aligned}\big|\langle \xi, \Upsilon_1 \xi\rangle\big| &\leq CN^{2\alpha-1}\int_{\Lambda^2} dxdy\, \chi_\ell(x-y)\|\check{b}(\check{\eta}_x)\check{b}_x\check{a}_y\xi\|\,[\|\check{a}_x\xi\| + \|\check{a}_y\xi\|]\\ &\leq CN^{2\alpha-1}\|\eta\|\int_{\Lambda^2} dxdy\,\chi_\ell(x-y)\|\check{b}_x\check{a}_y(\mathcal{N}_+ +1)^{1/2}\xi\|\|\check{a}_x\xi\|\\ &\leq CN^{-\alpha}(\log N)^{1/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\,.\end{aligned}$$

The expectation of Υ_2 is bounded following the same strategy used to show (4.69). For any $2 \leq q < \infty$ we have

$$\begin{aligned}&\big|\langle \xi, \Upsilon_2\xi\rangle\big|\\ &\leq CN^{2\alpha-1}\int_{\Lambda^3} dxdydz\chi_\ell(z-y)|\check{\eta}(z-x)|\|\check{a}_x\check{a}_y\xi\|\|\check{a}_z\check{a}_x\xi\|\\ &\leq CN^{2\alpha-1}\int_{\Lambda^2} dxdz|\check{\eta}(z-x)|\|\check{a}_z\check{a}_x\xi\|\\ &\qquad\times\left(\int_\Lambda dy\,\chi(|z-y|\leq N^{-\alpha})\right)^{1-1/q}\left(\int_\Lambda dy\|\check{a}_x\check{a}_y\xi\|^q\right)^{1/q}\\ &\leq Cq^{1/2}N^{2\alpha/q-1}\|\eta\|\|(\mathcal{N}_+ +1)\xi\|\left[\int_{\Lambda^2} dxdy\|\check{a}_x\nabla_y\check{a}_y\xi\|^2 + \int_{\Lambda^2} dxdy\|\check{a}_x\check{a}_y\xi\|^2\right]^{1/2}\\ &\leq CN^{-\alpha}(\log N)^{1/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\,,\end{aligned}$$

where in the last line we chose $q = \log N$. The term Υ_3 is of lower order; using that $\big|\sum_r \widehat{\omega}_N(r)\eta_r\big| \leq \|\widehat{\chi}(./N^\alpha)\|_2\|\eta\|_2 \leq C$ and Cauchy-Schwarz, we easily obtain

$$\big|\langle \xi, \Upsilon_3\xi\rangle\big| \leq CN^{-1}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2\,.$$

The term Υ_4 can be estimated as Υ_1 using (4.69):

$$\begin{aligned}\big|\langle \xi, \Upsilon_4\xi\rangle\big| &\leq CN^{2\alpha-1}\int_{\Lambda^2} dxdy\,\chi_\ell(x-y)\|\check{a}_x\check{a}_y\xi\|\|\check{a}(\check{\eta}_y)\check{a}_y\xi\|\\ &\leq CN^{2\alpha-1}\|\eta\|\int_{\Lambda^2} dxdy\,\chi_\ell(x-y)\|\check{a}_x\check{a}_y\xi\|\|\check{a}_y(\mathcal{N}_+ +1)^{1/2}\xi\|\\ &\leq CN^{-\alpha}(\log N)^{1/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\,.\end{aligned}$$

The term Υ_5 is bounded similarly to Υ_2; with $q = \log N$ we have

$$\begin{aligned}\big|\langle \xi, \Upsilon_5\xi\rangle\big| &\leq CN^{2\alpha-2}\|\eta\|\int_{\Lambda^3} dxdydz\,\chi_\ell(y-z)\|\check{a}_x\check{a}_y\check{a}_z\xi\|\|\mathcal{N}_+^{1/2}\check{a}_x\check{a}_y\xi\|\\ &\leq CN^{2\alpha-3/2}\|\eta\|\int_{\Lambda^2} dxdy\,\|\check{a}_x\check{a}_y\xi\|\\ &\qquad\times\left(\int_\Lambda dz\,\chi(|y-z|\leq N^{-\alpha})\right)^{1-1/q}\left(\int_\Lambda dz\,\|\check{a}_x\check{a}_y\check{a}_z\xi\|^q\right)^{1/q}\\ &\leq CN^{-\alpha}(\log N)^{1/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\,.\end{aligned}$$

The terms Υ_6 and Υ_7 are of smaller order and can be bounded with Cauchy-Schwarz; we have

$$\begin{aligned}
\big|\langle\xi,\Upsilon_6\xi\rangle\big| &\le CN^{2\alpha-2}\int_{\Lambda^2} dxdydz\,\chi_\ell(x-y)\|\check a_x\check a_y\xi\|\|\check a(\check\eta_x)\check a_y\xi\| \\
&\le CN^{\alpha-3/2}\left(\int_{\Lambda^2} dxdy\,\|\check a_x\check a_y\xi\|^2\right)^{1/2}\left(\int_{\Lambda^2} dxdy\,\chi(|x-y|\le N^{-\alpha})\|\check a_y\xi\|^2\right)^{1/2} \\
&\le CN^{-1}\|(\mathcal N_++1)^{1/2}\xi\|^2\,,
\end{aligned}$$

and

$$\begin{aligned}
\big|\langle\xi,\Upsilon_7\xi\rangle\big| &\le CN^{2\alpha-2}\int_{\Lambda^3} dxdydz\,\chi_\ell(y-z)|\check\eta(z-x)|\|\check a_x\check a_y\xi\|^2 \\
&\le CN^{2\alpha-2}\left(\int_{\Lambda^3} dxdydz\,\chi_\ell(y-z)\|\check a_x\check a_y\xi\|^2\right)^{1/2} \\
&\qquad\times\left(\int_{\Lambda^3} dxdydz\,|\check\eta(z-x)|^2\|\check a_x\check a_y\xi\|^2\right)^{1/2} \\
&\le CN^{-1}\|(\mathcal N_++1)^{1/2}\xi\|^2\,.
\end{aligned}$$

The terms $\Upsilon_8,\Upsilon_{11},\Upsilon_{12}$ are again bounded, as Υ_1, using (4.69). We find

$$\begin{aligned}
\big|\langle\xi,\big(\Upsilon_8+\Upsilon_{11}+\Upsilon_{12}\big)\xi\rangle\big| &\le CN^{2\alpha-1}\|\eta\|\int_{\Lambda^2} dxdy\,\chi_\ell(x-y)\|\mathcal N_+^{1/2}\check a_x\check a_y\xi\|\|\check a_x\xi\| \\
&\le CN^{-\alpha}(\log N)^{1/2}\|(\mathcal N_++1)^{1/2}\xi\|\|\mathcal K^{1/2}\xi\|\,.
\end{aligned}$$

It remains to bound Υ_9 and Υ_{10}. The term Υ_9 is bounded analogously to Υ_2:

$$\begin{aligned}
&\big|\langle\xi,\Upsilon_9\xi\rangle\big| \\
&\quad\le CN^{2\alpha-1}\int_{\Lambda^3} dxdydz\,\chi_\ell(x-z)|\check\eta(x-y)|\|\check a_x\check a_y\check a_z\xi\|\|\check a_y\xi\| \\
&\quad\le CN^{2\alpha-1}\int_{\Lambda^2} dxdy\,|\check\eta(x-y)|\|\check a_y\xi\|\left(\int_\Lambda dz\,\chi(|y-z|\le N^{-\alpha})\right)^{1-1/q} \\
&\qquad\qquad\times\left(\int_\Lambda dz\,\|\check a_x\check a_y\check a_z\xi\|^q\right)^{1/q} \\
&\quad\le Cq^{1/2}N^{2\alpha/q-1}\left[\int_{\Lambda^2} dxdy\,|\check\eta(x-y)|^2\|\check a_y\xi\|^2\right]^{1/2}\left[\int_{\Lambda^3} dxdy\,\Big\|\|\check a_x\check a_y\check a_z\xi\|\Big\|^2_{L^q_z}\right]^{1/2} \\
&\quad\le CN^{-\alpha}(\log N)^{1/2}\|(\mathcal N_++1)^{1/2}\xi\|\|\mathcal K^{1/2}\xi\|\,.
\end{aligned}$$

As for Υ_{10}, we find

$$\big|\langle\xi,\Upsilon_{10}\xi\rangle\big|\le CN^{2\alpha-1}\int_{\Lambda^3} dxdydz\,\chi_\ell(y-z)|\check\eta(x-z)|\|\check a_x\check a_y\xi\|^2$$

Proceeding as in (4.62), we obtain

$$\big|\langle\xi,\Upsilon_{10}\xi\rangle\big|\le CqN^{2\alpha}\|\chi_\ell*|\check\eta|\|_{q'}\|\mathcal K^{1/2}\xi\|^2\le Cq\|\check\eta\|_{q'}\|\mathcal K^{1/2}\xi\|^2$$

for any $q > 2$, and $q' < 2$ with $1/q + 1/q' = 1$. Since, for an arbitrary $q' < 2$, $\|\check{\eta}\|_{q'} \leq \|\check{\eta}\|_2 = \|\eta\|_2 \leq N^{-\alpha}$, we obtain

$$\big|\langle \xi, \Upsilon_{10}\xi\rangle\big| \leq CN^{-\alpha}\|\mathcal{K}^{1/2}\xi\|^2$$

We conclude that for any $\alpha > 1$

$$\big|\langle \xi, \sum_{j=1}^{12} \Upsilon_i \xi\rangle\big| \leq CN^{-\alpha}(\log N)^{1/2}\,\|(\mathcal{K}+1)^{1/2}\xi\|^2 + CN^{-1}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\,.$$

□

4.2.4 Analysis of $e^{-A}\mathcal{Z}_N e^{A}$

In this subsection, we consider contributions to $\mathcal{R}_{N,\alpha}$ arising from conjugation of $\mathcal{Z}_N$, as defined in (4.59).

Proposition 4.11. *Let A be defined in* (2.93). *Then, there exists a constant $C > 0$ such that*

$$e^{A}\mathcal{Z}_N e^{-A} = \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{\omega}_N(p)\,\big(b_p^* b_{-p}^* + b_p b_{-p}\big) + \delta_{\mathcal{Z}_N}$$

where

$$\pm\delta_{\mathcal{Z}_N} \leq CN^{1-\alpha}(\mathcal{H}_N + 1)$$

for all $\alpha > 0$, and $N \in \mathbb{N}$ large enough.

Proof. We have

$$\begin{aligned}\frac{1}{2}\sum_{p\in\Lambda_+^*} &\widehat{\omega}_N(p)\,\big[e^{-A}\big(b_p^* b_{-p}^* + b_p b_{-p}\big)e^{A} - \big(b_p^* b_{-p}^* + b_p b_{-p}\big)\big]\\ &= \frac{1}{2}\int_0^1 ds \sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)\; e^{-sA}\big[b_p^* b_{-p}^* + b_p b_{-p}, A\big]e^{sA}.\end{aligned} \tag{4.76}$$

We compute

$$\begin{aligned}\frac{1}{2}\sum_{p\in\Lambda_+^*}&\widehat{\omega}_N(p)\big[b_p^* b_{-p}^*, b_{r+v}^* a_{-r}^* a_v - a_v^* a_{-r} b_{r+v}\big]\\ &= -\,\widehat{\omega}_N(v) b_{r+v}^* b_{-v}^* b_{-r}^* + \widehat{\omega}_N(r) b_v^*\Big(b_r^* b_{r+v} - \frac{2}{N}a_r^* a_{r+v}\Big)\\ &\quad + \widehat{\omega}_N(r+v)\left(1 - \frac{\mathcal{N}_+}{N}\right) b_{-r-v}^* a_v^* a_{-r} - \frac{1}{N}\sum_{p\in\Lambda^*}\widehat{\omega}_N(p) b_p^* a_{-p}^* a_v^* a_{-r} a_{r+v}.\end{aligned} \tag{4.77}$$

With (4.77) we write

$$\frac{1}{2}\sum_{p\in\Lambda^*}\widehat{\omega}_N(p)\big[b_p^* b_{-p}^* + b_p b_{-p}, A\big] = \sum_{j=1}^{4}\Pi_j + \text{h.c.}$$

with

$$\begin{aligned}
\Pi_1 &= -\frac{1}{\sqrt{N}} \sum_{\substack{r,v\in\Lambda_+^*\\ r\neq -v}} \widehat{\omega}_N(v)\eta_r b^*_{r+v}b^*_{-v}b^*_{-r}\,,\\
\Pi_2 &= \frac{1}{\sqrt{N}} \sum_{\substack{r,v\in\Lambda_+^*:\\ r\neq -v}} \widehat{\omega}_N(r)\eta_r b^*_v\Big(b^*_r b_{r+v} - \frac{2}{N}a^*_r a_{r+v}\Big)\,,\\
\Pi_3 &= \frac{1}{\sqrt{N}} \sum_{\substack{r,v\in\Lambda_+^*\\ r\neq -v}} \widehat{\omega}_N(r+v)\eta_r \left(1-\frac{\mathcal{N}_+}{N}\right) b^*_{-r-v}a^*_v a_{-r}\,,\\
\Pi_4 &= -\frac{1}{N^{3/2}} \sum_{\substack{r,v,p\in\Lambda_+^*:\\ r\neq -v}} \widehat{\omega}_N(p)\eta_r b^*_p a^*_{-p}a^*_v a_{-r}a_{r+v}\,.
\end{aligned}$$

To bound the first term, we observe, with (2.101),

$$\begin{aligned}
|\langle\xi,\Pi_1\xi\rangle| &\leq \frac{\|\eta\|}{\sqrt{N}}\|\mathcal{K}^{1/2}\mathcal{N}_+^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|\left[\sum_{v\in\Lambda_+^*}\frac{|\widehat{\omega}_N(v)|^2}{v^2}\right]^{1/2}\\
&\leq CN^{-\alpha}(\log N)^{1/2}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|\,.
\end{aligned}$$

The term Π_3 can be bounded similarly to Π_1, with (2.101). We find

$$\big|\langle\xi,\Pi_3\xi\rangle\big| \leq CN^{-\alpha}(\log N)^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\,.$$

With $|\widehat{\omega}_N(r)|\leq C$, we similarly obtain

$$\begin{aligned}
|\langle\xi,\Pi_2\xi\rangle| &\leq N^{-1/2}\|\eta\|\|\mathcal{K}^{1/2}\mathcal{N}_+^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|\\
&\leq CN^{-\alpha}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|\,.
\end{aligned}$$

Finally, we estimate, using again (2.101),

$$\begin{aligned}
\big|\langle\xi,\Pi_4\xi\rangle\big| &\leq N^{-3/2}\Big(\sum_{r,v,p\in\Lambda_+^*} p^2|\eta_r|^2\|a_{-p}a_v(\mathcal{N}_++1)^{1/2}\xi\|^2\Big)^{1/2}\\
&\qquad\qquad\times\Big(\sum_{r,v,p\in\Lambda_+^*}\frac{|\widehat{\omega}_N(p)|^2}{p^2}\|a_{-r}a_{r+v}\xi\|^2\Big)^{1/2}\\
&\leq CN^{-3/2}\|\eta\|(\log N)^{1/2}\|\mathcal{K}^{1/2}(\mathcal{N}_++1)\xi\|\|(\mathcal{N}_++1)\xi\|\\
&\leq CN^{-\alpha}(\log N)^{1/2}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|\,.
\end{aligned}$$

With (4.76), we conclude that

$$\begin{aligned}
&\left|\frac{1}{2}\sum_{p\in\Lambda^*}\widehat{\omega}_N(p)\,\left[\langle\xi,e^{-A}\big(b^*_pb^*_{-p}+b_pb_{-p}\big)e^A\xi\rangle - \langle\xi,\big(b^*_pb^*_{-p}+b_pb_{-p}\big)\xi\rangle\right]\right|\\
&\qquad\leq CN^{-\alpha}(\log N)^{1/2}\int_0^1 ds\;\|\mathcal{K}^{1/2}e^{sA}\xi\|\|(\mathcal{N}_++1)^{1/2}e^{sA}\xi\|\,.
\end{aligned}$$

With Prop. 2.12, Lemma 2.13, we conclude that

$$\begin{aligned}\Big|\frac{1}{2}\sum_{p\in\Lambda^*}\widehat{\omega}_N(p)\,\big[\langle\xi,e^{-A}\big(b_p^*b_{-p}^*+b_pb_{-p}\big)e^A\xi\rangle-\langle\xi,\big(b_p^*b_{-p}^*+b_pb_{-p}\big)\xi\rangle\big]\Big|&\\ \leq CN^{-\alpha}(\log N)^{1/2}\Big[\|\mathcal{H}_N^{1/2}\xi\|+N^{1/2}\|\mathcal{N}_+^{1/2}\xi\|\Big]\,\|(\mathcal{N}_++1)^{1/2}\xi\|&\\ \leq CN^{1-\alpha}\|(\mathcal{H}_N+1)^{1/2}\xi\|^2\,.&\end{aligned}$$

□

4.2.5 Contributions from $e^{-A}\mathcal{C}_Ne^A$

In Section 4.2.6 we will analyse the contributions to $\mathcal{R}_{N,\alpha}$ arising from conjugation of the cubic operator $\mathcal{C}_N$ defined in (4.59). To this aim we will need some properties of the commutator $[\mathcal{C}_N,A]$, as established in the following proposition.

Proposition 4.12. *Let A be defined in (2.93). Then, there exists a constant $C>0$ such that*

$$\big[\mathcal{C}_N,A\big]=2\sum_{r,v\in\Lambda_+^*}\big[\widehat{V}(r/e^N)\eta_r+\widehat{V}((r+v)/e^N)\eta_r\big]a_v^*a_v\Big(1-\frac{\mathcal{N}_+}{N}\Big)+\delta_{\mathcal{C}_N}$$

where

$$|\langle\xi,\delta_{\mathcal{C}_N}\xi\rangle|\leq CN^{3/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\| \tag{4.78}$$

for all $\alpha>0$, $\xi\in\mathcal{F}_+^{\leq N}$, and $N\in\mathbb{N}$ large enough.

Proof. We consider the commutator

$$\big[\mathcal{C}_N,A\big]=\sum_{\substack{p,q\in\Lambda_+^*:p+q\neq0\\ r,v\in\Lambda_+^*}}\widehat{V}(p/e^N)\eta_r\big[b_{p+q}^*a_{-p}^*a_q,\,b_{r+v}^*a_{-r}^*a_v-a_v^*a_{-r}b_{r+v}\big]+\text{h.c. }.$$

As in the proof of Prop. 4.10, we use the commutators from the proof of Prop. 8.8 in [10] to conclude that

$$\big[\mathcal{C}_N,A\big]=2\sum_{r,v\in\Lambda_+^*}\big[\widehat{V}(r/e^N)\eta_r+\widehat{V}((r+v)/e^N)\eta_r\big]a_v^*a_v\frac{N-\mathcal{N}_+}{N}+\sum_{j=1}^{12}(\Xi_j+\text{h.c. })$$

where

$$\begin{aligned}\Xi_1&:=-\sum_{\substack{r,v,p\in\Lambda_+^*,\\ p\neq v}}\widehat{V}(p/e^N)\eta_rb_{r+v}^*b_{-r}^*a_{-p}^*a_{v-p},\\ \Xi_2&:=\sum_{\substack{r,v,p\in\Lambda_+^*\\ r\neq-p}}\widehat{V}(p/e^N)\eta_r(1-\mathcal{N}_+/N)a_v^*a_{-p}^*a_{-r-p}a_{r+v},\\ \Xi_3&:=\sum_{\substack{r,v,p\in\Lambda_+^*:\\ r+v\neq p}}\widehat{V}(p/e^N)\eta_r(1-\mathcal{N}_+/N)a_v^*a_{-p}^*a_{-r}a_{r+v-p},\end{aligned}$$

as well as

$$
\begin{aligned}
\Xi_4 &:= -\frac{1}{N}\sum_{r,v,p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/e^N)\eta_r a_v^* a_{p+q}^* a_{-p}^* a_{-r}a_{r+v}a_q,\\
\Xi_5 &:= -\frac{1}{N}\sum_{\substack{r,v,q\in\Lambda_+^*:\\ r+v\neq q}}\widehat{V}((r+v)/e^N)\eta_r a_v^* a_{q-r-v}^* a_{-r}a_q,\\
\Xi_6 &:= -\frac{1}{N}\sum_{\substack{r,v,q\in\Lambda_+^*:\\ r\neq -q}}\widehat{V}(r/e^N)\eta_r a_v^* a_{q+r}^* a_{r+v}a_q\\
\Xi_7 &:= \sum_{\substack{r,v,p\in\Lambda_+^*:\\ r+v\neq -p}}\widehat{V}(p/e^N)\eta_r b_{p+r+v}^* b_{-p}^* a_{-r}^* a_v,\\
\Xi_8 &:= \sum_{\substack{r,v,p\in\Lambda_+^*:\\ r\neq -p}}\widehat{V}(p/e^N)\eta_r b_{p-r}^* b_{r+v}^* a_{-p}^* a_v,\\
\Xi_9 &:= -\sum_{\substack{r,v,q\in\Lambda_+^*:\\ q\neq v}}\widehat{V}(v/e^N)\eta_r b_{q-v}^* b_{r+v}^* a_{-r}^* a_q,\\
\Xi_{10} &:= \sum_{\substack{r,v,q\in\Lambda_+^*:\\ r\neq -q}}\widehat{V}(r/e^N)\eta_r b_{q+r}^* a_v^* a_q b_{r+v},\\
\Xi_{11} &:= -\sum_{\substack{r,v,p\in\Lambda_+^*:\\ p\neq -v}}\widehat{V}(p/e^N)\eta_r b_{p+v}^* a_{-p}^* a_{-r} b_{r+v},\\
\Xi_{12} &:= \sum_{\substack{r,v,q\in\Lambda_+^*:\\ q\neq r+v}}\widehat{V}((r+v)/e^N)\eta_r b_{q-r-v}^* a_v^* a_{-r} b_q\,.
\end{aligned}
$$

To prove the proposition, we have to show that all terms Ξ_j, $j=1,\dots,12$, satisfy the bound (4.78). We bound Ξ_1 in position space, with Cauchy-Schwarz, by

$$
\begin{aligned}
\big|\langle\xi,\Xi_1\xi\rangle\big| &\leq C\int_{\Lambda^3} dxdydz e^{2N}V(e^N(x-y))|\check{\eta}(x-z)|\|\check{a}_x\xi\|\|\check{a}_x\check{a}_y\check{a}_z\xi\|\\
&\leq C\left[\int_{\Lambda^3} dxdydz\, e^{2N}V(e^N(x-y))\|\check{a}_x\check{a}_y\check{a}_z\xi\|^2\right]^{1/2}\\
&\qquad\qquad \times\left[\int_{\Lambda^3} dxdydz\, e^{2N}V(e^N(x-y))|\check{\eta}(x-z)|^2\|\check{a}_x\xi\|^2\right]^{1/2}\\
&\leq C\|\eta\|\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\mathcal{N}_+^{1/2}\xi\|\\
&\leq CN^{1/2-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|.
\end{aligned}
$$

We can proceed similarly to control Ξ_9. We obtain

$$
\big|\langle\xi,\Xi_9\xi\rangle\big| \leq CN^{1/2-\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|.
$$

The expectations of the terms Ξ_3 and Ξ_{12} can be bounded analogously:

$$\begin{aligned}
&\big|\langle \xi, \Xi_3 \xi\rangle\big| + \big|\langle \xi, \Xi_{12}\xi\rangle\big| \\
&\leq C \int_{\Lambda^3} dxdydz\, e^{2N} V(e^N(x-y))(|\eta(x-z)| + |\eta(y-z)|) \|\check{a}_x \check{a}_y \xi\| \|\check{a}_x \check{a}_z \xi\| \\
&\leq C \left[\int_{\Lambda^3} dxdydz\, e^{2N} V(e^N(x-y)) \|\check{a}_x \check{a}_y \xi\|^2 (|\eta(x-z)|^2 + |\eta(y-z)|^2)\right]^{1/2} \\
&\qquad\qquad \times \left[\int_{\Lambda^3} dxdydz\, e^{2N} V(e^N(x-y)) \|\check{a}_x \check{a}_z \xi\|^2\right]^{1/2} \\
&\leq C \|\eta\| \|(\mathcal{N}_+ + 1)\xi\| \|\mathcal{V}_N^{1/2} \xi\| \\
&\leq C N^{1/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{V}_N^{1/2}\xi\|.
\end{aligned}$$

As for Ξ_4, we find

$$\begin{aligned}
|\langle \xi, \Xi_4 \xi\rangle| &= \left| \frac{1}{N} \int_{\Lambda^2} dxdydz\, e^{2N} V(e^N(y-z)) \langle \xi, \check{a}_x^* \check{a}_y^* \check{a}_z^* \check{a}(\check{\eta}_x) \check{a}_x \check{a}_y \xi\rangle \right| \\
&\leq C N^{-1} \|\eta\| \int_{\Lambda^2} dxdydz\, e^{2N} V(e^N(y-z)) \|\check{a}_x \check{a}_y \check{a}_z \xi\| \|\mathcal{N}_+^{1/2} \check{a}_x \check{a}_y \xi\| \\
&\leq C N^{-1} \|\eta\| \left[\int_{\Lambda^2} dxdydz\, e^{2N} V(e^N(y-z)) \|\check{a}_x \check{a}_y \check{a}_z \xi\|^2\right]^{1/2} \\
&\qquad \times \left[\int_{\Lambda^2} dxdydz\, e^{2N} V(e^N(y-z)) \|\mathcal{N}_+^{1/2} \check{a}_x \check{a}_y \xi\|^2\right]^{1/2} \\
&\leq C N^{1/2-\alpha} \|\mathcal{V}_N^{1/2}\xi\| \|\mathcal{N}_+^{1/2}\xi\| \,.
\end{aligned}$$

The terms Ξ_5 and Ξ_6 can be bounded in momentum space, using (4.44). Hence,

$$\begin{aligned}
&|\langle \xi, \Xi_5 \xi\rangle| + |\langle \xi, \Xi_6 \xi\rangle| \\
&\leq C N^{-1} \sum_{r,v,q \in \Lambda_+^*} \Bigg(\frac{\widehat{V}((v+r)/e^N)}{|v|} |\eta_r| |v| \|a_v a_{q-r-v} \xi\| \|a_{-r} a_q \xi\| \\
&\qquad\qquad + \frac{\widehat{V}(r/e^N)}{|r+v|} |\eta_r| |r+v| \|a_{r+q} a_v \xi\| \|a_q a_{r+v} \xi\| \Bigg) \\
&\leq C N^{1/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\|.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
|\langle \xi, \Xi_2 \xi\rangle| + |\langle \xi, \Xi_{10} \xi\rangle| &\leq \sum_{r,v,p \in \Lambda_+^*} \Bigg(\frac{\widehat{V}(p/e^N)}{|p|} |\eta_r| |p| \|a_v a_{-p} \xi\| \|a_{r+v} a_{-r-p} \xi\| \\
&\qquad + \frac{\widehat{V}(r/e^N)}{|r+v|} |\eta_r| |r+v| \|a_q a_{r+v} \xi\| \|a_{r+q} a_v \xi\| \Bigg) \\
&\leq C N^{3/2-\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|\mathcal{K}^{1/2} \xi\|.
\end{aligned}$$

Next, we rewrite Ξ_7, Ξ_8 and Ξ_{11} as

$$\begin{aligned}\Xi_7 &= \int_{\Lambda^2} dxdy\; e^{2N}V(e^N(x-y))\check{b}_x^*\check{b}_y^* a^*(\check{\eta}_x)\check{a}_x\,,\\ \Xi_8 &= \int_{\Lambda^2} dxdydz\; e^{2N}V(e^N(x-y))\check{\eta}(z-x)\check{b}_x^*\check{b}_z^*\check{a}_y^*\check{a}_z\,,\\ \Xi_{11} &= -\int_{\Lambda^2} dxdy\; e^{2N}V(e^N(x-y))\check{b}_x^*\check{a}_y^*\check{a}(\check{\eta}_x)\check{b}_x\,.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|\langle\xi,\Xi_7\xi\rangle| &\le C\|\eta\|\int_{\Lambda^2} dxdy\; e^{2N}V(e^N(x-y))\,\|\mathcal{N}_+^{1/2}\check{a}_x\check{a}_y\xi\|\|\check{a}_x\xi\|\\ &\le C\|\eta\|\|\mathcal{N}_+^{1/2}\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|\\ &\le CN^{1/2-\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|\,,\end{aligned}$$

as well as

$$\begin{aligned}&|\langle\xi,\Xi_8\xi\rangle|\\ &\le C\int_{\Lambda^2} dxdydz\; e^{2N}V(e^N(x-y))|\check{\eta}(x-z)|\|\check{a}_x\check{a}_y\check{a}_z\xi\|\|\check{a}_z\xi\|\\ &\le C\left[\int_{\Lambda^2} dxdydz\; e^{2N}V(e^N(x-y))\|\check{a}_x\check{a}_y\check{a}_z\xi\|^2\right]^{1/2}\\ &\qquad\qquad\times\left[\int_{\Lambda^2} dxdydz\; e^{2N}V(e^N(x-y))|\eta(x-z)|^2\|\check{a}_z\xi\|^2\right]^{1/2}\\ &\le CN^{1/2-\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|,\end{aligned}$$

and

$$\begin{aligned}|\langle\xi,\Xi_{11}\xi\rangle| &\le C\|\eta\|\int_{\Lambda^2} dxdy\; e^{2N}V(e^N(x-y))\,\|\check{a}_x\check{a}_y\xi\|\|\mathcal{N}_+^{1/2}\check{a}_x\xi\|\\ &\le C\|\eta\|\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+\xi\| \le CN^{1/2-\alpha}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|.\end{aligned}$$

Collecting all the bounds above, we arrive at (4.78). □

4.2.6 Proof of Proposition 2.14

With the results of Sections 4.2.1-4.2.5, we can now show Proposition 2.14. We assume $\alpha > 2$. From Eq. (4.58), Prop. 4.9 and Prop. 4.11 we obtain that

$$\begin{aligned}\mathcal{R}_{N,\alpha} &= e^{-A}\mathcal{G}_{N,\alpha}^{\text{eff}}e^A\\ &= \frac{1}{2}\,\widehat{\omega}_N(0)(N-1)(1-\mathcal{N}_+/N) + \big[2N\widehat{V}(0) - \frac{1}{2}\widehat{\omega}_N(0)\big]\mathcal{N}_+(1-\mathcal{N}_+/N)\\ &\quad + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)\big[b_p^*b_{-p}^* + b_pb_{-p}\big] + \mathcal{K} + \mathcal{C}_N + \mathcal{V}_N\\ &\quad + \int_0^1 ds\; e^{-sA}\big[\mathcal{K}+\mathcal{C}_N+\mathcal{V}_N, A\big]e^{sA} + \mathcal{E}_{\mathcal{R}}^{(1)}\end{aligned}$$

with

$$\pm\mathcal{E}_{\mathcal{R}}^{(1)} \leq CN^{1-\alpha}(\mathcal{H}_N + 1)\,.$$

From Prop. 4.7, Prop. 4.10 and Prop. 4.12, we can write, for N large enough,

$$\begin{aligned}
&[\mathcal{K} + \mathcal{C}_N + \mathcal{V}_N, A] \\
&= \frac{1}{\sqrt{N}} \sum_{r,v\in\Lambda_+^*} \widehat{\omega}_N(r)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] - \sqrt{N} \sum_{\substack{r,v,\in\Lambda_+^*,\\ p\neq -q}} \widehat{V}(r/e^N)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] \\
&\quad + 2\sum_{r,v\in\Lambda_+^*} \big[\widehat{V}(r/e^N)\eta_r + \widehat{V}((r+v)/e^N)\eta_r\big] a^*_v a_v (1 - \mathcal{N}_+/N) + \mathcal{E}_{\mathcal{R}}^{(2)}
\end{aligned}$$

where

$$\begin{aligned}
|\langle \xi, \mathcal{E}_{\mathcal{R}}^{(2)} \xi\rangle| \leq & CN^{1/2-\alpha}(\log N)^{1/2} \|\mathcal{H}_N^{1/2}\xi\|^2 + CN^{3/2-\alpha}\|\mathcal{H}_N^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\
& + CN^{-1}(\log N)^{1/2}\|\mathcal{H}_N^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\|\,.
\end{aligned}$$

for all $\xi \in \mathcal{F}_+^{\leq N}$. From Prop. 2.12, Prop. 2.13 and recalling the definition (4.59) of the operator $\mathcal{C}_N$, we deduce that

$$\begin{aligned}
&\int_0^1 ds\; e^{-sA}[\mathcal{K} + \mathcal{C}_N + \mathcal{V}_N, A] e^{sA} \\
&= \int_0^1 ds\; e^{-sA}\Big[-\mathcal{C}_N + \frac{1}{\sqrt{N}} \sum_{r,v\in\Lambda_+^*} \widehat{\omega}_N(r)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] \\
&\qquad + 2\sum_{r,v\in\Lambda_+^*} \big[\widehat{V}(r/e^N)\eta_r + \widehat{V}((r+v)/e^N)\eta_r\big] a^*_v a_v \Big(1 - \frac{\mathcal{N}_+}{N}\Big)\Big] e^{sA} + \mathcal{E}_{\mathcal{R}}^{(3)}
\end{aligned} \tag{4.79}$$

with

$$\pm\mathcal{E}_{\mathcal{R}}^{(3)} \leq C[N^{2-\alpha} + N^{-1/2}(\log N)^{1/2}](\mathcal{H}_N + 1)$$

for $N \in \mathbb{N}$ sufficiently large.

We now rewrite

$$\begin{aligned}
&2\sum_{r,v\in\Lambda_+^*} \big[\widehat{V}(r/e^N)\eta_r + \widehat{V}((r+v)/e^N)\eta_r\big] a^*_v a_v \Big(1 - \frac{\mathcal{N}_+}{N}\Big) \\
&\quad = 4\sum_{r,v\in\Lambda_+^*} \widehat{V}(r/e^N)\eta_r a^*_v a_v \Big(1 - \frac{\mathcal{N}_+}{N}\Big) \\
&\qquad + 2\sum_{r,v\in\Lambda_+^*} \big[\widehat{V}((r+v)/e^N) - \widehat{V}(r/e^N)\big]\eta_r a^*_v a_v \Big(1 - \frac{\mathcal{N}_+}{N}\Big) := \mathrm{Q}_1 + \mathrm{Q}_2\,.
\end{aligned} \tag{4.80}$$

With Lemma 2.10, part iii) we get

$$\Big| 2\sum_{r\in\Lambda^*} \widehat{V}(r/e^N)\eta_r - \big[2\widehat{\omega}_N(0) - 2N\widehat{V}(0)\big]\Big| \leq \frac{C}{N}\,, \tag{4.81}$$

and therefore, using Lemma 4.8 and (4.81)

$$\pm\Big[e^{-sA}\mathrm{Q}_1e^{sA} - 2\big[2\widehat{\omega}_N(0) - 2N\widehat{V}(0)\big]\sum_{v\in\Lambda_+^*}a_v^*a_v\Big(1-\frac{\mathcal{N}_+}{N}\Big)\Big] \\ \leq CN^{1-\alpha}(\mathcal{N}_++1) + \frac{C}{N}\mathcal{N}_+\,. \tag{4.82}$$

On the other hand it is easy to check that $e^{-sA}\mathrm{Q}_2e^{sA}$ is an error term; to this aim we notice that

$$\Big|\sum_{r\in\Lambda^*}\big[\widehat{V}(r/e^N)\eta_r - \widehat{V}((r+v)/e^N)\eta_r\big]\Big| \leq CN|v|e^{-N}\,.$$

Hence with Props. 2.12 and 2.13 we find

$$\pm\big[e^{-sA}\mathrm{Q}_2e^{sA}\big] \leq CNe^{-N}e^{-sA}\mathcal{N}_+^{1/2}\mathcal{K}^{1/2}e^{sA} \leq CN^2e^{-N}(\mathcal{H}_N+1)\,. \tag{4.83}$$

To handle the second term on the second line of (4.79), we apply Prop. 4.10 and then Prop. 2.12 and Prop. 2.13

$$\begin{aligned}\pm\bigg(\frac{1}{\sqrt{N}}\int_0^1 ds\sum_{r,v\in\Lambda_+^*}\widehat{\omega}_N(r)\Big[e^{-sA}b^*_{r+v}a^*_{-r}a_ve^{sA} - b^*_{r+v}a^*_{-r}a_v\Big] + \text{h.c.}\bigg)& \\ = \pm\bigg(\frac{1}{\sqrt{N}}\int_0^1 ds\int_0^s dt\sum_{r,v\in\Lambda_+^*}\widehat{\omega}_N(r)e^{-tA}\Big[b^*_{r+v}a^*_{-r}a_v, A\Big]e^{tA}\bigg)& \\ \leq C\int_0^1 ds\int_0^s dt\, e^{-tA}\big(N^{-\alpha}(\log N)\,\mathcal{K} + N^{-1}(\mathcal{N}_++1)\big)e^{tA}& \\ \leq CN^{1-\alpha}\log N(\mathcal{H}_N+1)\,.&\end{aligned} \tag{4.84}$$

As for the first term on the second line of (4.79), we use again Prop. 4.12. Using (4.80), (4.82) and (4.83) we have

$$\begin{aligned}\int_0^1 ds\; e^{-sA}\mathcal{C}_Ne^{sA} - \mathcal{C}_N &= \int_0^1 ds\int_0^s dt\; e^{-tA}[\mathcal{C}_N, A]e^{tA} \\ &= \big[2\widehat{\omega}_N(0) - 2N\widehat{V}(0)\big]\sum_{p\in\Lambda_+^*}a_p^*a_p\Big(1-\frac{\mathcal{N}_+}{N}\Big) + \mathcal{E}_{\mathcal{R}}^{(4)}\end{aligned} \tag{4.85}$$

with $\pm\mathcal{E}_{\mathcal{R}}^{(4)} \leq CN^{2-\alpha}(\mathcal{H}_N+1) + CN^{-1}(\mathcal{N}_++1)$.

Inserting the bounds (4.82), (4.83), (4.84) and (4.85) into (4.79) we arrive at

$$\begin{aligned}\mathcal{R}_{N,\alpha} = &\frac{1}{2}(N-1)\,\widehat{\omega}_N(0)(1-\mathcal{N}_+/N) + \frac{1}{2}\widehat{\omega}_N(0)\,\mathcal{N}_+\,(1-\mathcal{N}_+/N) \\ &+ \widehat{\omega}_N(0)\sum_{p\in\Lambda_+^*}a_p^*a_p\Big(1-\frac{\mathcal{N}_+}{N}\Big) + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)\big[b_p^*b_{-p}^* + b_pb_{-p}\big] \\ &+ \frac{1}{\sqrt{N}}\sum_{\substack{r,v\in\Lambda_+^*:\\ r\neq -v}}\widehat{\omega}_N(r)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] + \mathcal{H}_N + \mathcal{E}_{\mathcal{R}}\end{aligned}$$

with

$$\pm\mathcal{E}_{\mathcal{R}} \leq C[N^{2-\alpha} + N^{-1/2}(\log N)^{1/2}](\mathcal{H}_N + 1)$$

for $N \in \mathbb{N}$ sufficiently large.

Chapter 5

Conclusion and Perspectives

In the thesis we presented a proof of Bose-Einstein condensation for 2d bosons interacting through positive potentials both in the Gross-Pitaevskii regime and in intermediate scaling limits interpolating between the mean-field and the Gross-Pitaevskii scaling. Our results provide an optimal rate of convergence, thus extending previous results available in the literature [47, 51, 46, 59, 59, 41, 42].

The main idea behind these results (borrowed from [7, 9, 10]) is to use a Fock space setting to describe excitations out of the condensate and unitary operators to implement correlations among excitations. This leads to a renormalization of the original Hamiltonian, where the singular interaction is replaced by a softer potential.

In the case of the singular interaction, the Hamilton operator is reduced to a mean-field Hamiltonian, for which standard arguments are available. Differently, in the Gross-Pitaevskii regime the slowly decaying GP potential $\widehat{V}(p/e^N)$ is replaced by the potential $\widehat{\omega}_N(p)$ (see definition (2.88)) which decays faster and we are able to control (see Section 2.3).

The strategy developed to show Theorem 1.1 and 1.3 can be used as a starting point to investigate the validity of Bogoliubov theory for two dimensional bosons in the corresponding scaling limits, following the strategy developed in [8] for the three dimensional case in the GP regime.

In the following we focus only on the Gross-Pitaevskii setting, even though a similar analysis also holds for less singular regimes. What follows is a ongoing project with Serena Cenatiempo and Benjamin Schlein.

Bogoliubov Theory

Bogoliubov theory [11] was the first rigorous treatment concerning Bose-Einstein condensation. In physics, it is used to approximate with high accuracy the ground state energy and the excitation energies of a dilute system of weakly interacting bosons [63].

Bogoliubov worked in a periodic box Λ in the thermodynamic limit, in which, we recall, the side length of the box L and the number of particles N go to infinity while the density is kept fixed $\rho = N/L^3$. Under the assumption that the system exhibits complete Bose-Einstein condensation in the zero momentum mode, and neglecting processes involving more than two excited particles, Bogoliubov derived an expression for the ground state and excitation energies of the system, which is believed to be correct in the dilute limit (for a review on Bogoliubov method we address the reader to [49, Appendix A] or [16]).

More precisely, he predicted that in the thermodynamic limit the ground state energy of weakly interacting Bose gas is given by

$$E_0 = 4\pi\rho\mathfrak{a}N\left(1+\frac{128}{15\sqrt{\pi}}\sqrt{\rho\mathfrak{a}^3}+o\big(\sqrt{\rho\mathfrak{a}^3}\big)\right), \tag{5.1}$$

where $\mathfrak{a}$ is the scattering length associated to the interaction potential. The equation above is the so-called *Lee-Huang-Yang formula* [40]. The first result for the upper bound was achieved by Dyson [22] (for hard-sphere interactions). Later in [54] Lieb-Yngvason proved the lower bound at a first order. For the second order correction Lieb-Solovej verified the validity of Eq.(5.1) in [52] and for the two-component charged Bose gas in [53] as well as [72].

Erdös-Schlein-Yau in [27] proved an upper bound for the Lee-Huang-Yang formula, up to errors that are subleading for small potentials. Their result was later improved in [73] by Yau-Yin. Another important result was achieved by Giuliani-Seiringer in [32], where they proved a lower bound matching the Lee-Huang-Yang formula for systems of interacting Bose gases in a regime of weak coupling and high density. This result was improved by Brietzke-Solovej [19]. Very recently, Fournais-Solovej in [30] eventually proved the Lee-Huang-Yang formula from below for L^1 potentials. The best available lower bound for general potentials, including hard-core, is due to Brietzke-Fournais-Solovej in [18], where the leading order has been shown with an error of the order of the LHY correction.

Bogoliubov's approximation was originally proposed in three dimensions, however it also leads to a prediction for the two-dimensional case. The analogous of the Lee-Huang-Yang formula (5.1) is given in Eq. (1.18), for which, we recall, the second order correction has not yet been proved. However, results based on the restriction to quasi-free states have been obtained in Fournais-Napiorkowski-Reuvers-Solovej [29, Theorem 1].

In an ongoing project with Serena Cenatiempo and Benjamin Schlein we aim to prove the second order correction of the ground state energy and excitation energies of our Hamiltonian H_N^{GP} (1.10). With the same strategy used to show the bound in Eq. (2.124) one could also get an estimate for the energy operator $\mathcal{H}_N$. We are able to prove the following statement.

Proposition 5.1. *Let $V \in L^3(\mathbb{R}^2)$ be non-negative, compactly supported and spherically symmetric. Let $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ belong to the spectral subspace of H_N with energies below $E_N + K$, i.e.*

$$\psi_N = \mathbf{1}_{(-\infty;2\pi N+K]}(H_N)\psi_N\,.$$

Let $\xi_N = e^{-A}e^{-B}U_N\psi_N$ be the renormalized excitation vector associated with ψ_N. Then, for any $j \in \mathbb{N}$ there exists a constant $C > 0$ such that

$$\langle \xi_N, (\mathcal{N}_+ + 1)^j(\mathcal{H}_N + 1)\,\xi_N\rangle \le C(1+K)(\log N)^{j+1}\big[(1+K)^2 + (\log N)^2\big]^{j/2}\,.$$

The proof is obtained by induction and it is similar as in the three-dimensional setting [8, Proposition 4.1] (although the authors of [8] worked over excitated states of the form $\xi_N = e^{-B}U_N\psi_N$, we need to deal with excitated states $\xi_N = e^{-A}e^{-B}U_N\psi_N$, and so we use properties of the renormalized Hamiltonian $e^{-A}\mathcal{G}_{N,\alpha}e^{A}$).

The proposition above allows us to improve the estimates involving the excitation Hamiltonian $\mathcal{R}_{N,\alpha}$ introduced in Chapter 4 and obtain the following result.

Proposition 5.2. *Let $V \in L^3(\mathbb{R}^2)$ be compactly supported, pointwise non-negative and spherically symmetric. Let $\mathcal{R}_{N,\alpha} = e^{-A}\mathcal{G}_{N,\alpha}e^{A}$, with $\mathcal{G}_{N,\alpha}$ defined as in Eq. (2.86). Then for any $\alpha > 2$*

$$\mathcal{R}_{N,\alpha} = C_{\mathcal{R}} + Q_{\mathcal{R}} + \mathcal{V}_N + \mathcal{E}'_{\mathcal{R}}.$$

with

$$\begin{aligned} C_{\mathcal{R}} &= \frac{N}{2}\big(\widehat{V}(\cdot/e^N) * \widehat{f}_{N,\ell}\big)(0)(N-1) + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)\eta_p \\ Q_{\mathcal{R}} &= \sum_{p\in\Lambda_+^*}(p^2 + \widehat{\omega}_N(p))b_p^*b_p + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{\omega}_N(p)\big[b_p^*b_{-p}^* + b_pb_{-p}\big]. \end{aligned} \tag{5.2}$$

where and $\mathcal{E}'_{\mathcal{R}}$ such that

$$\pm\,\mathcal{E}'_{\mathcal{R}} \le CN^{2-\alpha}(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + CN^{-1/2}(\log N)^{1/2}(\mathcal{N}_+ + 1)(\mathcal{K} + 1)\,. \tag{5.3}$$

Proof. We start from the expression of $\mathcal{R}^{\text{eff}}_{N,\alpha}$ as in Eq. (2.96). The constant $C_{\mathcal{R}}$ is obtained in the same way as in the proof of Prop. 2.14 (as well as Prop. 2.11), keeping track of the order $\mathcal{O}(1)$ terms. To estimate the cubic term $\mathcal{C}_N$ in (2.96) we use the bound in Eq. (2.102), namely

$$\Big|\frac{1}{\sqrt{N}}\sum_{\substack{r,v\in\Lambda_+^*\\ r\neq -v}}\widehat{\omega}_N(r)\langle\xi, b_{r+v}^*a_{-r}^*a_v\xi\rangle\Big| \le \frac{C(\log N)^{1/2}}{\sqrt{N}}\,\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}+1)\xi\|\,. \tag{5.4}$$

With Prop. 5.1, we get (5.3). □

If now we neglect the potential energy operator $\mathcal{V}_N$, as in Eq. (2.87), which we recall to be defined as

$$\mathcal{V}_N = \frac{1}{2}\sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*:\\ r\neq -p,-q}}\widehat{V}(r/e^N)a_{p+r}^*a_q^*a_{q+r}a_p$$

we obtain a quadratic Hamiltonian which can be explicitly diagonalized (similarly as in [8, Section 5]), acting with a unitary operator quadratic in terms of annihilation and creation operators. We find the following lower bound.

Proposition 5.3. *Let $V \in L^3(\mathbb{R}^2)$ be non-negative, compactly supported and spherically symmetric. Let E_{Bog} be defined as*

$$E_{Bog} := \frac{1}{2}\sum_{p\in\Lambda_+^*}\Big(\sqrt{p^4 + 8\pi p^2} - p^2 - 4\pi + \frac{(4\pi)^2}{2p^2}\Big).$$

Then

$$e^{-B_\tau}\mathcal{R}_{N,\alpha}e^{B_\tau} \geq C_N + Q_N + \mathcal{E}''_{\mathcal{R}},$$

where C_N *and* Q_N *are respectively of the form*

$$C_N = 2\pi(N-1) + \pi\log(2\mathfrak{a}^2) + E_{Bog} - 4\pi^2 \sum_{p\in\Lambda_+^*} \frac{J_0(|p|/\sqrt{2})}{|p|^2},$$

with $\mathfrak{a}$ *the scattering length defined in Eq.* (1.11)*, and* J_0 *is the Bessel function of the first kind, and*

$$Q_N = \sum_{p\in\Lambda_+^*} \sqrt{p^4 + 8\pi p^2}\; a_p^* a_p,$$

with $\mathcal{E}''_{\mathcal{R}}$ *satisfying*

$$\begin{aligned}\pm\mathcal{E}''_{\mathcal{R}} \;&\leq CN^{2-\alpha}(\log N)^2(\mathcal{H}_N+1)\\ &\quad + CN^{-1/2}(\log N)^{3/2}(\mathcal{N}_+ +1)(\mathcal{K}+1) + C(\log N)^2 N^{-1}\,,\end{aligned}$$

for any $\alpha > 2$ *and* N *large enough.*

Getting an upper bound is, on the contrary, non-trivial. In fact, differently from the 3d case, the potential energy operator $\mathcal{V}_N$ is of order $\mathcal{O}(1)$, hence it cannot be neglected. In particular, its expectation on the trial state $e^{B_\tau}\Omega$ with B_τ the generalized Bogoliubov transformation which allows us to diagonalize the quadratic part $\mathcal{Q}_R$ as in Eq. (5.2), is of order $\mathcal{O}(\log N)$. This suggest that to obtain an upper bound up to $\mathcal{O}(1)$ the excitation Hamiltonian $\mathcal{R}_{N,\alpha}$ has to be further renormalized to cancel out the large energy contributions hidden in $\mathcal{V}_N$. We are currently working on the project of finding a unitary operator which allows us to reduce $\mathcal{R}_{N,\alpha}$ to a quadratic excitation Hamiltonian (up to lower order terms). This will also allow to get information on the low-energy spectrum of the system and also to provide a norm-approximation to the many-body low energy wave function in the spirit of [8, Equation 6.7].

Obtaining a norm-approximation for the ground-state wave function would open the way to investigate fluctuations of observables measured on the ground state, with respect to their expected value provided by the knowledge of the condensate wave function. In fact, it is a natural question whether one may adapt the strategy followed in [64, 65, 38] to investigate the validity of a central limit theorem for one particle observables measured on the condensate.

Last but not least, one can ask if the methods used in this thesis may be adapted to investigate the properties of 2d bosons in the thermodynamic limit, in the same spirit of recent results [2].

Appendix A

Properties of the Scattering Function in the GP scaling

In this appendix we are going to show some uselful properties of the scattering function stated in Chap. 2, Sec. 2.3.1. For practical reason we prove before properties for the Gross-Pitaevskii scaling, while in Appendix B we prove similar properties for the scattering equation with singular interacting potential, corresponding to Lemma 2.10

Let V be a potential with finite range $R_0 > 0$ and scattering length $\mathfrak{a}$. For a fixed $R > R_0$, we study properties of the ground state f_R of the Neumann problem

$$\Big(-\Delta + \frac{1}{2} V(x) \Big) f_R(x) = \lambda_R \, f_R(x) \tag{A.1}$$

on the ball $|x| \leq R$, normalized so that $f_R(x) = 1$ for $|x| = R$. Lemma 2.10, parts i)-iv), follows by setting $R = e^N \ell$ in the following lemma.

Lemma A.1. *Let $V \in L^3(\mathbb{R}^2)$ be non-negative, compactly supported and spherically symmetric, and denote its scattering length by $\mathfrak{a}$. Fix $R > 0$ sufficiently large and denote by f_R the Neumann ground state of* (A.1)*. Set $w_R = 1 - f_R$. Then we have*

$$0 \leq f_R(x) \leq 1$$

Moreover, for R large enough there is a constant $C > 0$ independent of R such that

$$\left| \lambda_R - \frac{2}{R^2 \log(R/\mathfrak{a})} \left(1 + \frac{3}{4} \frac{1}{\log(R/\mathfrak{a})} \right) \right| \leq \frac{C}{R^2} \frac{1}{\log^3(R/\mathfrak{a})} \,. \tag{A.2}$$

and

$$\left| \int dx \, V(x) f_R(x) - \frac{4\pi}{\log(R/\mathfrak{a})} \right| \leq \frac{C}{\log^2(R/\mathfrak{a})} \,. \tag{A.3}$$

Finally, there exists a constant $C > 0$ such that

$$\begin{aligned} |w_R(x)| &\leq \chi(|x| \leq R_0) + C \, \frac{\log(|x|/R)}{\log(\mathfrak{a}/R)} \, \chi(R_0 \leq |x| \leq R) \\ |\nabla w_R(x)| &\leq \frac{C}{\log(R/\mathfrak{a})} \frac{\chi(|x| \leq R)}{|x| + 1} \end{aligned} \tag{A.4}$$

for R large enough.

To show Lemma A.1 we adapt to the two dimensional case the strategy used in [26, Lemma A.1] for the three dimensional problem. We will use some well known properties of the zero energy scattering equation in two dimensions, summarized in the following lemma.

Lemma A.2. *Let $V \in L^3(\mathbb{R}^2)$ non-negative, with supp $V \subset B_{R_0}(0)$ for an $R_0 > 0$. Let $\mathfrak{a} \leq R_0$ denote the scattering length of V. For $R > R_0$, let $\phi_R : \mathbb{R}^2 \to \mathbb{R}$ be the radial solution of the zero energy scattering equation*

$$\left[-\Delta + \frac{1}{2}V\right]\phi_R = 0 \tag{A.5}$$

normalized such that $\phi_R(x) = 1$ for $|x| = R$. Then

$$\phi_R(x) = \frac{\log(|x|/\mathfrak{a})}{\log(R/\mathfrak{a})} \tag{A.6}$$

for all $|x| > R_0$. Moreover, $|x| \to \phi_R(x)$ is monotonically increasing and there exists a constant $C > 0$ (depending only on V) such that

$$\phi_R(x) \geq \phi_R(0) \geq \frac{C}{\log(R/\mathfrak{a})} \tag{A.7}$$

for all $x \in \mathbb{R}^2$. Furthermore, there exists a constant $C > 0$ such that

$$|\nabla\phi_R(x)| \leq \frac{C}{|\log(R/\mathfrak{a})|}\frac{1}{|x|+1} \tag{A.8}$$

for all $x \in \mathbb{R}^2$.

Proof. The existence of the solution of (A.5), the expression (A.6), the fact that $\phi_R(x) \geq 0$ and the monotonicity are standard (see, for example, Theorem C.1 and Lemma C.2 in [49]). The bound (A.7) for $\phi_R(0)$ follows from (A.6), comparing $\phi_R(0)$ with $\phi_R(x)$ at $|x| = R_0$, with Harnack's inequality (see [71, Theorem C.1.3]). Finally, (A.8) follows by rewriting (A.5) in integral form

$$\phi_R(x) = 1 - \frac{1}{4\pi}\int_{\mathbb{R}^2} \log\Big(R/|x-y|\Big)V(y)\phi_R(y)\mathrm{d}y\,.$$

For $|x| \leq R_0$, this leads (using that $\phi_R(y) \leq \log(R_0/\mathfrak{a})/\log(R/\mathfrak{a})$ for all $|y| \leq R_0$ and the local integrability of $|.|^{-3/2}$) to

$$|\nabla\phi_R(x)| \leq C\int \frac{V(y)\phi_R(y)}{|x-y|}dy \leq \frac{C\|V\|_3}{\log(R/\mathfrak{a})}$$

Combining with the bound for $|x| > R_0$ obtained differentiating (A.6), we obtain the desired estimate. □

Proof of Lemma A.1. By standard arguments (see for example [49, proof of theorem C1]), $f_R(x)$ is spherically symmetric and non-negative. We now start by proving an upper bound for λ_R, consistent with (A.2). To this end, we calculate the energy of a suitable trial function. For $k \in \mathbb{R}$ we define

$$\psi_k(x) = J_0(k|x|) - \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} Y_0(k|x|)\,.$$

with J_0 and Y_0 the zero Bessel functions of first and second type, respectively. Note that

$$-\Delta\psi_k(x) = k^2\psi_k(x)\,.$$

and $\psi_k(x) = 0$ if $|x| = a$. We define $k = k(R)$ to be the smallest positive real number satisfying $\partial_r \psi_R(x) = 0$ for $|x| = R$. One can check that

$$\left| k^2 - \frac{2}{R^2 \log(R/\mathfrak{a})}\left(1 + \frac{3}{4}\frac{1}{\log(R/\mathfrak{a})}\right)\right| \leq \frac{C}{R^2}\frac{1}{\log^3(R/\mathfrak{a})} \tag{A.9}$$

in the limit $R \to \infty$. To prove (A.9), we observe that

$$\partial_r \psi_k(x)\Big|_{|x|=R} = -kJ_1(kR) + k\frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} Y_1(kR)\,, \tag{A.10}$$

and we expand for $kR, k\mathfrak{a} \ll 1$ using (with γ the Euler constant)

$$\begin{aligned}
&\left| J_0(r) - 1 + \frac{r^2}{4}\right| \leq Cr^4\,, \qquad \left| J_1(r) - \frac{r}{2}\Big(1 - \frac{r^2}{8}\Big)\right| \leq Cr^5\,,\\
&\left| Y_0(r) - \frac{2}{\pi}\log(re^\gamma/2)\right| \leq Cr^2\log(r)\,,\\
&\left| Y_1(r) + \frac{2}{\pi}\frac{1}{r}\left(1 - \frac{r^2}{2}\Big(1 - \frac{r^2}{8}\Big)\log(re^\gamma/2) + \frac{r^2}{4}\right)\right| \leq Cr^3.
\end{aligned} \tag{A.11}$$

With (A.11) one finds that (A.10)

$$\begin{aligned}
&\partial_r \psi_R(x)\Big|_{|x|=R}\\
&\quad = -\frac{1}{2kR\log(k\mathfrak{a}e^\gamma/2)}\\
&\qquad \cdot \left\{ \frac{(kR)^4}{8}\log(R/\mathfrak{a}) - (kR)^2\left[\log(R/\mathfrak{a}) - \frac{1}{2}\right] + 2 + \mathcal{O}((kR)^4 + (k\mathfrak{a})^2)\right\}
\end{aligned} \tag{A.12}$$

The smallest solution of

$$\frac{(kR)^4}{8}\log(R/\mathfrak{a}) - (kR)^2\left[\log(R/\mathfrak{a}) - \frac{1}{2}\right] + 2 = 0$$

is such that

$$(kR)^2 = \frac{2}{\log(R/\mathfrak{a})}\left[1 + \frac{3}{4\log(R/\mathfrak{a})}\right] + \mathcal{O}(\log^{-3}(R/\mathfrak{a})) \tag{A.13}$$

in the limit of large R. Inserting in (A.12), we find that the r.h.s. changes sign around the value of k defined in (A.13). By the intermediate value theorem, we conclude that there is a $k = k(R) > 0$ satisfying (A.9), such that $\partial_r \psi_{k(R)}(x) = 0$ if $|x| = R$.

Now, let $\phi_R(x)$ be the solution of the zero energy scattering equation (A.5), with $\phi_R(x) = 1$ for $|x| = R$. We set

$$\Psi_R(x) := \psi_k(m_R(x)) = J_0(km_R(x)) - \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} Y_0(km_R(x)) \,, \tag{A.14}$$

with $k = k(R)$ satisfying (A.9) and

$$m_R(x) := \mathfrak{a} \exp\big(\log(R/\mathfrak{a})\phi_R(x)\big) \,.$$

With this choice we have $m_R(x) = |x|$ outside the range of the potential; hence $\Psi_R(x) = \psi_k(x)$ for $R_0 \le |x| \le R$. In particular, Ψ_R satisfies Neumann boundary conditions at $|x| = R$.

From (A.6), (A.7) and the monotonicity of ϕ_R, we get

$$C\mathfrak{a} \le m_R(x) \le R_0 \qquad \text{for all} \quad 0 \le |x| \le R_0 \tag{A.15}$$

and for a constant $C > 1$, independent of R. From (A.8) we also get

$$|\nabla m_R(x)| \le C \qquad \text{for all} \quad 0 \le |x| \le R \,. \tag{A.16}$$

With the notation $\mathfrak{h} = -\Delta + \frac{1}{2}V$, we now evaluate $\langle \Psi_R, \mathfrak{h}\Psi_R \rangle$. To this end we note that

$$\langle \Psi_R, \mathfrak{h}\Psi_R \rangle = \int_{|x|<R_0} \overline{\Psi_R(x)}(\mathfrak{h}\Psi_R(x))dx + k^2 \int_{|x|\ge R_0} |\Psi_R(x)|^2 \, dx \,. \tag{A.17}$$

Let us consider the region $|x| < R_0$. From (A.14) and (A.11) we find, first of all,

$$\left| \Psi_R(x) + \frac{\log(m_R(x)/\mathfrak{a})}{\log(k\mathfrak{a}e^\gamma/2)} \right| \le C(km_R(x))^2 \,, \tag{A.18}$$

Next, we compute $-\Delta\Psi_R(x)$. With

$$\begin{aligned} J_0'(r) &= -J_1(r) & J_1'(r) &= \frac{1}{2}\big(J_0(r) - J_2(r)\big) \\ Y_0'(r) &= -Y_1(r) & Y_1'(r) &= \frac{1}{2}\big(Y_0(r) - Y_2(r)\big) \,. \end{aligned}$$

we obtain (here, we use the notation m_R' and m_R'' for the radial derivatives of the radial function m_R)

$$\begin{aligned} -\Delta\Psi_R(x) = &\; -\partial_r^2\Psi_R(x) - \frac{1}{|x|}\partial_r\Psi_R(x) \\ = &\; -km_R''(x)\Big[-J_1(km_R(x)) + \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})}Y_1(km_R(x))\Big] \\ &- \frac{1}{2}k^2\big(m_R'(x)\big)^2\Big[J_2(km_R(x)) - \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})}Y_2(km_R(x))\Big] \\ &- \frac{1}{2}k^2\big(m_R'(x)\big)^2\Big[-J_0(km_R(x)) + \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})}Y_0(km_R(x))\Big] \\ &- \frac{km_R'(x)}{|x|}\Big[-J_1(km_R(x)) + \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})}Y_1(km_R(x))\Big] \,. \end{aligned}$$

We note that, using the scattering equation (A.5),

$$m_R'' - \frac{(m_R')^2}{m_R} + \frac{1}{|x|} m_R' = \frac{1}{2} V m_R \, \phi_R \log(R/\mathfrak{a}) = \frac{1}{2} V m_R \log(m_R/\mathfrak{a})\,. \qquad \text{(A.19)}$$

Now we write

$$\begin{aligned} &- \Delta\Psi_R(x) \\ &= \Big[- k\Big(m_R''(x) + \frac{m_R'(x)}{|x|}\Big) Y_1(k m_R(x)) + \frac{k^2}{2} (m_R'(x))^2 Y_2(k m_R(x))\Big] \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} \\ &\quad + g_R(x) \end{aligned} \qquad \text{(A.20)}$$

where $g_R(x) = \sum_{i=1}^{3} g_R^{(i)}(x)$ with

$$\begin{aligned} g_R^{(1)}(x) &= k\left(m_R''(x) + \frac{m_R'(x)}{|x|}\right) J_1(k m_R(x)) \\ g_R^{(2)}(x) &= -\frac{1}{2} k^2 (m_R'(x))^2 J_2(k m_R(x)) \\ g_R^{(3)}(x) &= -\frac{1}{2} k^2 (m_R'(x))^2 \Big(- J_0(k m_R(x) + \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} Y_0(k m_R(x))\Big) \\ &= \frac{k^2}{2} (m_R'(x))^2 \Psi_R(x)\,. \end{aligned}$$

With (A.19), (A.11) and (A.15), (A.16), we find

$$|g_R^{(1)}(x)| \le C k^2 \Big((m_R'(x))^2 + \frac{1}{2} V(x) m_R^2(x) \log(m_R(x)/\mathfrak{a}) \Big) \le C k^2 (1 + V(x))\,.$$

Next, with $|J_2(r) - r^2/8| \le C r^4$ we get

$$|g_R^{(2)}(x)| \le C k^4 (m_R'(x))^2 (m_R(x))^2 \le C k^4\,.$$

With (A.18), we can also bound

$$|g_R^{(3)}(x)| \le C k^2 (m_R'(x))^2 \frac{\log(m_R(x)/\mathfrak{a})}{\log(k\mathfrak{a})} \le C k^2 \log^{-1}(k\mathfrak{a})\,.$$

We conclude that $|g_R(r)| \le C(1 + V(x))k^2$ for all $r \le R_0$ and R large enough. Finally, using Eq. (A.19), the expansion for $Y_1(r)$ in Eq. (A.11), and the bound

$$\left| Y_2(r) + \frac{4}{\pi} \frac{1}{r^2} \right| \le C\,,$$

we can rewrite the first term on the r.h.s. of (A.20) as

$$\begin{aligned} &\Big[- k\Big(m_R''(x) + \frac{m_R'(x)}{|x|}\Big) Y_1(k m_R(x)) + \frac{k^2}{2} (m_R'(x))^2 Y_2(k m_R(x))\Big] \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} \\ &\quad = \frac{1}{\pi} V(x) \log(m_R(x)/\mathfrak{a}) \frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})} + h_R(x) \end{aligned} \qquad \text{(A.21)}$$

with $|h_R(x)| \leq C(1+V(x))k^2$ for all $r \leq R_0$, R large enough. With the identities (A.20) and (A.21) we obtain

$$\left| -\Delta\Psi_R(x) - \frac{1}{\pi}\frac{J_0(k\mathfrak{a})}{Y_0(k\mathfrak{a})}V(x)\log(m_R(x)/\mathfrak{a})\right| \leq C(1+V(x))k^2\,,$$

for all $|x| \leq R_0$ and for R sufficiently large. With (A.18), we conclude that, for $0 \leq |x| \leq R_0$,

$$\left|(-\Delta + \frac{1}{2}V)\Psi_R(x)\right| \leq C(1+V(x))k^2\,. \tag{A.22}$$

With (A.17), (A.22) and the upper bound

$$|\Psi_R(r)| \leq \frac{C}{|\log(k\mathfrak{a})|} \tag{A.23}$$

for all $|x| \leq R_0$ (which follows from (A.18) and (A.15)), we get

$$\langle \Psi_R, \mathfrak{h}\Psi_R\rangle \leq k^2\langle \Psi_R, \Psi_R\rangle + \frac{Ck^2}{|\log(k\mathfrak{a})|}\int_{|x|\leq R_0}(1+V(x))\,\mathrm{d}x\,.$$

On the other hand, Eq.(A.18), together with $m_R(x) = |x|$ for $|x| \geq R_0$, implies the lower bound

$$\langle \Psi_R, \Psi_R\rangle \geq \int_{R_0\leq|x|\leq R}|\Psi_R(x)|^2\mathrm{d}x \geq \frac{C}{|\log(k\mathfrak{a})|^2}\int_{R_0\leq|x|\leq R}\log^2(|x|/\mathfrak{a})\mathrm{d}x \geq CR^2\,.$$

Hence, with (A.9), we conclude that

$$\begin{aligned}\lambda_R \leq \frac{\langle \Psi_R, \mathfrak{h}\Psi_R\rangle}{\langle \Psi_R, \Psi_R\rangle} &\leq k^2\left(1+\frac{C\,|\log(k\mathfrak{a})|}{R^2}\right)\\ &\leq \frac{2}{R^2\log(R/\mathfrak{a})}\left(1+\frac{3}{4}\frac{1}{\log(R/\mathfrak{a})}+\frac{C}{\log^2(R/\mathfrak{a})}\right)\end{aligned} \tag{A.24}$$

in agreement with (A.2).

To prove the lower bound for λ_R it is convenient to show some upper and lower bounds for f_R. We start by considering f_R outside the range of the potential. We denote $\varepsilon_R = \sqrt{\lambda_R}\,R$. Keeping into account the boundary conditions at $|x| = R$, we find, for $R_0 \leq |x| \leq R$,

$$f_R(x) = A_R\,J_0(\varepsilon_R\,|x|/R) + B_R\,Y_0(\varepsilon_R\,|x|/R)\,,$$

with

$$A_R = \left(J_0(\varepsilon_R) - J_1(\varepsilon_R)\,\frac{Y_0(\varepsilon_R)}{Y_1(\varepsilon_R)}\right)^{-1}\,,$$

and

$$B_R = \left(Y_0(\varepsilon_R) - \frac{J_0(\varepsilon_R)}{J_1(\varepsilon_R)}Y_1(\varepsilon_R)\right)^{-1}\,.$$

From (A.24), we have $|\varepsilon_R| \leq C\,|\log(R/\mathfrak{a})|^{-1/2}$. Thus, we can expand f_R for large R, using (A.11) and, for Y_0, the improved bound

$$\left|Y_0(r) - \frac{2}{\pi}\log(re^{\gamma}/2)\left(1 - \frac{1}{4}r^2\right)\right| \leq C\,r^2\,,$$

we find

$$\begin{aligned} \left|A_R - 1 + \frac{\varepsilon_R^2}{4}\Big(2\log(\varepsilon_R e^{\gamma}/2) - 1\Big)\right| &\leq C\varepsilon_R^4(\log\varepsilon_R)^2\,, \\ \left|B_R - \frac{\pi}{4}\varepsilon_R^2\left(1 - \frac{\varepsilon_R^2}{8}\right)\right| &\leq C\varepsilon_R^6\,. \end{aligned} \tag{A.25}$$

which leads to

$$\begin{aligned} \left|f_R(x) - 1 + \frac{\varepsilon_R^2}{4}\left(2\log(R/|x|) - 1 + \frac{x^2}{R^2}\right) - \frac{\varepsilon_R^4}{16}\log(R/|x|)\left(1 + \frac{2x^2}{R^2}\right)\right| \\ \leq C\varepsilon_R^4(\log\varepsilon_R)^2\,. \end{aligned} \tag{A.26}$$

We can also compute the radial derivative

$$\partial_r f_R(x) = -\frac{\varepsilon_R}{R}\Big(A_R\,J_1(\varepsilon_R\,r/R) + B_R\,Y_1(\varepsilon_R\,r/R)\Big)\,.$$

With the expansions (A.11) and (A.25) we conclude that for all $R_0 \leq |x| < R$ we have

$$\left|\partial_r f_R(x) - \frac{\varepsilon_R^2}{2|x|}\left(1 - \frac{x^2}{R^2} + \frac{\varepsilon_R^2 x^2}{2R^2}\log(R/|x|)\right)\right| \leq C\varepsilon_R^4\log\varepsilon_R\,. \tag{A.27}$$

The bound (A.27) shows that $\partial_r f_R(x)$ is positive, for, say, $R_0 < |x| < R/2$. Since $\partial_r f_R(x)$ must have its first zero at $|x| = R$, we conclude that f_R is increasing in $|x|$, on $R_0 \leq |x| \leq R$. From the normalization $f_R(x) = 1$, for $|x| = R$, we conclude therefore that $f_R(x) \leq 1$, for all $R_0 \leq |x| \leq R$.

From (A.26) and (A.24) we obtain, on the other hand, the lower bound

$$\begin{aligned} f_R(x) &\geq 1 - \frac{\varepsilon_R^2}{2}\log(R/|x|) - C\varepsilon_R^4(\log\varepsilon_R)^2 \\ &\geq 1 - \frac{\log(R/|x|)}{\log(R/\mathfrak{a})}\left(1 + \frac{3}{4}\frac{1}{\log(R/\mathfrak{a})} + \frac{C}{\log^2(R/\mathfrak{a})}\right) - C\,\frac{(\log\log(R/\mathfrak{a}))^2}{\log^2(R/\mathfrak{a})} \\ &\geq \frac{\log(|x|/\mathfrak{a})}{\log(R/\mathfrak{a})} - \frac{3}{4}\frac{\log(R/|x|)}{\log^2(R/\mathfrak{a})} - C\frac{\log(R/|x|)}{\log^3(R/a)} - C\,\frac{(\log\log(R/\mathfrak{a}))^2}{\log^2(R/\mathfrak{a})}\,, \end{aligned} \tag{A.28}$$

for R sufficiently large. Let $R_* = \max\{R_0, e\mathfrak{a}\}$. Then Eq. (A.28) implies in particular that, for R large enough,

$$f_R(x) \geq \frac{C}{\log(R/\mathfrak{a})}\,. \tag{A.29}$$

for all $R_* < |x| \leq R$.

Finally, we show that $f_R(x) \leq 1$ also for $|x| \leq R_0$. First of all, we observe that, by elliptic regularity, as stated for example in [45, Theorem 11.7, part iv)], there exists $0 < \alpha < 1$ and $C > 0$ such that

$$|f_R(x) - f_R(y)| \leq C\|(V - 2\lambda_R)f_R\|_2 \, |x-y|^\alpha$$

With $\|Vf_R\|_2 \leq \|V\|_3\|f_R\|_6 \leq C\|f_R\|_{H^1} \leq C(1+\lambda_R)\|f_R\|_2$, we conclude that $0 \leq f_R(x) \leq 1 + C\|f\|_2$ for all $|x| \leq R_0$ (because we know that $f_R(x) \leq 1$ for $R_0 \leq |x| \leq R$). To improve this bound, we go back to the differential equation (A.1), to estimate

$$\Delta f_R = \frac{1}{2}Vf_R - \lambda_R f_R \geq -\lambda_R(1 + C\|f\|_2) \tag{A.30}$$

This implies that $f_R(x) + \lambda_R(1 + C\|f\|_2)x^2/2$ is subharmonic. Using (A.26), we find $f_R(x) \leq 1 - C\varepsilon_R^2$ for $|x| = R_0$. From the maximum principle, we obtain therefore that

$$f_R(x) \leq 1 - C\varepsilon_R^2 + C\lambda_R(1 + C\|f_R\|_2) \tag{A.31}$$

for all $|x| \leq R_0$. In particular, this implies that $\|f_R\mathbf{1}_{|x|\leq R_0}\|_2 \leq C + C\lambda_R\|f_R\|_2$, and therefore that

$$\|f_R\mathbf{1}_{R_0\leq|x|\leq R}\|_2 \geq \|f_R\|_2(1 - C\lambda_R) - C$$

With $f_R(x) \leq 1$ for $R_0 \leq |x| \leq R$, we find, on the other hand, that $\|f_R\mathbf{1}_{R_0\leq|x|\leq R}\|_2 \leq CR$. We conclude therefore that $\|f_R\|_2 \leq CR$ and, from (A.31), that $f_R(x) \leq 1 - C\varepsilon_R^2 + C/R \leq 1$, for all $|x| \leq R_0$, if R is large enough.

We are now ready to prove the lower bound for λ_R. We use now that any function Φ satisfying Neumann boundary conditions at $|x| = R$ can be written as $\Phi(x) = q(x)\Psi_R(x)$, with $\Psi_R(x)$ the trial function used for the upper bound and $q > 0$ a function that satisfies Neumann boundary condition at $|x| = R$ as well. This is in particular true for the solution $f_R(x)$ of (A.1). In the following we write

$$f_R(x) = q_R(x)\Psi_R(x)$$

where q_R satisfies Neumann boundary conditions at $|x| = R$. From (A.18), we find $|\Psi_R(x)| \geq C/\log(ka)$. The bound $f_R(x) \leq 1$ implies therefore that there exists $c > 0$ such that

$$q_R(x) \leq C\log(k\mathfrak{a}) \qquad \forall\, |x| \leq R_0\,. \tag{A.32}$$

From the identity

$$\mathfrak{h}f_R = (\mathfrak{h}\Psi_R)q_R - (\Delta q_R)\Psi_R - 2\nabla q_R\nabla\Psi_R$$

we have

$$\int_{|x|\leq R} \mathrm{d}x\, f_R\mathfrak{h}f_R = \int_{|x|\leq R} \mathrm{d}x\, |\nabla q_R|^2\Psi_R^2 + \int_{|x|\leq R} \mathrm{d}x\, |q_R|^2\Psi_R\mathfrak{h}\Psi_R\,.$$

From (A.22) and (A.23), we have

$$\left|\Psi_R(x)(\mathfrak{h}\Psi_R)(x) - k^2\Psi_R^2(x)\right| \le C\frac{k^2}{|\log ka|}(1+V(x))\chi(|x|\le R_0)\,.$$

Hence

$$\int_{|x|\le R} \mathrm{d}x\, f_R \mathfrak{h} f_R \ge \; k^2\|f_R\|^2 - \frac{Ck^2}{|\log k|}\int_{|x|\le R_0} \mathrm{d}x\,(1+V(x))|q_R(x)|^2\,. \qquad \text{(A.33)}$$

With (A.32), we obtain

$$\int_{|x|\le R} \mathrm{d}x\, f_R \mathfrak{h} f_R \ge k^2\|f_R\|^2 - Ck^2\log(k\mathfrak{a})\,.$$

With (A.29) (recalling that $R_* = \max\{R_0, e\mathfrak{a}\}$), we bound

$$\|f_R\|^2 \ge \int_{R_*\le|x|\le R} |f_R(x)|^2\,\mathrm{d}x \ge \frac{CR^2}{\log^2(R/\mathfrak{a})}$$

and, inserting in (A.33), we conclude that

$$\begin{aligned}\lambda_R = \frac{\langle f_R, \mathfrak{h} f_R\rangle}{\langle f_R, f_R\rangle} &\ge k^2\left(1 - \frac{C\log^3(R/\mathfrak{a})}{R^2}\right)\\ &\ge \frac{2}{R^2\log(R/\mathfrak{a})}\left(1+\frac{3}{4}\frac{1}{\log(R/\mathfrak{a})} - \frac{C}{\log^2(R/\mathfrak{a})}\right)\,,\end{aligned}$$

where in the last inequality we used (A.9).

To prove (A.3) we use the scattering equation (A.1) to write

$$\int \mathrm{d}x\, V(x) f_R(x) = 2\int_{|x|\le R}\mathrm{d}x\,\Delta f_R(x) + 2\int_{|x|\le R}\mathrm{d}x\lambda_R f_R(x)\,.$$

Passing to polar coordinates, and using that $\Delta f_R(x) = |x|^{-1}\partial_r|x|\partial_r f_R(x)$, we find that the first term vanishes. Hence

$$\int \mathrm{d}x\, V(x) f_R(x) = 2\lambda_R\int \mathrm{d}x\, f_R(x)\,.$$

With the upper bound $f_R(r)\le 1$ and with (A.2), we find

$$\int \mathrm{d}x\, V(x) f_R(x) \le 2\pi R^2\lambda_R \le \frac{4\pi}{\log(R/\mathfrak{a})}\left(1+\frac{C}{\log(R/\mathfrak{a})}\right)\,.$$

To obtain a lower bound for the same integral we use that $f_R(r)\ge 0$ inside the range of the potential. Outside the range of V, we use (A.26). We find

$$\int \mathrm{d}x\, V(x) f_R(x) \ge 4\pi\lambda_R\int_{R_0}^R \mathrm{d}r\, r\,(1 - C\varepsilon_R^2\log(R/r)) \ge \frac{4\pi}{\log(R/\mathfrak{a})}\left(1-\frac{C}{\log(R/\mathfrak{a})}\right)$$

We conclude that

$$\left| \int \mathrm{d}x\, V(x) f_R(x) - \frac{4\pi}{\log(R/\mathfrak{a})} \right| \leq \frac{C}{\log^2(R/\mathfrak{a})}.$$

Finally, we show the bounds in (A.4). For $r \in [R_0, R]$, from (A.26) we have

$$\left| w_R(x) - \frac{\log(R/|x|)}{\log(R/\mathfrak{a})} \right| \leq \frac{C}{\log(R/\mathfrak{a})}\,. \tag{A.34}$$

As for the derivative of w_R we use (A.27) to compute

$$|\partial_r f_R(x)| \leq \frac{C}{|x|} \frac{1}{\log(R/\mathfrak{a})}\,.$$

Moreover $\partial_r f_R(x) = 0$ if $|x| = R$, by construction.

On the other hand, if $|x| \leq R_0$, we have $w_R(x) = 1 - f_R(x) \leq 1$. As for the derivative, we define $\widetilde{f}_R$ on $\mathbb{R}_+$ through $\widetilde{f}_R(r) = f_R(x)$, if $|x| = r$, and we use the representation

$$\widetilde{f}_R'(r) = \frac{1}{r} \int_0^r \mathrm{d}s \big(\widetilde{f}_R''(s) s + \widetilde{f}_R'(s) \big)\,.$$

With (A.1), we have (with $\widetilde{V}$ defined on $\mathbb{R}_+$ through $V(x) = \widetilde{V}(r)$, if $|x| = r$)

$$\widetilde{f}_R''(r) + \frac{1}{r} \widetilde{f}_R'(r) = \lambda_R \widetilde{f}_R(r) - \frac{1}{2} \widetilde{V}(r) \widetilde{f}_R(r)\,,$$

By (A.34), we can estimate $\widetilde{f}_R(R_0) \leq C/\log(R/\mathfrak{a})$. From (A.30), we also recall that

$$\widetilde{f}_R(r) \leq \widetilde{f}_R(R_0) + CR\lambda_R \leq C/\log(R/\mathfrak{a})$$

for any $r < R_0$. We conclude therefore that

$$\begin{aligned} |\widetilde{f}_R'(r)| &= \Big| \frac{1}{r} \int_0^r \mathrm{d}s s \big(\lambda_R \widetilde{f}_R(s) - \frac{1}{2} \widetilde{V}(s) \widetilde{f}_R(s) \big) \Big| \\ &\leq \frac{\lambda_R}{r} \int_0^r r dr + \frac{C}{r \log(R/\mathfrak{a})} \int_0^r dr\, r \widetilde{V}(r) \\ &\leq \frac{C}{\log(R/\mathfrak{a})} + C \left\| V \right\|_2 \frac{\log(R_0/\mathfrak{a})}{\log(R/\mathfrak{a})} \leq \frac{C}{\log(R/\mathfrak{a})}\,. \end{aligned}$$

□

Appendix B

Properties of the Scattering Function through singular potentials

In this appendix, we give a proof of Lemma 2.10, containing basic properties of the ground state f_R of the Neumann problem

$$\Big(- \Delta + \frac{1}{2N} V(x) \Big) f_R(x) = \lambda_R \, f_R(x) \tag{B.1}$$

on the ball $|x| \leq R$, with boundary condition $\partial_{|x|} f_R(|x|) = 0$ for $|x| = R$, normalized so that $f_R(x) = 1$ for $|x| = R$. Here (and in Lemma 2.10) we are using the notation $R := N^\beta \ell$. Note that due to the factor $1/N$ in front of the potential in (B.1) the proof of Lemma B.1 is easier than the corresponding Lemma in the GP regime.

In the course of the proof we will use some well known properties of the zero-energy scattering equation in two dimensions, namely $\Big(-\Delta + \frac{1}{2N} V(x) \Big) \phi_R^{(N)}(x) = 0$ with $\phi_R^{(N)}(x) = 1$ as $|x| = R$ on the ball of radius R, that we recall in the following lemma.

Lemma B.1. *Let $V \in L^2(\mathbb{R}^2)$ non-negative, with supp $V \subset B_{R_0}(0)$ for an $R_0 > 0$. Let $\mathfrak{a} \leq R_0$ denote the scattering length of V. For $R > R_0$, let $\phi_R : \mathbb{R}^2 \to \mathbb{R}$ be the radial solution of the zero energy scattering equation*

$$\Big[- \Delta + \frac{1}{2N} V \Big] \phi_R^{(N)} = 0 \tag{B.2}$$

normalized so that $\phi_R^{(N)}(x) = 1$ for $|x| = R$. Then for all $|x| \geq R_0$,

$$\phi_R^{(N)}(x) = \frac{\log(x/\mathfrak{a}_N)}{\log(R/\mathfrak{a}_N)} \,, \tag{B.3}$$

where $\mathfrak{a}_N = \mathfrak{a}/N$. Moreover, $|x| \to \phi_R^{(N)}(x)$ is monotonically increasing and there exists a constant $C > 0$ (depending only on V) such that

$$\phi_R^{(N)}(x) \geq \phi_R^{(N)}(0) \geq \frac{C}{\log(R/\mathfrak{a}_N)} \tag{B.4}$$

for all $x \in \mathbb{R}^2$. Furthermore, there exists a constant C such that

$$\left| \frac{1}{\log(R/\mathfrak{a}_N)} - \frac{\widehat{V}(0)}{4\pi N} \right| \leq \frac{C}{N^2} \tag{B.5}$$

The proof of (B.3)-(B.4) follows exactly as in Lemma A.2 in Appendix A. We give a proof for Eq. (B.5).

Remark. Notice that we ask for $V \in L^2(\mathbb{R}^2)$, differently from the Lemma A.2, the proof of Eq. (B.8) does not require properties on the gradient of $\phi_R^{(N)}$. Indeed, it would be sufficient to choose $V \in L^{1+\varepsilon}$ for $\varepsilon > 0$, which is necessary for the proof of Eq.(B.4), based on the Harnack inequality (see [71, Theorem C.1.3]). We eventually need $V \in L^2(\mathbb{R}^2)$ in the proof of Eq. (B.5) as we can see below.

Proof. Rewriting (B.2) in the integral form we have

$$\phi_R(x) = 1 - \frac{1}{4\pi N}\int_{\mathbb{R}^2} \log\Big(R/|x-y|\Big) V(y)\phi_R(y)\mathrm{d}y\,. \tag{B.6}$$

Using the definition of scattering length, and the integral form (B.6) we get

$$\frac{1}{\log(R/\mathfrak{a}_N)} = \frac{1}{4\pi N}\int_{\mathbb{R}^2} V(y)\phi_R(y)\mathrm{d}y = \frac{\widehat{V}(0)}{4\pi N} + \varepsilon,$$

with

$$\begin{aligned}|\varepsilon| &\leq \frac{1}{(4\pi N)^2}\int_{\mathbb{R}^2} V(y)dy \int_{\mathbb{R}^2} \log(R/|z-y|)V(z)\phi_R^{(N)}(z)dz \\ &\leq \frac{\widehat{V}(0)}{(4\pi N)^2}\|V\|_2\frac{R^2}{4},\end{aligned}$$

where we used that $\phi_R^{(N)} \leq 1$. Thus Eq. (B.5) directly follows . □

We are now ready to prove Lemma 2.10. We adapt to the two dimensional case the strategy used in [26, Lemma A.1] for the three dimentional problem. For completeness we recall the statement of the lemma.

Lemma B.2. *Let $V \in L^2(\mathbb{R}^2)$ be non-negative, compactly supported and spherically symmetric, and denote its scattering length by $\mathfrak{a}$. Fix $R > 0$ sufficiently large and denote by f_R the Neumann ground state of* (B.1)*. Set $w_R = 1 - f_R$. Then we have*

$$0 \leq f_R(x) \leq 1 \tag{B.7}$$

for all $x \in \mathbb{R}^2$. Moreover,

$$\left|\lambda_R - \frac{1}{R^2}\frac{\widehat{V}(0)}{2\pi N}\left(1 - \frac{\widehat{V}(0)}{4\pi N}\log(R)\right)\right| \leq \frac{1}{R^2}\frac{C}{N^2} \tag{B.8}$$

and

$$\left|\frac{1}{N}\int dx\, V(x)f_R(x) - \frac{\widehat{V}(0)}{N}\left(1 - \frac{\widehat{V}(0)}{4\pi N}\log(R)\right)\right| \leq \frac{C}{N^2}. \tag{B.9}$$

Finally, there exists a constant $C > 0$ *such that red*

$$\begin{aligned} |w_R(x)| &\le C \qquad \textit{if} \quad |x| \le R_0, \\ \left| w_R(x) - \frac{\widehat{V}(0)}{4\pi N}\log(R/|x|) \right| &\le \frac{C}{N} \qquad \textit{if} \quad R_0 \le |x| \le R, \\ |\nabla w_R(x)| &\le \frac{C}{N}\frac{1}{|x|+1} \qquad \forall\ |x| \le R. \end{aligned} \tag{B.10}$$

Proof of Lemma B.2. By standard arguments (see for example [49, proof of theorem C1] it follows that $f_R(x)$ is spherically symmetric and non negative. We start by proving an upper bound, consistent with (B.8). To obtain this upper bound for λ_R we compute the energy of a suitable trial function. In this direction, we consider as a trial function

$$\psi_R = 1 - \frac{\widehat{V}(0)}{4\pi N}\log(R/|x|)\chi(|x| \ge 1).$$

Set $\mathfrak{h} = -\Delta + \frac{1}{2N}V$, then we have

$$\begin{aligned} \langle \psi_R, \mathfrak{h}\ \psi_R \rangle \le\ & \frac{\widehat{V}(0)^2}{16\pi^2 N^2}\int_{1\le|x|\le R} |\nabla \log(R/x)|^2 dx + \frac{1}{2N}\int_{|x|\le R_0} V(x)dx \\ & - \frac{\widehat{V}(0)}{4\pi N^2}\int_{1\le|x|\le R_0} V(x)\log R\, dx + \frac{\widehat{V}(0)}{4\pi N^2}\int_{1\le|x|\le R_0} V(x)\log|x|\, dx \\ & + \frac{C}{N^3}\int_{1\le|x|\le R_0} V(x)\log^2(R/|x|)\, dx \\ \le\ & \frac{\widehat{V}(0)^2}{8\pi^2 N^2}\log R + \frac{\widehat{V}(0)}{2N} - \frac{\widehat{V}(0)^2}{4\pi N^2}\log R + \frac{C\|V\|_2}{N^2}R_0\log R_0 + \frac{C\|V\|_2}{N^3}R_0 \\ \le\ & \frac{\widehat{V}(0)}{2N} - \frac{\widehat{V}(0)^2}{8\pi^2 N^2}\log R + \frac{C}{N^2} \\ \le\ & \frac{\widehat{V}(0)}{2N}\left(1 - \frac{\widehat{V}(0)}{4\pi N}\log R + \frac{C}{N}\right) \end{aligned} \tag{B.11}$$

while

$$\begin{aligned} \langle \psi_R,\ \psi_R \rangle &= \int_{|x|\le R} \Big(1 - \frac{\widehat{V}(0)}{4\pi R}\log(R/|x|)\chi(|x| \ge 1)\Big)^2 \mathrm{d}x \\ &\ge \int_{|x|\le R} dx - \frac{\widehat{V}(0)}{2\pi N}\int_{1\le|x|\le R} \log(R/|x|)dx \\ &\ge \pi R^2\left(1 - \frac{\widehat{V}(0)}{4\pi N} - \frac{C}{N}\frac{\log R}{R^2}\right). \end{aligned} \tag{B.12}$$

Putting together Eq.s (B.11) and (B.12) we end up with

$$\lambda_R = \frac{\langle \psi_R, \mathfrak{h}\, \psi_R \rangle}{\langle \psi_R, \psi_R \rangle} \leq \frac{1}{R^2} \frac{\widehat{V}(0)}{2\pi N} \left(1 - \frac{\widehat{V}(0)}{4\pi N} \log R + \frac{C}{N} \right). \tag{B.13}$$

To prove (B.7) and (B.10) we proceed as in Appendix A. We recall here only the main steps of the proof for the reader convenience. First, we get an explicit expression of f_R outside the range of the potential. Indeed solving the equation

$$-\Delta f_R(x) = k^2 f_R(x)$$

with $k^2 = \lambda_R R^2$, we find, for $R_0 \leq |x| \leq R$

$$f_R(x) = A_R\, J_0(k\,|x|/R) + B_R\, Y_0(k\,|x|/R)\,, \tag{B.14}$$

with

$$A_R = \left(J_0(k) - J_1(k)\, \frac{Y_0(k)}{Y_1(k)} \right)^{-1}, \quad B_R = \left(Y_0(k) - \frac{J_0(k)}{J_1(k)} Y_1(k) \right)^{-1},$$

where J_0 and Y_0 are the zero Bessel functions of first and second type respectively.

From the upper bound found before we have $k = \sqrt{\lambda_R} R \leq C N^{-1/2}$. This allows us to expand $f_R(x)$ for large R; using the bounds in (A.11), we find A_R and B_R as in (A.25), which leads to

$$\left| f_R(x) - 1 + \frac{k^2}{4} \left(2 \log(R/|x|) - 1 + \frac{x^2}{R^2} \right) \right| \leq C k^4 (\log k)^2 \tag{B.15}$$

for k small enough. The argument to bound f_R from above is the same as in Appendix A.

On the other hand, from (B.15) and (B.13) we obtain the lower bound

$$f_R(x) \geq 1 - \frac{k^2}{2} \log(R/|x|) - C k^4 (\log k)^2 \geq 1 - \frac{\widehat{V}(0)}{4\pi N} - C\, \frac{(\log N)^2}{N^2} > 0, \tag{B.16}$$

for N sufficiently large.

Now we want to prove the bounds for w_R in Eq. (B.10). For $R_0 \leq |x| \leq R$, using (B.15) we see that

$$\left| f_R(x) - 1 + \frac{k^2}{2} \log(R/|x|) \right| \leq C k^2,$$

hence, it follows that

$$\left| w_R(x) - \frac{k^2}{2} \log(R/|x|) \right| \leq C k^2. \tag{B.17}$$

While from (B.14), taking the derivative (this leads to one expression similar to (A.27) but with ε_R replaced by k) we find

$$|\partial_r f_R(x)| \leq C \frac{k^2}{|x|}. \tag{B.18}$$

Moreover $\partial_r f_R(x) = 0$ if $|x| = R$, by construction. Which proves the second bound in (B.10). On the other hand, if $|x| \le R_0$, we have $w_R(x) = 1 - f_R(x) \le 1$. As for the derivative, the proof for $|\nabla w_R| \le C/N$ inside the range of the potential follows the one in Appendix A.

Now we are ready to prove the lower bound for λ_R. With $f_R = 1 - w_R$, and using (B.17) and (B.18) to bound $|\nabla w_R|$, we get

$$\begin{aligned}
\langle f_R, \mathfrak{h}\, f_R \rangle &= \Big\langle (1-w_R), \Big(-\Delta + \frac{1}{2N} V \Big)(1-w_R) \Big\rangle \\
&\ge \int_{1\le |x| \le R} d^2x\, |\nabla w_R(x)|^2 + \frac{1}{2N} \int_{|x|\le R} d^2x\, V(x)(1 - 2w_R(x)) \\
&\ge \frac{k^4}{4} \int_{1\le|x|\le R} d^2x\, \frac{1}{|x|^2} + \frac{1}{2N}\int_{|x|\le R} d^2x\, V(x) \\
&\qquad - \frac{1}{N} \int_{1\le |x| \le R_0} d^2x V(x) \left[\frac{k^2}{2} \log(R/|x|) + Ck^2 \right] \\
&\ge \frac{\widehat{V}(0)}{2N} - \frac{k^2}{2}\log(R) \left(\frac{\widehat{V}(0)}{N} - k^2\pi \right) - \frac{C}{N} k^2.
\end{aligned} \tag{B.19}$$

On the other hand we have (still with (B.17))

$$\langle f_R,\, f_R \rangle \le \int_{|x|\le R} d^2x\, (1 + w_R^2(x)) \le \pi R^2 \left(1 + Ck^4\right).$$

This allows us to conclude, using the upper bound $k^2 \le C/N$

$$\begin{aligned}
\lambda_R = \frac{\langle f_R, \mathfrak{h}\, f_R\rangle}{\langle f_R, f_R\rangle} &\ge \left[\frac{\widehat{V}(0)}{2N} - \frac{k^2}{2}\log(R) \left(\frac{\widehat{V}(0)}{N} - k^2\pi \right) - \frac{C}{N} k^2 \right] \left[\pi R^2 \left(1 + Ck^4\right)\right]^{-1} \\
&\ge \frac{\widehat{V}(0)}{2\pi N} \frac{1}{R^2} \left(1 - C \frac{\log R}{N} \right).
\end{aligned} \tag{B.20}$$

With the help of (B.20) we have

$$k^2 = \left| \lambda_R R^2 - \frac{\widehat{V}(0)}{2\pi N} \right| \le C \frac{\log R}{N^2}, \tag{B.21}$$

thus we can improve the lower bound of λ_R. Indeed, using the improved bound (B.21) into (B.19), we end up with

$$\lambda_R \ge \frac{\widehat{V}(0)}{2\pi N} \frac{1}{R^2} \left(1 - \frac{\widehat{V}(0)}{4\pi N} \log(R) - \frac{C}{N} \right).$$

This concludes (B.8). Moreover, using once again (B.17), (B.18), and (B.21) we conclude the bounds in (B.10).

To prove Eq. (B.9) we use the scattering equation (B.1):

$$\frac{1}{N}\int_{|x|\leq R} V(x)f_R(x)\,\mathrm{d}x \;= 2\int_{|x|\leq R} \Delta f_R(x)\,\mathrm{d}x \;+ 2\int_{|x|\leq R} \lambda_R f_R(x)\,\mathrm{d}x$$

Passing to spherical coordinates (and denoting the radial coordinate as $|x| = r$) we get:

$$\frac{1}{N}\int_{|x|\leq R} V(x)f_R(x)\,\mathrm{d}x \;= 4\pi\int_0^R \frac{d}{dr}\Big(r\frac{d}{dr}f_R(r)\Big)\,\mathrm{d}r \;+ 4\pi\lambda_R\int_0^R rf_R(r)\,\mathrm{d}r\,. \tag{B.22}$$

The first integral on the r.h.s. of (B.22) is zero due to the boundary condition $\partial_r f_R(r)|_{r=R} = 0$. So it is sufficient to find upper and lower bounds for the second integral on the r.h.s. of (B.22) . To obtain an upper bound we use $f_R(r) \leq 1$ together with the upper bound for λ_R in (B.8). We get

$$\begin{aligned}\frac{1}{N}\int_{|x|\leq R} V(x)f_R(x)\,\mathrm{d}x &= 4\pi\lambda_R\int_0^R rf_R(r)\,\mathrm{d}r\\ &\leq \frac{4\pi\widehat{V}(0)}{2\pi R^2 N}\left(1-\frac{\widehat{V}(0)}{4\pi N}\log(R)\right)\int_0^R r\,\mathrm{d}r\\ &\leq \frac{\widehat{V}(0)}{N}\left(1-\frac{\widehat{V}(0)}{4\pi N}\log(R)+\frac{C}{N}\right).\end{aligned}$$

To obtain a lower bound for the same integral we use that $f_R(r) \geq 0$ inside the range of the potential, and (B.16) for $r \in (R_0, R]$. Hence:

$$\int_{|x|\leq R} V(x)f_R(x)\,\mathrm{d}x \geq 4\pi\lambda_R\int_{R_0}^R r\,\mathrm{d}r\,\Big(1-\frac{C}{N}\Big) \geq \frac{\widehat{V}(0)}{N}\left(1-\frac{\widehat{V}(0)}{4\pi N}\log(R)-\frac{C}{N}\right).$$

□

Appendix C

Ground state for many-bosons system through singular potentials

C.1 Introduction

In this appendix we will show Theorem 1.2, adapting the proof for three dimensional bosons in the Gross-Pitaevskii regime from [60]. Even though for the purpose of the present paper we only need to consider bosons in a box with periodic boundary conditions, in the following we will describe the arguments leading to the result in Theorem 1.2 in the more general case of bosons trapped by an external potential.

In the following we consider N interacting bosons in the two-dimensional space $\mathbb{R}^2$, described by the many-body Hamiltonian

$$H_N^\beta = \sum_{j=1}^N h_j + \sum_{1\leq j<k\leq N} V_\beta(x_j - x_k) \tag{C.1}$$

acting on the space $L^2_s(\mathbb{R}^{2N})$. The one-body operator is given by

$$h := -\Delta + V_{\text{ext}}(x)$$

with $V_{\text{ext}}(x)$ an external potential satisfying

$$0 \leq V_{\text{ext}} \in L^1_{\text{loc}}(\mathbb{R}^2); \quad \lim_{|x|\to\infty} V_{\text{ext}}(x) = +\infty \tag{C.2}$$

The particles interact pairwise via a repulsive potential V_β given by

$$V_\beta(x) = N^{2\beta-1} V(N^\beta x), \tag{C.3}$$

with $\beta > 0$ such that $\lim_{N\to\infty} \log N^\beta / N = 0$. Here V is a non-negative, radially symmetric and finite range function, i.e. $V(x) \equiv 0$ for $|x| > R_0$, with scattering length $\mathfrak{a} \leq R_0$. We are going to use some properties of the zero-energy scattering equation $\phi_R^{(N)}(x)$, *i.e.* the solution of (B.2) on a ball of radius $R = N^\beta \ell$. From (B.3) we recall that for any $R_0 \leq |x| \leq R$

$$\phi_R^{(N)}(x) = \frac{\log(x/\mathfrak{a}_N)}{\log(R/\mathfrak{a}_N)}\,, \tag{C.4}$$

where $\mathfrak{a}_N = \mathfrak{a}/N$. Moreover for any β such that $\lim_{N\to\infty} \log N^\beta/N = 0$ we have

$$\int_{|x|\leq R} \Delta\phi_R^{(N)}(x)\, dx = \int_{|x|=R} \nabla\phi_R^{(N)}(x)\cdot \mathbf{n}\, d\sigma_R = \frac{2\pi}{\log(R/\mathfrak{a}_N)} \tag{C.5}$$

where we denoted with $d\sigma_R$ the surface measure of the ball of radius R. From (B.2) and the divergence theorem it also follows

$$\int_{|x|\leq R} \Big[|\nabla\phi_R^{(N)}|^2 + \frac{1}{2N}V(x)|\phi_R^{(N)}|^2\Big] dx = \frac{2\pi}{\log(R/\mathfrak{a}_N)}. \tag{C.6}$$

Moreover, with (B.5) we also have

$$\lim_{N\to\infty}\left[N\int_{|x|\leq R} \Big[|\nabla\phi_R^{(N)}|^2 + \frac{1}{2N}V(x)|\phi_R^{(N)}|^2\Big] dx = \frac{\widehat{V}(0)}{2}\right].$$

We are going to show that the ground state energy and ground states of H_N^β converge to the ones of the non linear functional,

$$\mathcal{E}_{\mathrm{NLS}}[u] := \langle u, hu\rangle + \frac{\widehat{V}(0)}{2}\int_{\mathbb{R}^2} |u(x)|^4 dx. \tag{C.7}$$

Since we are not considering magnetic fields and V is radially symmetric, the minimizer of (C.7) exists and is unique by well-known arguments, see for instance [31]. Aim of this appendix is to show the following theorem.

Theorem C.1. *Let H_N^β be defined in* (C.1) *with V_{ext} satisfying* (C.2) *and V_β defined in* (C.3)*. Then for all $\beta > 3/4$ s.t.* $\lim_{N\to\infty} \beta\log N/N = 0$ *we have*

$$\lim_{N\to\infty}\inf_{\|\psi\|=1} \frac{\langle\psi, H_N^\beta\psi\rangle}{N} = \inf_{\|u\|_{L^2(\mathbb{R}^2)}=1} \mathcal{E}_{NLS}[u]\,. \tag{C.8}$$

Moreover, if ψ_N is an approximate ground state for H_N^β, namely

$$\lim_{N\to\infty} \frac{\langle\psi_N, H_N^\beta\psi_N\rangle}{N} = \inf_{\|u\|_{L^2(\mathbb{R}^2)}=1} \mathcal{E}_{NLS}[u],$$

and $\gamma_{N,m}^{(k)} = \mathrm{Tr}_{k+1\to N}|\psi_N\rangle\langle\psi_N|$ *is the k- particle reduced density matrix of ψ_N, then*

$$\lim_{N\to\infty} \mathrm{Tr}\left|\gamma_N^{(k)} - |\varphi_0^{\otimes k}\rangle\langle\varphi_0^{\otimes k}|\right| = 0, \quad \forall k\in\mathbb{N}\,,$$

where φ_0 is the minimizer of (C.7).

Actually, the theorem above holds for any $\beta > 0$ such that $\lim_{N\to\infty}\beta\log N/N = 0$. A proof valid for $\beta < 1$ can be found in [59]. Our proof follows closely the proof of condensation for approximate minimizers of the three dimensional Gross-Pitaevskii Hamiltonian, obtained in [60]. However, we reproduce below the main steps of the proof in our setting, for the reader convenience. We also refer the reader to the arxiv version of [10] for an adaptation of the proof by [60] to the translation invariant case, which is also the setting of relevance for this thesis.

C.2 Generalized Dyson Lemma

Using the same notations as [60] we set $\theta : \mathbb{R}^2 \to \mathbb{R}$ a radial smooth Heaviside-like function, i.e.

$$0 \leq \theta \leq 1; \;\; \theta(x) = 0 \text{ for } |x| \leq 1, \quad \theta(x) = 1 \text{ for } |x| \geq 2.$$

Let $U : \mathbb{R}^2 \to \mathbb{R}$ a radial smooth function supported on the annulus $1/2 \leq |x| \leq 1$ such that

$$U(x) \geq 0, \qquad \int_{\mathbb{R}^2} U(x) \log\left(|x|/\mathfrak{a}_N\right) dx = 2\pi N$$

with $\mathfrak{a}_N$ the scattering length of the potential $\frac{1}{N}V$. Using the monotonicity of the logarithm we clearly have

$$\frac{2\pi N}{|\log(\mathfrak{a}_N)|} \leq \int U(x) d^2x \leq \frac{2\pi N}{|\log(2\mathfrak{a}_N)|} \tag{C.9}$$

and therefore we conclude with (B.5) that for any $R = N^\beta$ s.t. $\lim_{N\to\infty} \log N^\beta / N = 0$ there exist constants $c_2 > c_1 > 0$ s.t.

$$c_1 \leq \int U(x) d^2x \leq c_2 \,. \tag{C.10}$$

In the application we will choose $U(x) = const.$ for all x s.t. $1/2 \leq |x| \leq 1$ and zero otherwise, so that $\|U\|_\infty \leq C$. For every $\ell > 0$ we now define

$$\theta_\ell(x) = \theta(x/\ell), \qquad U_\ell(x) = \frac{1}{\ell^2} U(x/\ell). \tag{C.11}$$

The first step of the proof by [60] consists in using the Dyson lemma to replace the Gross-Pitaevskii interaction in the original Hamiltonian with a less singular potential. In our setting, we use a generalized version of Dyson lemma to replace the original potential V_β in H_N^β by the potential U_ℓ, with some $\ell = N^{-\gamma}$, for some suitably chosen $\gamma \in (0;1)$. To this aim we will make use of the following Lemma.

Lemma C.1. *Let V compactly supported in the ball of radius $R_0 < 1/2$, with scattering length $\mathfrak{a}$ defined through the solution of the scattering equation* (B.2) *on the disk of radius R, with $R_0 < R < 1/2$. Let $\chi_R(x)$ be the characteristic function of a disc of radius R centered at the origin, $\theta(p)$ a radial function, such that $h(x) = \widehat{(1-\theta)}(x)$ bounded and integrable. Let*

$$g_R(x) = \sup_{|y|\leq R} |h(x-y) - h(x)|$$

and

$$\omega_R(x) = \frac{2}{\pi} g_R(x) \int_{\mathbb{R}^2} g_R(y) dy$$

Let $\mathfrak{a}_N = \mathfrak{a}/N$ and $u(x)$ a positive radial function supported in the annulus $R_0 \leq |x| \leq R$ such that

$$\int_{\mathbb{R}^2} u(x) \log(|x|/\mathfrak{a}_N) dx = 2\pi N(1+\delta_N)$$

with δ_N *such that* $\lim_{N\to\infty}\delta_N = 0$*; then for all* $\varepsilon > 0$

$$-\nabla\theta(p)\chi_R(x)\theta(p)\nabla + \frac{1}{2N}V(x) \geq \frac{(1-\varepsilon)}{N(1+\delta_N)}\,u(x) - \frac{1}{2\pi N(1+\delta_N)\varepsilon}\left(\int u(x)d^2x\right)\omega_R(x). \tag{C.12}$$

Proof. We adapt the proof in [48, Appendix A]. We first notice that it suffices to show that the operator inequality (C.12) holds for the expectation value with any smooth function $\psi(x)$ with compact support. Given $\psi(x)$ we define $\xi(x)$ by its Fourier transform $\widehat{\xi}(p) = \theta(p)\widehat{\psi}(p)$. We thus have to show that

$$\int_{|x|\leq R}\left[|\nabla\xi(x)|^2 + \frac{1}{2N}V(x)|\psi(x)|^2\right]dx \geq \frac{1}{N(1+\delta_N)}\int_{|x|\leq R}\left[(1-\varepsilon)\,u(x)|\psi(x)|^2 - \frac{\|u\|_1}{2\pi\varepsilon}|\psi(x)|^2\omega_R(x)\right]dx.$$

Let $\phi_R^{(N)}(x)$ be the solution to the zero-energy scattering equation (B.2), subject to boundary condition $\phi_R^{(N)}(x) = 1$ for $|x| = R$. Let ν be a complex-valued function on the unit disk $\mathcal{D}_1$, such that $\int_{\mathcal{D}_1}|\nu|^2 = 1$. Now consider the following expression, with ψ as above,

$$A := \int_{|x|\leq R}\nu(x)\nabla\xi^*(x)\cdot\nabla\phi_R^{(N)}(x)dx + \frac{1}{2N}\int_{|x|\leq R}V(x)\psi^*(x)\phi_R^{(N)}(x)\nu(x)dx.$$

By Cauchy-Schwarz inequality we can bound

$$|A|^2 \leq \left(\int_{|x|\leq R}\left[|\nabla\xi(x)|^2 + \frac{1}{2N}V(x)|\psi(x)|^2\right]dx\right) \times \left(\int_{|x|\leq R}\left[|\nabla\phi_R^{(N)}(x)|^2 + \frac{1}{2N}V(x)|\phi_R^{(N)}(x)|^2\right]|\nu(x)|^2dx\right).$$

Since $\phi_R^{(N)}$ is a radial function, the angular integration in the last term can be performed by using the condition on ν. The remaining expression is then bounded using (C.6). Hence we end up with

$$\int_{|x|\leq R}\left[|\nabla\xi(x)|^2 + \frac{1}{2N}V(x)|\psi(x)|^2\right]dx \geq |A|^2\left(\frac{1}{\log(R/\mathfrak{a}_N)}\right)^{-1}. \tag{C.13}$$

Now we need a lower bound on $|A|^2$. Note that $\phi_R^{(N)}$ is a radial function and that $\nabla\phi_R^{(N)}(x)|_{|x|=R} = \frac{1}{R}\frac{1}{\log(R/\mathfrak{a}_N)}$ from (C.4). Therefore, by partial integration we get

$$\int_{|x|\leq R}\nu(x)\nabla\xi^*(x)\cdot\nabla\phi_R^{(N)}(x)dx = -\int_{|x|\leq R}\xi^*(x)\nu(x)\Delta\phi_R^{(N)}(x)dx + \frac{1}{R}\frac{1}{\log(R/\mathfrak{a}_N)}\int_{|x|=R}\xi^*(x)\nu(x)d\sigma_R, \tag{C.14}$$

where $d\sigma_R$ denotes the surface measure of the ball of radius R, and we used that $\nabla\nu(x)\cdot\nabla\phi_R^{(N)}(x) = 0$. Recall now that by definition of $h(x)$ we have $\xi(x) = \psi(x) - (2\pi)^{-1}(h*\psi)(x)$, hence we can rewrite (C.14)

$$\begin{aligned}\int_{|x|\leq R}\nu(x)\nabla\xi^*(x)\cdot\nabla\phi_R^{(N)}(x)dx = &-\int_{|x|\leq R}\psi^*(x)\Delta\phi_R^{(N)}(x)\nu(x)dx\\&+\frac{1}{2\pi}\int_{|x|\leq R}(h*\psi)^*(x)\nu(x)\Delta\phi_R^{(N)}(x)dx\\&+\frac{1}{R}\frac{1}{\log(R/\mathfrak{a}_N)}\int_{|x|=R}\psi^*(x)\nu(x)d\sigma_R\\&-\frac{1}{2\pi R}\frac{1}{\log(R/\mathfrak{a}_N)}\int_{|x|=R}(h*\psi)^*(x)\nu(x)d\sigma_R\,.\end{aligned}$$

Hence, with (A.5)

$$\begin{aligned}A = &\frac{1}{2\pi}\int_{|x|\leq R}(h*\psi)^*(x)\nu(x)\Delta\phi_R^{(N)}(x)dx\\&+\frac{1}{R}\frac{1}{\log(R/\mathfrak{a}_N)}\int_{|x|=R}\psi^*(x)\nu(x)d\sigma_R\\&-\frac{1}{2\pi R}\frac{1}{\log(R/\mathfrak{a}_N)}\int_{|x|=R}(h*\psi)^*(x)\nu(x)d\sigma_R.\end{aligned}\tag{C.15}$$

We rewrite the first and the last term as

$$\frac{1}{2\pi}\int\psi^*(x)\Big[\int h(x-y)d\mu(y)\Big]\,dx,\tag{C.16}$$

where we defined the measure $d\mu$ supported in the ball of radius R, such that $d\mu(y) = \nu(y)\Delta\phi_R^{(N)}(y)dy - \frac{1}{R}\frac{1}{\log(R/\mathfrak{a}_N)}\nu(y)\delta(|y|-R)dy$. Note that, by integration by parts, radial integration and (C.5)

$$\int d\mu(y) = \int\nu(y)\Delta\phi_R^{(N)}(y)dy - \frac{1}{R}\frac{1}{\log(R/\mathfrak{a}_N)}\int\nu(x)\delta(|y|-R)dy = 0.$$

Moreover,

$$\begin{aligned}\int d|\mu(y)| &= \int|\nu(y)\Delta\phi_R^{(N)}(y)|dy - \frac{1}{R}\frac{1}{\log(R/\mathfrak{a}_N)}\int|\nu(x)\delta(|y|-R)|dy\\&=\int_{S^1}|\nu|\int_0^R\frac{d}{dr}[r(\phi_R^{(N)})'(x)]dr + \frac{1}{\log(R/\mathfrak{a}_N)}\int_{S^1}|\nu(y)|dy\\&=\frac{2}{\log(R/\mathfrak{a}_N)}\int_{S^1}|\nu(y)|dy,\end{aligned}$$

using Cauchy-Schwarz can be bounded by

$$\int d|\mu(y)| \leq \frac{2(2\pi)^{1/2}}{\log(R/\mathfrak{a}_N)}.$$

This bound allows us to estimate the integral in (C.16) with h, using the definition of g_R

$$\left|\int h(x-y)d\mu(y)\right| \leq \frac{2(2\pi)^{1/2}}{\log(R/\mathfrak{a}_N)} g_R(x).$$

Now, using again Cauchy-Schwarz, the definition of ω_R we get

$$\begin{aligned}\int |\psi(x)| g_R(x)dx &\leq \left(\int |\psi(x)|^2 g_R(x)dx\right)^{1/2}\left(\int g_R(x)dx\right)^{1/2}\\ &\leq \frac{\pi^{1/2}}{2^{1/2}}\left(\int |\psi(x)|^2\omega_R(x)dx\right)^{1/2}.\end{aligned}$$

Thus we can conclude that (C.16)

$$\frac{1}{2\pi}\int \psi^*(x)\Big[\int h(x-y)d\mu(y)\Big]dx \geq -\frac{1}{\log(R/\mathfrak{a}_N)}\left(\int |\psi(x)|^2\omega_R(x)dx\right)^{1/2},$$

which is independent of ν. We still have to bound the second term on the r.h.s. of (C.15). To do so we choose $\nu(x)$ to be the restriction of $\psi(x)$ to the disk of radius R, normalized, namely

$$\nu(x) = \psi(x)\chi(|x|\leq R)\left(\int_{\mathcal{D}_1} |\psi(Rx)|^2\right)^{-1/2}$$

where we denote with $\mathcal{D}_r$ the disk of radius r. We obtain

$$A \geq \frac{1}{\sqrt{R}}\frac{1}{\log(R/\mathfrak{a}_N)}\left(\int_{|x|=R} |\psi(x)|^2 d\sigma_R\right)^{1/2} - \frac{1}{\log(R/\mathfrak{a}_N)}\left(\int |\psi(x)|^2\omega_R(x)dx\right)^{1/2}.$$

Again, using Cauchy-Schwarz we get that, for any $\varepsilon > 0$

$$|A|^2 \geq \frac{1}{\log^2(R/\mathfrak{a}_N)}\left[\frac{1}{R}(1-\varepsilon)\int_{|x|=R} |\psi(x)|^2 d\sigma_R - \frac{1}{\varepsilon}\int |\psi(x)|^2\omega_R(x)dx\right].$$

Now, this equation, together with (C.13), imply that

$$\begin{aligned}&\int_{|x|\leq R}\Big[|\nabla\xi(x)|^2 + \frac{1}{2N}V(x)|\psi(x)|^2\Big]dx\\ &\qquad \geq |A|^2\left(\frac{1}{\log(R/\mathfrak{a}_N)}\right)^{-1}\\ &\qquad \geq \frac{1}{\log(R/\mathfrak{a}_N)}\left[\frac{1}{R}(1-\varepsilon)\int_{|x|=R} |\psi(x)|^2 d\sigma_R - \frac{1}{\varepsilon}\int |\psi(x)|^2\omega_R(x)dx.\right]\end{aligned}$$

Namely,

$$\begin{aligned}&\int_{|x|\leq R}\Big[|\nabla\xi(x)|^2 + \frac{1}{2N}V(x)|\psi(x)|^2\Big]dx\\ &\qquad \geq \frac{1}{\log(R/\mathfrak{a}_N)}\left[\frac{1}{R}(1-\varepsilon)\int_{|x|=R} |\psi(x)|^2 d\sigma_R - \frac{1}{\varepsilon}\int |\psi(x)|^2\omega_R(x)dx\right],\end{aligned}$$

which allows us to conclude the proof replacing R by s, multiplicating both sides by $u(s)s\log(s/\mathfrak{a}_N)$, where $u(s) = u(x)$ for $|x| = s$, and integrating over $s \in [0; R]$.

$$\int_0^R u(s)s\log(s/\mathfrak{a}_N)\,ds\int_{|x|\le R}\Big[|\nabla\xi(x)|^2 + \frac{1}{2N}V(x)|\psi(x)|^2\Big]dx$$
$$\ge (1-\varepsilon)\int_0^R u(s)\,ds\int_{|x|=s}|\psi(x)|^2 d\sigma_s - \frac{1}{\varepsilon}\int_0^R u(s)sds\int|\psi(x)|^2\omega_R(x)dx.$$

Using the assumption (C.1) we end up with

$$N(1+\delta_N)\int_{|x|\le R}\big[|\nabla\xi(x)|^2 + \frac{1}{2N}V(x)|\psi(x)|^2\big]dx$$
$$\ge (1-\varepsilon)\int_{|x|\le R}u(x)|\psi(x)|^2 d^2x - \frac{1}{2\pi\varepsilon}\left(\int u(x)d^2x\right)\int|\psi(x)|^2\omega_R(x)dx,$$

which concludes the proof of (C.12).

□

Let $l(p)$ be a smooth, radial, positive function with $l(p) = 0$ for $|p| \le 1$ and $l(p) = 1$ for $|p| \ge 2$, and $0 \le l(p) \le 1$ in between, let $\theta_s(p) = l(p/s)$ so that the h defined in the theorem is such that $h_s(x) = \widehat{(1-\theta_s)}(x)$, then the corresponding potential ω_R, for $R \le Cs$ satisfies

$$|\omega_R(x)| \le CR^2s^4 \quad \text{and} \quad \int_{\mathbb{R}^2}|\omega_R(x)|d^2x \le CR^2s^2. \tag{C.17}$$

We can easily extend Lemma C.1 with the following corollary

Corollary C.2. *If $y_1, \dots, y_N$, N points in $\mathbb{R}^{2N}$, s.t. $|y_i - y_j| \ge 2R$ for any $i \ne j$ then, under the assumptions of Lemma C.1, we have*

$$\begin{aligned}&-\nabla\theta(p)^2\nabla + \frac{1}{2N(1+\delta_N)}\sum_{i=1}^N V(x-y_i)\\&\ge \frac{1}{N(1+\delta_N)}\sum_{i=1}^N\left[(1-\varepsilon)\,u(x-y_i) - \frac{\|u\|_1}{2\pi\varepsilon}\omega_R(x-y_i)\right].\end{aligned} \tag{C.18}$$

We can now state the equivalent of Lemma 2.1 in [60] in our setting. The aim of this lemma is to smooth out the singular interaction potential V_β with a softer one with a larger range of interaction, namely U_ℓ defined in (C.11).

From now on we set $R = N^\beta\ell$. It is important to notice that in this appendix we are considering $\ell = N^{-\gamma}$, while in Appendix B the parameter ℓ was small but of order one. With a slight abuse of notation we keep the same notation.

Lemma C.2. *Let H_N^β and U_ℓ be defined in* (C.1) *and* (C.11) *respectively. Let $\theta_s(p)$ be defined before* (C.17). *Then, for all $s > 0$, $0 < \varepsilon < 1$ and $\ell > 2R_0/N^\beta$, we have*

$$H_N^\beta \ge \sum_{i=1}^N\big(h_i - (1-\varepsilon)p_i^2\theta_s(p_i)\big) + \frac{\alpha_N(1-\varepsilon)^2}{N}W_\beta - \frac{CN\ell^2s^4}{\varepsilon}, \tag{C.19}$$

where α_N *is a constant bounded uniformly in* N *and s.t.* $\lim_{N\to\infty}\alpha_N = 1$ *and*

$$W_\beta := \sum_{i\neq j}^{N} U_\ell(x_i - x_j) \prod_{k\neq i,j} \theta_{2\ell}(x_j - x_k)\,, \tag{C.20}$$

with $C > 0$ *generic constant.*

Proof. The proof is an application of Eq. C.18. We first notice that by scaling the solution of the scattering equation for the potential V_β on the disk of radious $\ell = N^{-\beta}R$ is given by $\phi_\ell^{(\beta)}(x) := \phi_R^{(N)}(N^\beta x)$. Hence, Eq. C.18 holds with $(\frac{1}{N}V,\, \mathfrak{a}_N, R_0, R)$ substituted by $(V_\beta, N^{-\beta}\mathfrak{a}_N,\, N^{-\beta}R_0, \ell)$ and a potential $u(x)$ satisfying the assumptions in Lemma C.1.

To conclude it is then sufficient to notice that the potential $U_\ell(x)$ is different from zero for $\ell/2 \le |x| \le \ell$, and satisfies

$$\int U_\ell(x)\log(|x|/N^{-\beta}\mathfrak{a}_N)d^2x = 2\pi N + \log(N^\beta \ell)\int U(x)d^2x\,.$$

Therefore with (C.10) we have

$$U_\ell(x)\log(|x|/N^{-\beta}\mathfrak{a}_N)d^2x = 2\pi N(1+\delta_N)\,, \qquad \text{with } \pm\,\delta_N \le \frac{C\log(N^\beta\ell)}{N}\,.$$

Then (C.18) and (C.17) imply that for all ℓ s.t. $\ell > N^{-\beta}R_0/2$ (so that $U_\ell(x)$ is supported in the annulus $N^{-\beta}R_0 \le |x| \le \ell$) we have

$$p^2\theta_s(p) + \frac{1}{2}\sum_{i=1}^{N-1} V_\beta(x - y_i) \ge \frac{\alpha_N(1-\varepsilon)}{N}\sum_{i=1}^{N-1} U_\ell(x-y_i) - \frac{C\ell^2 s^4}{\varepsilon}$$

on $L^2(\mathbb{R}^2)$, for all given points y_i satisfying $\min_{j\neq k}|y_j - y_k| \ge 2\ell$ and $\alpha_N = (1+\delta_N)^{-1}$. Since the left hand side is non-negative we can relax the condition on the distance of points by multiplying the r.h.s. with $\prod_{k\neq j}\theta_{2\ell}(y_j - y_k)$. Thus for every $i = 1,\ldots,N$

$$p^2\theta_s(p) + \frac{1}{2}\sum_{j\neq i}^{N} V_\beta(x_i - x_j) \ge \frac{\alpha_N(1-\varepsilon)}{N}\sum_{i\neq j}^{N} U_\ell(x_i - x_j)\prod_{k\neq j}\theta_{2\ell}(y_j - y_k) - \frac{C\ell^2 s^4}{\varepsilon}$$

Now, multiplying both sides with $1-\varepsilon$ and summing over i we obtain

$$\begin{aligned}\sum_{i=1}^{N}(1-\varepsilon)p_i^2\theta_s(p_i) &+ \frac{1}{2}(1-\varepsilon)\sum_{j\neq i}^{N} V_\beta(x_i - x_j) \\ &\ge \frac{\alpha_N(1-\varepsilon)^2}{N}\sum_{i\neq j}^{N} U_\ell(x_i - x_j)\prod_{k\neq j}\theta_{2\ell}(y_j - y_k) - \frac{CN\ell^2 s^4}{\varepsilon}\,.\end{aligned} \tag{C.21}$$

With the definition $H_N^\beta = \sum_{j=1} h_j + \sum_{1\le j<k\le N} V_\beta(x_j - x_k)$ we have

$$H_N^\beta \ge \sum_{i=1}^{N}\big(h_i - (1-\varepsilon)p_i^2\theta_s(p_i)\big) + (1-\varepsilon)\sum_{i=1}^{N} p_i^2\theta_s(p_i) + \frac{(1-\varepsilon)}{2}\sum_{j\neq i}^{N} V_\beta(x_i - x_j)$$

hence with (C.21) we conclude. □

C.3 Second moment estimate

In the next step we will focus on the Hamilton operator

$$\tilde{H}_N^\beta := \sum_{i=1}^N \tilde{h}_i + \frac{\alpha_N(1-\varepsilon)^2}{N} W_\beta \tag{C.22}$$

where W_β has been defined in (C.20)

$$\tilde{h}_i := p_i^2(1-(1-\varepsilon)\theta_s(p_i)) + V_{\text{ext}} + 1\,. \tag{C.23}$$

and $\alpha_N = (1+\delta_N)^{-1}$ is bounded uniformly in N and s.t. $\lim_{N\to\infty}\alpha_N = 1$. Here we are adding a constant to make sure that $\tilde{h}_i \geq 1$. We will remove it when we will compare $\tilde{H}_N^\beta$ with H_N^β later on. Our goal is to bound the second moment of $\tilde{H}_N$ from below in terms of the second moment of $\sum_i^N \tilde{h}_i$. To this end we use the following lemma, which is an adaptation to the two dimensional case of [60, Lemma 3.2]. A statement of the corresponding lemma in two and three dimensions can be found in [67, Lemma 4.4].

Lemma C.3. *For every $0 \leq W \in L^1 \cap L^2(\mathbb{R}^2)$, the multiplication operator $W(x-y)$ on $L^2(\mathbb{R}^2)$ satisfies*

$$0 \leq W(x-y) \leq C\|W\|_{L^{3/2}(\mathbb{R}^2)}(1-\Delta_x), \tag{C.24}$$

for any $0 \leq \delta < 1/2$

$$0 \leq W(x-y) \leq C_\delta \|W\|_{L^1(\mathbb{R}^2)}(1-\Delta_x)^{1-\delta}(1-\Delta_y)^{1-\delta}. \tag{C.25}$$

Moreover, for all $1 > \varepsilon > 0$, $s > 0$ and $0 \geq V_{ext} \in L^1_{loc}(\mathbb{R}^2)$

$$\tilde{h}_x W(x-y) + W(x-y)\tilde{h}_x \geq -C\big[\|W\|_{L^2(\mathbb{R}^2)} + (1+s^2)\|W\|_{L^{3/2}(\mathbb{R}^2)}\big](1-\Delta_x)(1-\Delta_y) \tag{C.26}$$

Proof of (C.24). To prove (C.24) we use Hölder and Sobolev inequality in two-dimensions. For any function $f \in H^1((\mathbb{R}^2)^2)$

$$\begin{aligned}
\langle f, W(x-y)f\rangle &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} W(x-y)|f(x,y)|^2 dxdy \\
&\leq \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} W(x-y)^{3/2}dx\right)^{2/3}\left(\int |f(x,y)|^6 dx\right)^{1/3} dy \\
&\leq C\|W\|_{L^{3/2}(\mathbb{R}^2)}\int_{\mathbb{R}^2}\|f\|^2_{L^6_x} dy \\
&\leq C\|W\|_{L^{3/2}(\mathbb{R}^2)}\bigg[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|f(x-y)|^2 dxdy \\
&\qquad\qquad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\nabla_x f(x-y)|^2 dxdy\bigg] \\
&\leq C\|W\|_{L^{3/2}(\mathbb{R}^2)}(\langle f,f\rangle + \langle f, -\Delta_x f\rangle) \\
&\leq C\|W\|_{L^{3/2}(\mathbb{R}^2)}\langle f, (1-\Delta_x)f\rangle
\end{aligned}$$

□

Proof of (C.25). A proof of the estimate with $\delta = 0$ can be found in [28] in 3d and stated in 2d in [39]. Note that for every operator K, $K^*K \leq 1$ if and only if $KK^* \leq 1$. Therefore, (C.25) is equivalent to

$$\sqrt{W(x-y)}(1-\Delta_x)^{\delta-1}(1-\Delta_y)^{\delta-1}\sqrt{W(x-y)} \leq C_\delta \|W\|_{L^1}. \tag{C.27}$$

Let G be the Green function of $(1-\Delta)^{\delta-1}$, which Fourier transform is given by

$$\widehat{G}(k) := \int_{\mathbb{R}^2} e^{-2\pi i x\cdot k} G(x)dx = \frac{1}{(1+4\pi^2|k|^2)^{1-\delta}}.$$

For every function $f \in L^2((\mathbb{R}^2)^2)$ we have

$$\begin{aligned}
&\langle f, (\sqrt{W(x-y)}(1-\Delta_x)^{\delta-1}(1-\Delta_y)^{\delta-1}\sqrt{W(x-y)})f\rangle \\
&= \int \overline{f(x,y)}\sqrt{W(x-y)}G(x-x')G(y-y')\sqrt{W(x'-y')}f(x',y')dxdydx'dy' \\
&\leq \frac{1}{2}\int \big[W(x-y)|G(x-x')|^2|f(x',y')|^2 \\
&\qquad\qquad +W(x'-y')|G(y-y')|^2|f(x,y)|^2\big]\, dxdydx'dy' \\
&\leq C_\delta \|W\|_{L^1}\langle f, f\rangle,
\end{aligned}$$

where

$$C_\delta := \int |G|^2 = \int |\widehat{G}|^2 = \int_{\mathbb{R}^2} \frac{dk}{(1+4\pi^2|k|^2)^{2(1-\delta)}}$$

which is finite for all $0 \leq \delta < 1/2$. Thus (C.27) holds. □

Proof of (C.26). Before we prove a simpler version of (C.26), namely

$$(-\Delta_x)W(x-y) + W(x-y)(-\Delta_x) \geq -C\|W\|_{L^2}(1-\Delta_x)(1-\Delta_y). \tag{C.28}$$

By an approximation argument one can assume that W is smooth. For every $f \in H^2(\mathbb{R}^2 \times \mathbb{R}^2)$, a straightforward calculation using integration by parts and the identity $\nabla_x(W(x-y)) = -\nabla_y(W(x-y))$ gives us

$$\begin{aligned}
&\langle f, \big((-\Delta_x)W(x-y) + W(x-y)(-\Delta_x)\big)f\rangle \\
&\quad = 2\mathrm{Re}\int\int \nabla_x\overline{f(x,y)}\nabla_x(W(x-y)f(x,y))dxdy \\
&\quad = 2\int\int |\nabla_x f(x,y)|^2 W(x-y)dxdy \\
&\qquad\qquad + 2\mathrm{Re}\int\int \nabla_x\overline{f(x,y)}\nabla_x(W(x-y))f(x,y)dxdy \\
&\quad \geq -2\mathrm{Re}\int\int \nabla_x\overline{f(x,y)}\nabla_y(W(x-y))f(x,y)dxdy \\
&\quad = 2\mathrm{Re}\int\int \nabla_y\big((\nabla_x\overline{f(x,y)})f(x,y)\big)W(x-y)dxdy \\
&\quad = 2\mathrm{Re}\int\int \Big[\nabla_x\overline{f(x,y)}\nabla_y f(x,y) + \nabla_y(\nabla_x\overline{f(x,y)})f(x,y)\Big]W(x-y)dxdy.
\end{aligned}$$

Using Cauchy-Schwarz and Sobolev inequality (C.24) we get

$$\begin{aligned}
&\left|\int\int \nabla_x \overline{f(x,y)}\nabla_y f(x,y)W(x-y)dxdy\right| \\
&\qquad\qquad \leq \frac{1}{2}\int\int \left[|\nabla_x f(x,y)|^2 + |\nabla_y f(x,y)|^2\right]|W(x-y)|dxdy \\
&\qquad\qquad \leq C\|W\|_{L^{3/2}}\langle f,(1-\Delta_x)(1-\Delta_y)f\rangle.
\end{aligned}$$

Moreover, by Cauchy-Schwarz inequality and (C.25) with $\delta = 0$ and W replaced by W^2 we get

$$\begin{aligned}
&\left|\int\int (\nabla_y\nabla_x \overline{f(x,y)})f(x,y)W(x-y)dxdy\right| \\
&\leq \left(\int\int |\nabla_y\nabla_x f(x,y)|^2 dxdy\right)^{1/2}\left(\int\int |f(x,y)|^2|W(x-y)|^2dxdy\right)^{1/2} \\
&\leq C\|W\|_{L^2}\langle f,(1-\Delta_x)(1-\Delta_y)f\rangle.
\end{aligned}$$

Thus we can conclude

$$\begin{aligned}
&\langle f, \big((-\Delta_x)W(x-y) + W(x-y)(-\Delta_x)\big)f\rangle \\
&\qquad\qquad \geq -C\left[\|W\|_{L^2} + \|W\|_{L^{3/2}}\right]\langle f,(1-\Delta_x)(1-\Delta_y)f\rangle.
\end{aligned}$$

Now if we want to prove (C.26), we just need to estimate the second part of the operator $\tilde{h}$. Using the Cauchy-Schwarz estimate for operators, namely

$$\pm\left(XY + Y^*X^*\right) \leq \delta XX^* + \delta^{-1}Y^*Y \qquad \forall\delta > 0 \tag{C.29}$$

and using (C.24)

$$\begin{aligned}
&p_x^2(1-\theta_s(p_x))W(x-y) + W(x-y)p_x^2(1-\theta_s(p_x)) \\
&\quad \geq -\delta p_x^2(1-\theta_s(p_x))W(x-y)p_x^2(1-\theta_s(p_x)) - \delta^{-1}W(x-y) \\
&\quad \geq -C\|W\|_{L^{3/2}}(\delta p_x^4(1-\theta_s(p_x))^2 + \delta^{-1})(1-\Delta_x)
\end{aligned}$$

for all $\delta > 0$. Now since $1-\theta_s(p_x) \leq \chi(|p| \leq 2s)$ and choosing $\delta \sim s^{-2}$ we end up with

$$\begin{aligned}
&p_x^2(1-\theta_s(p_x))W(x-y) + W(x-y)p_x^2(1-\theta_s(p_x)) \\
&\quad \geq -C\|W\|_{L^{3/2}}(s^{-2}p_x^4(1-\theta_s(p_x))^2 + s^2)(1-\Delta_x) \\
&\quad \geq -C\|W\|_{L^{3/2}}s^2(1-\Delta_x).
\end{aligned} \tag{C.30}$$

Putting together (C.28),(C.30) we conclude the proof for (C.26). □

Now we are ready to prove the following key bound for $\tilde{H}_N^\beta$.

Lemma C.4 (Second moment estimate)**.** *Let* $\tilde{H}_N^\beta$ *and* $\tilde{h}_i$ *be defined in* (C.22) *and* (C.23) *respectively. Then, for every* $0 < \varepsilon < 1$ *and* $s > 0$*, and* $\ell = \ell(N) \gg N^{-1}$ *when* $N \to \infty$*, then*

$$(\tilde{H}_N^\beta)^2 \geq \frac{1}{3}\left(\sum_{i=1}^N \tilde{h}_i\right)^2, \tag{C.31}$$

for N *large enough.*

Proof. The proof follows similarly to [60, Lemma 3.1]. We have

$$(\tilde{H}_N^\beta)^2 - \left(\sum_{i=1}^N \tilde{h}_i\right)^2 = \frac{\alpha_N(1-\varepsilon)^2}{N}\sum_{m=1}^N(\tilde{h}_m W_\beta + W_\beta \tilde{h}_m) + \frac{\alpha_N^2(1-\varepsilon)^4}{N^2}W_\beta^2 \quad \text{(C.32)}$$

The goal is to bound the "mixed" term $\tilde{h}_1 W_\beta + W_\beta \tilde{h}_1$ from below. In order to do it we first decompose W_β as

$$W_\beta = W_a + W_b$$

where

$$W_a = \sum_{1\in\{i,j\}} U_\ell(x_i - x_j)\prod_{k\neq i,j}\theta_{2\ell}(x_j - x_k),$$
$$W_b = \sum_{i,j\geq 2} U_\ell(x_i - x_j)\prod_{k\neq i,j}\theta_{2\ell}(x_j - x_k).$$

First we estimate W_a. By the Cauchy-Schwarz inequality (C.29) we get

$$\pm\,(\tilde{h}_1 W_a + W_a \tilde{h}_1) \leq N^{-1}\tilde{h}_1 W_a \tilde{h}_1 + N W_a \quad \text{(C.33)}$$

having chosen $\delta = N^{-1}$. Let us show that

$$W_a \leq \frac{C}{\ell^2}. \quad \text{(C.34)}$$

Indeed, for every given $(x_1, x_2, \ldots, x_N) \in (\mathbb{R}^2)^N$, the product

$$U_\ell(x_1 - x_j)\prod_{k\neq 1,j}\theta_{2\ell}(x_j - x_k)$$

is bounded by $\|U_\ell\|_{L^\infty} \leq C\ell^{-2}$ and it is zero except in the case

$$|x_1 - x_j| < \ell < 2\ell < \min_{k\neq 1,j}|x_j - x_k|.$$

By triangle inequality last condition implies

$$|x_1 - x_j| < \ell < \min_{k\neq 1,j}|x_1 - x_k|$$

and it is satisfied by at most one index $j \neq 1$. Therefore,

$$\sum_{j\geq 2} U_\ell(x_1 - x_j)\prod_{k\neq 1,j}\theta_{2\ell}(x_j - x_k) \leq \frac{C}{\ell^2}.$$

Similarly we have

$$\sum_{i\geq 2} U_\ell(x_i - x_1)\prod_{k\neq 1,i}\theta_{2\ell}(x_i - x_k) \leq \frac{C}{\ell^2}$$

and so (C.34) holds true. From (C.33) and (C.34) we obtain

$$\pm\,(\tilde{h}_1 W_a + W_a \tilde{h}_1) \leq \frac{C}{N\ell^2}(\tilde{h}_1)^2 + 2N \sum_{1\in\{i,j\}} U_\ell(x_i - x_j) \prod_{k\neq i,j} \theta_{2\ell}(x_j - x_k). \quad \text{(C.35)}$$

Now we can proceed with W_b, we need to split it again

$$W_b = W_c + W_d$$

with

$$\begin{aligned} W_c &= \sum_{i,j\geq 2} U_\ell(x_i - x_j) \prod_{k\neq 1,i,j} \theta_{2\ell}(x_j - x_k) \\ W_d &= \sum_{i,j\geq 2} U_\ell(x_i - x_j)\big(1 - \theta_{2\ell}(x_j - x_1)\big) \prod_{k\neq i,j} \theta_{2\ell}(x_j - x_k). \end{aligned}$$

Note that

$$W_c \geq 0, \quad W_d \geq 0, \quad \text{and} \quad \tilde{h}_1 W_c = W_c \tilde{h}_1 \geq 0.$$

On the other hand by Cauchy-Schwarz inequality (C.29)

$$\pm\,(\tilde{h}_1 W_d + W_d \tilde{h}_1) \leq \delta \tilde{h}_1 W_d \tilde{h}_1 + \delta^{-1} W_d. \quad \text{(C.36)}$$

We have two different ways to bound W_d. First by (C.24) and $\tilde{h}_i \geq 1$,

$$(1 - \theta_{2\ell}(x_j - x_1)) \leq C\|1 - \theta_{2\ell}\|_{L^{3/2}} \leq C_\varepsilon \ell^{4/3} \tilde{h}_1,$$

because

$$\|1 - \theta_{2\ell}\|_{L^{3/2}} = \left(\int_{\mathbb{R}^2} (1 - \theta(x/2\ell))^{3/2} d^2x\right)^{2/3} \leq C\ell^{4/3}$$

Since here $i, j \geq 2$, both sides of the last estimate commute with

$$U_\ell(x_i - x_j) \prod_{k\neq 1,i,j} \theta_{2\ell}(x_j - x_k),$$

and we deduce that, multiplying both side by this term,

$$\begin{aligned} (1 - \theta_{2\ell}(x_j - x_1))\, U_\ell(x_i - x_j) &\prod_{k\neq 1,i,j} \theta_{2\ell}(x_j - x_k) \\ &\leq C_\varepsilon \ell^{4/3} \tilde{h}_1 U_\ell(x_i - x_j) \prod_{k\neq 1,i,j} \theta_{2\ell}(x_j - x_k). \end{aligned}$$

Taking the sum over $i, j \geq 2$ we obtain

$$W_d \leq C_\varepsilon \ell^{4/3} \tilde{h}_1 W_c. \quad \text{(C.37)}$$

On the second hand we can show that

$$W_d \leq \frac{C}{\ell^2}. \quad \text{(C.38)}$$

Indeed, for every given $(x_1, \dots, x_N) \in \mathbb{R}^{2N}$, the product

$$U_\ell(x_i - x_j)\,(1 - \theta_{2\ell}(x_j - x_1)) \prod_{k \neq 1,i,j} \theta_{2\ell}(x_j - x_k)$$

is zero except in the case

$$|x_i - x_j| < \ell,\ |x_j - x_1| < 4\ell,\ \min_{k \neq 1,i,j} |x_j - x_k| > 2\ell. \tag{C.39}$$

By the triangle inequality, equation above implies that the ball $B(x_1, 5\ell)$ contains $B(x_i, \ell/2), B(x_j, \ell/2)$, and the balls $B(x_i, \ell/2), B(x_j, \ell/2)$ do not intersect with $B(x_k, \ell/2)$ for all $k \neq 1, i, j$. Since $B(x_1, 5\ell)$ can contain only a finite number of disjoint balls of radius $\ell/2$, we see that there are only a finite number of pairs (i, j) satisfying (C.39). Thus we can conclude that

$$W_d \leq C\|U_\ell\|_{L^\infty} \leq C\ell^{-2}.$$

From (C.36),(C.37),(C.38), we obtain

$$\begin{aligned} \tilde{h}_1 W_b + W_b \tilde{h}_1 &= \tilde{h}_1 W_d + W_d \tilde{h}_1 + 2\tilde{h}_1 W_c \\ &\geq -\frac{C\delta}{\ell^2}(\tilde{h}_1)^2 + \left(2 - \frac{C_\varepsilon \ell^{4/3}}{\delta}\right) \tilde{h}_1 W_c. \end{aligned}$$

Choosing $\delta \sim \ell^{4/3}$ we get

$$\tilde{h}_1 W_b + W_b \tilde{h}_1 \geq -\frac{C_\varepsilon}{\ell^{2/3}} (\tilde{h}_1)^2. \tag{C.40}$$

We can now put together the pieces and conclude the proof. From (C.35) and (C.40) we get

$$\tilde{h}_1 W_\beta + W_\beta \tilde{h}_1 \geq -\left(\frac{C}{N\ell^2} + \frac{C_\varepsilon}{\ell^{2/3}}\right)(\tilde{h}_1)^2 - 2N \sum_{1 \in \{i,j\}} U_\ell(x_i - x_j) \prod_{k \neq i,j} \theta_{2\ell}(x_j - x_k).$$

Summing the similar estimates with 1 replaced by m and using

$$\sum_{m=1}^{N} \sum_{m \in \{i,j\}} U_\ell(x_i - x_j) \prod_{k \neq i,j} \theta_{2\ell}(x_j - x_k) = 2W_\beta$$

we find that

$$\sum_{m=1}^{N} \left(\tilde{h}_m W_\beta + W_\beta \tilde{h}_m\right) \geq -\left(\frac{C}{N\ell^2} + \frac{C_\varepsilon}{\ell^{2/3}}\right) \sum_{m=1}^{N} (\tilde{h}_m)^2 - 2N W_\beta.$$

Therefore, coming back to our original equation (C.32) we can conclude that

$$
\begin{aligned}
&(\tilde{H}_N^\beta)^2 - \left(\sum_{i=1}^N \tilde{h}_i\right)^2 \\
&= \frac{\alpha_N(1-\varepsilon)^2}{N}\sum_{m=1}^N(\tilde{h}_m W_\beta + W_\beta\tilde{h}_m) + \frac{\alpha_N^2(1-\varepsilon)^4}{N^2}W_\beta^2 \\
&\geq -\frac{\alpha_N(1-\varepsilon)^2}{N}\left(\frac{C}{N\ell^2}+\frac{C_\varepsilon}{\ell^{2/3}}\right)\sum_{m=1}^N(\tilde{h}_m)^2 \\
&\qquad\qquad\qquad - 2\alpha_N(1-\varepsilon)^2 W_\beta + \frac{\alpha_N^2(1-\varepsilon)^4}{N^2}W_\beta^2 \pm N^2 \\
&\geq -\left(\frac{C}{N^2\ell^2}+\frac{C_\varepsilon}{\ell^{2/3}N}\right)\sum_{m=1}^N(\tilde{h}_m)^2 + \left(N - \frac{\alpha_N(1-\varepsilon)^2}{N}W_\beta\right)^2 - N^2 \\
&\geq -\left(\frac{C}{N^2\ell^2}+\frac{C_\varepsilon}{\ell^{2/3}N}\right)\sum_{m=1}^N(\tilde{h}_m)^2 - N^2.
\end{aligned}
$$

When $\ell \gg N^{-1}$ we have

$$\frac{C}{N^2\ell^2}+\frac{C_\varepsilon}{\ell^{2/3}N} \ll 1$$

and hence

$$(\tilde{H}_N^\beta)^2 \geq 2\sum_{1\leq i<j\leq N}\tilde{h}_i\tilde{h}_j + (1-o(1))\sum_{i=1}^N(\tilde{h}_i)^2 - N^2,$$

which yields the result, recalling that $\tilde{h} \geq 1$, i.e. $\sum_{1\leq i<j\leq N}\tilde{h}_i\tilde{h}_j \geq N^2$. □

C.4 Three body-estimate

Goal of this step is to remove the cut-off $\prod_{k\neq i,j}\theta_{2\ell}(x_j-x_k)$ in the potential W_β to obtain a lower bound in terms of a two-body potential. Using the elementary inequality

$$\prod_i(1-s_i) \geq 1 - \sum_i s_i$$

with $0 \leq s_i \leq 1$ for all i, we get

$$
\begin{aligned}
\prod_{k\neq i,j}\left[\theta_{2\ell}(x_j-x_k) \pm 1\right] &= \prod_{k\neq i,j}\left[1-(1-\theta_{2\ell}(x_j-x_k))\right] \\
&\geq 1 - \sum_{k\neq i,j}(1-\theta_{2\ell}(x_j-x_k)).
\end{aligned}
$$

Therefore, we have

$$W_\beta \geq \sum_{i\neq j}^N U_\ell(x_i-x_j) - \sum_{k\neq i\neq j}U_\ell(x_i-x_j)(1-\theta_{2\ell}(x_j-x_k)). \tag{C.41}$$

This implies that we have only a three-boby term to estimate. To remove this interaction, we make use of the second moment estimate lemma.

Lemma C.5. *For every* $0 < \varepsilon < 1$ *and* $s > 0$*, if* $\ell = \ell(N) \gg N^{-1}$*, then*

$$\sum_{i \neq j} U_\ell(x_i - x_j) \sum_{k \neq i,j} (1 - \theta_{2\ell}(x_j - x_k)) \leq C_{\varepsilon,s} \frac{\ell^{4/3}}{N} (\tilde{H}_N^\beta)^4 \tag{C.42}$$

Hence,

$$\tilde{H}_N^\beta \geq \sum_{j=1}^N \tilde{h}_j + \frac{\alpha_N (1-\varepsilon)^2}{N} \sum_{i \neq j} U_\ell(x_i - x_j) - C_{\varepsilon,s} \frac{\ell^{4/3}}{N^2} (\tilde{H}_N^\beta)^4 \tag{C.43}$$

Proof. By (C.24) and $\tilde{h}_i \geq 1$ we have

$$(1 - \theta_{2\ell}(x_2 - x_k)) \leq C \|(1 - \theta_{2\ell})\|_{L^{3/2}} (1 - \Delta_x) \leq C_{\varepsilon,s} \ell^{4/3} \tilde{h}_k$$

for $k \geq 3$. Since $U_\ell(x_1 - x_2)$ commutes with both sides, we get

$$\begin{aligned}
U_\ell(x_1 - x_2) &\sum_{k \geq 3} (1 - \theta_{2\ell}(x_2 - x_k)) \\
&\leq C_{\varepsilon,s} \ell^{4/3} U_\ell(x_1 - x_2) \sum_{k \geq 3} \tilde{h}_k \\
&= \frac{1}{2} C_{\varepsilon,s} \ell^{4/3} \big(\tilde{H}_N^\beta - \tilde{h}_1 - \tilde{h}_2 - (1-\varepsilon)^2 \frac{1}{N} W_\beta\big) U_\ell(x_1 - x_2) \\
&\quad + \frac{1}{2} C_{\varepsilon,s} \ell^{4/3} U_\ell(x_1 - x_2) \big(\tilde{H}_N^\beta - \tilde{h}_1 - \tilde{h}_2 - (1-\varepsilon)^2 \frac{1}{N} W_\beta\big) \\
&\leq \frac{1}{2} C_{\varepsilon,s} \ell^{4/3} \big(\tilde{H}_N^\beta U_\ell(x_1 - x_2) + U_\ell(x_1 - x_2) \tilde{H}_N^\beta\big) \\
&\quad - \frac{1}{2} C_{\varepsilon,s} \ell^{4/3} \sum_{j=1}^2 \big(\tilde{h}_j U_\ell(x_1 - x_2) + U_\ell(x_1 - x_2) \tilde{h}_j\big),
\end{aligned} \tag{C.44}$$

where in the last estimate we used that $W_\beta \geq 0$. Thanks to (C.26) and $\tilde{h}_i \geq 1$ we get for all $j = 1, 2$,

$$\begin{aligned}
\tilde{h}_j U_\ell(x_1 - x_2) &+ U_\ell(x_1 - x_2) \tilde{h}_j \\
&\geq -C_{\varepsilon,s} \Big[\|U_\ell\|_{L^2} + (1+s^2) \|U_\ell\|_{L^{4/3}}\Big] (1 - \Delta_1)(1 - \Delta_2) \\
&\geq -C_{\varepsilon,s} (\ell^{-1} + (1+s^2) \ell^{-2/3}) (1 - \Delta_1 + V)(1 - \Delta_2 + V) \\
&\geq -C_{\varepsilon,s} \ell^{-1} \tilde{h}_1 \tilde{h}_2.
\end{aligned} \tag{C.45}$$

On the other hand, by the Cauchy-Schwarz inequality (C.29), with $X = \tilde{H}_N^\beta U_\ell(x_1 -$

$x_2)^{1/2}$ and $Y = U_\ell(x_1 - x_2)^{1/2}$, $\tilde{h}_i \geq 1$ and (C.25) (with $\delta = 0$ and $W = U_\ell$)

$$\begin{aligned}\tilde{H}_N^\beta U_\ell(x_1 - x_2) + U_\ell(x_1 - x_2)\tilde{H}_N^\beta &\leq \delta \tilde{H}_N^\beta U_\ell(x_1 - x_2)\tilde{H}_N^\beta + \delta^{-1}U_\ell(x_1 - x_2) \\ &\leq C_{\varepsilon,s}\delta\|U_\ell\|_{L^1}\tilde{H}_N^\beta(1 - \Delta_1)(1 - \Delta_2)\tilde{H}_N^\beta \\ &\qquad + \delta^{-1}(1 - \Delta_1)(1 - \Delta_2) \\ &\leq C_{\varepsilon,s}\delta\tilde{H}_N^\beta\tilde{h}_1\tilde{h}_2\tilde{H}_N^\beta + C_{\varepsilon,s}\delta^{-1}\tilde{h}_1\tilde{h}_2\end{aligned} \tag{C.46}$$

for all $\delta > 0$. Choosing $\delta = N^{-1}$ and using that $\ell^{-1} \ll N$, then from (C.44),(C.45) and (C.46) we end up with

$$\begin{aligned}U_\ell(x_1 - x_2)\sum_{k\geq 3}(1 - \theta_{2\ell}(x_2 - x_k)) &\leq \frac{1}{2}C_{\varepsilon,s}\ell^{4/3}\big(\delta\tilde{H}_N^\beta\tilde{h}_1\tilde{h}_2\tilde{H}_N^\beta + \delta^{-1}\tilde{h}_1\tilde{h}_2\big) \\ &\qquad + \frac{1}{2}C_{\varepsilon,s}\ell^{4/3}\ell^{-1}\tilde{h}_1\tilde{h}_2 \\ &\leq C_{\varepsilon,s}\ell^{4/3}\big(N^{-1}\tilde{H}_N^\beta\tilde{h}_1\tilde{h}_2\tilde{H}_N^\beta + N\tilde{h}_1\tilde{h}_2\big).\end{aligned}$$

By symmetrization with respect to the indices, we find that

$$\sum_{i\neq j}U_\ell(x_i - x_j)\sum_{k\neq i,j}(1 - \theta_{2\ell}(x_j - x_k)) \leq C_{\varepsilon,s}\ell^{4/3}\Big(N^{-1}\tilde{H}_N^\beta\sum_{i\neq j}\tilde{h}_i\tilde{h}_j\tilde{H}_N^\beta + N\sum_{i\neq j}\tilde{h}_i\tilde{h}_j\Big).$$

Combining with the second moment estimate (C.31) we obtain

$$\begin{aligned}\sum_{i\neq j}U_\ell(x_i - x_j)\sum_{k\neq i,j}(1 - \theta_{2\ell}(x_j - x_k)) &\leq C_{\varepsilon,s}\ell^{4/3}\Big(N^{-1}\tilde{H}_N^\beta(\tilde{H}_N^\beta)^2\tilde{H}_N^\beta + N(\tilde{H}_N^\beta)^2 \\ &\qquad - N^{-1}\tilde{H}_N^\beta\sum_i\tilde{h}_i^2\tilde{H}_N^\beta - N\sum_i\tilde{h}_i^2\Big).\end{aligned}$$

and hence with $H_N^\beta \geq N$ and neglecting the negative terms on the second line we find (C.42). From the three-body estimate (C.42) and the inequality(C.41) we get the second bound (C.43), namely

$$\begin{aligned}\tilde{H}_N^\beta &= \sum_{i=1}^N\tilde{h}_i + \frac{\alpha_N(1-\varepsilon)^2}{N}W_\beta \\ &\geq \sum_{i=1}^N\tilde{h}_i + \frac{\alpha_N(1-\varepsilon)^2}{N}\sum_{i\neq j}^N U_\ell(x_i - x_j) \\ &\qquad - \frac{\alpha_N(1-\varepsilon)^2}{N}\sum_{k\neq i\neq j}U_\ell(x_i - x_j)(1 - \theta_{2\ell}(x_j - x_k)) \\ &\geq \sum_{i=1}^N\tilde{h}_i + \frac{\alpha_N(1-\varepsilon)^2}{N}\sum_{i\neq j}^N U_\ell(x_i - x_j) - C_{\varepsilon,s}\frac{\ell^{4/3}}{N^2}(\tilde{H}_N^\beta)^4.\end{aligned}$$

□

C.5 Energy lower bound and convergence of states

Using Lemma C.2, C.4 and C.5 we are able to prove the convergences in Theorem C.1. In particular we can eventually justify the mean-field approximation for the new Hamiltonian with the two-body interaction potential $U_\ell(x-y)$, which converges to a Dirac delta much slower than the original potential V_β. The proof follows directly from [60], however the analysis is simplified since we do not have a magnetic field. We recall the main result used here, namely Quantum de Finetti theorem, stated as in [60, Theorem 2.2].

Theorem C.3 (Quantum de Finetti). *Let $\mathfrak{h}$ be an arbitrary Hilbert space and let $\psi_N \in \bigotimes_{sym}^N \mathfrak{h}$ with $\|\psi_N\| = 1$. Assume that the sequence of one-particle density matrices $\gamma_{\psi_N}^{(1)}$ converges strongly in trace class when $N \to \infty$. Then, up to a subsequence, there exists a (unique) Borel probability measure μ on the unit sphere $S\mathfrak{h}$ in $\mathfrak{h}$, invariant under the action of S^1, such that*

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_{\psi_N}^{(k)} - \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u)\right| = 0, \tag{C.47}$$

for all $\kappa \in \mathbb{N}$.

Before, we define the energy functional $\mathcal{E}_{\mathrm{NLS}}^{\varepsilon,s}$ for $0<\varepsilon<1$ and $s>0$ by

$$\mathcal{E}_{\mathrm{NLS}}^{\varepsilon,s}[u] = \langle u, \tilde{h}u\rangle + (1-\varepsilon)^2 \frac{\widehat{V}(0)}{2}\int_{\mathbb{R}^2} |u|^4,$$

again one can prove that there exists a unique, positive minimizer φ_0. We aim to prove the following proposition

Proposition C.6 (Mean-field approximation). *Let $\tilde{H}_N^\beta$ be defined in* (C.22). *Assume $N^{-1} \ll \ell = \ell(N) \ll N^{-3/4}$ then for any $0<\varepsilon<1$ and $s>0$*

$$\lim_{N\to\infty} \frac{\inf \sigma(\tilde{H}_N^\beta)}{N} = \inf_{\|u\|_{L^2(\mathbb{R}^2)}=1} \mathcal{E}_{NLS}^{\varepsilon,s}. \tag{C.48}$$

We adapt to the 2d case the proof in [60, Proposition 4.1], with some simplification due to the fact that we do not include magnetic fields.

Proof. The upper bound in (C.48) can be obtained easily using trial states of the form $\varphi_0^{\otimes N}$, since $\lim_{N\to\infty}\int U_\ell(x)dx = \widehat{V}(0)/2$ from (C.9) and $\lim_{N\to\infty}\alpha_N = 1$. For the lower bound, let us consider a ground state $\tilde{\psi}_N$ of $\tilde{H}_N^\beta$ (which exists because $\tilde{h}$ has compact resolvent). Using the ground state equation, we find that

$$\langle \tilde{\psi}_N, (\tilde{H}_N^\beta)^k \tilde{\psi}_N\rangle = (\inf\sigma(\tilde{H}_N^\beta))^k \le (C_{\varepsilon,s}N)^k \tag{C.49}$$

for all $k\in\mathbb{N}$. In particular the second moment estimate (C.31) implies that

$$\langle \tilde{\psi}_N, \tilde{h}_1\tilde{h}_2\tilde{\psi}_N\rangle \le C_{\varepsilon,s} \tag{C.50}$$

and the operator estimate (C.43) implies that

$$\liminf_{N\to\infty} \frac{\langle \tilde{\psi}_N, \tilde{H}_N^{\beta} \tilde{\psi}_N \rangle}{N} \geq \liminf_{N\to\infty} \Big(\mathrm{Tr}(\tilde{h}\gamma^{(1)}_{\tilde{\psi}_N}) + \alpha_N (1-\varepsilon)^2 \mathrm{Tr}(U_\ell \gamma^{(2)}_{\tilde{\psi}_N}) \Big). \tag{C.51}$$

Here $\gamma^{(k)}_{\tilde{\psi}_N}$ is the k-particle density matrices of $\tilde{\psi}_N$ and U_ℓ is understood as the multiplication operator $U_\ell(x-y)$ on $L^2((\mathbb{R}^2)^2)$. The proof of the lower bound in (C.48) will be obtained by showing that

$$\begin{aligned} &\liminf_{N\to\infty} \Big(\mathrm{Tr}(\tilde{h}\gamma^{(1)}_{\tilde{\psi}}) + \alpha_N (1-\varepsilon)^2 \mathrm{Tr}(U_\ell \gamma^{(2)}_{\tilde{\psi}}) \Big) \\ &\qquad \geq \int_{S(L^2(\mathbb{R}^2))} \left(\langle u, \tilde{h} u \rangle + (1-\varepsilon)^2 \frac{\widehat{V}(0)}{2} \int_{\mathbb{R}^2} |u|^4 \right) d\tilde{\mu}(u)\,. \end{aligned} \tag{C.52}$$

We first notice that since $\mathrm{Tr}(\tilde{h}\gamma^{(1)}_{\tilde{\psi}_N})$ is bounded uniformly in N and $\tilde{h}$ has compact resolvent, up to a subsequence we can assume $\tilde{h}\gamma^{(1)}_{\tilde{\psi}_N}$ converges strongly in trace class. By the quantum de Finetti theorem up to a subsequence we can find a Borel probability measure $\tilde{\mu}$ on the unit sphere $S(L^2(\mathbb{R}^2))$ such that equation (C.47) holds. Since $\tilde{h}$ is positive and independent of N, (C.47) and Fatou's lemma imply

$$\liminf_{N\to\infty} \mathrm{Tr}(\tilde{h}\gamma^{(1)}_{\tilde{\psi}}) \geq \int_{S(L^2(\mathbb{R}^2))} \langle u, \tilde{h} u \rangle d\tilde{\mu}(u). \tag{C.53}$$

It remains to prove the lower bound for the term involving the two particle reduced density, i.e.

$$\liminf_{N\to\infty} \mathrm{Tr}(\alpha_N U_\ell \gamma^{(2)}_{\tilde{\psi}}) \geq \frac{\widehat{V}(0)}{2} \int_{\mathbb{R}^2} |u|^4 d\tilde{\mu}(u). \tag{C.54}$$

Since U_ℓ does depend on ℓ, and so on N, we cannot conclude immediately as before using quantum de Finetti and Fatou lemma. We proceed replacing U_ℓ by an operator bounded independently on N. In order to do it we localize the problem onto energy levels of the one-body Hamiltonian $\tilde{h}$ lying below a chosen cut-off Λ. Indeed, since $\tilde{h}$ has compact resolvent, for every $\Lambda \geq 1$ the projection

$$P_\Lambda := \mathbb{1}(\tilde{h} \leq \Lambda)$$

has finite rank. Let us denote

$$\Pi := \mathbb{1}_{L^2((\mathbb{R}^2)^2)} - P_\Lambda^{\otimes 2}.$$

Since $U_\ell \geq 0$, we can apply the Cauchy-Schwarz inequality for operators with

$$X = P_\Lambda^{\otimes 2} U_\ell^{1/2}, \quad Y = U_\ell^{1/2} \Pi$$

to obtain

$$\begin{aligned} U_\ell &= (P_\Lambda^{\otimes 2} + \Pi) U_\ell (P_\Lambda^{\otimes 2} + \Pi) \\ &= P_\Lambda^{\otimes 2} U_\ell P_\Lambda^{\otimes 2} + \Pi U_\ell \Pi + P_\Lambda^{\otimes 2} U_\ell \Pi + \Pi U_\ell P_\Lambda^{\otimes 2} \\ &\geq P_\Lambda^{\otimes 2} U_\ell P_\Lambda^{\otimes 2} - \delta^{-1} \Pi U_\ell \Pi + \delta P_\Lambda^{\otimes 2} U_\ell P_\Lambda^{\otimes 2} \end{aligned}$$

for all $\delta > 0$. Using the operator bound (C.25), and the fact that the 4/5-th power is operator monotone [6, Chapter 5] we have

$$U_\ell(x_1 - x_2) \leq C\|U_\ell\|_{L^1}(1-\Delta_1)^{4/5}(1-\Delta_2)^{4/5} \leq C_{\varepsilon,s}(\tilde{h}_1)^{4/5}(\tilde{h}_2)^{4/5}. \quad \text{(C.55)}$$

Therefore,

$$P_\Lambda^{\otimes 2} U_\ell P_\Lambda^{\otimes 2} \leq C_{\varepsilon,s}\tilde{h}_1\tilde{h}_2 \quad \text{and} \quad \Pi U_\ell \Pi \leq C_{\varepsilon,s}\Lambda^{-1/5}\tilde{h}_1\tilde{h}_2.$$

Here, in the second estimate we have used that $\Pi := \mathbb{1}_{L^2((\mathbb{R}^2)^2)} - P_\Lambda^{\otimes 2} \leq C_{\varepsilon,s}\Lambda^{-1/5}\tilde{h}^{1/5}$, which is a consequence of the definition of P_Λ. Thus,

$$U_\ell - P_\Lambda^{\otimes 2} U_\ell P_\Lambda^{\otimes 2} \geq -C_{\varepsilon,s}(\delta + \delta^{-1}\Lambda^{-1/5})\tilde{h}_1\tilde{h}_2.$$

If we choose $\delta = \Lambda^{-1/10}$ and take the trace against $\gamma^{(2)}_{\tilde{\psi}_N}$, then by the a-priori estimate (C.50) we find

$$\operatorname{Tr}(U_\ell \gamma^{(2)}_{\tilde{\psi}_N}) - \operatorname{Tr}(P_\Lambda^{\otimes 2} U_\ell P_\Lambda^{\otimes 2}\gamma^{(2)}_{\tilde{\psi}_N}) \geq -C_{\varepsilon,s}\Lambda^{-1/10}.$$

On the othere hand, from (C.55) and the definition of P_Λ it follows that the operator norm $P_\Lambda^{\otimes 2} U_\ell P_\Lambda^{\otimes 2}$ is bounded uniformly in N for Λ fixed. Since α_N is bounded uniformly in N, the strong convergence (C.47) implies that

$$\lim_{N\to\infty}\left(\operatorname{Tr}(P_\Lambda^{\otimes 2}\alpha_N U_\ell P_\Lambda^{\otimes 2}\gamma^{(2)}_{\tilde{\psi}_N}) - \int_{S(L^2(\mathbb{R}^2))}\langle (P_\Lambda^{\otimes 2}u), \alpha_N U_\ell (P_\Lambda^{\otimes 2}u)\rangle d\tilde{\mu}(u)\right) = 0.$$

Since the left side of (C.53) is finite, every function u in the support of $d\tilde{\mu}$ belongs to the quadratic form domain $Q(\tilde{h})$ of $\tilde{h}$ and hence $P_\Lambda u \to u$ strongly in $Q(\tilde{h})$. Using (C.25) and the continuous embeddings $Q(\tilde{h}) \subset H^1 \subset L^4$ we get

$$\lim_{\Lambda\to\infty}\lim_{\ell\to 0}\langle (P_\Lambda^{\otimes 2}u), \alpha_N U_\ell (P_\Lambda^{\otimes 2}u)\rangle = \lim_{\Lambda\to\infty}\|P_\Lambda u\|_{L^4}^4 = \|u\|_{L^4}^4.$$

By Fatou's lemma,

$$\liminf_{\Lambda\to\infty}\liminf_{N\to\infty}\int \langle (P_\Lambda^{\otimes 2}u), \alpha_N U_\ell (P_\Lambda^{\otimes 2}u)\rangle d\tilde{\mu}(u) \geq \frac{\widehat{V}(0)}{2}\int \|u\|_{L^4}^4 d\tilde{\mu}\,,$$

where we used that $\lim_{N\to\infty}\|U_\ell\|_1 = \widehat{V}(0)/2$ and $\lim_{N\to\infty}\alpha_N = 1$. So the convergence (C.54) follows. □

We are now ready to prove the convergence of the ground state energy stated in Theorem C.1.

Proof of energy convergence (C.8). The proof of the upper bound for a generic external potential V_{ext} can be obtained as in [51]. In our setup, with the bosons trapped in a torus, where the minimizer is exactly $\widehat{V}(0)/2$, it is sufficient to test the excitation Hamiltonian $\mathcal{G}^\beta_{N,\ell}$ in Eq. (2.45) on the vacuum in $\mathcal{F}_+^{\leq N}$.

We consider now the lower bound. We are considering $N^{-1} \ll \ell \ll N^{-3/4}$. From lemma (C.19) (which requires $\ell > N^{-\beta}$) and Proposition C.6 it follows that for every $0 < \varepsilon < 1$, $s > 0$

$$\liminf_{N\to\infty}\frac{\inf\sigma(H_N^\beta)}{N} \geq \liminf_{N\to\infty}\frac{\inf\sigma(\tilde{H}_N^\beta)}{N} - 1 = \inf_{\|u\|_{L^2}=1}\mathcal{E}^{\varepsilon,s}_{\text{NLS}}.$$

□

C.5.1 Convergence of density matrices

We are left with the proof of the convergence for approximate minimizers of H_N^β. Following [60] we use the Hellmann-Feynman principle. For $v \in L^2(\mathbb{R}^2)$ and $k \in \mathbb{N}$ we will perturb H_N^β by

$$S_{v,k} := \frac{k!}{N^{k-1}} \sum_{1 \le i_1 < \ldots < i_k \le N} |v^{\otimes k}\rangle\langle v^{\otimes k}|_{i_1,\ldots,i_k}.$$

Here $|v^{\otimes k}\rangle\langle v^{\otimes k}|_{i_1,\ldots,i_k}$ acting on the k-body Hilbert space of the i_1-th,..., i_k-th variables. We have the following extension of (C.8).

Lemma C.7 (Energy lower bound for perturbed Hamiltonian)**.** *Assumption as before. For every $v \in L^2(\mathbb{R}^2)$ and $k \in \mathbb{N}$, we have*

$$\liminf_{N\to\infty} \frac{\sigma(H_N^\beta - S_{v,k})}{N} \ge \inf_{\|u\|_{L^2}=1} \Big(\mathcal{E}_{NLS}[u] - |\langle v, u\rangle|^{2k} \Big).$$

Proof. Let $0 < \varepsilon < 1$ and $s > 0$ and $\ell \gg N^{-1}$. Recall that from (C.19) we have

$$H_N^\beta - S_{v,k} \ge \tilde{H}_N^\beta - S_{v,k} + N - C_{\varepsilon,s} N \ell^2. \tag{C.56}$$

Let ϕ_N be a ground state for $\tilde{H}_N^\beta - S_{v,k}$. Since $\|S_{v,k}\|/N$ is bounded uniformly in N, then for any $\ell = N^{-\gamma}$ for some $\gamma > 0$, Eq. (C.49) still holds with $\tilde{\psi}_N$ replaced by ϕ_N, namely

$$\langle \phi_N, (\tilde{H}_N^\beta)^n \phi_N \rangle \le (C_{\varepsilon,s} N)^n \tag{C.57}$$

for all $n \in \mathbb{N}$. Combining (C.57) with the three-body lemma C.5 we get the following analogue of (C.51)

$$\begin{aligned} \liminf_{N\to\infty} \frac{\sigma(\tilde{H}_N^\beta - S_{v,k})}{N} &= \liminf_{N\to\infty} \frac{\langle \phi_N, (\tilde{H}_N^\beta - S_{v,k})\phi_N\rangle}{N} \\ &\ge \liminf_{N\to\infty} \Big(\mathrm{Tr}(\tilde{h}\gamma_{\phi_N}^{(1)}) + \alpha_N (1-\varepsilon)^2 \mathrm{Tr}(U_\ell \gamma_{\phi_N}^{(2)}) - \mathrm{Tr}(|v^{\otimes k}\rangle\langle v^{\otimes k}|\gamma_{\phi_N}^{(k)}) \Big). \end{aligned} \tag{C.58}$$

Moreover, (C.57) and the second moment estimate (C.31) imply the a-priori estimate $\langle \phi_N, \tilde{h}_1 \tilde{h}_2 \phi_N \rangle \le C_{\varepsilon,s}$. Therefore, we can estimate the right-hand side of (C.58) by proceeding exactly as in the proof of Prop. C.6. More precisely, by the quantum de Finetti Theorem C.3 we can find a Borel probability measure μ_ϕ on the unit sphere $S\mathfrak{h}$ such that, up to a subsequence,

$$\lim_{N\to\infty} \mathrm{Tr} \left| \gamma_{\phi_N}^{(k)} - \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu_{\phi_N}(u) \right| = 0,$$

for all $k \in \mathbb{N}$. Using (C.52) with $\tilde{\psi}_N$ replaced by ϕ_N and employing the fact that $|v^{\otimes \ell}\rangle\langle v^{\otimes \ell}|$ is bounded, we obtain

$$\begin{aligned} \liminf_{N\to\infty} &\Big(\mathrm{Tr}(\tilde{h}\gamma_{\phi_N}^{(1)}) + \alpha_N(1-\varepsilon)^2 \mathrm{Tr}(U_\ell \gamma_{\phi_N}^{(2)}) - \mathrm{Tr}(|v^{\otimes k}\rangle\langle v^{\otimes k}|\gamma_{\phi_N}^{(k)}) \Big) \\ &\ge \int \left(\langle u, \tilde{h} u\rangle + (1-\varepsilon)^2 \frac{\widehat{V}(0)}{2} \int_{\mathbb{R}^2} |u|^4 - |\langle v, u\rangle|^{2k} \right) d\tilde{\mu}_{\phi_N}(u). \end{aligned} \tag{C.59}$$

From (C.56), (C.58) and (C.59), it follows that

$$\liminf_{N\to\infty} \frac{\sigma(H_N^\beta - S_{v,k})}{N}$$
$$\geq \inf_{\|u\|_{L^2}=1} \left(\langle u, (\tilde{h}-1)u\rangle + (1-\varepsilon)^2 \frac{\widehat{V}(0)}{2} \int_{\mathbb{R}^2} |u|^4 - |\langle v,u\rangle|^{2k} \right) d\tilde{\mu}_{\phi_N}(u).$$

Finally, by a standard compactness argument from [47] we have that $-\Delta+1$ has compact resolvent, then the limit

$$\lim_{\varepsilon\to 0}\lim_{s\to\infty}\left(\inf_{\|u\|_{L^2}=1}\left(\langle u,(\tilde{h}-1)u\rangle + (1-\varepsilon)^2\frac{\widehat{V}(0)}{2}\int_{\mathbb{R}^2}|u|^4 - |\langle v,u\rangle|^{2k}\right)d\tilde{\mu}_{\phi_N}(u)\right)$$
$$= \inf_{\|u\|_{L^2}=1}\left(\mathcal{E}_{\mathrm{NLS}}[u] - |\langle v,u\rangle|^{2k}\right).$$

□

Now we prove convergence of density matrices.

Proof of state convergence. Let ψ_N be an approximate ground state for H_N^β as in Theorem C.1. For every $v \in L^2(\mathbb{R}^2)$ and $k \in \mathbb{N}$, from the upper bound in (C.8) and the lower bound from the previous lemma we have

$$\begin{aligned}\limsup_{N\to\infty} \mathrm{Tr}(|v^{\otimes k}\rangle\langle v^{\otimes k}|\gamma_N^{(k)}) &= \limsup_{N\to\infty}\left(\frac{\langle\psi_N, H_N^\beta\psi_N\rangle}{N} - \frac{\langle\psi_N,(H_N^\beta - S_{v,k})\psi_N\rangle}{N}\right)\\ &\leq \limsup_{N\to\infty}\left(\frac{\inf\sigma(H_N^\beta)}{N} - \frac{\inf\sigma(H_N^\beta - S_{v,k})}{N}\right)\\ &\leq \inf_{\|u\|_{L^2}=1}\mathcal{E}_{\mathrm{NLS}} - \inf_{\|u\|_{L^2}=1}\left(\mathcal{E}_{\mathrm{NLS}} - |\langle v,u\rangle|^{2k}\right).\end{aligned}$$

Here v is not necessarily normalized. Therefore we can replace v by $\lambda^{1/(2k)}v$ with $\lambda > 0$ and obtain

$$\limsup_{N\to\infty}\mathrm{Tr}(|v^{\otimes k}\rangle\langle v^{\otimes k}|\gamma_{\psi_N}^{(k)}) \leq \frac{1}{\lambda}\left(\inf_{\|u\|_{L^2}=1}\mathcal{E}_{\mathrm{NLS}} - \inf_{\|u\|_{L^2}=1}\left(\mathcal{E}_{\mathrm{NLS}} - \lambda|\langle v,u\rangle|^{2k}\right)\right).$$

With given v and k, for every $\lambda > 0$ let u_λ be a normalized minimizer for $u \mapsto \mathcal{E}_{\mathrm{NLS}} - \lambda|\langle v,u\rangle|^{2k}$. Since $\langle u_\lambda, hu_\lambda\rangle$ is bounded and h has compact resolvent, there exists a subsequence $\lambda_j \to 0$ such that u_{λ_j} converges to u_0 in L^2. By Fatou's lemma, u_0 is a minimizer of $\mathcal{E}_{\mathrm{NLS}}$, hence

$$\limsup_{j\to\infty}\frac{1}{\lambda_j}\left(\inf_{\|u\|_{L^2}=1}\mathcal{E}_{\mathrm{NLS}} - \inf_{\|u\|_{L^2}=1}\left(\mathcal{E}_{\mathrm{NLS}}[u] - \lambda_j|\langle v,u\rangle|^{2k}\right)\right)$$
$$\leq \limsup_{j\to\infty}\frac{1}{\lambda}\left(\mathcal{E}_{\mathrm{NLS}}[u_{\lambda_j}] - \left(\mathcal{E}_{\mathrm{NLS}}[u_{\lambda_j}] - \lambda_j|\langle v,u_{\lambda_j}\rangle|^{2k}\right)\right) = |\langle v,\varphi_0\rangle|^{2k}.$$

We used the uniqueness of the minimizer of $\mathcal{E}_{\rm NLS}$. This implies that for any $v \in L^2(\mathbb{R}^2)$ and $k \in \mathbb{N}$

$$\limsup_{N\to\infty} \operatorname{Tr}(|v^{\otimes k}\rangle\langle v^{\otimes k}|\gamma_{\psi_N}^{(k)}) \le |\langle v,\varphi_0\rangle|^{2k}.$$

Now we can conclude the convergence of density matrices using the quantum de Finetti theorem. In fact, by Theorem C.3, up to a subsequence of ψ_N, there exists a probability measure μ on the unit sphere $S(L^2(\mathbb{R}^2))$ such that

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_N^{(k)} - \int_{S(L^2(\mathbb{R}^2))} |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u)\right| = 0,$$

for all $k \in \mathbb{N}$. To conclude the proof we will show that μ is supported on the set of minimizers of $\mathcal{E}_{\rm NLS}$, called $\mathcal{M}_{NLS}$. To show it, we assume by contradiction that there exists $v_0 \in L^2(\mathbb{R}^2)$ in the support of μ but $v_0 \notin \mathcal{M}_{\rm NLS}$. Denoting by B the set of all points in the support of μ within a L^2-distance less than δ from v_0, i.e. $\|v - v_0\|_2 \le \delta$. We claim that we could then find $\delta \in (0, 1/2)$ such that

$$|\langle v,\varphi_0\rangle| \le 1 - 3\delta^2, \tag{C.60}$$

for all $v \in B$. Indeed, if that was not the case, we would have two sequences in the support of μ strongly converging in $L^2(\mathbb{R}^2)$

$$v_n \to v_0, \quad u_n \to \varphi_0 \in \mathcal{M}_{NLS}$$

with $\|u_n - v_n\| \to 0$, and thus $v_0 \in \mathcal{M}_{\rm NLS}$. Here we have used that $\mathcal{M}_{\rm NLS}$ is a compact subset of $L^2(\mathbb{R}^2)$. On the other hand by triangle inequality,

$$|\langle v,u\rangle| \ge \frac{1}{2}(\|u\|^2 + \|v\|^2 - \|u-v\|^2) \ge 1 - 2\delta^2, \tag{C.61}$$

for all $u, v \in B$. From (C.60) and (C.61) we find that

$$\begin{aligned}(\mu(B))^2(1-2\delta^2)^{2k} &\le \int_B\int_B |\langle v,u\rangle|^{2k} d\mu(u)d\nu(v)\\ &\le \int_B |\langle v,\varphi_0\rangle|^{2k} d\mu(v) \le \mu(B)(1-3\delta^2)^{2k}\end{aligned}$$

for all $k \in \mathbb{N}$. Taking the limit $k \to \infty$, we have $\mu(B) = 0$. However, it contradicts the fact that v_0 belongs to the support of μ and μ is a Borel measure. Thus we can conclude that μ is supported on $\mathcal{M}_{\rm NLS}$ and the proof is complete. □

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