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A quasineutral type limit for the Navier–Stokes–Poisson system with large data

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Abstract
In this paper we investigate a quasineutral type limit for the Navier–Stokes–Poisson system. We prove that the projection of the approximating velocity fields on the divergence-free vector field is relatively compact and converges to a Leray weak solution of the incompressible Navier–Stokes equation. By exploiting the wave equation structure of the density fluctuation we achieve the convergence of the approximating sequences by means of a dispersive estimate of the Strichartz type.

Mathematics Subject Classification: 35Q35, 35Q30 (76D05, 76W05, 76X05)

1. Introduction

This paper is concerned with the analysis of a vanishing Debye length type limit for a coupled Navier–Stokes–Poisson system in 3D. Namely, we investigate the behaviour of the solutions of the following initial value problem on the whole $\mathbb{R}^3$, when λ vanishes to zero,

$$\partial_s \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0, \tag{1}$$

$$\partial_s(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + \frac{1}{\gamma}\nabla(\rho^\lambda)^\gamma = \overline{\mu}\Delta u^\lambda + (\overline{\nu} + \overline{\mu})\nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda, \tag{2}$$

$$\lambda^2 \Delta V^\lambda = \rho^\lambda - 1. \tag{3}$$

We denote by $x \in \mathbb{R}^3$, $s \geqslant 0$, the space and time variable, $\rho(x,t)$ the *negative charge density*, $m(x,t) = \rho(x,t)u(x,t)$ the *current density*, $u(x,t)$ the *velocity vector density*, $V(x,t)$ the *electrostatic potential* and μ, ν the *shear viscosity* and *bulk viscosity*, respectively. The parameter λ is the so-called *Debye length* (up to a constant factor).

The Navier–Stokes–Poisson system is a simplified model (for instance, the temperature equation is not taken into account) to describe the dynamics of a plasma where the compressible electron fluid interacts with its own electric field against a constant charged ion background (see Degond [5]). We recall that the Debye length is a characteristic physical parameter related to the phenomenon of the so-called 'Debye shielding'. Any charged particle inside a plasma attracts other particles with opposite charge and repels those with the same charge, thereby creating a net cloud of opposite charges around itself. This cloud shields the particle's own charge from external view; it causes the particle's Coulomb field to fall off exponentially at large radii, rather than falling off as $1/r^2$. This phenomenon was studied by Debye (1912). The physical meaning of the Debye length λ is the 'screening' distance or the distance over which the usual Coulomb field $1/r$ is killed off exponentially by the polarization of the plasma.

This type of limit has been studied by many authors. In the case of the Euler–Poisson system by Cordier and Grenier [4], Grenier [17], Cordier *et al* [3], Loeper [23], Peng *et al* [25], in the case of a Navier–Stokes–Poisson system by Wang [32] and Jiang and Wang [18] and in the context of a combined quasineutral and relaxation time limit by Gasser and Marcati in [13–15]. This paper is still a mathematical theoretical approach to this complicated physical problem which however removes many regularity and smallness assumptions of various papers in the literature, see for instance Wang [32] and Jiang and Wang [18].

Our approach is based on the idea of estimating the behaviour of the acoustic waves as the parameter λ goes to zero; in particular, we exploit the structure of the wave equations satisfied by the fluctuation density. Our singular analysis has some similarities with the low Mach number limit, see the paper by Lions and Masmoudi [22], Desjardins *et al* [7], Desjardins and Grenier [6]. The limiting behaviour analysis is very hard because of the presence of very stiff terms due to the scaled electric field. In fact, because of the incompressible limit regime it is necessary to introduce a time scaling but the singularity introduced by the coupling electric field leads to the acoustic waves. In order to handle these difficulties the system (1)–(3) will be studied as a semilinear wave equation and we will get uniform estimates in λ by the use of the L^p-type estimates due to Strichartz [16, 19, 30]. The particular type of Strichartz estimates that we are going to use here can be found in the book of Sogge [28] or deduced by the so-called bilinear estimates of Klainerman and Machedon [20] and Foschi and Klainerman [12]. In this way we get sufficient bounds in order to study the limiting behaviour of the velocity vector field. In particular we will separately analyse the limiting behaviour of the divergence free part and the gradient part of u^λ. Similar techniques have already been used in [9]. This paper is organized as follows. In section 2 we recall the mathematical tools needed in the paper and recall same basic definitions. In section 3 we set up our problem, explain our approach and state our main result. Section 4 is devoted to recovering the *a priori* estimates needed to get the strong convergence of the approximating sequences and to prove the main theorem. In section 5 we prove the strong convergence of the velocity vector field. Finally, in section 6 we give the proof of the main result.

2. Preliminaries

For the convenience of the reader we establish some notation and recall some basic facts that will be useful in the following.

If F, G are functions we denote by $F \lesssim G$ the fact that there exists $c \in \mathbb{R}$ such that $F \leqslant G$.

We will denote by $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}_+)$ the space of test function $C_0^\infty(\mathbb{R}^d \times \mathbb{R}_+)$, by $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+)$ the space of Schwartz distributions and $\langle \cdot, \cdot \rangle$ the duality bracket between $\mathcal{D}'$ and $\mathcal{D}$. Moreover $W^{k,p}(\mathbb{R}^d) = (I - \Delta)^{-\frac{k}{2}} L^p(\mathbb{R}^d)$ and $H^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d)$ denote the nonhomogeneous Sobolev spaces, for any $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{R}$. $\dot{W}^{k,p}(\mathbb{R}^d) = (-\Delta)^{-\frac{k}{2}} L^p(\mathbb{R}^d)$ and $\dot{H}^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d)$ denote the homogeneous Sobolev spaces. The notation $L_t^p L_x^q$ and $L_t^p W_x^{k,q}$ will abbreviate, respectively, the spaces $L^p([0,T]; L^q(\mathbb{R}^d))$, and $L^p([0,T]; W^{k,q}(\mathbb{R}^d))$. We denote by $L^p_2(\mathbb{R}^d)$ the Orlicz space defined as follows:

$$L^p_2(\mathbb{R}^d) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \mid |f|\chi_{|f|\leqslant\frac{1}{2}} \in L^2(\mathbb{R}^d),\ |f|\chi_{|f|>\frac{1}{2}} \in L^p(\mathbb{R}^d)\}, \tag{4}$$

see [1, 21] for more details. We shall denote by Q and P, respectively, Leray's projectors Q on the space of gradient vector fields and P on the space of divergence-free vector fields. Namely,

$$Q = \nabla\Delta^{-1}\operatorname{div} \qquad P = I - Q. \tag{5}$$

It is well known that Q and P can be expressed in terms of Riesz multipliers; therefore, they are bounded linear operators on every $W^{k,p}$ $(1 < p < \infty)$ space (see [29]).

Let us recall that if w is a (weak) solution of the following wave equation in the space $[0,T]\times\mathbb{R}^d$

$$\begin{cases} \left(-\dfrac{\partial^2}{\partial t} + \Delta\right) w(t,x) = F(t,x), \\ w(0,\cdot) = f, \qquad \partial_t w(0,\cdot) = g, \end{cases}$$

for some data f, g, F and $0 < T < \infty$, then w satisfies the following Strichartz estimates (see [16, 19]):

$$\|w\|_{L^q_t L^r_x} + \|\partial_t w\|_{L^q_t W^{-1,r}_x} \lesssim \|f\|_{\dot H^\gamma_x} + \|g\|_{\dot H^{\gamma-1}_x} + \|F\|_{L^{\tilde q'}_t L^{\tilde r'}_x}, \tag{6}$$

where (q,r), $(\tilde q, \tilde r)$ are *wave admissible* pairs, namely, they satisfy

$$\frac{2}{q} \leqslant (d-1)\left(\frac{1}{2} - \frac{1}{r}\right) \qquad \frac{2}{\tilde q} \leqslant (d-1)\left(\frac{1}{2} - \frac{1}{\tilde r}\right),$$

and moreover the following conditions hold:

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma = \frac{1}{\tilde q'} + \frac{d}{\tilde r'} - 2.$$

Beside the Strichartz estimate (6) in the case of $d = 3$ (see [28]), there is a less standard estimate, related to an earlier linear Strichartz [30] estimate, namely,

$$\|w\|_{L^4_{t,x}} + \|\partial_t w\|_{L^4_t W^{-1,4}_x} \lesssim \|f\|_{\dot H^{1/2}_x} + \|g\|_{\dot H^{1/2}_x} + \|F\|_{L^1_t L^2_x}. \tag{7}$$

This estimate follows from the homogeneous case by a standard application of the Duhamel principle. It was first obtained by Strichartz [30] (see [28, p 72, formula (4.6)]. Alternatively, it could be derived by using the bilinear estimates in Foschi and Klainerman [12, corollary 13.4] by following step by step the same computations as in Klainerman and Machedon [20] (see the proof of theorem 2.2, p 1237) by replacing the inequality in corollary 2.8 with the one of corollary 13.4 of [12].

It is straightforward to observe that for any $s \geqslant 0$ this estimate also holds:

$$\|w\|_{L^4_t W^{-s,4}_x} + \|\partial_t w\|_{L^4_t W^{-1-s,4}_x} \lesssim \|f\|_{H^{1/2-s}_x} + \|g\|_{H^{-1/2-s}_x} + \|F\|_{L^1_t H^{-s}_x}. \tag{8}$$

(It is sufficient to apply the operator $(I-\Delta)^{-s/2}$ to (7)). In conclusion we state the following elementary lemma that will be used later on.

Lemma 2.1. *Let us consider a smoothing kernel $j \in C^\infty_0(\mathbb{R}^d)$, such that $j \geqslant 0$, $\int_{\mathbb{R}^d} j\,\mathrm{d}x = 1$, and let us define*

$$j_\alpha(x) = \alpha^{-d} j\left(\frac{x}{\alpha}\right).$$

Then for any $f \in \dot H^1(\mathbb{R}^d)$, one has

$$\|f - f * j_\alpha\|_{L^p(\mathbb{R}^d)} \leqslant C_p \alpha^{1-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|\nabla f\|_{L^2(\mathbb{R}^d)}, \tag{9}$$

where

$$p \in [2, \infty) \quad \text{if } d = 2, \qquad p \in [2, 6] \quad \text{if } d = 3.$$

Moreover the following Young type inequality holds:

$$\| f * j_\alpha \|_{L^p(\mathbb{R}^d)} \leqslant C \alpha^{s - d\left(\frac{1}{q} - \frac{1}{p}\right)} \| f \|_{W^{-s,q}(\mathbb{R}^d)}, \tag{10}$$

for any $p, q \in [1, \infty]$, $q \leqslant p$, $s \geqslant 0$, $\alpha \in (0, 1)$.

3. Statement of the problem and main result

We rewrite here the compressible Navier–Stokes equation coupled with the Poisson equation:

$$\begin{cases} \partial_s \rho^\lambda + \operatorname{div}(\rho^\lambda u^\lambda) = 0, \\ \partial_s(\rho^\lambda u^\lambda) + \operatorname{div}(\rho^\lambda u^\lambda \otimes u^\lambda) + \dfrac{1}{\gamma} \nabla (\rho^\lambda)^\gamma = \overline{\mu} \Delta u^\lambda + (\overline{\nu} + \overline{\mu}) \nabla \operatorname{div} u^\lambda + \rho^\lambda \nabla V^\lambda, \\ \lambda^2 \Delta V^\lambda = \rho^\lambda - 1. \end{cases} \tag{11}$$

As already discussed in the introduction our aim is to study the limiting behaviour of the system (11) as $\lambda \to 0$, namely, a *quasineutral type limit*. Formally, if we set $\lambda = 0$, then we obtain $\rho = 1$ which is the so-called quasineutrality regime in plasma physics and the behaviour of the fluid can be described by the incompressible Navier–Stokes system. The present limit analysis has a very strong analogy with the theory of incompressible limits widely investigated on mathematical fluid dynamics. In particular low Mach number limits have been studied by several authors, among which we recall [7, 22, 24]. The quasineutral limit yields to the introduction of a time scaling because of the incompressible limit regime; in addition there is an electric potential scaling which is responsible for a very singular term which requires a more careful analysis of the acoustic waves. The incompressible limit scaling is given by

$$\rho^\varepsilon(x, t) = \rho^\lambda\left(x, \frac{t}{\varepsilon}\right), \quad u^\varepsilon = \frac{1}{\varepsilon} u^\lambda \left(x, \frac{t}{\varepsilon}\right), \quad V^\varepsilon = V^\lambda\left(x, \frac{t}{\varepsilon}\right), \quad \overline{\mu} = \varepsilon\mu, \quad \overline{\nu} = \varepsilon\nu. \tag{12}$$

With scaling (12) system (11) becomes

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \dfrac{\nabla(\rho^\varepsilon)^\gamma}{\gamma \varepsilon^2} = \mu \Delta u^\varepsilon + (\nu + \mu) \nabla \operatorname{div} u^\varepsilon + \dfrac{\rho^\varepsilon}{\varepsilon^2} \nabla V^\varepsilon, \\ \lambda^2 \Delta V^\varepsilon = \rho^\varepsilon - 1. \end{cases} \tag{13}$$

Our analysis is performed under the assumption that the previous small parameter ε is related to the Debye length λ (after suitable renormalization of the physical units) by the power law

$$\varepsilon^\beta = \lambda^2, \qquad \text{where } \beta > 0. \tag{14}$$

To simplify our notation from now on we will set

$$\pi^\varepsilon = \frac{(\rho^\varepsilon)^\gamma - 1 - \gamma(\rho^\varepsilon - 1)}{\varepsilon^2 \gamma(\gamma - 1)}.$$

System (13) is endowed with the following initial conditions:

$$\begin{aligned} & \rho^\varepsilon|_{t=0} = {\rho^\varepsilon}_0 \geqslant 0, \qquad V^\varepsilon|_{t=0} = V_0^\varepsilon, \\ & \rho^\varepsilon u^\varepsilon|_{t=0} = m_0^\varepsilon, \qquad m_0^\varepsilon = 0 \text{ on } \{x \in \mathbb{R}^3 \mid {\rho^\varepsilon}_0(x) = 0\}, \\ & \int_{\mathbb{R}^3} \left(\pi^\varepsilon|_{t=0} + \frac{|m_0^\varepsilon|^2}{2{\rho^\varepsilon}_0} + \varepsilon^{\beta-2} |V_0^\varepsilon|^2 \right) \mathrm{d}x \leqslant C_0, \\ & \frac{m_0^\varepsilon}{\sqrt{{\rho^\varepsilon}_0}} \rightharpoonup u_0 \quad \text{weakly in } L^2(\mathbb{R}^3). \end{aligned} \tag{ID}$$

The existence of global weak solutions for fixed $\varepsilon > 0$ for system (13) has been proved in the case of a bounded domain in [8] and in the case of a whole domain in [10] and [11]. We summarize this existence result in the following theorem.

Theorem 3.1. *Assume (ID) and let $\gamma > 3/2$, then there exists a global weak solution $(\rho^\varepsilon, u^\varepsilon, V^\varepsilon)$ to (13) such that $\rho^\varepsilon - 1 \in L^\infty((0,T); L_2^\gamma(\mathbb{R}^3))$, $\sqrt{\rho^\varepsilon} u^\varepsilon \in L^\infty((0,T); L^2(\mathbb{R}^3))$, $u^\varepsilon \in L^2((0,T); W^{1,2}(\mathbb{R}^3))$. Furthermore*

- *the energy inequality holds for almost every $t \geqslant 0$,*

$$\int_{\mathbb{R}^3} \left(\rho^\varepsilon \frac{|u^\varepsilon|^2}{2} + \pi^\varepsilon + \varepsilon^{\beta-2} |\nabla V^\varepsilon|^2 \right) \mathrm{d}x + \int_0^t \int_{\mathbb{R}^3} \left(\mu |\nabla u^\varepsilon|^2 + (\nu + \mu) |\mathrm{div}\, u^\varepsilon|^2 \right) \mathrm{d}x\, \mathrm{d}s \leqslant C_0, \tag{15}$$

- *the continuity equation is satisfied in the sense of renormalized solutions, i.e.*

$$\partial_t b(\rho^\varepsilon) + \mathrm{div}\,(b(\rho^\varepsilon) u) + (b'(\rho^\varepsilon)\rho^\varepsilon - b(\rho^\varepsilon))\, \mathrm{div}\, u^\varepsilon = 0,$$

  for any $b \in C^1(\mathbb{R}^3)$ such that

$$b'(z) = \text{constant}, \qquad \text{for any } z \text{ large enough, say } z \geqslant M,$$

- *system (13) holds in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$.*

For the sake of completeness we recall here some definitions and results concerning our limiting system, namely, the incompressible Navier–Stokes equations,

$$\begin{cases} \partial_t u + \mathrm{div}\,(u \otimes u) - \mu \Delta u = \nabla p + f, \\ \mathrm{div}\, u = 0, \\ u(x,0) = u_0, \end{cases} \tag{16}$$

where $(x,t) \in \mathbb{R}^3 \times [0,T]$, $u \in \mathbb{R}^3$ denotes the velocity vector field, $p \in \mathbb{R}$ the pressure of the fluid, $f \in \mathbb{R}^3$ is a given external force and μ is the kinematic viscosity. Before stating our main result, let us recall (see Lions [21] and Temam [31]) the notion of the Leray weak solution.

Definition 3.2. *We say that u is a Leray weak solution of the Navier–Stokes equation if it satisfies (16) in the sense of distributions, namely,*

$$\int_0^T \int_{\mathbb{R}^d} \left(\nabla u \cdot \nabla \varphi - u_i u_j \partial_i \varphi_j - u \cdot \frac{\partial \varphi}{\partial t} \right) \mathrm{d}x\, \mathrm{d}t = \int_0^T \langle f, \varphi \rangle_{H^{-1} \times H_0^1} \mathrm{d}x\, \mathrm{d}t + \int_{\mathbb{R}^d} u_0 \cdot \varphi\, \mathrm{d}x,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d \times [0,T])$, $\mathrm{div}\, \varphi = 0$ and

$$\mathrm{div}\, u = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^d \times [0,T]),$$

and the following energy inequality holds:

$$\frac{1}{2} \int_{\mathbb{R}^d} |u(x,t)|^2 \mathrm{d}x + \mu \int_0^t \int_{\mathbb{R}^d} |\nabla u(x,t)|^2 \mathrm{d}x\, \mathrm{d}s \leqslant \frac{1}{2} \int_{\mathbb{R}^d} |u_0|^2 \mathrm{d}x + \int_0^t \langle f, u \rangle_{H^{-1} \times H_0^1} \mathrm{d}s, \qquad \text{for all } t \geqslant 0. \tag{17}$$

There exist in the mathematical literature several results concerning the existence of Leray weak solutions to the Navier–Stokes equations; for example, we can refer to the books of Lions [21] and Temam [31]. The case $d = 3$ is a major open problem and a considerably more difficult case than the case $d = 2$, since the bound on the L^2 norm (kinetic energy) provides only a control on a supercritical norm and does not provide any information concerning the critical controlling (and scaling invariant) norm L^3. Hence we do not know (opposite to the case $d = 2$) whether or not the Leray weak solutions are unique, unless (see Serrin [26]) we assume a control on the L^3 norm. Some important regularity results can be found in [2].

Now we are ready to state our main result. The convergence of $\{u^\varepsilon\}$ will be described by analysing the convergence of the associated Hodge decomposition.

Theorem 3.3. *Let $(\rho^\varepsilon, u^\varepsilon, V^\varepsilon)$ be a sequence of weak solutions in $\mathbb{R}^3$ of system (13); assume that the initial data satisfy (ID). Then*

(i) $\rho^\varepsilon \longrightarrow 1$ strongly in $L^\infty([0,T]; L^k_2(\mathbb{R}^3))$,

(ii) there exists $u \in L^\infty([0,T]; L^2(\mathbb{R}^3)) \cap L^2([0,T]; \dot{H}^1(\mathbb{R}^3))$ *such that*

$$u^\varepsilon \rightharpoonup u \qquad \text{weakly in } L^2([0,T]; \dot{H}^1(\mathbb{R}^3)),$$

(iii) the gradient component Qu^ε of the vector field u^ε satisfies

$$Qu^\varepsilon \longrightarrow 0 \qquad \text{strongly in } L^2([0,T]; L^p(\mathbb{R}^3)), \quad \text{for any } p \in [4,6),$$

provided $\beta < 1/2$,

(iv) the divergence-free component Pu^ε of the vector field u^ε satisfies

$$Pu^\varepsilon \longrightarrow Pu = u \qquad \text{strongly in } L^2([0,T]; L^2_{\text{loc}}(\mathbb{R}^3)),$$

(v) $u = Pu$ is a Leray weak solution to the incompressible Navier–Stokes equation:

$$P(\partial_t u - \Delta u + (u\cdot\nabla)u) = 0 \quad \text{in } \mathcal{D}'([0,T]\times\mathbb{R}^3), \tag{18}$$

provided that

$$\begin{cases} \beta = \min\left\{\dfrac{1}{2}, -2 + \dfrac{4}{\gamma}\right\} & \text{if } \gamma < 2,\\[2ex] 0 < \beta < \min\left\{\dfrac{1}{2}, \mu\left(\dfrac{3}{q} - \dfrac{1}{2}\right), \dfrac{1}{6} - \dfrac{2}{3}\mu\left(s_0 + \dfrac{7}{4} - \dfrac{3}{q}\right)\right\} & \text{if } \gamma \geqslant 2,\end{cases}$$

where $\mu > 0$, $s_0 \geqslant 3/2$ and $4 \leqslant q < 6$.

Remark 3.4. The hypotheses (ID) do not allow us to recover the energy inequality (17) for the limiting solution u of the incompressible Navier–Stokes system (16). Moreover if we assume the following conditions on the initial data, namely, that

$$\pi^\varepsilon|_{t=0} \to 0 \qquad \text{strongly in } L^\infty([0,T]; L^1(\mathbb{R}^3)), \quad \text{as } \varepsilon\gamma \to 0, \tag{19}$$

$$\varepsilon^{\frac{\beta}{2}-1}\nabla V_0 \to 0 \qquad \text{strongly in } L^\infty([0,T]; L^2(\mathbb{R}^3)), \quad \text{as } \varepsilon \to 0, \tag{20}$$

then, we are able to recover the energy inequality (17). In fact, now, by using the hypotheses (ID) with (19), (20) and the weak lower semicontinuity of the weak limits we get

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{1}{2}|u(x,t)|^2\,\mathrm{d}x + \int_0^T\!\!\int_{\mathbb{R}^3} \mu|\nabla u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t &\leqslant \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^3}\left(\rho^\varepsilon\frac{|u^\varepsilon|^2}{2} + \pi^\varepsilon + \varepsilon^{\beta-2}|\nabla V^\varepsilon|^2\right)\mathrm{d}x\\ &\quad + \liminf_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^3}\left(\mu|\nabla u^\varepsilon|^2 + (\nu+\mu)|\operatorname{div} u^\varepsilon|^2\right)\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^3}\left(\pi^\varepsilon|_{t=0} + \frac{|m_0^\varepsilon|^2}{2\rho^\varepsilon{}_0} + \varepsilon^{\beta-2}|\nabla V_0|^2\right)\mathrm{d}x = \int_{\mathbb{R}^3}\frac{1}{2}|u_0|^2\,\mathrm{d}x.\end{aligned} \tag{21}$$

4. *A priori* estimates

In this section we wish to establish all the *a priori* estimates, independent of ε, for the solutions of system (13) which are necessary to prove theorem 3.3. First of all we recover the *a priori* bounds that come as a direct consequence of the energy inequality (15). Then we get stronger estimates by exploiting the structure of the system. As we will see later on, the density fluctuation $\frac{\rho^\varepsilon-1}{\varepsilon}$ satisfies a wave equation. The use of the dispersive estimate (8) will give us further bounds.

4.1. *Consequences of the energy estimate*

In this section we recover all the *a priori* bounds that are a consequence of the energy inequality (15). Before going on let us define the density fluctuation

$$\sigma^\varepsilon = \frac{\rho^\varepsilon - 1}{\varepsilon}. \tag{22}$$

Proposition 4.1. *Let us consider the solution $(\rho^\varepsilon, u^\varepsilon, V^\varepsilon)$ of the Cauchy problem for system (13). Assume that the hypotheses (ID) hold, then it follows that*

$$\sigma^\varepsilon \quad \text{is bounded in } L^\infty([0,T]; L^k_2(\mathbb{R}^3)), \qquad \text{where } k = \min(\gamma, 2), \tag{23}$$

$$\nabla u^\varepsilon \text{ is bounded in } L^2([0,T]\times\mathbb{R}^3), \tag{24}$$

$$u^\varepsilon \quad \text{is bounded in } L^2([0,T]\times\mathbb{R}^3) \cap L^2([0,T]; L^6(\mathbb{R}^3)), \tag{25}$$

$$\sigma^\varepsilon u^\varepsilon \text{ is bounded in } L^2([0,T]; H^{-1}(\mathbb{R}^3)). \tag{26}$$

Proof. From (15) it follows that $\pi^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}^3))$. By taking into account that the function $z \to z^\gamma - 1 - \gamma(z-1)$ is convex and by following the same line of arguments as in [22] we get when $\gamma < 2$ that

$$\sup_{t\geqslant 0} \int_{\mathbb{R}^3} \left\{ |\rho^\varepsilon - 1|^2 \chi_{|\rho^\varepsilon-1|\leqslant 1/2} + |\rho^\varepsilon - 1|^\gamma \chi_{|\rho^\varepsilon-1|\geqslant 1/2} \right\}(t,x)\,\mathrm{d}x \leqslant C\varepsilon^2 \tag{27}$$

and when $\gamma \geqslant 2$,

$$\sup_{t\geqslant 0} \int_{\mathbb{R}^3} |\rho^\varepsilon - 1|^2 (t,x)\,\mathrm{d}x \leqslant C\varepsilon^2, \tag{28}$$

so we can conclude that σ^ε is uniformly bounded in ε in $L^\infty([0,T]; L^k_2(\mathbb{R}^3))$, where $k = \min(\gamma, 2)$. Equation (24) is a consequence of (15). The fact that $u^\varepsilon \in L^2([0,T]; L^6(\mathbb{R}^3))$ follows from (24) and by Sobolev's embeddings. Now we prove $u^\varepsilon \in L^2([0,T]\times\mathbb{R}^3)$:

$$\begin{aligned} \int_{\mathbb{R}^3} |u^\varepsilon|^2\,\mathrm{d}x &= \int_{\mathbb{R}^3} \left\{ |u^\varepsilon|^2 \chi_{|\rho^\varepsilon-1|\leqslant 1/2} + |u^\varepsilon|^2 \chi_{|\rho^\varepsilon-1|\geqslant 1/2} \right\}(x)\,\mathrm{d}x \\ &\leqslant 2\int_{\mathbb{R}^3} \rho^\varepsilon |u^\varepsilon|^2\,\mathrm{d}x + 2\|\rho^\varepsilon - 1\|_{L^k_x} \|u^\varepsilon\|^2_{L^{2k/(k-1)}_x} \\ &\leqslant C_0 + \varepsilon^{2/k} \|u^\varepsilon\|^{2-\frac{3}{k}}_{L^2_x} \|\nabla u^\varepsilon\|^{\frac{3}{k}}_{L^2_x}. \end{aligned} \tag{29}$$

Now, by using (24), we get from (29) the estimate (25). Recalling that $\gamma > 3/2$ and by interpolating we get that $u^\varepsilon \in L^2([0,T]; L^4(\mathbb{R}^3) \cap L^{2\gamma/(\gamma-1)}(\mathbb{R}^3))$. By using (23) we obtain that $\rho^\varepsilon u^\varepsilon$ is uniformly bounded in $L^2([0,T]; L^{4/3}(\mathbb{R}^3) + L^{2k/(k+1)}(\mathbb{R}^3))$. Therefore by Sobolev's embeddings we get (26). □

We want to complete this paragraph with a remark concerning the regularity of the initial data.

Remark 4.2. With the same procedure as for σ^ε, taking into account (ID) we get that $\sigma^\varepsilon{}_0$ is bounded in $L^k_2(\mathbb{R}^3)$, hence in $H^{-1}(\mathbb{R}^3)$, since $\gamma > 3/2$. If we rewrite m_0^ε in the following way

$$m_0^\varepsilon = \frac{m_0^\varepsilon}{\sqrt{\rho^\varepsilon{}_0}}\sqrt{\rho^\varepsilon{}_0}\chi_{|\rho^\varepsilon{}_0-1|\leqslant 1/2} + \frac{m_0^\varepsilon}{\sqrt{\rho^\varepsilon{}_0}}\frac{\sqrt{\rho^\varepsilon{}_0}}{\sqrt{|\rho^\varepsilon{}_0-1|}}\sqrt{|\rho^\varepsilon{}_0-1|}\chi_{|\rho^\varepsilon{}_0-1|>1/2},$$

we get that m_0^ε is bounded in $L^2(\mathbb{R}^3) + L^{2k/(k+1)}(\mathbb{R}^3)$ and hence in $H^{-1}(\mathbb{R}^3)$. Finally we can conclude that

$$\sigma_0^\varepsilon,\ m_0^\varepsilon \text{ are bounded in } H^{-1}(\mathbb{R}^3) \text{ uniformly in } \varepsilon. \tag{30}$$

4.2. Density fluctuation wave equation

In this section we wish to recover more refined bounds on σ^ε. As we will see, σ^ε will satisfy a wave equation; this will allow us to use the Strichartz estimate (8). First of all we rewrite system (13) in the following way:

$$\partial_t\sigma^\varepsilon + \frac{1}{\varepsilon}\mathrm{div}\,(\rho^\varepsilon u^\varepsilon) = 0, \tag{31}$$

$$\partial_t(\rho^\varepsilon u^\varepsilon) + \frac{1}{\varepsilon}\nabla\sigma^\varepsilon = \mu\Delta u^\varepsilon + (\nu+\mu)\nabla\,\mathrm{div}\,u^\varepsilon - \mathrm{div}\,(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon) - (\gamma-1)\nabla\pi^\varepsilon + \frac{\sigma^\varepsilon}{\varepsilon}\nabla V^\varepsilon + \frac{1}{\varepsilon^2}\nabla V^\varepsilon, \tag{32}$$

$$\varepsilon^{\beta-1}\Delta V^\varepsilon = \sigma^\varepsilon. \tag{33}$$

Then, by differentiating with respect to time equation (31) and taking the divergence of (32) we get that σ^ε satisfies the following nonhomogeneous wave equation:

$$\varepsilon^2\partial_{tt}\sigma^\varepsilon - \Delta\sigma^\varepsilon = -\varepsilon^2\mathrm{div}\,(\mu\Delta u^\varepsilon + (\nu+\mu)\nabla\mathrm{div}\,u^\varepsilon) + \varepsilon^2\mathrm{div}\,\mathrm{div}\,(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon) + \varepsilon^2(\gamma-1)\,\mathrm{div}\,\nabla\pi^\varepsilon - \varepsilon\,\mathrm{div}\,(\sigma^\varepsilon\nabla V^\varepsilon) - \mathrm{div}\,\nabla V^\varepsilon. \tag{34}$$

Now we rescale the time variable, the density fluctuation, the velocity and the potential in the following way:

$$\tau = \frac{t}{\varepsilon}, \tag{35}$$

$$\tilde{u}(x,\tau) = u^\varepsilon(x,\varepsilon\tau), \qquad \tilde{\rho}(x,t) = \rho^\varepsilon(x,\varepsilon\tau),$$
$$\tilde{\sigma}(x,\tau) = \sigma^\varepsilon(x,\varepsilon\tau), \qquad \tilde{V}(x,\tau) = V^\varepsilon(x,\varepsilon\tau). \tag{36}$$

As a consequence of this scaling wave equation (34) becomes

$$\partial_{\tau\tau}\tilde{\sigma} - \Delta\tilde{\sigma} = -\varepsilon^2\,\mathrm{div}\,(\mu\Delta\tilde{u} + (\nu+\mu)\nabla\,\mathrm{div}\,\tilde{u}) + \varepsilon^2\,\mathrm{div}\,(\mathrm{div}\,(\tilde{\rho}\tilde{u}\otimes\tilde{u}) + (\gamma-1)\nabla\tilde{\pi}) - \varepsilon\,\mathrm{div}\,(\tilde{\sigma}\nabla\tilde{V}) - \mathrm{div}\,\nabla\tilde{V}. \tag{37}$$

Now we consider $\tilde{\sigma} = \tilde{\sigma}_1 + \tilde{\sigma}_2 + \tilde{\sigma}_3 + \tilde{\sigma}_4$ where $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\sigma}_4$ solve the following wave equations:

$$\begin{cases} \partial_{\tau\tau}\tilde{\sigma}_1 - \Delta\tilde{\sigma}_1 = -\varepsilon^2\,\mathrm{div}\,(\mu\Delta\tilde{u} + (\nu+\mu)\nabla\mathrm{div}\,\tilde{u}) = \varepsilon^2 F_1, \\ \tilde{\sigma}_1(x,0) = \tilde{\sigma}(x,0) = \tilde{\sigma}_0 \qquad \partial_\tau\tilde{\sigma}_1(x,0) = \partial_\tau\tilde{\sigma}(x,0) = \partial_t\tilde{\sigma}_0, \end{cases} \tag{38}$$

$$\begin{cases} \partial_{\tau\tau}\tilde{\sigma}_2 - \Delta\tilde{\sigma}_2 = \varepsilon^2\,\mathrm{div}\,(\mathrm{div}\,(\tilde{\rho}\tilde{u}\otimes\tilde{u}) + (\gamma-1)\nabla\tilde{\pi}) = \varepsilon^2 F_2, \\ \tilde{\sigma}_2(x,0) = \partial_\tau\tilde{\sigma}_2(x,0) = 0, \end{cases} \tag{39}$$

$$\begin{cases} \partial_{\tau\tau}\tilde\sigma_3 - \Delta\tilde\sigma_3 = -\varepsilon \operatorname{div}(\tilde\sigma\nabla\tilde V) = \varepsilon F_3, \\ \tilde\sigma_3(x,0) = \partial_\tau\tilde\sigma_3(x,0) = 0, \end{cases} \tag{40}$$

$$\begin{cases} \partial_{\tau\tau}\tilde\sigma_4 - \Delta\tilde\sigma_4 = -\operatorname{div}(\nabla\tilde V) = F_4, \\ \tilde\sigma_4(x,0) = \partial_\tau\tilde\sigma_4(x,0) = 0. \end{cases} \tag{41}$$

We are able to prove the following theorem.

Theorem 4.3. *Let us consider the solutions $(\rho^\varepsilon, u^\varepsilon, V^\varepsilon)$ of the Cauchy problem for system (13). Assume that the hypotheses (ID) hold. Then for any $s_0 \geqslant 3/2$, the following estimate holds:*

$$\begin{aligned} \varepsilon^{-\frac14+\frac\beta2}\|\sigma^\varepsilon\|_{L^4_tW^{-s_0-2,4}_x} + \varepsilon^{\frac34+\frac\beta2}\|\partial_t\sigma^\varepsilon\|_{L^4_tW^{-s_0-3,4}_x} &\lesssim \varepsilon^{\frac\beta2}\|\sigma^\varepsilon{}_0\|_{H^{-1}_x} + \varepsilon^{\frac\beta2}\|m^\varepsilon_0\|_{H^{-1}_x} \\ &+ \varepsilon^{1+\frac\beta2}T\|\operatorname{div}(\operatorname{div}(\sigma^\varepsilon u^\varepsilon\otimes u^\varepsilon) - (\gamma-1)\nabla\pi^\varepsilon)\|_{L^\infty_tH^{-s_0-2}_x} \\ &+ \varepsilon^{1+\frac\beta2}\|\operatorname{div}\Delta u^\varepsilon + \nabla\operatorname{div} u^\varepsilon\|_{L^2_tH^{-2}_x} \\ &+ T\|\operatorname{div}\nabla V^\varepsilon\|_{L^\infty_tH^{-1}_x} + \varepsilon^{1+\frac\beta2}T\|\varepsilon^{\beta-2}\operatorname{div}(\sigma^\varepsilon V^\varepsilon)\|_{L^\infty_tH^{-s_0-1}_x}. \end{aligned} \tag{42}$$

Proof. Since $\tilde\sigma_1$, $\tilde\sigma_2$, $\tilde\sigma_3$, $\tilde\sigma_4$ are solutions of the wave equations (38)–(41) we can apply the Strichartz estimate (8) with $(x,\tau)\in\mathbb{R}^3\times(0,T/\varepsilon)$. We start with $\tilde\sigma_1$. From (24) we deduce that $F_1\in L^2_tH^{-2}_x$, so by using (8) with $s=2$ we get

$$\|\tilde\sigma_1\|_{L^4_\tau W^{-2,4}_x} + \|\partial_\tau\tilde\sigma_1\|_{L^4_\tau W^{-3,4}_x} \lesssim \|\tilde\sigma_0\|_{H^{-3/2}_x} + \|\partial_\tau\tilde\sigma_0\|_{H^{-5/2}_x} + \varepsilon^{3/2}T\|\operatorname{div}\Delta\tilde u + \nabla\operatorname{div}\tilde u\|_{L^2_\tau H^{-2}_x}. \tag{43}$$

From estimate (13) we have that $\tilde\sigma|\tilde u|^2$, $\tilde\pi\in L^\infty_\tau L^1_x$, but L^1 is continuously embedded in H^{-s_0}, $s_0\geqslant 3/2$, so we have that $F_2\in L^\infty_\tau H^{-s_0-2}_x$. If we apply (8) to $\tilde\sigma_2$ we obtain for any $s_0\geqslant 3/2$

$$\|\tilde\sigma_2\|_{L^4_\tau W^{-s_0-2,4}_x} + \|\partial_\tau\tilde\sigma_2\|_{L^4_\tau W^{-s_0-3,4}_x} \lesssim \varepsilon^{3/2}T\|\operatorname{div}(\operatorname{div}(\tilde\rho\tilde u\otimes\tilde u) + \nabla\tilde\pi)\|_{L^\infty_\tau H^{-s_0-2}_x}. \tag{44}$$

Using Poisson equation (33) we can rewrite F_3 as $F_3 = \varepsilon^{\beta-1}(\operatorname{div}(\nabla\tilde V\otimes\nabla\tilde V) + \frac12\nabla|\nabla\tilde V|^2)$. Taking into account (15), as for F_2, we get $\varepsilon^{-1}F_3\in L^\infty_\tau H^{-s_0-1}_x$, for any $s_0\geqslant 3/2$. Hence $\tilde\sigma_3$ satisfies

$$\|\tilde\sigma_3\|_{L^4_\tau W^{-s_0-1,4}_x} + \|\partial_\tau\tilde\sigma_3\|_{L^4_\tau W^{-s_0-2,4}_x} \lesssim \varepsilon T\|\varepsilon^{\beta-2}\operatorname{div}(\nabla\tilde V\otimes\nabla\tilde V) + \tfrac12\nabla|\nabla\tilde V|^2\|_{L^\infty_\tau H^{-s_0-1}_x}. \tag{45}$$

For $F_4 = -\operatorname{div}\nabla\tilde V$, using again (13), we have $\varepsilon^{\beta/2-1}F_4\in L^\infty_\tau H^{-1}_x$ and so $\tilde\sigma_4$ verifies the following estimate:

$$\|\tilde\sigma_4\|_{L^4_\tau W^{-1,4}_x} + \|\partial_\tau\tilde\sigma_4\|_{L^4_\tau W^{-2,4}_x} \lesssim \varepsilon^{-\frac\beta2}T\|\varepsilon^{\frac\beta2-1}\operatorname{div}(\nabla V)\|_{L^\infty_\tau H^{-1}_x}. \tag{46}$$

Summing up (43), (44), (45) and (46), $\tilde\sigma$ verifies

$$\begin{aligned} \|\sigma^\varepsilon\|_{L^4_tW^{-s_0-2,4}_x} + \|\partial_t\sigma^\varepsilon\|_{L^4_tW^{-s_0-3,4}_x} &\lesssim \|\tilde\sigma_1\|_{L^4_\tau W^{-2,4}_x} + \|\partial_\tau\tilde\sigma_1\|_{L^4_\tau W^{-3,4}_x} \\ &+ \|\tilde\sigma_2\|_{L^4_\tau W^{-s_0-2,4}_x} + \|\partial_\tau\tilde\sigma_2\|_{L^4_\tau W^{-s_0-3,4}_x} \\ &+ \|\tilde\sigma_3\|_{L^4_\tau W^{-s_0-1,4}_x} + \|\partial_\tau\tilde\sigma_3\|_{L^4_\tau W^{-s_0-2,4}_x} \\ &+ \|\tilde\sigma_4\|_{L^4_\tau W^{-1,4}_x} + \|\partial_\tau\tilde\sigma_4\|_{L^4_\tau W^{-2,4}_x} \\ &\lesssim \|\tilde\sigma_0\|_{H^{-3/2}_x} + \|\partial_\tau\tilde\sigma_0\|_{H^{-5/2}_x} \\ &+ \varepsilon^{3/2}T\|\operatorname{div}\Delta\tilde u + \nabla\operatorname{div},\tilde u\|_{L^2_\tau H^{-2}_x} \\ &+ \varepsilon^{3/2}T\|\operatorname{div}(\operatorname{div}(\tilde\rho\tilde u\otimes\tilde u) + \nabla\tilde\pi)\|_{L^\infty_\tau H^{-s_0-2}_x} \\ &+ \varepsilon T\|\varepsilon^{\beta-2}\operatorname{div}(\nabla\tilde V\otimes\nabla\tilde V) + \frac12\nabla|\nabla\tilde V|^2\|_{L^\infty_\tau H^{-s_0-1}_x} \\ &+ \varepsilon^{-\frac\beta2}T\|\varepsilon^{\frac\beta2-1}\operatorname{div}(\nabla V)\|_{L^\infty_\tau H^{-1}_x}. \end{aligned} \tag{47}$$

Finally, since

$$\|\tilde{\sigma}\|_{L^p((0,T/\varepsilon);L^q(\mathbb{R}^3))} = \varepsilon^{-1/p}\|\sigma^\varepsilon\|_{L^p([0,T];L^q(\mathbb{R}^3))}$$

and using (30), we end up with (42). □

5. Strong convergence

In this section we will study the strong convergence of the velocity field u^ε. This will be achieved by separately studying the convergence of the divergence free vector field Pu^ε and the gradient vector field Qu^ε.

5.1. Strong convergence of Qu^ε

Here we prove the convergence of Qu^ε to 0. In particular we will use estimate (42) combined with the Young type inequalities (9) and (10). As we will see, to get this strong convergence we need to impose some restrictions on the values of β.

Proposition 5.1. *Let us consider the solution $(\rho^\varepsilon, u^\varepsilon, V^\varepsilon)$ of the Cauchy problem for system (13). Assume that the hypotheses (ID) hold and $\beta < 1/2$. Then as $\varepsilon \downarrow 0$,*

$$Qu^\varepsilon \longrightarrow 0 \quad \text{strongly in } L^2([0,T];L^p(\mathbb{R}^3)) \text{ for any } p \in [4,6)\,. \tag{48}$$

Proof. In order to prove proposition 5.1 we split Qu^ε as follows:

$$\|Qu^\varepsilon\|_{L_t^2L_x^p} \leqslant \|Qu^\varepsilon - Qu^\varepsilon * j_\alpha\|_{L_t^2L_x^p} + \|Qu^\varepsilon * j_\alpha\|_{L_t^2L_x^p} = J_1 + J_2,$$

where j_α is the smoothing kernel defined in lemma 2.1. Now we separately estimate J_1 and J_2. For J_1 using (9) we get

$$J_1 \leqslant \alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|\nabla u^\varepsilon\|_{L_{t,x}^2}. \tag{49}$$

To estimate J_2 we take into account definition (22) and so we split J_2 as

$$J_2 \leqslant \varepsilon\|Q(\sigma^\varepsilon u^\varepsilon) * j_\alpha\|_{L_t^2L_x^p} + \|Q(\rho^\varepsilon u^\varepsilon) * j_\alpha\|_{L_t^2L_x^p} = J_{2,1} + J_{2,2}. \tag{50}$$

For $J_{2,1}$ we use (26) and (10), so we have

$$J_{2,1} \leqslant \varepsilon\alpha^{-1-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|\sigma^\varepsilon u^\varepsilon\|_{L_t^2H_x^{-1}}. \tag{51}$$

From the identity $Q(\rho^\varepsilon u^\varepsilon) = \varepsilon\nabla\Delta^{-1}\partial_t\sigma^\varepsilon$ and by inequality (10) we get that $J_{2,2}$ satisfies the following estimate:

$$\begin{aligned} J_2 &= \varepsilon^{\frac{1}{4}-\frac{\beta}{2}}\|\nabla\Delta^{-1}\varepsilon^{\frac{3}{4}+\frac{\beta}{2}}\partial_t\sigma^\varepsilon * j\|_{L_t^2L_x^p} \\ &\leqslant \varepsilon^{\frac{1}{4}-\frac{\beta}{2}}\alpha^{-s_0-4-3\left(\frac{1}{4}-\frac{1}{p}\right)}\|\varepsilon^{\frac{1}{4}-\frac{\beta}{2}}\partial_t\sigma^\varepsilon\|_{L_t^2W_x^{-s_0-4,4}} \\ &\leqslant \varepsilon^{\frac{1}{4}-\frac{\beta}{2}}\alpha^{-s_0-4-3\left(\frac{1}{4}-\frac{1}{p}\right)}T^{1/4}\|\varepsilon^{\frac{1}{4}-\frac{\beta}{2}}\partial_t\sigma^\varepsilon\|_{L_t^4W_x^{-s_0-4,4}}. \end{aligned} \tag{52}$$

Now, summing up (50), (51) and (52) we get

$$\|Qu^\varepsilon\|_{L_t^2L_x^p} \lesssim C\alpha^{1-3\left(\frac{1}{2}-\frac{1}{p}\right)} + C_T\varepsilon^{\frac{1}{4}-\frac{\beta}{2}}\alpha^{-s_0-4-3\left(\frac{1}{4}-\frac{1}{p}\right)}, \tag{53}$$

where $1/4 - \beta/2 > 0$ since $\beta < 1/2$. Finally, we choose α in terms of ε, for example, in a way that the two terms on the right-hand side of the inequality (53) are of the same order, namely,

$$\alpha = \varepsilon^{\frac{1-2\beta}{17+s_0}}. \tag{54}$$

Therefore, we obtain

$$\|Qu^\varepsilon\|_{L_t^2L_x^p} \leqslant C_T\varepsilon^{\frac{p-1}{6p}\frac{1-2\beta}{17+s_0}} \qquad \text{for any } p \in [4,6).$$

□

5.2. Strong convergence of Pu^ε

It remains to prove the strong compactness of the incompressible component of the velocity field. To achieve this goal we need to recall here the following theorem (see [27]).

Theorem 5.2. *Let $\mathcal{F} \subset L^p([0,T]; B)$, $1 \leqslant p < \infty$, B be a Banach space. $\mathcal{F}$ is relatively compact in $L^p([0,T]; B)$ for $1 \leqslant p < \infty$ or in $C([0,T]; B)$ for $p = \infty$ if and only if*

(i) $\left\{\int_{t_1}^{t_2} f(t)\mathrm{d}t,\ f \in \mathcal{F}\right\}$ is relatively compact in B, $0 < t_1 < t_2 < T$,
(ii) $\lim_{h\to 0} \|f(t+h) - f(t)\|_{L^p([0,T-h];B)} = 0$ uniformly for any $f \in \mathcal{F}$.

The compactness can be obtained by looking at some time regularity properties of Pu^ε and by using theorem 5.2, but before that we need to prove the following lemma.

Lemma 5.3. *Let us consider the solution $(\rho^\varepsilon, u^\varepsilon)$ of the Cauchy problem for system (13). Assume that the hypotheses (ID) hold. Then for all $h \in (0,1)$, we have*

$$\|Pu^\varepsilon(t+h) - Pu^\varepsilon(t)\|_{L^2([0,T]\times\mathbb{R}^3)} \leqslant C_T(h^{1/5} + \varepsilon^{1/2}). \tag{55}$$

Proof. Let us set $z^\varepsilon = u^\varepsilon(t+h) - u^\varepsilon(t)$; we have

$$\begin{aligned}\|Pu^\varepsilon(t+h) - Pu^\varepsilon(t)\|^2_{L^2_{t,x}} &= \int_0^T\int_{\mathbb{R}^3} \mathrm{d}t\,\mathrm{d}x(Pz^\varepsilon)\cdot(Pz^\varepsilon - Pz^\varepsilon * j_\alpha)\\ &\quad + \int_0^T\int_{\mathbb{R}^3} \mathrm{d}t\,\mathrm{d}x(Pz^\varepsilon)\cdot(Pz^\varepsilon * j_\alpha) = I_1 + I_2.\end{aligned} \tag{56}$$

Using (9) together with (25) we can estimate I_1 in the following way:

$$I_1 \leqslant \|Pz^\varepsilon\|_{L^2_{t,x}}\|Pz^\varepsilon(t) - (Pz^\varepsilon * j_\alpha)(t)\|_{L^2} \lesssim \alpha\|u^\varepsilon\|_{L^2_{t,x}}\|\nabla u^\varepsilon\|_{L^2_{t,x}}. \tag{57}$$

In order to estimate I_2 we split it as follows:

$$\begin{aligned}I_2 &= \int_0^T\int_{\mathbb{R}^3} \mathrm{d}t\,\mathrm{d}x\,P(\rho^\varepsilon z^\varepsilon)\cdot(Pz^\varepsilon * j_\alpha) + \varepsilon\int_0^T\int_{\mathbb{R}^3} \mathrm{d}t\,\mathrm{d}x\,P(\sigma^\varepsilon z^\varepsilon)\cdot(Pz^\varepsilon * j_\alpha)\\ &= I_{2,1} + I_{2,2}.\end{aligned} \tag{58}$$

$I_{2,2}$ can be estimated by taking into account (25) and (26), so we have

$$I_{2,2} = \varepsilon\|u^\varepsilon\|_{L^2([0,T];L^4(\mathbb{R}^3)\cap L^{2k/k-1}(\mathbb{R}^3))}\|\sigma^\varepsilon u^\varepsilon\|_{L^2([0,T];L^{4/3}(\mathbb{R}^3)+L^{2k/k+1})} \lesssim \varepsilon. \tag{59}$$

Now we estimate $I_{2,1}$. Let us reformulate $P(\rho^\varepsilon z^\varepsilon)$ in the integral form by using equation $(13)_2$ and Poisson equation $(13)_3$; hence

$$\begin{aligned}I_{2,1} &\leqslant \left|\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\mathrm{d}x\int_t^{t+h}\mathrm{d}s(\mathrm{div}(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon) + \Delta u^\varepsilon)(s,x)\cdot(Pz^\varepsilon * j_\alpha)(t,x)\right|\\ &\quad + \left|\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\mathrm{d}x\int_t^{t+h}\mathrm{d}s\,P\left(\frac{\sigma^\varepsilon}{\varepsilon}\nabla V^\varepsilon\right)(s,x)\cdot(Pz^\varepsilon * j_\alpha)(t,x)\right|\\ &= \left|\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\mathrm{d}x\int_t^{t+h}\mathrm{d}s(\mathrm{div}(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon) + \Delta u^\varepsilon)\cdot(Pz^\varepsilon * j_\alpha)(t,x)\right|\\ &\quad + \left|\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\mathrm{d}x\int_t^{t+h}\mathrm{d}s\,\varepsilon^{\beta-2}\,\mathrm{div}(\nabla V^\varepsilon\otimes\nabla V^\varepsilon))(s,x)\cdot(Pz^\varepsilon * j_\alpha)(t,x)\right|.\end{aligned} \tag{60}$$

Then, by integrating by parts, using (10) with $s = 0$, $p = \infty$, $q = 2$, we deduce

$$I_{2,1} \leqslant h\|\nabla u^\varepsilon\|^2_{L^2_{t,x}} + \int_0^T \|\nabla z^\varepsilon * j_\alpha\|_{L^\infty_x}(t,x)\,\mathrm{d}x \int_{\mathbb{R}^3}\int_t^{t+h} (\rho^\varepsilon|u^\varepsilon|^2 + \varepsilon^{\beta-2}|\nabla V^\varepsilon|^2)(s,x)\,\mathrm{d}s$$

$$\leqslant h\|\nabla u^\varepsilon\|^2_{L^2_{t,x}} + C\alpha^{-3/2}T^{1/2}h\|\nabla u^\varepsilon\|_{L^2_{t,x}}\left(\|\rho^\varepsilon|u^\varepsilon|^2\|_{L^\infty_t L^1_x} + \|\varepsilon^{\beta-2}|\nabla V^\varepsilon|^2\|_{L^\infty_t L^1_x}\right). \quad (61)$$

Summing up I_1, $I_{2,1}$, $I_{2,2}$ and taking into account (15) we have

$$\|Pu^\varepsilon(t+h) - Pu^\varepsilon(t)\|^2_{L^2([0,T]\times\mathbb{R}^3)} \leqslant C(\alpha + h + h\alpha^{-3/2}T^{1/2} + \varepsilon), \quad (62)$$

by choosing $\alpha = h^{2/5}$, we end up with (55). □

Corollary 5.4. *Let us consider the solution $(\rho^\varepsilon, u^\varepsilon)$ of the Cauchy problem for system (13). Assume that the hypotheses (ID) hold. Then as $\varepsilon \downarrow 0$*

$$Pu^\varepsilon \longrightarrow Pu, \qquad \text{strongly in } L^2(0,T; L^2_{\mathrm{loc}}(\mathbb{R}^3)). \quad (63)$$

Proof. Using lemma 5.3, theorem 5.2 and proposition 5.1 we get (63). □

6. Proof of the main theorem (3.3)

(i) It follows from (23).
(ii) It follows from (24).
(iii) It is a consequence of proposition 5.1.
(iv) By taking into account the decomposition $u^\varepsilon = Pu^\varepsilon + Qu^\varepsilon$, corollary 5.4 and proposition 5.1 we have that

$$Pu^\varepsilon \longrightarrow u \qquad \text{strongly in } L^2([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^3)).$$

(v) First of all, let us apply the Leray projector P to the momentum equation (13_2), then it follows that

$$\partial_t P(\rho^\varepsilon u^\varepsilon) + P\,\mathrm{div}\,(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) = \mu\Delta Pu^\varepsilon + P\left(\frac{\rho^\varepsilon - 1}{\varepsilon^2}\nabla V^\varepsilon\right). \quad (64)$$

It is a straightforward computation to pass into the limit in the terms $\partial_t P(\rho^\varepsilon u^\varepsilon)$, $\mu\Delta Pu^\varepsilon$, so, for any $\varphi \in \mathcal{D}([0,T]\times\mathbb{R}^3)$ as $\varepsilon \downarrow 0$, we get

$$\langle P(\partial_t(\rho^\varepsilon u^\varepsilon) - \mu\Delta u^\varepsilon), \varphi\rangle \longrightarrow \langle P(\partial_t u - \mu\Delta u), \varphi\rangle. \quad (65)$$

For the part $P\mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon)$, if we take into account (i) and (48), we have, as $\varepsilon \downarrow 0$,

$$\begin{aligned}\langle P\mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon), \varphi\rangle &= \langle \mathrm{div}((\rho^\varepsilon - 1)u^\varepsilon \otimes u^\varepsilon), P\varphi\rangle \\ &\quad + \langle \mathrm{div}(Pu^\varepsilon \otimes Pu^\varepsilon), P\varphi\rangle + \langle \mathrm{div}(Qu^\varepsilon \otimes Qu^\varepsilon), P\varphi\rangle \\ &\quad + \langle \mathrm{div}(Pu^\varepsilon \otimes Qu^\varepsilon), P\varphi\rangle + \langle \mathrm{div}(Qu^\varepsilon \otimes Qu^\varepsilon), P\varphi\rangle \\ &\rightarrow \langle \mathrm{div}(u \otimes u), P\varphi\rangle = \langle P\mathrm{div}(u\otimes u), \varphi\rangle. \end{aligned} \quad (66)$$

The only term missing in the convergence is $P\left(\frac{\rho^\varepsilon - 1}{\varepsilon^2}\nabla V^\varepsilon\right)$. In order to study it we have to proceed in a different way if $\gamma < 2$ or if $\gamma \geqslant 2$.

Case $\frac{3}{2} \leqslant \gamma < 2$. From (27) we have that $\frac{\rho^\varepsilon - 1}{\varepsilon} \in L^\infty_t L^\gamma(\mathbb{R}^3)$, so it follows that

$$\left\|\frac{\rho^\varepsilon - 1}{\varepsilon^2}\nabla V^\varepsilon\right\|_{L^\infty_t L^{\frac{2\gamma}{\gamma+2}}_x} \leqslant \frac{1}{\varepsilon^2}\|\rho^\varepsilon - 1\|_{L^\infty_t L^\gamma_x}\|\nabla V^\varepsilon\|_{L^\infty_t L^2_x} \leqslant C\varepsilon^{-2+\frac{2}{\gamma}+1-\frac{\beta}{2}}$$

$$\lesssim \varepsilon^{-1+\frac{2}{\gamma}+\frac{\beta}{2}} \longrightarrow 0 \qquad \text{as } \varepsilon \to 0, \quad (67)$$

provided that $\beta < -2 + 4/\gamma$. It is obvious now that as $\varepsilon \downarrow 0$

$$\langle P\left(\frac{\rho^\varepsilon - 1}{\varepsilon^2}\nabla V^\varepsilon\right), \varphi\rangle \longrightarrow 0 \tag{68}$$

Case $\gamma \geqslant 2$. First of all let us apply inequalities (9) and (10) to $f = \Delta^{-1/2}\sigma^\varepsilon$; in the case $s = s_0 + 1$, $s_0 > 3/2$, $p = 4$, for any $4 \leqslant q < 6$, we have

$$\|\sigma^\varepsilon\|_{W_x^{-1,q}} \leqslant \alpha^{1-3\left(\frac{1}{2}-\frac{1}{q}\right)}\|\sigma^\varepsilon\|_{L_x^2} + \alpha^{-s_0-1-3\left(\frac{1}{4}-\frac{1}{q}\right)}\|\sigma^\varepsilon\|_{W_x^{-s_0-2,4}}. \tag{69}$$

By taking into account (23) and (42) we have

$$\|\sigma^\varepsilon\|_{L_t^4 W_x^{-1,q}} \leqslant C\alpha^{1-3\left(\frac{1}{2}-\frac{1}{q}\right)} + C\varepsilon^{\frac{1}{4}-\frac{\beta}{2}}\alpha^{-s_0-1-3\left(\frac{1}{4}-\frac{1}{q}\right)}. \tag{70}$$

Now, if $\varphi \in \mathcal{D}([0, T] \times \mathbb{R}^3)$, we have

$$\begin{aligned}\left|\left\langle P\left(\frac{\sigma^\varepsilon}{\varepsilon}\nabla V^\varepsilon\right), \varphi\right\rangle\right| &\leqslant \frac{1}{\varepsilon}\|\sigma^\varepsilon\|_{L_t^4 W_x^{-1,q}}\|\nabla V^\varepsilon\varphi\|_{L_t^{4/3}W_x^{1,q'}} \\ &\leqslant \frac{1}{\varepsilon}\|\sigma^\varepsilon\|_{L_t^4 W_x^{-1,q}} + (\|\nabla V^\varepsilon\|_{L_t^\infty L_x^2} + \|\Delta V^\varepsilon\|_{L_t^\infty L_x^2})\|\varphi\|_{L_t^{4/3}L_x^{2q/q-1}}.\end{aligned} \tag{71}$$

Using (15) and the elliptic regularity, (71) becomes

$$\begin{aligned}\left|\left\langle P\left(\frac{\sigma^\varepsilon}{\varepsilon}\nabla V^\varepsilon\right), \varphi\right\rangle\right| &\leqslant \frac{1}{\varepsilon}\left(\alpha^{1-3\left(\frac{1}{2}-\frac{1}{q}\right)} + C\varepsilon^{\frac{1}{4}-\frac{\beta}{2}}\alpha^{-s_0-1-3\left(\frac{1}{4}-\frac{1}{q}\right)}\right)(\varepsilon^{1-\frac{\beta}{2}} - \varepsilon^{1-\beta}) \\ &\lesssim \varepsilon^{-\beta}\alpha^{1-3\left(\frac{1}{2}-\frac{1}{q}\right)} + \varepsilon^{\frac{1}{4}-\frac{3}{2}\beta}\alpha^{-s_0-1-3\left(\frac{1}{4}-\frac{1}{q}\right)}.\end{aligned} \tag{72}$$

Now, if we choose α in terms of ε, namely, $\alpha = \varepsilon^\mu$, $\mu > 0$ and β is such that

$$0 < \beta < \min\left\{\frac{1}{2}, \mu\left(\frac{3}{q} - \frac{1}{2}\right), \frac{1}{6} - \frac{2}{3}\mu\left(s_0 + \frac{7}{4} - \frac{3}{q}\right)\right\},$$

we get

$$\left|\left\langle P\left(\frac{\sigma^\varepsilon}{\varepsilon}\nabla V^\varepsilon\right), \varphi\right\rangle\right| \to 0, \qquad \text{as } \varepsilon \to 0, \tag{73}$$

provided that $\beta < 1/(22 + 16s_0)$. So, using (65) and (66) together with (68) or (73), we have that u satisfies the following equation in $\mathcal{D}'([0, T] \times \mathbb{R}^3)$:

$$P(\partial_t u - \Delta u + (u \cdot \nabla)u) = 0. \tag{74}$$

provided that

$$\begin{cases}\beta = \min\left\{\frac{1}{2}, -2 + \frac{4}{\gamma}\right\} & \text{if } \gamma < 2, \\ 0 < \beta < \min\left\{\frac{1}{2}, \mu\left(\frac{3}{q} - \frac{1}{2}\right), \frac{1}{6} - \frac{2}{3}\mu\left(s_0 + \frac{7}{4} - \frac{3}{q}\right)\right\} & \text{if } \gamma \geqslant 2.\end{cases}$$

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